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Abstract. Square matrices are shown to be diagonalizable over all known classes of (von Neumann) regular rings. This diagonalizability is equivalent to a cancellation property for finitely generated projective modules which conceivably holds over all regular rings. These results are proved in greater generality, namely for matrices and modules over exchange rings, where attention is restricted to regular matrices.

Introduction

The aim of this paper is to study the question of diagonalizability for matrices over regular rings, and somewhat more generally, for regular matrices over exchange rings. The theme of the paper is that diagonalizability properties are equivalent to cancellation conditions for finitely generated projective modules.

Let us say that an $m \times n$ matrix A over a ring R *admits a diagonal reduction* if there exist invertible matrices $P \in \mathrm{GL}_m(R)$ and $Q \in \mathrm{GL}_n(R)$ such that PAQ is a diagonal matrix. Following Henriksen [**11**, p. 133], R is called an *elementary divisor ring* provided all square matrices over R admit diagonal reductions. This is less stringent than Kaplansky's definition of an elementary divisor ring [**12**, p. 465], since Kaplansky requires a stronger form of diagonal reduction. The central problem we address is the question of whether every (von Neumann) regular ring is an elementary divisor ring (cf. [**16**, Question 6]). Henriksen [**11**, Theorem 3] has proved that every unit-regular ring is an elementary divisor ring.

The diagonalizability question for rectangular matrices was answered by Menal and Moncasi [**15**, Theorem 9], who showed that all rectangular matrices over a given regular ring R admit diagonal reductions if and only if the finitely generated projective R-modules enjoy the following cancellation law:

$$2R \oplus A \cong R \oplus B \quad \Longrightarrow \quad R \oplus A \cong B.$$

This condition does not hold in general: For instance, if $2R \cong R \neq 0$, the condition fails in the case $A = B = 0$. Further, the stable rank (in the sense of K-theory) of a regular ring satisfying the above condition is at most 2 [**15**, Proposition 8].

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We prove that a regular ring R is an elementary divisor ring if its finitely generated projective modules satisfy the following cancellation law, which we call *separativity*:

$$A \oplus A \cong A \oplus B \cong B \oplus B \quad \Longrightarrow \quad A \cong B.$$

In fact, separativity is equivalent to the assumption that all corner rings eRe (for idempotents $e \in R$) are elementary divisor rings. It can be shown that all known classes of regular rings enjoy separativity, and thus are elementary divisor rings. No non-separative regular rings are known, and hence it is conceivable that all regular rings are elementary divisor rings. In particular, our results make it is easy to exhibit regular elementary divisor rings which are not unit-regular, and which do not satisfy the Menal-Moncasi conditions. Thus we provide a very strong answer to Henriksen's question whether a regular ring can be an elementary divisor ring without being unit-regular [**11**, Section 3(F)]. Our results also provide a large class of regular rings over which all square matrices are diagonalizable, but some rectangular matrices are not. The corresponding phenomenon for matrices over serial rings was exhibited by Levy in [**14**].

The methods of Menal and Moncasi mix module-theoretic and matrix-theoretic techniques, as do those of other work on regular matrices in the literature, such as [**7**, **8**, **9**, **10**]. We were unable to adapt these kinds of methods to the problem of diagonalizing square matrices over regular rings. Instead, we work almost entirely in the context of modules and homomorphisms. The methods we develop apply equally well to rectangular as to square matrices, and they easily yield a new proof of the Menal-Moncasi theorem.

All our proofs carry over, with no extra effort, to the case of exchange rings (cf. Section 1 for the definition), provided we restrict attention to (von Neumann) regular matrices. Hence, we derive our main results for regular matrices over exchange rings.

We consider only unital rings and unital modules. Modules will be right modules unless otherwise specified, and homomorphisms will act on the left of their arguments. Our notation is standard; see for instance [**6**]. In particular, we write nA for the direct sum of n copies of a module A.

1. Exchange rings and separative cancellation

Definition. A module M has the *exchange property* (see [**5**]) if for every module A and any decompositions

$$A = M' \oplus N = \bigoplus_{i \in I} A_i$$

with $M' \cong M$, there exist submodules $A'_i \subseteq A_i$ such that

$$A = M' \oplus \left(\bigoplus_{i \in I} A'_i \right).$$

(It follows from the modular law that A'_i must be a direct summand of A_i for all i.) If the above condition is satisfied whenever the index set is finite, M is said to satisfy the *finite exchange property*. Clearly a finitely generated module satisfies the exchange property if and only if it satisfies the finite exchange property. It should be emphasized that the direct sums in the definition of the exchange property are internal direct sums of submodules of A. One advantage of the resulting internal direct sum decompositions (as opposed to isomorphisms with external direct sums) rests on the fact that direct summands with common complements are isomorphic – e.g., $N \cong \bigoplus_{i\in I} A'_i$ above since each of these summands of A has M' as a complementary summand.

Definition. Following [**18**], we say that a ring R is an *exchange ring* if the module R_R satisfies the (finite) exchange property. By [**18**, Corollary 2], this definition is left-right symmetric. If R is an exchange ring, then every finitely generated projective R-module has the exchange property (by [**5**, Lemma 3.10], the exchange property passes to finite direct sums and to direct summands), and so the endomorphism ring of any such module is an exchange ring.

The class of exchange rings is quite large. It includes all semiregular rings (i.e., rings which modulo the Jacobson radical are regular and have idempotent-lifting), all π-regular rings, and more; see [**18**, **17**]. It also includes all C*-algebras with real rank zero [**2**].

Proposition 1.1. *Assume that R is an exchange ring. If $A_1, \dots, A_m$ and $B_1, \dots, B_n$ are finitely generated projective R-modules such that $A_1 \oplus \cdots \oplus A_m \cong B_1 \oplus \cdots \oplus B_n$, then there exist decompositions $A_i = C_{i1} \oplus \cdots \oplus C_{in}$ for each i such that $C_{1j} \oplus \cdots \oplus C_{mj} \cong B_j$ for each j.*

Proof. This is a special case of [**5**, Theorem 4.1]. (Cf. [**6**, Theorem 2.8] for the case of regular rings.) We give the proof since it is easy and it illustrates the use of the exchange property. An obvious induction reduces the problem to the case $m = n = 2$.

It suffices to consider the case of an internal direct sum decomposition $P = A_1 \oplus A_2 = B_1 \oplus B_2$. Since B_1 has the exchange property, $P = B_1 \oplus C_{12} \oplus C_{22}$ for some submodules $C_{i2} \subseteq A_i$; moreover, $A_i = C_{i1} \oplus C_{i2}$ for some C_{i1}. Now $P = B_1 \oplus (C_{12} \oplus C_{22}) = B_1 \oplus B_2$, whence $C_{12} \oplus C_{22} \cong B_2$. Further, $P = (C_{11} \oplus C_{21}) \oplus (C_{12} \oplus C_{22}) = B_1 \oplus (C_{12} \oplus C_{22})$, and thus $C_{11} \oplus C_{21} \cong B_1$. $\square$

Definition. Let R be a ring, and let $\mathrm{FP}(R)$ denote the class of finitely generated projective R-modules. We shall say that R is *separative* if for all $A, B \in \mathrm{FP}(R)$,

$$A \oplus A \cong A \oplus B \cong B \oplus B \qquad \Longrightarrow \qquad A \cong B.$$

(Since the categories of left and right finitely generated projective R-modules are equivalent, separativity is a left-right symmetric condition.) In describing alternate forms of this condition, it is convenient to use the following notation, adapted from [**19**, Section 2]. For modules A and B, we write $A \propto B$ if there exists a positive integer n such that A is isomorphic to a direct summand of nB.

Proposition 1.2. *Let R be a ring. The following conditions are equivalent:*

(i) *R is separative.*

(ii) *For $A, B \in \mathrm{FP}(R)$, if $2A \cong 2B$ and $3A \cong 3B$, then $A \cong B$.*

(iii) *For $A, B \in \mathrm{FP}(R)$, if there exists $n \in \mathbb{N}$ such that $nA \cong nB$ and $(n+1)A \cong (n+1)B$, then $A \cong B$.*

(iv) *For $A, B, C \in \mathrm{FP}(R)$, if $A \oplus C \cong B \oplus C$ and $C \propto A$ and $C \propto B$, then $A \cong B$.*

In case R is an exchange ring, separativity is also equivalent to the following:
(v) *For $A, B, C \in \mathrm{FP}(R)$, if $A \oplus 2C \cong B \oplus 2C$, then $A \oplus C \cong B \oplus C$.*

Proof. The implication (iii) $\Longrightarrow$ (iv) is based on an argument of Kimura and Tsai [**13**, Theorem 1] (cf. [**3**, Theorem 2.1.9]).

(i) $\Longrightarrow$ (ii). Observe that $2(2A) \cong 2(A \oplus B) \cong 2A \oplus (A \oplus B)$. Then by (i), we have $2A \cong A \oplus B$. Since $2A \cong 2B$ also, we conclude using (i) again that $A \cong B$.

(ii) $\Longrightarrow$ (iii). If $n \in \mathbb{N}$ such that $nA \cong nB$ and $(n+1)A \cong (n+1)B$, then $nA \oplus A \cong nA \oplus B$. It follows that $nA \oplus kA \cong nA \oplus kB \cong nB \oplus kB$ for all $k \in \mathbb{N}$. If $n > 1$, then $2n - 2 \geq n$ and so $2(n-1)A \cong 2(n-1)B$ and $3(n-1)A \cong 3(n-1)B$. We conclude using (ii) that $(n-1)A \cong (n-1)B$. Therefore by induction on n, we obtain $A \cong B$.

(iii) $\Longrightarrow$ (iv). Assume that $A \oplus C \cong B \oplus C$ with $kA \cong C \oplus C'$ and $kB \cong C \oplus C''$ for some $k \in \mathbb{N}$ and $C', C'' \in \mathrm{FP}(R)$. We have

$$(k+1)A \cong A \oplus C \oplus C' \cong B \oplus C \oplus C' \cong kA \oplus B.$$

Then $(k+2)A \cong (k+1)A \oplus B \cong kA \oplus 2B$, and so on: $(k+r)A \cong kA \oplus rB$ for all $r \in \mathbb{N}$. By symmetry, $(k+r)B \cong kB \oplus rA$ for all $r \in \mathbb{N}$. In particular, taking $r = k$ we obtain $2kA \cong kA \oplus kB \cong 2kB$. Further, $(2k+1)A \cong kA \oplus (k+1)A \cong 2kA \oplus B \cong (2k+1)B$, and therefore $A \cong B$ using (iii).

(iv) $\Longrightarrow$ (i). Obvious.

Now assume that R is an exchange ring. The implication (iv) $\Longrightarrow$ (v) is obvious. For the converse, consider $A, B, C \in \mathrm{FP}(R)$ such that $A \oplus C \cong B \oplus C$ while $C \propto A$ and $C \propto B$. Since C is isomorphic to a direct summand of kA for some $k \in \mathbb{N}$, Proposition 1.1 implies that $C = C_1 \oplus \cdots \oplus C_k$ where each C_i is isomorphic to a direct summand of A. It suffices to cancel the C_i successively from the isomorphism $A \oplus C_1 \oplus \cdots \oplus C_k \cong B \oplus C_1 \oplus \cdots \oplus C_k$, and so there is no loss of generality in assuming that C is isomorphic to a direct summand of A. Similarly, we may reduce to the case that C is also isomorphic to a direct summand of B. Now write $A \cong A' \oplus C$ and $B \cong B' \oplus C$ for some $A', B' \in \mathrm{FP}(R)$. Then $A' \oplus 2C \cong B' \oplus 2C$ and so $A' \oplus C \cong B' \oplus C$ by (v), that is, $A \cong B$. This shows that (v) $\Longrightarrow$ (iv).

2. Cancellation implies diagonalization

Definition. The standard concept of equivalence for matrices translates into module theoretic language as follows: homomorphisms $f, g : N \to M$ are *equivalent* if $g = ufv$ for some automorphisms $u \in \operatorname{Aut} M$ and $v \in \operatorname{Aut} N$. A homomorphism $f : N \to M$ is *(von Neumann) regular* provided f has a generalized inverse, i.e., there exists a homomorphism $h : M \to N$ such that $fhf = f$. Recall that in this case fh and hf are idempotent endomorphisms of M and N respectively, and so $\operatorname{im} f = \operatorname{im} fh$ is a direct summand of M while $\ker f = \ker hf$ is a direct summand of N.

The following elementary lemma is perhaps well known, but we were unable to locate a reference in the literature. One implication is observed in [**4**, Definition 1.6ff].

Lemma 2.1. *Let $f_1, f_2 : N \to M$ be regular homomorphisms. Then f_1 and f_2 are equivalent if and only if f_1 and f_2 have isomorphic kernels, isomorphic images, and isomorphic cokernels.*

Proof. Suppose first that $f_2 = uf_1v$ for some $u \in \operatorname{Aut} M$ and $v \in \operatorname{Aut} N$. First, $\ker f_2 = \ker(f_1v) = v^{-1}(\ker f_1)$, which is isomorphic to $\ker f_1$ via v. Second, $f_2N = uf_1N$, which is isomorphic to f_1N via u^{-1}. Third, $M/f_2N = M/uf_1N$, and u^{-1} induces an isomorphism of this module onto M/f_1N.

Conversely, assume that f_1 and f_2 have isomorphic kernels, images, and cokernels. Since f_1 and f_2 are regular, there exist decompositions $N = K_j \oplus K'_j$ and $M = I_j \oplus I'_j$ for $j = 1, 2$ where $K_j = \ker f_j$ and $I_j = \operatorname{im} f_j$. Further, each $K'_j \cong I_j$ via f_j, and each $I'_j \cong \operatorname{coker} f_j$.

By assumption, $K_1 \cong K_2$ and $K'_1 \cong K'_2$. Hence, there exists $v \in \operatorname{Aut} N$ such that $vK_2 = K_1$ and $vK'_2 = K'_1$, and $\ker(f_1v) = v^{-1}K_1 = K_2$. After replacing f_1 by f_1v, we may assume that $K_1 = K_2$ and $K'_1 = K'_2$. We also have $I_1 \cong I_2$ and $I'_1 \cong I'_2$, and so there exists $u \in \operatorname{Aut} M$ such that $uI_1 = I_2$ and $uI'_1 = I'_2$. After replacing f_1 by uf_1, we may assume that $I_1 = I_2$ and $I'_1 = I'_2$.

Now f_1 and f_2 both restrict to isomorphisms of K'_1 onto I_1. There exists $w \in \operatorname{Aut} M$ such that $w = 1$ on I'_1 and $w = f_2f_1^{-1}$ on I_1, and $wf_1 = f_2$. □

For any ring R and any positive integers m, n, we identify the set $M_{m\times n}(R)$ of all $m \times n$ matrices over R with $\operatorname{Hom}_R(nR, mR)$ in the standard manner. (This is consistent with our convention that homomorphisms act on the left of their arguments, and requires that we view elements of nR and mR as column vectors.) In the case $m = n$, this becomes an identification of $M_n(R)$ with $\operatorname{End}_R(nR)$, and restricts to an identification of $GL_n(R)$ with $\operatorname{Aut}_R(nR)$.

Proposition 2.2. *Let R be an exchange ring, and let $f \in M_{m\times n}(R)$ be regular.*

(a) *Suppose that $n \geq m$. Then f admits a diagonal reduction if and only if the following condition holds:*

(*) *There are decompositions*

$$\ker f = K_1 \oplus \cdots \oplus K_n, \qquad \operatorname{im} f = I_1 \oplus \cdots \oplus I_m, \qquad \operatorname{coker} f = C_1 \oplus \cdots \oplus C_m$$

such that $K_j \oplus I_j \cong C_j \oplus I_j \cong R$ for $j = 1, \dots, m$ and $K_j \cong R$ for $j = m+1, \dots, n$.

(b) *Suppose that $n \leq m$. Then f admits a diagonal reduction if and only if the following condition holds:*

(**) *There are decompositions*

$$\ker f = K_1 \oplus \cdots \oplus K_n, \qquad \operatorname{im} f = I_1 \oplus \cdots \oplus I_n, \qquad \operatorname{coker} f = C_1 \oplus \cdots \oplus C_m$$

such that $K_j \oplus I_j \cong C_j \oplus I_j \cong R$ for $j = 1, \dots, n$ and $C_j \cong R$ for $j = n+1, \dots, m$.

Proof. Set $N = nR = N_1 \oplus \cdots \oplus N_n$ and $M = mR = M_1 \oplus \cdots \oplus M_m$ where N_i (respectively, M_i) is the direct summand of N (respectively, M) generated by the i-th standard basis vector. Since f is regular, we can write $N = K \oplus K'$ and $M = I \oplus C$ with $K = \ker f$, $I = \operatorname{im} f$, and $C \cong \operatorname{coker} f$. Note that I and K have the exchange property.

(a) Assume first that we have decompositions $K = K_1 \oplus \cdots \oplus K_n$, $I = I_1 \oplus \cdots \oplus I_m$, and $C = C_1 \oplus \cdots \oplus C_m$ as in (*). Since f maps K' isomorphically onto I, we also have $K' = K'_1 \oplus \cdots \oplus K'_m$ such that f maps each K'_j isomorphically onto I_j. By assumption, $K_j \oplus K'_j \cong R$ for $j \leq m$ and $K_j \cong R$ for $j > m$, and hence there exists $v \in GL_n(R)$ such that $vN_j = K_j \oplus K'_j$ for $j \leq m$ and $vN_j = K_j$ for $j > m$. Similarly, there exists

$u \in GL_m(R)$ such that $u(C_j \oplus I_j) = M_j$ for all $j \le m$. Then $ufvN_j = ufK'_j = uI_j \subseteq M_j$ for $j \le m$ and $ufvN_j = 0$ for $j > m$. It follows that ufv is diagonal. Namely, if $\nu_1, \dots, \nu_n$ and $\mu_1, \dots, \mu_m$ are the standard bases for N and M, then there exist $r_1, \dots, r_m \in R$ such that $ufv(\nu_j) = \mu_j r_j$ for $j \le m$ and $ufv(\nu_j) = 0$ for $j > m$. Therefore

$$ufv = \begin{pmatrix} r_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & r_m & 0 & \cdots & 0 \end{pmatrix}.$$

Conversely, suppose that ufv is diagonal for some $u \in \operatorname{Aut} M$ and $v \in \operatorname{Aut} N$. In view of Lemma 2.1, it suffices to find decompositions as in (*) for the kernel, image, and cokernel of ufv. Hence, we may assume that f is diagonal, that is, $fN_j \subseteq M_j$ for $j \le m$ and $fN_j = 0$ for $j > m$.

Now $M = I \oplus C = M_1 \oplus \cdots \oplus M_m$. By the exchange property, each $M_i = M_{i1} \oplus M_{i2}$ such that $M = I \oplus M_{12} \oplus \cdots \oplus M_{m2}$. Since the only property required of C is that it be a complement for I, there is no loss of generality in assuming that $C = M_{12} \oplus \cdots \oplus M_{m2}$. Similarly, since $N = K \oplus K' = N_1 \oplus \cdots \oplus N_n$, each $N_i = N_{i1} \oplus N_{i2}$ such that $N = K \oplus N_{12} \oplus \cdots \oplus N_{n2}$, and there is no loss of generality in assuming that $K' = N_{12} \oplus \cdots \oplus N_{n2}$. Note that $K \cong N_{11} \oplus \cdots \oplus N_{n1}$ (since both of these submodules of N are complements for K'). Hence, there is a decomposition $K = K_1 \oplus \cdots \oplus K_n$ such that $K_j \cong N_{j1}$ for all j. Further, since $N_j \subseteq K$ for $j > m$ (recall that $fN_j = 0$), we have $N_{j2} = 0$ for $j > m$.

Since f maps K' isomorphically onto I, we have $I = I_1 \oplus \cdots \oplus I_m$ with each $I_j = fN_{j2} \cong N_{j2}$. Note that $fN_{j2} \subseteq fN_j \subseteq M_j$ for all $j \le m$. Since fN_{j2} is a direct summand of I, which is a direct summand of M, it follows that fN_{j2} is also a direct summand of M_j, say $M_j = fN_{j2} \oplus F_j$. Now

$$M = M_1 \oplus \cdots \oplus M_m = fN_{12} \oplus F_1 \oplus \cdots \oplus fN_{m2} \oplus F_m = I \oplus F_1 \oplus \cdots \oplus F_m.$$

Since C and $F_1 \oplus \cdots \oplus F_m$ are both complements for I in M, they must be isomorphic. Thus $C = C_1 \oplus \cdots \oplus C_m$ with each $C_j \cong F_j$. Finally, we have

$$K_j \oplus I_j \cong N_{j1} \oplus N_{j2} = N_j \cong R \qquad \text{and} \qquad C_j \oplus I_j \cong F_j \oplus fN_{j2} = M_j \cong R$$

for $j = 1, \dots, m$ and $K_j \cong N_{j1} = N_{j1} \oplus N_{j2} = N_j \cong R$ for $j = m+1, \dots, n$. Therefore (*) is proved.

(b) The proof is an easy modification of the proof of part (a), and is left to the reader. □

Definition. Consider decompositions $nR \cong K \oplus I$ and $mR \cong I \oplus C$, with $n \ge m$. Just for the purposes of the next few proofs, let us define a *diagonal refinement* of the given decompositions to be a set of decompositions $K = K_1 \oplus \cdots \oplus K_n$, $I = I_1 \oplus \cdots \oplus I_m$, and $C = C_1 \oplus \cdots \oplus C_m$ such that $K_j \oplus I_j \cong C_j \oplus I_j \cong R$ for $j \le m$ and $K_j \cong R$ for $j > m$.

Lemma 2.3. *Let R be an exchange ring. Consider decompositions $nR \cong K \oplus I$ and $mR \cong I \oplus C$ with $n \ge m$, and suppose that $K \cong K^* \oplus X$ and $C \cong C^* \oplus X$ for some* *modules K^*, C^*, X. If the decompositions $nR \cong K^* \oplus (I \oplus X)$ and $mR \cong (I \oplus X) \oplus C^*$ have a diagonal refinement, so do the original decompositions $nR \cong K \oplus I$ and $mR \cong I \oplus C$.*

Proof. By assumption, there is a diagonal refinement

$$K^* = K_1^* \oplus \cdots \oplus K_n^*, \qquad I \oplus X = I_1^* \oplus \cdots \oplus I_m^*, \qquad C^* = C_1^* \oplus \cdots \oplus C_m^*.$$

By Proposition 1.1, $I = I_1 \oplus \cdots \oplus I_m$ and $X = X_1 \oplus \cdots \oplus X_m$ with $I_j \oplus X_j \cong I_j^*$ for all $j \leq m$. We can then write decompositions

$$K \cong (K_1^* \oplus X_1) \oplus \cdots \oplus (K_m^* \oplus X_m) \oplus K_{m+1}^* \oplus \cdots \oplus K_n^*$$

and $C \cong (C_1^* \oplus X_1) \oplus \cdots \oplus (C_m^* \oplus X_m)$. Together with the decomposition $I = I_1 \oplus \cdots \oplus I_m$, this provides the desired diagonal refinement.

We can now show that diagonalizability of square matrices follows from separativity, and in fact from a somewhat weaker cancellation law. Recall that an R-module A is a *generator* (in the category of R-modules) provided R is isomorphic to a direct summand of nA for some n, that is, $R \propto A$ in the notation of Section 1.

Theorem 2.4. *Let R be an exchange ring, and assume that $2R \oplus A \cong R \oplus B$ implies $R \oplus A \cong B$ for any finitely generated projective R-modules A and B such that B is a generator. Then every regular square matrix over R admits a diagonal reduction.*

Proof. In view of Proposition 2.2, it suffices to show that every decomposition $nR \cong K \oplus I \cong I \oplus C$ (with $n \geq 2$) has a diagonal refinement.

By Proposition 1.1, $K = X_1 \oplus X_2$ and $I = Y_1 \oplus Y_2$ such that $X_1 \oplus Y_1 \cong I$ and $X_2 \oplus Y_2 \cong C$. In view of Lemma 2.3, it suffices to find a diagonal refinement for the decompositions $nR \cong X_1 \oplus (I \oplus X_2) \cong (I \oplus X_2) \oplus Y_2$. Hence, we may replace K, I, C by X_1, $I \oplus X_2$, Y_2. Thus there is no loss of generality in assuming that K is isomorphic to a direct summand of I, whence nR is isomorphic to a direct summand of $2I$. In particular, I is now a generator.

Since $nR \oplus C \cong K \oplus I \oplus C \cong (n-1)R \oplus (R \oplus K)$ with $R \oplus K$ a generator, our cancellation hypothesis (applied $n-1$ times) implies that $R \oplus C \cong R \oplus K$. By Proposition 1.1, $R = R_1 \oplus R_2$ and $C = Z_1 \oplus Z_2$ such that $R_1 \oplus Z_1 \cong R$ and $R_2 \oplus Z_2 \cong K$. In view of Lemma 2.3, it now suffices to find a diagonal refinement for the decompositions $nR \cong R_2 \oplus (I \oplus Z_2) \cong (I \oplus Z_2) \oplus Z_1$. Since $R_1 \oplus R_2 \cong R_1 \oplus Z_1 \cong R$, we may now assume that $W \oplus K \cong W \oplus C \cong R$ for some W.

At this point, we have $2R \oplus (n-2)R \oplus W \cong K \oplus I \oplus W \cong R \oplus I$. Since I is a generator, it follows from our hypothesis that $(n-1)R \oplus W \cong I$. Therefore the decompositions

$$K = K \oplus 0 \oplus \cdots \oplus 0, \qquad I \cong W \oplus R \oplus \cdots \oplus R, \qquad C = C \oplus 0 \oplus \cdots \oplus 0$$

form a diagonal refinement for the decompositions $nR \cong K \oplus I \cong I \oplus C$. $\square$

Of course, when R is regular all matrices over R are regular (cf. [**6**, Theorem 1.7]), and we obtain our main result:

Theorem 2.5. *If R is a separative regular ring, then every square matrix over R admits a diagonal reduction.* □

Theorem 2.5 conceivably applies to all regular rings, since no non-separative regular rings are known. (In fact, no non-separative exchange rings are known.) As a particular application of the theorem, we note the following result of Moncasi and the second author:

Corollary 2.6. [**16**, Teorema 2.19] *Square matrices admit diagonal reductions over any right self-injective regular ring R.*

Proof. It is known that $2A \cong 2B$ implies $A \cong B$ for $A, B \in \mathrm{FP}(R)$ [**6**, Theorem 10.34]. Hence, R is separative. □

We mention that the class of separative regular rings includes all unit-regular rings, all right or left $\aleph_0$-continuous regular rings [**1**, Theorem 2.13], and all regular rings satisfying general comparability [**6**, Theorem 8.16]. It is not difficult to show that this class is closed under taking corners, finite matrix rings, arbitrary direct products, direct limits, and factor rings. It is also closed under extensions in the sense that if R is a regular ring with an ideal I such that R/I and eRe are separative for all idempotents $e \in I$, then R is separative [**2**].

We now turn to diagonal reduction for non-square matrices. This will lead, in the next section, to the promised generalization of the Menal-Moncasi theorem.

Proposition 2.7. *Let R be an exchange ring, and let $f \in M_{m\times n}(R)$ be regular.*

(a) $nR \oplus \operatorname{coker} f \cong mR \oplus \ker f$.

(b) *Suppose that $n > m$. Then f admits a diagonal reduction if and only if* $\ker f \cong (n-m)R \oplus \operatorname{coker} f$.

(c) *Suppose that $n < m$. Then f admits a diagonal reduction if and only if* $\operatorname{coker} f \cong (m-n)R \oplus \ker f$.

Proof. Write $nR = K \oplus K'$ and $mR = I \oplus C$ where, as usual, $K = \ker f$, $I = \operatorname{im} f$, and $C \cong \operatorname{coker} f$.

(a) Since $K' \cong I$ via f, we have $nR \cong K \oplus I$, whence $nR \oplus C \cong K \oplus I \oplus C \cong K \oplus mR$.

(b) ($\Longrightarrow$): By Proposition 2.2, there exists a diagonal refinement

$$K = K_1 \oplus \cdots \oplus K_n, \qquad I = I_1 \oplus \cdots \oplus I_m, \qquad C = C_1 \oplus \cdots \oplus C_m.$$

For $j \le m$, we have $K_j \oplus I_j \cong C_j \oplus I_j \cong R$, whence $K_j \oplus R \cong K_j \oplus I_j \oplus C_j \cong C_j \oplus R$. Consequently,

$$\begin{aligned} K_1 \oplus \cdots \oplus K_m \oplus R &\cong C_1 \oplus K_2 \oplus \cdots \oplus K_m \oplus R \cong C_1 \oplus C_2 \oplus K_3 \oplus \cdots \oplus K_m \oplus R \\ &\cong \cdots \cong C_1 \oplus \cdots \oplus C_m \oplus R = C \oplus R. \end{aligned}$$

Since $n > m$ and $K_j \cong R$ for $j > m$, we thus obtain

$$K \cong K_1 \oplus \cdots \oplus K_m \oplus (n-m)R \cong C \oplus (n-m)R.$$

($\Longleftarrow$): By Proposition 2.2, it suffices to find a diagonal refinement for the decompositions $nR \cong K \oplus I$ and $mR = I \oplus C$. We have $K \cong (n-m)R \oplus C$ by assumption, and so Lemma 2.3 shows that it is enough to find a diagonal refinement for the decompositions

$$nR \cong (n-m)R \oplus (I \oplus C) \qquad \text{and} \qquad mR \cong (I \oplus C) \oplus 0.$$

However, this is easy: take

$$(n-m)R \cong 0 \oplus \cdots \oplus 0 \oplus R \oplus \cdots \oplus R, \qquad I \oplus C \cong R \oplus \cdots \oplus R, \qquad 0 = 0 \oplus \cdots \oplus 0.$$

(c) This is very similar to (b), and is left to the reader.

Theorem 2.8. *Let R be an exchange ring, and assume that $2R \oplus A \cong R \oplus B$ implies $R \oplus A \cong B$ for any finitely generated projective R-modules A and B. Then every regular matrix over R admits a diagonal reduction.*

Proof. Theorem 2.4 immediately implies that every regular square matrix over R admits a diagonal reduction. It follows from our hypotheses that $nR \oplus C \cong mR \oplus K$ implies $(n-m)R \oplus C \cong K$ for $n > m$ and any finitely generated projective R-modules C and K. Hence, Proposition 2.7 implies that regular $m \times n$ matrices over R admit diagonal reduction for all $n > m$. Finally, diagonal reduction for regular $m \times n$ matrices with $n < m$ likewise follows from Proposition 2.7. □

3. Diagonalization implies cancellation

The cancellation condition used in Theorem 2.8 actually characterizes diagonalizability of regular matrices over exchange rings, as follows.

Theorem 3.1. *For an exchange ring R, the following conditions are equivalent:*

(a) *Every regular matrix over R admits a diagonal reduction.*

(b) *Every 1×2 regular matrix over R admits a diagonal reduction.*

(c) *Every 2×1 regular matrix over R admits a diagonal reduction.*

(d) *$2R \oplus A \cong R \oplus B$ implies $R \oplus A \cong B$ for any finitely generated projective R-modules A and B.*

Proof. We have (d)$\Longrightarrow$(a) by Theorem 2.8, and (a)$\Longrightarrow$(b),(c) a priori.

(b)$\Longrightarrow$(d): Apply Proposition 1.1 to the given isomorphism $2R \oplus A \cong R \oplus B$. Thus, there exist decompositions $2R = N_1 \oplus N_2$ and $A = A_1 \oplus A_2$ such that $N_1 \oplus A_1 \cong R$ and $N_2 \oplus A_2 \cong B$. Write $R = M_1 \oplus M_2$ with $M_1 \cong N_1$ and $M_2 \cong A_1$. Since $N_1 \cong M_1$, there is a regular homomorphism $f : 2R \to R$ such that $\ker f = N_2$ and f maps N_1 isomorphically onto M_1. Note that $M_2 \cong \operatorname{coker} f$. We identify f with a regular 1×2 matrix, which admits a diagonal reduction by assumption. Consequently, Proposition 2.7 implies that $R \oplus \operatorname{coker} f \cong \ker f$, that is, $R \oplus M_2 \cong N_2$. Therefore

$$R \oplus A = R \oplus A_1 \oplus A_2 \cong R \oplus M_2 \oplus A_2 \cong N_2 \oplus A_2 \cong B.$$

(c)$\Longrightarrow$(d): This is proved in the same manner as the implication above.

Kaplansky defined a ring R to be *right (left) Hermite* provided every 1×2 (2×1) matrix over R admits a diagonal reduction [**12**, p. 465]. Thus the specialization of Theorem 3.1 to the case of a regular ring yields a new proof of the following version of the Menal-Moncasi theorem:

Theorem 3.2. [**15**, Theorem 9] *For a regular ring R, the following conditions are equivalent:*

(a) *Every matrix over R admits a diagonal reduction.*

(b) *R is right Hermite.*

(c) *R is left Hermite.*

(d) *$2R \oplus A \cong R \oplus B$ implies $R \oplus A \cong B$ for any finitely generated projective R-modules A and B.* $\square$

It is easy to find regular rings which are not Hermite, for instance because Hermite regular rings have stable range at most 2 [**15**, Proposition 8]. To give a more specific example, let R be any nonzero right self-injective regular ring which is *purely infinite* in the sense of [**6**], that is, $2R \cong R$. (For instance, the endomorphism ring of any infinite dimensional vector space has these properties.) Since $2R \oplus 0 \cong R \oplus 0$ while $R \oplus 0 \not\cong 0$, Theorem 3.2 shows that R is not Hermite. In fact, it follows from Proposition 2.7 that the 1×2 matrix corresponding to any isomorphism $2R \to R$ cannot admit a diagonal reduction. On the other hand, all square matrices over R admit diagonal reductions, by Corollary 2.6. Therefore the class of regular rings exhibits the same distinction between diagonalizability of square and rectangular matrices that Levy proved for serial rings [**14**].

We conclude by proving that separativity for an exchange ring R is in fact characterized by diagonalizability of square matrices. However, the characterization involves square matrices not only over R but also over corner rings eRe, where e is any idempotent in R. For this purpose, we recall a few standard observations about the relations between projective modules over R and eRe. First, if $A \in \mathrm{FP}(R)$, then $Ae \in \mathrm{FP}(eRe)$. Conversely, if $B \in \mathrm{FP}(eRe)$, then $B \otimes_{eRe} eR \in \mathrm{FP}(R)$, and $(B \otimes_{eRe} eR)e \cong B$. However, if $A \in \mathrm{FP}(R)$, then $Ae \otimes_{eRe} eR$ need not be isomorphic to A; in fact, $Ae \otimes_{eRe} eR \cong A$ if and only if A is isomorphic to a direct summand of $n(eR)$ for some n.

Proposition 3.3. *Assume that R is an exchange ring, and that all regular matrices in $M_2(R)$ admit diagonal reductions. If A, B, C are finitely generated projective R-modules such that $A \oplus C \cong B \oplus C$ and R is isomorphic to direct summands of both A and B, then $A \cong B$.*

Proof. We are given that $A \cong R \oplus A'$ and $B \cong R \oplus B'$ for some A', B'. Further, $C \oplus C' \cong nR$ for some C' and some $n \in \mathbb{N}$. Hence, it suffices to show that $(n+1)R \oplus A' \cong (n+1)R \oplus B'$ implies $R \oplus A' \cong R \oplus B'$ for any finitely generated projective R-modules A' and B'. By an obvious induction on n, this reduces to the case $n = 2$.

Therefore, assume that $2R \oplus A' \cong 2R \oplus B'$. Set $M = 2R$. Since $M \oplus A' \cong M \oplus B'$, Proposition 1.1 implies that there exist decompositions $M = C_{11} \oplus C_{12}$ and $A' = C_{21} \oplus C_{22}$ such that $C_{11} \oplus C_{21} \cong M$ and $C_{12} \oplus C_{22} \cong B'$. It suffices to show that $R \oplus C_{12} \cong R \oplus C_{21}$, since then $R \oplus B' \cong R \oplus C_{12} \oplus C_{22} \cong R \oplus C_{21} \oplus C_{22} \cong R \oplus A'$. Thus, we have decompositions $M = K \oplus K' = I \oplus C$ with $K = C_{12}$ and $K' = C_{11}$ while $I \cong C_{11}$ and $C \cong C_{21}$, and it suffices to show that $R \oplus K \cong R \oplus C$.

As usual, we identify $M_2(R)$ with $\mathrm{End}_R(M)$. Since $K' = C_{11} \cong I$, there is a regular matrix $f \in M_2(R)$ such that $\ker f = K$ and f maps K' isomorphically onto I; then $C \cong \operatorname{coker} f$. By hypothesis, f admits a diagonal reduction. We then obtain decompositions $K = K_1 \oplus K_2$, $I = I_1 \oplus I_2$, and $C = C_1 \oplus C_2$ as in condition (*) of Proposition 2.2.

Therefore

$$R \oplus K \cong C_1 \oplus I_1 \oplus K_1 \oplus K_2 \cong R \oplus C_1 \oplus K_2 \cong C_2 \oplus I_2 \oplus C_1 \oplus K_2 \cong C \oplus R.$$

Theorem 3.4. *An exchange ring R is separative if and only if for all idempotents $e \in R$, every regular matrix in $M_2(eRe)$ admits a diagonal reduction.*

Proof. Assume first that R is separative, and let e be an idempotent in R. If A and B are any finitely generated projective right eRe-modules such that $2A \cong A \oplus B \cong 2B$, then $A \otimes_{eRe} eR$ and $B \otimes_{eRe} eR$ are finitely generated projective right R-modules such that

$$2(A \otimes_{eRe} eR) \cong (A \otimes_{eRe} eR) \oplus (B \otimes_{eRe} eR) \cong 2(B \otimes_{eRe} eR).$$

Since R is separative, $A \otimes_{eRe} eR \cong B \otimes_{eRe} eR$, and thus $A \cong (A \otimes_{eRe} eR)e \cong (B \otimes_{eRe} eR)e \cong B$. This shows that eRe is separative, and therefore Theorem 2.4 implies that all regular square matrices over eRe admit diagonal reductions.

Conversely, assume that all regular matrices in each $M_2(eRe)$ admit diagonal reductions. We shall show that for any idempotent $e \in R$ and any $A, B \in FP(R)$, the implication

$$2(eR) \oplus A \cong 2(eR) \oplus B \quad \Longrightarrow \quad eR \oplus A \cong eR \oplus B$$

holds. It follows that for all $A, B, C \in FP(R)$, if $2C \oplus A \cong 2C \oplus B$, then $C \oplus A \cong C \oplus B$ (use the fact that $C \cong e_1R \oplus \cdots \oplus e_nR$ for some idempotents $e_1, \ldots, e_n \in R$ [**6**, Proposition 2.6]). Therefore R is separative by Proposition 1.2.

Thus, suppose that $2(eR) \oplus A \cong 2(eR) \oplus B$ for some idempotent $e \in R$ and some $A, B \in \mathrm{FP}(R)$. By Proposition 1.1, there exist decompositions $2(eR) = C_{11} \oplus C_{12}$ and $A = C_{21} \oplus C_{22}$ such that $C_{11} \oplus C_{21} \cong 2(eR)$ and $C_{12} \oplus C_{22} \cong B$. Now

$$2(eR) \oplus C_{12} \cong C_{11} \oplus C_{21} \oplus C_{12} \cong 2(eR) \oplus C_{21},$$

and so $2(eRe) \oplus C_{12}e \cong 2(eRe) \oplus C_{21}e$. In view of Proposition 3.3 (applied over the ring eRe), it follows that $eRe \oplus C_{12}e \cong eRe \oplus C_{21}e$. Since C_{12} and C_{21} are isomorphic to direct summands of $2(eR)$, we obtain

$$eR \oplus C_{12} \cong (eRe \oplus C_{12}e) \otimes_{eRe} eR \cong (eRe \oplus C_{21}e) \otimes_{eRe} eR \cong eR \oplus C_{21},$$

and therefore $eR \oplus B \cong eR \oplus C_{12} \oplus C_{22} \cong eR \oplus C_{21} \oplus C_{22} \cong eR \oplus A$, as desired.

If one could show that all regular 2×2 matrices over all exchange rings (or over all regular rings) admit diagonal reductions, Theorem 3.4 would then imply that all exchange rings (or all regular rings) are separative.

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