RESONANCES IN COMPOUND PROCESSES

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Abstract

The first-exit time of a compound process with strictly positive jumps reaching a horizontal barrier is considered. The first-exit time distribution for the specific case of Poisson arrivals and gamma distributed jump sizes is derived. If the jump size distribution converges weakly to a Dirac delta function as the variance tends to zero, the process tends to a compound process with constant jump size. In the case when the barrier is an exact multiple of the constant jump size a small peculiarity arises; the first-exit time distribution with general jumps does not tend to the first-exit time distribution with constant jumps. The first-exit time distribution for M/G/1 queues with gamma distributed service times is shown to have the same peculiarity.

Keywords: first-exit time, compound G/G processes, resonance, M/G/1 queue,

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1 Introduction

Consider a compound process, G/G. The interarrival times, $\{T_i\}_{i\in\mathbb{Z}^+}$, are independent, identically distributed (iid) random variables from a general distribution, F, with density $f(t) = F'(t)$ and expectation $E(T) = \frac{1}{\lambda}$. The jump sizes, $\{X_i\}_{i\in\mathbb{Z}^+}$, are strictly positive, iid, random variables from a general distribution G, with density $g(x)$, expectation $\mathrm{E}(X) = \mu$ and variance $\mathrm{Var}(X) = \sigma^2$.

The first-exit time of the compound process hitting a horizontal barrier at X_{exit} is

$$T_{\text{exit}} = \inf\{t > 0 : X_1 + \cdots X_n \geq X_{\text{exit}}\} . \tag{1}$$

Perry et al. (1999) and Stadje and Zacks (2003, 2005) considered the first-exit time problems of M/G and G/M processes with more general boundaries and exponential jump size distributions. Here the first-exit time distribution for Poisson arrivals and a gamma distributed jump size is derived, *i.e.* M/Γ. This allows the limiting case of $\sigma^2 \to 0$ while μ remains fixed and positive to be studied, *i.e.* the jump size density converges weakly (or in distribution) to the Dirac delta function

$$g(x) \xrightarrow{d} \delta(x - \mu) .$$

In this limit the compound process becomes M/D because all jumps have the same size. It is shown that, in this case, a resonant phenomenon when $\frac{X_{\text{exit}}}{\mu} \in \mathbb{Z}^+$ is seen. This yields a discontinuity in the expected first-exit time. That is, the expected first-exit time for the M/Γ case does not tend to that for the M/D case

$$\mathrm{E}(T_{exit})_\Gamma \neq \mathrm{E}(T_{exit})_D \qquad \text{as } \sigma^2 \to 0 .$$

Here, the subscripts "D" and "Γ" refer to fixed (or deterministic) and gamma jump distributions respectively, the "G" subscript is also used to denote a general jump distribution. It is argued that this phenomena is generic to any G/G process where the jump size distribution tends to a delta function. This problem arose in the context of environmental variance in individual growth models (Mullowney and James, 2005).

In §2, the first-exit time pdf is derived for the G/G case. In §3, the specific distribution for the M/Γ case with gamma distributed jumps is presented and examples of the resonance phenomena are shown. §4 shows the same phenomena occurring in the first-exit time distribution for an M/Γ/1 queue with gamma distributed service times. Finally, the results are discussed and it is argued that they can be extended to general distributions which converge weakly to the Dirac delta function in §5.

2 First-Exit Time Distribution

Define the exit condition as the number of jumps required to terminate the compound process:

$$n = \inf\{i \in \mathbb{Z}^+ : X_1 + \cdots X_i \geq X_{\text{exit}}\} \ . \qquad (2)$$

Let $\Phi_G(n)$ $(n = 1, \cdots, \infty)$ be the discrete distribution for the number of jumps required to satisfy the exit condition (2). Formally, $\Phi_G(n)$ is written as

$$\Phi_G(n) \quad = \quad \mathbf{Pr}\left\{X_n + \cdots + X_1 \geq X_{\text{exit}} | X_{n-1} + \cdots + X_1 < X_{\text{exit}}\right\} \ .$$

$\sum_{i=1}^{n} X_i$ has density function given by the n-fold convolution of the jump density, $g^{(n)}(y)$. Since X_n and $\sum_{i=1}^{n-1} X_i$ are independent, the joint density function is separable and $\Phi_G(n)$ is reduced to a single integral:

$$\begin{aligned}\Phi_G(n) &= \int_0^{X_{\text{exit}}} g^{(n-1)}(y) \left(\int_{X_{\text{exit}}-y}^{\infty} g(x)\mathrm{d}x\right) \mathrm{d}y \\ &= \int_0^{X_{\text{exit}}} g^{(n-1)}(y) \left(1 - G\left(X_{\text{exit}} - y\right)\right) \mathrm{d}y \ . \qquad (3)\end{aligned}$$

The pdf of the first-exit times (1) for general arrivals and jump distributions is

$$\Omega_G(t) \;=\; \sum_{n=1}^{\infty} \Phi_G(n) f^{(n)}(t) \; , \tag{4}$$

where $f^{(n)}(t)$ is the n-fold convolution of the arrival density, $f(t)$. The expected first-exit time is

$$\begin{aligned} \mathrm{E}(T_{\mathrm{exit}})_G &= \int_0^{\infty} t\Omega_G(t)\mathrm{d}t = \sum_{n=0}^{\infty} \Phi_G(n) \int_0^{\infty} t f^{(n)}(t)\mathrm{d}t \\ &= \sum_{n=0}^{\infty} n\Phi_G(n) \int_0^{\infty} t f(t)\mathrm{d}t = \sum_{n=0}^{\infty} n\Phi_G(n)\mathrm{E}(T) \\ &= \frac{1}{\lambda}\mathrm{E}(\Phi_G) \; . \end{aligned} \tag{5}$$

In the G/D case where all jumps are of size μ, the number of jumps needed to reach the exit condition is always $\left\lceil \frac{X_{\mathrm{exit}}}{\mu} \right\rceil$, *i.e.* the smallest integer greater than or equal to $\frac{X_{\mathrm{exit}}}{\mu}$. Then, $\Phi_D\left(\left\lceil \frac{X_{\mathrm{exit}}}{\mu} \right\rceil\right) = 1$, the first-exit time density is

$$\Omega_D(t) = f^{\left(\left\lceil \frac{X_{\mathrm{exit}}}{\mu} \right\rceil\right)}(t) \; , \tag{6}$$

and its expectation is

$$\mathrm{E}(T_{\mathrm{exit}})_D = \frac{1}{\lambda}\left\lceil \frac{X_{\mathrm{exit}}}{\mu} \right\rceil \; . \tag{7}$$

3 Compound Poisson process with Gamma distributed jumps

For the case of an exponential arrivals distribution and a gamma jump size distribution it is possible to calculate the first-exit time distribution analytically. Here

$$\begin{aligned} f(t) &= \lambda e^{-\lambda t} \\ g(x, \alpha, \beta) &= H(x)\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} \; , \end{aligned}$$

where $H(x)$ is the Heaviside function. The mean and variance of the jump sizes are $\mu = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$ with $\alpha > 1$. In the limit that $\alpha \to \infty$ and $\beta \to 0$ such that μ remains constant, $\sigma^2 \to 0$. In this situation, the gamma distribution is asymptotic to the normal distribution, $N(\mu, \sigma^2)$ (Balkema et al., 1999). Since the normal distribution converges weakly to the Dirac delta function (Strauss, 1992), the gamma distribution converges weakly to the Dirac delta function as required. For small variance, $g(x, \alpha, \beta)$ resembles a tall, narrow spike centered about μ; *i.e.* an approximation to $\delta(x - \mu)$. The n-fold convolutions of f and g are

$$f^{(n)}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{\Gamma(n)} \tag{8}$$

$$g^{(n)}(x, \alpha, \beta) = g(x, n\alpha, \beta) . \tag{9}$$

Therefore, the probability of satisfying the exit condition with the n^{th} jump is calculated from (3) to give

$$\Phi_\Gamma(n) = \Gamma\left((n-1)\alpha, \alpha\frac{X_{\text{exit}}}{\mu}\right) - \int_0^{\alpha\frac{X_{\text{exit}}}{\mu}} \frac{y^{(n-1)\alpha-1}e^{-y}}{\Gamma((n-1)\alpha)} \Gamma\left(\alpha, \alpha\frac{X_{\text{exit}}}{\mu} - y\right) \mathrm{d}y .$$

Here, $\Gamma(x)$ and $\Gamma(\cdot, x)$ are the complete and lower incomplete gamma functions respectively (Abramovich and Stegun, 1968). In the case where α is an integer, $\Phi_\Gamma(n)$ is

$$\Phi_\Gamma(n) = \Gamma\left((n-1)\alpha, \alpha\frac{X_{\text{exit}}}{\mu}\right) - \Gamma\left(\alpha, \alpha\frac{X_{\text{exit}}}{\mu}\right) + e^{-\alpha\frac{X_{\text{exit}}}{\mu}} \sum_{i=0}^{(n-1)\alpha-1} \frac{\left(\alpha\frac{X_{\text{exit}}}{\mu}\right)^{n\alpha-1-i}}{\Gamma(n\alpha - i)} .$$

For non-integer α there is no closed form expression but, as $\Gamma(\cdot, x)$ is a continuous function, this is of no consequence. Substituting

$$\Gamma(n, x) = 1 - e^{-x}\sum_{i=0}^{n-1}\frac{x^i}{i!} \qquad n \in \mathbb{Z}^+$$

into $\Phi_\Gamma(n)$ gives

$$\Phi_\Gamma(n) = \Gamma\left((n-1)\alpha, \alpha\frac{X_{\text{exit}}}{\mu}\right) - \Gamma\left(n\alpha, \alpha\frac{X_{\text{exit}}}{\mu}\right) . \tag{10}$$

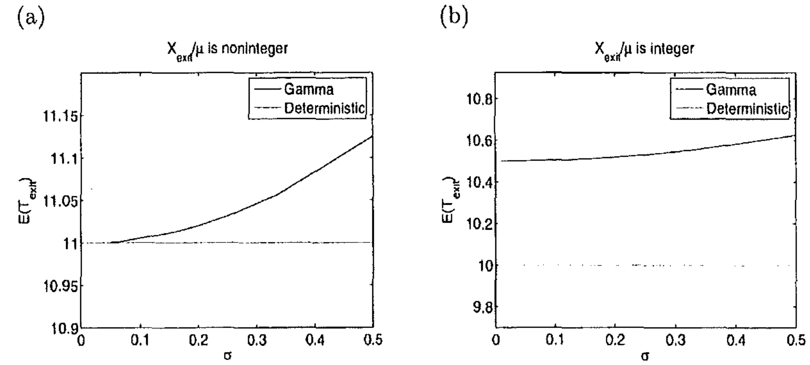

Figure 1: $\mathrm{E}(T_{\text{exit}})_\Gamma$ (black) and $\mathrm{E}(T_{\text{exit}})_D$ (grey) versus σ, $\lambda = 1$. In (a) $X_{\text{exit}}/\mu = 10.5$ and the gamma case tends to the constant jumps case as expected. In (b) $X_{\text{exit}}/\mu = 10$ and the gamma case does not tend to the constant jumps case.

The first-exit time density and expectation are

$$\Omega_\Gamma(t) = \sum_{n=1}^{\infty}\left(\Gamma\left((n-1)\alpha, \alpha\frac{X_{\text{exit}}}{\mu}\right) - \Gamma\left(n\alpha, \alpha\frac{X_{\text{exit}}}{\mu}\right)\right)\frac{\lambda^n t^{n-1}e^{-\lambda t}}{\Gamma(n)}$$

$$\mathrm{E}(T_{\text{exit}})_\Gamma = \frac{1}{\lambda}\sum_{n=1}^{\infty} n\left(\Gamma\left((n-1)\alpha, \alpha\frac{X_{\text{exit}}}{\mu}\right) - \Gamma\left(n\alpha, \alpha\frac{X_{\text{exit}}}{\mu}\right)\right).$$

Figure 1 shows the expected first-exit times of both the gamma and constant size jump distributions for 2 cases. In (a), $X_{\text{exit}}/\mu = 10.5$; as $\sigma \to 0$ the expected first-exit time for the M/G case tends to the expected first-exit time for the M/D case (horizontal line). In (b), $X_{\text{exit}}/\mu = 10$ and the expectations do not converge as $\sigma \to 0$.

Figure 2 shows the absolute relative difference of the first-exit times

$$\Delta\mathrm{E}\left(T_{\text{exit}}\right) = \left|\frac{\mathrm{E}(T_{\text{exit}})_\Gamma - \mathrm{E}(T_{\text{exit}})_D}{\mathrm{E}(T_{\text{exit}})_D}\right| \tag{11}$$

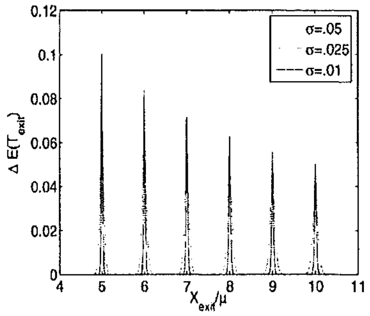

Figure 2: $\Delta \mathrm{E}(T_{\text{exit}})$ versus X_{exit}/μ for $\sigma = .05, .025, .01$.

against X_{exit}/μ for different values of σ. As $\sigma \to 0$, $\Delta \mathrm{E}\,(T_{\text{exit}})$ tends to zero for every value of X_{exit}/μ except the integers. The magnitude of the difference at the integer values decreases as X_{exit}/μ increases.

4 Queues

This phenomena is not limited to compound processes. It can also be seen in the first-exit time of an $\mathrm{M}/\Gamma/1$ queueing process when the service distribution converges weakly to $\delta(x-\mu)$. For queues $\{X_i\}_{i\in\mathbb{Z}^+}$ are the service times, $\mathrm{E}(X) = \mu$ is the mean service time and $\rho = \lambda\mu$ is the traffic density of the queue. The distribution function for (1) represents the total time it takes the server to process for X_{exit} arrivals, including the time when the server is inactive. The first exit time distribution and expectation are

$$\Omega(t) = H(t - X_{\text{exit}}) \sum_{i=1}^{\infty} \frac{\Phi(i)\lambda^i}{\Gamma(i)} (t - X_{\text{exit}})^{i-1} e^{-\lambda(t - X_{\text{exit}})} \tag{12}$$

$$\mathrm{E}(T_{\text{exit}}) = X_{\text{exit}} + \frac{1}{\lambda}\mathrm{E}(\Phi)\,. \tag{13}$$

$H(t - X_{\text{exit}})$ is the Heaviside function which ensures that it always takes at least X_{exit} time to exit. For this case the discrete distribution, $\Phi(i)$, gives the probability of reaching X_{exit} during the i^{th} busy period rather than the i^{th} jump. Let Ω_Γ and Ω_D be the first-exit time distributions corresponding to gamma and deterministic service distributions respectively. Define the discrete distributions Φ_Γ and Φ_D in the same manner (see Mullowney and James (2005) for a full derivation). Then, the error in the expected first-exit time is:

$$\Delta \mathrm{E}\left(T_{\text{exit}}\right) = \left| \frac{\mathrm{E}(\Phi_\Gamma) - \mathrm{E}(\Phi_D)}{\lambda X_{\text{exit}} + \mathrm{E}(\Phi_D)} \right| . \tag{14}$$

The requirement for the resonance phenomena remains $\frac{X_{\text{exit}}}{\mu} \in \mathbb{Z}^+$. Fig. 3 shows $\Delta \mathrm{E}\left(T_{\text{exit}}\right)$ versus $\frac{X_{\text{exit}}}{\mu}$ for $\sigma = .05, .025, .01$. The traffic densities are (a) $\rho = .1$ and (b) $\rho = .5$ respectively. As $\rho \longrightarrow 0$, the interarrival time is much greater than the mean service time. Then, the queue is almost always empty and the M/Γ/1 queue should behave like an M/Γ compound Poisson process. Moreover, it can be shown that Φ_Γ limits onto (10) and Φ_D limits onto $\left\lceil \frac{X_{\text{exit}}}{\mu} \right\rceil$ as $\rho \longrightarrow 0$. Then, the only difference between the relative errors of the compound process and the queue will be the extra term in the denominator of (14). In the light traffic limit (μ fixed, $\lambda \longrightarrow 0$), this term is small and the size of error should be comparable to the M/Γ process. This can be seen by comparing the size of the spikes in Fig. 3a and Fig. 2. As ρ increases, the resonance is still present however it is less substantial as noted by the y-axis scale (Fig. 3b).

5 Discussion

A simple explanation for this phenomenon is found by considering a jump size distribution with mean 1 and an exit barrier at $n \in \mathbb{Z}^+$ and any arrivals distribution. When the jumps are of constant size 1, the process will exit after n jumps with probability 1. Moreover, there is no "excess" in the compound process when the barrier is crossed. In fact, one could say that the barrier is reached rather than crossed. When the jumps are gamma distributed with very

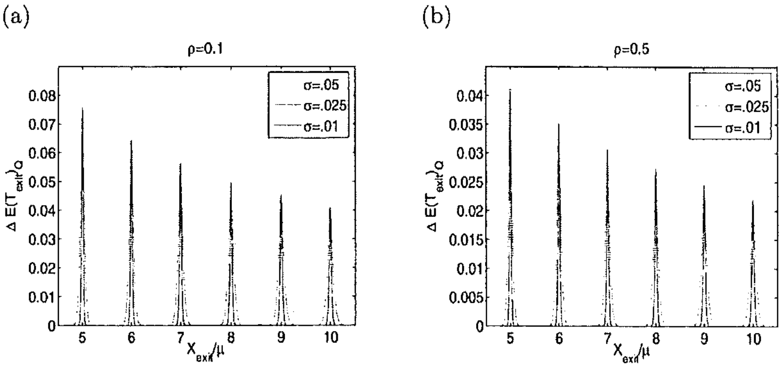

Figure 3: $\Delta\mathrm{E}(T_{\text{exit}})$ versus X_{exit}/μ for $\lambda = 1$ and $\sigma = .05, .025, .01$. (a) traffic density of queue is $\rho = .1$, (b) traffic density of queue is $\rho = .5$

small variance, the probability of having a jump of exactly integer size is 0 for any $\sigma^2 > 0$. The i^{th} jump will be of size $1 + \epsilon_i$ where $\epsilon_i \in N(0, \sigma^2)$ for σ^2 small. The sum of n jumps will be

$$n + \sum_{i=1}^{n} \epsilon_i \, .$$

Whenever $\sum_{i=1}^{n} \epsilon_i < 0$, $n+1$ jumps are required to exit the compound process and the excess will be large. When $\sum_{i=1}^{n} \epsilon_i > 0$, n jumps are required to exit and the excess will be small. For σ^2 small but nonzero, either of these outcomes will occur with roughly equal probability since the ϵ_i are normally distributed about 0. Thus, the process will not limit to the G/D case.

When n is non-integer, a process with constant jumps will always exit after $\lceil n \rceil$ jumps and there will be some excess. That is, the barrier is crossed rather than reached. When σ^2 is small,

$$\lfloor n \rfloor + \sum_{i=1}^{\lfloor n \rfloor} \epsilon_i$$

will almost never cross the boundary and

$$\lceil n \rceil + \sum_{i=1}^{\lceil n \rceil} \epsilon_i$$

almost always will. Hence the process will limit to the G/D case. By this argument, the discontinuities should be observed for any jump distribution that satisfies $g(x) \xrightarrow{d} \delta(x - \mu)$ regardless of the arrivals distribution. Numerical explorations show that the phenomena can be seen with both normally distributed jump sizes and Pareto arrivals processes.

It is not surprising that the phenomenon is most significant in the light traffic limit of the queue. In this limit, almost all busy periods contain at most one arrival. This process should behave identically to the M/G compound process with the exception that service times are accounted for in the queue. This gives rise to the additional term in the expected first-exit time error in (14). The resonance will also occur in higher traffic densities but its effect will be less substantial. Often, the barrier will be almost reached in the middle of a busy period. Then, the server will cross the barrier well before the next arrival. When the barrier is almost reached at the end of a busy period, another arrival will be necessary. It is these events which give rise to the discontinuity in the expected first-exit time. As traffic density increases, the average lengths of busy periods increase and this effect will be less common.

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