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SOME REMARKS ON A FAMILY OF T-DESIGNS

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Abstract

In a $(2m-2) - (4m-1, 2m, m)$ design there can be no repeated blocks, every two blocks have at least two points in common, and no two blocks can have more than $2m-2$ points in common.

A *t - (v,k,λ) design* on v *varieties* (or points or symbols consists of subsets of size k, called *blocks*, chosen from the set of v varieties in such a way that each t-subset of varieties is contained in exactly λ of the blocks. It is usual to require that no block is ever repeated although in this note this requirement will not be insisted upon initially. If every k-subset of the variety set is a block then the design is said to be *trivial*.

For non-trivial designs examples are known with $t \leqslant 5$. It is not at present known if non-trivial t-designs can exist for $t \geqslant 6$.

In a t-design let λ_i, $(0 \leqslant i \leqslant t)$, be the number of times each i-subset of varieties occurs. Then it is well known that (e.g. see [1],p56)

$$\lambda_i = \frac{(v-i)(v-i-1)(v-i-2)\dots(v-t+1)}{(k-i)(k-i-1)(k-i-2)\dots(k-t+1)}\,\lambda\,. \tag{1}$$

In particular $\lambda_t = \lambda$, and $\lambda_0 = b$ the number of blocks. It is a necessary condition for the existence of the design that all the λ_i be non-negative integers.

In a t-design let n_i, $(0 \leqslant i \leqslant k)$, be the number of blocks which contain exactly i points of a given block. The n_i are called *block intersection numbers* of the given block. It can be shown that the n_i and λ_i are connected through the equations

$$\sum_{i=0}^{i=k} \binom{i}{j} n_i = \binom{k}{j} \lambda_j \quad ; \qquad 0 \leqslant j \leqslant t. \tag{2}$$

Now consider the family of t-designs with $v = 4m-1$, $k = 2m$ $\lambda = m$ and $t = 2m-2$. There are at least two such (2m-2) - (4m-1, 2m, m) designs known. For $m = 2$ there is a 2 - (7,4,2) design which is the complement of the 2 - (7,3,1) design i.e. the finite projective plane of order 2. (e.g. see [1] p37 and p61). For

$m = 3$ there is a $4 - (11,6,3)$ design which is the complement of the $4 - (11,5,1)$ design i.e. the Steiner system associated with the Mathieu group M_{11}. (e.g. see [1] p22 and p24). For $m > 3$ some possibilities are ruled out by the integrity demands of the λ_i. Thus, since λ_{2m-3} is an integer, $m \not\equiv 1 \pmod 3$. However even with this restriction there are values of m for which all the λ_i are integers; for example $m = 8$.

We establish the following result

Theorem: *In a $(2m-2) - (4m-1,\ 2m,\ m)$ design $n_0 = 0$, $n_1 = 0$, $n_{2m-1} = 0$ and $n_{2m} = 1$.*

Proof: For a $(2m-2) - (4m-1,\ 2m,\ m)$ design by equation (1)

$$\lambda_p = \frac{1}{(2m+1)} \binom{4m-1-p}{2m-1} . \tag{3}$$

Let $B(t) \equiv \sum_{i=0}^{2m} n_i t^i$ and call B the *block intersection polynomial*. Consider the derivatives of B evaluated at $t = 1$. Then from Taylor's theorem and equations (2) we have the identity

$$B(t) \equiv \sum_{i=0}^{2m} n_i t^i \equiv n_{2m}(t-1)^{2m} + (2m\, n_{2m} + n_{2m-1})(t-1)^{2m-1} + \sum_{p=0}^{2m-2} \binom{2m}{p} \lambda_p . \tag{4}$$

Now $n_0 = 0$ because each block contains more than half the varieties. Therefore on evaluating B at $t = 0$ we have

$$\sum_{p=0}^{2m-2} (-1)^p \binom{2m}{p} \lambda_p = n_{2m-1} + (2m-1)\, n_{2m}. \tag{5}$$

But by equation (3) the series in (5) is $\frac{1}{(2m+1)} \sum_{p=0}^{2m-2} \binom{2m}{p} (-1)^{2p} \binom{-2m}{2m-p}$ which is the coefficient of t^{2m} in $[(1-t)^{2m} + 2mt^{2m-1} - t^{2m}][(2m+1)(1-t)^{2m}]^{-1}$.

This coefficient is $(2m-1)$.

Therefore

$$n_{2m-1} = (2m-1)(1-n_{2m}). \tag{6}$$

Since $n_{2m} \geqslant 1$ and $n_{2m-1} \geqslant 0$ we must have $n_{2m} = 1$ (which means that no repeated blocks are possible even if we allowed them). Then $n_{2m-1} = 0$.

Then when B is differentiated once and t put equal to 0 we find

$$n_1 = 4m(m-1) - \sum_{p=0}^{2m-2} (-1)^{p-1}\, p \binom{2m}{p} \lambda_p . \tag{7}$$

The series is the coefficient of t^{2m} in

$[-2m\,t(1-t)^{2m-1} + 2m(2m-1)t^{2m-1} - 2mt^{2m}][(2m+1)(1-t)^{2m}]^{-1}$

which is $4m(m-1)$ so $n_1 = 0$. □

It is perhaps asking too much that non-trivial t-designs should exist for large values of t. However any t-design can be reduced to a (t-1)-design by discarding a variety and all blocks not containing that variety so even if an allowable parameter set does not lead to a t-design we can still generate allowable parameter sets for designs with smaller values of t. In particular a (2m-2) - (4m-1 , 2m, m) design can be contracted in this way 2m-8 times to produce a 6 - (2m+7,8,m) design. It may be possible to show that such 6-designs do not exist for any value of m in which case the theorem of this note becomes pointless. The notion of a block intersection polynomial is however useful in other contexts.

Reference

[1] N.L. BIGGS and A.T. WHITE, *Permutation groups and combinatorial structures*, Lond. Math. Soc. Lecture Note Series, 33, Cambridge University Press, 1979.