Multiple Minimum Coverings of K_n with Copies of $K_4 - e$

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Abstract – This paper is the last of a trilogy completely solving the maximum packing and minimum covering problems for the complete graph on n vertices, K_n, with copies of $K_4 - e$, that is, the complete graph on four vertices with one edge missing.

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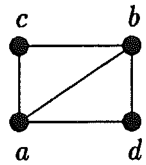
1 Introduction

A $K_4 - e$ *design* of *order* n is a pair (S, B), where B is an edge-disjoint decomposition of the edge set of K_n (the complete undirected graph on n vertices) with vertex set S, into copies of

$K_4 - e =$ 

In what follows we will denote $K_4 - e$ by any one of (a, b, c, d), (a, b, d, c), (b, a, c, d), or (b, a, d, c). In other words (x, y, z, w) denotes the complete graph K_4 with vertex set $\{x, y, z, w\}$ with the edge $\{z, w\}$ "missing". It is well-known (see [1] for example) that the *spectrum* for $K_4 - e$ designs (= the set of all n such that a $K_4 - e$ design of order n exists) is precisely the set of all $n \equiv 0$ or $1 \pmod 5 \geq 6$, and that if (S, B) is a $K_4 - e$ design of order n then $|B| = n(n-1)/10$.

A *packing* of K_n with copies of $K_4 - e$ (PK) is a pair (S, B), where S is the vertex set of K_n and B is a collection of edge disjoint copies of $K_4 - e$. The *number* n is called the *order* of the PK (S, B) and the collection of *unused* edges L is called the *leave*. If $|B|$ is as large as possible (or L is as small as possible), (S, B) is called a *maximum packing* (MPK). So, a $K_4 - e$ design is a MPK with leave $L = \emptyset$. In [2] a *complete solution* of the maximum packing problem for K_n with copies of $K_4 - e$ is given. More precisely, for each $n \geq 4$, MPKs of order n are constructed having all possible leaves.

A *covering* of K_n with copies of $K_4 - e$ is a pair (S, C), where S is the vertex set of K_n and C is a collection of edge disjoint copies of $K_4 - e$ which partition $E(K_n) \cup P$, where $P \subseteq E(\lambda K_n)$. The collection of edges belonging to P is called the *padding* and, of course, the number n is called the *order* of (S, C). A *minimum covering* of K_n with copies of $K_4 - e$ (MCK) is a covering (S, C) where $|P|$ is as small as possible. If $\lambda = 1$ the covering is said to be *simple* (SMCK). In other words, the padding is a simple graph and is as small as possible. Of course, a $K_4 - e$ design is a SMCK with padding $P = \emptyset$.

In [4] a *complete solution* of the *simple minimum covering problem* of K_n with copies of $K_4 - e$ is given. To be precise, for each $n \geq 4$ SMCKs of order n are constructed with *all possible* paddings.

Not too surprisingly, minimum coverings are possible with $\lambda \geq 2$. Such coverings are called *multiple* minimum coverings (MMCK). The object of this paper is the completion of the maximum packing, minimum covering trilogy by giving a complete solution of the *multiple minimum covering problem* of K_n with copies of $K_4 - e$. In particular, we construct for every $n \geq 4$ MMCKs of order

n with *all possible* paddings.

So that there is no confusion in what follows it is worth emphasizing that if (S, C) is a minimal covering of order n, the padding P must be as small as possible *regardless* of λ. In other words, if (S, C_1) and (S, C_2) are minimum coverings with paddings P_1 and P_2, then regardless of whether they are SMCKs or MMCKs or one of each, we must have $|P_1| = |P_2|$. In what follows we will refer to the padding of a MMCK as a *multiple* padding (and the padding of a SMCK as a *simple* padding).

2 Some small examples

We begin with some small examples of orders $n = 5, 7, 8$, and 9. There are two good reasons for doing this. The first is the obvious reason that we will eventually have to handle these cases if we are to give a complete solution. The second reason is that $n = 5, 7$, and 9 (though not 8) are anomalies. Certainly, the best (= smallest) possible padding for $n = 5$ would be the empty set. But since there *does not exist* a $K_4 - e$ design of order 5, the best we can hope for is a multiple padding of size 5. There are 18 candidates for such paddings but just 3 are possible, and these are given in Example 2.1. For $n = 7$ and 9 there are 12 candidates for minimal multiple paddings. However, only 6 are possible for 7 while 11 are possible for 9. These are given in Examples 2.2 and 2.4. As previously mentioned $n = 8$ is not an anomaly. Trivially, the only multiple minimum padding for $n = 8$ is a double edge and this is given in Example 2.3.

Example 2.1 (3 MMCKs of order 5).
$S = \{1, 2, 3, 4, 5\}$;
$C_1 = \{(2, 5, 1, 4), (2, 5, 1, 3), (1, 4, 5, 3)\}$,
$P_1 = \{\{1, 2\}, \{1, 5\}, \{1, 5\}, \{2, 5\}, \{4, 5\}\}$.

$C_2 = \{(1, 5, 2, 4), (1, 3, 4, 5), (2, 4, 1, 3)\}$,
$P_2 = \{\{1, 4\}, \{1, 4\}, \{1, 2\}, \{1, 5\}, \{3, 4\}\}$.

$C_3 = \{(1, 2, 3, 5), (1, 4, 2, 5), (1, 3, 4, 5)\}$,
$P_3 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 5\}\}$.

Example 2.2 (6 MMCKs of order 7).

$S = \{1,2,3,4,5,6,7\}$;

$C_1 = \{(1,4,6,7),(2,3,1,6),(5,7,1,6),(2,7,1,3),(4,5,2,3)\}$,

$P_1 = \{\{1,2\},\{2,3\},\{1,7\},\{1,7\}\}$.

$C_2 = \{(3,5,4,7),(1,5,4,6),(2,5,4,6),(4,6,3,7),(1,2,3,7)\}$,

$P_2 = \{\{4,5\},\{4,5\},\{3,4\},\{5,6\}$.

$C_3 = \{(1,7,2,4),(1,7,3,5),(1,6,2,7),(4,6,3,5),(2,5,3,4)\}$,

$P_3 = \{\{1,7\},\{1,7\},\{1,2\},\{4,5\}\}$.

$C_4 = \{(1,5,4,7),(1,6,2,3),(2,4,5,7),(4,5,3,6),(3,7,2,6)\}$,

$P_4 = \{\{2,7\},\{3,6\},\{4,5\},\{4,5\}\}$.

$C_5 = \{(1,2,3,7),(1,6,2,7),(4,5,1,2),(4,7,3,5),(5,6,3,4)\}$,

$P_5 = \{\{1,2\},\{1,7\},\{4,5\},\{4,5\}\}$.

$C_6 = \{(1,4,3,5),(1,5,2,7),(5,6,1,3),(2,7,1,3),(4,6,2,7)\}$,

$P_4 = \{\{1,5\},\{1,5\},\{1,2\},\{1,7\}\}$.

Example 2.3 (MMCK of order 8).

$S = \{(1,2,3,4,5,6,7,8\}$;

$C_1 = \{(1,2,3,4),(1,2,7,8),(1,2,5,6),(3,4,7,8),(5,6,3,4),(7,8,5,6)\}$,

$P_1 = \{\{1,2\},\{1,2\}\}$.

Example 2.4 (11 MMCKs of order 9).

$S = \{1,2,3,4,5,6,7,8,9\}$;

$C_1 = \{(1,2,5,9),(3,4,1,9),(5,6,3,9),(7,8,6,9),(2,7,1,3),(2,8,1,3),(2,6,1,4),$
$(4,5,7,8)\}$,

$P_1 = \{\{1,2\},\{1,2\},\{1,2\},\{2,3\}\}$.

$C_2 = \{(1,5,2,8),(1,9,2,3),(1,6,2,4),(2,7,1,4),(3,8,2,4),(4,5,3,9),(6,7,3,5),$
$(8,9,6,7)\},$
$P_2 = \{\{1,2\},\{1,2\},\{1,2\},\{3,4\}\}.$

$C_3 = \{(1,9,2,8),(1,3,2,9),(2,4,1,5),(6,9,1,4),(5,7,1,9),(2,7,6,8),(3,4,7,8),$
$(5,6,3,8)\},$
$P_3 = \{\{1,2\},\{1,2\},\{1,9\},\{1,9\}\}.$

$C_4 = \{(4,5,8,9),(2,4,1,5),(3,5,1,4),(6,7,4,5),(6,8,1,7),(7,9,1,6),(2,3,6,7),$
$(8,9,2,3)\},$
$P_4 = \{\{6,7\},\{6,7\},\{4,5\},\{4,5\}\}.$

$C_5 = \{(1,2,3,4),(1,5,6,7),(2,5,8,9),(3,4,5,6),(3,7,8,9),(8,9,1,4),(6,7,2,4),$
$(6,8,4,9)\},$
$P_5 = \{\{4,6\},\{4,6\},\{4,8\},\{8,9\}\}.$

$C_6 = \{(1,2,3,4),(1,5,6,7),(2,6,7,8),(5,8,3,4),(6,9,3,4),(8,9,1,7),(2,5,3,9),$
$(3,7,2,4)\},$
$P_6 = \{\{2,3\},\{2,3\},\{2,7\},\{3,5\}\}.$

$C_7 = \{(1,2,3,4),(1,5,6,7),(2,5,8,9),(3,4,5,6),(3,8,7,9),(6,7,2,9),(4,8,7,9),$
$(1,8,6,9)\},$
$P_7 = \{\{8,9\},\{8,9\},\{7,8\},\{1,6\}\}.$

$C_8 = \{(1,2,5,6),(1,7,8,9),(2,9,1,8),(3,4,1,8),(3,5,6,9),(4,6,5,9),(5,6,7,8),(2,7,3,4)\},$
$P_8 = \{\{5,6\},\{5,6\},\{1,2\},\{1,9\}\}.$

$C_9 = \{(1,2,3,4),(1,5,6,7),(2,6,7,8),(3,4,6,7),(3,8,5,9),(5,9,2,4),(7,9,6,8),$
$(1,8,4,9)\},$
$P_9 = \{\{8,9\},\{8,9\},\{1,4\},\{6,7\}\}.$

$C_{10} = \{(1,2,3,4),(1,5,6,7),(2,6,7,8),(3,6,4,9),(3,8,5,7),(5,9,2,4),(7,8,4,9),(8,9,1,7)\},$
$P_{10} = \{\{7,8\},\{7,8\},\{7,9\},\{8,9\}\}.$

$C_{11} = \{(1,2,3,4),(1,5,6,7),(2,6,7,8),(3,6,4,9),(3,8,5,7),(5,9,2,4),(1,4,7,8),$
$(1,9,7,8)\}$,
$P_{11} = \{\{1,7\},\{1,7\},\{1,4\},\{1,8\}\}$.

The object of this paper is to validate the information in the accompanying tables, Tables 2.5 and 2.6.

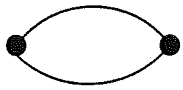
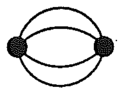
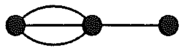
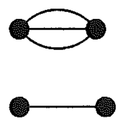
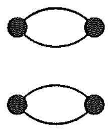
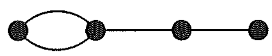
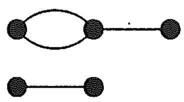
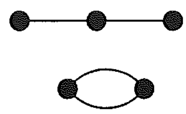
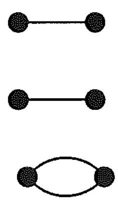
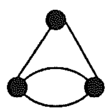
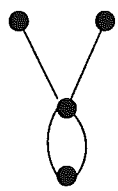
K_n n (mod 10)	Paddings of MMCKs		
3 or 8 $n \geq 8$			
2, 4, 7, 9 $n \geq 12$	A 	B 	C 
	D 	E 	F 
	G 	H 	I 
	J 	K 	L 

Table 2.5 Possible paddings of MMCKs for $n \geq 8$, $n \neq 9$.

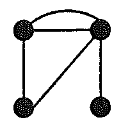
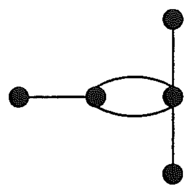
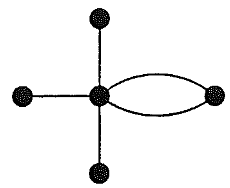
K_n	Paddings of MMCKs		
5			
7	F, G, H, I, J, and L		
9	B, C, D, E, F, G, H, I, J, K, and L		

Table 2.6 Possible paddings of MMCKs for $n = 5, 7$ and 9.

3 The cases $n = 4, 5, 7, 8$, and 9.

n = 4. Suppose that P is a 4-edge padding of K_4. Then $K_4 \cup P$ must decompose into two copies of $K_4 - e$. Hence no edge in $K_4 \cup P$ can be repeated three times, so only simple paddings are possible; see [4].

n = 8. Trivially the only relevant multiple padding consists of one double edge, and Example 2.3 shows this is possible.

To deal with the other cases conveniently, we need another definition. Since $K_4 - e$ has vertices of degrees 2 and 3 only, a vertex of degree 4 in $K_n \cup P$ must occur in two of the copies of $K_4 - e$ in the MMCK, and with degree 2 in each. We say that the vertex has *type* $2 + 2$. Similarly vertices of degrees 5 and 7 have types $2 + 3$ and $2 + 2 + 3$ respectively. Vertices of other degrees may be of several types: thus a vertex of degree 6 may have type $2 + 2 + 2$ or $3 + 3$, and so on.

Suppose that x copies of $K_4 - e$ appear in the decomposition of $K_n \cup P$. Each copy of $K_4 - e$ contains two vertices each of degrees 2 and 3. Thus in the sequence of types of the vertices of $K_n \cup P$,

$$\text{the number of } 2s = 2.x = \text{the number of } 3\text{s}. \qquad (R_1)$$

If two vertices always appear with degree 2 in the copies of $K_4 - e$, then no edge joining them in $K_n \cup P$ can be covered in the decomposition. Hence:

$$\text{at most one vertex has type } 2 + 2 + \cdots + 2. \qquad (R_2)$$

If edge (x, y) appears m times in $K_n \cup P$, then:

$$\text{vertices } x \text{ and } y \text{ must each have a type containing at least } m \text{ terms}. \qquad (R_3)$$

n = 5. In this case P must be a 5-edge padding of K_4 and hence a graph on at most five vertices. Since $K_5 \cup P$ decomposes into three copies of $K_4 - e$, no triple edge can occur in P. But P is a multiple minimum padding so it contains either one double edge (and three single edges) or two double edges (and one single edge).

The three multiple paddings which actually occur are given in Example 2.1 and Table 2.6. The 15 remaining graphs which might occur as padding are shown in Table 3.1; we proceed to rule them out!

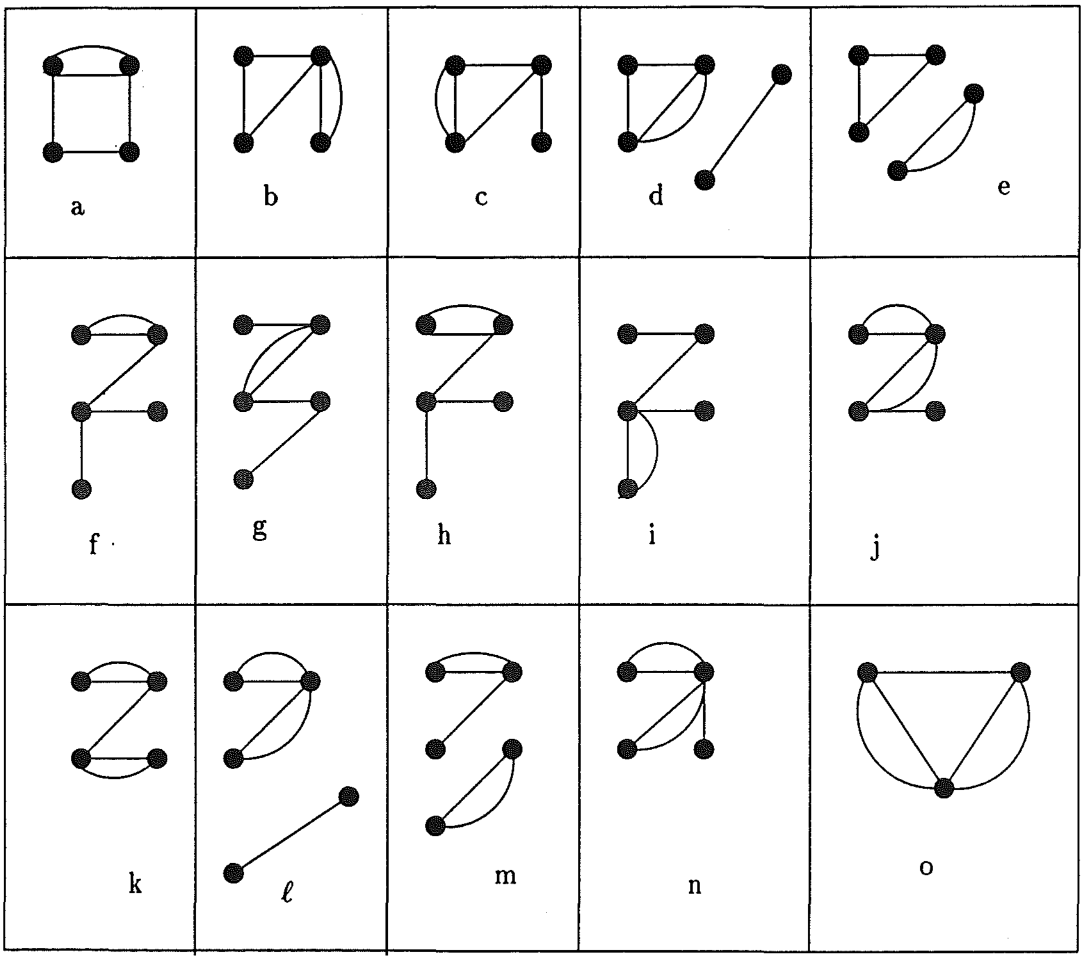


Table 3.1: 15 graphs, each on at most 5 vertices, with one double and three single edges, or two double and one single edges.

(a) $P = \{(2,3),(2,5),(2,5),(3,4),(4,5)\}$.

Vertex 1 has type $2+2$; vertices 3 and 4 have type $2+2+3$. By R_1, vertices 2 and 5 have type $3+3$, contradicting R_3. Hence (a) is impossible.

(b) $P = \{(1,2),(1,5),(2,3),(2,3),(2,5)\}$.

Vertex 4 has type $2+2$. By R_3, vertex 2 has type $2+3+3$. Since edge $(2,3)$ appears in each copy of $K_4 - e$, vertex 3 has type $2+2+2$, contradicting R_2. Hence (b) is impossible.

(c) $P = \{(1,2),(1,4),(1,5),(2,5),(2,5)\}$.

(d) $P = \{(1,2),(1,5),(2,5),(2,5),(3,4)\}$.

In both cases vertices 2 and 5 have type $2+2+3$ and hence cannot be adjacent in all three copies of $K_4 - e$. This rules out (c) and (d).

(e) $P = \{(1,2),(1,5),(2,5),(3,4),(3,4)\}$.

All vertices have degree 6, so by R_1, three must have type $3+3$ and the other two have type $2+2+2$. This contradicts R_2, so (e) cannot occur.

(f) $P = \{(1,2),(1,3),(2,5),(2,5),(3,4)\}$.

Vertex 2 has type $2+2+3$; vertex 5 has type $2+2+2$ or $3+3$. Whichever type vertex 5 has, edge $(2,5)$ cannot occur in all three copies of $K_4 - e$, so (f) is impossible.

(g) $P = \{(1,2),(1,2),(1,5),(2,3),(3,4)\}$.

Vertices 1 and 2 have type $2+2+3$, so they cannot be adjacent to each other in all three copies of $K_4 - e$. Hence (g) is impossible.

(h) $P = \{(1,2),(1,4),(1,5),(3,4),(3,4)\}$.

Vertices 1 and 4 have type $2+2+3$, and vertices 2 and 5 have type $2+3$. By R_1, vertex 3 has type $3+3$, contradicting R_3 and ruling out (h).

(i) $P = \{(1,2),(1,2),(1,4),(1,5),(3,4)\}$.

Since only three copies of $K_4 - e$ appear in the decomposition, vertex 1 has type $2+3+3$. By R_3, vertex 2 has type $2+2+2$, but still cannot be adjacent to vertex 1 in all three $K_4 - e$'s. Hence (i) is impossible.

(j) $P = \{(1,2),(1,),(1,5),(1,5),(2,3)\}$.

Vertices 1, 2, 3 and 4 have types $2+3+3$, $2+2+3$, $2+3$ and $2+2$ respectively. By R_1, vertex 5 has type $3+3$, contradicting R_3 and ruling out (j).

(k) $P = \{(1,2),(1,5),(1,5),(2,3),(2,3)\}$.

Vertices 1 and 2 have type $2+2+3$ and vertex 4 has type $2+2$. By R_1, vertices 3 and 5 have type $3+3$ contradicting R_3 and ruling out (k).

(ℓ) $P = \{(1,2),(1,2),(1,5),(1,5),(3,4)\}$.

Vertex 1 has type $2+3+3$. Whether vertex 2 has type $2+2+2$ or type $3+3$, it cannot be adjacent to vertex 1 in all three copies of $K_4 - e$, so (ℓ) is impossible.

(m) $P = \{(1,2),(1,2),(1,5),(3,4),(3,4)\}$.

Vertex 1 has type $2+2+3$. Whether vertex 2 has type $2+2+2$ or type $3+3$, it cannot be adjacent to vertex 1 in all three copies of $K_4 - e$, so (m) is impossible.

(n) $P = \{(1,2),(1,2),(1,3),(1,5),(1,5)\}$.

Vertex 1 has type $3+3+3$, and by R_3, vertices 2 and 5 have type $2+2+2$. But vertices 3 and 4 have types $2+3$ and $2+2$ respectively, contradicting R_1. Hence (n) cannot occur.

(o) $P = \{(1,2),(1,2),(1,5),(1,5),(2,5)\}$.

Vertex 1 has type $2+3+3$; vertices 2 and 5 have type $2+2+3$; vertices 3 and 4 have type $2+2$. This contradicts R_2 so (o) is impossible.

This completes the possibilities for $n = 5$.

n = 7. Here P must be a 4-edge padding of K_7 and hence a graph on at most seven vertices. The 12 possible paddings are shown in Table 2.5 and paddings F, G, H, I, J and L are given in Example 2.2. We now rule out the remaining six cases.

(A) $P = \{(4,5),(4,5),(4,5),(4,5)\}$.

Vertices 4 and 5 have degree 10 and, by R_3, must both have type $2+2+2+2+2$. This contradicts R_1, ruling out (A).

(B) $P = \{(3,4),(4,5),(4,5),(4,5)\}$.

Vertex 4 has degree 10 and type $2+2+2+2+2$ or $2+2+3+3$. Vertex 5 has degree 9 and, by R_3, must have type $2+2+2+3$. But with these types, edge $(4,5)$ cannot appear in four copies of $K_4 - e$ so (B) is impossible.

(C) $P = \{(3,6),(4,5),(4,5),(4,5)\}$.

By R_3, vertices 4 and 5 must have type $2+2+2+3$. But edge $(4,5)$ can only occur in two copies of $K_4 - e$ so (C) is impossible.

(D) $P = \{(1,2),(1,2),(1,7),(1,7)\}$.

By R_3, vertex 1 must have type $2+2+3+3$ and vertices 2 and 7 must have type $2+3+3$. By R_1, two of the remaining vertices must have type $3+3$ and two must have type $2+2+2$. This contradicts R_2, so (D) cannot occur.

(E) $P = \{(2,7),(2,7),(3,6),(3,6)\}$.

Vertices 2, 3, 6 and 7 have degree 8. To ensure that edges $(2,7)$ and $(3,6)$ appear in three copies of $K_4 - e$ each, they must all be of type $2+3+3$. Vertices 1, 4 and 5 have degree 6 each and by R_2, two of them must have type $2+2+2$. This contradicts R_3, and rules out (E).

(K) $P = \{(1,2),(2,7),(2,7),(1,7)\}$.

Vertices 2 and 7 have degree 9 and, to ensure that edge $(2,7)$ appears in three copies of $K_4 - e$, at least one of then must have type $3+3+3$.

(ii) If both have type $3+3+3$, then by R_2 at least two of the vertices 3, 4, 5 and 6 must have type $2+2+2+2$, contradicting R_1.

(ii) If one of the vertices 2 and 7 has type $3+3+3$ and the other has type $2+2+2+3$ then, by R_2, either at least two vertices of degree 6 have type $2+2+2$, or vertex 1 has degree $2+2+2+2$ and at least one vertex of degree 6 has type $2+2+2$. This also contradicts R_1.

Hence K is impossible.

This completes the possibilities for $n = 7$.

n = 9. In this case P must be a 4-edge padding of K_9 and hence a graph on at most nine vertices. All possible multiple minimal paddings are shown in Table 2.5, and all but one occur, as shown by Example 2.4. We now check that case A cannot occur.

Let $P = \{(4,6),(4,6),(4,6),(4,6)\}$ so that vertices 4 and 6 have degree 12. By R_3, each must have type $2+2+2+3+3$ or $2+2+2+2+2+2$. But even so, edge $(4,6)$ can appear in at most four copies of $K_4 - e$, so (A) is impossible.

This completes the argument for $n = 9$.

4 The cases between 12 and 29 inclusive.

The cases for $n = 4, 5, 7, 8$, and 9 are handled in Sections 2 and 3. We will concern ourselves here with $12 \leq n \leq 29$ and $n \not\equiv 0, 1, 5$, or 6 (mod 10). Some of these results are necessary, as well, for the recursive constructions which follow in Section 5. As was pointed out in Section 3, the graphs in Table 2.5 are the only possibilities for multiple paddings with 2 or 4 edges (= Table 2.5 is a distinct listing of all graphs with multiple edges containing 2 or 4 edges).

Before proceeding we will need a well-known definition. A $K_4 - e$ design of order n with a *hole* of size h is a triple (S, H, B), where B is an edge-disjoint collection of copies of $K_4 - e$ which partition the edge set of $K_n \setminus K_h$, where S is the vertex set of K_n and $H \subseteq S$ is the vertex set of K_h.

Example 4.1 (12 MMCKs of order 12).

$S = \{1,2,3,4,5,6,7,8,9,10,11,12\}$;

Padding of type A

$C_1 = \{(1,2,3,4),(3,4,5,6),(5,6,1,2),(7,8,1,2),(9,10,1,2),(7,9,3,4),(8,10,3,4),(7,10,5,6),$
$(8,9,5,6),(11,12,1,2),(11,12,3,4),(11,12,5,6),(11,12,7,8),(11,12,9,10)\}$,

$P_1 = \{\{11,12\},\{11,12\},\{11,12\},\{11,12\}\}$.

Padding of type B

$C_2 = \{(3,10,1,2),(5,10,4,6),(7,9,8,10),(1,4,7,11),(5,8,2,11),(3,9,6,11),(1,9,5,12),(2,6,7,12),$
$(3,4,8,12),(5,7,3,12),(10,12,8,11),(1,6,2,8),(2,4,6,9),(2,11,6,7)\}$,

$P_2 = \{\{2,6\},\{2,6\},\{2,6\},\{2,7\}\}$.

Padding of type C

$C_3 = \{(1,2,3,4),(3,4,5,6),(5,6,1,2),(7,8,1,2),(9,10,1,2),(7,9,3,4),(8,10,3,4),(11,12,1,2),$
$(11,12,3,4),(11,12,5,6),(11,12,7,8),(9,10,11,12),(7,10,5,6),(8,9,5,6)\}$,

$P_3 = \{\{11,12\},\{11,12\},\{11,12\},\{9,10\}\}$.

Padding of type D

$C_4 = \{(1,3,2,10),(4,5,6,10),(7,9,8,10),(1,7,4,11),(2,5,8,11),(3,6,9,11),(1,9,5,12),(2,6,7,12),$
$(3,4,8,12),(4,9,2,11),(5,7,3,12),(11,12,8,10),(2,10,6,8),(6,8,1,2)\}$,

$P_5 = \{\{2,8\},\{2,8\},\{2,6\},\{2,6\}\}$.

Padding of type E

$C_5 = \{(7,10,8,11),(8,11,9,12),(9,12,10,7),(1,2,11,12),(3,4,11,12),(5,6,11,12),(7,8,1,6),(7,8,4,5),$
$(2,3,7,8),(9,10,2,5),(9,10,3,6),(1,4,9,10),(1,3,5,6),(2,4,5,6)\}$,

$P_5 = \{\{7,8\},\{7,8\},\{9,10\},\{9,10\}\}$.

Padding of type F

$C_6 = \{(3,10,1,2),(5,6,4,10),(7,8,9,10),(4,7,1,11),(2,5,8,11),(3,9,6,11),(1,9,5,12),(6,7,2,12),$
$(4,8,3,12),(4,9,2,10),(6,8,1,11),(3,5,7,12),(2,12,10,11),(1,11,2,10)\}$,

$P_6 = \{\{2,11\},\{2,11\},\{2,10\},\{1,10\}\}$.

Padding of type G

$C_7 = \{(7,10,8,11),(8,11,9,12),(9,12,10,7),(1,3,7,8),(2,5,7,8),(4,6,7,8),(1,4,9,10),(2,6,9,10),$
$(3,5,9,10),(1,5,11,12),(2,4,11,12),(3,6,11,12),(1,3,2,4),(4,6,1,5)\}$,
$P_7 = \{\{1,3\},\{1,4\},\{1,4\},\{4,6\}\}$.

Padding of type H

$C_8 = \{(7,10,8,11),(8,11,9,12),(9,12,10,7),(7,10,1,4),(1,5,9,11),(1,6,8,12),(2,4,9,12),(2,5,8,10),$
$(2,6,7,11),(3,4,8,11),(3,5,7,12),(3,6,9,10),(1,2,3,4),(4,5,2,6)\}$,
$P_8 = \{\{2,4\},\{2,4\},\{2,5\},\{7,10\}\}$.

Padding of type I

$C_9 = \{(7,10,8,11),(8,11,9,12),(9,12,10,7),(7,10,1,4),(7,11,2,6),(1,5,9,11),(1,6,8,12),(2,4,9,12),$
$(2,5,8,10),(3,4,8,11),(3,5,7,12),(3,6,9,10),(1,2,3,4),(4,6,2,5)\}$,
$P_9 = \{\{2,4\},\{2,4\},\{7,10\},\{7,11\}\}$.

Padding of type J

$C_{10} = \{(1,2,3,4),(3,4,5,6),(5,6,1,2),(7,8,1,2),(9,10,1,2),(7,9,3,4),(8,10,3,4),(7,10,5,6),$
$(8,9,5,6),(11,12,1,2),(11,12,3,4),(11,12,5,6),(7,8,11,12),(9,10,11,12)\}$,
$P_{10} = \{\{11,12\},\{11,12\},\{7,8\},\{9,10\}\}$.

Padding of type K

$C_{11} = \{(2,3,1,10),(4,5,6,10),(7,9,8,10),(7,11,1,4),(5,8,2,11),(3,6,9,11),(1,9,5,12),(2,6,7,12),$
$(3,12,4,8),(10,12,1,11),(1,8,4,12),(6,8,1,10),(2,9,4,11),(5,7,3,12)\}$,
$P_{11} = \{\{1,12\},\{1,12\},\{1,8\},\{8,12\}\}$.

Padding of type L

$C_{12} = \{(7,10,8,11),(8,11,9,12),(9,12,10,7),(1,3,7,8),(2,5,7,8),(4,6,7,8),(1,4,9,10),(2,6,9,10),$
$(3,5,9,10),(1,5,11,12),(2,4,11,12),(3,6,11,12),(1,3,2,4),(1,5,4,6)\}$,
$P_{12} = \{\{1,4\},\{1,4\},\{1,3\},\{1,5\}\}$.

Example 4.2 (MMCK of order 13).

$S = \{(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$;

$C_1 = \{(1, 3, 2, 10), (4, 6, 5, 10), (7, 8, 9, 10), (4, 7, 1, 11), (8, 11, 2, 5), (6, 9, 3, 11), (5, 9, 1, 12), (2, 7, 6, 12), (4, 8, 3, 12), (8, 13, 1, 6), (4, 13, 2, 9), (5, 7, 3, 13), (10, 12, 11, 13), (1, 12, 3, 6), (3, 11, 1, 13), (2, 10, 5, 9)\}$,

$P_1 = \{\{1, 3\}, \{1, 3\}\}$.

Example 4.3 (12 MMCKs of order 14).

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$;

Paddings of type F, G, H, I, J, and L. An example of a $K_4 - e$ design of order 14 with a hole of size 7 is given in [4] (Example 4.3). Fill in the hole with the MMCKs of order 7 in Example 2.2.

Padding of type A

$C_1 = \{((4, 5, 6, 10), (8, 9, 7, 10), (7, 11, 1, 4), (5, 11, 2, 8), (3, 9, 6, 11), (9, 12, 1, 5), (6, 12, 2, 7), (4, 8, 3, 12), (1, 8, 6, 13), (2, 13, 4, 9), (3, 13, 5, 7), (10, 12, 11, 13), (2, 14, 1, 8), (3, 14, 1, 12), (10, 14, 1, 6), (4, 14, 1, 9), (5, 14, 1, 7), (11, 13, 6, 14), (2, 10, 3, 7)\}$,

$P_1 = \{\{1, 14\}, \{1, 14\}, \{1, 14\}, \{1, 14\}\}$.

Padding of type B

$C_2 = \{(3, 10, 1, 2), (4, 6, 5, 10), (8, 10, 7, 9), (7, 11, 1, 4), (5, 11, 2, 8), (3, 6, 9, 11), (9, 12, 1, 5), (2, 12, 6, 7), (4, 8, 3, 12), (8, 13, 1, 6), (2, 9, 4, 13), (5, 7, 3, 13), (10, 11, 12, 13), (2, 14, 1, 8), (4, 14, 1, 13), (5, 14, 1, 10), (6, 14, 1, 7), (9, 14, 7, 11), (3, 12, 13, 14)\}$,

$P_2 = \{\{1, 14\}, \{1, 14\}, \{1, 14\}, \{7, 14\}\}$.

Padding of type C

$C_3 = \{(3, 10, 1, 2), (4, 13, 2, 9), (7, 11, 1, 4), (3, 8, 4, 12), (9, 12, 1, 5), (2, 8, 5, 11), (8, 13, 1, 6), (2, 12, 6, 7), (5, 7, 3, 13), (6, 9, 3, 11), (10, 12, 11, 13), (7, 9, 8, 10), (4, 6, 5, 10), (2, 14, 1, 9), (4, 14, 1, 12), (5, 14, 1, 11), (6, 14, 1, 7), (3, 13, 11, 14), (8, 10, 5, 14)\}$,

$P_3 = \{\{1, 14\}, \{1, 14\}, \{1, 14\}, \{5, 8\}\}$.

Padding of type D

$C_4 = \{(4, 6, 5, 10), (7, 9, 8, 10), (4, 7, 1, 11), (5, 8, 2, 11), (6, 9, 3, 11), (1, 5, 9, 12), (2, 6, 7, 12), (4, 8, 3, 12), (1, 8, 6, 13), (2, 9, 4, 13), (5, 13, 3, 7), (10, 13, 11, 12), (12, 14, 9, 11), (10, 14, 5, 8), (13, 14, 4, 6), (1, 2, 3, 14), (3, 7, 12, 14), (2, 10, 1, 3), (2, 11, 1, 3)\}$,

$P_4 = \{\{1, 2\}, \{1, 2\}, \{2, 3\}, \{2, 3\}\}$.

Padding of type E

$C_5 = \{(7,10,8,9),(5,8,2,11),(3,11,6,9),(6,12,2,7),(3,8,4,12),(7,13,3,5),(10,11,12,13),(2,9,4,13),$
$(6,8,1,13),(9,12,1,5),(6,10,4,5),(7,11,1,4),(2,14,1,11),(3,14,1,5),(10,14,1,7),(9,14,6,8),$
$(12,14,4,13),(2,10,3,7),(1,4,5,13)\}$,
$P_5 = \{\{1,14\},\{1,14\},\{7,10\},\{7,10\}\}$.

Padding of type K

$C_6 = \{(4,6,5,10),(7,9,8,10),(4,7,1,11),(5,8,2,11),(6,9,3,11),(1,5,9,12),(2,6,7,12),(4,8,3,12),$
$(1,8,6,13),(2,9,4,13),(5,13,3,7),(10,13,11,12),(12,14,9,11),(10,14,5,8),(13,14,4,6),(1,2,3,14),$
$(3,7,12,14),(2,10,1,3),(1,11,2,3)\}$,
$P_6 = \{\{1,2\},\{1,2\},\{1,3\},\{2,3\}\}$.

Example 4.4 (12 MMCKs of order 17).

$S = \{1,2,3,\ldots,17\}$;

Paddings of type F, G, H, I, J, and L. An example of a $K_4 - e$ design of order 17 with a hole of size 7 is given in [2] (Example 4.5). Fill in the hole with the MMCKs of order 7 in Example 2.2.

Padding of type A

$C_1 = \{(3,4,1,2),(5,7,6,8),(11,12,9,10),(14,16,13,15),(9,13,1,5),(11,16,1,6),(12,14,1,7),$
$(10,15,1,8),(5,15,2,12),(10,14,2,6),(7,9,2,16),(11,13,2,8),(3,5,10,16),(6,12,3,13),$
$(7,11,3,15),(3,9,8,14),(4,14,5,11),(9,15,4,6),(4,10,7,13),(4,12,8,16),(2,17,1,12),$
$(5,17,1,11),(6,17,1,4),(7,17,1,13),(8,17,1,14),(10,17,9,16),(3,15,13,17),(2,8,6,16)\}$,
$P_1 = \{\{1,17\},\{1,17\},\{1,17\},\{1,17\}\}$.

Padding of type B

$C_2 = \{(3,4,1,2),(5,7,6,8),(11,12,9,10),(14,16,13,15),(9,13,1,5),(11,16,1,6),(12,14,1,7),$
$(1,15,8,10),(5,15,2,12);(10,14,2,6),(7,9,2,16),(11,13,2,8),(3,5,10,16),(6,12,3,13),$
$(7,11,3,15),(3,9,8,14),(4,14,5,11),(9,15,4,6),(4,10,7,13),(4,12,8,16),(2,17,1,12),$
$(5,17,1,11),(6,17,1,4),(7,17,1,13),(8,17,10,14),(10,17,9,16),(3,15,13,17),(2,8,6,16)\}$,
$P_2 = \{\{1,17\},\{1,17\},\{1,17\},\{10,17\}\}$.

Padding of type C

$C_3 = \{(3,4,1,2),(5,6,7,8),(9,12,10,11),(13,16,14,15),(9,13,1,5),(11,16,1,6),(12,14,1,7),$
$(1,8,10,15),(2,5,12,15),(2,14,6,10),(2,7,9,16),(8,11,2,13),(5,10,3,16),(3,13,6,12),$
$(3,7,11,15),(3,9,8,14),(4,11,5,14),(9,15,4,6),(7,13,4,10),(4,16,8,12),(2,17,1,13),$
$(5,17,1,14),(6,17,1,12),(7,17,1,8),(16,17,3,9),(4,10,6,17),(11,15,10,17),(8,15,12,14)\},$
$P_3 = \{\{1,17\},\{1,17\},\{1,17\},\{8,15\}\}.$

Padding of type D

$C_4 = \{(6,8,5,7),(11,12,9,10),(13,15,14,16),(9,13,1,5),(1,6,11,16),(1,7,12,14),(1,15,8,10),$
$(12,15,2,5),(2,14,6,10),(2,7,9,16),(2,11,8,13),(10,16,3,5),(6,13,3,12),(3,7,11,15),$
$(3,8,9,14),(11,14,4,5),(6,15,4,9),(4,7,10,13),(8,12,4,16),(5,17,1,7),(10,17,6,9),$
$(12,17,3,14),(11,17,15,16),(8,13,10,17),(2,4,3,17),(9,16,4,14),(1,3,2,4),(3,5,2,4)\},$
$P_5 = \{\{2,3\},\{2,3\},\{3,4\},\{3,4\}\}.$

Padding of type E

$C_5 = \{(13,14,15,16),(5,9,1,13),(7,12,1,14),(10,15,1,8),(12,15,2,5),(2,6,10,14),(2,7,9,16),$
$(2,13,8,11),(10,16,3,5),(12,13,3,6),(3,9,8,14),(11,14,4,5),(10,13,4,7),(4,16,8,12),$
$(2,17,1,5),(3,17,1,6),(4,17,1,7),(8,17,11,12),(16,17,9,15),(3,4,2,5),(1,14,13,8),$
$(6,11,7,9),(14,17,10,13),(5,8,6,7),(10,12,9,11),(1,16,6,11),(3,15,7,11),(4,15,6,9)\},$
$P_5 = \{\{1,17\},\{1,17\},\{13,14\},\{13,14\}\}.$

Padding of type K

$C_6 = \{(6,8,5,7),(11,12,9,10),(13,15,14,16),(9,13,1,5),(1,6,11,16),(1,7,12,14),(1,15,8,10),$
$(12,15,2,5),(2,14,6,10),(2,7,9,16),(2,11,8,13),(10,16,3,5),(6,13,3,12),(3,7,11,15),$
$(3,8,9,14),(11,14,4,5),(6,15,4,9),(4,7,10,13),(8,12,4,16),(5,17,1,7),(10,17,6,9),(12,17,3,14),$
$(11,17,15,16),(8,13,10,17),(2,4,3,17),(9,16,4,14),(1,3,2,4),(2,5,3,4)\},$
$P_6 = \{\{2,3\},\{2,3\},\{2,4\},\{3,4\}\}.$

Example 4.5 (MMCK of order 18).

$S = \{1,2,3,4,\ldots,18\};$

An example of a $K_4 - e$ design of order 18 with a hole of size 8 is given in [2] (Example 4.6). Fill in the hole with the MMCK of order 8 in Example 2.3.

Example 4.6 (12 MMCKs of order 19).

$S = \{1, 2, 3, \ldots, 19\}$;

Paddings of all types *except* A. An example of a $K_4 - e$ design of order 19 with a hole of size 9 is given in [2] (Example 4.9). Fill in the hole with the 11 MMCKs of order 9 in Example 2.4.

Padding of type A

$C_1 = \{(5,17,1,2), (9,16,1,5), (12,14,2,5), (16,17,6,11), (2,10,6,16), (3,7,11,16), (15,17,3,4),$
$(6,12,3,15), (5,11,4,15), (9,17,7,8), (2,15,7,9), (3,14,8,9), (10,17,12,13), (9,11,10,12),$
$(3,5,10,13), (13,14,4,11), (15,16,13,14), (2,8,11,13), (1,6,11,14), (6,9,4,13), (12,16,4,8),$
$(12,13,1,7), (10,15,1,8), (5,6,7,8), (2,3,1,4), (14,17,18,19), (1,8,7,19), (4,19,7,10),$
$(4,18,1,8), (7,10,14,18), (18,19,2,3), (18,19,5,6), (18,19,9,11), (18,19,12,13), (18,19,15,16)\}$,
$P_1 = \{\{18,19\}, \{18,19\}, \{18,19\}, \{18,19\}\}$.

Example 4.7 (12 MMCKs of order 22).

$S = \{1, 2, 3, \ldots, 22\}$;

Paddings of *all* types. An example of a $K_4 - e$ design of order 22 with a hole of size 14 in given in [4] (Example 4.8). Fill in the hole with the 12 MMCKs of order 14 in Example 4.3.

Example 4.8 (MMCK of order 23).

$S = \{1, 2, 3, \ldots, 23\}$;

An example of a $K_4 - e$ design of order 23 with a hole of size 13 is given in [2] (Example 4.11). Fill in the hole with the MMCK of order 13 in Example 4.2.

Example 4.9 (12 MMCKs of order 24).

$S = \{1, 2, 3, \ldots, 24\}$.

Paddings of *all* types. An example of a $K_4 - e$ design of order 24 with a hole of size 12 is given in [4] (Example 4.11). Fill in the hole with the 12 MMCKs in Example 4.1.

Example 4.10 (12 MMCKs of order 27).

$S = \{1, 2, 3, \ldots, 27\}$;

Paddings of all types *except* A. An example of a $K_4 - e$ design of order 27 with a hole of size 9 is given in [4] (Example 4.13). Fill in the hole with the 11 MMCKs of order 9 in Example 2.4.

Padding of type A

$C_1 = \{(1,18,15,16),(22,24,7,16),(2,25,16,20),(13,23,7,15),(4,10,7,20),(17,25,8,23),$
$(2,10,8,18),(9,20,8,13),(5,19,18,23),(14,25,13,18),(11,15,20,22),(6,21,20,23),$
$(5,17,14,20),(2,9,22,23),(1,4,14,23),(12,13,17,24),(7,9,1,17),(4,18,17,22),(19,20,1,24),$
$(8,21,1,22),(6,9,3,19),(1,11,3,25),(10,23,3,24),(10,22,19,25),(4,6,24,25),(11,19,2,17),$
$(4,12,2,21),(5,15,2,24),(3,16,17,21),(9,18,21,24),(7,16,15,20),(8,18,13,23),(20,23,14,22),$
$(1,17,22,24),(3,25,19,24),(2,21,17,24),(2,13,1,6),(7,11,6,18),(2,14,3,7),(5,22,3,13),$
$(9,11,4,5),(6,16,5,8),(10,16,9,14),(19,21,14,15),(13,21,10,11),(16,23,11,12),(10,12,1,5),$
$(21,25,5,7),(3,20,12,18),(7,19,8,12),(8,15,3,4),(16,19,4,13),(12,22,6,14),(8,24,11,14),$
$(12,25,9,15),(6,17,10,15),(26,6,1,18),(26,3,7,13),(26,5,4,8),(26,14,9,15),(26,11,10,12),$
$(27,5,1,7),(27,12,8,18),(27,4,3,13),(27,14,6,11),(27,15,9,10),(26,27,2,16),(26,27,17,19),$
$(26,27,20,21),(26,27,22,23),(26,27,24,25)\},$
$P_1 = \{\{26,27\},\{26,27\},\{26,27\},\{26,27\}\}.$

Example 4.11 (MMCK of order 28).

$S = \{1,2,3,\ldots,28\};$

An example of a $K_4 - e$ design of order 28 with a hole of size 18 is given in [2] (Example 4.15). Fill in the hole with the MMCK of order 18 in Example 4.5.

Example 4.12 (12 MMCKs of order 29).

$S = \{1,2,3,4,\ldots,29\};$

Paddings of *all* types. An example of a $K_4 - e$ design of order 29 with a hole of size 17 in given in [2] (Example 4.7). Fill in the hole with the 12 MMCKs of order 17 in Example 4.4.

5 The MMCK Construction

The following construction takes care of all cases $n \geq 32$. We will need the following well-known ingredient for the construction. Let X be a set of size $2k \geq 6$ and $\pi = \{x_1, x_2, \ldots, x_k\}$ a partition of X into 2-element subsets. The subsets in π are called *holes* (of size 2). Let $(X, \circ)$ be a *commutative quasigroup* with the property that for each hole $x_i \in \pi$, $(x_i, \circ)$ is a subquasigroup of order 2. Such a quasigroup is called a *commutative quasigroup with holes* π (of size 2), and such quasigroups exist for every order $2k \geq 6$ (see [3] for example).

The MMCK Construction. Let X be a set of size $2n \geq 6$, $\pi = \{x_1, x_2, \ldots, x_n\}$ a partition of X into holes of size 2, $(X, \circ)$ a commutative quasigroup with holes π, and H a set of size $h \in \{2,3,4,7,8,9\}$ such that $H \cap X = \emptyset$. Set $S = H \cup (X \times \{1,2,3,4,5\})$ and define a collection C of copies of $K_4 - e$ as follows:

(1) let $(H \cup (x_1 \times \{1,2,3,4,5\}), h_1)$ be a MMCK of order $10 + h$ with padding P and place the graphs of h_1 in C;

(2) for each $x_i \in \pi$, $x_i \neq x_1$, let $(H \cup (x_i \times \{1,2,3,4,5\}), H, h_i)$ be a $K_4 - e$ design with hole H and place the graphs of h_i in C; and

(3) for each x, y in different holes, place the 5 graphs $((x,1),(y,1),(x \circ y,2),(x \circ y,3))$, $((x,2),(y,2),(x \circ y,3),(x \circ y,4))$, $((x,3),(y,3),(x \circ y,4),(x \circ y,5))$, $((x,4),(y,4),(x \circ y,5),(x \circ y,1))$, and $((x,5),(y,5),(x \circ y,1),(x \circ y,2))$ in C.

Then (S, C) is a MMCK of order $h + 10n$ with padding P.

Lemma 5.1 *If* $m \equiv 2,3,4,7,8,$ *or* 9 *(mod 10)* ≥ 32, *there exists a MMCK of order* m. *All paddings are possible.*

Proof: Write $m = h + 2n \geq 32$, where $h \in \{2,3,4,7,8,9\}$ and $2n \geq 6$. Since there exist MMCKs of order 12, 13, 14, 17, 18 and 19 with all possible paddings (Examples 4.1, 4.2, 4.3, 4.4, 4.5, and 4.6), and $K_4 - e$ designs of order p with holes of size q for all $(p,q) = (12,2), (13,3), (14,4), (17,7), (18,8)$, and $(19,9)$ (Examples 1.5, 4.1, 4.2, 4.5, 4.6, and 4.9 in [2]) the statement of the lemma follows from the MMCK Construction.

Theorem 5.2 *The information in the tables in Section 2 is correct.*

Proof: The cases $n = 4,5,7,8,$ and 9 are handled in Sections 2 and 3. The cases $n \equiv 0,1,5$ or 6 (mod 10) are well-known and can be found in [1], for example. All of the remaining cases are taken care of in Section 4 if $12 \leq n \equiv 2,3,4,7,8$ or 9 (mod 10) ≤ 29 *and* in this section if $n \equiv 2,3,4,7,8$ or 9 (mod 10) ≥ 32.

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