RESEARCH REPORT

ON THE NON-EXISTENCE OF 5-(24,12,6) AND 4-(23,11,6) DESIGNS

by

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Abstract

It is shown that 5-(24,12,6) and 4-(23,11,6) designs cannot exist and that consequently a non-trivial 2-(2n+1,n,n-1) design can be extended to a 4-(2n+3,n+2,n-1) design if and only if $n = 4$.

ON THE NON-EXISTENCE OF 5-(24,12,6) AND 4-(23,11,6) DESIGNS

DEFINITIONS AND NOTATION

A t-(v,k,λ) design is formed from a set of v symbols (or points, or varieties) by taking b subsets, called blocks, each of size k, subject to the conditions that every unordered t-tuple must appear just λ times and no two blocks are identical. Let λ_i, $(0 \leqslant i \leqslant t)$, be the number of times each unordered i-tuple appears in the design. Then $\lambda_0 = b$ the number of blocks; $\lambda_1 = r$, the number of replications of each symbol; and $\lambda_t = \lambda$. The complete set of the λ_i's is given by the standard equations

$$\lambda_i = \lambda \frac{(v-i)(v-i-1)(v-i-2)\dots(v-t+1)}{(k-i)(k-i-1)(k-i-2)\dots(k-t+1)}; \quad 0 \leqslant i \leqslant t-1. \qquad (1)$$

Let n_i, $(0 \leqslant i \leqslant k)$, be the number of blocks intersecting a given block in exactly i points. Note that $n_k = 1$. Another standard set of equations connects the λ_i's and the n_i's through

$$\sum_{i=0}^{k} n_i = b = \lambda_0;$$

and $$\sum_{i=0}^{k} i(i-1)\dots(i-j)n_i = [k(k-1)\dots(k-j+1)]\lambda_j; \quad 1 \leqslant j \leqslant t. \qquad (2)$$

Given a t-design a $(t-1)$-design can be formed by rejecting all blocks not containing a given symbol which is then deleted from the remaining blocks. The smaller design is called a restriction or contraction of the larger. The reverse process is sometimes possible, in which case the same new symbol is added to all the blocks of (t-1)-design and the further blocks not containing the new symbol are adjoined to make a t-design. The new design is called an extension of the original design.

The complement of a t-design is also a t-design and is formed by replacing each block by the set of symbols not contained in it.

BACKGROUND

It is well-known that any $2-(2n+1,n,\lambda)$ design can be extended to a $3-(2n+2),n+1,\lambda)$ design by complementation; that is to say by adding the same new symbol to each original block and then adjoining the complement of each block with respect to the $2n+2$ symbols then available. For the Hadamard 2-designs, in which $\lambda = \frac{1}{2}$ and n is odd, it is known that this is the only way of extending to a 3-design. Furthermore, if a Hadamard design has a repeated extension to a $4-(2n+3,2n+2, \frac{n}{2}-\frac{1}{2})$ design then from (1), $(n+2)|6$. Since n has to be odd the only possibility is $n=1$ corresponding to a vacuous situation. Therefore a Hadamard design never extends to a 4-design.

For the family of designs with the larger value $\lambda = n-1$ however, it is sometimes possible to extend to a $3-(2n+2,n+1,n-1)$ design other than by complementation. In particular it has been shown (Breach, [1]) that among the eleven $2-(9,4,3)$ designs there are two which have extensions other than by complementation and one of these extends to a $4-(11,6,3)$ design. Are there any other values of n for which an extension to a 4-design is possible? If so the condition $(n+2)|12$ must be satisfied. Thus $n = 1,2,4$ or 10. The case $n=1$ is vacuous; the case $n=2$ is trivial since a 4-(7,4,1) design contains all possible combinations of 4 symbols from 7 symbols; the case $n=4$ is already decided; thus the only further possibility is that there exists a 2-(21,10,9) design which extends to a 4-(23,12,9) design. This note shows that such a 4-design cannot exist. This is achieved by showing that the complementary design 4-(23,11,6), and its possible extension to a 5-(24,12,6) design cannot exist.

THE PROOF OF NON-EXISTENCE

(a) *Any 4-(23,11,6) design can be extended to a 5-(24,12,6) design by complementation.*

For if N_5 is the number of blocks of the 4-design which contain all of 5 given symbols and N_0 is the number containing none of them, then by the principle of inclusion and exclusion,

$$N_0 = \lambda_0 - \binom{5}{1}\lambda_1 + \binom{5}{2}\lambda_2 - \binom{5}{3}\lambda_3 + \binom{5}{4}\lambda_4 - N_5 . \qquad (3)$$

But for a 4-(23,11,6) design, from (1),

$$\lambda_0 = 161, \ \lambda_1 = 77, \ \lambda_2 = 35, \ \lambda_3 = 15, \ \lambda_4 = 6. \qquad (4)$$

Therefore $N_0 = 6 - N_5$ and the result follows.

(b) *A self-complementary 5-(24,12,6) design cannot exist.*

Consider two blocks A and B. Then if A intersects B in i points, A intersects the complement of B in $12-i$ points. Thus the block intersection numbers for A are such that $n_i = n_{12-i}$. In particular $n_0 = n_{12} = 1$.

Let (λ_i) stand for the block intersection equation of (2) in which λ_i appears. Then the linear combination $(\lambda_5) - 2(\lambda_4) - 360(\lambda_2) + 1800(\lambda_1)$ reduces to

$$21{,}600\, n_1 + 8{,}640\, n_2 + 2{,}592\, n_3 + 432\, n_4 = -12^2.10.9.$$

This has no solution in non-negative integers. Therefore a self-complementary 5-(24,12,6) design cannot exist.

(c) *There are no 5-(24,12,6) or 4-(23,11,6) designs.*

This follows by combining (a) and (b) since any such 5-design would have a restriciton to a 4-design which in turn would have an extension to a self-complementary 5-design.

REMARKS.

(i) Since a 4-(23,12,9) design cannot exist, being the complement of a 4-(23,11,6) design, it follows that *a non-trivial 2-(2n+1, n, n-1) design can be extended to a 4-design if and only if n = 4. The resulting 4-(11,6,3) design is unique.*

A construction and proof of uniqueness is given in Breach [1].

(ii) Although a 4-(23,12,9) design cannot exist there are 3-(22,11,9) designs. The writer has constructed some by ad hoc methods.

(iii) A chain of restrictions from 5-(24,12,6) leads to a 2-(21,9,6) design. There is an example of such a design listed in Hall [3]. This raises the question, is it ever possible to extend a 2-(21,9,6) design to a 3-(22,10,6) design?

(iv) It has been shown that a 5-(24,12,λ) design cannot exist if $\lambda = 6$ but there does exist a 5-design with a larger value of λ, namely 48. This is described by Conway [2] who shows that the Mathieu group M_{24} is quintuply transitive on a set of 2576 dodecads. Is $\lambda = 48$ the smallest value for which a 5-(24,12,λ) exists?

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