THE TRANSITIVE 3 - (12,6,4) AND 2 - (11,5,4) DESIGNS

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ABSTRACT

The automorphism groups of each of the eight transitive 3 - (12,6,4) designs and the three transitive 2 - (11,5,4) designs are described. Repeated blocks are allowed in the designs.

I: INTRODUCTION

A $t-(v,k,\lambda)$ *design* on a set of v *points* is a collection of subsets of size k called the *blocks* which between them contain every t-subset of points exactly λ times. In this paper repeated blocks are permitted.

A permutation of the point labels of a $t-(v,k,\lambda)$ design D which maps blocks onto blocks is called an *automorphism* of D. The set of all automorphisms of D under successive applications is the *automorphism group* of D, Aut D. If the permutation group Aut D maps any ordered i-tuple of points onto any other i-tuple then Aut D is said to be *i-transitive*. 'One-transitive' is usually contracted to 'transitive'. By an extension of notation the design D is said to be transitive whenever Aut D is. The subgroup of Aut D which fixes the point x is the *stabilizer* of x, written Aut $D_{(x)}$.

If the blocks of a $t-(v,k,\mu+\lambda)$ design can be partitioned into two collections to make a $t-(v,k,\mu)$ design and a $t-(v,k,\lambda)$ design then the original design is said to be *decomposable*.

In an enumeration of non-isomorphic $3-(12,6,4)$ designs, with or without repeated blocks and decomposable or not decomposable, Thompson [4] has found just eight which are transitive. A program which gives a complete listing of the automorphisms for each $3-(12,6,4)$ design D was developed so the groups of the transitive $3-(12,6,4)$ designs are known in the sense that all their elements are known as point permutations. Nevertheless there is some interest in showing that these permutation groups (or their larger subgroups) are isomorphic to known groups. To this end the tables given in Coxeter and Moser [3] (hereafter contracted to C & M) are invaluable. For each case, except the first which is well known, a complete list of blocks is given together with enough automorphisms to generate Aut D.

From any $3-(12,6,4)$ design D a $2-(11,5,4)$ design can be obtained by discarding all blocks not containing a given element x which is then deleted from the remaining blocks. The 2 - (11,5,4) design then has the stabilizer Aut $D_{(x)}$ as its automorphism group. The reverse process is always possible; any 2 - (11,5,4) design can be extended to a 3 - (12,6,4) design by adding the same new point x to each block and then creating further blocks by taking complements with respect to the point set. In fact any 3 - (12,6,4) design must have its blocks in complementary pairs [2]. The extension and restriction procedures need not preserve transitivity. There are transitive 2 - (11,5,4) designs which do not extend to transitive 3 - (12,6,4) designs. After listing the eight transitive 3 - (12,6,4) designs we list the three transitive 2 - (11,5,4) designs of which only one extends to a transitive 3-design.

For a particular block X of any t-design let n_i be the number of blocks having exactly i points in common with the point set of X. The n_i need not be the same for all choices of X. For a 3 - (12,6,4) design the possible *block types* are given by [2];

	n_0	n_1	n_2	n_3	n_4	n_5	n_6
Type AC:	1	1	5	30	5	1	1
Type B :	1	0	9	24	9	0	1
Type R :	2	0	0	40	0	0	2

For 2 - (11,5,4) designs we have [2];

	n_0	n_1	n_2	n_3	n_4	n_5
Type A:	1	0	15	5	0	1
Type B:	0	3	12	6	0	1
Type C:	0	2	15	3	1	1
Type R:	0	0	20	0	0	2

Thus blocks of type R are repeated blocks. Blocks of types A and C in a 2 - (11,5,4) design generate blocks of type AC in the 3 - (12,6,4) design which is the extension of the 2-design. Similar remarks hold for blocks of types B and R. For each 3 - (12,6,4) design listed a breakdown of the

44 blocks into the various types is given; likewise for the 22 blocks of each 2 - (11,5,4) design. S_n, A_n, D_n and C_n are respectively the symmetric alternating, dihedral and cyclic groups of degree n.

II: A LIST OF TRANSITIVE 3 - (12,6,4) DESIGNS

CASE I: Repeated blocks; decomposable.

AC = 0, B = 0, R = 44.

As starter blocks take [1 3 4 5 9 ∞] and [2 6 7 8 10 0] and develop them cyclically modulo 11 with ∞ fixed. Then repeat each block. This gives a 3 - (12,6,4) design which is trivial in that it is the double of the well known (and unique) 3 - (12,6,2) design obtained as the extension of the Hadamard 2 - (11,5,2) design. The 2-design is 2-transitive; the 3-design is 3-transitive with a group of order 12.11.10.6 = 7920.

CASE II: Repeated blocks, decomposable.

AC = 0, B = 40, R = 4.

1	2	3	4	5	12	6	7	8	9	10	11
1	2	6	9	11	12	3	4	5	7	8	10
1	3	7	9	10	12	2	4	5	6	8	11
1	4	6	7	8	12	2	3	5	9	10	11
1	5	8	10	11	12	2	3	4	6	7	9
2	3	6	8	10	12	1	4	5	7	9	11
2	4	7	10	11	12	1	3	5	6	8	9
2	5	7	8	9	12	1	3	4	6	10	11
3	4	8	9	11	12	1	2	5	6	7	10
3	5	6	7	11	12	1	2	4	8	9	10
4	5	6	9	10	12	1	2	3	7	8	11
1	2	3	4	5	12	6	7	8	9	10	11
1	2	7	8	10	12	3	4	5	6	9	11
1	3	6	8	11	12	2	4	5	7	9	10
1	4	9	10	11	12	2	3	5	6	7	8
1	5	6	7	9	12	2	3	4	8	10	11
2	3	7	9	11	12	1	4	5	6	8	10
2	4	6	8	9	12	1	3	5	7	10	11
2	5	6	10	11	12	1	3	4	7	8	9
3	4	6	7	10	12	1	2	5	8	9	11
3	5	8	9	10	12	1	2	4	6	7	11
4	5	7	8	11	12	1	2	3	6	9	10

This design has an automorphism group of order 1440 containing the permutation (1 8 12 6 3 10 4 11 2 7)(5 9) and (5 12)(6 8)(7 10)(9 11) so Aut D is transitive. The stabilizer Aut D has two point orbits, (1, 2, 3, 4, 5) and (6, 7, 8, 9, 10, 11). Aut $D_{(12)}$ contains

24 elements like (1 2 3 4 5)(7)(6 8 9 10 11),

30 elements like (1)(2 3 4 5)(6)(7 9 11 8)(10),

10 elements like (1)(2)(3)(4 5)(6 7)(8 9)(10 11),

20 elements like (1)(2)(3 4 5)(6 11 9)(7 8 10),

20 elements like (1 2)(3 4 5)(6 10 9 8 11 7),

15 elements like (1)(2 3)(4 5)(6 10)(7 11)(8)(9),

and the identity. Thus $|\text{Aut } D_{(12)}| = 120$. Furthermore each permutation of 1, 2, 3, 4, 5 occurs just once. Therefore $\text{Aut}_{(12)} \cong S_5$. The permutations on 6, 7, 8, 9, 10, 11 provide another representation of S_5 as a permutation group acting on six elements.

CASE III: Repeated blocks; non-decomposable.

AC = 0, B = 40, R = 4.

1	2	3	4	5	12	6	7	8	9	10	11
1	2	3	4	5	12	6	7	8	9	10	11
1	2	6	7	8	12	3	4	5	9	10	11
1	2	9	10	11	12	3	4	5	6	7	8
1	3	6	7	9	12	2	4	5	8	10	11
1	3	8	10	11	12	2	4	5	6	7	9
1	4	6	7	10	12	2	3	5	8	9	11
1	4	8	9	11	12	2	3	5	6	7	10
1	5	6	7	11	12	2	3	4	8	9	10
1	5	8	9	10	12	2	3	4	6	7	11
2	3	6	8	9	12	1	4	5	7	10	11
2	3	7	10	11	12	1	4	5	6	8	9
2	4	6	8	10	12	1	3	5	7	9	11
2	4	7	9	11	12	1	3	5	6	8	10
2	5	6	8	11	12	1	3	4	7	9	10
2	5	7	9	10	12	1	3	4	6	8	11
3	4	6	9	10	12	1	2	5	7	8	11
3	4	7	8	11	12	1	2	5	6	9	10
3	5	6	9	11	12	1	2	4	7	8	10
3	5	7	8	10	12	1	2	4	6	9	11
4	5	6	10	11	12	1	2	3	7	8	9
4	5	7	8	9	12	1	2	3	6	10	11

Here $|\text{Aut } D| = 1440$, The group contains the elements

$$\alpha = (1\ 2)(3)(4)(5)(6)(7\ 8)(9)(10)(11)(12)$$

and $\beta = (1\ 2\ 3\ 4\ 5\ 12)(6\ 7\ 8\ 9\ 10\ 11)$. These satisfy $\beta^6 = (\alpha\beta)^5 = e$ and therefore generate a subgroup of Aut D isomorphic to S_6 (C & M, p 137).

The element

$\gamma = (1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12)$ commutes with both α and β.

Therefore Aut D $\cong S_6 \times C_2$. Also Aut D permutes the rows and columns of the array

1	2	3	4	5	12
7	8	9	10	11	6

in all possible ways.

CASE IV: Repeated blocks; non-decomposable

AC = 0, B = 40, R = 4

1	2	3	4	5	12	6	7	8	9	10	11
1	2	3	4	5	12	6	7	8	9	10	11
1	2	6	7	8	12	3	4	5	9	10	11
1	2	9	10	11	12	3	4	5	6	7	8
1	3	6	7	9	12	2	4	5	8	10	11
1	3	8	10	11	12	2	4	5	6	7	9
1	4	6	7	10	12	2	3	5	8	9	11
1	4	8	9	11	12	2	3	5	6	7	10
1	5	6	7	11	12	2	3	4	8	9	10
1	5	8	9	10	12	2	3	4	6	7	11
2	3	6	8	9	12	1	4	5	7	10	11
2	3	7	10	11	12	1	4	5	6	8	9
2	4	6	8	10	12	1	3	5	7	9	11
2	4	7	9	11	12	1	3	5	6	8	10
2	5	6	9	10	12	1	3	4	7	8	11
2	5	7	8	11	12	1	3	4	6	9	10
3	4	6	9	11	12	1	2	5	7	8	10
3	4	7	8	10	12	1	2	5	6	9	11
3	5	6	8	11	12	1	2	4	7	9	10
3	5	7	9	10	12	1	2	4	6	8	11
4	5	6	10	11	12	1	2	3	7	8	9
4	5	7	8	9	12	1	2	3	6	10	11

Here $|\text{Aut D}| = 12$ and Aut D is generated by

$$\alpha = (1\ 2\ 5\ 12\ 3\ 4)(6\ 9\ 10\ 7\ 8\ 11)$$

and $\beta = (1\ 6)(2\ 11)(3\ 10)(4\ 9)(5\ 8)(7\ 12)$

which obey the relations $\alpha^6 = \beta^2 = (\alpha\beta)^2 = e$. Thus Aut D $\cong D_6 \cong S_2 \times D_3$ (see C & M, p 134). Aut D acts imprimitively on the sextuples (1, 2, 3, 4, 5, 12) and (6, 7, 8, 9, 10, 11). The stabilizer Aut $D_{(1)}$ is the identity.

CASE V: Repeated blocks; non-decomposable.

AC = 0, B = 36, R = 8.

1	2	3	4	5	12	6	7	8	9	10	11
1	2	3	4	5	12	6	7	8	9	10	11
1	2	6	7	8	12	3	4	5	9	10	11
1	2	6	7	8	12	3	4	5	9	10	11
1	3	6	9	10	12	2	4	5	7	8	11
1	3	7	9	11	12	2	4	5	6	8	10
1	4	6	9	11	12	2	3	5	7	8	10
1	4	8	10	11	12	2	3	5	6	7	9
1	5	7	10	11	12	2	3	4	6	8	9
1	5	8	9	10	12	2	3	4	6	7	11
2	3	6	10	11	12	1	4	5	7	8	9
2	3	8	9	11	12	1	4	5	6	7	10
2	4	7	10	11	12	1	3	5	6	8	9
2	4	8	9	10	12	1	3	5	6	7	11
2	5	6	9	11	12	1	3	4	7	8	10
2	5	7	9	10	12	1	3	4	6	8	11
3	4	6	7	10	12	1	2	5	8	9	11
3	4	7	8	9	12	1	2	5	6	10	11
3	5	6	8	10	12	1	2	4	7	9	11
3	5	7	8	11	12	1	2	4	6	9	10
4	5	6	7	9	12	1	2	3	8	10	11
4	5	6	8	11	12	1	2	3	7	9	10

For this design $|\text{Aut } D| = 48$. The group is generated by

$$\alpha = (1\ 7\ 10\ 5\ 2\ 6\ 9\ 4\ 12\ 8\ 11\ 3),$$

$$\beta = (1\ 7\ 12\ 8\ 2\ 6)(3\ 9\ 4\ 10\ 5\ 11),$$

and $$\gamma = (1\ 3\ 9\ 6)(2\ 4\ 11\ 7)(5\ 10\ 8\ 12).$$

For these $\alpha^{12} = \beta^6 = \gamma^4 = e$, $\beta^2 = \alpha^8$ and $\gamma^2 = \alpha^6$. Aut D acts imprimitively on the triples (1, 12, 2), (5, 3, 4), (9, 10, 11), (8, 6, 7) and also on the quadruples (1, 5, 9, 8), (12, 3, 10, 6), (2, 4, 11, 7). Therefore Aut D simultaneously permutes the rows and columns of the array

	R	S	T
a:	1	12	2
b:	5	3	4
c:	9	10	11
d:	8	6	7 .

The columns are permuted under the action of the symmetric group S_3 on R,S,T. The rows move under the action of the group generated by (a b c d) and (ab)(cd) which is the dihedral group D_4 with order 8. Each of the 48 elements of Aut D provides a unique permutation of the array and Aut D $\cong S_3 \times D_4$. The stabilizer $D_{(1)}$ is Klein's four group.

CASE VI: No repeated blocks; non-decomposable.

AC = 0, B = 44, R = 0.

1	2	3	4	5	6	7	8	9	10	11	12
1	2	9	10	11	12	3	4	5	6	7	8
1	2	7	8	11	12	3	4	5	6	9	10
1	2	7	8	9	10	3	4	5	6	11	12
3	4	8	10	11	12	1	2	5	6	7	9
3	5	8	9	11	12	1	2	4	6	7	10
3	6	7	8	9	10	1	2	4	5	11	12
4	5	7	8	9	10	1	2	3	6	11	12
4	6	7	9	11	12	1	2	3	5	8	10
5	6	7	10	11	12	1	2	3	4	8	9
1	3	4	7	8	11	2	5	6	9	10	12
1	3	4	7	10	12	2	5	6	8	9	11
1	3	5	7	10	11	2	4	6	8	9	12
1	3	5	9	10	12	2	4	6	7	8	11
1	3	6	7	9	11	2	4	5	8	10	12
1	3	6	8	9	12	2	4	5	7	10	11
1	4	5	7	9	12	2	3	6	8	10	11
1	4	5	8	9	11	2	3	6	7	10	12
1	4	6	8	10	12	2	3	5	7	9	11
1	4	6	9	10	11	2	3	5	7	8	12
1	5	6	7	8	12	2	3	4	9	10	11
1	5	6	8	10	11	2	3	4	7	9	12

This design has $\alpha = (1\ 3)(2\ 6)(4\ 12)(5\ 11)(7\ 10)(8\ 9)$

and $\beta = (1\ 11\ 2\ 11)(3\ 8)(4\ 10\ 5\ 9)(6\ 7)$

as automorphisms. These satisfy $\alpha^4 = \beta^2 = (\alpha\beta)^3 = e$. Therefore Aut D $\cong S_4$ (see C & M, p 134). Also $|\text{Aut D}| = 24$. The group S_4 is also known as the octahedral group and is the group of rotations of the regular octahedron or its dual the cube. It is to be expected therefore that the points of the design can be associated with geometrical elements of the cube. Now Aut D acts imprimitively on the sextuples (3, 6, 9, 10, 11, 12) and (1, 2, 4, 5, 7, 8). The twelve face diagonals of a cube can be partitioned into two sets of six, each set forming the edges of a regular tetrahedron. These two tetrahedra form the well known compound, Kepler's stella octangula. Figure 1 is the net of a cube with the face diagonals labelled with the elements of the

3-design. The heavy diagonals correspond to one of the imprimitivity sextuples, the dotted diagonals correspond to the other; the two sets of edges belong to the two tetrahedra of the stella octangula. With this labelling all the elements of Aut D are generated as edge cycles of the stella octangula under symmetry rotations.

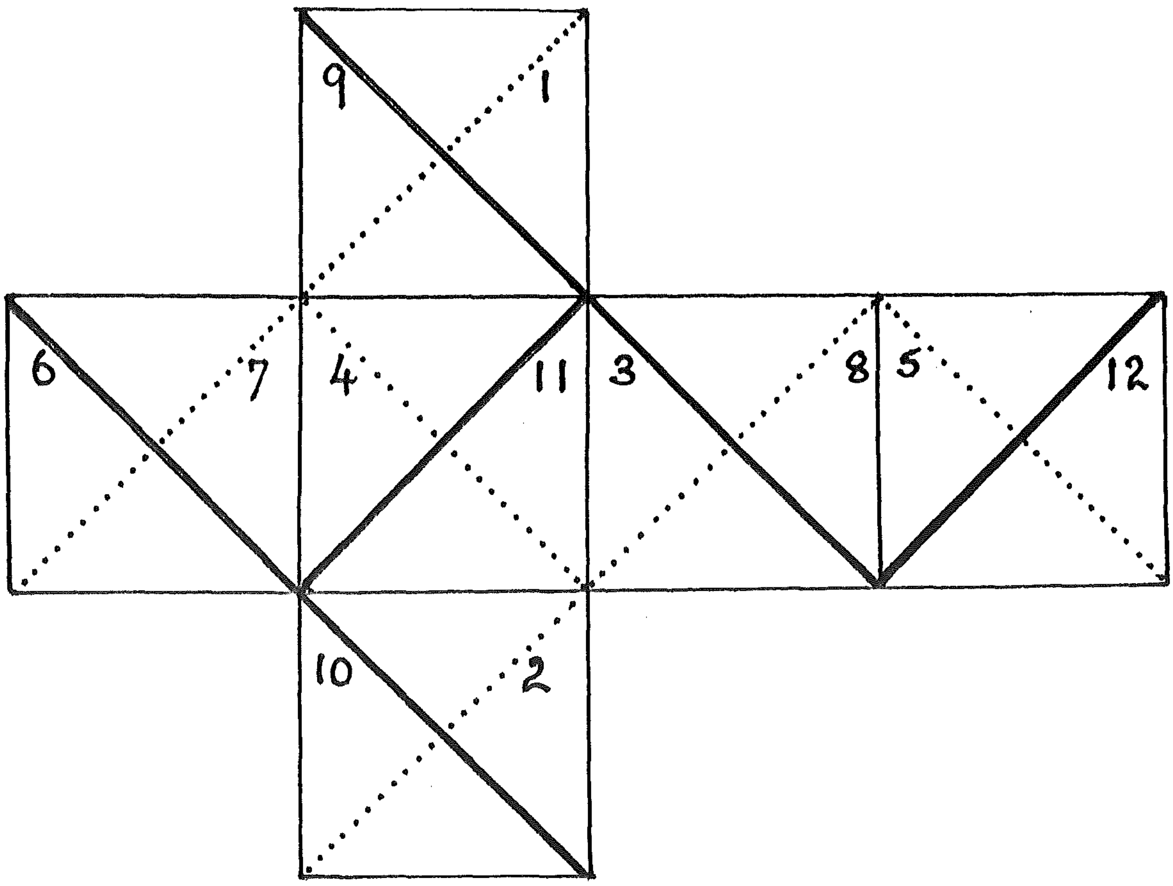

Figure 1

CASE VII: No repeated blocks; non-decomposable.

AC = 0, B = 44, R = 0.

1	2	3	4	5	6	7	8	9	10	11	12
1	2	9	10	11	12	3	4	5	6	7	8
1	2	7	8	11	12	3	4	5	6	9	10
1	2	7	8	9	10	3	4	5	6	11	12
3	4	8	10	11	12	1	2	5	6	7	9
3	4	7	9	11	12	1	2	5	6	8	10
5	6	8	9	11	12	1	2	3	4	7	10
5	6	7	10	11	12	1	2	3	4	8	9
3	5	7	8	9	10	1	2	4	6	11	12
4	6	7	8	9	10	1	2	3	5	11	12
1	3	4	7	8	11	2	5	6	9	10	12
1	3	5	7	9	11	2	4	6	8	10	12
1	3	5	8	10	12	2	4	6	7	9	11
1	3	6	7	10	12	2	4	5	8	9	11
1	3	6	8	9	12	2	4	5	7	10	11
1	3	6	9	10	11	2	4	5	7	8	12
1	4	5	7	10	12	2	3	6	8	9	11
1	4	5	8	9	12	2	3	6	7	10	11
1	4	5	9	10	11	2	3	6	7	8	12
1	4	6	7	9	12	2	3	5	8	10	11
1	4	6	8	10	11	2	3	5	7	9	12
1	5	6	7	8	11	2	3	4	9	10	12

Two automorphisms are $\alpha = (1)(2)(11)(12)(3\ 4)(5\ 6)(7\ 8)(9\ 10)$

and $\beta = (1\ 3\ 10)(2\ 5\ 9)(4\ 8\ 11)(6\ 7\ 12)$.

For these $\alpha^2 = \beta^3 = (\beta^{-1}\alpha\beta\alpha)^2 = e$. Therefore Aut D $\cong A_4 \times C_2$ (see C & M, p 134) where A_4 is the alternating group on four elements, the tetrahedral group. Consequently $|\text{Aut D}| = 24$. Also Aut $D_{(1)} = <\alpha>$. Aut D acts imprimitively on the quadruples (1, 2, 11, 12), (3, 4, 5, 6) and (7, 8, 9, 10).

CASE VIII: No repeated blocks; non-decomposable.

AC = 24, B = 20, R = 0.

3	4	5	6	7	8	1	2	9	10	11	12
3	5	6	7	8	10	1	2	4	9	11	12
1	4	5	6	8	9	2	3	7	10	11	12
1	4	6	8	9	11	2	3	5	7	10	12
1	2	5	6	9	10	3	4	7	8	11	12
2	5	6	7	9	10	1	3	4	8	11	12
1	2	3	6	10	11	4	5	7	8	9	12
1	3	6	8	10	11	2	4	5	7	9	12
2	3	4	6	7	11	1	5	8	9	10	12
2	4	6	7	9	11	1	3	5	8	10	12
6	7	8	9	10	11	1	2	3	4	5	12
1	2	3	4	5	6	7	8	9	10	11	12

1	2	6	7	8	12	3	4	5	9	10	11
2	3	6	8	9	12	1	4	5	7	10	11
3	4	6	9	10	12	1	2	5	7	8	11
4	5	6	10	11	12	1	2	3	7	8	9
1	5	6	7	11	12	2	3	4	8	9	10
1	3	6	7	9	12	2	4	5	8	10	11
2	4	6	8	10	12	1	3	5	7	9	11
3	5	6	9	11	12	1	2	4	7	8	10
1	4	6	7	10	12	2	3	5	8	9	11
2	5	6	8	11	12	1	3	4	7	9	10

Aut D, which has order 240, contains the elements

$$\alpha = (1\ 2\ 3\ 4\ 5)(7\ 8\ 9\ 10\ 11),$$

$$\beta = (2\ 10\ 9\ 5\ 6)(4\ 3\ 11\ 12\ 8),$$

and $$\gamma = (1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12).$$

If $R = \beta^3\alpha$ and $S = \beta^2\alpha$ then

$$R^2 = S^3 = (RS)^5 = e$$

so $<\alpha,\beta> \cong A_5$ (see C & M, p 67), the icosahedral group. The element γ commutes with both α and β so $<\alpha,\beta,\gamma> \cong A_5 \times C_2$. The element γ also commutes with the automorphism $\delta = (1\ 11\ 2\ 4\ 3\ 12\ 7\ 5\ 8\ 10\ 9\ 6)$. In fact Aut D $\cong S_5 \times C_2$. Aut D acts imprimitively on the pairs (1, 7), (2, 8), (3, 9), (4, 10), (5, 11), (6, 12).

In [1] this design is associated with a regular dodecahedron and is generalised to give a family of 3-designs.

III: THE TRANSITIVE 2 - (11,5,4) DESIGNS

If a 2 - (11,5,4) design has a transitive automorphism group then its order is a multiple of 11^{α} where $\alpha \geqslant 1$. The Sylow theorems then assure us that a subgroup of order 11 exists. This means that any transitive 2 - (11,5,4) design can be generated from a pair of supplementary difference sets developed cyclically modulo 11.

CASE (i): Repeated blocks; decomposable.

A = B = C = 0, R = 22.

Take {1, 3, 4, 5, 9} repeated to give the two supplementary difference sets. The cyclic development modulo 11 gives a trivial doubling of the well known 2 - (11,5,2) design with a 2-transitive group of order 660. This design extends to the 3-transitive 3 - (12,6,4) design, Case I of Section II.

CASE (ii): No repeated blocks; decomposable.

A = 22, B = 0, C = 0, R = 0.

As supplementary difference sets take {1, 3, 4, 5, 9} and {2, 6, 7, 8, 10} consisting of squares and non-squares modulo 11. The automorphism group of the resulting design contains the permutations $x \to ax + b \pmod{11}$, $a \neq 0$ and so is 2-transitive. The stabilizer of a pair of points is the identity. Therefore the design has a sharply 2-transitive group of order 110. The extension to a 3 - (12, 6, 4) design is not transitive.

CASE (iii): No repeated blocks: not decomposable.

A = 0, B = 22, C = 0, R = 0.

The supplementary difference sets are {1, 2, 4, 7, 8} and {4, 6, 7, 8, 10}. The automorphism group is cyclic and isomorphic to C_{11}. It is generated by $x \to x + 1 \pmod{11}$. The design is transitive but its extension to a 3-design is not transitive.

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