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Introduction

The theory of symmetric functions plays a natural role in the construction of certain basic representations of affine Lie algebras, based either on Clifford or on Heisenberg algebras. The vertex operator construction[1] provides a link between such realizations (the so-called fermion-boson correspondence), and is intimately connected with the formulation and properties of solitonic integrable hierarchies[2]. Recently the fermion-boson correspondence has been used to investigate various classes of symmetric functions. Thus, parallels to the correspondence for the Schur S-functions

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(and the KP hierarchy) have been developed for the Schur Q-functions (related to the BKP hierarchy), and more generally for the Hall-Littlewood functions[3]. Matrix elements of products of (standard bosonic) vertex operators in the Schur function basis are essentially the composite, or rational, Schur functions[4]. Parallel statements exist for Q-functions, Hall-Littlewood functions, or even symmetric functions with arbitrarily weighted[5] inner products related to realizations of q-deformed Heisenberg algebras[6]. Finally, the fermion-boson correspondence has been used to prove new determinantal identities for Q-functions[7] as well as composite Schur S- and Schur Q-functions[8], along with new approaches to symmetric function products and plethysms[9].

In this note we consider a general class of new determinantal forms for (skew) Schur functions based on outer strip decompositions of (skew Ferrers) diagrams (when talking about decompositions we will use the words "diagram" and "tableau" interchangeably). These forms were introduced by Hamel and Goulden[10] and their equivalence to Schur functions was proved combinatorially using lattice paths. Similar determinantal and Pfaffian forms have been derived for Schur Q-functions[11], and symplectic and orthogonal Schur functions[12]. Here an extended fermion-boson correspondence is introduced, and it is suggested that the new determinantal forms are manifestations of Wick's theorem in the extended space. This is demonstrated explicitly for certain special cases. We follow the symmetric function notation of Macdonald[13].

The Fermion-Boson Correspondence

The infinite-dimensional Clifford algebra of (charged) free fermions is generated by elements ψ_i, ψ_i^* with $i \in \mathbb{Z}$ and defining relations

$$\begin{array}{l} \{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0, \\ \{\psi_i, \psi_j^*\} = \delta_{ij}, \quad i, j \in \mathbb{Z} \end{array} \tag{1}$$

The Fock representation $\mathcal{F}$ is defined by the choice of vacuum $|0>$ such that $\psi_i|0>= \psi_j^*|0>= 0$, $i \leq 0, j > 0$, with states $\psi_{i_1} \ldots \psi^*_{-j_1} \ldots |0>$, $i_k > 0, j_k \geq 0$ orthonormal with respect to an inner product which makes ψ_i, ψ_i^* adjoint. The space $\mathcal{F}$ carries a representation of an algebra $gl(\infty)$ generated by $E_{ij} =: \psi_i \psi_j^* :$, where $: vv' :\equiv vv' - < vv' >$, and the vacuum expectation value is $< vv' >= (|0>, vv'|0>)$. In particular, the operators $H_n = \sum_{i \in \mathbb{Z}} : \psi_i \psi_{i+n}^* :$ satisfy a Heisenberg algebra, and the eigenvalues of $ad\, H_0$ provide a $\mathbb{Z}$-grading on $\mathcal{F} \equiv \oplus_n \mathcal{F}_n$ which counts the total 'charge'.

There is an intimate connection between the space $\mathcal{F}$ and the universal ring $\Lambda = \mathbb{C}[x_1, x_2, x_3, \ldots]$ of symmetric polynomials in indeterminates $x_1, x_2, x_3, \ldots$. For a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$, we define in the usual way various generating sets such as the *elementary*, *complete*, or *power sum* symmetric functions[13], for example

$$\sum_{m=1}^{\infty} p_m(x) t^{m-1} \quad = \quad \frac{d}{dt} \log \prod_{i=1}^{\infty} (1 - x_i t)^{-1}.$$

Λ has the structure of a Hilbert space under the following inner product: if $p_\lambda \equiv p_{\lambda_1} p_{\lambda_2} \cdots$, etc., then

$$(p_\lambda, p_\mu)_\Lambda = z_\lambda \delta_{\lambda\mu}, \quad z_\lambda = \prod_i i^{m_i} i!,$$

where $\lambda = (n^{m_n}, \cdots, 2^{m_2}, 1^{m_1})$. The connection with 'free field representations' is seen from the fact that Λ carries a representation of a Heisenberg algebra: $p_n \leftrightarrow a_{-n}, D(p_n) \leftrightarrow a_n$.

A fundamental part of the fermion-boson correspondence is that the following 'vertex operators',

$$E(u,v) = \tfrac{u}{u-v}(Z(u,v)-1),$$
$$Z(u,v) = exp(\textstyle\sum_1^\infty (u^n - v^n)X_n)\, exp\Big(-\sum_1^\infty \frac{1}{n}(u^{-n}-v^{-n})\frac{\partial}{\partial X_n}\Big),$$

acting on functions of variables X_n, when expanded in Laurent modes with respect to u and v^{-1}, provide the generators E_{ij} of the algebra of $gl(\infty)$ realized above on the space of free fermions. After the change of variables $nX_n \to p_n(x)$, $\partial/\partial X_n \to D(p_n(x))$, we can consider equally the vertex operators acting on Λ. A key result[4] is that the matrix element of $Z(u,v)$ is in fact a well-known symmetric function:

$$\begin{aligned}
&Z(u,v)s_\alpha(x)\\
&= exp\left(\textstyle\sum \frac{1}{n}p_n(u|v)p_n(x)\right) exp\left(-\sum \frac{1}{n}p_n(\bar u|\bar v)D(p_n(x))\right) s_\alpha\\
&= (-1)^{|\alpha|}\textstyle\sum_{\beta,\rho} s_{\alpha'/\rho'}(\bar u|\bar v)(-1)^{|\rho|}s_{\beta/\rho}(u|v)s_\beta(x) \equiv (-1)^{|\alpha|}\sum_\beta s_{\bar\alpha';\beta}(u|v)s_\beta(x),
\end{aligned}$$

where the summation (over skew Schur functions of the arguments $\bar u \equiv u^{-1}, \bar v \equiv v^{-1}$, as well as u,v) can be recognised as the 'composite' Schur function with supersymmetric argument[4]. Crucial to the proof of this property are the exponential or so-called Cauchy identities,

$$exp\left(\sum_n \frac{1}{n}p_n(x)p_n(y)\right) = \sum_\lambda s_\lambda(x)s_\lambda(y).$$

In fact, the formulae are valid for *any number* of arguments u, v, so that (after normal ordering) a formally identical expression also holds for the matrix element of a product of N vertex operators. These results also admit considerable generalization. For example, symmetric polynomials can be defined for a generic class of inner products[5] on Λ, labelled by a sequence of positive real numbers $\alpha = (\alpha_1, \alpha_2, \cdots)$. Define $(p_\lambda, p_\mu) = \delta_{\lambda\mu}\, z_\lambda\, \alpha_\lambda$, where $\alpha_\lambda = \alpha_n^{m_n}\cdots\alpha_1^{m_1}$ for a partition $\lambda = (n^{m_n}\cdots 1^{m_1})$. Then there is an orthogonal basis (defined by Gram-Schmidt orthogonalization from the monomial symmetric functions) for which the above properties of vertex operators generalize. For example, if we take $\alpha_n = \alpha(q^{\kappa n} - q^{-\kappa n})/(q^{2n} - q^{-2n})$, then for appropriate choices of α and κ one can build level k realizations of affine $\widehat{sl}_q(2)$ and give explicit trace and matrix element expressions[6] using the q-deformed vertex operators.

The link between $\mathcal{F}$ and Λ is made formally by introducing the operator $e^{H(x)}$, well-defined on $\mathcal{F}$, where $H(x) = \sum_{n=1}^\infty p_n(x)H_n/n$. Then for $\mathcal{F}_0$ there is an isomorphism σ_0 such that

$$\begin{aligned}
\sigma_0(a|0>) &= <e^{H(x)}a>,\\
(\sigma_0(a|0>), \sigma_0(b|0>)_\Lambda &= (a|0>, b|0>)
\end{aligned}\tag{2}$$

which also extends to arbitrary charge ℓ. As a result, the symmetric polynomials related to any desired elementary basis of Λ can be found in $\mathcal{F}$. For example the Schur functions are recovered for a partition $\lambda = (\lambda_1 - 1 \ldots \lambda_r - r|\lambda_1' - 1 \ldots \lambda_r' - r)$ in Frobenius notation as:

$$s_\lambda(x) = \sigma_0\left(e^{H(x)}\psi_{\lambda_1}\psi_{\lambda_2-1}\cdots\psi_{\lambda_r-r+1}\psi^*_{-\lambda_1'+1}\cdots\psi^*_{-\lambda_r'+r}|0>\right).$$

Fermionic representations of symmetric polynomials of this type allow powerful algebraic machinery to be brought to bear in verifying properties and suggesting new interrelationships. One of the major tools is Wick's theorem. For v_i either ψ_i or ψ_i^*, this gives the vacuum expectation value of a monomial in (an even number of) v_i as a Pfaffian,

$$< v_1 \cdots v_n >= \mathrm{Pf}(< v_i v_j >).$$

For the case of s_λ above, the Pfaffian simplifies to a determinant (since $< \psi\psi' >=\ < \psi^*\psi'^* >= 0$) and produces the Giambelli formula[13] for a Schur function in terms of its Frobenius parts,

$$s_\lambda = \det\left(s_{(\lambda_i - i|\lambda'_j - j)}\right)$$

This can easily be generalized to give a new formula[8] for the corresponding composite Schur functions (associated with rational tableaux), which may be proved in essentially the same way, using the general vertex operator matrix element noted above, and mapping back to the fermionic space:

$$s_{\bar{\lambda};\mu}(x) = \det\begin{pmatrix} s_{(\lambda_i - i|\lambda'_j - j)}(\frac{1}{x}) & s_{\overline{\lambda_i - i};1^{\mu'_j - j}}(x) \\ s_{\overline{\mu_i - i};1^{\lambda'_j - j}}(\frac{1}{x}) & s_{(\mu_i - i|\mu'_j - j)}(x) \end{pmatrix}.$$

The equivalent of this formula also for Q-functions has recently been developed[8].

Determinantal Forms, Strip Decompositions and an Extended Fermion-Boson Correspondence

We now turn to a general class of Schur function determinantal forms which have recently been established combinatorially using Gessel–Viennot lattice path techniques, and which are based on planar decompositions of the underlying skew shape tableau[10]. A decomposition divides the tableau λ/μ into a class of edgewise connected sets of boxes called strips θ_i such that the strips are "nested," are nonoverlapping, and each intersect the left or bottom, *and* the top or right perimeter of the tableau (see Figure 1). The determinant $s_{\lambda/\mu} = \det(s_{\theta_i\#\theta_j})$ involves strips $\theta_i\#\theta_j$ which are determined from θ_i and θ_j by a series of slide moves along top–left–to–bottom–right diagonals, with appropriate in-filling and boundary conditions for cases where the overlaying of θ_i on θ_j is not in general position, i.e. superimpose θ_i on θ_j such that the box of content k (where content is the column index minus the row index) in θ_i is superimposed on the box of content k in θ_j for all k. Then $\theta_i\#\theta_j$ is the strip defined by the boxes between the last box of θ_i (that is, the box on the top or right perimeter) and the first box of θ_j (that is, the box on the left or bottom perimeter). For example, in Figure 1, if $\theta_1 = 2$ and $\theta_2 = 64/3$, then $\theta_1\#\theta_2 = 4$ and $\theta_2\#\theta_1 = 42/1$ (note the operation is noncommutative). For the general case, see Hamel and Goulden[10].

Thus for the decomposition of the tableau $8,6,2,1/3,2$ in Figure 1, the determinant reads[10]

$$s_{8,6,2,1/3,2} = \det\begin{pmatrix} s_2 & s_4 & s_{71} \\ s_{42/1} & s_{64/3} & s_{971/6} \\ 0 & 0 & s_{21} \end{pmatrix}.$$

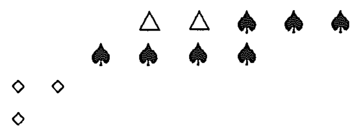

Figure 1: An outside decomposition of 8, 6, 6, 2, 1/3, 2.

The existence of such generalized determinantal forms poses a challenge for algebraic techniques based on Wick's theorem and the fermion-boson correspondence as outlined in the previous section. Additional generality is provided by a sequence of isomorphisms at each charge sector, $\sigma_\ell : \mathcal{F}_\ell \to \Lambda$ such that

$$\sigma_\ell(a|0>) =< \ell|e^{H(x)}a|0>,$$

for suitable ground states $|\ell>$. For example, an alternative to the Giambelli form of $s_\lambda(x)$ is the Foulkes form, which reads $s_\lambda = \det(s_{\lambda_i - i+1,1^{m-j}})$ where $m \geq \max(\lambda_1, \lambda'_1)$ and derives via Wick's theorem from

$$s_\lambda =< 0|e^{H(x)}\psi_{\lambda_1}\psi_{\lambda_2-1}\cdots\psi_{\lambda_m-m+1}\psi^*_0\psi^*_{-1}\cdots\psi^*_{-m}|0> .$$

Thus it could be anticipated that more general determinantal forms for $s_\lambda(x)$ might derive from a systematic treatment of the fermion-boson correspondence at different charge sectors (see also [7,9]).

Here however we propose an extended construction based on the view that strip decompositions of arbitrary shape should be accommodated in a more general algebraic scheme. In fact, the scheme is demonstrated only for the $\lambda_1 + \lambda'_1 + 1$ outer decompositions interpolating between the Jacobi-Trudi, rows-only determinantal form $(\det(s_{\lambda_i - i+j}))$, and the dual Jacobi-Trudi, columns-only forms $(\det(s_{1^{\lambda'_i - i+j}}))$, with the Giambelli principal hook decomposition as a non-extended case. We call these *rows-first* or *columns-first*[12] outside decompositions. They were known to Littlewood[14] (p. 114) who used algebraic techniques to prove the determinants they generated were equal to the Schur function.

We conjecture that the extended construction described here applies more generally to yield also the determinantal forms given in Hamel and Goulden[10]. These determinantal forms are based on arbitrary outside decompositions and include the skew Giambelli and rim ribbon results of Lascoux and Pragacz[15,16]. Note further that other determinantal forms due to Lascoux and Pragacz[16] (and not representable in terms of outside decompositions) are also candidates for investigation by these methods.

The basic additional structure needed is a representation of a skew Schur function, $s_{\lambda/\mu}$, in terms of an extended fermionic space. Noting that $s_{\lambda/\mu} = D(s_\mu)s_\lambda$ and the fact that $D(\cdot)$ provides an algebra (anti-)homomorphism, it follows from the linearity of $H(x)$ that

$$s_{\lambda/\mu} =< 0|e^{D(H(x))}a_\mu|0>< 0|e^{H(x)}a_\lambda|0>

and a_λ, a_μ are the operators (at charge 0) associated with λ, μ respectively. At this point we may introduce an independent fermionic space $\widetilde{\mathcal{F}} \simeq \mathcal{F}$ generated by χ_i, χ_j^*, $i, j \in \mathbb{Z}$, with associated vacuum $|\tilde{0}>$, and Heisenberg generators $\widetilde{H}_n$, such that

$$s_{\lambda/\mu} =< e^{\widetilde{H}(x)} e^{H(x)} \widetilde{a}_\mu a_\lambda >$$

where $\widetilde{H}(x) = \sum_{n=1}^{\infty} \widetilde{H}_n D(p_n(x))/n$ and the vacuum expectation value is with respect to the tensor product $|0> \otimes |\tilde{0}>$. Substituting now the explicit monomials in ψ, χ for $\widetilde{a}_\mu$ and a_λ gives an expression which may be manipulated directly using Wick's theorem for the tensor product of the two Clifford algebras of free fermions. Moreover, it is possible in certain cases to define *composite* 'fermion' operators ϕ_i, ϕ_j^* for which Wick's theorem still applies, as we now show.

Given a partition λ, consider the 'rows first outside decomposition' where as strips the first k rows of λ are taken for any fixed k, $1 \leq k \leq \lambda_1'$, and the remainder are hooks, i.e. the decomposition is $\theta_1 = \lambda_1$, $\theta_2 = \lambda_2, \ldots \theta_k = \lambda_k, \theta_{k+1} = \lambda_{k+1}, 1^{\lambda_1'-k}, \ldots \theta_{k+i} = \lambda_{k+i} - i, 1^{\lambda_i'-k-i+1} \ldots$. The number c of such hook strips will be equal to the number of boxes in λ of content $-k$. As far as this decomposition is concerned, the hook diagonal has been shifted down k positions. However, the decomposition can equally be regarded as originating from a *principal* hook (Giambelli) decomposition of a tableau *augmented* by k columns of length λ_1', and skewed by the block $k^{\lambda_1'}$. For this case the product $\widetilde{a_\mu} a_\lambda$ is, up to an overall sign resulting from rearranging the ψ and ψ^* terms,

$$\prod_{i=1}^{k} \chi_{k-i+1} \chi^*_{-(\lambda_1'-i)} \prod_{i=1}^{k} \psi_{\lambda_i+k-i+1} \psi^*_{-(\lambda_1'-i)} \prod_{i=1}^{c} \psi_{\lambda_{k+i}-i+1} \psi^*_{-(\lambda_i'-k-i)}$$

Set

$$\phi_i = \psi_{\lambda_i+k-i+1}, \; \phi_i^* = \psi^*_{-(\lambda_1'-i)} \chi_{k-i+1} \chi^*_{-(\lambda_1'-i)}$$

for $1 \leq i \leq k$, and

$$\phi_{k+i} = \psi_{\lambda_{k+i}-i+1}, \; \phi^*_{k+i} = \psi^*_{-(\lambda_i'-k-i)}$$

for $1 \leq i \leq c$. Then if Wick's theorem is assumed for the ϕ and ϕ^* fermions the determinantal form

$$\det(< e^{\widetilde{H}} e^H \phi_i \phi_j^* >)_{(k+c)\times(k+c)}$$

is suggested. Remarkably, if the entries are interpreted separately as individual skew tableaux using the above rules, the resulting determinant coincides with the strip decomposition form[10] for the present case, where θ_i, $1 \leq i \leq k$ are rows, and θ_j, $k+1 \leq j \leq k+c$ are hooks as above, and the vacuum expectation values of the $\phi_i \phi_j^*$ products precisely reproduce the $\theta_i \# \theta_j$ strip calculus[10] for this case.

Moreover, for a skew tableau λ/μ for any $k > \mu_1'$, with the same augmentation, but allowing for the additional skewing of λ itself by adjusting the labels on the χ, χ^* operators, the $\widetilde{a_\mu} a_\lambda$ product

$$\prod_{i=1}^{k} \psi_{\lambda_i+k-i+1} \psi^*_{-(\lambda_1'-i)} \prod_{i=1}^{c} \psi_{\lambda_{k+i}-i+1} \psi^*_{-(\lambda_i'-k-i)} \prod_{i=1}^{k} \chi_{k-i+\mu_i+1} \chi^*_{-(\lambda_1'-i)}$$

$$\begin{array}{cccc}
 & & \triangle & \triangle \\
 & & \diamondsuit & \diamondsuit \\
 & & \heartsuit & \heartsuit \\
\bigtriangledown & \bigtriangledown & \bigtriangledown & \\
\spadesuit & \spadesuit & \spadesuit & \\
\spadesuit & \clubsuit & \clubsuit & \\
\spadesuit & \clubsuit & & \\
\spadesuit & \clubsuit & &
\end{array}$$

Figure 2: A rows–first decomposition for $s_{44433322/222}$, $k = 4$.

allows for composite operators ϕ_i, ϕ_j^* to be defined in a precisely analogous way to the above, but with χ_{k-i+1} replaced by $\chi_{k-i+\mu_i+1}$. For the example given in Figure 2,

$$\phi_1 = \psi_8,\ \phi_2 = \psi_7,\ \phi_3 = \psi_6,\ \phi_4 = \psi_4,\ \phi_5 = \psi_3,\ \phi_6 = \psi_2$$

$$\phi_1^* = \psi_{-7}^*\chi_6\chi_{-7},\ \phi_2^* = \psi_{-6}^*\chi_5\chi_{-6},\ \phi_3^* = \psi_{-5}^*\chi_4\chi_{-5},\ \phi_4^* = \psi_{-4}^*\chi_1\chi_{-4},\ \phi_5^* = \psi_{-3}^*,\ \phi_6^* = \psi_{-2}^*$$

and the determinant produced is

$$\begin{aligned}
s_{44433322/222} &= \det \begin{pmatrix}
s_{81^7/61^7} & s_{81^6/51^6} & s_{81^5/41^5} & s_{81^4/11^4} & s_{81^3} & s_{81^2} \\
s_{71^7/61^7} & s_{71^6/51^6} & s_{71^5/41^5} & s_{71^4/11^4} & s_{71^3} & s_{71^2} \\
s_{61^7/61^7} & s_{61^6/51^6} & s_{61^5/41^5} & s_{61^4/11^4} & s_{61^3} & s_{61^2} \\
s_{41^7/61^7} & s_{41^6/51^6} & s_{41^5/41^5} & s_{41^4/11^4} & s_{41^3} & s_{41^2} \\
s_{31^7/61^7} & s_{31^6/51^6} & s_{31^5/41^5} & s_{31^4/11^4} & s_{31^3} & s_{31^2} \\
s_{21^7/61^7} & s_{21^6/51^6} & s_{21^5/41^5} & s_{21^4/11^4} & s_{21^3} & s_{21^2}
\end{pmatrix} \\
&= \det \begin{pmatrix}
s_2 & s_3 & s_4 & s_7 & s_{81^3} & s_{81^2} \\
s_1 & s_2 & s_3 & s_6 & s_{71^3} & s_{71^2} \\
1 & s_1 & s_2 & s_5 & s_{61^3} & s_{61^2} \\
0 & 0 & 1 & s_3 & s_{41^3} & s_{41^2} \\
0 & 0 & 0 & s_2 & s_{31^3} & s_{31^2} \\
0 & 0 & 0 & s_1 & s_{21^3} & s_{21^2}
\end{pmatrix},
\end{aligned}$$

which is exactly the determinant produced from the strip decomposition approach of Hamel and Goulden[10] with $\theta_1 = 2$ (the $\triangle$), $\theta_2 = 2$ (the $\diamondsuit$), $\theta_3 = 2$ (the $\heartsuit$), $\theta_4 = 3$ (the $\bigtriangledown$), $\theta_5 = 3111$ (the $\spadesuit$), and $\theta_6 = 211$ (the $\clubsuit$).

The procedure also generalizes to a 'columns-first outside decomposition' where the first strips are the first k columns and the remaining strips are hooks. Here the augmentation involves adding k rows to the top of the tableau, and skewing by the block λ_1^k. For such a case, the $\widetilde{a_\mu} a_\lambda$ monomial is

$$\prod_{i=1}^{k} \chi_{\lambda_1-i+1}\chi^*_{-(k-i)} \prod_{i=1}^{k} \psi_{\lambda_1-i+1}\psi^*_{-(\lambda_i+k-i)} \prod_{i=1}^{c} \psi_{\lambda_i-k-i+1}\psi^*_{-(\lambda'_{k+i}-i)}.$$

$$\begin{array}{cccccccc}
\triangle & \triangle & \triangle & \triangle & \triangle & \triangle & \triangle & \triangle \\
\triangle & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond \\
\triangle & \diamond & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit \\
\triangle & \diamond & \heartsuit & \nabla & \nabla & \nabla & \nabla & \\
\triangle & \diamond & \heartsuit & \nabla & \spadesuit & \spadesuit & \spadesuit & \\
\triangle & \diamond & \heartsuit & \nabla & \spadesuit & \clubsuit & \clubsuit & \\
\triangle & \diamond & \heartsuit & \nabla & \spadesuit & \clubsuit & & \\
\triangle & \diamond & \heartsuit & \nabla & \spadesuit & \clubsuit & &
\end{array}$$

Figure 3: Figure 2 augmented.

$$\begin{array}{ccccc}
\diamond & \spadesuit & \triangle & \triangle & \triangle \\
\diamond & \spadesuit & \triangle & & \\
\diamond & \spadesuit & \triangle & & \\
\diamond & & & &
\end{array}$$

Figure 4: A decomposition for 5331.

Setting

$$\phi_i = \psi_{\lambda_1-i+1}\chi_{\lambda_1-i+1}\chi^*_{-(k-i)},\ \phi_i^* = \psi^*_{-(\lambda_i'+k-i)}$$

for $1 \leq i \leq k$, and

$$\phi_{k+i} = \psi_{\lambda_i-k-i+1}, \phi_i^* = \psi^*_{-(\lambda'_{k+i}-i)}$$

for $1 \leq i \leq c$, and again the determinant obtained by invoking Wick's theorem for the *composite* 'fermions' ϕ_i, ϕ_j^* reproduces the strip decomposition form[10]. For example, for Figure 4, the determinant is

$$\det \begin{pmatrix} s_{51^5/51} & s_{51^3/51} & s_{51^2/51} \\ s_{41^5/4} & s_{41^3/4} & s_{41^2/4} \\ s_{31^5} & s_{31^3} & s_{31^2} \end{pmatrix}.$$

The Giambelli principal hook decomposition thus emerges as a special case ($k = 0$) for which *no* augmentation to an extended fermionic space is needed in the corresponding determinantal form. By contrast, the Jacobi-Trudi form is associated with a rows-only decomposition corresponding to augmentation of λ by a maximal block $(\lambda_1' - 1) \times \lambda_1'$ to the left, while the dual Jacobi-Trudi form is associated with a columns-only decomposition corresponding to augmentation of λ by a maximal block $\lambda_1 \times (\lambda_1 - 1)$ on the top.

Conclusions

The fermion-boson correspondence has been reviewed in the context of algebraic approaches to the theory of symmetric polynomials. In particular, new classes of determinantal forms for Schur functions associated with skew tableaux[10] have been considered in the framework of Wick's theorem. The new formulae generalize existing determinants by allowing planar 'outer' decompositions of tableaux into congruent strips of arbitrary shape. A restricted class, interpolating between the rows-only Jacobi-Trudi case, through the principal hook Giambelli case, to the columns-only dual Jacobi-Trudi case, has been shown to be accommodated through Wick's theorem for *composite* 'fermion' operators acting in an extended space, and which is mapped to the bosonic space of symmetric polynomials. The fact that the new determinantal forms exist for orthogonal and symplectic tableaux[12] strongly suggests[4] a generalised fermion-boson correspondence for these cases also.

1. V. G. Kac, D. A. Kazhdan, J. Lepowsky, and R. L. Wilson, Realization of the basic representations of the Euclidean Lie algebras, *Adv. in Math.*, 42 (1981) 83; I. B. Frenkel and V. G. Kac, Basic representations of affine Lie algebras and dual resonance models, *Invent. Math.*, 62 (1980) 23; G. Segal, Unitary representations of some infinite-dimensional groups, *Commun. Math. Phys.*, 80 (1981) 301
2. I. B. Frenkel, Two constructions of affine Lie algebra representations and the fermion-boson correspondence in quantum field theory, *J. Funct. Anal.*, 44 (1981) 259; M. Jimbo and T. Miwa, Solitons and Infinite-Dimensional Lie algebras, *Publ. RIMS, Kyoto Univ.*, 19 (1983) 943
3. N. Jing, Vertex operators, symmetric functions and the spin group Γ_n, *J. Alg.*, 138 (1991) 340; Vertex operators and Hall-Littlewood symmetric functions, *Adv. Math.*, 87 (1991) 226
4. P. D. Jarvis and C. M. Yung, The Schur function realization of vertex operators, *Lett. Math. Phys.*, 26 (1992) 115; Vertex operators and composite supersymmetric S-functions, *J. Phys. A: Math. Gen.*, 26 (1993) 1881; Symmetric functions and the KP and BKP hierarchies, *J. Phys. A: Math. Gen.*, 28 (1995) 589; T. H. Baker, P. D. Jarvis, D. S. McAnally and C. M. Yung, Symmetric functions, vertex operators and applications, *Proc. XXthICTGMP*, World Scientific, (to appear)
5. S. V. Kerov, Hall-Littlewood functions and orthogonal polynomials, *Funct. Anal. Appl.*, 25 (1991) 65
6. T H Baker, *Symmetric Functions and Infinite-Dimensional Algebras*, PhD Thesis, University of Tasmania, (Hobart, 1994)
7. Yuching You, On some identities of Schur Q-functions, *J. Alg.*, 145 (1992) 349
8. P. D. Jarvis and C. M. Yung, Determinantal forms for composite Schur and Q-functions via the boson-fermion correspondence, *J. Phys. A: Math. Gen.*, 27 (1994) 903
9. T. H. Baker, Symmetric function products and plethysms and the boson-fermion correspondence, *J. Phys. A: Math. Gen.*, 28 (1995) 589
10. A.M. Hamel and I.P. Goulden, Planar decompositions of tableaux and Schur function determinants, *Europ. J. of Combin*, to appear.
11. A.M. Hamel, Pfaffians and determinants for Schur Q–functions, University of Canterbury Research Report, No. 111.
12. A.M. Hamel, Determinantal forms for symplectic and orthogonal Schur functions, *Can. J. Math.*, to appear.
13. I.G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, Oxford, 1979.
14. D.E. Littlewood, *The Theory of Group Characters*, 2nd. ed., Oxford University Press, Oxford, 1950.

15. A. Lascoux and P. Pragacz, Équerres et fonctions de Schur, *C.R. Acad. Sci. Paris, Série I, Math*, 299 (1984), 955–958.
16. A. Lascoux and P. Pragacz, Ribbon Schur functions, *Europ. J. Combin.*, 9 (1988), 561–574.