A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces

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Abstract

We give characterizations of weighted Besov-Lipschitz and Triebel-Lizorkin spaces with A_∞ weights via a smooth kernel which satisfies "minimal" moment and Tauberian conditions. The results are stated in terms of the mixed norm of certain maximal function of a distribution in these weighted spaces.

1. Introduction

We recall the definitions of the weighted Besov-Lipschitz and Triebel-Lizorkin spaces. We refer to [1] for references to the relevant literature as well as proofs. Throughout this paper let $0 < p < \infty$, $0 < q \leq \infty$, $-\infty < \alpha < \infty$, and $w \in A_\infty$, where A_∞ is the Muckenhoupt weight class. All functions and distributions are defined on $\mathbf{R}^n$ and explicit reference to $\mathbf{R}^n$ in the notation will be dropped. $\mathcal{S}$ is the usual space of test functions for the space of tempered distributions $\mathcal{S}'$.

To define the scales of spaces choose a function $\theta \in \mathcal{S}$ such that

$$\operatorname{supp} \hat{\theta} \subseteq \{\tfrac{1}{2} \leq |\xi| \leq 2\}; \sum_{j=-\infty}^{\infty} \hat{\theta}(2^{-j}\xi) = 1, |\xi| \neq 0.$$

For each integer j, let $\psi_j \in \mathcal{S}$ be given by $\hat{\psi}_j(\xi) = \hat{\theta}(2^{-j}\xi)$. Following J. Peetre and H. Triebel, we define two scales of function spaces as follows:

$$\dot{B}^{\alpha,w}_{p,q} = \left\{ f \in \mathcal{S}' : \|f\|_{\dot{B}^{\alpha,w}_{p,q}} = \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\psi_j * f\|_{p,w})^q \right)^{1/q} < \infty \right\},$$

$$\dot{F}^{\alpha,w}_{p,q} = \left\{ f \in \mathcal{S}' : \|f\|_{\dot{F}^{\alpha,w}_{p,q}} = \left\| \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} |\psi_j * f(\cdot)|)^q \right)^{1/q} \right\|_{p,w} < \infty \right\}, \tag{1.1}$$

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where $\|g\|_{p,w} = \left(\int_{\mathbf{R}^n} |g(x)|^p w(x)dx\right)^{1/p}$ is the quasi-norm for the weighted Lebesgue space L^p_w. Since the functions $\hat{\psi}_j$ vanish in a neighbourhood of the origin, we see that these spaces are quasi-Banach spaces that are continuously embedded in $\mathcal{S}'/\mathcal{P}$, the space of tempered distributions modulo (all) polynomials. Different choices of θ lead to the same spaces with equivalent quasi-norms.

These two scales of function spaces and their inhomogeneous counterparts (see Section 5) play an important role in the various branches of analysis. In particular,

$$\dot{F}^{0,w}_{p,2} = H^p_w \ , \ 0 < p < \infty \ ,$$

where H^p_w denotes the weighted Hardy space of $f \in \mathcal{S}'$ for which

$$\|f\|_{H^p_w} = \| \sup_{0<t<\infty} |\phi_t * f(\cdot)| \|_{p,w} < \infty \ ,$$

where ϕ is a fixed function in $\mathcal{S}$ with $\int_{\mathbf{R}^n} \phi(x)dx \neq 0$, and $\phi_t(x) = t^{-n}\phi(x/t)$.

By the fundamental work of C. Fefferman and E.M. Stein [6] adapted to the weighted case, H^p_w (or its local version h^p_w given in Section 5) does not depend on the function ϕ used in its definition (see also [1, Theorem 1.2],[12]). For the Besov-Lipschitz and Triebel-Lizorkin spaces, a basic result by J. Peetre [11, Theorem 3.1] showed that they are independent of the sequence $\{\psi_j\}$ entering in their definitions.

In our results we have restricted p to be finite, but this is a technicality since for a non-trivial A_∞ weight w, $L^\infty_w = L^\infty$. Observe that

$$\dot{B}^\alpha_{p,\infty} = \dot{\Lambda}^p_\alpha, \quad 0 < \alpha < \infty, 1 \leq p \leq \infty,$$

where $\dot{\Lambda}^p_\alpha$ is the homogeneous Hölder-Zygmund space of order α, and so the results are well-known. See [9] for a treatment of the unweighted Besov-Lipschitz spaces.

There is an equivalent family of quasi-norms in which the sums in (1.1) are replaced by integrals. Thus if $\varphi \in \mathcal{S}$,

$$\operatorname{supp} \hat{\varphi} \subseteq \{1/2 \leq |\xi| \leq 2\} \quad \text{and} \quad |\hat{\varphi}(\xi)| \geq c > 0, 3/5 \leq |\xi| \leq 5/3$$

for some $c > 0$, then the weighted homogeneous Besov-Lipschitz and Triebel-Lizorkin spaces are characterized by

$$\begin{aligned} \|f\|_{\dot{B}^{\alpha,w}_{p,q}} &\sim \left(\int_0^\infty (t^{-\alpha}\|\varphi_t * f\|_{p,w})^q \frac{dt}{t}\right)^{1/q}, \\ \|f\|_{\dot{F}^{\alpha,w}_{p,q}} &\sim \left\| \left(\int_0^\infty (t^{-\alpha}|\varphi_t * f|)^q \frac{dt}{t}\right)^{1/q} \right\|_{p,w} \end{aligned} \tag{1.2}$$

for all $f \in \mathcal{S}'/\mathcal{P}$. The fact that these quasi-norms are independent of the choice of φ and are equivalent to those given in (1.1) follows by standard arguments that mimic the proof that the quasi-norms in (1.1) are independent of the choice of θ. See [11] or [7] for details of this argument. More can be found in Section 6(d).

It is our purpose to find characterizations such as (1.2) but for kernels that occur naturally and satisfy conditions simply stated and easily verified. One such characterization is well-known and it will be used in this paper. We shall use $C, c, \ldots$ to denote positive constants which may depend on the parameters concerned, such as, $\alpha, p, q, w, \ldots$, but not on the variable quantity, usually a distribution f.

Theorem 1.1 (See [2],[3]). *Let $-\infty < \alpha < \infty$, $0 < p < \infty$, $0 < q \le \infty$, $w \in A_\infty$, and $r_0 = \inf\{r : w \in A_r\}$. Assume that k is a non-negative integer with $2k > \alpha$, and $\phi \in \mathcal{S}$ is given by $\hat{\phi}(\xi) = (-|\xi|^2)^k e^{-|\xi|^2}$. Then*

$$\begin{aligned} c\left(\int_0^\infty (t^{-\alpha}\|\phi_t * f\|_{H^p_w})^q \frac{dt}{t}\right)^{1/q} &\le \|f\|_{\dot{B}^{\alpha,w}_{p,q}} \le C\left(\int_0^\infty (t^{-\alpha}\|\phi_t * f\|_{p,w})^q \frac{dt}{t}\right)^{1/q}, \\ c\left\|\left(\int_0^\infty (t^{-\alpha}\phi_t^* f)^q \frac{dt}{t}\right)^{1/q}\right\|_{p,w} &\le \|f\|_{\dot{F}^{\alpha,w}_{p,q}} \le C\left\|\left(\int_0^\infty (t^{-\alpha}|\phi_t * f|)^q \frac{dt}{t}\right)^{1/q}\right\|_{p,w} \end{aligned} \tag{1.3}$$

for all $f \in \mathcal{S}'/\mathcal{P}$, where $\lambda > \max(nr_0/p, nb/q)$ and

$$\phi_t^* f(x) = \phi_{t,\lambda}^* f(x) = \sup_{y \in \mathbf{R}^n} |\phi_t * f(x-y)| \left(1 + \frac{|y|}{t}\right)^{-\lambda}.$$

REMARKS. (i) Let $W_t(x) = W(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ be the Gauss-Weierstrass kernel on $\mathbf{R}^{n+1}_+$. If we set $v(x,t) = (\partial/\partial t)^k W_t * f(x)$, then, since $\widehat{W_t}(\xi) = e^{-t|\xi|^2}$, $\phi_t * f(x) = t^{2k} v(x, t^2)$, and the theorem above gives characterizations of the weighted homogeneous Besov-Lipschitz and Triebel-Lizorkin spaces via temperatures; i.e., solutions to the heat equation.

(ii) Since convolution with ϕ in (1.3), (unlike convolution with φ in (1.2)) does not annihilate all polynomials, one should interpret the left-hand side inequalities in (1.3) as being valid for some representative in the equivalence class in $\mathcal{S}'/\mathcal{P}$. Similar conventions hold for the conclusions of Theorems 3.1 and 4.1. However, if $\hat{f}$ vanishes in a neighbourhood of the origin, then the proof in [2] showed that these inequalities hold for the same representative f.

(iii) Notice that the left-hand side inequality for the Besov-Lipschitz quasi-norms in (1.3) is stronger than that suggested by (1.2). Ideally one would want the quasi-norm on the left-hand side to be as strong as possible and that on the right-hand side as weak as possible. The role of the Hardy quasi-norm in (1.3) is described in Section 4. A similar comment holds for the Triebel-Lizorkin quasi-norms in (1.3).

(iv) Versions of Theorem 1.1 are valid for harmonic functions. That is, derivatives of the Gauss-Weierstrass kernel can be replaced by derivatives of the Poisson kernel, provided high enough orders of differentiation are used. See Section 6(b) for details. This is an example of how removing the kernel from $\mathcal{S}$ requires the imposition of "unnatural conditions". These additional conditions are imposed so the various integrals and sums that occur in the proof converge.

There is near universal agreement on the minimum "natural conditions" to be satisfied by a kernel ϕ in order to yield a characterization as in (1.2). There are a "moment condition" to get size estimates, as in Lemma 2.1, used to get the left-hand side inequalities and a "Tauberian condition" (a non-degeneracy condition on the Fourier transform) as in Lemma 2.3, used to get the right-hand side inequalities. Further requirements that the kernel be in $\mathcal{S}$ or that it is a measure with compact support are imposed so that the kernel and its derivatives have controlled growth at infinity and so that its Fourier transform is a multiplier on $\mathcal{S}$. One hopes to get a version of (1.2) for kernels in $\mathcal{S}$ that satisfy these minimal conditions.

However, an examination of our main results: Theorems 3.1, 4.1, and 5.1 shows that we have fallen short of our goal. We have obtained versions of (1.2) with $\varphi_t * f$ replaced by the maximal function of Peetre and Triebel, $\varphi_t^* f = \varphi_{t,\lambda}^* f$. (See the first paragraph of Section 3 below for the definition.) Since $(\varphi_t^* f)(x)$ dominates $(\varphi_t * f)(x)$ pointwise, the left-hand side inequalities are better than what we are looking for, but the right-hand side inequalities are worse. The significant open problem is to close this gap. More on this subjet can be found in Section 6(d).

We noted above that the independence of θ for the definitions of the Triebel-Lizorkin and Besov-Lipschitz spaces in (1.1) depends on a basic result of Peetre. The weighted version of Peetre's result is essential in what follows. It is stated in the next theorem.

Theorem 1.2 (See [1], [10]). *Let* $-\infty < \alpha < \infty$, $0 < p < \infty$, $0 < q \le \infty$, $w \in A_\infty$, *and* $r_0 = \inf\{r : w \in A_r\}$. *Let* $a > 0$, *and assume that* $\{\phi_j\}_{j=-\infty}^{\infty}$ *is a sequence of functions in* $\mathcal{S}$ *such that* $\operatorname{supp}\hat{\phi}_j \subseteq \{2^{j-a} \le |\xi| \le 2^{j+a}\}$, *and* $|D^\kappa \hat{\phi}_j(\xi)| \le C_\kappa 2^{-j|\kappa|}$ *for all* j, κ, ξ. *For* $\lambda > 0$ *and* $j = 0, \pm 1, \pm 2, \dots$, *define*

$$\phi_{j,\lambda}^* f(x) = \phi_j^* f(x) = \sup_{y \in \mathbf{R}^n} |\phi_j * f(x-y)|(1+2^j|y|)^{-\lambda} ,$$

$f \in \mathcal{S}'$, $x \in \mathbf{R}^n$.

(*i*) *If* $\lambda > \max(nr_0/p,\ n/q)$, *then*

$$\left\| \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} \phi_j^* f(x))^q \right)^{1/q} \right\|_{p,w} \le C \|f\|_{\dot{F}_{p,q}^{\alpha,w}}$$

for all $f \in \mathcal{S}'$.

(*ii*) *If* $\lambda > nr_0/p$, *then*

$$\left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\phi_j^* f\|_{p,w})^q \right)^{1/q} \le C \|f\|_{\dot{B}_{p,q}^{\alpha,w}}$$

for all $f \in \mathcal{S}'$.

The rest of the paper is organized as follows: Section 2 contains five technical lemmas needed for the proofs of the main results. Section 3 gives a characterization of the weighted homogeneous Triebel-Lizorkin spaces, and Section 4 a characterization of the weighted homogeneous Besov-Lipschitz spaces. In Section 5 these results are extended to the inhomogeneous spaces. Section 6 is devoted to remarks and further results.

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2. Technical lemmas

In this section we shall gather a number of lemmas needed in the proofs of our main results in Sections 3 and 4.

Let μ be a measure either with density in $\mathcal{S}$ or with compact support and ℓ be an integer. We say that μ has moments of order up to ℓ that vanish if

$$\int_{\mathbf{R}^n} x^\kappa d\mu(x) = 0$$

for all $|\kappa| \leq \ell$, with the convention that no moment condition is required when $\ell < 0$.

Lemma 2.1 (Size estimates of Heideman type). *Let k, m and r be non-negative integers, and assume that $\lambda \geq 0$.*

(i) Let $\eta \in \mathcal{S}$, and let μ be a measure either with density in $\mathcal{S}$ or with compact support. Assume that μ has moments of order up to $k-1$ that vanish. Then there exists $C > 0$ such that

$$\int_{\mathbf{R}^n} \left(1 + \frac{|y|}{t}\right)^\lambda |\eta_t * \mu_s(y)| dy \leq C \left(\frac{s}{t}\right)^k$$

for all $0 < s \leq t < \infty$.

(ii) If $\mu \in \mathcal{S}$, and $\eta \in \mathcal{S}$ has moments of order up to $m - 1 + r$ that vanish and $0 \leq \lambda \leq r$, then there exists $C > 0$ such that

$$\int_{\mathbf{R}^n} \left(1 + \frac{|y|}{t}\right)^\lambda |\eta_t * \mu_s(y)| dy \leq C \left(\frac{t}{s}\right)^m$$

for all $0 < t \leq s < \infty$.

(iii) If $\eta \in \mathcal{S}$ and μ is a measure with compact support, then there exists $C > 0$ such that

$$\int_{\mathbf{R}^n} \left(1 + \frac{|y|}{t}\right)^\lambda |\eta_t * \mu_s(y)| dy \leq C \left(\frac{s}{t}\right)^\lambda ,$$

for all $0 < t \leq s < \infty$.

PROOF. We shall prove (i) in the case μ has density in $\mathcal{S}$, since the compactly supported case can be similarly handled. By Taylor's formula and the moment condition on μ, we have, for every $y \in \mathbf{R}^n$,

$$\begin{aligned}
\eta_t * \mu_s(y) &= \int_{\mathbf{R}^n} t^{-n}\eta\left(\frac{y}{t} - z\right)\mu\left(\frac{t}{s}z\right)\left(\frac{t}{s}\right)^n dz \\
&= \sum_{|\kappa|=k} c_\kappa \int_{\mathbf{R}^n} t^{-n}\left(\int_0^1 \rho^{k-1} D^\kappa \eta\left(\frac{y}{t} - \rho z\right) d\rho\right) z^\kappa \mu\left(\frac{t}{s}z\right)\left(\frac{t}{s}\right)^n dz \\
&= \sum_{|\kappa|=k} c_\kappa I_\kappa(y,s,t) \ .
\end{aligned}$$

Since $\eta \in \mathcal{S}$, we see that

$$\left|D^\kappa \eta\left(\frac{y}{t} - \rho z\right)\right| \leq C(\kappa,\eta,\lambda)\left(1 + \frac{|y|}{2t}\right)^{-\lambda-n-1}$$

for all $|z| \leq |y|/2t$ and $0 < \rho < 1$, and

$$\left|D^\kappa \eta\left(\frac{y}{t} - \rho z\right)\right| \leq C(\kappa,\eta)$$

for all y, ρ, z and t. It follows that, for each κ,

$$\begin{aligned}
&\int_{\mathbf{R}^n}\left(1 + \frac{|y|}{t}\right)^\lambda |I_\kappa(y,s,t)| dy \leq \\
&\left(\frac{s}{t}\right)^k \int_{\mathbf{R}^n}\left(1 + \frac{|y|}{t}\right)^\lambda t^{-n}\left\{\left(\int_{\{|z|\leq|y|/2t\}} + \int_{\{|z|>|y|/2t\}}\right)\right. \\
&\left.\left(\int_0^1 \rho^{k-1}\left|D^\kappa\eta\left(\frac{y}{t} - \rho z\right)\right| d\rho\right)\left|\frac{t}{s}z\right|^k \left|\mu\left(\frac{t}{s}z\right)\right|\left(\frac{t}{s}\right)^n dz\right\} dy \\
\leq\ & C\left(\frac{s}{t}\right)^k \left\{\left(\int_{\mathbf{R}^n}\left(1 + \frac{|y|}{t}\right)^\lambda \left(1 + \frac{|y|}{2t}\right)^{-\lambda-n-1} t^{-n} dy\right)\right. \\
&\left(\int_{\mathbf{R}^n}\left|\frac{t}{s}z\right|^k \left|\mu\left(\frac{t}{s}z\right)\right|\left(\frac{t}{s}\right)^n dz\right) \\
&\left. + \int_{\mathbf{R}^n}\left(\int_{\{|y|/t<2|z|\}} t^{-n}\left(1 + \frac{|y|}{t}\right)^\lambda dy\right)\left|\frac{t}{s}z\right|^k \left|\mu\left(\frac{t}{s}z\right)\right|\left(\frac{t}{s}\right)^n dz\right\} \\
\leq\ & C\left(\frac{s}{t}\right)^k \int_{\mathbf{R}^n}(1+2|z|)^{\lambda+n}\left|\frac{t}{s}z\right|^k \left|\mu\left(\frac{t}{s}z\right)\right|\left(\frac{t}{s}\right)^n dz \\
\leq\ & C\left(\frac{s}{t}\right)^k
\end{aligned}$$

as $\mu \in \mathcal{S}$ and $s \leq t$. Thus (i) follows.

Using Taylor's formula and the moment condition of η in a similar way as that in the proof of (i), we get

$$\int_{\mathbf{R}^n}\left(1+\frac{|y|}{t}\right)^{\lambda}|\eta_t*\mu_s(y)|dy \leq$$
$$\left(\frac{t}{s}\right)^m\int_{\mathbf{R}^n}\left(1+\frac{|y|}{t}\right)^{\lambda}\left(\frac{t}{s}\right)^r s^{-n}\left\{\sum_{|\kappa|=m+r}\left(\int_{\{|z|\leq|y|/2s\}}+\int_{\{|z|>|y|/2s\}}\right)\right.$$
$$\left.\left(\int_0^1\left|D^{\kappa}\mu\left(\frac{y}{s}-\rho z\right)\right|d\rho\right)\left|\frac{s}{t}z\right|^{m+r}\left|\eta\left(\frac{s}{t}z\right)\right|\left(\frac{s}{t}\right)^n dz\right\}dy$$
$$\leq C\left(\frac{t}{s}\right)^m\int_{\mathbf{R}^n}\left(1+\frac{|y|}{t}\right)^{\lambda}\left(\frac{t}{s}\right)^{\lambda}s^{-n}\{\cdots\}dy$$
$$\leq C\left(\frac{t}{s}\right)^m\int_{\mathbf{R}^n}\left(\frac{t}{s}+\frac{|y|}{s}\right)^{\lambda}s^{-n}\{\cdots\}dy$$
$$\leq C\left(\frac{t}{s}\right)^m\int_{\mathbf{R}^n}\left(1+\frac{|y|}{s}\right)^{\lambda}s^{-n}\{\cdots\}dy$$

for $0 < t/s \leq 1$ and $r \geq \lambda \geq 0$. The last integral can be seen to be dominated by a positive constant by an argument similar to that in the proof of (i) (by interchanging t with s and η with μ). Note that, although the proofs of (i) and (ii) are rather similar, part (ii) does not follow from part (i).

For (iii), note that

$$\int_{\mathbf{R}^n}\left(1+\frac{|y|}{t}\right)^{\lambda}|\eta_t*\mu_s(y)|dy \leq$$
$$\int_{\mathbf{R}^n}\left(1+\frac{|y|}{t}\right)^{\lambda}\left(\int_{\mathbf{R}^n}t^{-n}\left|\eta\left(\frac{y}{t}-\frac{s}{t}z\right)\right|d|\mu|(z)\right)dy$$
$$\leq\int_{\mathbf{R}^n}\left(1+\frac{s}{t}|z|\right)^{\lambda}\left(\int_{\mathbf{R}^n}t^{-n}\left|\eta\left(\frac{y}{t}-\frac{s}{t}z\right)\right|\left(1+\left|\frac{y}{t}-\frac{s}{t}z\right|\right)^{\lambda}dy\right)d|\mu|(z)$$
$$=\left(\int_{\mathbf{R}^n}|\eta(y)|(1+|y|)^{\lambda}dy\right)\left(\int_{\mathbf{R}^n}\left(1+\frac{s}{t}|z|\right)^{\lambda}d|\mu|(z)\right)$$
$$\leq C(\eta,\lambda,\mu)\left(\frac{s}{t}\right)^{\lambda}$$

as $\eta\in\mathcal{S}$ and $s/t \geq 1$.

REMARKS. (a) Parts (i) and (iii) when $\lambda = 0$ are due to N.J.H. Heideman [8].

(b) By mimicking the proof of the above Lemma, we can prove that, under the same assumptions as in part (ii), and for every $b > 0$, there exists a constant C such that

$$|\eta_t*\mu_s(y)| \leq Cs^{-n}\left(\frac{t}{s}\right)^m\left(1+\frac{|y|}{t}\right)^{-r}$$

for all $0 < t \le bs$ and $y \in \mathbf{R}^n$. In particular, if η has infinitely many vanishing moments, then for every $N > 0$, there exists a constant C such that

$$|\eta_t * \mu_s(y)| \le C s^{-n} \left(\frac{t}{s}\right)^N \left(1 + \frac{|y|}{t}\right)^{-N}$$

for all $0 < t \le bs$ and $y \in \mathbf{R}^n$.

Lemma 2.2 ([3, Proposition 1.1]). *Let* $-\infty < \alpha < \infty$, $0 < p < \infty$, $0 < q \le \infty$ *and* $w \in A_\infty$. *If* $f \in \dot{B}^{\alpha,w}_{p,q}$ *or* $f \in \dot{F}^{\alpha,w}_{p,q}$, *then there exist polynomials* $P, P_1, P_2, P_3, \ldots$ *such that*

$$f - P = \lim_{m \to \infty} \left(\sum_{j=-m}^{\infty} \psi_j * f - P_m \right)$$

in $\mathcal{S}'$, *and* $\deg(P_m) \le [\alpha]$ *for all* m.

Lemma 2.3 (Calderón representation theorem ([8], [9])). *Let* μ *be a measure either with density in* $\mathcal{S}$ *or of compact support. Assume that* μ *satisfies the Tauberian condition; i.e.,*

$$\forall \xi \ne 0 \quad \exists t > 0 \text{ such that } \hat{\mu}(t\xi) \ne 0 \ .$$

Then there exists $\eta \in \mathcal{S}$ *with* $\operatorname{supp} \hat{\eta}$ *contained in an annulus about the origin such that*

$$\int_0^\infty \hat{\eta}(t\xi) \hat{\mu}(t\xi) \frac{dt}{t} = 1 \quad \forall \xi \ne 0 \ ,$$

and for every $f \in \mathcal{S}'$, *there is a non-negative integer* k *for which*

$$f = \int_0^\infty \eta_t * \mu_t * f \frac{dt}{t}$$

in $\mathcal{S}'/\mathcal{P}_k$, *the space of tempered distributions modulo polynomials of degree at most* k.

REMARK. The use of the Calderón representation theorem in the theory of function spaces originated with A.P. Calderón, and the idea was developed further by N.J. Heideman ([8]), by A.P. Calderón and A. Torchinsky ([5]), and by S. Janson and M.H. Taibleson ([9]). The formulation of our Lemma 2.3 is taken from [9]. Note also that the integer k in the lemma depends on the order of f as a distribution or on the growth of f at infinity. See [9] for details.

We shall need the following special case of a result by J-O. Strömberg and A. Torchinsky (see [12, Chap.V, Theorem 2(b)]).

Lemma 2.4 (Sub-mean value property). *Let* $\varphi \in \mathcal{S}$ *satisfy the Tauberian condition in the sense of Lemma 2.3. Assume that* $\hat{\varphi}$ *is supported in an annulus about the origin. Then for every* $r > 0$ *and* $N > 0$, *there exists* $C > 0$ *for which*

$$|\varphi_t * g(x)|^r \le C \int_0^\infty \int_{\mathbf{R}^n} |\varphi_s * g(y)|^r \left(1 + \frac{|x-y|}{s}\right)^{-Nr} s^{-n} \left(\min(s/t, t/s)\right)^{Nr} dy \frac{ds}{s}$$

for all $g \in \mathcal{S}'$, $x \in \mathbf{R}^n$ *and* $t > 0$.

PROOF. Let $\eta \in \mathcal{S}$ be the function given in Lemma 2.3 for φ. Let $u > 0$. Since φ_u has infinitely many vanishing moments, Lemma 2.3 implies that

$$\varphi_u * g(z) = \int_0^\infty \varphi_u * \eta_s * \varphi_s * g(z) \frac{ds}{s}$$

for all $z \in \mathbf{R}^n$. By considering the supports of $\widehat{\varphi_u}$ and $\widehat{\eta_s}$, we can find $a, b > 0$ such that $\varphi_u * \eta_s = 0$ unless $au \leq s \leq bu$. By using Remark (b) after Lemma 2.1, we deduce that

$$|\varphi_u * \eta_s(\xi)| \leq C u^{-n} \left(\frac{s}{u}\right)^N \left(1 + \frac{|\xi|}{u}\right)^{-N}$$

for all $0 < s \leq bu$ and $\xi \in \mathbf{R}^n$. It follows that

$$\begin{aligned} |\varphi_u * g(z)| &\leq \int_{au}^{bu} \int_{\mathbf{R}^n} |\varphi_u * \eta_s(z-y)| |\varphi_s * g(y)| dy \frac{ds}{s} \\ &\leq C \int_{au}^{bu} \int_{\mathbf{R}^n} \left(1 + \frac{|y-z|}{u}\right)^{-N} |\varphi_s * g(y)| \left(\frac{s}{u}\right)^N u^{-n} dy \frac{ds}{s}. \end{aligned} \tag{2.1}$$

If $r \geq 1$, then by using (2.1) for $N + n + 1$ (in place of N) and Hölder's inequality, we obtain

$$|\varphi_t * g(x)|^r \leq C \int_0^{bt} \int_{\mathbf{R}^n} |\varphi_s * g(y)|^r \left(1 + \frac{|x-y|}{s}\right)^{-Nr} s^{-n} \left(\frac{s}{t}\right)^{Nr} dy \frac{ds}{s},$$

which implies the conclusion of the lemma in this case.

Assume next that $0 < r < 1$. For $x \in \mathbf{R}^n$ and $t > 0$, define

$$M_N(x,t) = \sup_{y \in \mathbf{R}^n, s > 0} |\varphi_s * g(y)| \left(1 + \frac{|x-y|}{s}\right)^{-N} \min(s/t, t/s)^N.$$

Then by using this "maximal function" M_N, together with (2.1) and the obvious inequality

$$\left(\frac{s}{u}\right)^N \left[\frac{1 + |x-y|/s}{(1 + |x-z|/u)(1 + |z-y|/u)}\right]^N \left[\frac{\min(u/t, t/u)}{\min(t/s, s/t)}\right]^N \leq C$$

for all $au \leq s \leq bu, t > 0$, and x, y, z in $\mathbf{R}^n$, we can mimic the proof of Theorems 1 and 5 in [12] to obtain the desired result in this case.

REMARK. Note that the vector-valued versions of Lemmas 2.1, 2.3 and 2.4 also hold.

Let $W(\cdot, t) = W_t$ be the Gauss-Weierstrass kernel on $\mathbf{R}_+^{n+1}$ (as in Section 1). For $\lambda \geq 0$ and $g \in \mathcal{S}'$, following C. Fefferman and E.M. Stein we define

$$g_\lambda^{**}(x) = g^{**}(x) = \sup_{y \in \mathbf{R}^n, \rho > 0} |W(\cdot, \rho^2) * g(y)| \left(1 + \frac{|x-y|}{\rho}\right)^{-\lambda},$$

$x \in \mathbf{R}^n$. Note then that $\|g^{**}\|_{p,w} \approx \|g\|_{H^p_w}$ if $\lambda > nr_0/p,\ r_0 = \inf\{r : w \in A_r\}$ (see e.g., [1, p.584]).

Lemma 2.5 *(i) Let k be a non-negative integer, and let $\phi \in \mathcal{S}$ be given by $\hat{\phi}(\xi) = (-|\xi|^2)^k e^{-|\xi|^2}$. Suppose $\lambda \geq 0$ and $M \geq 1$. Then there exists $C > 0$ such that*

$$(\phi_s * f)^{**}(x) \leq C(\phi_t * f)^{**}(x)$$

for all $t \leq s \leq Mt$, $f \in \mathcal{S}'$ and $x \in \mathbf{R}^n$.

(ii) Let θ and η be functions in $\mathcal{S}$. assume that $0 < a \leq b < \infty$ and $\lambda \geq 0$. Then there exists $C > 0$ such that

$$\sup_{y \in \mathbf{R}^n} |\theta_s * \eta_t * g(y)| \left(1 + \frac{|x-y|}{t}\right)^{-\lambda} \leq C \sup_{y \in \mathbf{R}^n} |\eta_t * g(y)| \left(1 + \frac{|x-y|}{t}\right)^{-\lambda}$$

for all $at \leq s \leq bt$, $g \in \mathcal{S}'$ and $x \in \mathbf{R}^n$.

PROOF. We shall prove (i) only as the proof of (ii) is similar. As $W_u * W_v = W_{u+v}$ for all $u, v > 0$, by writing $s = \sqrt{t^2 + a^2t^2}$ with $0 \leq a \leq \sqrt{M^2 - 1}$, we deduce that

$$\begin{aligned} W(\cdot, \rho^2) * \phi_s * f &= (1 + a^2)^k W(\cdot, \rho^2 + a^2t^2) * \phi_t * f \\ &= (1 + a^2)^k W(\cdot, \sigma^2) * W(\cdot, \sigma^2) * \phi_t * f \ , \end{aligned}$$

where $\sigma > 0$ is given by $2\sigma^2 = \rho^2 + a^2t^2$. It follows that

$$\begin{aligned} & |W(\cdot, \rho^2) * \phi_s * f(y)| \left(1 + \frac{|x-y|}{\rho}\right)^{-\lambda} \leq \\ & M^{2k} \int_{\mathbf{R}^n} W(z, \sigma^2) |W(\cdot, \sigma^2) * \phi_t * f(y - z)| \left(1 + \frac{|x-y+z|}{\sigma}\right)^{-\lambda} \\ & \left(1 + \frac{|x-y+z|}{\sigma}\right)^{\lambda} \left(1 + \frac{|x-y|}{\rho}\right)^{-\lambda} dz \\ \leq \ & C(\phi_t * f)^{**}(x) \int_{\mathbf{R}^n} \left(1 + \frac{|z|}{\sigma}\right)^{-\lambda-n-1} \left(1 + \frac{|z|}{\sigma}\right)^{\lambda} \left(1 + \frac{|x-y|}{\sigma}\right)^{\lambda} \\ & \left(1 + \frac{|x-y|}{\rho}\right)^{-\lambda} \sigma^{-n} dz \\ \leq \ & C(\phi_t * f)^{**}(x) \ . \end{aligned}$$

Taking the supremum of the left-hand side with respect to y and ρ, we obtain (i).

3. Characterization of the weighted homogeneous Triebel-Lizorkin spaces

Let $\lambda \geq 0$ and $\mu \in \mathcal{S}$. For $f \in \mathcal{S}'$ and $t > 0$, following J. Peetre and H. Triebel, we define a version of the Fefferman-Stein maximal function by

$$\mu_{t,\lambda}^* f(x) = \sup_{y \in \mathbf{R}^n} |\mu_t * f(x-y)| \left(1 + \frac{|y|}{t}\right)^{-\lambda} \ ,$$

$x \in \mathbf{R}^n$. In the sequel, as we shall fix a λ (satisfying some additional assumption), we write $\mu_t^* f$ for $\mu_{t,\lambda}^* f$.

Theorem 3.1 *Let $-\infty < \alpha < \infty$, $0 < p < \infty$, $0 < q \leq \infty$, $w \in A_\infty$, $r_0 = \inf\{r : w \in A_r\}$, and $\lambda > \max(nr_0/p, n/q)$. Assume that $\mu \in \mathcal{S}$ satisfies the moment condition; i.e.,*

$$\int_{\mathbf{R}^n} x^\kappa \mu(x)dx = 0$$

for all $|\kappa| \leq [\alpha]$, and that $\nu \in \mathcal{S}$ satisfies the Tauberian condition; i.e.,

$$\forall \xi \neq 0 \quad \exists t > 0 \text{ such that } \hat{\nu}(t\xi) \neq 0 \ .$$

Then there exist positive constants c and C for which

$$\begin{aligned} c\left\| \left(\int_0^\infty (t^{-\alpha}\mu_t^* f(x))^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w} &\leq \|f\|_{\dot{F}_{p,q}^{\alpha,w}} \\ &\leq C\left\| \left(\int_0^\infty (t^{-\alpha}\nu_t^* f(x))^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w} \qquad (3.1) \end{aligned}$$

for all $f \in \mathcal{S}'/\mathcal{P}$.

Proof. Assume that $f \in \dot{F}_{p,q}^{\alpha,w}$. By subtracting a suitable polynomial from f and using Lemma 2.2, we deduce that there exists a sequence of polynomials $\{P_m\}$ such that

$$f = \lim_{m\to\infty} \left\{ \sum_{j=-m}^{\infty} \psi_j * f - P_m \right\}$$

in $\mathcal{S}'$, and $\deg(P_m) \leq [\alpha]$ for all m. Hence it follows from the fact that $\mu \in \mathcal{S}$ and the moment condition of μ that

$$f * \mu_t(x) = \sum_{j=-\infty}^{\infty} \mu_t * \psi_j * f(x)$$

for all $x \in \mathbf{R}^n$ and $t > 0$. Let ϕ be a function in $\mathcal{S}$ with the following properties: $\hat{\phi}(\xi) = 1$ for $1/2 \leq |\xi| \leq 2$; $\operatorname{supp}\hat{\phi} \subseteq \{1/3 \leq |\xi| \leq 3\}$. For $j = 0, \pm1, \pm2, \ldots$, let $\phi_j \in \mathcal{S}$ be given by $\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi)$. Then, since $\psi_j = \psi_j * \phi_j$ for all j, the above can be rewritten as

$$f * \mu_t(x) = \sum_{j=-\infty}^{\infty} \mu_t * \psi_j * \phi_j * f(x) \ .$$

By an argument similar to the Gaussian case given in [2, p.56] we see that, for each integer ℓ, and $2^{-\ell-1} \leq t \leq 2^{-\ell}$,

$$t^{-\alpha}\mu_t^* f(x) \leq C \sum_{j=-\infty}^{\infty} a_{j-\ell}(2^{(\ell-j)\alpha} + 2^{(\ell-j)(\alpha-\lambda)})2^{j\alpha}\phi_j^* f(x) \ ,$$

where

$$a_j = \sup_{\frac{1}{2}\le s\le 1} \int_{\mathbf{R}^n} |\mu_s * \psi_j(y)|(1+2^j|y|)^\lambda dy,\ j=0,\pm1,\pm2,\ldots$$

and

$$\phi_j^* f(x) = \sup_{y\in\mathbf{R}^n} \{|\phi_j * f(x-y)|(1+2^j|y|)^{-\lambda}\}$$

(as in Theorem 1.2). It follows that, with $\rho = \min(1,q)$,

$$\begin{aligned}\left(\int_0^\infty (t^{-\alpha}\mu_t^* f(x))^q \frac{dt}{t}\right)^{1/q} &= \left(\sum_{\ell=-\infty}^{\infty} \int_{2^{-\ell-1}}^{2^{-\ell}} (t^{-\alpha}\mu_t^* f(x))^q \frac{dt}{t}\right)^{1/q} \\ &\le C\left(\sum_{\ell=-\infty}^{\infty} [a_\ell(2^{-\ell\alpha}+2^{-\ell(\alpha-\lambda)})]^\rho\right)^{1/\rho} \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha}\phi_j^* f(x))^q\right)^{1/q}\end{aligned}$$

by using well-known inequalities. The left-hand side inequality in (3.1) follows from Theorem 1.2 if we can show that

$$S = \sum_{\ell=-\infty}^{\infty} [a_\ell(2^{-\ell\alpha}+2^{-\ell(\alpha-\lambda)})]^\rho < \infty\ .$$

Noting that $\hat{\psi}_\ell(\xi) = \hat{\theta}(2^{-\ell}\xi)$ with $\hat{\theta}\in\mathcal{S}$ supported in $\{\frac{1}{2}\le|\xi|\le 2\}$ (cf. Section 1), we deduce from Lemma 2.1 (i), (ii) that

$$\begin{aligned} a_\ell &= \sup_{\frac{1}{2}\le s\le 1} \int_{\mathbf{R}^n} |\mu_s * \psi_\ell(y)|(1+2^\ell|y|)^\lambda dy \\ &= \sup_{\frac{1}{2}\le s\le 1} \int_{\mathbf{R}^n} |\mu_s * \theta_{2^{-\ell}}(y)| \left(1+\frac{|y|}{2^{-\ell}}\right)^\lambda dy \\ &\le C\begin{cases} 2^{\ell k} & \text{if } \ell\le 0 \\ 2^{-\ell m} & \text{if } \ell > 0\ , \end{cases}\end{aligned}$$

where $k=[\alpha]+1$ if $\alpha\ge 0$, $k=0$ if $\alpha<0$, and m is chosen so that $m+\alpha-\lambda>0$. The above estimates imply the finiteness of S.

Next we shall prove the second inequality in (3.1). Let $\eta\in\mathcal{S}$ be the function given by Lemma 2.3 for ν. Assume that supp $\hat{\eta}\subseteq\{2^{-A+1}\le|\xi|\le 2^{A-1}\}$, where $A>1$. As each ψ_j has moments of all orders which vanish, Lemma 2.3 implies that, for every $x\in\mathbf{R}^n$,

$$\begin{aligned}\psi_j * f(x) &= \int_0^\infty \nu_t * \eta_t * \psi_j * f(x)\frac{dt}{t} \\ &= \int_{I_j} \nu_t * \eta_t * \psi_j * f(x)\frac{dt}{t}\ ,\end{aligned} \tag{3.2}$$

because $\eta_t * \psi_j = 0$ unless $t\in I_j = [2^{-j-A}, 2^{-j+A}]$. Note also that $\nu * \eta$ satisfies the Tauberian condition by Lemma 2.3. Let $r>0$ and $t\in I_j$. Choose $N=\lambda+(n+1)/r$. Applying Lemma 2.4 to $\varphi = \nu*\eta$ and $g=\psi_j * f$, we obtain

$$|\nu_t * \eta_t * \psi_j * f(x)|^r \leq C \int_0^\infty \int_{\mathbf{R}^n} |\nu_s * \eta_s * \psi_j * f(y)|^r \min(s/t, t/s)^{Nr}$$

$$\left(1 + \frac{|x-y|}{s}\right)^{-\lambda r} \left(1 + \frac{|x-y|}{s}\right)^{-n-1} s^{-n} dy \frac{ds}{s} = C \int_{I_j} \int_{\mathbf{R}^n} \cdots dy \frac{ds}{s}$$

because $\eta_s * \psi_j = 0$ unless $s \in I_j$. Since $t \approx s \approx 2^{-j}$ for $t, s \in I_j$, Lemma 2.5 (ii) implies that

$$\sup_y \left\{ |\nu_s * \eta_s * \psi_j * f(y)| \left(1 + \frac{|x-y|}{s}\right)^{-\lambda} \right\}$$
$$\leq C \sup_y \left\{ |\nu_s * f(y)| \left(1 + \frac{|x-y|}{s}\right)^{-\lambda} \right\} = C \nu_s^* f(x) \ .$$

It follows from all the above that

$$|\psi_j * f(x)| \leq C \int_{I_j} \left(\int_{I_j} (\nu_s^* f(x))^r \frac{ds}{s} \right)^{1/r} \frac{dt}{t} \tag{3.3}$$

for every x and j.

Assume first that $0 < q \leq 1$. Letting $r = q$ in (3.3) and using Minkowski's inequality we obtain

$$\begin{aligned} |\psi_j * f(x)|^q \ &\leq \ C \int_{I_j} \left(\int_{I_j} (\nu_s^* f(x)) \frac{dt}{t} \right)^q \frac{ds}{s} \\ &\leq \ C \int_{I_j} (\nu_s^* f(x))^q \frac{ds}{s} \ . \end{aligned}$$

On the other hand if $q > 1$, then by taking $r = 1$ in (3.3) we get

$$\begin{aligned} |\psi_j * f(x)| \ &\leq \ C \int_{I_j} \nu_s^* f(x) \frac{ds}{s} \\ &\leq \ C \left(\int_{I_j} (\nu_s^* f(x))^q \frac{ds}{s} \right)^{1/q} \ . \end{aligned}$$

Hence it follows that

$$\left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} |\psi_j * f(x)|)^q \right)^{1/q} \leq C \left(\sum_{j=-\infty}^{\infty} \int_{I_j} (s^{-\alpha} \nu_s^* f(x))^q \frac{ds}{s} \right)^{1/q}$$
$$\leq C(2A)^{1/q} \left(\int_0^\infty (s^{-\alpha} \nu_s^* f(x))^q \frac{ds}{s} \right)^{1/q}$$

for all $q > 0$. Thus we obtain the desired inequality by taking the L^p_w-quasi-norms on both sides of the above.

REMARK. Note that if the right-hand side inequality in (3.1) holds for a λ_0, then it holds for all $\lambda \leq \lambda_0$. It follows that the right-hand side inequality holds for all $\lambda \in \mathbf{R}$.

4. Characterization of the weighted homogeneous Besov-Lipschitz spaces

We shall keep the same notations as in Section 3 with regard to the maximal functions $\mu_t^* f$, $\nu_t^* f$.

Theorem 4.1 *Let $-\infty < \alpha < \infty$, $0 < p < \infty$, $0 < q \leq \infty$, $w \in A_\infty$, $r_0 = \inf\{r : w \in A_r\}$, and $\lambda > nr_0/p$. Assume that $\mu \in \mathcal{S}$ satisfies the moment condition and $\nu \in \mathcal{S}$ satisfies the Tauberian condition (as in Theorem 3.1). Then there exist positive constants c and C for which*

$$c\left(\int_0^\infty (t^{-\alpha}\|\mu_t * f\|_{H^p_w})^q \frac{dt}{t}\right)^{1/q} \leq \|f\|_{\dot{B}^{\alpha,w}_{p,q}} \leq C\left(\int_0^\infty (t^{-\alpha}\|\nu_t^* f\|_{p,w})^q \frac{dt}{t}\right)^{1/q} \tag{4.1}$$

for all $f \in \mathcal{S}'/\mathcal{P}$.

Remark. Since

$$\begin{aligned} |\mu_t * f(x-y)|\left(1+\frac{|y|}{t}\right)^{-\lambda} &= \lim_{\substack{\rho\to 0\\ \rho\leq t}} |W(\cdot,\rho^2) * \mu_t * f(x-y)|\left(1+\frac{|y|}{t}\right)^{-\lambda} \\ &\leq \sup_{\rho\leq t} |W(\cdot,\rho^2) * \mu_t * f(x-y)|\left(1+\frac{|y|}{\rho}\right)^{-\lambda} \\ &\leq (\mu_t * f)^{**}(x) \end{aligned}$$

for every x, y, t, we deduce that

$$\|\mu_t^* f\|_{p,w} \leq C\|\mu_t * f\|_{H^p_w} ,$$

where C is independent of $t > 0$ and $f \in \mathcal{S}'$. Hence the left-hand side inequality in (4.1) is stronger than the corresponding result for the Triebel-Lizorkin spaces in Theorem 3.1.

Proof of Theorem 4.1. Since the proof of the right-hand side inequality in (4.1) is similar to the proof of the corresponding result in Theorem 3.1, we shall only give the details for the other inequality in (4.1). Let k be a non-negative integer such that $2k > \alpha$, and let $\phi \in \mathcal{S}$ be given by $\hat{\phi}(\xi) = (-|\xi|^2)^k e^{-|\xi|^2}$. Clearly ϕ satisfies the Tauberian condition. Let η be the function given by Lemma 2.3 for ϕ. Let $f \in \dot{B}^{\alpha,w}_{p,q}$. Assume first that $\hat{f} = 0$ in a neighbourhood of the origin. Then it is easy to see from Lemma 2.3 that we have the representation

$$f = \int_0^\infty \phi_t * \eta_t * f \frac{dt}{t}$$

in $\mathcal{S}'$. It follows from the above and the semi-group property of the Gaussian kernel (as in the proof of Lemma 2.5 (i)) that, for every $\rho > 0$, $s > 0$ and $x, z \in \mathbf{R}^n$,

$$|W(\cdot,\rho^2) * \mu_s * f(z)| = \left| 4^k \int_0^\infty W(\cdot,\rho^2 + 3t^2/4) * \phi_{t/2} * f * \eta_t * \mu_s(z) \frac{dt}{t} \right|$$

$$\leq C \int_0^\infty \left\{ \int_{\mathbf{R}^n} |\eta_t * \mu_s(z)| \, |W(\cdot,\rho^2 + 3t^2/4) * (\phi_{t/2} * f)(z-y)| \right.$$

$$\left. \left(1 + \frac{|x-z+y|}{\sqrt{\rho^2 + 3t^2/4}}\right)^{-\lambda} \left(1 + \frac{|x-z|}{\rho}\right)^{\lambda} \left(1 + \frac{|y|}{t}\right)^{\lambda} \right\} dy \frac{dt}{t} ,$$

and

$$(\mu_s * f)^{**}(x) = \sup_{z,\rho} |W(\cdot,\rho^2) * (\mu_s * f)(z)| \left(1 + \frac{|x-z|}{\rho}\right)^{-\lambda}$$

$$\leq C \int_0^\infty (\phi_{t/2} * f)^{**}(x) \left\{ \int_{\mathbf{R}^n} \left(1 + \frac{|y|}{t}\right)^{\lambda} |\mu_s * \eta_t(y)| dy \right\} \frac{dt}{t}$$

$$= C \int_0^\infty u_{t/2}^{**}(x) \left\{ \int_{\mathbf{R}^n} \left(1 + \frac{|y|}{t}\right)^{\lambda} |\mu_s * \eta_t(y)| dy \right\} \frac{dt}{t}$$

$$= C \int_0^s \cdots + C \int_s^\infty \cdots = I_1(x,s) + I_2(x,s) ,$$

where we have set $u_t = \phi_t * f$ for simplicity. Assume that $0 < p < 1$. Choose a non-negative integer m with $m + \alpha > 0$. Using Lemma 2.1 (ii) and the "monotonicity" of u_t^{**} (Lemma 2.5 (i)), we obtain

$$|I_1(x,s)|^p \leq C \left(\int_0^s u_{t/2}^{**}(x) \left(\frac{t}{s}\right)^m \frac{dt}{t} \right)^p$$

$$\leq C \int_0^s \left(u_t^{**}(x) \left(\frac{t}{s}\right)^m \right)^p \frac{dt}{t} .$$

If $q \geq p$, the above and Hardy's inequality imply that

$$\left(\int_0^\infty (s^{-\alpha} \|I_1(\cdot,s)\|_{p,w})^q \frac{ds}{s} \right)^{p/q} \leq$$

$$C \left\{ \int_0^\infty \left(s^{-\alpha p} \int_0^s \|u_t^{**}\|_{p,w}^p \left(\frac{t}{s}\right)^{mp} \frac{dt}{t} \right)^{q/p} \frac{ds}{s} \right\}^{p/q}$$

$$\leq C \left(\int_0^\infty (s^{-\alpha} \|u_t^{**}\|_{p,w})^q \frac{ds}{s} \right)^{p/q} .$$

On the other hand, if $0 < q < p$, then by using Lemma 2.5 (i) again, we get

$$\left(\int_0^\infty (s^{-\alpha}\|I_1(\cdot,s)\|_{p,w})^q \frac{ds}{s}\right)^{1/q} \le C\left\{\int_0^\infty \left(s^{-\alpha p}\int_0^s \|u_t^{**}\|_{p,w}^p \left(\frac{t}{s}\right)^{mp}\frac{dt}{t}\right)^{q/p}\frac{ds}{s}\right\}^{1/q}$$

$$\le C\left\{\int_0^\infty \left(s^{-\alpha q}\int_0^s \|u_t^{**}\|_{p,w}^q \left(\frac{t}{s}\right)^{mq}\frac{dt}{t}\right)\frac{ds}{s}\right\}^{1/q}$$

$$= \frac{C}{((m+\alpha)q)^{1/q}}\left(\int_0^\infty (t^{-\alpha}\|u_t^{**}\|_{p,w})^q \frac{dt}{t}\right)^{1/q} .$$

Using Lemma 2.1 (i) and Lemma 2.5(i) we obtain

$$|I_2(x,s)|^p \le C\int_s^\infty \left(u_t^{**}(x)\left(\frac{s}{t}\right)^k\right)^p \frac{dt}{t} .$$

Then by arguments similar to those immediately above, one can show that

$$\left(\int_0^\infty (s^{-\alpha}\|I_2(\cdot,s)\|_{p,w})^q \frac{ds}{s}\right)^{1/q} \le C\left(\int_0^\infty (t^{-\alpha}\|u_t^{**}\|_{p,w})^q \frac{dt}{t}\right)^{1/q} .$$

The above estimates for I_1 and I_2 imply that

$$\left(\int_0^\infty (s^{-\alpha}\|(\mu_s * f)^{**}\|_{p,w})^q \frac{dt}{t}\right)^{1/q} \le C\left(\int_0^\infty (t^{-\alpha}\|(\phi_t * f)^{**}\|_{p,w})^q \frac{dt}{t}\right)^{1/q} .$$

Since $\|(\mu_s * f)^{**}\|_{p,w} \approx \|\mu_s * f\|_{H_w^p}$ and $\|(\phi_t * f)^{**}\|_{p,w} \approx \|\phi_t * f\|_{H_w^p}$, the proof of the desired result is completed by invoking Theorem 1.1.

Next we shall remove the restriction that $\hat{f} = 0$ in a neighbourhood of the origin. Using Lemma 2.2 and the moment condition of μ as in the proof of Theorem 3.1, we obtain

$$W_\rho * \mu_s * f(x) = \lim_{m\to\infty} W_\rho * \mu_s * f_m(x) ,$$

for all $\rho, s > 0$ and $x \in \mathbf{R}^n$ (after subtracting a suitable polynomial from f), where $f_m = \sum_{j=-m}^{\infty} \psi_j * f$. The desired inequality then follows from the above representation of $W_\rho * \mu_s * f$ by a limit argument similar to the Gaussian case (cf. [2, p.61]). The proof in the case $p < 1$ is thus complete.

The case $p \ge 1$ can be handled in a similar way but is generally simpler than the case $p < 1$ and so details are omitted.

5. Characterizations of inhomogeneous spaces

In this section we shall give the inhomogeneous version of the results in Sections 3 and 4. As the proofs are similar to the homogeneous case we shall be brief and indicate only the necessary modifications.

Let $\{\psi_j\}$ be as in Section 1 and let Ψ be the function in $\mathcal{S}$ given by $\hat{\Psi}(\xi)+\sum_{j=1}^{\infty}\hat{\psi}_j(\xi)=1$ for all $\xi\in\mathbf{R}^n$. Let $0<p<\infty$, $0<q\leq\infty$, $-\infty<\alpha<\infty$, and $w\in A_\infty$. The weighted inhomogeneous Besov and Triebel-Lizorkin spaces are defined by

$$B_{p,q}^{\alpha,w}=\left\{f\in\mathcal{S}':\|f\|_{B_{p,q}^{\alpha,w}}=\|\Psi*f\|_{p,w}+\left(\sum_{j=1}^{\infty}(2^{j\alpha}\|\psi_j*f\|_{p,w})^q\right)^{1/q}<\infty\right\},$$

$$F_{p,q}^{\alpha,w}=\left\{f\in\mathcal{S}':\|f\|_{F_{p,q}^{\alpha,w}}=\|\Psi*f\|_{p,w}+\left\|\left(\sum_{j=1}^{\infty}(2^{j\alpha}|\psi_j*f(\cdot)|)^q\right)^{1/q}\right\|_{p,w}<\infty\right\}.$$

Let $\{\phi_j\}$ and $a>0$ be as in Theorem 1.2, and let $\Phi\in\mathcal{S}$ be such that $\operatorname{supp}\hat{\Phi}\subseteq\{|\xi|\leq 2^a\}$. For $\lambda>0$ and $f\in\mathcal{S}'$, define $\phi_{j,\lambda}^*f=\phi_j^*f$ as in Theorem 1.2 and let

$$\Phi_\lambda^*f(x)=\Phi^*f(x)=\sup_{y\in\mathbf{R}^n}|\Phi*f(x-y)|(1+|y|)^{-\lambda},$$

$x\in\mathbf{R}^n$. Let $r_0=\inf\{r:w\in A_r\}$. The maximal inequalities (Theorem 1.2) for inhomogeneous spaces are as follows:

If $\lambda>\max(nr_0/p,n/q)$, then

$$\|\Phi^*f\|_{p,w}+\left\|\left(\sum_{j=1}^{\infty}(2^{j\alpha}\phi_j^*f(x))^q\right)^{1/q}\right\|_{p,w}\leq C\|f\|_{F_{p,q}^{\alpha,w}} \tag{5.1}$$

for all $f\in\mathcal{S}'$.

If $\lambda>nr_0/p$, then

$$\|\Phi^*f\|_{p,w}+\left(\sum_{j=1}^{\infty}(2^{j\alpha}\|\phi_j^*f\|_{p,w})^q\right)^{1/q}\leq C\|f\|_{B_{p,q}^{\alpha,w}} \tag{5.2}$$

for all $f\in\mathcal{S}'$. It is useful to note that (5.2) implies a stronger result:

$$\|\Phi*f\|_{h_w^p}+\left(\sum_{j=1}^{\infty}(2^{j\alpha}\|\phi_j*f\|_{h_w^p})^q\right)^{1/q}\leq C\|f\|_{B_{p,q}^{\alpha,w}}. \tag{5.2$'$}$$

Here, for $g\in\mathcal{S}'$,

$$\|g\|_{h_w^p}=\|\sup_{0<t<1}|\phi_t*g(x)|\,\|_{p,w},$$

where $\phi\in\mathcal{S}$ with $\int_{\mathbf{R}^n}\phi(x)dx\neq 0$.

We refer again to [1] for the above and related properties of the inhomogeneous spaces as well as references to the relevant literature.

In the rest of this section let $\mu \in \mathcal{S}$ satisfy the moment condition and let $\nu \in \mathcal{S}$ satisfy the Tauberian condition (as in Theorem 3.1). For $f \in \mathcal{S}'$ and $\lambda > 0$, let $\mu^*_{t,\lambda} f = \mu^*_t f$ and $\nu^*_{t,\lambda} f = \nu^*_t f$ be defined as at the beginning of Section 3, and for ζ, $\varphi \in \mathcal{S}$, let $\zeta^* f$ and $\varphi^* f$ be defined similarly to $\Phi^* f$ above.

Theorem 5.1 *Let $0 < p < \infty$, $0 < q \le \infty$, $-\infty < \alpha < \infty$, $a > 0$, $w \in A_\infty$ and $r_0 = \inf\{r : w \in A_r\}$. Assume that $\zeta \in \mathcal{S}$, and that $\varphi \in \mathcal{S}$ satisfies the strong Tauberian condition $\hat{\varphi}(0) = \int_{\mathbf{R}^n} \varphi(x)dx \neq 0$.*

(i) If $\lambda > \max(nr_0/p, n/q)$, then there exist positive constant b, c and C for which

$$c\left(\|\zeta^* f\|_{p,w} + \left\| \left(\int_0^a (t^{-\alpha}\mu^*_t f(x))^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w} \right) \le \|f\|_{F^{\alpha,w}_{p,q}}$$
$$\le C\left(\|\varphi^* f\|_{p,w} + \left\| \left(\int_0^b (t^{-\alpha}\nu^*_t f(x))^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w} \right) \tag{5.3}$$

for all $f \in \mathcal{S}'$.

(ii) If $\lambda > nr_0/p$, then there exist positive constants b, c and C for which

$$c\left(\|\zeta * f\|_{h^p_w} + \left(\int_0^a (t^{-\alpha}\|\mu_t * f\|_{h^p_w})^q \frac{dt}{t} \right)^{1/q} \right) \le \|f\|_{B^{\alpha,w}_{p,q}}$$
$$\le C\left(\|\varphi^* f\|_{p,w} + \left(\int_0^b (t^{-\alpha}\|\nu^*_t f\|_{p,w})^q \frac{dt}{t} \right)^{1/q} \right) \tag{5.4}$$

for all $f \in \mathcal{S}'$.

Proof. Observe that in the inhomogeneous case, we have the representation

$$f = \Psi * f + \sum_{j=1}^{\infty} \psi_j * f \qquad \text{in } \mathcal{S}' .$$

Let $\{\phi_j\}$ be as in the proof of Theorem 3.1 and let $\Phi \in \mathcal{S}$ be such that supp $\hat{\Phi} \subseteq \{|\xi| \le 3\}$ and $\hat{\Phi} = 1$ on supp $\hat{\Psi}$. We start with the proof of the left-hand side inequality in (5.3). The above representation then implies that

$$\mu_t * f(x) = \mu_t * \Phi * \Psi * f(x) + \sum_{j=1}^{\infty} \mu_t * \phi_j * \psi_j * f(x) \tag{5.5}$$

for all x and t. By using (5.1) the infinite sum on the right-hand side of (5.5) can be estimated in the same way as in the proof of Theorem 3.1. We shall next deal with the first term. For each x and y in $\mathbf{R}^n$ and $0 < t \le a$,

$$|\mu_t * \Phi * \Psi * f(x-y)| \le C \int_{\mathbf{R}^n} |\mu_t * \Phi(z)|\, |\Psi * f(x-y-z)|$$
$$(1+|y+z|)^{-\lambda} \left(1 + \frac{|z|}{a}\right)^{\lambda} \left(1 + \frac{|y|}{t}\right)^{\lambda} dz .$$

It follows that

$$\sup_y |\mu_t * \Phi * \Psi * f(x-y)| \left(1+\frac{|y|}{t}\right)^{-\lambda} \leq C\Psi^* f(x) \int_{\mathbf{R}^n} |\mu_t * \Phi(z)| \left(1+\frac{|z|}{a}\right)^{\lambda} dz \leq Ct^k \Psi^* f(x)$$

for all $0 < t \leq a$ by Lemma 2.1 (i), because we can write $\Phi = (\Phi_{1/a})_a$. This estimate and the estimate for the infinite sum observed above give

$$\left\| \left(\int_0^a (t^{-\alpha} \mu_t^* f(x))^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w} \leq C \|f\|_{F_{p,q}^{\alpha,w}} .$$

It is easy to see that

$$\|\zeta^* f\|_{p,w} \leq C \|\zeta * f\|_{h_w^p} ,$$

and by an argument similar to the Gaussian case [2, (18)], using a multiplier theorem on the weighted Hardy space h_w^p [1, Lemma 4.8], we obtain

$$\|\zeta * f\|_{h_w^p} \leq C \|f\|_{F_{p,q}^{\alpha,w}} .$$

The proof of the left-hand side inequality in (5.3) is thus complete.

To prove the right-hand side inequality in (5.3) let η be the function given by Lemma 2.3 for ν. Let $\delta > 0$ be such that $\hat{\varphi}(\xi) \neq 0$ for $|\xi| \leq \delta$. Following [9] we set

$$\hat{\gamma}(\xi) = \begin{cases} \displaystyle\int_b^\infty \hat{\eta}(t\xi)\hat{\nu}(t\xi)\frac{dt}{t} & \text{if } \xi \neq 0 \\ 1 & \text{if } \xi = 0 . \end{cases}$$

Then $\hat{\gamma} \in \mathcal{S}$, and as $\hat{\eta}$ is supported in an annulus about the origin, if we choose b large enough, we see that supp $\hat{\gamma} \subseteq \{|\xi| \leq \delta\}$. It follows that $\hat{\gamma} = (\hat{\gamma}/\hat{\varphi})\hat{\varphi}$ with $(\hat{\gamma}/\hat{\varphi}) \in \mathcal{S}$, so that

$$\|\psi^* f\|_{p,w} \leq C \|\varphi^* f\|_{p,w} .$$

Using the above inequality and the representation

$$f = \gamma * f + \int_0^b \eta_t * \nu_t * f \frac{dt}{t} \tag{5.6}$$

in $\mathcal{S}'$ (cf. [9]), we obtain the right-hand side inequality in (5.3) by an argument similar to the proof of Theorem 3.1.

The right-hand side inequality in (5.4) can be verified in a similar way to the corresponding inequality in (5.3). To prove the other inequality in (5.4), let k be a non-negative integer with $2k > \alpha$ and let $\hat{\phi}(\xi) = (-|\xi|^2)^k e^{-|\xi|^2}$ as in the proof of Theorem 4.1. By [2, Theorem 1], we have the following inequality

$$\|W_1 * f\|_{h_w^p} + \left(\int_0^a (t^{-\alpha} \|\phi_t * f\|_{h_w^p})^q \frac{dt}{t} \right)^{1/q} \leq C \|f\|_{B_{p,q}^{\alpha,w}} ,$$

where $W_1(x) = (4\pi)^{-n/2}e^{-|x|^2/4}$ as in Section 1. Similarly to (5.6), letting η be the function given by Lemma 2.3 for ϕ, we can find $\gamma \in \mathcal{S}$ such that supp $\hat{\gamma}$ is compact, and that

$$f = \gamma * f + \int_0^a \eta_t * \phi_t * f \frac{dt}{t}$$

in $\mathcal{S}'$. As $\hat{\gamma}/\widehat{W_1} \in \mathcal{S}$ and $\hat{\gamma} = (\hat{\gamma}/\widehat{W_1})\widehat{W_1}$, we deduce that

$$\|\gamma * f\|_{h_w^p} \leq C\|W_1 * f\|_{h_w^p}$$

(by a multiplier theorem for the weighted Hardy spaces h_w^p).

Using all the above, we can modify the proof of Theorem 4.1 to obtain the left-hand side inequality in (5.4).

6. Remarks and further results

(a) When μ equals ν and satisfies both the moment condition and the Tauberian condition, our results in Sections 3–5 give necessary and sufficient conditions for memberships in the corresponding weighted spaces and also equivalent quasi-norms on them. In the unweighted case ($w = 1$), there are results by H. Triebel where he obtained equivalent quasi-norms on the unweighted Besov-Lipschitz and Triebel-Lizorkin spaces (see [13],[14]). Triebel's methods seem designed to apply to kernels that are not smooth (i.e., not in $\mathcal{S}$) and so growth conditions at infinity and behaviours near zero of the kernel and its Fourier transform are expressed in rather complicated forms. These conditions seem stronger than the moment condition when p is close to 0. Moreover, his Tauberian condition takes the form $\hat{\mu}(\xi) = \hat{\nu}(\xi) \neq 0$ for $1/2 \leq |\xi| \leq 2$ (so that the quotient $\hat{\theta}/\hat{\mu}$ is defined). His approach is based on an inequality of Plancherel-Polya-Nikolskij type and seems difficult to extend to the weighted case without some restriction on the weight function w.

We note also that Triebel's results are stated in terms of the mixed norm of $|\nu_t * f(x)|$, but his results do not give a sufficient condition for a distribution to be in the relevant unweighted space. See (d) below for a discussion in the weighted case.

(b) Though Lemma 2.1 (i) and (ii) were stated under the assumption $\mu \in \mathcal{S}$, a close examination of the proof shows that (i) holds if

$$\int_{\mathbf{R}^n} (1 + |z|)^{\lambda+n+k}|\mu(z)|dz < \infty ,$$

while (ii) holds if there exists $\delta > 0$ for which

$$|D^\kappa \mu(z)| \leq C(1 + |z|)^{-\lambda-n-\delta}$$

for all $|\kappa| = m + r$. It follows that our main results in Sections 3–5 remain valid for the Poisson kernel; i.e., for $\hat{\mu}(\xi) = (-|\xi|)^\ell e^{-|\xi|}$, $\ell > \lambda + n + \max(\alpha, 0)$, where $\mu_t * f$ is interpreted appropriately.

(c) Our main results in Sections 3–5 hold for vector-valued μ or ν, where we require the moment condition for each component and the Tauberian condition as a vector-valued function. Moreover, they remain valid for compactly supported (vector-valued) measures ν, and also for compactly supported (vector-valued) measures μ if $\alpha > \max(nr_0/p,\ n/q)$ in the Triebel-Lizorkin case and $\alpha > nr_0/p$ in the Besov case (see Lemma 2.1) provided the function f satisfies an appropriate growth condition at infinity (see Remark after Lemma 2.3). A special case of the compactly supported vector-valued measures worth noting here is when $\mu = (\mu^{(1)}, \dots, \mu^{(n)}) = \nu$, where $\mu^{(j)} = (\delta e_j - \delta_0) * \cdots * (\delta e_j - \delta_0)$ (k times), $k > \alpha$, δe_j is the Dirac measure concentrated at $e_j = (0, \dots, 1, \dots 0)$ and δ_0 is the Dirac measure concentrated at the origin. Note that each $\mu^{(j)}$ does not satisfy the Tauberian condition, but μ does. Observe that

$$(\delta e_j - \delta_0)_t * f(x) = f(x - te_j) - f(x) ,$$

so that our results in this case give characterizations of the corresponding weighted spaces by means of difference operators.

(d) It would be of interest to replace $\nu_t^* f(x)$ by $|\nu_t * f(x)|$ in our results. By [2], this is certainly possible for the Gauss-Weierstrass kernel (and the proof given there also works for the Poisson kernel). For a general ν, we can prove the above in the following two cases: either

(i) $1 < p < \infty$, $w \in A_p$, $1 < q \leq \infty$ (in the Besov case, we can let $0 < q \leq \infty$); or

(ii) $\hat{\nu}$ is compactly supported and ν also satisfies the moment condition.

We shall give proof only for the homogeneous Triebel spaces $\dot{F}_{p,q}^{\alpha,w}$, as the other cases can be handled similarly.

Assume that the conditions of (i) are satisfied. Then by (3.2) and Hölder's inequality,

$$\left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} |\psi_j * f(x)|)^q \right)^{1/q} \leq C \left(\int_0^\infty (M(t^{-\alpha} |\nu_t * f|)(x))^q \ \frac{dt}{t} \right)^{1/q} ,$$

where Mg denotes the (Hardy-Littlewood) maximal function of g. Hence it follows from the weighted estimate for the vector-valued maximal function [10] that

$$\|f\|_{\dot{F}_{p,q}^{\alpha,w}} \leq C \left\| \left(\int_0^\infty (t^{-\alpha} |\nu_t * f(x)|)^q \ \frac{dt}{t} \right)^{1/q} \right\|_{p,w}$$

if $q < \infty$, while for $q = \infty$ we need only use the weighted estimate for the maximal function.

By using a version of Heideman type estimate, which is different from Lemma 2.1, we had proved the above result in the unweighted case ([4]), and the proof given there works also for the weighted case. Note that the result in [4] for the unweighted Besov spaces holds also for $p = 1$.

Assume next that the conditions of (ii) are satisfied. Since f is a tempered distribution, we can choose $\lambda = \lambda_f$ large enough so that

$$\sup_y |\nu_s * f(x - y)| \left(1 + \frac{|y|}{s} \right)^{-\lambda} = \nu_s^* f(x) < \infty$$

for all s and x. By a variation of the Fefferman-Stein maximal function techniques used by J. Peetre [11, Lemma 2.1 and Proof of Theorem 3.1], we can show that

$$\nu_s^* f(x) \le c\delta^{-n/r}(M(|\nu_s * f|^r)(x))^{1/r} + c\delta\nu_s^* f(x) ,$$

where λ satisfies the additional condition, $\lambda > \max(nr_0/p,\ n/q,\ n)$, $r = n/\lambda$, $0 < \delta < 1$, and c is independent of f, s and δ (see also [3, Lemma 2.1], [13, Theorem 1.3.1]). Choosing δ such that $c\delta < 1/2$ in the above, we obtain

$$\nu_s^* f(x) \le C(M(|\nu_s * f|^r)(x))^{1/r} ,$$

and putting this into (3.3) and using well-known inequalities, we get

$$|\psi_j * f(x)|^q \le C \int_{I_j} (M(|\nu_s * f|^r)(x))^{q/r} \frac{ds}{s} .$$

It follows that

$$\left(\sum_{j=-\infty}^{\infty} (2^{j\alpha}|\psi_j * f(x)|)^q\right)^{1/q} \le C\left(\int_0^\infty (M((s^{-\alpha}|\nu_s * f|)^r)(x))^{q/r} \frac{ds}{s}\right)^{1/q} .$$

Hence, similarly to case (i), we obtain

$$\|f\|_{\dot{F}_{p,q}^{\alpha,w}} \le C \left\| \left(\int_0^\infty (s^{-\alpha}|\nu_s * f(x)|)^q \frac{ds}{s}\right)^{1/q} \right\|_{p,w} . \tag{6.1}$$

However, as C depends on λ and hence implicitly on f, we have proved that if the right-hand side of (6.1) is finite, then $f \in \dot{F}_{p,q}^{\alpha,w}$. Fix $\lambda > \max(nr_0/p,\ n/q,\ n)$. As $\dot{F}_{p,q}^{\alpha,w} \subseteq \dot{F}_{p,\infty}^{\alpha,w}$, and ν satisfies the moment condition, the left-hand side inequality in (3.1) implies that

$$\nu_s^* f(x) < \infty$$

for all s and x. Repeating the above proof of (6.1) we obtain the independence of C on f.

REMARK. The above use of the moment condition seems artificial. It is used to show that the inclusion map, which is into, is continuous. If there were a direct proof that the space of distributions for which the quasi-norm on the right-hand side of (6.1) is finite, is complete, we would get the continuity by means of the closed graph theorem.

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