Aspects of Matroid Connectivity

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Abstract

Connectivity is a fundamental tool for matroid theorists, which has become increasingly important in the eventual solution of many problems in matroid theory. Loosely speaking, connectivity can be used to help describe a matroid's structure. In this thesis, we prove a series of results that further the knowledge and understanding in the field of matroid connectivity. These results fall into two parts.

First, we focus on 3-connected matroids. A chain theorem is a result that proves the existence of an element, or elements, whose deletion or contraction preserves a predetermined connectivity property. We prove a series of chain theorems for 3-connected matroids where, after fixing a basis B, the elements in B are only eligible for contraction, while the elements not in B are only eligible for deletion. Moreover, we prove a splitter theorem, where a 3-connected minor is also preserved, resolving a conjecture posed by Whittle and Williams (2013).

Second, we consider k-connected matroids, where $k \geq 3$. A certain tree, known as a k-tree, can be used to describe the structure of a k-connected matroid. We present an algorithm for constructing a k-tree for a k-connected matroid M. Provided that the rank of a subset of $E(M)$ can be found in unit time, the algorithm runs in time polynomial in $|E(M)|$. This generalises Oxley and Semple's (2013) polynomial-time algorithm for constructing a 3-tree for a 3-connected matroid.

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Chapter 1

Introduction

Matroid connectivity has been an integral part of the theory of matroids since Tutte's (1966) seminal paper. Having proved that every 3-connected simple graph can be constructed from a wheel graph by splitting a vertex or adding an edge between non-adjacent vertices, Tutte examined whether this result could be generalised to matroids. Tutte proved what is now known as the Wheels-and-Whirls Theorem. Moreover, in the process he introduced the concept of connectivity in matroids.

Let M be a matroid with ground set E. The *connectivity function* of M, denoted by λ_M, or simply λ, is defined on all subsets $X \subseteq E$ by

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

We follow the recent convention of excluding the "$+1$" that was present in Tutte's original definition. A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation* if $|X| \geq k$ and $|E - X| \geq k$. The matroid M is *n-connected* if, for all $k < n$, it has no k-separations. A k-separating set X, a k-separating partition $(X, E-X)$, or a k-separation $(X, E-X)$ is *exact* if $\lambda_M(X) = k-1$. An exactly 3-separating partition (X, Y) is *sequential* if there is an ordering $(e_1, e_2, \ldots, e_k)$ of X or Y such that $\{e_1, e_2, \ldots, e_i\}$ is 3-separating for all $i \in \{1, 2, \ldots, k\}$; otherwise, it is *non-sequential*.

Connectivity in matroids is closely related to connectivity in graphs, but additionally incorporates duality, a fundamental concept in the theory of matroids. In particular, the connectivity function is invariant under duality; that is, $\lambda_M(X) = \lambda_{M^*}(X)$, where M^* is the matroid dual of M. As a consequence, matroid connectivity is different to graph connectivity in one crucial way: small circuits limit the connectivity of a matroid, whereas small cycles do not limit the connectivity of a graph.

Historically, a significant proportion of research in matroid theory has focussed on 3-connected matroids. This is partly due to decomposition results that allow an arbitrary matroid to be broken into a collection of smaller 3-connected matroids, where the original matroid can be reconstructed from the components. More specifically, the 1-separations of a matroid induce a partition of the ground set, where each part consists of the elements in a component of the decomposition, and the original matroid can be reconstructed via direct sum. Cunningham and Edmonds (1980) showed that a 2-connected matroid can be decomposed into 3-connected components, and the original 2-connected matroid can be reconstructed using the matroid operation of 2-sum. Moreover, we can obtain a labelled tree that describes precisely how the 3-connected components are put together in the reconstruction. A number of matroid properties have been shown to hold precisely if the property holds for each of the 3-connected components in the 2-sum decomposition; that is, the property is closed under 2-sums. One example of such a property is matroid representability over a field.

Another reason for the focus on 3-connected matroids is the existence of satisfactory chain theorems for these matroids. A *chain theorem* is a result that asserts the existence of an element, or elements, that can be either deleted or contracted from a matroid while a predetermined connectivity condition is preserved. These theorems are important tools that enable inductive arguments to be made in order to derive matroid structure results.

The primordial example of a chain theorem is Tutte's aforementioned Wheels-and-Whirls Theorem:

Theorem 1.0.1 (Tutte, 1966)**.** *Let M be a 3-connected matroid with at least one element. Then, the following are equivalent:*

(i) *There exists an element $e \in E(M)$ such that either $M \backslash e$ or M/e is 3-connected.*

(ii) *M is not isomorphic to a wheel or a whirl of rank at least three.*

An even stronger result is Seymour's (1980) Splitter Theorem. This result was also proved independently by Tan (1981).

Theorem 1.0.2 (Seymour, 1980)**.** *Let M be a 3-connected matroid, and let N be a 3-connected proper minor with $|E(N)| \geq 4$ where if N is a wheel, then M has no larger wheel as a minor, while if N is a whirl, then M has no larger whirl as a minor. Then there exists an element $e \in E(M)$ such that either $M\backslash e$ or M/e is 3-connected and has an N-minor.*

These foundational theorems have had a profound influence on matroid structure theory (Seymour, 1995; Oxley, 1996), and, over time, a number of variants and extensions have been found (for example, Coullard and Oxley, 1992; Whittle, 1999; Oxley et al., 2012). The research in the first part of this thesis also falls into this category.

Let M be a 3-connected matroid, and fix a basis B for M. In Part I, we present some chain theorems, and a Splitter Theorem, where the removed element e can only be contracted if $e \in B$, and can only be deleted if $e \in E(M) - B$. We say that an element e removed in this way is *removed relative to* B. When M is a representable matroid, it has a standard matrix representation of the form $[I_r|D]$, where I_r is the $r \times r$ identity matrix. A natural choice for B is the set of elements corresponding to columns of I_r. With this choice of basis, deleting an element $e \in E(M) - B$ corresponds to removing a column from the representation, while contracting an element $e \in B$ corresponds to removing a row and a column. In either case, the resulting representation remains in standard form without the need for a pivot operation. Thus, any information visible in the original representation is preserved. The benefit of such an approach is illustrated by the arguments of Geelen et al. (2000) in their proof of the excluded minors for $GF(4)$. Indeed, the results in the first part of the thesis are already being used as tools to prove results in matroid representation theory.

Oxley et al. (2008*a*) and Whittle and Williams (2013) have previously studied the existence of elements that can be removed relative to a fixed basis and preserve a connectivity condition. In particular, Oxley et al. (2008*a*) proved a Splitter Theorem that ensures a single element can be removed relative to a fixed basis while preserving 3-connectivity and retaining a 3-connected N-minor. We extend this result by ensuring the presence of more than one such element. However, to do so requires the use of a slightly weaker connectivity condition. We discuss the previously known results and how they relate to our findings in the introduction to Part I.

It was once a predominant line of thought that focussing on 3-connected matroids was sufficient to avoid degeneracies that arise in less structured matroids. In particular, Kahn (1988) conjectured that, for some prime power q and 3-connected matroid M, the number of inequivalent $GF(q)$-representations of M is bounded by some integer $n(q)$. Although Kahn's conjecture is true for $q \leq 5$, Oxley et al. (1996) showed that it is false for any larger prime power q. However, all known counterexamples have one feature in common: the presence of mutually interacting 3-separations. This inspired Oxley et al. (2004) to investigate the structure of 3-separations in a 3-connected matroid. They showed that for any 3-connected matroid there exists a tree, known as a 3-tree, that describes all the non-sequential 3-separations, up to a natural equivalence.

Due to mounting evidence that restricting one's attention to 3-connected matroids is sometimes insufficient, there has been a recent interest in further understanding higher connectivity. Beavers (2006), and, independently, Chen and Xiang (2012), showed that a 3-connected representable matroid consisting of at least nine elements can be decomposed into sequentially 4-connected matroids and sporadic matroids of three types, where the original matroid can be reconstructed from the components. Aikin and Oxley (2012) showed that a 4-connected matroid with at least 17 elements has a "4-tree" that describes its non-trivial 4-separations, up to an equivalence. Clark and Whittle (2013) extended this to the abstract setting of tangles in a connectivity system. As a specialisation, their result shows that a k-connected matroid, with at least $8k - 15$ elements, has a k-tree that describes the matroid's non-trivial k-separations.

However, there are a number of complications when using the notion of k-connectivity for $k \geq 4$. To begin with, strict 4-connectivity is often too strong to be useful in practice. For example, neither the cycle matroid $M(K_{r+1})$ of a complete graph, nor the finite projective geometry $GF(r-1, q)$ is 4-connected. As these highly structured matroids are, in a sense, the maximal members in the class of rank-r graphic matroids, or rank-r representable matroids respectively, a more reasonable approach is to use one of the various weaker forms of 4-connectivity. Although the results in the second part of the thesis apply to strictly k-connected matroids, it is conceivable that a similar approach could be used for weaker forms of k-connectivity. However, we do not address this further in the remainder of the thesis.

Another complication is that the concept of a non-sequential k-separation needs to be generalised in order to make sense for more than just the case when $k = 3$. This issue is well explained by Aikin and Oxley (2012), and we follow their approach for $k = 4$. More generally, our approach for all k is consistent with Clark and Whittle (2013). Let (X, Y) be a k-separation in a k-connected matroid M. We say that (X, Y) is *k-sequential* if there is an ordered partition $(Z_1, Z_2, \ldots, Z_k)$ of X or Y where each Z_i consists of at most $k - 2$ elements and $Z_1 \cup Z_2 \cup \cdots \cup Z_i$ is k-separating for all $i \in \{1, 2, \ldots, k\}$; otherwise, (X, Y) is *non-sequential.* Under this definition, the so-called "non-trivial" k-separations in the tree decomposition results of Aikin and Oxley (2012) and Clark and Whittle (2013) are, more precisely, the non-sequential k-separations.

Although Oxley et al. (2004) proved the existence of a 3-tree for a 3-connected matroid, the approach taken in their proof of this result does not appear to elicit an efficient algorithm for finding such a 3-tree. However, Oxley and Semple (2013) presented such an algorithm, thereby reproving the result using a different approach. Provided that the rank of any subset of the ground set E of a 3-connected matroid M can be found in unit time, this algorithm finds a 3-tree for M, with running time polynomial in the size of E. Similarly, although Clark and Whittle's (2013) result ensures the existence of a k-tree for a k-connected matroid M, it does not guarantee the existence of a polynomial-time algorithm for finding such a k-tree. In the second part of this thesis, we present a polynomial-time algorithm for constructing a k-tree for a k-connected matroid. We describe our approach in proving the correctness of this algorithm in the introduction to Part II.

We would hope that the existence of an algorithm for constructing a k-tree for a k-connected matroid is useful in its own right. That said, we close this section with a short remark to demonstrate it may have other applications. Recall that for a prime power $q > 5$, Oxley et al. (1996) gave counterexamples to the conjecture by Kahn that a 3-connected matroid M has at most $n(q)$ inequivalent $GF(q)$-representations. A common feature of these counterexamples is that they have what are known as swirl-like flowers. Recently, Geelen and Whittle (2013) showed that for a 3-connected matroid with no swirl-like flowers of order j, where $j \geq 5$, there is a function $\gamma(j, p)$ that provides an upper bound on the number of inequivalent representations over $GF(p)$, where p is prime. Such a result gives an indication of the value of a polynomial-time algorithm for constructing a k-tree for a given matroid. For example, consider an arbitrary 3-connected matroid M for which the rank of a subset of $E(M)$ can be found in unit time, and let $j \geq 5$. As a straightforward consequence of Oxley and Semple's (2013) algorithm for constructing a 3-tree, one can find, in polynomial time, whether M is a member of the class of matroids with no swirl-like flowers of order at least j, where any matroid in this class has at most $\gamma(j,p)$ inequivalent representations over $GF(p)$.

1.1 Overview

The research in this thesis falls into two parts. In the first part we focus on 3-connected matroids, proving a series of results that are analogous to the Wheels-and-Whirls Theorem (Theorem 1.0.1) or Splitter Theorem (Theorem 1.0.2), but where elements are removed relative to a fixed basis. Chapter 2 contains two analogues of the Wheels-and-Whirls Theorem; while in Chapter 3 we present a Splitter Theorem relative to a fixed basis. The results in Sections 2.2 and 2.3 and Chapter 3 are new unless otherwise stated. The key results of Sections 2.3, 3.2 and 3.3 are published in "Annals of Combinatorics" (Brettell and Semple, 2014*a*).

In the second part of this thesis, we turn our attention to matroids that may be more highly connected. The main result of this part of the thesis is a polynomial-time algorithm for constructing a k-tree for a k-connected matroid. In Chapter 4, we cover the concepts required in order to describe the algorithm and prove its correctness; in particular, k-connectivity, k-flowers, k-trees, and k-paths. In Chapter 5, we describe the algorithm. Finally, in Chapter 6, we prove the correctness of the algorithm and that it runs in polynomial time. Our approach, in this part, was inspired by Oxley and Semple (2013), but there are a number of additional hurdles to overcome for our more general result. We clearly state the results that are obtained by a straightforward generalisation. Sections 4.2 and 4.4 and Chapters 5 and 6 contain new results. This work in this part of the thesis has also been submitted for publication (Brettell and Semple, 2014*b*).

More detailed overviews of the individual chapters are given at the beginning of each of the two parts.

Throughout the thesis, we assume a basic understanding of matroid theory. We refer the uninitiated reader to Chapters 1–6 of Oxley's (2011) text

"Matroid Theory". We follow the notation and terminology of this text unless otherwise specified.

Part I

Preserving 3-connectivity relative to a fixed basis

Let M be a 3-connected matroid, let B be a basis of M, and let N be a 3-connected minor of M. We say that an element $e \in E(M)$ is (N, B)-*robust* if either

(i) $e \in B$ and M/e has an N-minor, or

(ii) $e \in E(M) - B$ and $M\backslash e$ has an N-minor.

Furthermore, an element $e \in E(M)$ is *strictly removable with respect to* B, or *strictly* B*-removable*, if either

(i) $e \in B$ and M/e is 3-connected, or

(ii) $e \in E(M) - B$ and $M\backslash e$ is 3-connected.

Oxley et al. (2008a) were the first to investigate the presence of elements that can be removed relative to a fixed basis so that 3-connectivity is preserved. They proved the following:

Theorem 2.0.1 (Oxley et al., 2008)**.** *Let M be a 3-connected matroid with no 4-element fans, let N be a 3-connected minor of M, and let B be a basis of M. Suppose that M has an (N, B)-robust element. Then M has an element that is both strictly B-removable and (N, B)-robust.*

A 4-*element fan* is a set of four elements consisting of a triangle (a 3-element circuit) that meets a triad (a 3-element cocircuit); we discuss fans further in

Section 2.3.1. Oxley et al. (2008a) also demonstrated that the requirement that M has some (N, B)-robust element is necessary, by giving an example of a 3-connected matroid M with 3-connected minor N and basis B such that M has no (N, B)-robust elements.

In one sense, Theorem 2.0.1 is best possible; we cannot guarantee the presence of more than one strictly B-removable (N, B)-robust element, as we shall demonstrate in Section 3.1. However, if we are not concerned about retaining an N-minor, two strictly B-removable elements can be found. This is our first main result.

Theorem 2.0.2. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 2$, and let B be a basis of M. Then M has at least two strictly B-removable elements.*

We prove this theorem in Section 2.2.1.

Note that, if $|E(M)| \geq 4$, then an element of B that is in a triangle is not strictly B-removable, as the resulting matroid has a non-trivial parallel class. Dually, an element of $E(M) - B$ that is in a triad is not strictly B-removable, as the resulting matroid has a non-trivial series class. If these elements are the only obstacles to maintaining 3-connectivity, a natural question is whether we can extend Theorem 2.0.1 or Theorem 2.0.2 to find more elements that can be removed relative to a fixed basis.

Whittle and Williams (2013) addressed this question when 3-connectivity is preserved, but no N-minor is retained. Their result extends Theorem 2.0.2 when considering 3-connectivity up to simplification or cosimplification. Following their example, we say that an element $e \in E(M)$ is *removable with respect to B*, or *B-removable*, if either

(i) $e \in B$ and $\text{si}(M/e)$ is 3-connected, or

(ii) $e \in E(M) - B$ and $\text{co}(M \backslash e)$ is 3-connected.

Theorem 2.0.3 (Whittle and Williams, 2013)**.** *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 4$, and let B be a basis of M. Then M has at least four B-removable elements.*

The requirement that M has no 4-element fans is consistent with the work of Oxley et al. (2008a), but is not strictly necessary when taking into account which elements of the fan are in the basis B. Indeed, we prove a stronger form of Theorem 2.0.3, as Corollary 2.3.12, where 4-element fans are permitted unless the fan has one of two particular labellings relative to B.

Our second main result is an analogue of the Splitter Theorem when considering 3-connectivity up to simplification or cosimplification. We say that an element $e \in E(M)$ is (N, B)-*strong* if either

(i) $e \in B$, and $\mathrm{si}(M/e)$ is 3-connected and has an N-minor, or

(ii) $e \in E(M) - B$, and $\mathrm{co}(M \backslash e)$ is 3-connected and has an N-minor.

Theorem 2.0.4. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 5$, let N be a 3-connected minor of M, and let B be a basis of M. If M has at least two (N, B)-robust elements, then M has at least two (N, B)-strong elements.*

We prove this theorem in Section 3.2.1.

This result resolves Whittle and Williams' conjecture (2013, Conjecture 6.1). It is worth noting that the theorem differs from the conjecture in that M is required to have at least five elements, and at least two (N, B)-robust elements. These are both necessary assumptions. To see that M must have at least five elements, consider the matroid $U_{2,4}$ with $U_{1,3}$- or $U_{2,3}$-minor. Furthermore, we give an example, in Section 3.2.2, of a 3-connected matroid with a 3-connected proper minor N that has only one (N, B)-robust element. In Section 3.2.3, we give an example of a 3-connected matroid with precisely two (N, B)-strong elements, which demonstrates that Theorem 2.0.4 is, in a sense, the best we can hope for.

However, as with Theorem 2.0.3, we are able to strengthen Theorem 2.0.4 by considering the labellings of the 4-element fans relative to the fixed basis. The stronger result, Theorem 3.2.10, demonstrates that M still has the two desired elements when a 4-element fan is present, unless the fan has a particular labelling relative to B. In Section 3.2.4, we give an example to illustrate that when a matroid has a 4-element fan with this particular labelling, we cannot guarantee the presence of even a single (N, B)-strong element.

Having established a lower bound on the number of strong elements, it is natural to consider what can be said about matroids that have the minimum number of such elements. A matroid has *path-width three* if its ground set is sequential; that is, there is an ordering $(e_1, e_2, \ldots, e_n)$ of $E(M)$ such that $\{e_1, e_2, \ldots, e_i\}$ is 3-separating for all $i \in \{1, 2, \ldots, n\}$. Whittle and Williams (2013) proved that a matroid with precisely four removable elements, with respect to some fixed basis, has path-width three. In Section 3.3, we prove the following theorem.

Theorem 2.0.5. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 5$, let N be a 3-connected minor of M, and let B be a basis of M. Let P denote the set of (N, B)-robust elements of M. If M has precisely two (N, B)-strong elements, then $(P, E(M) - P)$ is a sequential 3-separation.*

A stronger result, where we consider the 4-element fans' labellings relative to the fixed basis, is presented as Theorem 3.3.1.

This part of the thesis is structured as follows. In Chapter 2, we prove two analogues of the Wheels-and-Whirls Theorem: first, in Section 2.2, we prove Theorem 2.0.2; then, in Section 2.3, we prove an upgrade of Theorem 2.0.3. In Chapter 3, we focus on removable elements that also retain a copy of a specified 3-connected minor. In Section 3.1, we give an example to show that Theorem 2.0.1 is best possible in the sense that we cannot guarantee more than one element that is strictly removable and robust. Section 3.2 culminates in Theorem 3.2.10, a generalisation of Theorem 2.0.4. Finally, in Section 3.3, we prove Theorem 3.3.1, a generalisation of Theorem 2.0.5.

We write $x \in \mathrm{cl}^{(*)}(Y)$ to denote that either $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$. The phrase *by orthogonality* refers to the fact that a circuit and a cocircuit cannot intersect in exactly one element. Lastly, we remark that the 3-connectivity conclusions in the theorems in Sections 2.3 and 3.2 are up to parallel and series classes. However, with the help of Lemma 2.1.8 in the next section, it is easily seen that these conclusions are really up to parallel and series couples, where a *parallel couple* (respectively, *series couple*) is a parallel (respectively, series) class of size two.

Chapter 2

Chain Theorems

In this chapter we prove two chain theorems: Theorem 2.0.2 and a generalisation of Theorem 2.0.3. These theorems are analogues of Tutte's Wheels-and-Whirls Theorem (Theorem 1.0.1) that ensure the existence of elements that can be removed relative to a fixed basis.

The chapter is structured as follows. The next section contains some necessary preliminaries regarding 3-connectivity and vertical 3-separations that are used throughout Part I of the thesis. In each of the two subsequent sections we prove a chain theorem. In Section 2.2, the result ensures the existence of two strictly removable elements, which preserve strict 3-connectivity, while the result in Section 2.3 ensures the existence of four removable elements, which preserve 3-connectivity up to simplification or cosimplification.

2.1 Preliminaries

The following lemma is a consequence of the easily verified fact that the connectivity function is submodular.

Lemma 2.1.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*

(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

When Lemma 2.1.1 is applied in this part of the thesis, we refer to it as "uncrossing". The next corollary follows by a routine induction argument.

Corollary 2.1.2. *Let M be a 3-connected matroid, and let $\mathcal{X}$ be a finite set of 3-separating subsets of $E(M)$. If $\left|E(M) - \left(\bigcup_{X\in\mathcal{X}} X\right)\right| \geq 2$, then $\bigcap_{X\in\mathcal{X}} X$ is 3-separating.*

The following two lemmas are used frequently in this part of the thesis. The first is well known (see, for example, Proposition 2.1.12, Oxley, 2011) and is a consequence of orthogonality; the second is a consequence of the first.

Lemma 2.1.3. *Let e be an element of a matroid M, and let X and Y be disjoint sets whose union is $E(M) - \{e\}$. Then $e \in \mathrm{cl}(X)$ if and only if $e \notin \mathrm{cl}^*(Y)$.*

Lemma 2.1.4. *Let X be an exactly 3-separating set in a 3-connected matroid with ground set E, and suppose that $e \in E - X$. Then*

(i) *$X \cup \{e\}$ is 3-separating if and only if $e \in \mathrm{cl}^{(*)}(X)$, and*

(ii) *$X \cup \{e\}$ is exactly 3-separating if and only if e is in exactly one of $\mathrm{cl}(X) \cap \mathrm{cl}(E - (X \cup \{e\}))$ and $\mathrm{cl}^*(X) \cap \mathrm{cl}^*(E - (X \cup \{e\}))$.*

The next lemma was established by Oxley et al. (2008*b*).

Lemma 2.1.5. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M. If $|X| \geq 3$ and $x \in X$, then $x \in \mathrm{cl}^{(*)}(X - \{x\})$.*

A 3-separation $(X, E - X)$ of a matroid M with ground set E is *vertical* if $r(X) \geq 3$ and $r(E - X) \geq 3$. We also say a partition $(X, \{e\}, Y)$ of E is a *vertical 3-separation* when $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are both vertical 3-separations and $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. The next three lemmas will be used frequently; a proof of the first is given by Oxley et al. (2008*a*), the second follows from a result established by Oxley et al. (2008*b*), while the third is elementary.

Lemma 2.1.6. *Let M be a 3-connected matroid and let $z \in E(M)$. If $\mathrm{si}(M/z)$ is not 3-connected, then M has a vertical 3-separation $(X, \{z\}, Y)$.*

Lemma 2.1.7. *Let $(X, \{z\}, Y)$ be a vertical 3-separation of a 3-connected matroid M. Then there exists a vertical 3-separation $(X', \{z\}, Y')$ such that $X' \subseteq X$, and $Y' \cup \{z\}$ is closed.*

Lemma 2.1.8. *Let M be a 3-connected matroid and let L be a rank-2 subset with at least four elements. If $l \in L$, then $M \backslash l$ is 3-connected.*

Let $(X', \{b'\}, Y')$ be a vertical 3-separation in a matroid M, and let $B \subseteq E(M)$. We say that X' is *minimal in* $(X', \{b'\}, Y')$ *with respect to* B, where $b' \in B$, if for any other vertical 3-separation $(X, \{b\}, Y)$ on M such that $b \in X' \cap B$, we have $X \nsubseteq X' \cup \{b'\}$ and $Y \nsubseteq X' \cup \{b'\}$. If our choice of B is clear, we will just say X' is minimal in $(X', \{b'\}, Y')$.

The next two lemmas are contained in results from the literature; the proofs are provided for completeness. In particular, the second lemma is extracted from proofs by both Oxley et al. (2008*a*, Lemma 3.2), and Whittle and Williams (2013, Lemma 3.1).

Lemma 2.1.9. *Let $(X, \{b\}, Y)$ be a vertical 3-separation of a matroid M with $B \subseteq E(M)$ and $b \in B$. There exists a vertical 3-separation $(X', \{b'\}, Y')$ such that X' is minimal in $(X', \{b'\}, Y')$ with respect to B, the set $Y' \cup \{b'\}$ is closed, $X' \cup \{b'\} \subseteq X \cup \{b\}$, and $b' \in (X \cup \{b\}) \cap B$.*

Proof. By Lemma 2.1.7, there exists a vertical 3-separation $(X_1, \{b\}, Y_1)$ such that $Y_1 \cup \{b\}$ is closed, and $X_1 \subseteq X$. Suppose that X_1 is not minimal in $(X_1, \{b\}, Y_1)$. Then there exists a vertical 3-separation $(X_2, \{b_2\}, Y_2)$ with $b_2 \in X_1 \cap B$ such that $X_2 \subseteq (X_1 - \{b_2\}) \cup \{b\}$. If $X_2 = (X_1 - \{b_2\}) \cup \{b\}$, then $Y_2 = Y_1$, so $b_2 \in \mathrm{cl}(Y_1)$, contradicting the fact that $Y_1 \cup \{b\}$ is closed. So $X_2 \subsetneqq (X_1 - \{b_2\}) \cup \{b\}$.

If X_2 is not minimal in $(X_2, \{b_2\}, Y_2)$, then we can pick a $b_3 \in X_2 \cap B$ such that $(X_3, \{b_3\}, Y_3)$ is a vertical 3-separation and repeat the process. Since $|X_j| < |X_i|$ for $i < j$, we will eventually obtain a vertical 3-separation $(X_n, \{b_n\}, Y_n)$ such that X_n is minimal in $(X_n, \{b_n\}, Y_n)$. By Lemma 2.1.7, there exists a vertical 3-separation $(X', \{b'\}, Y')$ that also satisfies the criterion that $Y' \cup \{b'\}$ is closed. □

Lemma 2.1.10. *Let M be a 3-connected matroid with a vertical 3-separation $(X_1, \{b_1\}, Y_1)$ such that $Y_1 \cup \{b_1\}$ is closed, X_1 is minimal in $(X_1, \{b_1\}, Y_1)$ with respect to some $B \subseteq E(M)$, and $b_1 \in B$. If $(X_2, \{b_2\}, Y_2)$ is a vertical 3-separation of M with $Y_2 \cup \{b_2\}$ closed, $b_2 \in X_1 \cap B$, and $b_1 \in Y_2$, then all of the following hold:*

(i) *$X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ are all non-empty,*

(ii) $r((X_1 \cap X_2) \cup \{b_2\}) = 2$, *and*

(iii) *if* $|Y_1 \cap X_2| \geq 2$, *then* $r((X_1 \cap Y_2) \cup \{b_1, b_2\}) = 2$.

Proof. Suppose that $X_1 \cap X_2 = \emptyset$; then $X_2 \subseteq Y_1 \cup \{b_1\}$ and thus $b_2 \in \mathrm{cl}(X_2) \subseteq \mathrm{cl}(Y_1 \cup \{b_1\}) = Y_1 \cup \{b_1\}$, contradicting $b_2 \in X_1$. Likewise, if $X_1 \cap Y_2 = \emptyset$, then $Y_2 \subseteq Y_1 \cup \{b_1\}$ and $b_2 \in Y_1 \cup \{b_1\}$; a contradiction. Now suppose that $Y_1 \cap X_2 = \emptyset$; then $X_2 \subseteq X_1 \cup \{b_1\}$, but since $b_2 \in X_1 \cap B$, this contradicts the minimality of X_1 in $(X_1, \{b_1\}, Y_1)$. Likewise if $Y_1 \cap Y_2 = \emptyset$, then $Y_2 \subseteq X_1 \cup \{b_1\}$; a contradiction. So (i) holds.

As $E(M) - (X_1 \cup X_2) = (Y_1 \cap Y_2) \cup \{b_1\}$, we have $|E(M) - (X_1 \cup X_2)| \geq 2$. Thus, as X_1 and X_2 are 3-separating, $X_1 \cap X_2$ is 3-separating, by uncrossing. If $|X_1 \cap X_2| = 1$, then (ii) holds. It remains to consider when $|X_1 \cap X_2| \geq 2$. In this case, since $|E(M) - (X_1 \cap X_2)| \geq 2$ and M is 3-connected, $X_1 \cap X_2$ is an exact 3-separation. Since X_1 and $X_2 \cup \{b_2\}$ are 3-separating, and $|E(M) - (X_1 \cup X_2 \cup \{b_2\})| = |Y_1 \cap Y_2| + 1 \geq 2$, it follows, by uncrossing, that $X_1 \cap (X_2 \cup \{b_2\})$, which equals $(X_1 \cap X_2) \cup \{b_2\}$, is 3-separating. By Lemma 2.1.4(i), $b_2 \in \mathrm{cl}^{(*)}(X_1 \cap X_2)$. If $b_2 \in \mathrm{cl}^*(X_1 \cap X_2)$, then, by Lemma 2.1.3, $b_2 \notin \mathrm{cl}(Y_1 \cup Y_2)$; a contradiction, as $b_2 \in \mathrm{cl}(Y_2)$. So $b_2 \in \mathrm{cl}(X_1 \cap X_2)$. But then, if $r(X_1 \cap X_2) \geq 3$, the partition $(X_1 \cap X_2, \{b_2\}, Y_1 \cup Y_2)$ is a vertical 3-separation of M, contradicting the minimality of X_1 in $(X_1, \{b_1\}, Y_1)$. Thus (ii) holds.

Finally, to prove (iii), let $|Y_1 \cap X_2| \geq 2$. Since $X_1 \cup \{b_1\}$ and $Y_2 \cup \{b_2\}$ are 3-separating, and $|E(M) - ((X_1 \cup \{b_1\}) \cup (Y_2 \cup \{b_2\}))| = |Y_1 \cap X_2| \geq 2$, it follows, by uncrossing, that $(X_1 \cup \{b_1\}) \cap (Y_2 \cup \{b_2\})$ is 3-separating. But $|(X_1 \cap Y_2) \cup \{b_1, b_2\}| \geq 2$ and $|E(M) - ((X_1 \cap Y_2) \cup \{b_1, b_2\})| \geq 2$, so, since M is 3-connected, $(X_1 \cap Y_2) \cup \{b_1, b_2\}$ is exactly 3-separating. By Lemma 2.1.5, $b_2 \in \mathrm{cl}^{(*)}((X_1 \cup \{b_1\}) \cap Y_2)$. But noting that $X_2 \subseteq E(M) - ((X_1 \cap Y_2) \cup \{b_1, b_2\})$, we have $b_2 \in \mathrm{cl}(E(M) - ((X_1 \cap Y_2) \cup \{b_1, b_2\}))$, thus, by Lemma 2.1.3, $b_2 \notin \mathrm{cl}^*((X_1 \cup \{b_1\}) \cap Y_2)$. So $b_2 \in \mathrm{cl}((X_1 \cup \{b_1\}) \cap Y_2)$. If $r((X_1 \cap Y_2) \cup \{b_1, b_2\}) \geq 3$, then it follows that $((X_1 \cup \{b_1\}) \cap Y_2, \{b_2\}, E(M) - ((X_1 \cap Y_2) \cup \{b_1, b_2\}))$ is a vertical 3-separation that contradicts the minimality of X_1 in $(X_1, \{b_1\}, Y_1)$. So (iii) holds, completing the proof of the lemma. □

2.2 The existence of strictly removable elements

Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M. Recall that an element $e \in E(M)$ is *strictly B-removable* if $e \in B$ and M/e is 3-connected, or $e \in E(M) - B$ and $M\backslash e$ is 3-connected. The matroid M has at least one strictly B-removable element (Oxley et al., 2008*a*, Theorem 1.1). In Section 2.2.1, we show that there are at least two such elements, thus proving Theorem 2.0.2. However, we cannot guarantee more than two such elements, as we show in Section 2.2.2.

2.2.1 The proof of Theorem 2.0.2

We start with two simple lemmas. A proof of the dual of the first is given by Oxley (2011, Proposition 8.2.7), and a proof of the dual of the second is given by Oxley et al. (2008*a*, Lemma 4.1).

Lemma 2.2.1. *Let e be an element of a matroid M. Suppose that M/e is 3-connected, but M is not. Then either e is a loop, e is a coloop, or e is contained in a series pair.*

When a matroid is 2-connected, we simply say it is *connected*; otherwise, we say it is *disconnected*.

Lemma 2.2.2. *Let M be a connected matroid with at least seven elements such that $\mathrm{si}(M)$ is 3-connected and all parallel classes of M have size at most two. Let $\{p_1, p_2\}$ and $\{q_1, q_2\}$ be distinct parallel pairs of M. Then $\{p_1, p_2, q_1, q_2\}$ is coindependent.*

The next lemma handles a special situation that arises regularly.

Lemma 2.2.3. *Let M be a 3-connected matroid with no 4-element fans, and containing the triangles $\{b_1, a, b_2\}$ and $\{b_2, d, d'\}$, and a cocircuit $\{a, b_2, d, d'\}$. Then $M\backslash d$ and $M\backslash d'$ are 3-connected.*

Proof. Towards a contradiction, suppose that $M\backslash d$ is not 3-connected. Then there exists a 2-separation (W, Z) of $M\backslash d$. Without loss of generality, $|W \cap \{b_1, a, b_2\}| \geq 2$. Since $r(\{b_1, a, b_2\}) = 2$, we have $r(W) = r(W \cup \{b_1, a, b_2\}) \leq r(M) - 1$, so either $(W \cup \{b_1, a, b_2\}, Z - \{b_1, a, b_2\})$ is a 2-separation of $M\backslash d$, or $|Z - \{b_1, a, b_2\}| < 2$. But if $|Z - \{b_1, a, b_2\}| < 2$, then it follows that Z is a series pair that meets the triangle $\{b_1, a, b_2\}$ in a single element, contradicting orthogonality. Since $\{a, b_2, d, d'\}$ is a cocircuit, $d' \in \mathrm{cl}^*_{M\backslash d}(W \cup \{b_1, a, b_2\})$ so either $(W \cup \{b_1, a, b_2, d'\}, Z - \{b_1, a, b_2, d'\})$ is a 2-separation of $M\backslash d$, or $|Z - \{b_1, a, b_2\}| = 2$. Since $d \in \mathrm{cl}(\{b_2, d'\})$, the first possibility implies that $(W \cup \{b_1, a, b_2, d, d'\}, Z - \{b_1, a, b_2, d'\})$ is a 2-separation of M; a contradiction. The second possibility implies that $(Z - \{b_1, a, b_2\}) \cup \{d\}$ is a triad that forms a 4-element fan with the triangle $\{b_2, d, d'\}$; a contradiction. Thus $M\backslash d$ is 3-connected, and, by symmetry, $M\backslash d'$ is also 3-connected. □

The approach taken to prove Theorem 2.0.2 is as follows. If M has an element $b \in B$ such that $\mathrm{si}(M/b)$ is not 3-connected, there is a vertical 3-separation $(X, \{b\}, Y)$, by Lemma 2.1.6. The set X contains an element $b_2 \in B$; Lemma 2.2.4 handles the case where $\mathrm{si}(M/b_2)$ is not 3-connected, while Lemmas 2.2.5 and 2.2.6 handle when $\mathrm{si}(M/b_2)$ is 3-connected. After compiling these, as Lemma 2.2.7, one special case remains, which is handled in Lemma 2.2.8.

Lemma 2.2.4. *Let M be a 3-connected matroid with no 4-element fans and let B be a basis of M. Let B' be a subset of B such that if $b \in B$ and $\mathrm{si}(M/b)$ is not 3-connected, then $b \in B'$. Suppose there exist elements $b_1 \in B'$ such that $(X_1, \{b_1\}, Y_1)$ is a vertical 3-separation with $Y_1 \cup \{b_1\}$ closed and X_1 minimal in $(X_1, \{b_1\}, Y_1)$ with respect to B', and $b_2 \in X_1 \cap B'$ such that $\mathrm{si}(M/b_2)$ is not 3-connected. Then either*

(i) *there exist distinct elements $d, d' \in X_1 \cap (E(M) - B)$ such that $M\backslash d$ and $M\backslash d'$ are 3-connected, or*

(ii) *$X_1 = \{a, b_2, b_3, d\}$ where $X_1 \cap B = \{b_2, b_3\}$, and $M\backslash d$ and $\mathrm{si}(M/b_3)$ are 3-connected, but $\mathrm{co}(M\backslash a)$, $\mathrm{si}(M/b_2)$ and M/b_3 are not.*

Proof. The matroid M has a vertical 3-separation $(X_2, \{b_2\}, Y_2)$, by Lemma 2.1.6, where $b_1 \in Y_2$ and, by Lemma 2.1.7, $Y_2 \cup \{b_2\}$ is closed. By Lemma 2.1.10, each of $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ is non-empty, and $r((X_1 \cap X_2) \cup \{b_2\}) = 2$. We consider two cases separately: when $|Y_1 \cap X_2| = 1$, and when $|Y_1 \cap X_2| \geq 2$.

First, consider the case when $|Y_1 \cap X_2| = 1$. As $|X_2| \geq 3$ and $b_1 \notin X_2$, $|X_1 \cap X_2| \geq 2$. If $|X_1 \cap X_2| = 2$, then $(X_1 \cap X_2) \cup \{b_2\}$ is a triangle and X_2 is a triad, so $X_2 \cup \{b_2\}$ is a 4-element fan; a contradiction. So $|X_1 \cap X_2| \geq 3$, in which case $(X_1 \cap X_2) \cup \{b_2\}$ is a rank-2 set of at least four elements. Thus, by Lemma 2.1.8, $M\backslash x$ is 3-connected for $x \in X_1 \cap X_2$. Since at most one element of $X_1 \cap X_2$ is in B, the set $X_1 \cap X_2$ contains distinct elements $d, d' \in X_1 \cap (E(M) - B)$ such that $M\backslash d$ and $M\backslash d'$ are 3-connected, satisfying (i).

Now consider the case when $|Y_1 \cap X_2| \geq 2$. By Lemma 2.1.10(iii), $r((X_1 \cap Y_2) \cup \{b_1, b_2\}) = 2$. We label the lines $L_1 = (X_1 \cap Y_2) \cup \{b_1\}$ and $L_2 = (X_1 \cap X_2) \cup \{b_2\}$. If $\mathrm{cl}(L_1)$ or $\mathrm{cl}(L_2)$ has cardinality at least four, then, by Lemma 2.1.8, there are two distinct elements in $E(M) - B$ whose deletion maintains 3-connectivity, again satisfying (i). Suppose instead that $|\mathrm{cl}(L_1)| = 3$ and $|\mathrm{cl}(L_2)| \in \{2, 3\}$. Let $X_1 \cap Y_2 = \{a\}$. If $|\mathrm{cl}(L_2)| = 2$, so $|X_1 \cap X_2| = 1$, then X_1 is a triad, $\mathrm{cl}(L_1)$ is a triangle, and both contain $\{a, b_2\}$ resulting in a 4-element fan; a contradiction. So let $|\mathrm{cl}(L_2)| = 3$ and, in particular, $X_1 \cap X_2 = \{c, d\}$ where $d \in E(M) - B$. Then, by Lemma 2.2.3, $M\backslash d$ and $M\backslash c$ are 3-connected. If $c \in E(M) - B$, then (i) holds. So we now assume that $c \in B$.

First, we consider the case where $\mathrm{si}(M/c)$ is 3-connected. Since c is in a triangle, M/c is not 3-connected. We also observe that $a \in E(M) - B$ since $b_1, b_2 \in \mathrm{cl}(L_2)$, and since $X_1 - \{a\}$ is a triangle, $\mathrm{co}(M\backslash a)$ is not 3-connected. Thus we have case (ii), with $b_3 = c$.

We now assume that $\mathrm{si}(M/c)$ is not 3-connected, in which case, by Lemma 2.1.6, M has a vertical 3-separation $(X_3, \{c\}, Y_3)$ with $b_1 \in Y_3$. We may assume that $Y_3 \cup \{c\}$ is closed, by Lemma 2.1.7. Then, as $c \in B'$, since $\mathrm{si}(M/c)$ is not 3-connected, each of $X_1 \cap X_3$, $X_1 \cap Y_3$, $Y_1 \cap X_3$ and $Y_1 \cap Y_3$ is non-empty and $r((X_1 \cap X_3) \cup \{c\}) = 2$, by Lemma 2.1.10. Since $X_1 = \{a, b_2, c, d\}$, we have $\{|X_1 \cap X_3|, |X_1 \cap Y_3|\} = \{1, 2\}$. If $|Y_1 \cap X_3| = 1$, then $|X_1 \cap X_3| = 2$, since $|X_3| \geq 3$, implying that $X_3 \cup \{c\}$ is a 4-element fan; a contradiction. So $|Y_1 \cap X_3| \geq 2$ and, by Lemma 2.1.10(iii), $r((X_1 \cap Y_3) \cup \{b_1, c\}) = 2$. Since $\{b_1, b_2, c\} \subseteq B$, it follows that $b_2 \in X_1 \cap X_3$. If $a \in X_1 \cap Y_3$, then $c \in \mathrm{cl}(\{a, b_1\}) \subseteq \mathrm{cl}(Y_2 \cup \{b_2\}) = Y_2 \cup \{b_2\}$; a contradiction. But then $a \in X_1 \cap X_3$, in which case $c \in \mathrm{cl}(\{a, b_2\}) \subseteq \mathrm{cl}(Y_2 \cup \{b_2\}) = Y_2 \cup \{b_2\}$; again, a contradiction. Thus the lemma holds. □

The approach taken in the proof of the next lemma is inspired by the proof of the main theorem in the paper by Oxley et al. (2008*a*, Theorem 1.2).

Lemma 2.2.5. *Let M be a 3-connected matroid with no 4-element fans and $|E(M)| \geq 7$. Let B be a basis of M such that $\mathrm{si}(M/b)$ is 3-connected for* *some* $b \in B$. *Then either*

(i) M/b *is* 3*-connected, or*

(ii) $M\backslash d$ *is* 3*-connected for some* $d \in E(M) - B$*, where* $\{b, d\}$ *is contained in a triangle of* M*.*

Proof. Suppose that (i) does not hold. The matroid M/b is not 3-connected, but $\mathrm{si}(M/b)$ is, so M/b has a non-trivial parallel class P, as M is 3-connected. Since at most one element of P is in B, there exists an element $d \in P \cap (E(M) - B)$. As $r_M(P \cup \{b\}) = 2$, if $|P| > 2$ then $M\backslash d$ is 3-connected, by Lemma 2.1.8, so (ii) holds.

Now we may assume that all parallel classes of M/b are parallel pairs. Let one such pair be $P = \{p_1, p_2\}$, with $p_1 \in E(M) - B$. If $M\backslash p_1$ is 3-connected, then, since $P \cup \{b\}$ is a triangle, (ii) holds; so we now assume that $M\backslash p_1$ is not 3-connected.

Suppose that b is in series with some other element s of $M\backslash p_1$; then, since b cannot be in series with s in M, $\{s, b, p_1\}$ is a triad in M. But $\{b, p_1, p_2\}$ is a triangle of M, so $\{s, b, p_1, p_2\}$ is a 4-element fan; a contradiction. Thus, b is not in series with any other element of $M\backslash p_1$.

Since M/b is 3-connected up to parallel pairs, and hence $M/b\backslash p_1$ is also, if $M/b\backslash p_1$ has no parallel pairs, then it is 3-connected. By the contrapositive of Lemma 2.2.1, $M\backslash p_1$ is also 3-connected, since b is not contained in a series pair in $M\backslash p_1$; a contradiction. So we may assume that $M\backslash p_1/b$ has at least one parallel pair Q.

If Q is a parallel pair of $M\backslash p_1$, it is a parallel pair of M; a contradiction. So, letting $Q = \{q_1, q_2\}$, we have that $\{q_1, q_2, b\}$ is a triangle of $M\backslash p_1$. Let (J, K) be a 2-separation of $M\backslash p_1$ where, without loss of generality, $b \in J$. If $|J| = 2$, then it follows that J is a series pair; a contradiction. Thus $|J| \geq 3$ and $(J - \{b\}, K)$ is a 2-separation of $M\backslash p_1/b$ since

$$\begin{aligned}\lambda_{M\backslash p_1/b}(J - \{b\}) &\leq (r_{M\backslash p_1}(J) - 1) + r_{M\backslash p_1}(K) - (r(M\backslash p_1) - 1)\\ &= 1.\end{aligned}$$

Because $M\backslash p_1/b$ is 3-connected up to parallel pairs, either $(J - \{b\}) \cap E(\mathrm{si}(M\backslash p_1/b))$ or $K \cap E(\mathrm{si}(M\backslash p_1/b))$ consists of a single element. Thus, either $r_{M\backslash p_1}(J) = 2$ or $r_{M\backslash p_1}(K \cup \{b\}) = 2$. If $b \in \mathrm{cl}_{M\backslash p_1}(K)$

then $r_{M\backslash p_1/b}(K) = r_{M\backslash p_1}(K) - 1$ and $M\backslash p_1/b$ is disconnected; a contradiction. Since K consists of at least two elements, it has rank at least two in $M\backslash p_1$, so $r_{M\backslash p_1}(K \cup \{b\}) > 2$ and $r_{M\backslash p_1}(J) = 2$, and it follows that $|J| = 3$. Hence $Q = J - \{b\}$ is the unique parallel pair of $M/b\backslash p_1$ and, by Lemma 2.1.3, $b \in \mathrm{cl}^*_{M\backslash p_1}(Q)$.

It follows that $\{b, p_1, q_1, q_2\}$ contains a cocircuit in M. Recalling that $\{q_1, q_2, b\}$ is a triangle of $M\backslash p_1$, and thus is also a triangle of M, if $\{b, p_1, q_1, q_2\}$ contains a triad, then we have a 4-element fan in M; a contradiction. So $\{b, p_1, q_1, q_2\}$ is a cocircuit.

Since the intersection of the circuit $\{q_1, q_2, b\}$ with the cobasis $E(M) - B$ is non-empty, we can assume that $q_1 \in E(M) - B$. Then, if $M\backslash q_1$ is 3-connected, (ii) is satisfied. If not, following the same argument as for when $M\backslash p_1$ is not 3-connected, we see that $M/b\backslash q_1$ has a unique parallel pair. But since Q is the only parallel pair in $M/b\backslash p_1$, the only parallel pairs in M/b are P and Q, and the unique parallel pair in $M/b\backslash q_1$ is P. Furthermore, $b \in \mathrm{cl}^*_{M\backslash q_1}(P)$. Thus $\{b, q_1, p_1, p_2\}$ contains a cocircuit—in fact it is a cocircuit since M has no 4-element fans. By the dual of the circuit elimination axiom, $\{p_1, p_2, q_1, q_2\}$ contains a cocircuit. Thus, by Lemma 2.2.2 and since $|E(M/b)| \geq 6$, we have a contradiction unless $|E(M/b)| = 6$.

In the exceptional case, $|E(M)| = 7$ and the only triangles of M containing b are $\{b, p_1, p_2\}$ and $\{b, q_1, q_2\}$. It follows that $|E(\mathrm{si}(M/b))| = 4$, thus $\mathrm{si}(M/b) \cong U_{2,4}$, since $\mathrm{si}(M/b)$ is 3-connected. Now $r(M) = 3$, and $E(M) - \{p_1, p_2, q_1, q_2\}$ is contained in a hyperplane with rank two; a contradiction. This completes the proof of the lemma. □

Lemma 2.2.6. *Let M be a 3-connected matroid with no 4-element fans and $|E(M)| \geq 7$. Let B be a basis of M and let $(X_1, \{b_1\}, Y_1)$ be a vertical 3-separation of M such that $\mathrm{si}(M/b)$ is 3-connected for some $b \in X_1 \cap B$, and $Y_1 \cup \{b_1\}$ is closed. Then one of the following holds:*

(i) *M/b is 3-connected,*

(ii) *there exists an element $d \in \mathrm{cl}(X_1) \cap (E(M) - B)$ such that $M\backslash d$ is 3-connected, and M has a triangle containing b and d, or*

(iii) *there exist distinct elements $d \in X_1 \cap (E(M) - B)$ and $d' \in E(M) - B$ such that both $M\backslash d$ and $M\backslash d'$ are 3-connected.*

Proof. It follows from Lemma 2.2.5 that either (i) holds, or there exists an element $d \in E(M) - B$ such that $M\backslash d$ is 3-connected and b and d are contained in a rank-2 set L of at least three elements. First suppose $|L| \geq 4$. Due to the rank of L, we have $|L \cap B| \leq 2$. Then, by Lemma 2.1.8, there are at least two elements $d, d' \in L \cap (E(M) - B)$ whose deletion maintains 3-connectivity. If $|L \cap (Y_1 \cup \{b_1\})| \geq 2$, then, since $Y_1 \cup \{b_1\}$ is closed, $L \subseteq Y_1 \cup \{b_1\}$, contradicting $b \in X_1 \cap L$. Thus $|L \cap (Y_1 \cup \{b_1\})| \leq 1$, so, without loss of generality, $d \in X_1$, and thus (iii) holds.

Now suppose $|L| = 3$ and let $L = \{d, b, q\}$. If $d \in X_1$, then (ii) holds, so assume that $d \in Y_1 \cup \{b_1\}$. Then, recalling $b \in X_1$, if $q \in Y_1 \cup \{b_1\}$, we have $b \in \text{cl}(Y_1 \cup \{b_1\})$, contradicting the fact $Y_1 \cup \{b_1\}$ is closed. So $q \in X_1$, and thus $d \in \text{cl}(X_1)$, satisfying (ii). □

Lemma 2.2.7. *Let M be a 3-connected matroid with no 4-element fans where B is a basis of M, and $|E(M)| \geq 7$. Suppose there exists an element $b_1 \in B$ such that $\text{si}(M/b_1)$ is not 3-connected, and let $(X_1, \{b_1\}, Y_1)$ be a vertical 3-separation of M. Then one of the following holds:*

(i) *M has at least two strictly B-removable elements, or*

(ii) *there exists an element $d \in \text{cl}(X_1) \cap \text{cl}(Y_1) \cap (E(M) - B)$ such that $M\backslash d$ is 3-connected. Moreover, there exist elements $b_x \in X_1 \cap B$ and $b_y \in Y_1 \cap B$ such that $\text{si}(M/b_x)$ and $\text{si}(M/b_y)$ are 3-connected, and M/b is not 3-connected for all $b \in B$.*

Proof. By Lemma 2.1.7, there exists a vertical 3-separation $(X', \{b_1\}, Y')$ such that $X' \subseteq X_1$ and $Y' \cup \{b_1\}$ is closed. There also exists a vertical 3-separation $(Y'', \{b_1\}, X'')$ where $X'' \cup \{b_1\}$ is closed and $Y'' \subseteq Y_1$, with $X' \cap Y'' = \emptyset$. We show that there exists a strongly B-removable element $b_x \in X' \cap B$ or $d_x \in \text{cl}(X') \cap (E(M) - B)$, and a strongly B-removable element $b_y \in Y'' \cap B$ or $d_y \in \text{cl}(Y'') \cap (E(M) - B)$. If the two elements we find are equal, we show that (ii) holds; otherwise (i) holds.

There exists an element $b_1' \in (X' \cup \{b_1\}) \cap B$ such that $(X_1', \{b_1'\}, Y_1')$ is a vertical 3-separation with X_1' minimal in $(X_1', \{b_1'\}, Y_1')$, and $X_1' \cup \{b_1'\} \subseteq X' \cup \{b_1\}$, by Lemma 2.1.9. First, suppose that $\text{si}(M/b)$ is 3-connected for all $b \in X_1' \cap B$. Since X_1' is an exactly 3-separating set of rank at least three, there exists at least one such b. Then, by Lemma 2.2.6, either (i) holds immediately, or there exists either a $b_x \in X_1' \cap B$ such that M/b_x is

3-connected, or a $d_x \in \text{cl}(X'_1) \cap (E(M) - B)$ such that $M\backslash d_x$ is 3-connected. Note that $b_x \neq b_1$, so $b_x \in X' \cap B$, and $\text{cl}(X'_1) \subseteq \text{cl}(X' \cup \{b_1\}) = \text{cl}(X')$, so $d_x \in \text{cl}(X') \cap (E(M) - B)$. Now suppose that $\text{si}(M/b_2)$ is not 3-connected for some $b_2 \in X'_1 \cap B$. Then, by Lemma 2.2.4, there exists an element $d_x \in X'_1 \cap (E(M) - B)$ such that $M\backslash d_x$ is 3-connected. Note that, in fact, $d_x \in X' \cap (E(M) - B)$.

By the same reasoning for the vertical 3-separation $(Y'', \{b\}, X'')$, there exists either a $b_y \in Y'' \cap B$ such that M/b_y is 3-connected, or a $d_y \in \text{cl}(Y'') \cap (E(M) - B)$ such that $M\backslash d_y$ is 3-connected. It is now clear that the lemma holds, apart from in the case where we have a d_x and d_y such that $d_x = d_y$. Consider this case. We relabel $d = d_x = d_y$. There exist vertical 3-separations $(X'_1, \{b'_1\}, Y'_1)$ and $(Y''_1, \{b''_1\}, X''_1)$ where $d \in \text{cl}(X'_1) \cap \text{cl}(Y''_1) \cap (E(M) - B)$, with $X'_1 \cup \{b'_1\} \subseteq X_1 \cup \{b_1\}$ and $Y''_1 \cup \{b''_1\} \subseteq Y_1 \cup \{b_1\}$. Since $\text{cl}(X') \cap Y'' = \emptyset = X' \cap \text{cl}(Y'')$, this is only possible when $\text{si}(M/b)$ is 3-connected for all $b \in X' \cup Y''$. If M/b is 3-connected for some $b \in B$, then (i) holds. Otherwise, letting $b_x \in X'_1 \cap B$ and $b_y \in Y''_1 \cap B$, we have case (ii). This completes the proof of the lemma. □

We require one more lemma in order to prove Theorem 2.0.2.

Lemma 2.2.8. *Let M be a 3-connected matroid with no 4-element fans and a basis B. Suppose there exists an element $b \in B$ such that $\text{si}(M/b)$ is 3-connected and b is in a triangle $\{b, x_1, x_2\}$, where $M\backslash x_1$ is not 3-connected. Then either*

(i) *there exist distinct elements $d, d' \in E(M) - B$ such that $M\backslash d$ and $M\backslash d'$ are 3-connected, and there exists a rank-2 set of at least four elements containing $\{b, d, d'\}$, or*

(ii) *b is contained in a triangle $\{b, d, x_3\}$, where $\{x_1, x_2\} \cap \{d, x_3\} = \emptyset$, the matroid $M\backslash d$ is 3-connected, and $d \in E(M) - B$.*

Proof. Since $M\backslash x_1$ is not 3-connected, it has a 2-separation (P, Q). Without loss of generality, let $b \in Q$. If $x_2 \in Q$, then $x_1 \in \text{cl}(Q)$ and $(P, Q \cup \{x_1\})$ is a 2-separation of M; a contradiction. So $x_2 \in P$. Also note that if $b \in \text{cl}(P)$, then $x_1 \in \text{cl}(P)$; a contradiction. So $b \notin \text{cl}(P)$.

Next we show that $(P \cup \{x_1\}, Q - \{b\})$ is 2-separating in M/b. Since $\{x_1, x_2\}$ is a parallel pair in M/b, and $b \notin \text{cl}(P)$, we have $r_{M/b}(P \cup \{x_1\}) =$

$r_{M/b}(P) = r_M(P)$. Also, $r_{M/b}(Q - \{b\}) = r_M(Q) - 1$. Thus,

$$\begin{aligned}\lambda_{M/b}(P \cup \{x_1\}) &= r_M(P) + (r_M(Q) - 1) - (r(M) - 1)\\ &= r_{M\backslash x_1}(P) + r_{M\backslash x_1}(Q) - r(M\backslash x_1)\\ &= 1.\end{aligned}$$

Since $\mathrm{si}(M/b)$ is 3-connected, either $|(Q - \{b\}) \cap E(\mathrm{si}(M/b))| = 1$ or $|(P \cup \{x_1\}) \cap E(\mathrm{si}(M/b))| = 1$. But since $b \notin \mathrm{cl}(P)$ and $|P| \geq 2$, the latter is not possible. Thus the former holds, so $r_M(Q) = 2$.

Because (P, Q) is a 2-separation of $M\backslash x_1$, we have $\lambda_M(P) = 2$, and $r_M(Q \cup \{x_1\}) = 3$. Thus, $r_M(P) = r(M) - 1$, so $Q \cup \{x_1\}$ contains a cocircuit. If $|Q| = 2$, then $Q \cup \{x_1, x_2\}$ is a 4-element fan; a contradiction. If, instead, $|Q| \geq 4$, then Q contains at most two elements of B, so (i) holds by Lemma 2.1.8. It remains to consider when $|Q| = 3$. If $Q \cup \{x_1\}$ contains a triad, then $Q \cup \{x_1\}$ is a contradictory 4-element fan; so $Q \cup \{x_1\}$ is a cocircuit. Given that $r(Q) = 2$, there is at least one element of Q not in B, so let $Q = \{b, d, x_3\}$ where $d \in E(M) - B$. By Lemma 2.2.3, $M\backslash d$ is 3-connected. Thus (ii) holds. □

Proof of Theorem 2.0.2. Suppose that $r(M) \leq 2$. Since the only 3-connected matroids of rank at most two are uniform, M is isomorphic to $U_{1,2}$, $U_{1,3}$ or $U_{2,n}$ for $n \geq 3$. Letting $E(U_{1,2}) = \{b, d\}$, where $\{b\}$ is a basis of $U_{1,2}$, we see that $U_{1,2}/b \cong U_{0,1}$ and $U_{1,2}\backslash d \cong U_{1,1}$, where both $U_{0,1}$ and $U_{1,1}$ are 3-connected, so the theorem holds when $M \cong U_{1,2}$. When $M \cong U_{1,3}$, the matroid $U_{1,3}\backslash d$ is isomorphic to $U_{1,2}$, which is 3-connected, for each $d \in E(U_{1,3}) - B$. Again, the theorem holds. Likewise, the theorem holds when $M \cong U_{2,3}$, by duality. Finally, if $M \cong U_{2,n}$ for $n \geq 4$, then $M\backslash x$ is 3-connected for any $x \in E(M)$, by Lemma 2.1.8, so the theorem holds in this case. We may now assume that $r(M) \geq 3$ and, by duality, $r^*(M) \geq 3$.

Suppose that $|E(M)| = 6$. Then $r(M) = r^*(M) = 3$, and it follows that since M has no 4-element fans, M is isomorphic to $U_{3,6}$ or P_6, where the latter is the 6-element rank-3 matroid that has a single triangle as its only non-spanning circuit. In $U_{3,6}$, we can delete any element of $E(M) - B$ to obtain the 3-connected matroid $U_{3,5}$, so the theorem holds when $M \cong U_{3,6}$. Now consider P_6. Deleting an element in the triangle results in a matroid isomorphic to $U_{3,5}$, so the theorem holds if at most one element in this triangle is in B. It remains to consider the case where there are two elements of B in this triangle. Suppose that the other element of B is b_3. Then $P_6/b_3 \cong U_{2,5}$, a 3-connected matroid, in which case the theorem holds.

We now assume that $|E(M)| \geq 7$ and consider two cases: the first is when there exists an element $b_1 \in B$ such that $\text{si}(M/b_1)$ is not 3-connected; and the second is when for every $b \in B$, the matroid $\text{si}(M/b)$ is 3-connected.

In the first case, Lemma 2.2.7 implies that either the theorem holds, or there exists an element $d \in \text{cl}(X_1) \cap \text{cl}(Y_1) \cap (E(M) - B)$ such that $M \backslash d$ is 3-connected, where $(X_1, \{b_1\}, Y_1)$ is a vertical 3-separation of M, there exist elements $b_x \in X_1 \cap B$ and $b_y \in Y_1 \cap B$ such that $\text{si}(M/b_x)$ and $\text{si}(M/b_y)$ are 3-connected, and M/b is not 3-connected for all $b \in B$. Since $\text{si}(M/b_x)$ is 3-connected but M/b_x is not, either b_x is in a rank-2 set of at least four elements, in which case the theorem holds by Lemma 2.1.8, or b_x is contained in a triangle $\{b_x, x, d_1\}$ where $M \backslash d_1$ is 3-connected, by Lemma 2.2.5. Likewise, when the theorem does not hold immediately, b_y is contained in a triangle $\{b_y, y, d_2\}$ where $M \backslash d_2$ is 3-connected. If $d \neq d_1$ or $d \neq d_2$, then the theorem holds, so assume otherwise. Now, since the union of these two triangles has rank three, either x or y is not in B. Without loss of generality, we may assume that $x \in E(M) - B$. If $M \backslash x$ is 3-connected, then the theorem holds; otherwise, by Lemma 2.2.8, b_x is contained in a triangle $\{b_x, x', d'\}$ where $M \backslash d'$ is 3-connected, $d' \in E(M) - B$, and $d' \neq d$, so the theorem holds in this case.

We now consider the second case. Suppose there exists an element $b_1 \in B$ such that M/b_1 is 3-connected. If there also exists an element $b_2 \in B - \{b_1\}$ such that M/b_2 is 3-connected, then clearly the theorem holds. Otherwise, for every $b_2 \in B - \{b_1\}$, of which there are at least two such elements, $\text{si}(M/b_2)$ is 3-connected, but M/b_2 is not. However, since $\text{si}(M/b_2)$ is 3-connected, Lemma 2.2.5 implies that $E(M) - B$ contains an element d such that $M \backslash d$ is 3-connected. Thus the theorem holds.

The only case that remains is when for every $b \in B$, the matroid $\text{si}(M/b)$ is 3-connected but M/b is not 3-connected. By Lemma 2.2.5, each $b_i \in B$ is contained in a triangle T_i that also contains an element $d_i \in E(M) - B$, where $M \backslash d_i$ is 3-connected. Since $r(M) \geq 3$, let b_1, b_2 and b_3 be distinct elements of B. Suppose that, for each T_i, we have $|T_i \cap B| \geq 2$. Without loss of generality, we may assume that $T_2 = T_3 = \{b_2, b_3, d_2\}$. If $b_2 \in T_1$ or $b_3 \in T_1$, then, as $r(T_1 \cup T_2) = 3$, the strictly B-removable elements d_1 and d_2

are distinct. So let $T_1 = \{b_1, b_4, d_1\}$ where $b_4 \in B$. If $d_1 = d_2$, then, by the circuit elimination axiom, $\{b_1, b_2, b_3, b_4\}$ contains a circuit; a contradiction. Thus, d_1 and d_2 are distinct strictly B-removable elements. We may now assume that $T_1 = \{b_1, x, d_1\}$, where $x \in E(M) - B$. If $M\backslash x$ is 3-connected, then the theorem is satisfied, so assume otherwise. By Lemma 2.2.8, either the theorem holds or $\{b_1, d, x_3\}$ is a triangle of M, where d and d_1 are distinct strictly B-removable elements. This completes the proof of the theorem. □

2.2.2 An example with two strictly removable elements

Every 3-connected matroid without 4-element fans has at least two strictly removable elements, by Theorem 2.0.2. In this section, we give an example to illustrate that we cannot guarantee more than two such elements. We shall describe how to construct a matroid $M_{k,j}$ of arbitrary rank with a basis B and precisely two strictly B-removable elements. Such a matroid is an "unpointed" variation of a pointed-flan as defined by Hall et al. (2005).

Let F be a flat of a matroid N. There is a unique extension N^+ of N on $E(N) \cup \{e\}$ such that the flats of N containing F are precisely the flats F' of N for which $F' \cup \{e\}$ is a flat of N^+ having the same rank as F' (Oxley, 2011, Theorem 7.2.3). We call this a *principal extension* of N and say that e has been *freely added to the flat* F. The rank function for N^+ is as follows: for all $X \subseteq E(N)$,

$$r_{N^+}(X) = r_N(X), \text{ and}$$
$$r_{N^+}(X \cup \{e\}) = \begin{cases} r_N(X) & \text{if } F \subseteq \mathrm{cl}_N(X), \\ r_N(X) + 1 & \text{if } F \not\subseteq \mathrm{cl}_N(X). \end{cases}$$

We now describe how to construct a matroid $M_{k,j}$ of rank $k + 2$ with precisely two strictly removable elements, where $k \geq 2$ and $0 < j < k$. Start with the free $(k + 2)$-element matroid $U_{k+2,k+2}$ with ground set $\{t, b_0, b_1, \ldots, b_k\}$. For each $i \in \{1, 2, \ldots, k\}$, freely add c_i to the flat $\{t, b_{i-1}, b_i\}$. Now, freely add g_i to the flat $\{b_i, t\}$ for each $i \in \{0, j, k\}$. Finally, we delete t to obtain $M_{k,j}$.

Observe that $B = \{b_0, b_1, \ldots, b_k, g_j\}$ is a basis for $M_{k,j}$. Every element in $E(M_{k,j}) - B$ is in a triad, so these elements are not strictly B-removable. Moreover, the only elements in B that, when contracted, do not open up a 2-separation are b_0 and b_k. Thus, these are the only two strictly removable elements of $M_{k,j}$.

In Figure 2.1, we illustrate the matroid $M_{4,2}$, of rank six. Solid black circles represent elements in B, while hollow circles represent elements in $E(M_{4,2}) - B$. A hollow square is used at the intersection of multiple lines to indicate that the intersection of the span of those lines is empty. We shall follow these rules for matroidal illustrations throughout Part I.

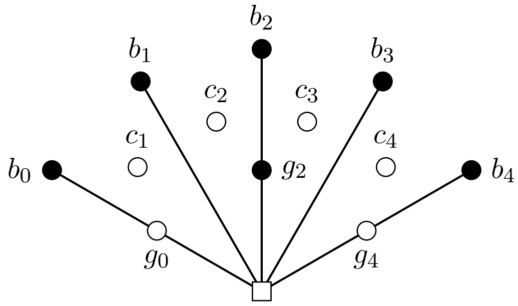


Figure 2.1: A 3-connected rank-6 matroid $M_{4,2}$ with two strictly removable elements b_0 and b_4.

2.3 The existence of removable elements

Let M be a 3-connected matroid and let B be a basis of M. We now turn our attention to the presence of B-removable elements in M; that is, elements $b \in B$ such that $\mathrm{si}(M/b)$ is 3-connected, or elements $d \in E(M) - B$ such that $\mathrm{co}(M\backslash d)$ is 3-connected. Whittle and Williams (2013) proved that M has at least four B-removable elements, provided that M has no 4-element fans and $|E(M)| \geq 4$ (Theorem 2.0.3). In this section, we strengthen this result by relaxing the requirement that no 4-element fans are present. However, a 4-element fan with one of two particular labellings, relative to B, requires special attention. We call these labelled fans either a Type I or Type II fan relative to B.

This section is structured as follows. In Section 2.3.1, we give some necessary preliminaries relating to fans; in particular, we define Type I and Type II fans. Section 2.3.2 contains two key results: the first, Corollary 2.3.11, shows that M has three B-removable elements provided that M has no Type I fans; while the second, Corollary 2.3.12, shows that M has four B-removable elements provided that M also has no Type II fans. The latter generalises Theorem 2.0.3. In Section 2.3.3, we give an example to illustrate that Corollary 2.3.12 is best possible in the sense that we cannot guarantee that M has more B-removable elements.

We begin with a well-known result, known as Bixby's Lemma (Bixby, 1982).

Lemma 2.3.1. *Let e be an element of a 3-connected matroid M. Then either* $\mathrm{si}(M/e)$ *or* $\mathrm{co}(M\backslash e)$ *is 3-connected.*

Proofs for the next three lemmas are given elsewhere; the first is due to Whittle and Williams (2013, Lemma 2.13), the second is due to Whittle (1999, Lemma 3.8), and the third is due to Oxley and Wu (2000, Lemma 3.4). A *segment* in a matroid M is a subset L of $E(M)$ such that $M|L \cong U_{2,k}$ for some $k \geq 2$, while a *cosegment* of M is a segment of M^*.

Lemma 2.3.2. *Let M be a 3-connected matroid with a triad $\{a, b, c\}$ and a circuit $\{a, b, c, d\}$. Then at least one of the following holds:*

(i) *either* $\mathrm{co}(M\backslash a)$ *or* $\mathrm{co}(M\backslash c)$ *is 3-connected, or*

(ii) *there exist elements $a', c' \in E(M)$ such that $\{a, a', b\}$ and $\{b, c, c'\}$ are triangles, or*

(iii) *there exists an element $z \in E(M) - \{a, b, c, d\}$ such that $\{a, b, c, z\}$ is a cosegment.*

Lemma 2.3.3. *Let C^* be a rank-3 cocircuit of a 3-connected matroid M. If $e \in C^*$ has the property that* $\mathrm{cl}_M(C^*) - \{e\}$ *contains a triangle of M/e, then* $\mathrm{si}(M/e)$ *is 3-connected.*

Lemma 2.3.4. *Let f_1, f_2, f_3, f_4, f_5 be distinct elements of a 3-connected matroid M that is not isomorphic to $M(\mathcal{W}_3)$. Suppose that $\{f_1, f_2, f_3\}$ and $\{f_3, f_4, f_5\}$ are triangles and $\{f_2, f_3, f_4\}$ is a triad of M. Then these two triangles and this one triad are the only triangles and triads of M containing f_3.*

2.3.1 Fans

Let M be a 3-connected matroid. A subset F of $E(M)$ having at least three elements is a *fan* if there is an ordering $(f_1, f_2, \ldots, f_k)$ of the elements of F such that

(i) for all $i \in \{1, 2, \ldots, k-2\}$, the triple $\{f_i, f_{i+1}, f_{i+2}\}$ is either a triangle or a triad, and

(ii) for all $i \in \{1, 2, \ldots, k-3\}$, if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triad, while if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triad, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triangle.

An ordering of F satisfying (i) and (ii) is a *fan ordering* of F. If F has a fan ordering $(f_1, f_2, \ldots, f_k)$ where $k \geq 4$, then f_1 and f_k are the *ends* of F, and $f_2, f_3, \ldots, f_{k-1}$ are the *internal elements* of F.

Let F be a fan with ordering $(f_1, f_2, \ldots, f_k)$ where $k \geq 5$, and let $i \in \{1, 2, \ldots, k\}$. An element f_i is a *spoke element* of F if $\{f_1, f_2, f_3\}$ is a triangle and i is odd, or if $\{f_1, f_2, f_3\}$ is a triad and i is even; otherwise f_i is a *rim element*. For a fan F with ordering (f_1, f_2, f_3, f_4), the element f_1 is a *spoke element* of F if $\{f_1, f_2, f_3\}$ is a triangle, otherwise it is a *rim element*; while f_4 is a *spoke element* if $\{f_1, f_2, f_3\}$ is a triad, otherwise it is a *rim element*.

The next lemma is a variant on a well-known result, which follows easily from Bixby's Lemma. We note that the requirement that $|E(M)| \geq 7$ is necessary; the rank-3 whirl $\mathcal{W}^3$ has six elements, and $\mathrm{si}(\mathcal{W}^3/e)$ is 3-connected for a spoke element $e \in E(\mathcal{W}^3)$, while $\mathrm{co}(\mathcal{W}^3 \backslash e)$ is 3-connected for a rim element $e \in E(\mathcal{W}^3)$.

Lemma 2.3.5. *Let M be a 3-connected matroid such that $|E(M)| \geq 7$. Suppose M has a fan F of at least four elements, and let f be an end of F.*

(i) *If f is a spoke element, then $\mathrm{co}(M \backslash f)$ is 3-connected and $\mathrm{si}(M/f)$ is not 3-connected.*

(ii) *If f is a rim element, then $\mathrm{si}(M/f)$ is 3-connected and $\mathrm{co}(M \backslash f)$ is not 3-connected.*

Proof. Since M has a triangle, $r(M) \geq 2$. But if $r(M) = 2$, then the 3-connected matroid M is isomorphic to $U_{2,n}$ for some $n \geq 7$, and such a matroid has no 4-element fans; a contradiction. By duality, we may now assume that $r(M) \geq 3$ and $r^*(M) \geq 3$. Next we show the following:

2.3.5.1. *When f is a spoke element, either $\mathrm{si}(M/f)$ is not 3-connected, or $\mathrm{si}(M/f) \cong U_{2,3}$. When f is a rim element, either $\mathrm{co}(M \backslash f)$ is not 3-connected, or $\mathrm{co}(M \backslash f) \cong U_{1,3}$.*

Let $f, f_2, f_3, f_4 \in F$, let $\{f, f_2, f_3\}$ be a triangle and let $\{f_2, f_3, f_4\}$ be a triad, so f is an end of F and a spoke element. The matroid $\mathrm{si}(M/f) \cong \mathrm{si}(M/f\backslash f_2)$ contains a series pair $\{f_3, f_4\}$. But the only 3-connected matroids with a series pair are $U_{1,2}$ and $U_{2,3}$, and the former also has a parallel pair. Thus, either $\mathrm{si}(M/f)$ is not 3-connected, or it is isomorphic to $U_{2,3}$, so (2.3.5.1) holds when f is a spoke element. By taking the dual, we see that (2.3.5.1) also holds when f is a rim element.

Now we assume, by duality, that $r(M) \geq 4$. By Bixby's Lemma, it suffices to prove that when f is a spoke element $\mathrm{si}(M/f)$ is not 3-connected, and when f is a rim element $\mathrm{co}(M\backslash f)$ is not 3-connected. Suppose that f is a spoke element. The matroid $\mathrm{si}(M/f)$ has rank at least three, so $\mathrm{si}(M/f) \not\cong U_{2,3}$ and by (2.3.5.1), $\mathrm{si}(M/f)$ is not 3-connected. Now suppose that f is a rim element. Note that, since $r^*(M) \geq 3$, the rank of the matroid $\mathrm{si}(M^*/f)$ is at least two, thus $r(\mathrm{co}(M\backslash f)) \geq 2$. It follows, by (2.3.5.1), that since $\mathrm{co}(M\backslash f) \not\cong U_{1,3}$, the matroid $\mathrm{co}(M\backslash f)$ is not 3-connected. □

Let M be a matroid and let B be a basis of M. We define a *Type I fan relative to* B in M to be a 4-element fan F with ordering (f_1, f_2, f_3, f_4) where $\{f_1, f_2, f_3\}$ is a triangle and $F \cap B = \{f_1, f_3\}$. We define a *Type II fan relative to* B in M to be a 4-element fan F with ordering (f_1, f_2, f_3, f_4) where $\{f_1, f_2, f_3\}$ is a triangle and $F \cap B = \{f_1, f_3, f_4\}$.

Let M be a 3-connected matroid M with a basis B, where $|E(M)| \geq 7$. By Lemma 2.3.5(ii), a Type II fan F in M, as described in the previous paragraph, contains a B-removable element f_4.

2.3.2 An upgrade of Theorem 2.0.3

The main result of this section is Corollary 2.3.12, a generalisation of Theorem 2.0.3 that relaxes the requirement that M has no 4-element fans. The crux is Proposition 2.3.9, which shows that if M has a B-removable element, then we can describe the "location" of either two other B-removable elements, or a Type I or Type II fan. We prove three corollaries of this result, each permitting different types of labelled fan. Corollary 2.3.10 shows that either M has two B-removable elements, or a Type I fan F where each element of F is not removable. Corollary 2.3.11 shows that if M has no Type I fans, then M has three removable elements. Finally, Corollary 2.3.12 generalises Theorem 2.0.3, showing that M has at least four removable elements provided M has no Type I or Type II fans.

We start with a series of lemmas.

Lemma 2.3.6. *Let M be a 3-connected matroid with $r(M) \geq 4$. Suppose that C^* is a rank-3 cocircuit of M such that $|C^*| \geq 4$.*

(i) *If there is no $T \subseteq C^*$ such that T is a triangle, then $\mathrm{co}(M\backslash d)$ is 3-connected for all $d \in C^*$.*

(ii) *If $T \subseteq C^*$ such that T is a triangle, then $\mathrm{co}(M\backslash d)$ is 3-connected for all $d \in T$.*

Proof. Suppose that $\mathrm{co}(M\backslash d)$ is not 3-connected for some d satisfying the hypothesis of either (i) or (ii). Then $M\backslash d$ has a 2-separation (U, V) in which neither U nor V is a series class. Clearly, $d \notin \mathrm{cl}(U)$ and $d \notin \mathrm{cl}(V)$; otherwise, M has a 2-separation. Thus $U \cap C^*$ and $V \cap C^*$ are both non-empty. Furthermore, either U or V contains two distinct elements $x_1, x_2 \in C^*$ such that $C^* \subseteq \mathrm{cl}(\{x_1, x_2, d\})$. Without loss of generality, we may assume that $\{x_1, x_2\} \subseteq U$. The set $U \cup \{d\}$ is exactly 3-separating. Therefore, by repeated applications of Lemma 2.1.4, for each subset D of $V \cap C^*$, the set $D \subseteq \mathrm{cl}(V - D)$ provided $|V - D| \geq 2$. Let $H = E(M) - C^*$. If $|V \cap H| \geq 2$, then $\mathrm{cl}(H) \cap C^*$ is non-empty, contradicting the fact that H is a hyperplane. Thus, $|V \cap H| \leq 1$. If $|V \cap H| = 0$, then $H \subseteq U$ and so, as $U \cap C^*$ is non-empty, $r(U) = r(M)$. This implies that V is a parallel class; a contradiction as M is 3-connected. Hence, $|V \cap H| = 1$ and $r(V) = 2$. Let $V \cap H = \{h\}$. If $|V \cap C^*| \geq 2$, then $h \in \mathrm{cl}(C^*)$ and so, by Lemma 2.1.4, $h \in \mathrm{cl}(H - \{h\})$. In particular, $H \subseteq \mathrm{cl}(U)$ and so $r(U) = r(M)$; a contradiction. Therefore, $|V \cap C^*| = 1$, and so V is a 2-element cocircuit, a contradiction. This completes the proof of the lemma. □

Lemma 2.3.7. *Let (X, Y) be a 3-separation of a 3-connected matroid M. If $X \cap \mathrm{cl}(Y) \neq \emptyset$ and $X \cap \mathrm{cl}^*(Y) \neq \emptyset$, then $|X \cap \mathrm{cl}(Y)| = 1$ and $|X \cap \mathrm{cl}^*(Y)| = 1$.*

Proof. Let $x \in X \cap \mathrm{cl}^*(Y)$, and consider $M\backslash x$. Since $x \in \mathrm{cl}^*(Y)$, it follows, by Lemma 2.1.3, that $x \notin \mathrm{cl}(X - \{x\})$. Therefore, as M is 3-connected,

$$\begin{aligned}\lambda_{M\backslash x}(X - \{x\}) &= r(X - \{x\}) + r(Y) - r(M\backslash x)\\ &= r(X) - 1 + r(Y) - r(M)\\ &= 1,\end{aligned}$$

so $(X-\{x\}, Y)$ is a 2-separation of $M\backslash x$. If $x \in \mathrm{cl}(Y)$, then $(X-\{x\}, Y\cup\{x\})$ is a 2-separation of M; a contradiction. Moreover, by the submodularity of the rank function, $r((X-\{x\}) \cap \mathrm{cl}(Y)) \leq \lambda_{M\backslash x}(X-\{x\}) = 1$. Hence, as M has no parallel pairs, $|X \cap \mathrm{cl}(Y)| \leq 1$. Thus $|X \cap \mathrm{cl}(Y)| = 1$.

By contracting an element in $X \cap \mathrm{cl}(Y)$ and applying a dual argument, we see that $|X \cap \mathrm{cl}^*(Y)| = 1$. □

The next lemma is straightforward, but it is used frequently in the proof of Proposition 2.3.9.

Lemma 2.3.8. *Let M be a matroid that is simple and cosimple, with $f_1, f_2, f_3, f_4 \in E(M)$. If the only triangle containing f_3 is $\{f_1, f_2, f_3\}$ and the only triad containing f_2 is $\{f_2, f_3, f_4\}$, then $\mathrm{si}(M/f_3)$ is 3-connected if and only if $\mathrm{co}(M\backslash f_2)$ is 3-connected.*

Proof. Since $\mathrm{si}(M/f_3) \cong M/f_3\backslash f_2$, and $\mathrm{co}(M\backslash f_2) \cong M\backslash f_2/f_3$, we see that $\mathrm{si}(M/f_3) \cong \mathrm{co}(M\backslash f_2)$. The result follows. □

Proposition 2.3.9. *Let M be a 3-connected matroid and let B be a basis of M. Suppose there exists an element $b \in B$ such that $\mathrm{si}(M/b)$ is not 3-connected, and let $(X, \{b\}, Y)$ be a vertical 3-separation of M. Then one of the following holds:*

(i) *there exist distinct elements $s_1, s_2 \in X$ that are B-removable, or*

(ii) *there exist distinct elements $s_1 \in X$ and $s_2 \in \mathrm{cl}^*(X) \cap B$ that are B-removable, and a vertical 3-separation $(X', \{b'\}, Y')$ of M such that $X' \cup \{s_2\}$ is a 4-element cosegment containing s_1, the element $b' \in B$ is not B-removable, and $X' \cup \{b'\} \subseteq X \cup \{b\}$, or*

(iii) *there exist distinct elements $s_1 \in X$ and $s_2, s_3 \in \mathrm{cl}(X) \cap (E(M) - B)$ that are B-removable, or*

(iv) *M has a Type I fan F relative to B where the internal elements of F are contained in X, or*

(v) *M has a Type II fan relative to B contained in $X \cup \{b\}$.*

Proof. In what follows, we shall assume that (iv) does not hold, in which case we will show that one of the other four cases holds. By Lemma 2.1.7, there exists a vertical 3-separation $(X', \{b\}, Y')$ such that $Y' \cup \{b\}$ is closed and

$X' \subseteq X$. If the proposition holds for the vertical 3-separation $(X', \{b\}, Y')$, then clearly it holds for the vertical 3-separation $(X, \{b\}, Y)$; so we may assume that $Y \cup \{b\}$ is closed. Note that $|X \cap B| \geq 1$. If $X \cap B$ contains two or more elements that are B-removable, then (i) holds; so we assume this is not the case. As a result, it is sufficient to consider the following two cases:

(I) $X \cap B = \{b_c\}$ and b_c is B-removable, or

(II) there exists an element $b_x \in X \cap B$ such that b_x is not B-removable.

We will show that in case (I), one of (i)–(iii) holds. Note that if one of (i)–(iii) holds for a vertical 3-separation $(X_1, \{b_1\}, Y_1)$ with $X_1 \cup \{b_1\} \subseteq X \cup \{b\}$, then one of (i)–(iii) holds for $(X, \{b\}, Y)$. Thus, we first prove the following, which includes the case where (I) holds.

2.3.9.1. *If there exists an element $b_1 \in B$ and a vertical 3-separation $(X_1, \{b_1\}, Y_1)$ such that $X_1 \cap B = \{b_c\}$, where b_c is B-removable, $X_1 \cup \{b_1\} \subseteq X \cup \{b\}$, and $Y_1 \cup \{b_1\}$ is closed, then one of* (i)–(iii) *holds.*

Since $|X_1 \cap B| = 1$ and $Y_1 \cup \{b_1\}$ is closed, $Y_1 \cup \{b_1\}$ is a hyperplane and X_1 is a rank-3 cocircuit. If $|X_1| \geq 4$, then, by Lemma 2.3.6, there exists a removable element in $X_1 \cap (E(M) - B)$, so (i) holds. Thus, assume that $|X_1| = 3$, and let $X_1 = \{a, b_c, c\}$. If a or c is removable with respect to B, then (i) is satisfied, so we may also assume neither $\text{co}(M \backslash a)$ nor $\text{co}(M \backslash c)$ is 3-connected.

Suppose that $X_1 \cup \{b_1\}$ is not a 4-element fan. Then $X_1 \cup \{b_1\}$ is a circuit. As neither $\text{co}(M \backslash a)$ nor $\text{co}(M \backslash c)$ is 3-connected, it follows, by Lemma 2.3.2, that either $\{a, b_c, c\}$ are the internal elements of a 5-element fan, or there exists an element $z \in E(M) - (X_1 \cup \{b_1\})$ such that $X_1 \cup \{z\}$ is a cosegment. For the latter, (ii) holds by the dual of Lemma 2.1.8. For the former, both ends of the 5-element fan, a' and c' say, are in $E(M) - B$, otherwise we have a Type I fan. It follows, by Lemma 2.3.5, that a' and c' are both removable. Since $b_c \in X$ is also removable, (iii) holds.

Now consider the case where $X_1 \cup \{b_1\}$ is a 4-element fan. Then, up to relabelling, either $\{a, b_c, b_1\}$ or $\{a, c, b_1\}$ is a triangle. If $\{a, b_c, b_1\}$ is a triangle, then $X_1 \cup \{b_1\}$ is a Type I fan; a contradiction. Thus, $\{a, c, b_1\}$ is a triangle. If c is contained in a triad T^* that is not $\{a, b_c, c\}$, then, by orthogonality, T^* contains either a or b_1. But if it contains a, then $X_1 \cup T^*$ is a cosegment of four elements, so by the dual of Lemma 2.1.8, (ii) holds. If instead $\{b_1, c\}$ is contained in T^*, then a is a spoke and an end element of a 4-element fan, so $\mathrm{co}(M\backslash a)$ is 3-connected by Lemma 2.3.5; a contradiction. It follows that the only triad containing c is $\{a, b_c, c\}$.

If the only triangle containing a is $\{a, c, b_1\}$, then $\mathrm{si}(M/a) \cong \mathrm{co}(M\backslash c)$ by Lemma 2.3.8, so $\mathrm{si}(M/a)$ is not 3-connected. But $\mathrm{co}(M\backslash a)$ is not 3-connected, contradicting Bixby's Lemma, so a is contained in a triangle other than $\{a, c, b_1\}$. By orthogonality, such a triangle contains either $\{a, b_c\}$ or $\{a, c\}$, but the latter is not possible since $\{a, c, b_1\}$ is a triangle and $Y_1 \cup \{b_1\}$ is closed. So a is contained in a triangle $\{a, b_c, a'\}$ say. Since $\mathrm{si}(M/b_c)$ is 3-connected but $\mathrm{co}(M\backslash a)$ is not 3-connected, either b_c is contained in a triangle other than $\{a, b_c, a'\}$, or a is contained in a triad other than $\{a, b_c, c\}$, by Lemma 2.3.8. But $\{b_1, c, a, b_c, a'\}$ is a 5-element fan and $r(M) \geq 4$, so, by Lemma 2.3.4, the only triad containing a is $\{a, b_c, c\}$. Thus, by orthogonality and since $Y_1 \cup \{b_1\}$ is closed, $\{b_c, c\}$ is contained in a triangle $\{b_c, c, c'\}$ say. Now (a', a, b_c, c, c') is a fan ordering of a 5-element fan. The elements $a', c' \in \mathrm{cl}(X)$ are both in $E(M) - B$, or this fan contains a Type I fan. It follows, by Lemma 2.3.5, that a' and c' are both removable, so (iii) holds, completing the proof of (2.3.9.1).

Now consider (II). By Lemma 2.1.9, there exists an element $b_1 \in B$ and a vertical 3-separation $(X_1, \{b_1\}, Y_1)$ such that $X_1 \cup \{b_1\} \subseteq X \cup \{b\}$, the subset $Y_1 \cup \{b_1\}$ is closed, and $X_1 \cup \{b_1\}$ is minimal in $(X_1, \{b_1\}, Y_1)$. If $X_1 \cap B = \{b_c\}$ where b_c is removable with respect to B, then (2.3.9.1) holds, so the proposition holds in this case. Otherwise, $X_1 \cap B$ contains an element, b_2 say, that is not B-removable.

By Lemma 2.1.6, M has a vertical 3-separation $(X_2, \{b_2\}, Y_2)$ where $b_2 \in X_1$. Without loss of generality, let $b_1 \in Y_2$ where, due to Lemma 2.1.7, we can assume that $Y_2 \cup \{b_2\}$ is closed. By Lemma 2.1.10, each of $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ is non-empty, and $r((X_1 \cap X_2) \cup \{b_2\}) = 2$. We consider two subcases: $|Y_1 \cap X_2| \geq 2$ and $|Y_1 \cap X_2| = 1$.

2.3.9.2. *The proposition holds when* $|Y_1 \cap X_2| \geq 2$.

If $|Y_1 \cap X_2| \geq 2$, then, by Lemma 2.1.10(iii), $r((X_1 \cap Y_2) \cup \{b_1, b_2\}) = 2$. Let $L_1 = (X_1 \cap Y_2) \cup \{b_1\}$ and $L_2 = (X_1 \cap X_2) \cup \{b_2\}$. If $|L_2| \geq 4$, then, by Lemma 2.1.8, (i) holds. Similarly, if $|\mathrm{cl}(L_1)| \geq 4$, then L_1 contains at least two removable elements, and these elements are in X_1 since $Y_1 \cup \{b_1\}$ is closed, thereby satisfying (i). Hence, since $X_1 \cap Y_2$ is non-empty, we may assume that $|\operatorname{cl}(L_1)| = 3$ and $|L_2| \in \{2, 3\}$.

Let $X_1 \cap Y_2 = \{a\}$ and $c \in X_1 \cap X_2$. Note that $a \in E(M) - B$. If $|L_2| = 2$, then $|X_1 \cap X_2| = 1$, $X_1 = \{a, b_2, c\}$ is a triad and $\operatorname{cl}(L_1) = \{b_1, a, b_2\}$ is a triangle, so $\{b_1, a, b_2, c\}$ is a 4-element fan. If $c \in E(M) - B$, then $X_1 \cup \{b_1\}$ is a Type I fan; a contradiction. But if $c \in B$, then $X_1 \cup \{b_1\}$ is a Type II fan, in which case (v) holds.

Now suppose $|L_2| = 3$ and, in particular, $X_1 \cap X_2 = \{c, d\}$. Since $r(L_2) = 2$, we may assume, without loss of generality, that $d \in E(M) - B$. By Lemma 2.3.6, $\operatorname{co}(M\backslash d)$ and $\operatorname{co}(M\backslash c)$ are 3-connected. If $c \in E(M) - B$, then (i) holds. Furthermore, if $c \in B$, then (i) also holds as $\operatorname{si}(M/c)$ is 3-connected by Lemma 2.3.3. Thus, (2.3.9.2) holds.

It remains to prove that the proposition holds when $|Y_1 \cap X_2| = 1$. First, we show that, in such a situation, if there is an element of B in $X_1 \cap X_2$, then the proposition holds.

2.3.9.3. *If, for some $b_z \in X_1 \cap B$ such that $\operatorname{si}(M/b_z)$ is not 3-connected, $(X_z, \{b_z\}, Y_z)$ is a vertical 3-separation of M where $b_1 \in Y_z$, the set $Y_z \cup \{b_z\}$ is closed, $|Y_1 \cap X_z| = 1$, and there exists an element $p \in (X_1 \cap X_z) \cap B$, then* (i) *holds.*

By Lemma 2.1.10, $r((X_1 \cap X_z) \cup \{b_z\}) = 2$, so if $|(X_1 \cap X_z) \cup \{b_z\}| \geq 4$, then (i) holds, by Lemma 2.1.8. If $|X_1 \cap X_z| = 1$, then, as $b_1 \in Y_z$, the set X_z consists of two elements; a contradiction. So let $X_1 \cap X_z = \{p, q\}$, where $p \in B$ and $q \in E(M) - B$, and let $Y_1 \cap X_z = \{y\}$. First, suppose that $\operatorname{si}(M/p)$ is not 3-connected. Then, by Lemmas 2.1.6 and 2.1.7, there exists a vertical 3-separation $(X_p, \{p\}, Y_p)$ such that $b_1 \in Y_p$ and $Y_p \cup \{p\}$ is closed. By Lemma 2.1.10, $(X_1 \cap X_p) \cup \{p\}$ is a rank-2 set, and if $|Y_1 \cap X_p| \geq 2$, then $r((X_1 \cap Y_p) \cup \{b_1, p\}) = 2$. If, indeed, $|Y_1 \cap X_p| \geq 2$, then $r(X_1) = 3$ and it follows, by Lemmas 2.3.3 and 2.3.6, that p and q are removable, satisfying (i). So assume that $|Y_1 \cap X_p| = 1$. Then $(X_1 \cap X_p) \cup \{p\}$ is a rank-2 set of at least three elements. If this set has four or more elements, then (i) holds by Lemma 2.1.8, so assume that $|X_1 \cap X_p| = 2$. Now $(X_1 \cap X_p) \cup \{p\}$ is a triangle contained in X_1, but since $Y_z \cup \{b_z\}$ is closed, this triangle contains q. Then either $(X_1 \cap X_p) \cup \{b_z, p\}$ is a rank-2 set of four elements, so (i) holds by Lemma 2.1.8, or $X_1 \cap X_p = \{q, b_z\}$. Since X_p is a triad, if $Y_1 \cap X_p = \{y\}$, then $\{y, p, q, b_z\}$ is a 4-element cosegment, and $\operatorname{si}(M/p)$ is 3-connected by the dual of Lemma 2.1.8; a contradiction. So $Y_1 \cap X_p = \{y'\}$

where $y' \neq y$, and $\{y, y'\} \subseteq \text{cl}^*(X_1)$. But, recalling that $b_1 \in \text{cl}(X_1)$, this contradicts Lemma 2.3.7.

Now suppose that $\text{si}(M/p)$ is 3-connected. If $\text{co}(M\backslash q)$ is also 3-connected, then (i) holds, so assume this is not the case. Now, $\text{si}(M/p) \not\cong \text{co}(M\backslash q)$, so, by Lemma 2.3.8, either p is contained in a triangle other than $\{p, q, b_z\}$, or q is contained in a triad other than $\{p, q, y\}$. Consider the former; by orthogonality and since $Y_z \cup \{b_z\}$ is closed, $\{p, y\}$ is contained in a triangle T. Let $T - \{p, y\} = \{y'\}$. Note that $y \in B$, otherwise $X_z \cup \{b_z\}$ is a Type I fan. Since $Y_1 \cup \{b_1\}$ is closed, and due to the rank of T, $y' \in X_1 \cap (E(M) - B)$. By Lemma 2.3.5, y' is removable so (i) holds. Now consider when q is in a triad T^* other than $\{p, q, y\}$. By orthogonality, T^* contains p or b_z. If $\{q, b_z\}$ is contained in T^*, then p is a spoke element and an end of a 4-element fan, so $\text{si}(M/p)$ is not 3-connected by Lemma 2.3.5; a contradiction. So assume that $\{p, q\}$ is contained in T^*. Then $T^* \cup \{y\}$ is a cosegment, and it contains a triad that intersects $\{b_z, p, q\}$ in a single element; a contradiction. Thus (2.3.9.3) holds.

2.3.9.4. *The proposition holds when* $|Y_1 \cap X_2| = 1$.

As $|X_2| \geq 3$ and $b_1 \notin X_2$, it follows that $|X_1 \cap X_2| \geq 2$. By Lemma 2.1.10, $r((X_1 \cap X_2) \cup \{b_2\}) = 2$. If $|X_1 \cap X_2| \geq 3$, then (i) holds by Lemma 2.1.8. Therefore, we may assume that $|X_1 \cap X_2| = 2$. At most one element in $X_1 \cap X_2$ is in B, but if there is such an element, then (i) holds by (2.3.9.3). So let $X_1 \cap X_2 = \{p, q\}$, where $\{p, q\} \subseteq E(M) - B$, and let $Y_1 \cap X_2 = \{y\}$ where $y \in B$.

We first show that either (i) holds, or there exists an element $b_3 \in X_1 \cap Y_2$ that is not removable with respect to B. If $r(X_1) = 3$, then p and q are removable by Lemma 2.3.6, satisfying (i). So assume that $r(X_1) \geq 4$, in which case $r(Y_1 \cup \{b_1\}) \leq r(M) - 2$, so $|X_1 \cap B| \geq 2$. Let $b_3 \in X_1 \cap B - \{b_2\}$, in which case $b_3 \in Y_2$. If $\text{si}(M/b_3)$ is not 3-connected, we have one of the desired outcomes. So assume that b_3 is removable. If either p or q is also removable, then (i) holds. Suppose neither p nor q is removable. Then, by Bixby's Lemma, $\text{si}(M/p)$ is 3-connected, so $\text{si}(M/p) \not\cong \text{co}(M\backslash q)$. It follows, by Lemma 2.3.8, that either p is contained in a triangle other than $\{p, q, b_2\}$ or q is contained in a triad other than $\{p, q, y\}$. If the latter, then, as in the last paragraph of (2.3.9.3), this leads to a contradiction. If the former, then by orthogonality and since $Y_2 \cup \{b_2\}$ is closed, such a triangle is $\{p, y, y'\}$

where $y' \in X_1$ since $Y_1 \cup \{b_1\}$ is closed. Furthermore, (y', y, p, q, b_2) is a fan ordering. By Lemma 2.3.5, if $y' \in B$, then y' is not removable, and choosing $b_3 = y'$ we have a desired outcome. So assume that $y' \in E(M) - B$, in which case y' is removable, thereby satisfying (i).

Now, by Lemmas 2.1.6 and 2.1.7, there exists a vertical 3-separation $(X_3, \{b_3\}, Y_3)$ such that $b_1 \in Y_3$ and $Y_3 \cup \{b_3\}$ is closed. By Lemma 2.1.10, $(X_1 \cap X_3) \cup \{b_3\}$ is a rank-2 set, and if $|Y_1 \cap X_3| \geq 2$, then $r((X_1 \cap Y_3) \cup \{b_1, b_3\}) = 2$. But if the latter holds, then p and q are removable by Lemma 2.3.6, satisfying (i). Furthermore, if $|(X_1 \cap X_3) \cup \{b_3\}| \geq 4$, then (i) holds by Lemma 2.1.8. So we may assume that $|Y_1 \cap X_3| = 1$ and $|X_1 \cap X_3| = 2$. Since X_2 and X_3 are triads, each with two elements contained in X_1, both y and the single element in $Y_1 \cap X_3$ are in the coclosure of X_1. But $b_1 \in \mathrm{cl}(X_1)$, so by Lemma 2.3.7, $Y_1 \cap X_3 = \{y\}$. If there exists an element $p' \in (X_1 \cap X_3) \cap B$, then (i) holds by (2.3.9.3). It remains to consider when $X_1 \cap X_3 \subseteq E(M) - B$. If $\{p, q\} \subseteq X_3$, then $\{p, q, b_3\}$ is a triangle, but $\{p, q, b_2\}$ is also a triangle, so p and q are removable, by Lemma 2.1.8, satisfying (i). Otherwise, since $Y_3 \cup \{b_3\}$ is closed, $\{p, q, b_2\} \subseteq Y_3$. Let $X_1 \cap X_3 = \{p', q'\}$. The two triads $\{p, q, y\}$ and $\{p', q', y\}$ intersect only at y, so $\{p, q, y, q', p'\}$ is a corank-3 set. But this set contains four cobasis elements; a contradiction. So (2.3.9.4) holds.

We deduce that the proposition holds. □

Corollary 2.3.10. *Let M be a 3-connected matroid such that $|E(M)| \geq 2$, and let B be a basis of M. Then, either*

(i) *M has at least two B-removable elements, or*

(ii) *M has a Type I fan F relative to B where each $f \in F$ is not B-removable.*

Proof. If every element $e \in E(M)$ is B-removable, then the corollary holds. Therefore, by duality, we may assume that there exists an element $b \in B$ such that $\mathrm{si}(M/b)$ is not 3-connected. By Lemmas 2.1.6 and 2.1.7, there exists a vertical 3-separation $(X, \{b\}, Y)$ of M such that $Y \cup \{b\}$ is closed, and thus $|E(M)| \geq 7$. By Proposition 2.3.9, either the corollary holds, or M has a fan F, where F is either a Type I fan whose internal elements are contained in X, or a Type II fan contained in $X \cup \{b\}$. We will show that when M has such a fan F, either the corollary holds, or there is a B-removable element in X.

Let $F = \{f_1, f_2, f_3, f_4\}$ where F has fan ordering (f_1, f_2, f_3, f_4) such that $\{f_1, f_2, f_3\}$ is a triangle. Suppose that F is a Type II fan, so $F \cap B = \{f_1, f_3, f_4\}$. Then $f_4 \in X$ is a B-removable element, by Lemma 2.3.5. Now suppose that F is a Type I fan. By Lemma 2.3.5, f_1 and f_4 are not B-removable. If the only triangle containing f_3 is $\{f_1, f_2, f_3\}$ and the only triad containing f_2 is $\{f_2, f_3, f_4\}$, then, by Lemma 2.3.8, either f_2 and f_3 are both B-removable, in which case (i) holds, or neither f_2 nor f_3 is B-removable, in which case (ii) holds. Suppose that f_3 is contained in a triangle T distinct from $\{f_1, f_2, f_3\}$. By orthogonality, T contains f_2 or f_4. When $f_2 \in T$, the set $T \cup \{f_1\}$ has rank two, so contains two B-removable elements by Lemma 2.1.8. When $f_4 \in T$, the set $T \cup \{f_1, f_2\}$ is a 5-element fan, so f_2 is not B-removable, by Lemma 2.3.5. If f_3 is not B-removable, (ii) holds. Otherwise, X contains the B-removable element f_3.

There exists a vertical 3-separation $(Y_2, \{b\}, X_2)$ of M such that $X_2 \cup \{b\}$ is closed, $X \subseteq X_2$ and $Y_2 \subseteq Y$, by Lemma 2.1.7. By a second application of Proposition 2.3.9, either the corollary holds, or M has a fan F, where F is either a Type I fan whose internal elements are contained in Y_2, or a Type II fan contained in $Y_2 \cup \{b\}$. By the same argument as in the previous paragraph, either the corollary holds, or Y_2 contains a B-removable element. In the exceptional case, X and Y_2 each contain a B-removable element, where $Y_2 \subseteq E(M) - X$, so the corollary holds. □

Corollary 2.3.11. *Let M be a 3-connected matroid, where $|E(M)| \geq 3$, and let B be a basis of M. Suppose that M has no Type I fans relative to B. Then M has at least three B-removable elements.*

Proof. If every element $e \in E(M)$ is B-removable, then the corollary holds. Therefore, by duality, we may assume that there exists an element $b \in B$ such that $\mathrm{si}(M/b)$ is not 3-connected. By Lemmas 2.1.6 and 2.1.7, there exists a vertical 3-separation $(X, \{b\}, Y)$ of M such that $Y \cup \{b\}$ is closed. By Lemma 2.1.7, there also exists a vertical 3-separation $(Y', \{b\}, X')$ of M such that $X' \cup \{b\}$ is closed, $X \subseteq X'$, and $Y' \subseteq Y$. By Proposition 2.3.9, X and Y' each contain a removable element, where if (v) holds, the Type II fan contains a removable element by Lemma 2.3.5. Thus, if Proposition 2.3.9(i) or Proposition 2.3.9(iii) holds for either vertical 3-separation, the corollary holds.

We may now assume that either Proposition 2.3.9(ii) or Proposition 2.3.9(v) holds for each of the vertical 3-separations. When Proposition 2.3.9(ii) holds for either vertical 3-separation, there are two removable elements $s_1, s_2 \in B$, by the dual of Lemma 2.1.8. On the other hand, if Proposition 2.3.9(v) holds for both vertical 3-separations, again there are two removable elements $s_1, s_2 \in B$, by Lemma 2.3.5. There exists an element $b^* \in E(M) - B$, as $|E(M)| \geq 3$ and M is 3-connected. If b^* is removable, the corollary holds. Otherwise, by the dual of Lemma 2.1.6, there is a vertical 3-separation $(P, \{b^*\}, Q)$ in M^*. Next we apply Proposition 2.3.9 to $(P, \{b^*\}, Q)$. If Proposition 2.3.9(i) or Proposition 2.3.9(iii) holds, then the corollary holds, noting in the former case that there is also a removable element in Q by an application of Proposition 2.3.9 to $(Q, \{b^*\}, P)$. But when Proposition 2.3.9(ii) or Proposition 2.3.9(v) holds for $(P, \{b^*\}, Q)$, there exists a removable element $s_1^* \in E(M) - B$ that is distinct from s_1 and s_2. Thus the corollary holds. □

Corollary 2.3.12. *Let M be a 3-connected matroid, where $|E(M)| \geq 4$, and let B be a basis of M. Suppose that M has no Type I or Type II fans relative to B. Then M has at least four B-removable elements.*

Proof. If every element $e \in E(M)$ is removable with respect to B, then the corollary holds. Therefore, by duality, we may assume that there exists an element $b \in B$ such that $\mathrm{si}(M/b)$ is not 3-connected. By Lemmas 2.1.6 and 2.1.7, there exists a vertical 3-separation $(X, \{b\}, Y)$ of M such that $Y \cup \{b\}$ is closed. There also exists a vertical 3-separation $(X_2, \{b\}, Y_2)$ of M such that $X_2 \cup \{b\}$ is closed, $X \subseteq X_2$ and $Y_2 \subseteq Y$.

We can now apply Proposition 2.3.9 using each of the two vertical 3-separations in turn, where Proposition 2.3.9(iv) and Proposition 2.3.9(v) cannot hold since M has no Type I or Type II fans. If Proposition 2.3.9(iii) holds for $(X, \{b\}, Y)$, then there exist distinct removable elements $s_1 \in X$ and $s_2, s_3 \in \mathrm{cl}(X)$. By an application of Proposition 2.3.9 to $(Y_2, \{b\}, X_2)$, there is at least one removable element in Y_2, and $\{s_2, s_3\} \subseteq X_2$ since $X_2 \cup \{b\}$ is closed, so the corollary holds in this case. By symmetry, we can now assume that Proposition 2.3.9(iii) does not hold for either vertical 3-separation. If Proposition 2.3.9(i) holds for both vertical 3-separations, then clearly the corollary holds, so it remains to consider when Proposition 2.3.9(ii) holds for at least one of the vertical 3-separations.

Now we may assume there exist a vertical 3-separation $(X', \{b'\}, Y')$

and removable elements $s_1 \in X'$ and $s_2 \in \text{cl}^*(X')$, where $X' \cup \{s_2\}$ is a 4-element cosegment. If $b' \in \text{cl}^*(X' \cup \{s_2\})$, then b' is removable by the dual of Lemma 2.1.8; a contradiction. So $b' \in \text{cl}(Y' - \{s_2\})$, by Lemma 2.1.3. It follows, by Lemma 2.1.4, that when $r(Y' - \{s_2\}) \geq 3$, the partition $(X' \cup \{s_2\}, \{b'\}, Y' - \{s_2\})$ is a vertical 3-separation. Then, by an application of Proposition 2.3.9 to $(Y' - \{s_2\}, \{b'\}, X' \cup \{s_2\})$, the corollary holds unless there exists an element $s_2' \in (X' \cup \{s_2\}) \cap B$ such that $(Y' - \{s_2\}) \cup \{s_2', b'\}$ contains a 4-element cosegment. This cosegment must contain s_2' and cannot contain b', by the dual of Lemma 2.1.8, as it is not removable. Thus, the two 4-element cosegments intersect at a single element s_2', so the union of these two cosegments has corank three. But $s_2' \in B$, so this union contains four elements of the cobasis $E(M) - B$; a contradiction. Now consider the case where $r(Y' - \{s_2\}) = 2$. If $|Y' - \{s_2\}| \geq 3$, then, recalling $b' \in \text{cl}(Y' - \{s_2\})$, there are two elements in $Y' - \{s_2\}$ that are removable by Lemma 2.1.8, so the corollary holds. It remains to consider when $|Y'| = 3$. Since $r(M) = 4$, precisely one element of $Y' - \{s_2\}$ is in B. But then $Y' \cup \{b_1\}$ is a Type II fan; a contradiction. So the corollary holds. □

2.3.3 An example with four removable elements

In this section, we give an example to demonstrate that the bound in Corollary 2.3.12 is sharp in the sense that a 3-connected matroid M with a basis B and no 4-element fans can have precisely four B-removable elements.

This example is similar to the one in Section 2.2.2, but extra care needs to be taken to ensure there are only two B-removable elements at each "end". Let $k \geq 4$. We will describe how to construct a matroid M_k of rank $k + 2$. The matroid M_6 is illustrated in Figure 2.2. Start with the free $(k+2)$-element matroid $U_{k+2,k+2}$ with ground set $\{t, b_0, b_1, \ldots, b_k\}$. For each $i \in \{3, 4, \ldots, k-2\} \cup \{1, k\}$, freely add c_i to the flat $\{t, b_{i-1}, b_i\}$. For each $i \in \{1, 2, k-1, k\}$, freely add x_i to the flat $\{b_{i-1}, b_i\}$. Finally, delete b_1 and b_{k-1} to obtain M_k. We fix a basis $B = \{b_0, x_1, b_2, b_3, \ldots, b_{k-2}, x_k, b_k, t\}$ for this matroid.

Note that M_k has no triangles, so it has no 4-element fans. We now show that the B-removable elements are $\{b_0, x_1, x_k, b_k\}$. Due to the lack of triangles, $\text{si}(M_k/b) \cong M_k/b$ for each $b \in B$. Thus, it is evident that $\text{si}(M_k/b_i)$ is not 3-connected for each $b_i \in \{b_2, b_3, \ldots, b_{k-2}\}$. Moreover, t is not removable as $(\{b_0, x_1, c_1\}, E(M_k) - \{b_0, x_1, c_1, t\})$ is a 2-separation of $\text{si}(M_k/t)$, for

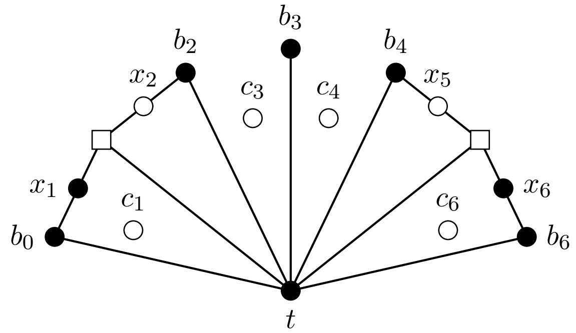


Figure 2.2: A 3-connected rank-8 matroid M_6 with precisely four B-removable elements: b_0, x_1, x_6 and b_6.

example. On the other hand, b_0, x_1, x_k and b_k are B-removable. Each c_i, for $i \in \{3,4,\ldots,k-2\}$, is in two triads, so $\mathrm{co}(M_k\backslash c_i) \cong M_k\backslash c_i/b_{i-1}/b_i$, which is not 3-connected, as $\{b_0,x_1,c_1,x_2\} \cup \{b_2,\ldots,b_{i-2}\} \cup \{c_3,\ldots,c_{i-1}\}$ is 2-separating in $\mathrm{co}(M_k\backslash c_i)$. Finally, $\{b_0,c_1,x_1,x_2\}$ is a cosegment in M_k, so $M_k\backslash c_1$, or $M_k\backslash x_2$, has a series class of three elements. But $M_k\backslash c_1/b_0/x_1$, or $M_k\backslash x_2/b_0/x_1$, has a parallel pair $\{b_2,x_2\}$, or $\{t,c_1\}$ respectively, so c_1 and x_2 are not B-removable. By symmetry, c_k and x_{k-1} are also not B-removable.

Although this example demonstrates that the bound in Corollary 2.3.12 is sharp when considering matroids with no Type I or Type II fans, it is unresolved whether the bound in Corollary 2.3.11 is sharp when considering matroids that may have Type II fans. In other words, does there exist a 3-connected matroid M with a basis B, no Type I fans relative to B, and precisely three B-removable elements? We leave this as an open question.

Chapter 3

A Splitter Theorem

In this chapter, we consider the existence of elements that can be removed, relative to a fixed basis, and also retain an N-minor. Recall that Oxley et al. (2008*a*) showed that, for a 3-connected matroid M with basis B and no 4-element fans, there is at at least one element that is strictly B-removable and (N, B)-robust. We give an example, in Section 3.1, to illustrate that we cannot guarantee more than one such element. However, relaxing our requirements slightly, we can guarantee the presence of two (N, B)-strong elements. This is the titular result of the chapter and is proved in Section 3.2. In the same section, we provide some examples to demonstrate that this result is, in a sense, best possible. We close the chapter with Section 3.3, where we consider the structure of matroids with the minimum number of (N, B)-strong elements. In particular, we prove that if P is the set of (N, B)-robust elements in such a matroid, then $(P, E(M)-P)$ is a sequential 3-separation.

3.1 An example with one strictly removable robust element

In this section, we describe the construction of a matroid, with arbitrary rank, that has precisely one element that is both strictly B-removable and (N, B)-robust.

Let $k \geq 1$. We describe how to construct a matroid M_k of rank $k+3$ with a single strictly B-removable (N, B)-robust element. In particular, M_3 is given in Figure 3.1, where $N = F_7^-$. Although we use F_7^- as the 3-

connected minor N in the construction, any sufficiently structured matroid with a triangle $\{t, t_1, t_2\}$ would do.

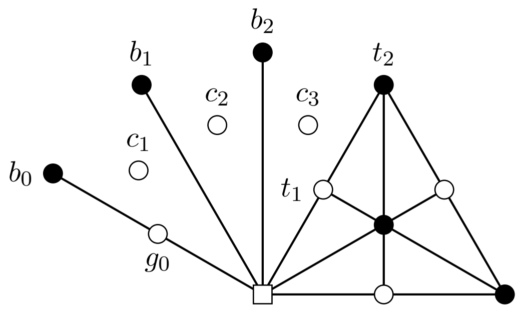


Figure 3.1: A 3-connected rank-6 matroid M_3 with only one element, b_0, that is both strictly B-removable and (F_7^-, B)-robust.

The matroid M_k is constructed as follows. Let $U_{k,k}$ be the free k-element matroid $U_{k,k}$ with ground set $\{b_0, b_1, \ldots, b_{k-1}\}$, and let F_7^- be the non-Fano matroid containing a triangle $\{t, t_1, t_2\}$. Construct the direct sum $U_{k,k} \oplus F_7^-$. For each $i \in \{1, 2, \ldots, k-1\}$, freely add c_i to the flat $\{t, b_{i-1}, b_i\}$, and freely add c_k to the flat $\{b_{k-1}, t, t_1, t_2\}$. We also freely add g_0 to the flat $\{b_0, t\}$. Finally, we delete t to obtain M_k.

Let A be a basis of $F_7^- \backslash t$. Then $B = A \cup \{b_0, b_1, \ldots, b_{k-1}\}$ is a basis for M_k. An element $e \in E(M_k)$ is (F_7^-, B)-robust if and only if $e \in \{b_0, b_1, \ldots, b_{k-1}\} \cup \{c_1, c_2 \ldots, c_k\}$. Thus, M_k has $2k$ elements that are (F_7^-, B)-robust, where $2k \geq 2$. Each c_i is in at least one triad, for $i \in \{1, 2, \ldots, k\}$, so $M_k \backslash c_i$ is not 3-connected. Moreover, each of b_1 and b_2 opens up a 2-separation when contracted, so these elements are not strictly B-removable. On the other hand, M_k / b_0 is 3-connected. Hence, b_0 is the only strictly B-removable (F_7^-, B)-robust element in M_k.

3.2 The existence of strong elements

Let M be a 3-connected matroid, let B be a basis of M, and let N be a 3-connected minor of M. Recall that an element $e \in E(M)$ is (N, B)-*strong* if either

(i) $e \in B$, and $\mathrm{si}(M/e)$ is 3-connected and has an N-minor, or

(ii) $e \in E(M) - B$, and $\mathrm{co}(M \backslash e)$ is 3-connected and has an N-minor.

In Section 3.2.1, we prove Theorem 3.2.10, a generalisation of Theorem 2.0.2. Informally, this theorem says we can find two (N, B)-strong elements in M provided that M has at least five elements, at least two (N, B)-robust elements, and no 4-element fans with a specific labelling with respect to B. In the remainder of the section, we give a series of examples to demonstrate that this theorem is the best we can hope for, in three ways: Section 3.2.2 shows that the existence of two (N, B)-robust elements is necessary; Section 3.2.3 shows that we cannot guarantee more than two (N, B)-strong elements; and Section 3.2.4 shows that if Type I fans are present, we cannot guarantee any (N, B)-strong elements at all.

3.2.1 The proof of Theorem 2.0.4

The proofs of the next two lemmas are straightforward.

Lemma 3.2.1. *Let e and f be distinct elements of a 3-connected matroid M, and suppose that $\mathrm{si}(M/e)$ is 3-connected. Then either*

(i) *$M/e\backslash f$ is connected, or*

(ii) *$\mathrm{si}(M/e) \cong U_{2,3}$ and M has no triangle containing $\{e, f\}$.*

Moreover, if no non-trivial parallel class of M/e contains f, then $M/e/f$ is connected.

Lemma 3.2.2. *Let (X, Y) be a 2-separation of a connected matroid M and let N be a 3-connected minor of M. Then $\{X, Y\}$ has a member U such that $|U \cap E(N)| \leq 1$. Moreover, if $u \in U$, then*

(i) *M/u has an N-minor if M/u is connected, and*

(ii) *$M\backslash u$ has an N-minor if $M\backslash u$ is connected.*

In the arguments that follow, we initially restrict our attention to a 3-connected N-minor with $|E(N)| \geq 4$, so that N is simple and cosimple. The next lemma illustrates this. We handle the case where $|E(N)| \leq 3$ in Lemma 3.2.8.

Lemma 3.2.3. *Let N be a 3-connected matroid such that $|E(N)| \geq 4$. If M has an N-minor, then $\mathrm{si}(M)$ has an N-minor.*

Proof. Since $|E(N)| \geq 4$, the matroid N is simple. Thus, removing parallel elements or loops from M cannot destroy the N-minor, so the lemma holds. □

Lemma 3.2.4. *Let N be a 3-connected minor of a 3-connected matroid M with $|E(N)| \geq 4$. Let $(X, \{z\}, Y)$ be a vertical 3-separation of M such that M/z has an N-minor, where $Y \cup \{z\}$ is closed and $|X \cap E(N)| \leq 1$. If $s \in \mathrm{cl}^*(X) - X$, then $(X', \{z\}, Y') = (X \cup \{s\}, \{z\}, Y - \{s\})$ is a vertical 3-separation where $Y' \cup \{z\}$ is closed and $|X' \cap E(N)| \leq 1$.*

Proof. Since X and $X \cup \{z\}$ are exactly 3-separating in M, and $s \in \mathrm{cl}^*(X)$, it follows, by Lemma 2.1.4(i), that X' and $X' \cup \{z\}$ are 3-separating. In particular, as $r(Y') \geq 2$, the sets X' and $X' \cup \{z\}$ are exactly 3-separating. By Lemma 2.1.4(ii), $z \in \mathrm{cl}(X')$ implies that $z \in \mathrm{cl}(Y')$. Now, since $|Y'| \geq 2$, the partition (X', Y') is a 2-separation of M/z. Since $s \in \mathrm{cl}^*(X)$, we have $s \notin \mathrm{cl}(Y')$ by Lemma 2.1.3. Therefore, $Y' \cup \{z\}$ is closed in M. By Lemma 3.2.2, either $|X' \cap E(N)| \leq 1$ or $|Y' \cap E(N)| \leq 1$. Suppose that $|X' \cap E(N)| \geq 2$. Then $|X \cap E(N)| = 1$ and $|Y \cap E(N)| \leq 2$, so $|E(N)| \leq 3$; a contradiction. So $|X' \cap E(N)| \leq 1$.

To see that $r(Y') \geq 3$, suppose that $r(Y') = 2$. Then $Y' \cup \{z\}$ is a line of at least three elements. But $|E(N)| \geq 4$, so N is simple, thus $\mathrm{si}(M/z)$ has an N-minor. Since $|X' \cap E(N)| \leq 1$, the matroid N is isomorphic to $U_{1,1}$ or $U_{1,2}$; a contradiction. Therefore, $r(Y') \geq 3$ and the lemma holds. □

Let M be a 3-connected matroid with a 3-connected minor N. An element x of M is *doubly N-labelled* if $M \backslash x$ has an N-minor and M/x has an N-minor. Now, let M_1 and M_2 be matroids, each with at least two elements, such that $E(M_1) \cap E(M_2) = \{p\}$ and p is not a loop or a coloop of either M_1 or M_2. Then the *2-sum* of M_1 and M_2 with *basepoint* p is the matroid whose ground set is $(E(M_1) \cup E(M_2)) - \{p\}$ and whose set of circuits consists of all circuits of $M_1 \backslash p$ together with all circuits of $M_2 \backslash p$ and all sets of the form $(C_1 \cup C_2) - \{p\}$ where each C_i is a circuit of M_i containing p.

Lemma 3.2.5. *Let N be a 3-connected minor of a 3-connected matroid M. Let $(X, \{z\}, Y)$ be a vertical 3-separation of M such that M/z has an N-minor, where $|X \cap E(N)| \leq 1$. If $Y \cup \{z\}$ is closed, then there is at most one element of X that is not doubly N-labelled. Moreover, if such an element x exists, then $x \in \mathrm{cl}^*(Y)$ and $z \in \mathrm{cl}(X - \{x\})$.*

Proof. The matroid M/z is the 2-sum of two matroids, M_X and M_Y with basepoint z' say. Note that $(M/z)|X = M_X\backslash z'$ and $(M/z)|Y = M_Y\backslash z'$. Let $x \in X$. Let C_x and C_x^* be a maximum-sized circuit and a maximum-sized cocircuit of M_X containing $\{x, z'\}$, respectively. If $|C_x| > 2$, then $M/z/x$, and hence M/x, has an N-minor. Dually, if $|C_x^*| > 2$, then $M\backslash x$ has an N-minor. Thus x is doubly N-labelled unless $|C_x| = 2$ or $|C_x^*| = 2$. But if $|C_x| = 2$, then $x \in \mathrm{cl}_{M/z}(Y)$, so $x \in \mathrm{cl}_M(Y \cup \{z\})$, contradicting the fact that $Y \cup \{z\}$ is closed. We deduce that x is doubly N-labelled unless $|C_x^*| = 2$. Moreover, $E(M_X) - \{z'\}$ cannot contain distinct elements s and t that are not doubly N-labelled otherwise $\{z', s, t\}$ is contained in a series class of M_X and so $\{s, t\}$ is a cocircuit of M; a contradiction. Thus X contains at most one element that is not doubly N-labelled. Moreover, when such an element x exists, $\{x, z'\}$ is a cocircuit of M_X, so $x \in \mathrm{cl}^*_{M/z}(Y)$ and $x \notin \mathrm{cl}_{M/z}(X - \{x\})$. Hence, $x \in \mathrm{cl}^*_M(Y)$ and $x \notin \mathrm{cl}_M((X - \{x\}) \cup \{z\})$. As $z \in \mathrm{cl}_M(X)$, it follows from the MacLane-Steinitz exchange condition that $z \in \mathrm{cl}_M(X - \{x\})$. □

Lemma 3.2.6. *Let M be a 3-connected matroid, let N be a 3-connected minor of M such that $|E(N)| \geq 4$, and let B be a basis of M with an element $b \in B$ such that b is (N, B)-robust, but not (N, B)-strong. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M such that $Y \cup \{b\}$ is closed and $|X \cap E(N)| \leq 1$. If there is a B-removable element $s \in X$, then s is an (N, B)-strong element.*

Proof. Let s be a B-removable element of X. By Lemma 3.2.5, at most one element in X is not doubly N-labelled, so we may assume s is the only element that is not (N, B)-robust in X, in which case $s \in \mathrm{cl}^*(Y)$ and $b \in \mathrm{cl}(X - \{s\})$. Therefore, $(X - \{s\}, Y \cup \{b\})$ is a 2-separation of $M\backslash s$. If $s \in E(M) - B$, then $\mathrm{co}(M\backslash s)$ is 3-connected, so $X - \{s\}$ is a series class in $M\backslash s$. But $b \in \mathrm{cl}(X - \{s\})$, so $\mathrm{co}(M\backslash s)$ contains a non-trivial parallel class; a contradiction. Thus $s \in B$.

Suppose that s and b are contained in a triangle $\{s, b, q\}$. If $q \in X$, then $s \in \mathrm{cl}((X - \{s\}) \cup \{b\})$, so $s \notin \mathrm{cl}^*(Y)$ by Lemma 2.1.3; a contradiction. But if $q \in Y$, then $s \in \mathrm{cl}(Y \cup \{b\}) = \mathrm{cl}(Y)$. Then $\{b, s\} \subseteq \mathrm{cl}(Y) - Y$ and $s \in \mathrm{cl}^*(Y) - Y$, contradicting Lemma 2.3.7. Since s and b are not contained in a triangle of M, no non-trivial parallel class of M/s contains b, and thus by Lemma 3.2.1, $M/s/b$ is connected. Since $|X \cap E(N)| \leq 1$ and $s \in X$, by

Lemma 3.2.2, $M/s/b$ has an N-minor and, therefore, M/s has an N-minor. Thus, by Lemma 3.2.3, the lemma holds. □

When M has no Type I fans, $|E(N)| \geq 4$, and at least one element $z \in E(M)$ is not B-removable, then there is a vertical 3-separation $(X, \{z\}, Y)$ of M or M^* such that "most" of the elements of N are contained in Y, by Lemmas 2.1.6 and 3.2.2. The next proposition demonstrates that, in this case, we can describe the location of the (N, B)-strong elements with respect to this vertical 3-separation.

Proposition 3.2.7. *Let M be a 3-connected matroid, let N be a 3-connected minor of M such that $|E(N)| \geq 4$, and let B be a basis of M. Suppose there exists an element $b \in B$ that is (N, B)-robust but not (N, B)-strong. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M such that $|X \cap E(N)| \leq 1$. Then one of the following holds:*

(i) *there exist distinct (N, B)-strong elements $s_1, s_2 \in X$, or*

(ii) *there exist distinct (N, B)-strong elements $s_1 \in X$ and $s_2 \in \text{cl}^*(X) \cap B$, or*

(iii) *there exist distinct (N, B)-strong elements $s_1 \in X$ and $s_2, s_3 \in \text{cl}(X) \cap (E(M) - B)$, or*

(iv) *M has a Type I fan F relative to B where the internal elements of F are contained in X, or*

(v) *$X \cup \{b\}$ contains a Type II fan F and an (N, B)-strong element $s_2 \in F \cap B$.*

Proof. By Lemma 2.1.7, there exists a vertical 3-separation $(X', \{b\}, Y')$ such that $Y' \cup \{b\}$ is closed and $X' \subseteq X$. If the proposition holds for the vertical 3-separation $(X', \{b\}, Y')$, then clearly it holds for the vertical 3-separation $(X, \{b\}, Y)$; so we may assume that $Y \cup \{b\}$ is closed. If $X \cup \{b\}$ contains a Type II fan F, then, by Lemma 2.3.5, there exists a removable element in $F \cap B$. By Lemma 3.2.6, such a removable element is (N, B)-strong, satisfying (v).

Now assume that $X \cup \{b\}$ does not contain a Type II fan. By Proposition 2.3.9, either (iv) holds, or there is an element $s_1 \in X$ and either a distinct element $s_2 \in X$, a distinct element $s_2 \in \text{cl}^*(X) \cap B$, or distinct elements $s_2, s_3 \in \mathrm{cl}(X) \cap (E(M) - B)$, where each s_i is B-removable for $i \in \{1, 2, 3\}$. By Lemma 3.2.6, the element s_1 is (N, B)-strong, and if $s_i \in X$ for $i \in \{2, 3\}$, then s_i is also (N, B)-strong, in which case (i) holds. Consider the case where $s_2 \in \mathrm{cl}^*(X) \cap B$. By Lemma 3.2.4, $(X \cup \{s_2\}, \{b\}, Y - \{s_2\})$ is a vertical 3-separation where $|(X \cup \{s_2\}) \cap E(N)| \leq 1$ and $(Y - \{s_2\}) \cup \{b\}$ is closed. By Lemma 3.2.6, s_2 is (N, B)-strong, so (ii) holds. It remains to consider the case where $s_2, s_3 \in (\mathrm{cl}(X) - X) \cap (E(M) - B)$. Now, $\{b, s_2, s_3\} \subseteq \mathrm{cl}(X) \cap \mathrm{cl}(Y)$, and, by submodularity, $r(\mathrm{cl}(X) \cap \mathrm{cl}(Y)) \leq 2$, so $r(\{b, s_2, s_3\}) = 2$. The matroid M/b has an N-minor, and N has no 2-circuits, but s_2 and s_3 are parallel elements in M/b. It follows that $M/b\backslash s_2$ and $M/b\backslash s_3$ have N-minors, so s_2 and s_3 are (N, B)-strong by Lemma 3.2.3, satisfying (iii). We deduce that the proposition holds. □

It remains to consider two "edge cases": when $|E(N)| \leq 3$, and when M has a Type I fan relative to B. First, we examine the case where the N-minor is small.

Lemma 3.2.8. *Let M be a 3-connected matroid with $|E(M)| \geq 5$, let B be a basis of M, and suppose that M has a 3-connected N-minor such that $|E(N)| \leq 3$. If $s \in E(M)$ is B-removable, then either*

(i) *s is an (N, B)-strong element, or*

(ii) *there exist distinct (N, B)-strong elements $s_1, s_2 \in E(M)$, and at least one of the following holds:*

  (a) *$r(M) = 2$,*

  (b) *$r^*(M) = 2$,*

  (c) *$s \in B$ and $\mathrm{si}(M/s) \cong U_{2,3}$, or*

  (d) *$s \in E(M) - B$ and $\mathrm{co}(M\backslash s) \cong U_{1,3}$.*

Proof. Since $|E(N)| \leq 3$, the matroid N is a minor of $U_{1,3}$ or $U_{2,3}$. By duality, we may assume that N is a minor of $U_{2,3}$. First, assume that $s \in B$, in which case $\mathrm{si}(M/s)$ is 3-connected. If $\mathrm{si}(M/s)$ has a $U_{2,3}$-minor, then (i) holds, so assume that $\mathrm{si}(M/s)$ does not have such a minor. Then $r(M) = 2$. In particular, $M \cong U_{2,n}$, where $n \geq 5$, in which case (ii) holds by Lemma 2.1.8.

Now assume that $s \in E(M) - B$, and so $\mathrm{co}(M\backslash s)$ is 3-connected. If $\mathrm{co}(M\backslash s)$ has a circuit of at least three elements, it has a $U_{2,3}$-minor, thus

(i) holds. Assuming otherwise, first consider the case where $\text{co}(M\backslash s)$ does not have a circuit of one or two elements. Then $\text{co}(M\backslash s) \cong U_{1,1}$, so $\text{si}(M^*/s) \cong U_{0,1}$; a contradiction. Consider the case where $\text{co}(M\backslash s)$ has a loop or a 2-circuit. If $\text{co}(M\backslash s)$ has a loop, then $\text{co}(M\backslash s) \cong U_{0,1}$. That is, $M^* \cong U_{2,n}$, where $n \geq 5$. For each element $e \in E(M)$, $M^*\backslash e$ is 3-connected and contains a $U_{1,3}$-minor. In particular, for each $e \in B$, M/e is 3-connected and contains a $U_{2,3}$-minor. Since $|B| \geq 2$, (ii) holds. If $\text{co}(M\backslash s)$ has a 2-circuit, then either $\text{co}(M\backslash s) \cong U_{1,2}$ or $\text{co}(M\backslash s) \cong U_{1,3}$. If $\text{co}(M\backslash s) \cong U_{1,2}$, then $\text{si}(M^*/s) \cong U_{1,2}$; a contradiction. Thus $\text{co}(M\backslash s) \cong U_{1,3}$, that is, $\text{si}(M^*/s) \cong U_{2,3}$.

Now, in M^*, every element lies on one of three lines intersecting at s and, as M^* is 3-connected, at least two of the lines contain three or more elements. Thus $|E(M)| \geq 6$. If one of the lines, L say, containing s has at least four elements, then, for each $e \in L$, we have $M^*\backslash e$ is 3-connected by Lemma 2.1.8, and it is straightforward to check that $M^*\backslash e$ contains a $U_{1,3}$-minor. Since at least two elements in L are in B, we deduce that (ii) holds. Therefore each of the lines containing s has at most three elements, so $|E(M)| \leq 7$.

If M^* has precisely six elements, it is isomorphic to $\mathcal{W}^3$, $M(K_4)$ or Q_6, in which case it is routine to check that (ii) holds. Now we assume that each of the three lines has precisely three elements. It follows that M^* does not contain a triad, so for all $b \in B$ the matroid $\text{co}(M^*\backslash b)$ is isomorphic to the 3-connected matroid $M^*\backslash b$. Thus, each such b is $(U_{2,3}, B)$-strong in M, so (ii) again holds. □

In the case where M contains a Type I fan, the next lemma shows that when we cannot guarantee that M has two (N, B)-strong elements, there is a Type I fan F for which either every $f \in F$ is not B-removable, or F is contained in a maximal 5-element fan containing a single B-removable element.

Lemma 3.2.9. *Let M be a 3-connected matroid, let N be a 3-connected minor of M such that $|E(N)| \geq 4$, and let B be a basis of M. Suppose that there exists an element $b \in B$ that is (N, B)-robust but not (N, B)-strong. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M such that $|X \cap E(N)| \leq 1$. Then one of the following holds:*

(i) *M has at least two (N, B)-strong elements contained in $\text{cl}(X)$ or*

$\mathrm{cl}^*(X)$, *or*

(ii) $X \cup \{b\}$ *contains a Type II fan* F *and an* (N, B)*-strong element* $s_2 \in F \cap B$, *or*

(iii) M *has a Type I fan* F *relative to* B *where the internal elements are contained in* X *and either*

(a) f *is not* B*-removable for all* $f \in F$, *or*

(b) *there exists an element* $f \in E(M) - F$ *such that* $F \cup \{f\}$ *is a maximal* 5*-element fan with fan ordering* (f, f_1, f_2, f_3, f_4) *and* f_2 *is the only* B*-removable element in* F

Proof. By Proposition 3.2.7, either (i) or (ii) holds unless M has a Type I fan relative to B. In the exceptional case, let F have fan ordering (f_1, f_2, f_3, f_4) where $F \cap B = \{f_1, f_3\}$, and $f_2, f_3 \in X$. By Lemma 2.3.5, f_1 and f_4 are not removable. If both f_2 and f_3 are removable, then (i) holds, while if neither f_2 nor f_3 is removable, then (iii) holds. So assume that precisely one of f_2 and f_3 is removable. By Lemma 2.3.8, f_2 is in a triad other than $T^* = \{f_2, f_3, f_4\}$ or f_3 is in a triangle other than $T = \{f_1, f_2, f_3\}$.

First consider the case where $\mathrm{co}(M\backslash f_2)$ is 3-connected. By orthogonality, if f_3 is in a triangle, it contains either f_2 or f_4. But $\{f_2, f_3\}$ is not contained in a triangle other than T since $E(M) - \{f_2, f_3, f_4\}$ is closed. Moreover, $\{f_3, f_4\}$ is also not contained in a triangle, otherwise f_2 is a rim and an end element in a 4-element fan, so is not removable by Lemma 2.3.5; a contradiction. So f_2 is contained in a triad other than T^*. By orthogonality, if f_2 is in a triad, it contains either f_1 or f_3. If a triad other than T^* contains f_3, then f_3 is removable by the dual of Lemma 2.1.8; a contradiction. So there exists an element f_0 such that $\{f_0, f_1, f_2\}$ is a triad. If $f_0 \in B$, then f_0 is B-removable, by Lemma 2.3.5. Since $f_2 \in X$ is a removable element, it is an (N, B)-strong element by Lemma 3.2.6. As N is simple, $\mathrm{co}(M\backslash f_2) \cong \mathrm{co}(M\backslash f_2/f_0)$ has an N-minor, so f_0 is strong, satisfying (i). So assume that $f_0 \in E(M) - B$. It follows that $(\{f_0, f_1, f_2\}, E(M) - \{f_0, f_1, f_2, f_3\})$ is a 2-separation of M/f_3, and $|\{f_0, f_1, f_2\} \cap E(N)| \leq 1$, due to Lemma 3.2.2 and since $|E(N)| \geq 4$. Now if $F \cup \{f_0\}$ is not maximal, then f_2 is a rim and an end of a 4-element fan; a contradiction. So (iii) holds.

Now we suppose that $\mathrm{si}(M/f_3)$ is 3-connected. By duality, we can again apply Lemmas 2.1.8 and 2.3.5, so $\{f_1, f_2\}$ is not contained in a triad other than T^*, and $\{f_2, f_3\}$ is not contained in a triangle other than T. If $\{f_2, f_3\}$ is contained in a triad other than T^*, then $\text{cl}^*(X)$ contains two B-removable elements and, by Lemmas 3.2.4 and 3.2.6, (i) holds. By orthogonality, in the only remaining case there exists an element f_5 such that $\{f_3, f_4, f_5\}$ is a triangle. If $f_5 \in E(M) - B$, then, by duality and the argument in the previous paragraph, f_5 is (N, B)-strong and (i) holds; whereas if $f_5 \in B$, then (iii) holds. □

We are now in a position where we can prove Theorem 2.0.4. In particular, it is a special case of the next theorem.

Theorem 3.2.10. *Let M be a 3-connected matroid such that $|E(M)| \geq 5$, let N be a 3-connected minor of M, and let B be a basis of M. If M has at least two (N, B)-robust elements, then either*

(i) *M has at least two (N, B)-strong elements, or*

(ii) *M has a Type I fan F for which either*

   (a) *f is not B-removable for all $f \in F$, or*

   (b) *there exists an internal element of F that is the only B-removable element in F, and there exists an element $f \in E(M) - F$ such that $F \cup \{f\}$ is a maximal 5-element fan.*

Proof. We may assume that M has at least two removable elements by Corollary 2.3.10. If $|E(N)| \leq 3$, then it follows, by Lemma 3.2.8, that (i) holds. So assume that $|E(N)| \geq 4$. Let p_1 and p_2 be distinct (N, B)-robust elements. If p_1 and p_2 are both (N, B)-strong elements, then (i) holds; so assume otherwise. By duality, we may assume that p_1, say, is not (N, B)-strong, and is a member of B. Since $\text{si}(M/p_1)$ is not 3-connected, there exists a vertical 3-separation $(X, \{p_1\}, Y)$ such that $|X \cap E(N)| \leq 1$, by Lemmas 2.1.6 and 3.2.2. Then, by Lemma 3.2.9, the theorem holds unless $X \cup \{p_1\}$ contains a Type II fan F.

Let (f_1, f_2, f_3, f_4) be a fan ordering of F such that $\{f_2, f_3, f_4\} \subseteq X$, $f_2 \in E(M) - B$, and $f_4 \in B$ is (N, B)-strong, by Lemma 3.2.6. We next show that $\text{co}(M \backslash f_2)$ has an N-minor. We may assume, by Lemma 2.1.7, that $Y \cup \{p_1\}$ is closed. By Lemma 3.2.5, at most one element of $\{f_2, f_3, f_4\}$ is not doubly N-labelled. If f_2 is doubly N-labelled, then $\text{co}(M \backslash f_2)$ has an N-minor. If f_2 is not doubly N-labelled, then f_3 is doubly N-labelled and so

$\mathrm{si}(M/f_3) \cong \mathrm{si}(M/f_3\backslash f_2)$ has an N-minor. But then $\mathrm{co}(M\backslash f_2)$ again has an N-minor. If $\mathrm{co}(M\backslash f_2)$ is 3-connected, then the theorem holds. So assume that f_2 is not strong, where f_2 is a member of the basis $E(M) - B$ of M^*. By Lemmas 2.1.6 and 3.2.2, there exists a vertical 3-separation $(P, \{f_2\}, Q)$ in M^* such that $|P \cap E(N)| \leq 1$. By Lemma 3.2.9, either M^* has at least two $(N^*, E(M) - B)$-strong elements, in which case (i) holds, or M^* has a Type II fan F^* containing an $(N^*, E(M) - B)$-strong element. But, in the latter case, the $(N^*, E(M) - B)$-strong element in M^* is an (N, B)-strong element in M, and is a member of the basis of M^*; that is, it is a member of $E(M) - B$. Since $f_4 \in B$ is also (N, B)-strong, (i) holds. □

3.2.2 An example with one robust element

In this section we demonstrate that Theorem 3.2.10 is best possible in the sense that a 3-connected matroid may only have a single element, relative to a fixed basis, that can be removed and retain an N-minor. In particular, we give an example of a matroid M_2, with a minor F_7, where both the size of the ground set of M_2 and the difference in the sizes of the ground sets of M_2 and F_7 is arbitrary, and M_2 has only a single element that can be removed relative to a fixed basis and retain the N-minor.

Let M and M^+ be matroids such that $M = M^+\backslash e$ where $e \in E(M^+)$. The matroid M^+ is a *free extension* of M if M^+ has the same rank as M and every circuit of M^+ containing e is spanning. In what follows, we base our argument on the Fano matroid F_7, but any sufficiently structured matroid would do. Our example is of a similar nature to that given by Oxley et al. (2008*a*, Section 5) that demonstrated that one can construct a matroid that has no elements, relative to a fixed basis, that can be removed and maintain an N-minor. As in that example, we make use of the following lemma.

Lemma 3.2.11. *Let M^+ be a free extension of M.*

(i) *If an element a of M is not a coloop of M, then $M^+\backslash a$ is a free extension of $M\backslash a$ and M^+/a is a free extension of M/a.*

(ii) *If M has no F_7-minor, then M^+ has no F_7-minor.*

Let k_1 be a positive integer and let M_1 be a matroid obtained by coextending F_7 k_1 times such that $r(M_1) = k_1 + 3$ and M_1 is 3-connected. One way to obtain such a matroid M_1 is to freely extend F_7^* k_1 times and dualise.

Note that $r^*(M_1) = r^*(F_7)$ so that, for all $a \in E(M_1)$, the matroid $M_1\backslash a$ does not have an F_7-minor. Let k_2 be an integer such that $0 \leq k_2 \leq k_1$, and let M_2 be the matroid obtained by freely extending M_1 $k_2 + 3$ times.

Let X be a $(k_1 - k_2)$-element subset of $E(M_1) - E(F_7)$ and let $B = (E(M_2) - E(M_1)) \cup X$. We can see that $|B| = k_1 + 3$. We will show that B is a basis of M_2. Suppose it is not. Then B contains a circuit C. If C contains an element in $E(M_2) - E(M_1)$, then, since every circuit containing this element is spanning, C is spanning, and thus $|C| = k_1 + 4$; a contradiction. So $C \subseteq E(M_1)$, and, since M_1 is a restriction of M_2, the set C is a circuit of M_1. It follows that $E(M_1) - C$ is a hyperplane of M_1^*, but $E(F_7) \subseteq E(M_1) - C$ and $E(F_7^*)$ spans M_1^*; a contradiction. So B is indeed a basis of M_2.

We can contract an element of X and retain the F_7-minor, since $X \subseteq E(M_1) - E(F_7)$ and $M_2/(E(M_1) - E(F_7))\backslash(E(M_2) - E(M_1)) = F_7$. However, as observed earlier, $M_1\backslash a$ has no F_7-minor for any $a \in E(M_1)$, so it follows, by Lemma 3.2.11, that $M_2\backslash d$ has no F_7-minor for $d \in E(M_2) - B$. Now let $b \in E(M_2) - E(M_1) = B - X$. To obtain a corank-4 minor of M_2/b, we must delete an element in $E(M_1)$, since $r^*(M_2/b\backslash(E(M_2) - (E(M_1) \cup \{b\}))) = 5$. But we have seen that if we delete such an element then the matroid has no F_7-minor. Thus we conclude that $M_2\backslash d$ has no F_7-minor for all $d \in E(M_2) - B$ and that M_2/b has no F_7-minor for all $b \in B - X$, but M_2/x has an F_7-minor for $x \in X$.

If we consider the case where $k_2 = k_1 - 1$, so that $|X| = 1$, we see that there is only a single element in $E(M_2) \cap B$ that can be contracted from M_2 and retain an F_7-minor, and there are no elements in $E(M_2) - B$ that can be deleted from M_2 and retain an F_7-minor. So M_2 has just one (F_7, B)-robust element.

3.2.3 An example with two strong elements

Now we provide an example to demonstrate that Theorem 3.2.10 is best possible in the sense that a 3-connected matroid M with a basis B and no Type I fans relative to B can have precisely two (N, B)-strong elements. The matroid, M_4, is illustrated in Figure 3.2.

This example is a cross between those in Sections 2.3.3 and 3.1. Let $k \geq 3$. The rank-$(k + 3)$ matroid M_k is constructed as follows. Let $U_{k,k}$ be the free k-element matroid $U_{k,k}$ with ground set $\{b_0, b_1, \ldots, b_{k-1}\}$, and let

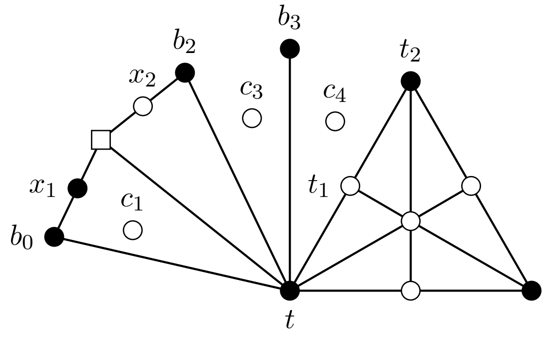


Figure 3.2: A 3-connected rank-7 matroid M_4 with only two (F_7^-, B)-strong elements: b_0 and x_1.

F_7^- be the non-Fano matroid containing a triangle $\{t, t_1, t_2\}$. Construct the direct sum $U_{k,k} \oplus F_7^-$. For each $i \in \{1, 2, \ldots, k-1\} - \{2\}$, freely add c_i to the flat $\{t, b_{i-1}, b_i\}$, and freely add c_k to the flat $\{b_{k-1}, t, t_1, t_2\}$. For each $i \in \{1, 2\}$, freely add x_i to the flat $\{b_{i-1}, b_i\}$. Finally, delete b_1 to obtain M_k.

Let A be a basis of F_7^- such that $t \in A$. Then $B = A \cup \{b_0, x_1\} \cup \{b_2, b_3, \ldots, b_{k-1}\}$ is a basis for M_k. The (F_7^-, B)-robust elements P are $E(M_k) - E(F_7^-)$, and $|P| = 2k \geq 6$. By the same argument as in Section 2.3.3, the elements b_0 and x_1 are B-removable, but every other element in P is not. Hence, b_0 and x_1 are the only (F_7^-, B)-strong elements in M_k.

3.2.4 An example with a Type I fan

In this section, we describe a 3-connected rank-4 matroid M_2 with an N-minor, a fixed basis B, and containing a Type I fan relative to B. Even though M_2 has more than two (N, B)-robust elements, it has no (N, B)-strong elements. More generally, we describe how to construct a matroid M_k of rank $k + 3$, for some $k \geq 2$, with no (N, B)-strong elements, but containing a Type I fan. We base our argument on the Fano matroid F_7, but any sufficiently structured matroid with a 3-point line would work.

We require some definitions for the construction. A flat X of a matroid N is a *modular flat* if, for every flat Y of N,

$$r(X) + r(Y) = r(X \cup Y) + r(X \cap Y).$$

Let N_1 and N_2 be matroids such that $E(N_1) \cap E(N_2) = T$, where T is a triangle of both N_1 and N_2, and T is a modular flat in N_1. Note that T is a modular flat when N_1 is binary. The *generalised parallel connection* of N_1 and N_2 across T is the matroid $P_T(N_1, N_2)$ on $E(N_1) \cup E(N_2)$ whose flats are the subsets $X \subseteq E(N_1) \cup E(N_2)$ such that $X \cap E(N_1)$ is a flat of N_1, and $X \cap E(N_2)$ is a flat of N_2 (Brylawski, 1975).

Let $M(\mathcal{W}_{k+2})$ be the rank-$(k+2)$ wheel, and let F_7 be the Fano matroid, where $M(\mathcal{W}_{k+2})$ and F_7 have a triangle $T = \{f_1, x, f_{2k+3}\}$ in common. We label the elements $E(M(\mathcal{W}_{k+2})) - T$ as $\{f_2, f_3, \ldots, f_{2k+2}\}$ such that $(f_1, f_2, f_3, \ldots, f_{2k+2}, f_{2k+3})$ is a fan ordering where $\{f_1, f_2, f_3\}$ is a triangle. Let M_k be the generalised parallel connection of $M(\mathcal{W}_{k+2})$ and F_7 across T. The matroid M_2 is illustrated in Figure 3.3.

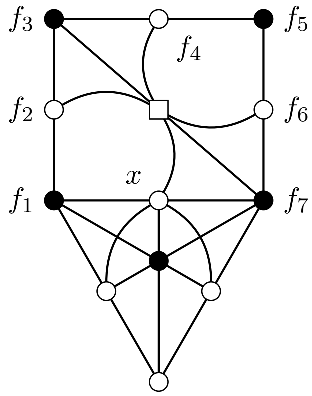


Figure 3.3: A 3-connected rank-5 matroid M_2 with a Type I fan, and no (N, B)-strong elements.

Let B be a basis of M_k that contains f_i if and only if i is odd. Note that $\{f_1, f_2, f_3, f_4\}$ is a Type I fan relative to B, for example. Evidently, f_i is an (F_7, B)-robust element for all $i \in \{2, 3, \ldots, 2k+2\}$, but every other element of M_k is not (F_7, B)-robust. By Lemma 2.3.5, none of the elements $\{f_2, f_3, \ldots, f_{2k+2}\}$ are (F_7, B)-strong. Thus, even though M_k has at least two (F_7, B)-robust elements, M_k has no (F_7, B)-strong elements.

3.3 The structure of matroids with two strong elements

In this section, we consider the structure of matroids that have the minimum number of strong elements. In particular, we establish the following:

Theorem 3.3.1. *Let M be a 3-connected matroid such that $|E(M)| \geq 5$, let B be a basis of M, and let N be a 3-connected minor of M. Suppose that M has no Type I or Type II fans relative to B, and let P denote the set of (N, B)-robust elements of M. If M has precisely two (N, B)-strong elements, then $(P, E(M) - P)$ is a sequential 3-separation.*

A *path of 3-separations* in a matroid M is an ordered partition $(P_0, P_1, \ldots, P_k)$ of $E(M)$ with the property that $\lambda(P_0 \cup P_1 \cup \cdots \cup P_i) = 2$ for all $i \in \{0, 1, \ldots, k-1\}$. Note that a vertical 3-separation $(X, \{z\}, Y)$ is a path of 3-separations. The following lemma is elementary.

Lemma 3.3.2. *A partition (X, Y) of a matroid M with $|X|, |Y| \geq 3$ is a sequential 3-separation if and only if, for some $U \in \{X, Y\}$, there is a path of 3-separations $(P_0, P_1, \ldots, P_k, U)$ in M such that $|P_0| = 2$, and $|P_i| = 1$ for all $i \in \{1, 2, \ldots, k\}$.*

We also make use of the following result (Whittle and Williams, 2013, Corollary 5.3).

Lemma 3.3.3. *Let $\mathbb{P} = (P_0, P_1, \ldots, P_k)$ be a path of 3-separations in a matroid M. Suppose that $e \in P_i$, for some $i \in \{1, 2, \ldots, k-1\}$, and there exists a path of 3-separations $(X, \{e\}, Y)$ in M with $P_0 \subseteq X$ and $P_k \subseteq Y$. Then $\mathbb{P}$ refines to a path of 3-separations $(P_0, \ldots, P_{i-1}, P_i', \{e\}, P_i'', P_{i+1}, \ldots, P_k)$, where $P_i' \cup \{e\} \cup P_i'' = P_i$.*

Lemma 3.3.4. *Let M be a 3-connected matroid with ground set E, and let s_1 and s_2 be distinct elements of M. Let Z be a subset of $E - \{s_1, s_2\}$ where $|E - (Z \cup \{s_1, s_2\})| \geq 2$ and, for each $z \in Z$, there exists a path of 3-separations $(X_z, \{z\}, Y_z)$ in M such that $\{s_1, s_2\} \subseteq X_z$ and $X_z \subseteq Z \cup \{s_1, s_2\}$. Then,*

$$(\{s_1, s_2\}, \{z_1\}, \{z_2\}, \ldots, \{z_k\}, E - (Z \cup \{s_1, s_2\}))$$

is a path of 3-separations in M.

Proof. Let $S = \{s_1, s_2\}$. If Z is empty, then the result holds immediately. So assume that Z is non-empty. Let $Z = \{z_1, z_2, \ldots, z_k\}$ be a subset of $E - S$ as described in the statement of the lemma. Now, for all $i \in \{1, 2, \ldots, k\}$, the tuple $(X_{z_i}, \{z_i\}, Y_{z_i})$ is a path of 3-separations in M. Since $S \subseteq X_{z_i}$ for all i, it follows that $S \subseteq E - (Y_{z_1} \cup Y_{z_2} \cup \cdots \cup Y_{z_k})$. In particular,

$|E-(Y_{z_1}\cup Y_{z_2}\cup\cdots\cup Y_{z_k})| \geq 2$ and so, by Corollary 2.1.2, $Y_{z_1}\cap Y_{z_2}\cap\cdots\cap Y_{z_k} = E-(Z\cup S)$ is 3-separating. Since $|S| \geq 2$ and $|E-(Z\cup S)| \geq 2$, the partition $(S, Z, E-(Z\cup S))$ is a path of 3-separations in M. By repeatedly applying Lemma 3.3.3, we deduce that the lemma holds. □

Lemma 3.3.5. *Let M be a 3-connected matroid, let B be a basis of M, and let N be a 3-connected minor of M such that $|E(N)| \geq 4$. Suppose that M has no Type I or Type II fans relative to B, and suppose there are precisely two distinct elements s_1 and s_2 that are (N, B)-strong in M. Let P denote the set of (N, B)-robust elements of M. Then, for every $z \in P - \{s_1, s_2\}$, there exists a path of 3-separations $(X_z, \{z\}, Y_z)$ such that $\{s_1, s_2\} \subseteq X_z$ and $X_z \subseteq P$.*

Proof. Let $S = \{s_1, s_2\}$. Consider an element $z \in P - S$. By duality, we may assume that $z \in B$. First we show the following:

3.3.5.1. *There exists a vertical 3-separation $(X', \{z\}, Y')$ such that $S \subseteq X'$, $|X' \cap E(N)| \leq 1$, and $Y' \cup \{z\}$ is closed.*

Since $\mathrm{si}(M/z)$ is not 3-connected, it follows by Lemmas 2.1.6 and 2.1.7 that there exists a vertical 3-separation $(X, \{z\}, Y)$ such that $Y \cup \{z\}$ is closed. By Lemma 3.2.2, either $|X \cap E(N)| \leq 1$ or $|Y \cap E(N)| \leq 1$. For the latter, by applying Lemma 2.1.7 and relabelling, we can obtain a vertical 3-separation $(X, \{z\}, Y)$ such that $Y \cup \{z\}$ is closed and $|X \cap E(N)| \leq 1$. By Proposition 2.3.9, there is an element $s_1' \in X$ and either a distinct element $s_2' \in \mathrm{cl}^*(X) \cap B$, or distinct elements $s_2', s_3' \in \mathrm{cl}(X) \cap (E(M) - B)$, where each s_i' is removable with respect to B for $i \in \{1, 2, 3\}$.

In the first case, there exists a vertical 3-separation $(X', \{z\}, Y')$ such that $\{s_1', s_2'\} \subseteq X'$, $|X' \cap E(N)| \leq 1$, and $Y' \cup \{z\}$ is closed by Lemma 3.2.4. By Lemma 3.2.6, s_1' and s_2' are (N, B)-strong, so $S = \{s_1', s_2'\}$, and (3.3.5.1) holds. In the second case, as M has precisely two (N, B)-strong elements, it follows by Lemma 3.2.6 that $\{s_2', s_3'\} \nsubseteq X$. If exactly one of s_2' and s_3' is in X, then this element is strong, so S consists of this element and s_1', satisfying (3.3.5.1). So we can assume that $\{s_2', s_3'\} \subseteq \mathrm{cl}(X) - X$. It follows that $\{z, s_2', s_3'\} \subseteq \mathrm{cl}(X) \cap \mathrm{cl}(Y)$, and so $r(\{z, s_2', s_3'\}) = 2$. Recall that $\{s_2', s_3'\} \subseteq E(M) - B$, the matroid M/z has an N-minor, and N has no 2-circuits. Now s_2' and s_3' are parallel in M/z, thus $M/z\backslash s_2'$ and $M/z\backslash s_3'$ have N-minors, and, by Lemma 3.2.3, s_2' and s_3' are (N, B)-strong. But

s_1' is also (N, B)-strong by Lemma 3.2.6; a contradiction. We deduce that (3.3.5.1) holds.

Now, by Lemma 3.2.5, at most one element in X' is not doubly N-labelled, and if such an element x exists, then $x \in \mathrm{cl}^*(Y')$. Suppose such an x exists, and x is not (N, B)-robust. By Lemma 2.1.4, $Y' \cup \{x\}$ and $Y' \cup \{x, z\}$ are 3-separating. Since $|X'| \geq 3$, these 3-separating sets are exact. It follows that $(X'', \{z\}, Y'') = (X' - \{x\}, \{z\}, Y' \cup \{x\})$ is a path of 3-separations such that $S \subseteq X''$ and $X'' \subseteq P$. Otherwise, when no such x exists or x is (N, B)-robust, every element in X is robust, so $X' \subseteq P$. This completes the proof of the lemma. □

Proof of Theorem 3.3.1. Let $S = \{s_1, s_2\}$ denote the set of (N, B)-strong elements of M. First suppose that $|E(N)| \geq 4$. By Lemma 3.3.5, for every $z \in P - S$ there exists a path of 3-separations $(X_z, \{z\}, Y_z)$ such that $S \subseteq X_z$ and $X_z \subseteq P$. By Corollary 2.3.12, M has at least four B-removable elements. However, only two of these elements are (N, B)-strong, so $|E(M) - P| \geq 2$. It follows, by Lemma 3.3.4, that $(S, \{z_1\}, \{z_2\}, \ldots, \{z_k\}, E(M) - P)$ is a path of 3-separations, where $P - S = \{z_1, z_2, \ldots, z_k\}$. Thus, by Lemma 3.3.2, $(P, E(M) - P)$ is a sequential 3-separation.

It remains to consider when $|E(N)| \leq 3$. We show that, in this case, M has more than two (N, B)-strong elements, thereby resulting in a contradiction. If $r(M) \leq 2$, then $M \cong U_{2,n}$ where $n \geq 5$, and it is easily checked that M has at least three (N, B)-strong elements. Thus, by duality, we can assume that $r(M) \geq 3$ and $r^*(M) \geq 3$. By Lemma 3.2.8 and Corollary 2.3.12, either

(I) there are at least three (N, B)-strong elements, or

(II) up to duality, there is a removable element $s \in B$ and $\mathrm{si}(M/s) \cong U_{2,3}$.

If (I) holds, then we obtain a contradiction. So assume that (II) holds. Then M consists of three lines that intersect at the element s, at least two of which contain three or more elements, since M is 3-connected. If one of these lines, L say, consists of at least four elements, then, by Lemma 2.1.8, the line L contains two elements in $E(M) - B$ that are B-removable and retain a $U_{1,3}$- or a $U_{2,3}$-minor. Furthermore, there exists at least one element $b \in B \cap (E(M) - L)$ such that $\mathrm{si}(M/b) \cong U_{2,n}$ for some $n \geq 4$. Thus, b is (N, B)-strong, so M has more than two (N, B)-strong elements; a contradiction. If $|E(M)| = 6$, then M is isomorphic to one of $\mathcal{W}^3$, $M(K_4)$ and

Q_6. But each of these matroids has at least three (N, B)-strong elements for every B and N such that $|E(N)| \leq 3$. So each of the three lines consists of precisely three elements that intersect in a single element s say. Now, M does not contain a triad, so $M\backslash e = \mathrm{co}(M\backslash e)$ for all $e \in E(M)$. Moreover, $M\backslash e$ is isomorphic to one of the 3-connected matroids $\mathcal{W}^3$, $M(K_4)$ and Q_6, which each have a $U_{2,3}$- and $U_{1,3}$-minor. So M has more than two (N, B)-strong elements; a contradiction. This completes the proof of the theorem. $\square$

Part II

Constructing a k-tree for a k-connected matroid

Oxley et al. (2004), in their seminal paper, showed that every 3-connected matroid M with at least nine elements has a 3-tree: a tree decomposition that displays, up to a natural equivalence, all non-sequential 3-separations of M. The approach taken in the proof of this result does not appear to elicit an efficient algorithm for finding such a 3-tree. However, by taking a different approach, and thereby reproving the result, Oxley and Semple (2013) presented such an algorithm. Provided the rank of a subset of $E(M)$ can be found in constant time, this algorithm finds a 3-tree for M with running time polynomial in the size of $E(M)$.

Clark and Whittle (2013) generalised the main result of Oxley et al. (2004), showing that every tangle of order k in a connectivity system that satisfies a certain "robustness" property has a tree decomposition, called a k-tree, that displays, up to equivalence, all the non-sequential k-separations of the connectivity system with respect to the tangle. In particular, this result specialises to k-connected matroids as follows:

Theorem 4.0.1 (Clark and Whittle, 2013)**.** *Let M be a k-connected matroid, where $k \geq 3$ and $|E(M)| \geq 8k - 15$. Then there is a k-tree T for M such that every non-sequential k-separation of M is equivalent to a k-separation displayed by T.*

As with the case where $k = 3$, although Theorem 4.0.1 ensures the existence of a k-tree for M, it does not guarantee the existence of a polynomial-time algorithm for finding such a tree. In this part of the thesis, we present an algorithm for finding a k-tree for M. The main result establishes that the algorithm indeed outputs a k-tree, thereby giving an independent proof of Theorem 4.0.1. Provided that the matroid M is specified in a way that enables the rank of any subset of $E(M)$ to be found in unit time, the algorithm runs in time polynomial in the size of $E(M)$. Such a matroid M is said to be specified by a *rank oracle*.

Theorem 4.0.2. *Let M be a k-connected matroid specified by a rank oracle, where $|E(M)| \geq 8k - 15$. Then there is a polynomial-time algorithm for finding a k-tree for M.*

Our overall approach is similar to that taken by Oxley and Semple (2013); however, there are a number of additional hurdles to overcome when $k \geq 4$.

This part of the thesis consists of three chapters, the first of which introduces the theory required to describe the algorithm and prove its correctness in the later chapters. In Section 4.1, we review the fundamental concepts relating to connectivity, flowers, and k-trees, each in the setting of k-connected matroids. In Section 4.2, we give an example to demonstrate why it is necessary for a k-connected matroid M to consist of at least $8k-15$ elements in order for M to have a k-tree. Section 4.3 contains a number of preliminary results concerning k-connectivity, k-flowers, and k-paths, where the latter are a generalisation of 3-paths introduced by Oxley and Semple (2013). In Section 4.4, we discuss one key situation that arises only when $k \geq 4$.

In Chapter 5, we present the algorithm for constructing a k-tree for a k-connected matroid. Throughout the algorithm, we repeatedly attempt to find non-sequential k-separations where each side of the separation contains certain subsets; in Section 5.1, we describe how to find such k-separations in polynomial time. Section 5.2 contains a formal description of the algorithm. We close the chapter by discussing, in Section 5.3, why a polynomial-time algorithm is not forthcoming from the proof of Theorem 4.0.1 given by Clark and Whittle (2013).

Finally, in Chapter 6, we prove the correctness of the algorithm and that it runs in polynomial time. The proof is in three parts: in Section 6.1, we prove that the algorithm outputs a conforming tree; in Section 6.2, we demonstrate that each flower vertex of this tree is maximal; and the proof of Theorem 4.0.2 is forthcoming in Section 6.3.

Throughout, we assume that the matroid M for which we wish to construct a k-tree is specified by a rank oracle.

Chapter 4

k-flowers, k-trees, and k-paths

This chapter is an introduction to the theory of flowers, trees, and k-paths, in the general setting of k-connected matroids. In Section 4.1, we define these terms and some related concepts. By Theorem 4.0.1, a k-connected matroid has a k-tree when its ground set consists of at least $8k - 15$ elements. We give an example in Section 4.2 to demonstrate why this constraint is necessary. Section 4.3 contains a number of prerequisite results regarding k-connectivity, k-flowers, and k-paths that will be used throughout Part II. Finally, in Section 4.4, we discuss a technical detail regarding the relationship between sequential petals of a k-flower, and ends of a k-path.

4.1 Definitions

4.1.1 k-connectivity

Let M be a k-connected matroid with ground set E, and let X be an exactly k-separating subset of E. A *partial k-sequence for X* is a sequence $(X_i)_{i=1}^m$ of pairwise-disjoint non-empty subsets of $E - X$ such that $|X_i| \leq k - 2$, for all $i \in \{1, 2, \ldots, m\}$, and $X \cup (\bigcup_{i=1}^j X_i)$ is k-separating, for all $j \in \{1, 2, \ldots, m\}$. A partial k-sequence $(X_i)_{i=1}^m$ for X is *maximal* if, for every partial k-sequence $(X'_i)_{i=1}^{m'}$ for X, we have $\bigcup_{i=1}^{m'} X'_i \subseteq \bigcup_{i=1}^m X_i$.

Let $(X_i)_{i=1}^m$ be a maximal partial k-sequence for the exactly k-separating set X. We define the *full k-closure of X*, denoted $\mathrm{fcl}_k(X)$, to be $X \cup \bigcup_{i=1}^m X_i$. For readers familiar with the work of Clark and Whittle (2013), note that this operator is a specialisation of the $\text{fcl}_{\mathcal{T}}$ operator, where $\mathcal{T}$ is the unique tangle for a k-connected matroid. The fcl_k operator is a well-defined closure operator on the set of exactly k-separating subsets of E (Clark and Whittle, 2013, Lemma 3.3). When $k = 3$, the operator is equivalent to the full closure operator for 3-connected matroids (as given by Oxley et al. (2004), for example) and, when $k = 4$, it is equivalent to the full 2-span operator (Aikin and Oxley, 2012). It is important to note that the full k-closure operator is only well-defined on exactly k-separating sets, where it follows from the definition of an exactly k-separating set that these are the k-separating sets with at least $k-1$ elements, but no more than $|E| - (k-1)$ elements.

An exactly k-separating set X is *k-sequential* if $\text{fcl}_k(E - X) = E$; otherwise, it is *not k-sequential.* An exact k-separation (X, Y) is *k-sequential* if X or Y is k-sequential; otherwise, when X and Y are not k-sequential, we say that (X, Y) is *not k-sequential.* When there is no ambiguity, we sometimes omit the "k-" and say that a k-separating set or k-separation is *sequential* or *non-sequential.* When X is k-sequential and $(X_1, X_2, \dots, X_m)$ is a maximal partial k-sequence for $E - X$, we say that $(X_m, X_{m-1}, \dots, X_1)$ is a *k-sequential ordering* of X.

Let (A_1, B_1) and (A_2, B_2) be exact k-separations of M; then (A_1, B_1) is *k-equivalent* to (A_2, B_2) if $\{\text{fcl}_k(A_1), \text{fcl}_k(B_1)\} = \{\text{fcl}_k(A_2), \text{fcl}_k(B_2)\}$.

4.1.2 k-flowers

The crossing k-separations of a k-connected matroid M are represented by the k-flowers of M.

Let M be a k-connected matroid for some $k \geq 3$ with ground set E. For $n > 1$, a partition $(P_1, P_2, \dots, P_n)$ of E is a *k-flower* with *petals* $P_1, P_2, \dots, P_n$ if each P_i is exactly k-separating, and each $P_i \cup P_{i+1}$ is k-separating, where subscripts are interpreted modulo n. We also view (E) as a k-flower with a single petal E; we call this k-flower *trivial.* In the remainder of this thesis, for a flower $(P_1, P_2, \dots, P_n)$, the subscripts will always be interpreted modulo n. A k-flower Φ *displays* a k-separating set X or a k-separation (X, Y) if X is a union of petals of Φ. Let Φ_1 and Φ_2 be k-flowers. Then $\Phi_1 \preccurlyeq \Phi_2$ if every non-sequential k-separation displayed by Φ_1 is k-equivalent to a k-separation displayed by Φ_2. We say that Φ_1 and Φ_2 are k-equivalent if $\Phi_1 \preccurlyeq \Phi_2$ and $\Phi_2 \preccurlyeq \Phi_1$. The *order* of a k-flower Φ is the minimum number of petals in a k-flower k-equivalent to Φ.

Let $\Phi = (P_1, P_2, \dots, P_n)$ be a k-flower of M. The k-flower Φ is a *k-anemone* if $\bigcup_{s \in S} P_s$ is k-separating for every subset S of $\{1, 2, \dots, n\}$; whereas Φ is a *k-daisy* if $P_i \cup P_{i+1} \cup \cdots \cup P_{i+j}$ is k-separating for all $i, j \in \{1, 2, \dots, n\}$, and no other union of petals is k-separating. Aikin and Oxley (2008) showed that every non-trivial k-flower is either a k-anemone or a k-daisy.

An element $e \in E$ is *loose* if $e \in \mathrm{fcl}_k(P_i) - P_i$ for some $i \in \{1, 2, \dots, n\}$, otherwise e is *tight*. A petal P_i, for some $i \in \{1, 2, \dots, n\}$, is *loose* if every $e \in P_i$ is loose; otherwise, P_i is *tight*. A flower of order at least three is *tight* if all its petals are tight; while a flower of order one or two is *tight* if it has one or two petals, respectively. A k-daisy Φ is *irredundant* if, for all $i \in \{1, 2, \dots, n\}$, there is a non-sequential k-separation (X, Y) displayed by Φ with $P_i \subseteq X$ and $P_{i+1} \subseteq Y$. A k-anemone Φ is *irredundant* if, for all distinct $i, j \in \{1, 2, \dots, n\}$, there is a non-sequential k-separation (X, Y) displayed by Φ with $P_i \subseteq X$ and $P_j \subseteq Y$. Note that a tight 3-flower is always irredundant, but this does not necessarily hold for tight k-flowers where $k \geq 4$ (Aikin and Oxley, 2012, Example 3.14). As the purpose of a k-tree is to describe the non-sequential k-separations of a matroid, it is most efficient to do so using irredundant flowers.

This definition of an irredundant k-flower Φ is stronger than that given by Aikin and Oxley (2012) when Φ is a k-anemone. The stronger definition ensures that for a tight irredundant k-anemone Φ with n petals, the order of Φ is n. This is illustrated by considering the 4-flower (P_1, P_2, P_4, P_3) as described by Aikin and Oxley (2012, Example 3.14), but with the last two petals interchanged; this 4-flower is "irredundant" under the weaker definition, but $(P_1, P_2 \cup P_3, P_4)$ is an equivalent 4-flower with fewer petals. It is also worth noting that our terminology differs from that used by Clark and Whittle (2013), where a k-flower in the unique tangle $\mathcal{T}$ for M is called $\mathcal{S}$-tight, where $\mathcal{S}$ is the set of all non-sequential k-separations of M, if no k-flower displaying the same k-separations contained in $\mathcal{S}$ has fewer petals. Thus, such an $\mathcal{S}$-tight k-flower must be not only tight, as defined here, but also irredundant.

4.1.3 k-trees

Let π be a partition of a finite set E. Let T be a tree such that every member of π labels exactly one vertex of T; some vertices may be unlabelled but no vertex is multiply labelled. We say that T is a *π-labelled tree*; labelled vertices are called *bag vertices* and members of π are called *bags*. If B is a bag vertex of T, then $\pi(B)$ denotes the subset of E that labels it. If the degree of B is at most one, then B is a *terminal* bag vertex; otherwise B is *non-terminal*.

Let G be a subgraph of T with components $G_1, G_2, \dots, G_m$. Let X_i be the union of those bags that label vertices of G_i. Then *the subsets of E displayed by G* are $X_1, X_2, \dots, X_m$. In particular, if $V(G) = V(T)$, then $\{X_1, X_2, \dots, X_m\}$ is the *partition of E displayed by G*. Let e be an edge of T. The *partition of E displayed by e* is the partition displayed by $T\backslash e$. If $e = v_1v_2$ for vertices v_1 and v_2, then (Y_1, Y_2) is the *(ordered) partition of $E(M)$ displayed by v_1v_2* if Y_1 is the union of the bags in the component of $T\backslash v_1v_2$ containing v_1. Let v be a vertex of T that is not a bag vertex. The *partition of E displayed by v* is the partition displayed by $T - v$. The edges incident with v correspond to the components of $T - v$, and hence to the members of the partition displayed by v. In what follows, if a cyclic ordering $(e_1, e_2, \dots, e_n)$ is imposed on the edges incident with v, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by v.

Let M be a k-connected matroid with ground set E. Let T be a π-labelled k-tree for M, where π is a partition of E such that:

(F1) For each edge e of T, the partition (X, Y) of E displayed by e is k-separating, and, if e is incident with two bag vertices, then (X, Y) is a non-sequential k-separation.

(F2) Every non-bag vertex v is labelled either D or A; if v is labelled D, then there is a cyclic ordering on the edges incident with v.

(F3) If a vertex v is labelled A, then the partition of E displayed by v is a k-anemone of order at least three.

(F4) If a vertex v is labelled D, then the partition of E displayed by v, with the cyclic order induced by the cyclic ordering on the edges incident with v, is a k-daisy of order at least three.

A vertex of T is referred to as a *daisy vertex* or an *anemone vertex* if it is labelled D or A, respectively. A vertex labelled either D or A is a *flower* *vertex*. By conditions (F3) and (F4), the partition displayed by a flower vertex v is a k-flower Φ of M; we say that Φ is the flower *corresponding to* v, and the k-separations displayed by Φ are the k-separations *displayed by* v. A k-separation is *displayed by* T if it is displayed by some edge or some flower vertex of T. A k-separation (R, G) of M *conforms* with T if either (R, G) is equivalent to a k-separation that is displayed by a flower vertex or an edge of T, or (R, G) is equivalent to a k-separation (R', G') with the property that either R' or G' is contained in a bag of T.

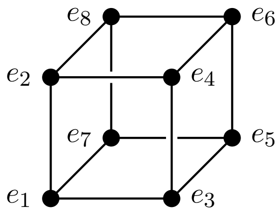
A π-labelled k-tree T for M satisfying (F1)–(F4) is a *conforming k-tree* for M if every non-sequential k-separation of M conforms with T. A conforming k-tree T is a *partial k-tree* if, for every flower vertex v of T, the partition of E displayed by v is a tight maximal k-flower of M.

We now define a quasi order on the set of partial k-trees for M. Let T_1 and T_2 be partial k-trees for M. Define $T_1 \preccurlyeq T_2$ if every non-sequential k-separation displayed by T_1 is equivalent to one displayed by T_2. If $T_1 \preccurlyeq T_2$ and $T_2 \preccurlyeq T_1$, then T_1 and T_2 are *equivalent* partial k-trees. A partial k-tree is *maximal* if it is maximal with respect to this quasi order. We call a maximal partial k-tree a *k-tree*.

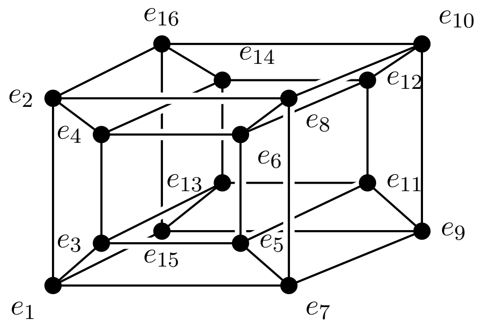
4.2 An example

In this section, we give a generic example to demonstrate that the constraint that $|E(M)| \geq 8k - 15$, in Theorems 4.0.1 and 4.0.2, is sharp. Clark and Whittle (2013, Section 5) showed that for each $k > 3$ there is a polymatroid that has a tangle $\mathcal{T}$ of order k with a non-sequential k-separation that does not conform with an $\mathcal{S}$-tight $\mathcal{S}$-maximal k-flower in $\mathcal{T}$. Restricting our attention to k-connected matroids, we show that for each $k \geq 3$ there is a k-connected matroid M with $8k - 16$ elements that has a non-sequential k-separation that does not conform with a tight irredundant maximal k-flower of M. This is consistent with examples in the literature: the 8-element 3-connected matroid R_8 given by Oxley et al. (2004, Section 9) and the 16-element 4-connected matroid H_{16} given by Aikin and Oxley (2012, Section 4).

Let H_{8k-16} be the $(8k - 16)$-element binary affine k-dimensional hypercube, or *k-cube*, of rank $k + 1$. The matroid H_{8k-16} is k-connected. For $k \in \{3, 4\}$, these matroids are illustrated in Figure 4.1. When $k = 4$, this



(a) The rank-4 binary affine 3-cube.



(b) The rank-5 binary affine 4-cube.

Figure 4.1: The binary affine k-cubes where $k \in \{3, 4\}$.

matroid coincides with the aforementioned example given by Aikin and Oxley (2012). A representation of H_{8k-16} can be constructed as follows. Let H'_8 be the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

over $GF(2)$. Let H'_8J be the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

over $GF(2)$ that is obtained by reversing the order of the columns of H'_8. Recursively, for all $k \geq 3$, define $H'_{8(k+1)-16}$ to be the matrix

$$\begin{pmatrix} H'_{8k-16} & H'_{8k-16}J \\ \mathbf{0}^T & \mathbf{1}^T \end{pmatrix}$$

over $GF(2)$ where $H'_{8k-16}J$ is the matrix obtained from H'_{8k-16} by reversing the order of the columns. Label the columns of H'_{8k-16} from e_1 to e_{8k-16}. We denote, for all $k \geq 2$, the vector matroid arising from H'_{8k-16} by H_{8k-16}. Then, the partition $\Phi = (\{e_1, e_2, \dots, e_{2k-4}\}, \{e_{2k-3}, \dots, e_{4k-8}\}, \{e_{4k-7}, \dots, e_{6k-12}\}, \{e_{6k-11}, \dots, e_{8k-16}\})$ is an irredundant tight k-flower. However, letting

$$X = \{e_1, e_2, \dots, e_{k-2}, e_{3k-5}, e_{3k-4}, \dots, e_{5k-10}, e_{7k-13}, e_{7k-12}, \dots, e_{8k-16}\},$$

the non-sequential k-separation $(X, E(H_{8k-16}) - X)$ does not conform with Φ. For example, when $k = 3$, the non-sequential 3-separation $(\{e_1, e_4, e_5, e_8\}, \{e_2, e_3, e_6, e_7\})$ does not conform with the 3-flower $(\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$; when $k = 4$, the non-sequential 4-separation

$$(\{e_1, e_2, e_7, e_8, e_9, e_{10}, e_{15}, e_{16}\}, \{e_3, e_4, e_5, e_6, e_{11}, e_{12}, e_{13}, e_{14}\})$$

does not conform with the 4-flower

$$(\{e_1, e_2, e_3, e_4\}, \{e_5, e_6, e_7, e_8\}, \{e_9, e_{10}, e_{11}, e_{12}\}, \{e_{13}, e_{14}, e_{15}, e_{16}\}).$$

4.3 Preliminaries

4.3.1 k-connectivity

Heretofore, we have referred to an application of Lemma 2.1.1 as "uncrossing". This lemma applies only to 3-connected matroids. More generally, for a k-connected matroid we have the following:

Lemma 4.3.1. *Let M be a k-connected matroid, and let X and Y be k-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq k - 1$, then $X \cup Y$ is k-separating.*

(ii) *If $|E(M) - (X \cup Y)| \geq k - 1$, then $X \cap Y$ is k-separating.*

In the remainder of the thesis, we use the phrase "by uncrossing" to refer to an application of Lemma 4.3.1.

The following results identify some elementary properties of sequential k-separating sets. The first is a generalisation of results by Oxley et al.

(2012, Lemma 2.7) and Aikin and Oxley (2012, Lemma 2.6). The proofs of the subsequent corollaries are straightforward.

Lemma 4.3.2. *In a k-connected matroid M, let X and Y be k-separating sets such that $|E(M) - X| \geq k - 1$ and $Y \subseteq X$. If X is k-sequential, then so is Y.*

Proof. Take a k-sequential ordering $(X_1, X_2, \ldots, X_t)$ of X. Then, by uncrossing, for all $i \in \{1, 2, \ldots, t\}$, the set $Y \cap (X_1 \cup X_2 \cup \cdots \cup X_i)$ is k-separating. □

Corollary 4.3.3. *Let (X, Y) be a k-separation in a k-connected matroid M and let Y' be a non-sequential k-separating set in M. If $Y' \subseteq Y$, then Y is non-sequential.*

Corollary 4.3.4. *Let M be a k-connected matroid, and let $\mathcal{F}$ be the collection of maximal k-sequential k-separating sets of M. Then, a k-separating set X is non-sequential if and only if no member of $\mathcal{F}$ contains X.*

The next lemma generalises a well-known property of non-sequential 3-separating sets (Oxley et al., 2004, Lemma 3.4(i)).

Lemma 4.3.5. *Let (X, Y) be exactly k-separating in a k-connected matroid M. If (X, Y) is not k-sequential, then $|X|, |Y| \geq 2k - 2$.*

Proof. Suppose that $|X| \leq 2k-3$. Clearly, $|X| \geq k-1$. Every $(k-1)$-element subset X_1 of X is trivially k-separating. Therefore, as $|X - X_1| \leq k - 2$, we have $\text{fcl}_k(E(M) - X) = \text{fcl}_k(E(M) - X_1) = E(M)$; a contradiction. □

An ordered partition $(Z_1, Z_2, \ldots, Z_t)$ of $E(M)$ is a *k-sequence* if, for all $i \in \{1, 2, \ldots, t - 1\}$, the set $\bigcup_{j=1}^{i} Z_j$ is k-separating.

Lemma 4.3.6. *Let U and Y be disjoint subsets of the ground set E of a k-connected matroid M. Suppose that U and $U \cup Y$ are k-separating and $Y \subseteq \text{fcl}_k(U)$. If $\text{fcl}_k(U) \neq E$, then there is a partition $(Y_1, Y_2, \ldots, Y_s)$ of Y such that $|Y_i| \leq k - 2$ for each $i \in \{1, 2, \ldots, s\}$ and $(U, Y_1, Y_2, \ldots, Y_s, E - (U \cup Y))$ is a k-sequence.*

Proof. Let $(U_1, U_2, \ldots, U_l)$ be a partition of $\text{fcl}_k(U) - U$ such that, for all $i \in \{1, 2, \ldots, l\}$, we have $|U_i| \leq k-2$ and $U \cup U_1 \cup U_2 \cup \cdots \cup U_i$ is k-separating. Let $(Y_1, Y_2, \ldots, Y_s)$ be the partition of the elements of Y induced by this partition of $\mathrm{fcl}_k(U) - U$. As $\mathrm{fcl}_k(U) \neq E$, we have $|E - \mathrm{fcl}_k(U)| \geq 2k - 2$ by Lemma 4.3.5. Thus, by uncrossing $U \cup Y$ and $U \cup U_1 \cup U_2 \cup \cdots \cup U_i$ for $i \in \{1, 2, \ldots, l\}$, we deduce that $U \cup Y_1 \cup Y_2 \cup \cdots \cup Y_j$ is k-separating for all j in $\{1, 2, \ldots, s\}$. In particular, $(U, Y_1, Y_2, \ldots, Y_s, E - (U \cup Y))$ is a k-sequence. □

The following corollary is a straightforward consequence of Lemma 4.3.6, where (ii) follows from a result by Clark and Whittle (2013, Lemma 3.7).

Corollary 4.3.7. *Let U and Y be disjoint subsets of the ground set E of a k-connected matroid M. Suppose that U and $U \cup Y$ are k-separating and $Y \subseteq \mathrm{fcl}_k(U)$. If $\mathrm{fcl}_k(U) \neq E$, then*

(i) *$Y \subseteq \mathrm{fcl}_k(E - (U \cup Y))$, and*

(ii) *$(U, E - U)$ is k-equivalent to $(U \cup Y, E - (U \cup Y))$.*

4.3.2 k-flowers

The following lemma is a generalisation of a result due to Aikin and Oxley (2012, Lemma 3.4). We say that a partial k-sequence $(X_i)_{i=1}^m$ for X is *fully refined* if, for every partial k-sequence $(X'_i)_{i=1}^{m'}$ for X such that $\bigcup_{i=1}^{m'} X'_i = \bigcup_{i=1}^{m} X_i$, we have $m \geq m'$.

Lemma 4.3.8. *Let $(P_1, P_2, \ldots, P_n)$ be a tight k-flower Φ of order at least three in a k-connected matroid M. Let $(Y_i)_{i=1}^m$ be a fully refined partial k-sequence of $P_1 \cup P_2 \cup \cdots \cup P_j$, where $j \leq n - 2$. Let d be the largest member of $\{1, 2, \ldots, m\}$ such that, for all $i \in \{1, 2, \ldots, d\}$, the set Y_i is contained in one of $P_{j+1}, P_{j+2}, \ldots, P_n$, or let $d = 0$ if there is no such member. Let $Y' = Y_1 \cup Y_2 \cup \cdots \cup Y_d$.*

(i) *If $d < m$, then*

  (a) *$j = n - 2$;*

  (b) *Y_{d+1} meets both P_{n-1} and P_n;*

  (c) *each of $P_{n-1} - (Y' \cup Y_{d+1})$ and $P_n - (Y' \cup Y_{d+1})$ has between 2 and $k - 2$ elements;*

  (d) *each of $P_{n-1} - Y'$ and $P_n - Y'$ has between $k - 1$ and $2k - 5$ elements; and*

(e) $\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) = E(M)$.

(ii) *When $i \le d$,*

(a) *if Y_i is contained in P_n, then $Y_i \subseteq \mathrm{fcl}_k(P_1) - P_1$; and*

(b) *if Y_i is not contained in P_n, then $Y_i \subseteq \mathrm{fcl}_k(P_j) - P_j$.*

(iii) *The k-flower Φ is k-equivalent to*

$$(P_1 \cup (Y' \cap P_n), P_2, \ldots, P_{j-1}, P_j \cup (Y' - P_n), P_{j+1} - Y', \ldots, P_n - Y').$$

Proof. We begin by establishing (ii) and (iii). As these hold trivially when $d = 0$, we may assume that $d \ge 1$. Suppose that $Y_1 \subseteq P_n$. The sets $P_1 \cup P_2 \cup \cdots \cup P_j \cup Y_1$ and $P_1 \cup P_n$ are k-separating, and their union avoids P_{n-1}, so their intersection, $P_1 \cup Y_1$, is k-separating by uncrossing. Thus $Y_1 \subseteq \mathrm{fcl}_k(P_1)$ if $Y_1 \subseteq P_n$. Now suppose that Y_1 is not contained in P_n. Then $P_n \cap Y_1 = \emptyset$. Since $P_1 \cup P_2 \cup \cdots \cup P_j \cup Y_1$ and $P_j \cup P_{j+1} \cup \cdots \cup P_{n-1}$ are k-separating, and their union avoids P_n, their intersection, $P_j \cup Y_1$, is k-separating by uncrossing; that is, $Y_1 \subseteq \mathrm{fcl}_k(P_j)$.

If $Y_1 \subseteq P_n$, then we replace $(P_1, P_2, \ldots, P_n)$ by $(P_1 \cup Y_1, P_2, P_3, \ldots, P_{n-1}, P_n - Y_1)$. If $Y_1 \subseteq P_m$, for $j+1 \le m \le n-1$, then we replace P_j by $P_j \cup Y_1$ and replace P_m by $P_m - Y_1$. In each case, as Φ is tight, the resulting k-flower $(P'_1, P'_2, \ldots, P'_n)$ is tight, and $\mathrm{fcl}_k(P'_1 \cup P'_2 \cup \cdots \cup P'_j) = \mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j)$ and $(Y_i)_{i=2}^m$ is a partial k-sequence for $P'_1 \cup P'_2 \cup \cdots \cup P'_j$. If $d = 1$, then (ii) and (iii) hold. Otherwise, when $d \ge 2$, we can repeat this process using Y_2 rather than Y_1 in our new k-flower, and we will get that Y_2 is contained in one of $P'_{j+1}, P'_{j+2}, \ldots, P'_n$. Hence Y_2 is contained in one of $P_{j+1}, P_{j+2}, \ldots, P_n$. Then, either $P'_1 \cup Y_2$ or $P'_j \cup Y_2$ is k-separating. Continuing in this way, using $Y_3, Y_4, \ldots, Y_d$, we obtain (ii) and (iii).

To prove (i), let $\Phi'' = (P''_1, P''_2, \ldots, P''_n)$

$$= (P_1 \cup (Y' \cap P_n), P_2, \ldots, P_{j-1}, P_j \cup (Y' - P_n), P_{j+1} - Y', \ldots, P_n - Y').$$

Recall that Y_{d+1} is not contained in any of $P_{j+1}, P_{j+2}, \ldots, P_n$. Let $s \in \{j+1, j+2, \ldots, n\}$ be the minimum index such that Y_{d+1} meets P''_s. The sets $P''_1 \cup P''_2 \cup \cdots \cup P''_j \cup Y_{d+1}$ and $P''_1 \cup P''_2 \cup \cdots \cup P''_s$ are k-separating. If their union avoids at least $k-1$ elements, then, by uncrossing, $P''_1 \cup P''_2 \cup \cdots \cup P''_j \cup (P''_s \cap Y_{d+1})$ is k-separating, where $P''_s \cap Y_{d+1}$ is a non-empty proper subset of Y_{d+1}, contradicting that the partial k-sequence is fully refined. Thus we may assume that $|(P''_{s+1} \cup P''_{s+2} \cup \cdots \cup P''_n) - Y_{d+1}| \leq k-2$. Since $|Y_{d+1}| \leq k-2$ and $P''_s \cap Y_{d+1} \neq \emptyset$, it follows that $|\bigcup_{i=s+1}^{n} P''_i| \leq 2k-5$. But $|P''_i| \geq k-1$ for all $i \in \{1,2,\ldots,n\}$, since Φ is tight. Thus $s+1 = n$, the set Y_{d+1} meets P''_n, and $k-1 \leq |P''_n| \leq 2k-5$. Likewise, by uncrossing $(\bigcup_{i=1}^{j} P''_i) \cup Y_{d+1}$ and $(\bigcup_{i=1}^{j} P''_i) \cup P''_n$, we deduce that $|(\bigcup_{i=j+1}^{n-1} P''_i) - Y_{d+1}| \leq k-2$, thus $s = j+1$ and $k-1 \leq |P''_s| \leq 2k-5$. Hence $j = n-2$ and, since $|P''_n - Y_{d+1}|, |P''_{n-1} - Y_{d+1}|, |Y_{d+1}| \leq k-2$, it follows that $|(P''_n \cup P''_{n-1}) - Y_{d+1}| \leq 2k-4$. Thus the k-separating set $(P''_{n-1} \cup P''_n) - Y_{d+1}$ is k-sequential, by Lemma 4.3.5. We deduce that $\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) = E(M)$. Thus (i) holds. □

We now give three corollaries of the previous lemma. The first is analogous to a result by Oxley and Semple (2013, Lemma 3.4(i)), which concerns only 3-flowers. The requirement that $\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) \neq E(M)$, not present in the $k = 3$ case, is necessary, as will become evident in Example 4.4.3. Corollary 4.3.10 generalises the corresponding results for $k = 3$ (Oxley et al., 2004, Corollary 5.10) and $k = 4$ (Aikin and Oxley, 2012, Corollary 3.15). Similarly, Corollary 4.3.11 is a generalisation of a result by Oxley and Semple (2013, Corollary 3.5).

Corollary 4.3.9. *Let $(P_1, P_2, \ldots, P_n)$ be a tight k-flower of order at least three in a k-connected matroid M. If $\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) \neq E(M)$ for some $1 \leq j \leq n-2$, then*

$$\mathrm{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_j) - (P_1 \cup P_2 \cup \cdots \cup P_j) \subseteq (\mathrm{fcl}_k(P_1) - P_1) \cup (\mathrm{fcl}_k(P_j) - P_j),$$

and every element of $(\mathrm{fcl}_k(P_1) - P_1) \cup (\mathrm{fcl}_k(P_j) - P_j)$ is loose.

Corollary 4.3.10. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight irredundant k-flower. Then the order of Φ is n.*

Proof. By definition, the order of Φ is at most n. Towards a contradiction, suppose Φ' is a k-flower with n' petals, where $n' < n$, that is k-equivalent to Φ. Without loss of generality, we may assume that Φ' is tight. If $n' = 1$, then Φ displays no non-sequential k-separations, and it follows that Φ is not tight; a contradiction. Thus $n' \geq 2$.

Let (U_1, V_1) be a non-sequential k-separation displayed by Φ. Then Φ' displays a k-separation (U'_1, V'_1) with $\mathrm{fcl}_k(U_1) = \mathrm{fcl}_k(U'_1)$ and $\mathrm{fcl}_k(V_1) =$

$\text{fcl}_k(V_1')$. Since Φ' has fewer petals than Φ, we may assume, without loss of generality, that U_1 is the union of p_1 petals of Φ, and U_1' is the union of p_1' petals of Φ', where $p_1' < p_1$. Suppose there is a petal P_1 of Φ contained in U_1 for which $(U_1 - P_1, V_1 \cup P_1)$ is a non-sequential k-separation. The k-flower Φ' displays an equivalent k-separation (U_2', V_2'), with $\text{fcl}_k(U_1 - P_1) = \text{fcl}_k(U_2')$ and $\text{fcl}_k(V_1 \cup P_1) = \text{fcl}_k(V_2')$, where U_2' is the union of p_2' petals of Φ'. Since Φ is tight, it follows, by Corollary 4.3.9, that $P_1 \not\subseteq \text{fcl}_k(V_1)$. Thus $\text{fcl}_k(V_1') = \text{fcl}_k(V_1) \subsetneq \text{fcl}_k(V_1 \cup P_1) = \text{fcl}_k(V_2')$. If there is a petal P' of Φ' contained in $V_1' - V_2'$, then $P' \subseteq \text{fcl}_k(V_2') - V_2'$. As $\text{fcl}_k(V_2') \neq E(M)$, the set U_2' contains a petal of Φ' other than P'. By Corollary 4.3.9, P' is loose; a contradiction. We deduce that $V_1' \subsetneq V_2'$. Since U_1' is the union of p_1' petals, it follows that U_2' is the union of at most $p_1' - 1$ petals; that is, $p_2' < p_1'$. Let $(U_2, V_2) = (U_1 - P_1, V_1 \cup P_1)$. If there is a petal P_2 contained in U_2 for which $(U_2 - P_2, V_2 \cup P_2)$ is a non-sequential k-separation, then we can repeat this process until, for some $i < n$, we obtain a non-sequential k-separation (U_i, V_i) where for each petal P_i of Φ contained in U_i, if $(U_i - P_i, V_i \cup P_i)$ is a k-separation, then it is k-sequential. We relabel this k-separation (U, V). Observe that Φ' displays a k-separation (U', V'), with $\text{fcl}_k(U) = \text{fcl}_k(U')$ and $\text{fcl}_k(V) = \text{fcl}_k(V')$, such that U' is the union of p' petals of Φ', and U is the union of p petals of Φ, with $p' < p$.

Suppose that $p' \geq 2$, so $p \geq 3$. Pick distinct petals P_a, P_b, and P_c of Φ contained in U. Since Φ is irredundant, there exists a non-sequential k-separation (A, B) displayed by Φ such that $P_a \subseteq A$ and $P_b \subseteq B$. Without loss of generality, we may assume that $P_c \subseteq B$. The k-flower Φ' displays a k-separation (A', B') equivalent to (A, B). We now consider petals of Φ' contained in U'. For any such petal P_a' contained in A', we have $P_a' \cap (P_b \cup P_c) \subseteq \text{fcl}_k(A) - A$, and these elements are loose in Φ by Corollary 4.3.9. As Φ is irredundant, there exists a non-sequential k-separation (B_2, C_2) displayed by Φ such that $P_b \subseteq B_2$ and $P_c \subseteq C_2$, with an equivalent k-separation (B_2', C_2') displayed by Φ'. Since $P_b \subsetneq U$ and (B_2, C_2) is non-sequential, B_2 contains a petal of Φ other than P_b. Likewise, C_2 contains a petal other than P_c. Let P_b' be a petal of Φ' contained in B' and U'. If $P_b' \subseteq C_2'$, then $P_b' \cap P_b \subseteq \text{fcl}_k(C_2) - C_2$, and these elements are loose in Φ by Corollary 4.3.9. Otherwise, $P_b' \subseteq B_2'$, in which case $P_b' \cap P_c \subseteq \text{fcl}_k(B_2) - B_2$, and, again, these elements are loose by Corollary 4.3.9. We deduce that all the elements of $U' \cap (P_b \cup P_c)$ are loose in Φ. If V' is a single petal of

Φ', then the only non-sequential k-separation displayed by Φ' is (U', V'), in which case (A', B') is an equivalent k-separation, contradicting the fact that Φ' is tight. Thus, by Corollary 4.3.9, the elements of $\text{fcl}_k(U') - U'$ are loose, so P_b and P_c are loose; a contradiction.

We may now assume that $p' = 1$. Let P_x and P_y be distinct petals of Φ contained in U such that $P_x \cup P_y$ is k-separating. Since Φ is irredundant, there exists a non-sequential k-separation (X, Y) displayed by Φ such that $P_x \subseteq X$ and $P_y \subseteq Y$. The k-flower Φ' displays an equivalent k-separation (X', Y') for which, without loss of generality, the petal U' is contained in X'. Thus $\text{fcl}_k(P_x \cup P_y) \subseteq \text{fcl}_k(U') \subseteq \text{fcl}_k(X') = \text{fcl}_k(X)$. Now $P_y \subseteq \text{fcl}_k(P_x \cup P_y) \subseteq \text{fcl}_k(X)$, and $P_y \subseteq Y$, so $P_y \subseteq \text{fcl}_k(X) - X$. Since Y is non-sequential, it contains a petal of Φ other than P_y. Thus, by Corollary 4.3.9, P_y is loose; a contradiction. This completes the proof of the corollary. □

Corollary 4.3.11. *Let Φ be a tight irredundant flower in a k-connected matroid M and let (U, V) be a non-sequential k-separation displayed by Φ. Then no petal of Φ is in the full k-closure of both U and V.*

Proof. Let P be a petal of Φ such that $P \subseteq U$ and $P \subseteq \text{fcl}_k(V)$. Then P is a proper subset of U as (U, V) is non-sequential. Hence Φ has at least three petals. Therefore, by Corollary 4.3.10, Φ has order at least three. Since $\text{fcl}_k(V) \neq E(M)$, it follows, by Corollary 4.3.9, that P is loose; a contradiction. □

The following lemma provides a straightforward way to verify that a petal is tight.

Lemma 4.3.12. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower in a k-connected matroid M. If, for some $i \in \{1, 2, \ldots, n\}$, the petal P_i is loose, then either $P_i \subseteq \text{fcl}_k(P_1 \cup P_2 \cup \cdots \cup P_{i-1})$ or $P_i \subseteq \text{fcl}_k(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n)$.*

Proof. Let $P_i^- = P_1 \cup P_2 \cup \cdots \cup P_{i-1}$ and $P_i^+ = P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n$. If $\text{fcl}_k(P_i^+) = E(M)$, then $P_i \subseteq \text{fcl}_k(P_i^+)$; so assume otherwise. Let $A = P_i \cap \text{fcl}_k(P_i^-)$ and $B = P_i - \text{fcl}_k(P_i^-)$. Since P_i is loose, $B \subseteq \text{fcl}_k(P_i^+)$. Then, there exists a set B' containing B where $B' \cup P_i^+$ is k-separating and $B' \subseteq \text{fcl}_k(P_i^+)$. By Corollary 4.3.7(i), $B' \subseteq \text{fcl}_k((P_i^- \cup P_i) - B') \subseteq \text{fcl}_k(P_i^- \cup A) \subseteq \text{fcl}_k(P_i^-)$. Thus $B \subseteq \text{fcl}_k(P_i^-)$. We deduce that $B = \emptyset$, completing the proof of the lemma. □

Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a k-flower of M. We can obtain a new flower Φ' from $\Phi = (P_1, P_2, \ldots, P_n)$ in the following way. Let $\Phi' = (P'_1, P'_2, \ldots, P'_m)$ where there are indices $0 = j_0 < j_1 < \cdots < j_m = n$ such that $P'_i = P_{j_{i-1}+1} \cup \cdots \cup P_{j_i}$ for all $i \in \{1, 2, \ldots, m\}$. Then we say that the flower Φ' is a *concatenation* of Φ, and that Φ *refines* Φ'.

4.3.3 k-paths

Oxley and Semple (2013) introduced the notion of a 3-path to facilitate describing inequivalent non-sequential 3-separations. Here, we generalise this notion to k-paths.

Let M be a k-connected matroid with ground set E. A *k-path* in M is an ordered partition $(X_1, X_2, \ldots, X_m)$ of E into non-empty sets, called *parts*, such that

(i) $\left(\bigcup_{j=1}^{i} X_j, \bigcup_{j=i+1}^{m} X_j\right)$ is a non-sequential k-separation of M for all $i \in \{1, 2, \ldots, m-1\}$; and

(ii) for all $i \in \{2, 3, \ldots, m-1\}$, the set X_i is not in the full k-closure of either $\bigcup_{j=1}^{i-1} X_j$ or $\bigcup_{j=i+1}^{m} X_j$.

Condition (ii) is equivalent to the assertion that the non-sequential k-separations $\left(\bigcup_{j=1}^{i} X_j, \bigcup_{j=i+1}^{m} X_j\right)$ and $\left(\bigcup_{j=1}^{i+1} X_j, \bigcup_{j=i+2}^{m} X_j\right)$ are inequivalent for all $i \in \{1, 2, \ldots, m-2\}$. We say X_1 and X_m are the *end parts* of the k-path. For each $i \in \{1, 2, \ldots, m\}$, we denote the sets $\bigcup_{j=1}^{i-1} X_j$ and $\bigcup_{j=i+1}^{m} X_j$ by X_i^- and X_i^+, respectively. In particular, $X_1^- = \emptyset = X_m^+$. Observe that each of X_1 and X_m has at least $2k-2$ elements, by Lemma 4.3.5, and each of $X_2, X_3, \ldots, X_{m-1}$ has at least $k-1$ elements by (ii).

For a subset X_0 of E, an *X_0-rooted k-path* is a k-path of the form $(X_0 \cup X_1, X_2, \ldots, X_m)$ where $X_0 \cap X_1 = \emptyset$. Thus a k-path is just a $\emptyset$-rooted k-path. An X_0-rooted k-path is *maximal* if

(I) none of the sets X_i with $i \geq 2$ can be partitioned into sets $X_{i,1}, X_{i,2}, \ldots, X_{i,k}$ for some $k \geq 2$ such that $(X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_{i,1}, X_{i,2}, \ldots, X_{i,k}, X_{i+1}, \ldots, X_m)$ is a k-path; and

(II) X_1 cannot be partitioned into sets $X_{1,1}, X_{1,2}, \ldots, X_{1,k}$ for some $k \geq 2$ such that $(X_0 \cup X_{1,1}, X_{1,2}, \ldots, X_{1,k}, X_2, \ldots, X_m)$ is a k-path.

Observe that, in (II), the set $X_{1,1}$ may be empty when X_0 is non-empty although all of $X_{1,2}, X_{1,3}, \ldots, X_{1,k}$ must be non-empty. An X_0-rooted k-path is *left-justified* if, for all $i \in \{2, 3, \ldots, m\}$, no element of X_i is in the full k-closure of $\bigcup_{j=0}^{i-1} X_j$.

In what follows, we shall frequently be referring to a k-separation (R, G) in a k-connected matroid M. In general, we shall view (R, G) as a colouring of the elements of $E(M)$, the elements in R and G being coloured *red* and *green*, respectively. A non-empty subset X of $E(M)$ is *bichromatic* if it meets both R and G; otherwise it is *monochromatic*. We shall view the empty set as being monochromatic. A proof of the following lemma is given by Clark and Whittle (2013, Lemma 3.7). We make repeated use of this result in the subsequent lemmas.

Lemma 4.3.13. *Let M be a k-connected matroid. If (R, G) is a non-sequential k-separation of M and (R', G') is a k-separation of M such that $\mathrm{fcl}_k(R') = \mathrm{fcl}_k(R)$ or $\mathrm{fcl}_k(R') = \mathrm{fcl}_k(G)$, then (R', G') is a non-sequential k-separation of M that is k-equivalent to (R, G).*

The following lemmas generalise the corresponding results for 3-paths (Oxley and Semple, 2013, Lemmas 3.8–3.12, 3.14, and 3.15). The proofs for Lemmas 4.3.14, 4.3.15 and 4.3.17–4.3.20 are uncomplicated upgrades, but have been provided for completeness. On the other hand, the proof for Lemma 4.3.16 is not a trivial upgrade, as the proof given by Oxley and Semple (2013, Lemma 3.10) relies on properties specific to 3-sequences, and Lemma 4.3.21 is new.

Lemma 4.3.14. *Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M. If, for some i in $\{2, 3, \ldots, m-1\}$, both X_i^- and X_i^+ contain at least $k-1$ red and at least $k-1$ green elements, then X_i is monochromatic.*

Proof. Assume that X_i is bichromatic. The set X_i^+ contains at least $k-1$ green elements by hypothesis. Thus, by uncrossing, as both $X_i^- \cup X_i$ and R are k-separating, so is their intersection $(X_i^- \cup X_i) \cap R$. Again by uncrossing, the union of the last set with X_i^-, which equals $X_i^- \cup (X_i \cap R)$, is k-separating. By maximality, $(X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_i \cap R, X_i \cap G, X_{i+1}, \ldots, X_m)$ is not a k-path. If $X_i \cap R \subseteq \mathrm{fcl}_k((X_i \cap G) \cup X_i^+)$, then $X_i \cap R \subseteq \mathrm{fcl}_k(X_i^-)$ by

Corollary 4.3.7(i). But the original k-path is left-justified, so it follows that $X_i \cap G \subseteq \mathrm{fcl}_k(X_i^+)$. By symmetry, $X_i^- \cup (X_i \cap G)$ is k-separating, yet $(X_0 \cup X_1, X_2, \ldots, X_{i-1}, X_i \cap G, X_i \cap R, X_{i+1}, \ldots, X_m)$ is not a k-path, so $X_i \cap R \subseteq \mathrm{fcl}_k(X_i^+)$. We conclude that $X_i \subseteq \mathrm{fcl}_k(X_i^+)$; a contradiction. □

Lemma 4.3.15. *Let $(X_1, X_2, \ldots, X_m)$ be a k-path in a k-connected matroid M. Let X_0 be a subset of X_1, and let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose that, for some i in $\{1, 2, \ldots, m\}$, the set X_i is bichromatic. If, for some Z in $\{X_i^-, X_i^+\}$, there is at least one red element in Z, then there are at least $k-1$ red elements in Z.*

Proof. Suppose first that $1 \leq |X_i^+ \cap R| \leq k-2$. As $(X_i^- \cup X_i, X_i^+)$ and (R, G) are non-sequential, $|X_i^+| \geq 2k-2$ and $|R| \geq 2k-2$ by Lemma 4.3.5. Thus $|R \cap (X_i^- \cup X_i)| \geq k-1$, and, by uncrossing, $G \cap X_i^+$ is k-separating. Since X_i^+ is also k-separating and $|X_i^+ \cap R| \leq k-2$, the red elements in X_i^+ can be recoloured green, producing a k-separation equivalent to (R, G) with fewer bichromatic parts; a contradiction. Hence $|X_i^+ \cap R| \geq k-1$. A symmetric argument establishes that if $|X_i^- \cap R| \geq 1$, then $|X_i^- \cap R| \geq k-1$, but additional care is needed to handle X_0. In particular, if $1 \leq |X_i^- \cap R| \leq k-2$ and this set has non-empty intersection with X_0, then $X_0 \subseteq X_i^- \cap R$ as X_0 is monochromatic. Thus X_0 stays monochromatic after recolouring and, as $X_0 \subseteq X_1$, we produce a k-separation equivalent to (R, G) with fewer bichromatic parts. □

Lemma 4.3.16. *Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose, for some $i \in \{2, 3, \ldots, m-1\}$, the set X_i is bichromatic. Then either X_i is not k-separating, or $X_i^- \cup X_i^+$ is monochromatic.*

Proof. Assume that X_i is k-separating and that $X_i^- \cup X_i^+$ is bichromatic. By Lemma 4.3.14, X_i^- or X_i^+ contains at most $k-2$ elements of some colour, red say. But if this set has at least one red element, then, by Lemma 4.3.15, it has at least $k-1$ red elements; a contradiction. We deduce that X_i^- or X_i^+ is green. Then, by Lemma 4.3.15, X_i^+ or X_i^-, respectively, contains at least

$k-1$ red elements. If X_i contains at most $k-2$ red elements, then, for some Y in $\{X_i^- \cup X_i, X_i \cup X_i^+\}$, there are at most $k-2$ red elements contained in Y. By uncrossing Y and G, we see that $Y \cup G$, which equals $X_i \cup G$, is k-separating, so $X_i \cap R$ can be recoloured green to produce a k-separation equivalent to (R, G) with fewer bichromatic parts. Thus X_i contains at least $k-1$ red elements. Suppose that X_i contains at most $k-2$ green elements. Now, by uncrossing, $X_i \cap R$ is k-separating, so $X_i \cap G \subseteq \mathrm{fcl}_k(X_i \cap R)$ as X_i is k-separating. Since $X_i \cup R$ is k-separating, by uncrossing, it follows that we can recolour the elements in $X_i \cap G$ red to obtain a k-separation that is k-equivalent to (R, G) and which reduces the number of bichromatic parts; a contradiction. We conclude that both $X_i \cap R$ and $X_i \cap G$ contain at least $k-1$ elements.

Recall that either X_i^- or X_i^+ is green. In the first case, by uncrossing $X_i^- \cup X_i$ and G, we deduce that $X_i^- \cup (X_i \cap G)$ is k-separating. As $(X_0 \cup X_1, X_2, \dots, X_{i-1}, X_i \cap G, X_i \cap R, X_{i+1}, \dots, X_m)$ is not a k-path, but $(X_0 \cup X_1, X_2, \dots, X_m)$ is a left-justified k-path, it follows, by Corollary 4.3.7(i), that $X_i \cap R \subseteq \mathrm{fcl}_k(X_i^+)$ or $X_i \cap R \subseteq \mathrm{fcl}_k(X_i^- \cup (X_i \cap G))$. Again by Corollary 4.3.7(i), $X_i \cap R \subseteq \mathrm{fcl}_k(X_i^- \cup (X_i \cap G)) \subseteq \mathrm{fcl}_k(G)$ in either case. Since $X_i \cup G$ is k-separating, $X_i \cap R$ can be recoloured green to give a k-separation that is equivalent to (R, G) but has fewer bichromatic parts; a contradiction. Similarly, if X_i^+ is green, then $(X_i \cap G) \cup X_i^+$ is k-separating by uncrossing G and $X_i \cup X_i^+$. As the original k-path is maximal and left-justified, it follows, by Corollary 4.3.7(i), that $X_i \cap G \subseteq \mathrm{fcl}_k(X_i^+) \subseteq \mathrm{fcl}_k(G - X_i)$, where $G - X_i$ is k-separating by uncrossing G and $E(M) - X_i$. It now follows that the elements in $X_i \cap G$ can be recoloured red to give a k-separation that is equivalent to (R, G) but has fewer bichromatic parts; a contradiction. This completes the proof of the lemma. □

Lemma 4.3.17. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. If, for some i in $\{2, 3, \dots, m-1\}$, the set X_i^- is monochromatic but X_i is bichromatic, then $X_i^- \cup X_i^+$ is monochromatic.*

Proof. Assume that X_i^- is green and X_i is bichromatic, but $X_i^- \cup X_i^+$ is bichromatic. Then, by Lemma 4.3.15, X_i^+ contains at least $k-1$ red elements. Thus, by uncrossing $X_i^- \cup X_i$ and G, the set $X_i^- \cup (X_i \cap G)$ is k-separating. As the k-path $(X_0 \cup X_1, X_2, \dots, X_m)$ is maximal and left-justified, it follows, by Corollary 4.3.7(i), that $X_i \cap R \subseteq \mathrm{fcl}_k(X_i^- \cup (X_i \cap G))$, so $X_i \cap R \subseteq \mathrm{fcl}_k(G)$. Moreover, $X_i \cup G$ is k-separating by uncrossing $X_i^- \cup X_i$ and G. It follows that (R, G) and $(R - X_i, G \cup X_i)$ are k-equivalent. Hence we can recolour all the elements in $X_i \cap R$ green thereby reducing the number of bichromatic parts; a contradiction. □

Lemma 4.3.18. *Let $(Z_0, Z_1, Z_2, \dots, Z_m)$ be a k-path in a k-connected matroid M where $m \geq 2$. Let (R, G) be a non-sequential k-separation of M such that*

(i) *each of $Z_1, Z_2, \dots, Z_{m-1}$ is monochromatic;*

(ii) *either*

  (a) *Z_0 is monochromatic but $Z_0 \cup Z_1$ is not, or*

  (b) *Z_0 is bichromatic and $\min\{|Z_0 \cap R|, |Z_0 \cap G|\} \geq k - 1$; and*

(iii) *either*

  (a) *Z_m is monochromatic but $Z_{m-1} \cup Z_m$ is not, or*

  (b) *Z_m is bichromatic and $\min\{|Z_m \cap R|, |Z_m \cap G|\} \geq k - 1$.*

Then M has a k-flower $(Z_0, Z_{i,1}, Z_{i,2}, \dots, Z_{i,s}, Z_m, Z_{j,t}, Z_{j,t-1}, \dots, Z_{j,1})$ where

(I) *both $Z_{i,1} \cup Z_{i,2} \cup \cdots \cup Z_{i,s}$ and $Z_{j,t} \cup Z_{j,t-1} \cup \cdots \cup Z_{j,1}$ are monochromatic;*

(II) *each of $(Z_{i,1}, Z_{i,2}, \dots, Z_{i,s})$ and $(Z_{j,1}, Z_{j,2}, \dots, Z_{j,t})$ is a subsequence of $(Z_1, Z_2, \dots, Z_{m-1})$; and*

(III) *$\{Z_1, Z_2, \dots, Z_{m-1}\} = \{Z_{i,1}, Z_{i,2}, \dots, Z_{i,s}\} \cup \{Z_{j,1}, Z_{j,2}, \dots, Z_{j,t}\}$.*

Moreover, when Z_m is bichromatic, this k-flower can be refined so that $(Z_0, Z_{i,1}, Z_{i,2}, \dots, Z_{i,s}, Z'_m, Z''_m, Z_{j,t}, Z_{j,t-1}, \dots, Z_{j,1})$ is a k-flower where $\{Z'_m, Z''_m\} = \{Z_m \cap R, Z_m \cap G\}$ and $Z_{i,s} \cup Z'_m$ and $Z''_m \cup Z_{j,t}$ are monochromatic. When Z_0 is also bichromatic, this k-flower can be refined so that $(Z'_0, Z''_0, Z_{i,1}, Z_{i,2}, \dots, Z_{i,s}, Z'_m, Z''_m, Z_{j,t}, Z_{j,t-1}, \dots, Z_{j,1})$ is a k-flower where $\{Z'_0, Z''_0\} = \{Z_0 \cap R, Z_0 \cap G\}$ and $Z''_0 \cup Z_{i,1}$ and $Z'_0 \cup Z_{j,1}$ are monochromatic.

Proof. If Z_m is bichromatic, let $(Z'_m, Z''_m) = (Z_m \cap R, Z_m \cap G)$; otherwise, let $(Z'_m, Z''_m) = (Z_{m-1}, Z_m)$. Without loss of generality, we may assume that $Z'_m \subseteq R$ and $Z''_m \subseteq G$. By assumption, $Z_0 \cup Z_1$ is bichromatic containing at least $k-1$ red elements and at least $k-1$ green elements. Let the subsequence of $(Z_2, Z_3, \ldots, Z_{m-1}, Z'_m, Z''_m)$ consisting of red sets be $(Z_{p_1}, Z_{p_2}, \ldots, Z_{p_u}, Z'_m)$. By uncrossing R and $Z_l \cup Z_{l+1} \cup \cdots \cup Z_{m-1} \cup Z'_m \cup Z''_m$, for appropriate $l \in \{2, 3, \ldots, m\}$, we deduce that Z'_m and $Z_{p_a} \cup Z_{p_{a+1}} \cup \cdots \cup Z_{p_u} \cup Z'_m$ are k-separating for all a in $\{1, 2, \ldots, u\}$. As $Z_0 \cup Z_1 \cup \cdots \cup Z_b$ is k-separating for all b in $\{1, 2, \ldots, m-1\}$, we deduce, by uncrossing, that each of $Z_{p_1}, Z_{p_2}, \ldots, Z_{p_u}, Z'_m, Z_{p_1} \cup Z_{p_2}, Z_{p_2} \cup Z_{p_3}, \ldots, Z_{p_{u-1}} \cup Z_{p_u}, Z_{p_u} \cup Z'_m$ is k-separating. Moreover, $Z'_m \cup Z''_m$ is either Z_m or $Z_{m-1} \cup Z_m$, so this set is also k-separating.

Now let the subsequence of $(Z_2, Z_3, \ldots, Z_{m-1}, Z'_m, Z''_m)$ consisting of green sets be $(Z_{q_1}, Z_{q_2}, \ldots, Z_{q_v}, Z''_m)$. Then Z''_m is k-separating and, by uncrossing again, we deduce that each of $Z_{q_1}, Z_{q_2}, \ldots, Z_{q_v}, Z_{q_1} \cup Z_{q_2}$, $Z_{q_2} \cup Z_{q_3}, \ldots, Z_{q_{v-1}} \cup Z_{q_v}, Z_{q_v} \cup Z''_m$ is k-separating.

As each of $Z_{p_1} \cup Z_{p_2}, Z_{p_2} \cup Z_{p_3}, \ldots, Z_{p_{u-1}} \cup Z_{p_u}, Z_{p_u} \cup Z'_m, Z'_m \cup Z''_m$, $Z''_m \cup Z_{q_v}, Z_{q_v} \cup Z_{q_{v-1}}, \ldots, Z_{q_2} \cup Z_{q_1}$ is k-separating, the union of all but the last of these sets is k-separating, and hence so is its complement $Z_0 \cup Z_1 \cup Z_{q_1}$. Similarly, $Z_0 \cup Z_1 \cup Z_{p_1}$ is k-separating. We deduce that $(Z_0 \cup Z_1, Z_{p_1}, Z_{p_2}, \ldots, Z_{p_u}, Z'_m, Z''_m, Z_{q_{v-1}}, \ldots, Z_{q_1})$ is a k-flower. If Z_1 is red, then, by uncrossing, $Z_1 \cup Z_{p_1} \cup \cdots \cup Z_{p_u} \cup Z'_m$ is k-separating, as are $Z_0 \cup Z_1$ and $Z_0 \cup Z_1 \cup Z_{p_1}$, so Z_1 and $Z_1 \cup Z_{p_1}$ are k-separating. Also, $E - (Z_1 \cup Z_{p_1} \cup \cdots \cup Z_{p_u} \cup Z'_m)$ is k-separating and, by uncrossing, so too is $Z_0 \cup Z_{q_1}$. Hence $(Z_0, Z_1, Z_{p_1}, Z_{p_2}, \ldots, Z_{p_u}, Z'_m, Z''_m, Z_{q_v}, Z_{q_{v-1}}, \ldots, Z_{q_1})$ is a k-flower. If Z_1 is green, then, as Z'_m is red, a similar argument gives that $(Z_0, Z_{p_1}, Z_{p_2}, \ldots, Z_{p_u}, Z'_m, Z''_m, Z_{q_v}, Z_{q_{v-1}}, \ldots, Z_{q_1}, Z_1)$ is a k-flower. We conclude, using the notation in the statement of the lemma, that $(Z_0, Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}, Z_m, Z_{j,t}, Z_{j,t-1}, \ldots, Z_{j,1})$ is a k-flower when Z_m is monochromatic, or $(Z_0, Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}, Z'_m, Z''_m, Z_{j,t}, Z_{j,t-1}, \ldots, Z_{j,1})$ is a k-flower when Z_m is bichromatic where $\{Z'_m, Z''_m\} = \{Z_m \cap R, Z_m \cap G\}$.

Finally, assume that Z_0 is bichromatic. Then, by uncrossing, $Z_0 \cap R$ and $Z_0 \cap G$ are both k-separating, and the argument at the end of the last paragraph implies that $(Z'_0, Z''_0, Z_{i,1}, Z_{i,2}, \ldots, Z_{i,s}, Z_m, Z_{j,t}, Z_{j,t-1}, \ldots, Z_{j,1})$ is a k-flower, where $\{Z'_0, Z''_0\} = \{Z_0 \cap G, Z_0 \cap R\}$. □

Lemma 4.3.19. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose that $\{2, 3, \dots, m-1\}$ contains an element j such that X_j and X_j^- are bichromatic, but X_j^+ is red. Then $R \cap X_j \subseteq \mathrm{fcl}_k(X_j^+)$. Furthermore, there is a k-separation (R', G') equivalent to (R, G) such that $R' \cap X_j = X_j \cap \mathrm{fcl}_k(X_j^+)$ while, for all $i \neq j$, the set $R' \cap X_i = R \cap X_i$ and $G' \cap X_i = G \cap X_i$.*

Proof. By Lemma 4.3.15, $|G \cap X_j^-| \geq k-1$ as $G \cap X_j^-$ is non-empty. Therefore, as R and $X_j \cup X_j^+$ are both k-separating and avoid $G \cap X_j^-$, it follows, by uncrossing, that $(X_j^- \cup (G \cap X_j), (R \cap X_j) \cup X_j^+)$ is a k-separation. By Corollary 4.3.3, this k-separation is non-sequential. But $(X_0 \cup X_1, X_2, \dots, X_m)$ is maximal and left-justified, so $R \cap X_j \subseteq \mathrm{fcl}_k(X_j^+)$. Now $(X_j \cap \mathrm{fcl}_k(X_j^+)) \cup X_j^+$ is k-separating, and thus, by uncrossing, $(X_j \cap \mathrm{fcl}_k(X_j^+)) \cup R$ is as well. Since the latter set is equal to $R \cup (G \cap X_j \cap \mathrm{fcl}_k(X_j^+))$, it follows, by Corollary 4.3.7(ii), that recolouring all the elements in $G \cap X_j \cap \mathrm{fcl}_k(X_j^+)$ red results in a k-separation equivalent to (R, G) with the desired properties. □

Lemma 4.3.20. *Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be a left-justified maximal X_0-rooted k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts. Suppose that $m \geq 2$, and that X_m and X_m^- are bichromatic. Then both $R \cap X_m$ and $G \cap X_m$ are sequential k-separating sets.*

Proof. By Lemma 4.3.15, $|R \cap X_m^-|, |G \cap X_m^-| \geq k-1$. Therefore, as R and X_m are k-separating, $R \cap X_m$ is k-separating by uncrossing. Similarly, $G \cap X_m$ is k-separating. If $(E(M) - (R \cap X_m), R \cap X_m)$ is non-sequential, then, as $(X_0 \cup X_1, X_2, \dots, X_m)$ is left-justified and maximal, $G \cap X_m \subseteq \mathrm{fcl}_k(R \cap X_m)$. But then, by Corollary 4.3.7(i), $G \cap X_m \subseteq \mathrm{fcl}_k(X_m^-)$; a contradiction. Thus $(E(M) - (R \cap X_m), R \cap X_m)$ is sequential; in particular, by Corollary 4.3.3, $R \cap X_m$ is sequential. Similarly, $G \cap X_m$ is sequential. □

Lemma 4.3.21. *Let (X_1, X_2) be a left-justified maximal k-path in a k-connected matroid M. Let (R, G) be a non-sequential k-separation in M for which X_1 and X_2 are bichromatic, and there is no equivalent k-separation* *where X_1 or X_2 is monochromatic. Then each of $R \cap X_1$, $G \cap X_1$, $R \cap X_2$, and $G \cap X_2$ are sequential k-separating sets.*

Proof. The sets $R \cap X_2$ and $G \cap X_2$ are sequential by Lemma 4.3.20. If $R \cap X_1$ is non-sequential, then as (X_1, X_2) is a maximal k-path, $G \cap X_1 \subseteq \mathrm{fcl}_k(R \cap X_1)$, and so $G \cap X_1 \subseteq \mathrm{fcl}_k(R)$. But $G \cap X_2$ is sequential, so $G \subseteq \mathrm{fcl}_k(R)$; a contradiction. We deduce that $R \cap X_1$, and similarly $G \cap X_1$, are sequential. □

4.4 Sequential petals at the ends of k-paths

In our algorithm for constructing a k-tree, we shall construct maximal k-flowers from k-paths. Although an end part of a k-path is a non-sequential k-separating set, a tight maximal k-flower may have k-sequential petals. When $k = 3$, Oxley and Semple (2013, Lemma 3.13) showed that a non-sequential 3-separating set displayed by an end part of a 3-path breaks into at most two petals in a tight 3-flower. However, the same does not necessarily hold for the ends of k-paths when $k \geq 4$, as we shall demonstrate in Examples 4.4.3 and 4.4.4. Nevertheless, the number of petals that such an end part breaks into does not depend on k. In this section, we will show that, for all $k \geq 3$, a non-sequential k-separating set displayed by an end part of a k-path breaks into at most three petals in a tight k-flower.

Let M be a k-connected matroid. The *truncation of* M, denoted $T(M)$, is the matroid obtained by freely adding an element e to M, and then contracting e. It can be shown that for a subset $X \subseteq E(M)$, the rank of X in $T(M)$ is given by $r_{T(M)}(X) = \min\{r_M(X), r(M) - 1\}$. We can truncate a k-connected matroid of sufficiently high rank, and with no "small" circuits, in order to obtain a $(k+1)$-connected matroid, as the next lemma demonstrates.

Lemma 4.4.1. *Let M be a k-connected matroid with $r(M) > k$ and no k-circuits. Then $T(M)$ is $(k+1)$-connected.*

Proof. Let $E = E(M)$. Towards a contradiction, suppose that $T(M)$ has a j-separation for $1 \leq j \leq k$. Then there exists a subset $X \subseteq E(M)$ such that $|X|, |E - X| \geq j$ and $\lambda_{T(M)}(X) \leq j - 1$; that is,

$$j - 1 \geq r_{T(M)}(X) + r_{T(M)}(E - X) - r(T(M)).$$

First, suppose that $r_M(X), r_M(E-X) \leq r(M)-1$. Then,

$$j-1 \geq r_M(X) + r_M(E-X) - (r(M)-1) = \lambda_M(X)+1,$$

so $(X, E-X)$ is a $(j-1)$-separation in M; a contradiction. Now, if $r_M(X) = r_M(E-X) = r(M)$, then $\lambda_{T(M)}(X) = r(M)-1$, so $r(M) \leq j \leq k$; a contradiction. Thus, we may assume that precisely one of $r_M(X)$ and $r_M(E-X)$ is equal to $r(M)$, so $r_{T(M)}(X) + r_{T(M)}(E-X) = r_M(X) + r_M(E-X) - 1$. Therefore,

$$j-1 \geq (r_M(X) + r_M(E-X) - 1) - (r(M)-1) = \lambda_M(X),$$

so $(X, E-X)$ is also a j-separation in M. Since M is k-connected, $(X, E-X)$ is an exact k-separation in M. As either X or $E-X$ has rank $r(M)$, either $E-X$ or X, respectively, has rank $k-1$, and consists of at least k elements. As M has no $(k-1)$-separations, this set contains a k-element subset of rank $k-1$; a contradiction. This completes the proof of the lemma. □

We can truncate a k-flower to obtain a $(k+1)$-flower, due to the following result of Aikin (2009, Lemma 2.5.2).

Lemma 4.4.2. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ in a k-connected matroid M, with $n \geq 3$. If $r(E(M)-P_i) < r(M)$ for all $i \in \{1, 2, \ldots, n\}$, then Φ is a $(k+1)$-flower in $T(M)$.*

Shortly, we give two examples of 4-connected matroids for which an end part of a maximal 4-path breaks into three petals in a tight irredundant 4-flower. In the first example we construct a 4-anemone by modifying a type of 3-anemone called a *paddle*. Informally, one can obtain a paddle by gluing together sufficiently structured matroids along a common line, called the *spine*. Further details are given by Oxley et al. (2004, Section 4). The *free (n, j)-swirl* is a 3-connected matroid obtained by beginning with a basis $\{1, 2, \ldots, n\}$, adding j points freely on each of the n lines spanned by $\{1, 2\}, \{2, 3\}, \ldots, \{n, 1\}$, and then deleting $\{1, 2, \ldots, n\}$. In the second example we construct a k-daisy from the free $(5, 3)$-swirl.

A set Z in a k-connected matroid M is a *k-pod* if $1 < |Z| \leq k-2$ and there is a partition (X, Z, Y) of $E(M)$ such that both X and Y are k-separating, but for all non-empty proper subsets Z_1 of Z, the set $X \cup Z_1$

is not k-separating. The partition (X, Z, Y) is a *k-pod partition.* A k-pod Z is *weak* if there is a non-empty proper subset Z_1 of Z such that M has a non-sequential k-separation (A, B) with $Z_1 \subseteq A$ and $Z - Z_1 \subseteq B$; otherwise it is *strong*. It is worth noting that the situation evident in the following examples arises due to the presence of a weak k-pod that crosses two petals, P_{n-1} and P_n say, of a k-flower $(P_1, P_2, \ldots, P_n)$. Moreover, in this situation Lemma 4.3.8(i) holds when $j = n - 2$.

Example 4.4.3. Let $(P_1, P_2, P_3, P_4, P_5)$ be a paddle in a 3-connected matroid N, where P_1 and P_2 each consist of eight points freely placed in rank four, the petal P_i is a triad $\{x_i, y_i, z_i\}$ for each $i \in \{3, 4, 5\}$, and each of $\{x_3, y_3, x_4, y_4\}$, $\{x_4, y_4, x_5, y_5\}$, and $\{x_3, y_3, x_5, y_5\}$ is a circuit of N. Then $\Phi = (P_1, P_2, P_3, P_4, P_5)$ is a tight 3-flower in N. A geometric representation of N is given in Figure 4.2, where the elements of P_1 and P_2 are suppressed. The rank-8 matroid $T(N)$ is 4-connected by Lemma 4.4.1, and Φ is a tight 4-flower in $T(N)$ by Lemma 4.4.2. It is easily verified that Φ is irredundant. The set $P_3 \cup P_4$ is 4-sequential, since it has a 4-sequential ordering $(\{x_3, y_3\}, \{x_4\}, \{y_4\}, \{z_3, z_4\})$; likewise, $P_4 \cup P_5$ and $P_3 \cup P_5$ are 4-sequential. Furthermore, $(P_1, P_2, P_3 \cup P_4 \cup P_5)$ is a left-justified maximal 4-path. We also note that, for $i, j \in \{3, 4, 5\}$ with $i \neq j$, the partition $(\{x_i, y_i, x_j, y_j\}, \{z_i, z_j\}, E(N) - (P_i \cup P_j))$ is a 4-pod partition where $\{z_i, z_j\}$ is a weak 4-pod.

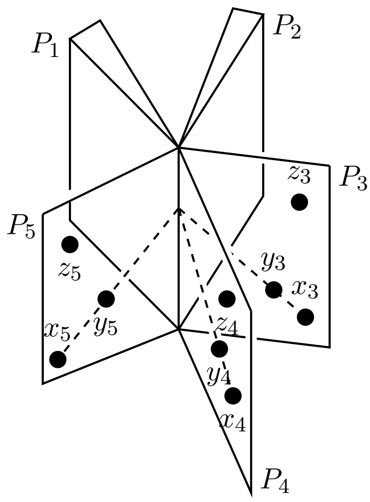


Figure 4.2: A representation of the 3-connected rank-9 paddle N.

Example 4.4.4. Let Ψ be the free $(5,3)$-swirl with $a_i, b_i, c_i \in E(\Psi)$ such that $r(\{a_i, b_i, c_i\}) = 2$ and $r(\{a_i, b_i, c_i, a_{i+1}, b_{i+1}, c_{i+1}\}) = 3$, for all $i \in \{1,2,3,4,5\}$, where the subscripts are interpreted modulo five. Let Ψ' be the coextension of this matroid by an element e where $\{a_1, b_1, a_2, b_2\}$, $\{a_2, b_2, a_3, b_3\}$ and $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ are the only dependent flats not containing e in the coextension. Let $M' = \Psi' \backslash e$. An illustration of the resulting rank-6 matroid M' is given in Figure 4.3, where the elements $\{a_i, b_i, c_i\}$ for $i \in \{4,5\}$ are suppressed. Take the direct sum of M' with a copy of $U_{2,2}$ having ground set $\{d_4, d_5\}$. Then, for each $i \in \{4,5\}$, freely add the elements e_i, f_i, g_i, and h_i, in turn, to the flat spanned by $\{a_i, b_i, c_i, d_i\}$. The resulting rank-8 matroid M is 4-connected, and $\Phi = (P_1, P_2, P_3, P_4, P_5)$ is a swirl-like 4-flower, where $P_i = \{a_i, b_i, c_i\}$ for $i \in \{1,2,3\}$ and $P_i = \{a_i, b_i, \ldots, h_i\}$ for $i \in \{4,5\}$.

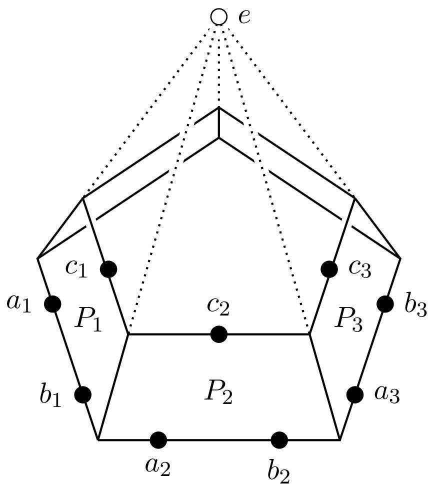


Figure 4.3: A representation of the 4-connected rank-6 matroid $M' = \Psi' \backslash e$.

It is easy to check that the 4-flower Φ is tight and irredundant. The set $P_1 \cup P_2$ is 4-sequential, since it has a 4-sequential ordering $(\{a_1, b_1\}, \{a_2\}, \{b_2\}, \{c_1, c_2\})$; likewise, $P_2 \cup P_3$ is 4-sequential. Furthermore, $(P_1 \cup P_2 \cup P_3, P_4, P_5)$ is a left-justified maximal 4-path. We also note that the partition $(\{a_1, b_1, a_2, b_2\}, \{c_1, c_2\}, E(M) - (P_1 \cup P_2))$ is a 4-pod partition where $\{c_1, c_2\}$ is a weak 4-pod, and, similarly, $(\{a_2, b_2, a_3, b_3\}, \{c_2, c_3\}, E(M) - (P_2 \cup P_3))$ is a 4-pod partition where $\{c_2, c_3\}$ is a weak 4-pod.

The k-flowers in these examples both have the property that a weak k-pod crosses two petals of the k-flower. It will become evident, in

Lemma 4.4.10, that this is precisely the situation where an end part of a k-path can break into three petals in a tight k-flower. By definition, a weak k-pod is only possible when $k \geq 4$. As a quick aside, the next lemma, which is a generalisation of a result of Aikin and Oxley (2012, Lemma 2.9), demonstrates that when (X, Z, Y) is a k-pod partition where Z is a weak k-pod, it is also necessary that either $|X| \leq 2k-4$ or $|Y| \leq 2k-4$.

Lemma 4.4.5. *Let M be a k-connected matroid. If (X, Z, Y) is a k-pod partition of $E(M)$ with $|X|, |Y| \geq 2k-3$, then Z is a strong k-pod.*

Proof. Suppose there is a k-separation (A, B) of M that is not k-sequential with $Z_1 \subseteq A$ and $Z - Z_1 \subseteq B$ for some non-empty proper subset Z_1 of Z. Let $Z_2 = Z - Z_1$. Without loss of generality, $|A \cap X| \geq k-1$. Thus, by uncrossing, $(B \cap Y) \cup Z_2$ is k-separating. If $|B \cap Y| \geq k-1$, then $Y \cup Z_2$ is k-separating, by uncrossing $(B \cap Y) \cup Z_2$ and Y, contradicting the fact that (X, Z, Y) is a k-pod partition. So $|B \cap Y| \leq k-2$. Now, since $|Y| \geq 2k-3$, we have $|A \cap Y| \geq k-1$. By symmetry, $|B \cap X| \leq k-2$. Recall that $(B \cap Y) \cup Z_2$ is k-separating; it follows that since $|B \cap X| \leq k-2$, we have $X \subseteq \mathrm{fcl}_k(A)$. But then $|B - \mathrm{fcl}_k(A)| \leq |(B \cap Y) \cup Z_2| \leq 2k-5$, so $B - \mathrm{fcl}_k(A)$ is k-sequential by Lemma 4.3.5; a contradiction. □

Corollary 4.4.6. *Let M be a k-connected matroid. If (X, Z, Y) is a k-pod partition of M where X and Y are non-sequential k-separating sets, then Z is a strong k-pod.*

Examples 4.4.3 and 4.4.4 showed that an end part of a 4-path can break into three petals of a tight k-flower, even if the k-flower is also irredundant. Recall that an end part of a 3-path can break into at most two petals of a tight 3-flower. Thus, one might expect that an end part of a k-path could break into $k-1$ petals in a tight k-flower. Fortunately, this is not the case; an end part cannot break into more than three petals, even when $k \geq 5$. This follows from the fact that, for all $k \geq 3$, the union of three consecutive petals in a tight k-flower is not k-sequential. We shall prove this as Corollary 4.4.9. First, we require the following two lemmas.

Lemma 4.4.7. *Let (U, Y, V) and (R, G) be partitions of the ground set E of a k-connected matroid. Suppose that U, V, and R are k-separating, $Y \subseteq \mathrm{fcl}_k(U) \cap R$, and $\mathrm{fcl}_k(U) \neq E$. If $|U \cap R|, |V \cap G| \geq k-1$, then $Y \subseteq \mathrm{fcl}_k(U \cap R)$.*

Proof. By Lemma 4.3.6, there exists a partition $(Y_1, Y_2, \dots, Y_s)$ of Y such that $(U, Y_1, Y_2, \dots, Y_s, V)$ is a k-sequence with $|Y_i| \le k-2$ for all $i \in \{1, 2, \dots, s\}$. As $|V \cap G| \ge k-1$, it follows, by uncrossing, that $U \cap R$ and $(U \cap R) \cup Y_1 \cup Y_2 \cup \dots \cup Y_i$ are k-separating for each i in $\{1, 2, \dots, s\}$. So $Y \subseteq \mathrm{fcl}_k(U \cap R)$. □

Lemma 4.4.8. *Let M be a k-connected matroid, and let A and B be k-separating subsets of $E(M)$ such that $|A \cap B|, |E(M) - (A \cup B)| \ge k-1$, and $A \cup B$ is a sequential k-separating set. Then, up to interchanging A and B, either*

(i) $B - A \subseteq \mathrm{fcl}_k(A \cap B)$, *where $A \cap B$ is k-separating, or*

(ii) $A \cap B \subseteq \mathrm{fcl}_k(B - A)$, *where $B - A$ is k-separating and $|B - A| \ge k-1$.*

Proof. Let $(Z_1, Z_2, \dots, Z_s)$ be a sequential ordering of $A \cup B$. We denote $Z_1 \cup Z_2 \cup \dots \cup Z_x$ as $Z_{[x]}$. Let i be the greatest index such that $|A \cap Z_{[i]}| \le k-2$ and $|B \cap Z_{[i]}| \le k-2$. Since $|A|, |B| \ge k-1$, we have $i \le s-1$. Without loss of generality, we may assume that $|A \cap Z_{[i+1]}| \ge k-1$. Suppose that $|(B - A) \cap Z_{[i+1]}| \le k-2$. By uncrossing, $A \cap Z_{[i+1]}$ is k-separating, so $(B - A) \cap Z_{[i+1]} \subseteq \mathrm{fcl}_k(A \cap Z_{[i+1]})$. Since $B - A \subseteq \mathrm{fcl}_k(Z_{[i+1]})$, we have that $B - A \subseteq \mathrm{fcl}_k(A \cap Z_{[i+1]}) \subseteq \mathrm{fcl}_k(A)$. It follows, by Lemma 4.4.7, that (i) holds. So we may assume that $|(B - A) \cap Z_{[i+1]}| \ge k-1$. Now, if $|(A - B) \cap Z_{[i+1]}| \le k-2$, then, as above, (i) holds but with the roles of A and B interchanged. Thus we may assume that $|(A - B) \cap Z_{[i+1]}| \ge k-1$. Then, by uncrossing B and $E(M) - A$, we deduce that $B - A$ is k-separating. Furthermore, since $|(A \cup B) \cap Z_{[i]}| = |B \cap Z_{[i]}| + |A \cap Z_{[i]}| - |B \cap A \cap Z_{[i]}| \le 2k-4$, and $|Z_{i+1}| \le k-2$, it follows that $|(A \cup B) \cap Z_{[i+1]}| \le 3k-6$. Thus $|A \cap B \cap Z_{[i+1]}| \le k-2$, in which case (ii) holds. □

The next corollary generalises a result of Aikin and Oxley regarding 4-flowers in 4-connected matroids (Aikin and Oxley, 2012, Corollary 3.5).

Corollary 4.4.9. *Let $(P_1, P_2, \dots, P_n)$ be a k-flower Φ of order at least three in a k-connected matroid. Then no union of three consecutive tight petals of Φ is a k-sequential set.*

Proof. Suppose that $(P_1, P_2, \dots, P_n)$ is a k-flower where $n \ge 3$, the petals P_1, P_2 and P_3 are tight, and $P_1 \cup P_2 \cup P_3$ is k-sequential. If $n = 3$, then, by Lemma 4.3.2, $P_2 \cup P_3$ is k-sequential, so $P_2 \cup P_3 \subseteq \mathrm{fcl}_k(P_1)$. Hence P_2 and P_3

are loose; a contradiction. So we may assume that $n \geq 4$. By Lemma 4.3.2, $P_1 \cup P_2$ and $P_2 \cup P_3$ are k-sequential sets. It follows, by Lemma 4.4.8, that $P_1 \subseteq \mathrm{fcl}_k(P_2)$ or $P_2 \subseteq \mathrm{fcl}_k(P_1)$, up to swapping P_1 and P_3. Thus one of P_1, P_2 or P_3 is loose; a contradiction. Hence the corollary holds. □

Recall that a non-sequential 3-separating set displayed by an end part of a 3-path breaks into at most two petals in a tight 3-flower (Oxley and Semple, 2013, Lemma 3.13). The following lemma is an analogue of this result for general k. When (ii) holds, the end part breaks into more than two petals in a tight k-flower. Corollary 4.4.11 shows that, in this case, the end part breaks into precisely three petals.

Lemma 4.4.10. *Let $(X_1, X_2, \ldots, X_m)$ be a maximal k-path in a k-connected matroid M with at least $8k-15$ elements. Let (U, V) be a non-sequential k-separation where $U \cap X_m$ and $V \cap X_m$ are k-separating sets, $U - X_m$ and $V - X_m$ are k-separating sets consisting of at least $k-1$ elements, and $U \cap X_m \nsubseteq \mathrm{fcl}_k(U - X_m)$ and $V \cap X_m \nsubseteq \mathrm{fcl}_k(V - X_m)$. Let (R, G) be a non-sequential k-separation such that both $R \cap X_m$ and $G \cap X_m$ are sequential k-separating sets. Then, by recolouring elements of X_m, there is a k-separation (R', G') equivalent to (R, G) such that either*

(i) *$U \cap X_m$ and $V \cap X_m$ are monochromatic, or*

(ii) *up to swapping R' and G', and swapping U and V, each of the following holds:*

  (a) *$U \cap X_m \subseteq R'$ and $V \cap X_m$ is bichromatic;*

  (b) *there is a sequential ordering $(Z_1, Z_2, \ldots, Z_q)$ of $R' \cap X_m$ where, for some $i \leq q$, the set Z_i is a weak k-pod, $\left|\bigcup_{j=1}^{i-1} Z_j \cap U\right|$, $\left|\bigcup_{j=1}^{i-1} Z_j \cap V\right| \leq k-2$ and $\left|\bigcup_{j=1}^{i} Z_j \cap U\right|, \left|\bigcup_{j=1}^{i} Z_j \cap V\right| \geq k-1$; and*

  (c) *for each $x \in R' \cap V \cap X_m$, $x \notin \mathrm{fcl}_k(G')$.*

Proof. We begin by proving two sublemmas.

4.4.10.1. *At least one of the sets $R \cap U \cap X_m$, $G \cap U \cap X_m$, $R \cap V \cap X_m$ and $G \cap V \cap X_m$ has at least $k-1$ elements.*

Suppose each of $R \cap U \cap X_m$, $G \cap U \cap X_m$, $R \cap V \cap X_m$, and $G \cap V \cap X_m$ has at most $k-2$ elements. Then $|X_m| \leq 4k-8$. Since $|E(M)| \geq 8k-15$, we may assume, without loss of generality, that $|U - X_m| \geq 2k - 3$ and $|R \cap (U - X_m)| \geq k - 1$. Suppose $|G \cap V| \leq k - 2$. If $|G \cap (U - X_m)| \leq k - 2$, then, by uncrossing R and $U - X_m$, it follows that $G \cap (U - X_m) \subseteq \mathrm{fcl}_k(R)$. Moreover, as $R \cup U$ is also k-separating, by uncrossing, $(G \cap (U - X_m), G \cap U \cap X_m, G \cap V)$ is a partial k-sequence for R, contradicting the fact that (R, G) is non-sequential. Thus $|G \cap (U - X_m)| \geq k - 1$. Since $|V| \geq 2k - 2$, by Lemma 4.3.5, $|R \cap V| \geq k - 1$, so $G \cap U$ is k-separating by uncrossing. It follows that $(G \cap U \cap X_m, G \cap V \cap X_m, R \cap U \cap X_m, R \cap V \cap X_m)$ is a partial k-sequence for X_m^-, so X_m^- is k-sequential; a contradiction. Now suppose $|G \cap V| \geq k - 1$. By uncrossing, $R \cap U$ is k-separating. Thus $X_m^- \cup (R \cap U)$ is k-separating. It follows that $(R \cap U \cap X_m, G \cap U \cap X_m, R \cap V \cap X_m, G \cap V \cap X_m)$ is a partial k-sequence for X_m^-; a contradiction. We deduce that (4.4.10.1) holds.

4.4.10.2. *If $|R \cap U \cap X_m| \geq k-1$ and $G \cap V \cap X_m \neq \emptyset$, then either $(U \cup R) \cap X_m$ is a sequential k-separating set, or $G \cap V \cap X_m$ can be recoloured red to obtain a k-separation equivalent to (R, G) where $V \cap X_m$ is monochromatic.*

Since $U \cap X_m$ and $R \cap X_m$ are k-separating, it follows, by uncrossing, that $(U \cup R) \cap X_m$ is k-separating. Suppose $(U \cup R) \cap X_m$ is non-sequential. As $(U \cup R) \cap X_m \subsetneq X_m$ and the k-path $(X_1, X_2, \ldots, X_m)$ is maximal, the non-empty set $G \cap V \cap X_m$ is contained in either $\mathrm{fcl}_k(X_m^-)$ or $\mathrm{fcl}_k((U \cup R) \cap X_m)$. By Corollary 4.3.7(i), $G \cap V \cap X_m$ is contained in both of these sets. If $|R \cap V \cap X_m| \leq k - 2$, then $R \cap V \cap X_m \subseteq \mathrm{fcl}_k(U \cap X_m)$. Since $G \cap V \cap X_m \subseteq \mathrm{fcl}_k((U \cup R) \cap X_m)$, we deduce that $V \cap X_m \subseteq \mathrm{fcl}_k(U \cap X_m) \subseteq \mathrm{fcl}_k(U)$. It follows, by Corollary 4.3.7(i), that $V \cap X_m \subseteq \mathrm{fcl}_k(V - X_m)$; a contradiction. So $|R \cap V \cap X_m| \geq k - 1$. Thus, since $G \cap V \cap X_m \subseteq \mathrm{fcl}_k((U \cup R) \cap X_m)$, and $|U - X_m| \geq k - 1$, it follows, by Lemma 4.4.7, that $G \cap V \cap X_m \subseteq \mathrm{fcl}_k(R \cap V \cap X_m) \subseteq \mathrm{fcl}_k(R)$. Thus $G \cap V \cap X_m$ can be recoloured red to obtain a k-separation equivalent to (R, G), thereby completing the proof of (4.4.10.2).

4.4.10.3. *Up to swapping U and V, there is a k-separation (R_1, G_1) equivalent to (R, G) such that $U \cap X_m$ is monochromatic.*

By (4.4.10.1), we can swap U and V, if necessary, so that either $R \cap U \cap X_m$ or $G \cap U \cap X_m$ consists of at least $k - 1$ elements. Without loss of generality, we assume that $|R \cap U \cap X_m| \geq k - 1$. If $G \cap V \cap X_m = \emptyset$, then

(4.4.10.3) holds. Thus we may assume, by (4.4.10.2), that $(U \cup R) \cap X_m$ is a sequential k-separating set. By Lemma 4.3.2, the k-separating set $U \cap X_m$ is also sequential. Hence, by Lemma 4.4.8, one of the following holds, where the set to which the full k-closure operator is applied is k-separating and consists of at least $k-1$ elements.

(I) $G \cap U \cap X_m \subseteq \mathrm{fcl}_k(R \cap U \cap X_m)$, or

(II) $R \cap U \cap X_m \subseteq \mathrm{fcl}_k(G \cap U \cap X_m)$, or

(III) $R \cap V \cap X_m \subseteq \mathrm{fcl}_k(R \cap U \cap X_m)$, or

(IV) $R \cap U \cap X_m \subseteq \mathrm{fcl}_k(R \cap V \cap X_m)$.

If (I) or (II) holds, then $G \cap U \cap X_m$ or $R \cap U \cap X_m$ is in the full k-closure of R or G respectively, in which case this set can be recoloured to obtain (R_1, G_1) where $U \cap X_m$ is monochromatic, satisfying (4.4.10.3).

We now consider (III) and (IV). If $G \cap U \cap X_m$ consists of at most $k-2$ elements, then this set can be recoloured red, satisfying (4.4.10.3); so assume otherwise. Suppose that (IV) holds. By uncrossing, $G \cup (U \cap X_m)$ is k-separating. Thus $R - (U \cap X_m)$ is k-separating. It follows that $R \cap U \cap X_m \subseteq \mathrm{fcl}_k(R \cap V \cap X_m) \subseteq \mathrm{fcl}_k(R - (U \cap X_m))$. Then, by Corollary 4.3.7(i), the set $R \cap U \cap X_m$ can be recoloured green, satisfying (4.4.10.3). In case (III), if $|G \cap V \cap X_m| \leq k-2$, then, by Corollary 4.3.7(i), $V \cap X_m \subseteq \mathrm{fcl}_k(U)$ implies that $V \cap X_m \subseteq \mathrm{fcl}_k(V - X_m)$; a contradiction. Now, by a similar argument as for (IV) but with U and V interchanged, the set $R - (V \cap X_m)$ is k-separating, $R \cap V \cap X_m \subseteq \mathrm{fcl}_k(R - (V \cap X_m))$, and hence $R \cap V \cap X_m$ can be recoloured green. This completes the proof of (4.4.10.3).

To complete the proof of the lemma, we may assume, by (4.4.10.3), that $U \cap X_m$ is red and $V \cap X_m$ is bichromatic with respect to (R_1, G_1). Now $|(R_1 - \mathrm{fcl}_k(G_1)) \cap X_m^-| \geq k-1$; otherwise, as $R_1 \cap X_m$ is k-sequential, $\mathrm{fcl}_k(G_1) = E(M)$. Therefore, by uncrossing, $\mathrm{fcl}_k(G_1) \cap X_m$ is k-separating. As $|U - X_m| \geq k-1$, the set $\mathrm{fcl}_k(G_1) \cap V \cap X_m$ is also k-separating, by uncrossing. If $|G_1 \cap X_m| \leq k-2$, then X_m is k-sequential; a contradiction. So $|G_1 \cap X_m| \geq k-1$, hence $G_1 \cup (\mathrm{fcl}_k(G_1) \cap V \cap X_m)$ is k-separating. Thus, we can recolour $(\mathrm{fcl}_k(G_1) - G_1) \cap V \cap X_m$ green to obtain an equivalent k-separation (R_2, G_2), where each $x \in R_2 \cap V \cap X_m$ has the property that $x \notin \mathrm{fcl}_k(G_2)$.

Let $(Z_1, Z_2, \dots, Z_t)$ be a sequential ordering of $R_2 \cap X_m$ such that $(Z_t, Z_{t-1}, \dots, Z_1)$ is a fully refined partial k-sequence for $E - (R_2 \cap X_m)$. In the remainder of this proof, we denote $Z_1 \cup Z_2 \cup \dots \cup Z_l$ as $Z_{[l]}$ for $l \in \{1, 2, \dots, t\}$. Pick the maximum $i \in \{1, 2, \dots, t-1\}$ such that $V \cap Z_{[i]} \subsetneqq V \cap Z_{[t]}$. Note that $|R_2 \cap V \cap X_m| \geq k-1$, otherwise (i) holds. Suppose $|U \cap Z_{[i]}| \geq k-1$ and $|V \cap Z_{[i]}| \leq k-2$. Since $|V - X_m| \geq k-1$, the set $U \cap Z_{[i]}$ is k-separating by uncrossing. Moreover, as $G_2 \cap X_m$, which is contained in V, has at least $k-1$ elements, $R_2 - (V \cap X_m)$ is k-separating. Thus, $V \cap Z_{[i]} \subseteq \mathrm{fcl}_k(U \cap Z_{[i]}) \subseteq \mathrm{fcl}_k(R_2 - (V \cap X_m))$. Since $R_2 \cap X_m \subseteq \mathrm{fcl}_k(Z_{[i]})$, it follows that $R_2 \cap V \cap X_m \subseteq \mathrm{fcl}_k(R_2 - (V \cap X_m))$. By Corollary 4.3.7(i), $R_2 \cap V \cap X_m \subseteq \mathrm{fcl}_k(G_2)$; a contradiction. Now suppose that $|U \cap Z_{[i]}| \leq k-2$ and $|V \cap Z_{[i]}| \geq k-1$. Then $U \cap Z_{[i]} \subseteq \mathrm{fcl}_k(V \cap Z_{[i]})$. As $U \cap X_m \subseteq \mathrm{fcl}_k(Z_{[i]})$, it follows that $U \cap X_m \subseteq \mathrm{fcl}_k(V)$. By Corollary 4.3.7(i), $U \cap X_m \subseteq \mathrm{fcl}_k(U - X_m)$; a contradiction.

Now suppose that $|U \cap Z_{[i]}|, |V \cap Z_{[i]}| \geq k-1$. Recall that $V \cap Z_{i+1}$ is non-empty, and $Z_{i+2}, Z_{i+3}, \dots, Z_t$ are contained in U. By uncrossing, $Z_{[i]} \cup (U \cap X_m)$ is k-separating, and $V \cap Z_{i+1} \subseteq \mathrm{fcl}_k(Z_{[i]} \cup (U \cap X_m))$. Since $|U \cap Z_{[i]}|, |U - X_m| \geq k-1$, it follows, by two applications of uncrossing, that $U \cup Z_{[i]} \cup X_m^-$ is k-separating. Thus the complement of this set, $(V \cap Z_{i+1}) \cup (G_2 \cap X_m)$, is k-separating. Again by uncrossing, $(V \cap Z_{i+1}) \cup G_2$ is k-separating. But then, as $|Z_{i+1}| \leq k-2$, the set $V \cap Z_{i+1}$ is contained in $\mathrm{fcl}_k(G_2)$; a contradiction. We deduce that $|U \cap Z_{[i]}| \leq k-2$ and $|V \cap Z_{[i]}| \leq k-2$.

Recall that $V \cap Z_{[i+1]} = R_2 \cap V \cap X_m$, and this set consists of at least $k-1$ elements. It follows, by uncrossing, that $V \cup Z_{[i+1]}$ is k-separating. Now, if $|U \cap Z_{[i+1]}| \leq k-2$, then $U \cap Z_{[i+1]} \subseteq \mathrm{fcl}_k(V)$, and hence $U \cap X_m \subseteq \mathrm{fcl}_k(V)$. Then, by Corollary 4.3.7(i), $U \cap X_m \subseteq \mathrm{fcl}_k(U - X_m)$; a contradiction. Thus, $|U \cap Z_{[i+1]}| \geq k-1$. Finally, we observe that Z_{i+1} is a k-pod, since $(Z_t, Z_{t-1}, \dots, Z_1)$ is a fully refined partial k-sequence for $E - (R_2 \cap X_m)$, and, since (U, V) is a non-sequential k-separation, the k-pod is weak. Thus (ii) holds, completing the proof of the lemma. □

Corollary 4.4.11. *Let $(X_1, X_2, \dots, X_m)$ be a maximal k-path in a k-connected matroid M with at least $8k-15$ elements. Let (U, V) be a non-sequential k-separation where $U \cap X_m$ and $V \cap X_m$ are k-separating sets, $U - X_m$ and $V - X_m$ are k-separating sets consisting of at least $k-1$ el-*

ements, and $U \cap X_m \not\subseteq \mathrm{fcl}_k(U - X_m)$ *and* $V \cap X_m \not\subseteq \mathrm{fcl}_k(V - X_m)$. *Let* (R, G) *be a non-sequential* k*-separation such that both* $R \cap X_m$ *and* $G \cap X_m$ *are sequential* k*-separating sets. Suppose there is no recolouring of elements of* X_m *that gives a* k*-separation equivalent to* (R, G) *such that both* $U \cap X_m$ *and* $V \cap X_m$ *are monochromatic. Then, up to swapping* U *and* V*, for some* (R', G') *equivalent to* (R, G) *obtained by recolouring elements of* X_m *and possibly swapping* R' *and* G'*:*

(i) $U \cap X_m \subseteq R'$ *and* $V \cap X_m$ *is bichromatic, and*

(ii) $(V \cap X_m^-, U \cap X_m^-, U \cap X_m, R' \cap V \cap X_m, G' \cap V \cap X_m)$ *is a* k*-flower where the last three petals are tight.*

Proof. By Lemma 4.4.10, and by swapping U and V, and R' and G', if necessary, (i) holds. Let $\Phi = (V \cap X_m^-, U \cap X_m^-, U \cap X_m, R' \cap V \cap X_m, G' \cap V \cap X_m)$. Since each of X_m, U, $R' \cap X_m$, and $V \cap X_m$ is k-separating, we deduce that Φ is a flower (Clark and Whittle, 2013, Lemma 4.2). If $U \cap X_m \subseteq \mathrm{fcl}_k(V)$, then $U \cap X_m \subseteq \mathrm{fcl}_k(U - X_m)$ by Corollary 4.3.7(i); a contradiction. Thus, by a cyclic shift of the petals and Lemma 4.3.12, $U \cap X_m$ is tight. Similarly, if $G' \cap V \cap X_m \subseteq \mathrm{fcl}_k(X_m^-)$, then $G' \cap V \cap X_m$ can be recoloured red by Corollary 4.3.7(i); a contradiction. Thus, by Lemma 4.3.12, $G' \cap V \cap X_m$ is tight. Since this petal consists of at least $k-1$ elements, $R' \cap U$ is k-separating by uncrossing. Suppose $R' \cap V \cap X_m \subseteq \mathrm{fcl}_k(V - (R' \cap X_m))$. Then $R' \cap V \cap X_m \subseteq \mathrm{fcl}_k(U)$, by Corollary 4.3.7(i), and it follows, by Lemma 4.4.7, that $R' \cap V \cap X_m \subseteq \mathrm{fcl}_k(R' \cap U)$. By uncrossing the sets $U \cup X_m^-$ and R', we deduce that $R' - (V \cap X_m)$ is k-separating. Hence $R' \cap V \cap X_m \subseteq \mathrm{fcl}_k(R' - (V \cap X_m))$, so $R' \cap V \cap X_m$ can be recoloured green by Corollary 4.3.7(i); a contradiction. Thus, by Lemma 4.3.12, $R' \cap V \cap X_m$ is tight, so (ii) holds. □

Chapter 5

A polynomial-time algorithm for constructing a k-tree

Let M be a k-connected matroid consisting of at least $8k-15$ elements, for a fixed constant k. In this chapter we present our algorithm for constructing a k-tree for M. We first address, in Section 5.1, the crucial task of finding a non-sequential k-separation satisfying certain criteria, in polynomial time. The algorithm is described, both informally and formally, in Section 5.2. Lastly, Section 5.3 discusses why an algorithm is not forthcoming from the proof of Theorem 4.0.1 (Clark and Whittle, 2013, Theorem 7.1).

5.1 Finding a non-sequential k-separation

Our approach for constructing a k-tree for a k-connected matroid depends on being able to repeatedly find non-sequential k-separations, in time polynomial in $|E(M)|$. We can do this by extending an algorithm of Cunningham and Edmonds that, in polynomial time, finds a k-separation if one exists. In order to find k-separations that are also non-sequential, we require a characterisation of non-sequential k-separations, which we prove as Lemma 5.1.3. Towards this result, we begin by considering the complexity of constructing maximal k-sequential k-separating sets.

Let M be a k-connected matroid, where $|E(M)| = n$, and let X be a subset of $E(M)$. Since there are $O(n^{k-2})$ subsets of $E(M)$ of size at most $k-2$, we can find a non-empty subset X_1 of $E(M)$ such that (X_1) is a partial k-sequence for X, or determine that no such X_1 exists, by making $O(n^{k-2})$

calls to the rank oracle. By repeating this process $O(n)$ times, we find a maximal partial k-sequence for X. Thus, we can find $\text{fcl}_k(X)$ by making at most $O(n^{k-1})$ calls to the rank oracle. We make use of this fact in the proof of the next lemma.

Lemma 5.1.1. *Let M be a k-connected matroid specified by a rank oracle, where $|E(M)| = n$. Then, the collection $\mathcal{F}$ of maximal k-sequential k-separating sets of M can be constructed in time polynomial in n.*

Proof. All $(k-1)$-element subsets of $E(M)$ are sequential k-separating sets, and every sequential k-separating set Y is a subset of $\text{fcl}_k(X)$ for some $(k-1)$-element set $X \subseteq E(M)$. Thus, the collection $\mathcal{F}$ consists of all the maximal members of $\{\text{fcl}_k(X) : |X| = k-1\}$. As there are $O(n^{k-1})$ subsets of $E(M)$ consisting of $k-1$ elements, and we can find the full k-closure of such a subset by making $O(n^{k-1})$ calls to the rank oracle, we deduce that the lemma holds. □

Recall that, up to k-equivalence, a k-tree displays each non-sequential k-separation of a k-connected matroid. From an algorithmic viewpoint, one reason we are interested in k-separations that are not sequential is that the sequential k-separations are easy to find, as shown by Lemmas 4.3.2 and 5.1.1.

We now work towards an efficient algorithm for finding a non-sequential k-separation. The following is due to Cunningham (1973), building on the Matroid Intersection Theorem of Edmonds (1970).

Theorem 5.1.2 (Cunningham, 1973)**.** *Let M be a k-connected matroid specified by a rank oracle, and let X' and Y' be disjoint subsets of $E(M)$ each having at least k elements. Then, there is a polynomial-time algorithm for either finding a k-separation (X, Y) such that $X' \subseteq X$ and $Y' \subseteq Y$, or identifying that no such k-separation exists.*

The algorithm referred to in Theorem 5.1.2 is known as the Matroid Intersection Algorithm. For details of the algorithm, we refer the reader to the book by Cook et al. (1998).

The Matroid Intersection Algorithm allows us to find a k-separation (X, Y) such that $X' \subseteq X$ and $Y' \subseteq Y$ for some disjoint sets X' and Y', or determine that none exists, in polynomial time. However, for our purposes we want to find, in polynomial time, such a k-separation (X, Y) that is non-sequential, if one exists. The next lemma, which characterises non-sequential k-separations, allows us to do this. The result generalises the characterisation of non-sequential 3-separations by Oxley and Semple (2013, Lemma 4.4). However, as the proof of that result relies on properties specific to 3-sequential sets, a different approach is taken in the proof below.

Lemma 5.1.3. *Let (U, V) be a k-separation in a k-connected matroid M, let $\mathcal{F}$ be the collection of maximal sequential k-separating sets of M, and let $j \in \{k, k+1, \ldots, 2k-2\}$. Then (U, V) is not k-sequential if and only if there are j-element subsets U' and V' of U and V, respectively, such that no member of $\mathcal{F}$ contains U' or V'.*

Proof. Suppose that (U, V) is not k-sequential. Then $(U - \mathrm{fcl}_k(V), \mathrm{fcl}_k(V))$ is also not k-sequential. We will show that there is a subset U' of $U - \mathrm{fcl}_k(V)$ satisfying the conditions of the lemma; then, symmetrically, there is a subset V' of $V - \mathrm{fcl}_k(U)$. Thus, in what follows, we may assume, without loss of generality, that V is fully closed.

By Lemma 4.3.5, $|U|, |V| \geq 2k-2$. Let U_1 be a j-element subset of U. Take $U' = U_1$, unless $U_1 \subseteq F_1$ for some $F_1 \in \mathcal{F}$. Consider the exceptional case. Let $i = 1$. If $|V - F_i| \leq k-2$, then $|V \cap F_i| \geq k-1$, so, by uncrossing, $V \subseteq \mathrm{fcl}_k(F_i)$; a contradiction. It follows that, since $|E(M) - (F_i \cup U)| = |V - F_i| \geq k-1$, the set $F_i \cap U$ is k-separating by uncrossing. Furthermore, $F_i \cap U$ is k-sequential, by Lemma 4.3.2. Thus there is a $(k-1)$-element subset Q_i of $F_i \cap U$ such that $F_i \cap U \subseteq \mathrm{fcl}_k(Q_i)$. Note that $|U - \mathrm{fcl}_k(Q_i)| \geq k-1$, otherwise $U \subseteq \mathrm{fcl}_k(Q_i)$ by uncrossing; a contradiction. Recall that j is fixed and $j - k + 1 \in \{1, 2, \ldots, k-1\}$. Let C_i be a $(j-k+1)$-element subset of $U - \mathrm{fcl}_k(Q_i)$ and let $U_{i+1} = C_i \cup Q_i$. If U_{i+1} is not contained in some $F_{i+1} \in \mathcal{F}$, then we have the desired $U' = U_{i+1}$. Otherwise, observe that for all $i \geq 1$ such that $U_{i+1} \subseteq F_{i+1} \in \mathcal{F}$, we have $F_i \cap U \subseteq \mathrm{fcl}_k(U_{i+1}) \subseteq F_{i+1}$ and $C_i \subseteq U_{i+1} - \mathrm{fcl}_k(U_i)$, so $|F_{i+1} \cap U| > |F_i \cap U|$. Therefore, we can repeat the process with $i = 2, 3, \ldots, i'$ until for $i' \leq |U| - k + 1$ either $U' = U_{i'}$ is not contained in F for all $F \in \mathcal{F}$, or $|U - \mathrm{fcl}_k(Q_{i'})| < j - k + 1$, contradicting the fact that U is not k-sequential.

The converse is a consequence of Corollary 4.3.4. □

Now to obtain a non-sequential k-separation of M, we apply Theorem 5.1.2 where the disjoint sets X' and Y' are chosen to be k-element sets that are not contained in any member of $\mathcal{F}$. Then, by Lemma 5.1.3, if there exists a k-separation (X, Y) such that $X' \subseteq X$ and $Y' \subseteq Y$, the k-separation (X, Y) is non-sequential. As k is fixed, there are polynomially many k-element subsets not contained in a member of $\mathcal{F}$. If, after searching through all such pairs of sets $\{X', Y'\}$, no k-separation (X, Y) with $X' \subseteq X$ and $Y' \subseteq Y$ is found, then M has no non-sequential k-separations.

5.2 The algorithm

At last we present the algorithm k-Tree for constructing a k-tree given a k-connected matroid M with $|E(M)| \geq 8k - 15$. We begin by describing the algorithm informally, then we give some additional definitions that are required for the subsequent formal description. We finish the section with an example to illustrate the algorithm.

Informally, the algorithm works as follows. Consider a k-connected matroid M with ground set E, for which we wish to construct a k-tree. We start with a single unmarked bag vertex labelled E as our π-labelled tree. The algorithm repeatedly selects an unmarked bag vertex B, and decides if there is a non-sequential k-separation (Y, Z) such that $Y \subseteq \pi(B)$ or $Z \subseteq \pi(B)$. If there is no such k-separation, the vertex is marked, another unmarked bag vertex B is selected, and the process repeats. If there is such a k-separation, the algorithm first finds a left-justified maximal $(E - \pi(B))$-rooted k-path by calling the first of its two subroutines, ForwardSweep. Starting with the k-path (Y, Z), this subroutine repeatedly finds non-sequential k-separations that are not equivalent to a k-separation currently displayed by the k-path. By refining the k-path methodically from the "rooted" end, outwards, we ensure that the k-path returned by ForwardSweep is maximal. Then the second subroutine, BackwardSweep, is called. This subroutine starts at the unrooted end of the k-path, and works towards the rooted end, uncovering flower structure along the way. We use a "generalised k-path" to represent the k-path together with the related uncovered flower structure. Loosely speaking, a generalised k-path allows us to describe a number of flowers in series; thus describing the k-tree structure in one direction. From the generalised k-path τ, we obtain the corresponding k-tree, which we call the "path realisation" of τ. We formally define these terms presently. The algorithm adjoins the path realisation to the bag vertex B, and then recursively proceeds by finding another unmarked bag vertex. Finally, when all bag vertices are marked, it outputs the k-tree for M.

We require some additional terminology in order to present the algorithm. Our definition of a generalised k-path is consistent with a generalised 3-path as defined by Oxley and Semple (2013); however, we need to allow for an end of a k-path to break into three petals, rather than just two, for the reasons discussed in Section 4.4.

Let M be a k-connected matroid with ground set E. Suppose that $\tau = (P_1, P_2, \dots, P_n)$ is an ordered tuple where, for each $i \in \{1, 2, \dots, n\}$, either

(i) P_i is a subset of E, or

(ii) $2 \le i \le n-1$ and $P_i = [(P_{i,1}, P_{i,2}, \dots, P_{i,j}), (P_{i,l}, P_{i,l-1}, \dots, P_{i,j+1})]$ for some $1 \le j \le l$, where the $P_{i,x}$ are mutually disjoint subsets of E for $x \in \{1, 2, \dots, l\}$.

We say that P_i is a *term* of τ for any $i \in \{2, 3, \dots, n-1\}$, and P_i is a *flower part* when (ii) holds for some $i \in \{2, 3, \dots, n-1\}$. Let $\mu = (X_1, X_2, \dots, X_n)$ be the ordered sequence obtained from τ by replacing each flower part P_i with the union X_i of all the sets enclosed by its square brackets; we say that μ is the *flattening* of τ. Suppose that for each flower part $P_i = [(P_{i,1}, P_{i,2}, \dots, P_{i,j}), (P_{i,l}, P_{i,l-1}, \dots, P_{i,j+1})]$, the partition $\Phi = (X_i^-, P_{i,1}, P_{i,2}, \dots, P_{i,j}, X_i^+, P_{i,j+1}, P_{i,j+2}, \dots, P_{i,l})$ is a k-flower, where $X_i^- = X_1 \cup X_2 \cup \cdots \cup X_{i-1}$ and $X_i^+ = X_{i+1} \cup X_{i+2} \cup \cdots \cup X_n$. We call X_i^- and X_i^+ the *entry* and *exit* petals, respectively, of Φ relative to τ, and we call $(P_{i,1}, P_{i,2}, \dots, P_{i,j})$ and $(P_{i,l}, P_{i,l-1}, \dots, P_{i,j+1})$ the *clockwise* and *anticlockwise* petals, respectively, of Φ relative to τ. If $j = l$, then the flower part P_i is of the form $[(P_{i,1}, P_{i,2}, \dots, P_{i,l})]$ and we say that Φ has no anticlockwise petals relative to τ. There are four variants of a generalised k-path. First, if μ is a k-path, then τ is a *generalised k-path.* Second, if μ is not a k-path, but P_1 is k-sequential and $P_2 = [(P_{2,1}, P_{2,2}, \dots, P_{2,j}), (P_{2,l}, P_{2,l-1}, \dots, P_{2,j+1})]$ is a flower part such that $(P_1 \cup P_{2,1}, X_2 - P_{2,1}, X_3, \dots, X_n)$ or $(P_1 \cup P_{2,1} \cup P_{2,2}, X_2 - (P_{2,1} \cup P_{2,2}), X_3, \dots, X_n)$ is a k-path, then τ is a *generalised k-path*, and we say that τ is obtained from the k-path via an *end move*, and $P_1 \cup P_{2,1}$ or $P_1 \cup P_{2,1} \cup P_{2,2}$, respectively, is the *split part*. Symmetrically, if P_n is k-sequential and $P_{n-1} = [(P_{n-1,1}, P_{n-1,2}, \dots, P_{n-1,j})(P_{n-1,l}, P_{n-1,l-1}, \dots, P_{n-1,j+1})]$ is a flower part such that either $(X_1, \dots, X_{n-2}, X_{n-1} - P_{n-1,j}, P_{n-1,j} \cup X_n)$ or

$(X_1, \dots, X_{n-2}, X_{n-1} - (P_{n-1,j-1} \cup P_{n-1,j}), P_{n-1,j-1} \cup P_{n-1,j} \cup X_n)$ is a k-path, then τ is also a *generalised k-path*, and again we say τ is obtained from the k-path via an end move, and $P_{n-1,j} \cup X_n$ or $P_{n-1,j-1} \cup P_{n-1,j} \cup X_n$, respectively, is the split part. A combination of the last two generalised k-paths also can arise: if $\tau = (P_1, [(P_{2,1}, P_{2,2}, \dots, P_{2,p})], P_3)$, where $p \in \{2, 3, 4\}$, and $(P_1 \cup P_{2,1} \cup P_{2,2} \cup \dots \cup P_{2,j}, P_{2,j+1} \cup \dots \cup P_{2,p} \cup P_3)$ is a k-path for some $j \in \{1, \dots, p-1\}$, then τ is a *generalised k-path*, we say τ is obtained from the k-path by *end moves*, and $P_1 \cup P_{2,1} \cup P_{2,2} \cup \dots \cup P_{2,j}$ and $P_{2,j+1} \cup \dots \cup P_{2,p} \cup P_3$ are the *split parts*.

Let τ be a generalised k-path. We say that τ is *left-justified* if the flattening of τ is left-justified. Let Z be a term in τ. We can then write τ as $(\tau(Z^-), Z, \tau(Z^+))$, so $\tau(Z^-)$ and $\tau(Z^+)$ denote, respectively, the portions of τ that occur before and after Z. In this case, as in a k-path, we shall denote by Z^- and Z^+ the union of all of the sets in τ that occur, respectively, before and after Z. If $\tau = (\tau(Z_i^-), Z_i, Z_{i+1}, \tau(Z_{i+1}^+))$, where Z_i and Z_{i+1} are terms for which (i) or (ii) holds, then we sometimes write $\tau(Z_{i+1}^+)$ as $\tau(Z_i^{++})$.

Let $\tau_1 = (P_1, P_2, \dots, P_n)$ be a generalised k-path of M. Suppose that τ_2 is obtained from τ_1 in one of the following ways:

(I) for some $1 \le i < i' \le n$, where each of $P_i, P_{i+1}, \dots, P_{i'}$ are subsets of E, $\tau_2 = (P_1, P_2, \dots, P_{i-1}, P_i \cup P_{i+1} \cup \dots \cup P_{i'}, P_{i'+1}, P_{i'+2}, \dots, P_n)$; or

(II) for some $2 \le i \le n-1$, where $P_i = [(P_{i,1}, P_{i,2}, \dots, P_{i,j}), (P_{i,l}, P_{i,l-1}, \dots, P_{i,j+1})]$ is a flower part, $\tau_2 = (P_1, P_2, \dots, P_{i-1}, P_{i,1} \cup P_{i,2} \cup \dots \cup P_{i,l}, P_{i+1}, P_{i+2}, \dots, P_n)$.

Clearly, τ_2 is a generalised k-path. We say that τ_m, for some $m \ge 1$, is a *concatenation* of τ_1 if there is a sequence $\tau_1, \tau_2, \dots, \tau_m$ where each τ_{i+1} is obtained from τ_i by either (I) or (II). Conversely, we say that τ_1 is a *refinement* of τ_m.

Let τ be a generalised k-path in a k-connected matroid M with ground set E, and let $\mu = (Y_1, Y_2, \dots, Y_p)$ be the flattening of τ. Note that μ is a k-path unless Y_1 or Y_p is k-sequential as may occur if we apply an end move or end moves. Let P denote the π-labelled tree consisting of a path of p bag vertices labelled, in order, $Y_1, Y_2, \dots, Y_p$. Now modify P as follows. For each Y_j that is the union of s clockwise petals and t anticlockwise petals of a flower, replace the bag vertex labelled Y_j with a flower vertex v and adjoin

$s + t$ new bag vertices to v, each via a new edge, so that the cyclic ordering induced by the cyclic ordering on the edges incident with v preserves the ordering of the flower Φ_j to which Y_j corresponds. Label the vertex v by D or A depending on whether Φ_j is a daisy or an anemone, respectively. We refer to the resulting modification of P as a *path realisation* of τ.

The algorithm k-Tree is given on the next page, while the subroutine ForwardSweep is on page 101, and the subroutine BackwardSweep begins on page 102. The algorithm follows the approach taken by Oxley and Semple (2013); indeed, it generalises their algorithm 3-tree. However, because of the additional hurdles in going from $k = 3$ to arbitrary k, modifications have been necessary, resulting in extra length in the description of the algorithm. These modifications are required in order to handle the more-complicated end moves, and to ensure the resulting k-flower is irredundant. The notable changes are in BackwardSweep, at lines 4–18, 29–32, and 67–70.

Algorithm 1 k-TREE(M)

Input: A k-connected matroid M with ground set E and $|E| \geq 8k-15$.
Output: A k-tree for M.

Construct the collection $\mathcal{F}$ of maximal sequential k-separating sets of M.
Let T_0 denote the π-labelled tree consisting of a single unmarked bag vertex labelled E.
if there exists a k-separation (U, V) for which U and V contain mutually disjoint k-element subsets U' and V', respectively, such that no member of $\mathcal{F}$ contains U' or V', **then**
    Set $X_0 = \emptyset$, set $X_1 = \mathrm{fcl}_k(U)$, set $X_2 = V - \mathrm{fcl}_k(U)$, and set $i = 1$.
    Call FORWARDSWEEP(M, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$) and let $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ be the resulting k-path.
    Call BACKWARDSWEEP(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$, $\mathcal{F}$), and let T_1 be the path realisation of the resulting generalised k-path, with each bag vertex unmarked.
    while there is an unmarked bag vertex B of T_i, **do**
        if B is a non-terminal bag vertex, **then**
            Find a k-separation (Y, Z) such that Y contains $\mathrm{fcl}_k(E - \pi(B))$, and Z contains a k-element subset $Z' \subseteq \pi(B) - \mathrm{fcl}_k(E - \pi(B))$ with no member of $\mathcal{F}$ containing Z'.
        else ▷ B is a terminal bag vertex
            Find a k-separation (Y, Z) such that Y contains $\mathrm{fcl}_k(E - \pi(B))$ and an element $y \in \pi(B) - \mathrm{fcl}_k(E - \pi(B))$, and Z contains a k-element subset $Z' \subseteq \pi(B) - \mathrm{fcl}_k(E - \pi(B)) - \{y\}$ with no member of $\mathcal{F}$ containing Z'.
        if there exists such a k-separation (Y, Z), **then**
            Set $X_0 = E - \pi(B)$, set $X_1 = \pi(B) \cap \mathrm{fcl}_k(Y)$, set $X_2 = \pi(B) - \mathrm{fcl}_k(Y)$, and increase i by 1.
            Call FORWARDSWEEP(M, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$), and let $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ be the resulting k-path.
            Call BACKWARDSWEEP(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$, $\mathcal{F}$).
            Find the path realisation T_i' of resulting generalised k-path.
            Identify the vertex $X_0 \cup Z_1$ of T_i' with the vertex B of T_{i-1}, label the resulting composite vertex Z_1, and, if $Z_1 = \emptyset$ and Z_1 has degree two, then suppress this vertex. Let T_i be the resulting tree, where each bag vertex originating from the path realisation, including the identified vertex, is unmarked.
        else ▷ There is no such k-separation (Y, Z)
            Mark B.
    output T_i.
else ▷ There is no such k-separation (U, V)
    Mark E and **output** T_0.

Algorithm 2 ForwardSweep(M, $(X_0 \cup X_1, X_2)$, $\mathcal{F}$)

Input: A k-connected matroid M with ground set E and $|E| \geq 8k-15$, a k-path $(X_0 \cup X_1, X_2)$ of M, and the collection $\mathcal{F}$ of maximal sequential k-separating sets of M.
Output: A k-path $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ of M that is a refinement of $(X_0 \cup X_1, X_2)$.

Let $\tau_0 = (X_0 \cup X_1, X_2)$, set $(i, s, m) = (0, 1, 2)$, and set $(X'_1, X'_2) = (X_1, X_2)$.
while $s \leq m$, **do**
 ▷ See if we can refine X'_s in $\tau_i = (X_0 \cup X'_1, X'_2, \ldots, X'_m)$
 if $s = 1$ **and** $X_0 = \emptyset$, **then**
  Find a k-separation (Y, Z) such that Y contains a k-element subset Y' of X'_1 with no member of $\mathcal{F}$ containing Y', and Z contains $X'_2 \cup \cdots \cup X'_m$ and an element z of X'_1 with $z \notin \mathrm{fcl}_k(X'_2 \cup \cdots \cup X'_m) \cup Y'$.
 else if $s = 1$ **and** $X_0 \neq \emptyset$, **then**
  Find a k-separation (Y, Z) such that Y contains $\mathrm{fcl}_k(X_0)$, and Z contains $X'_2 \cup \cdots \cup X'_m$ and an element z of X'_1 with $z \notin \mathrm{fcl}_k(X'_2 \cup \cdots \cup X'_m)$.
 else if $s < m$, **then**
  Find a k-separation (Y, Z) such that Y contains $X_0 \cup X'_1 \cup \cdots \cup X'_{s-1}$ and an element y of $X'_s - \mathrm{fcl}_k(X_0 \cup X'_1 \cup \cdots \cup X'_{s-1})$, and Z contains $X'_{s+1} \cup \cdots \cup X'_m$ and an element z of X'_s with $z \notin \mathrm{fcl}_k(X'_{s+1} \cup \cdots \cup X'_m) \cup \{y\}$.
 else ▷ $s = m$
  Find a k-separation (Y, Z) such that Y contains $X_0 \cup X'_1 \cup \cdots \cup X'_{s-1}$ and an element y of $X'_s - \mathrm{fcl}_k(X_0 \cup X'_1 \cup \cdots \cup X'_{s-1})$, and Z contains a k-element subset Z' of $X'_s - \mathrm{fcl}_k(X_0 \cup X'_1 \cup \cdots \cup X'_{s-1}) - \{y\}$ with no member of $\mathcal{F}$ containing Z'.
 if there exists such a k-separation (Y, Z), **then**
  Increase m by 1 and, for each $t > s$, set X'_t to be X'_{t+1}.
  Set X'_{s+1} to be $X'_s \cap (E - \mathrm{fcl}_k(Y))$ and set X'_s to be $X'_s \cap \mathrm{fcl}_k(Y)$.
  Increase i by 1 and set τ_i to be $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$.
 else
  Increase s by 1.
output τ_i.

Algorithm 3 BACKWARDSWEEP(M, $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$, $\mathcal{F}$)

Input: A k-connected matroid M with ground set E and $|E| \geq 8k-15$, a left-justified maximal k-path $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ of M, where $m \geq 2$, and the collection $\mathcal{F}$ of maximal sequential k-separating sets of M.

Output: A generalised k-path of M.

if $m = 2$, **then**

  if X_0 is empty **and** there exists a k-separation (U, V) for which U contains a subset U' and V contains a subset V' such that no member of $\mathcal{F}$ contains U' or V', and $|U' \cap Z_1| = |U' \cap Z_2| = |V' \cap Z_1| = |V' \cap Z_2| = k-1$, **then**

    ▷ See if Z_2 breaks into three petals.

    if there exists a k-separation (S, T) for which S contains $U \cap Z_2$ and an element $s' \in Z_2 - \mathrm{fcl}_k(U \cap Z_2)$, and T contains Z_1 and $|T \cap Z_2| \geq k-1$; **and** there exists a k-separation (S_1, T_1) for which S_1 contains S and an element $s \in Z_1 - \mathrm{fcl}_k(S)$, and T_1 contains a subset T' such that no member of $\mathcal{F}$ contains T' and $|T' \cap Z_1| = |T' \cap Z_2| = k-1$, **then**

      Set $\tau_2 = (Z_1, [(U \cap Z_2, S_1 \cap V)], T_1 \cap Z_2)$.

    else if there exists a k-separation (S, T) for which T contains $V \cap Z_2$ and an element $t' \in Z_2 - \mathrm{fcl}_k(V \cap Z_2)$, and S contains Z_1 and $|S \cap Z_2| \geq k-1$; **and** there exists a k-separation (S_1, T_1) for which T_1 contains T and an element $t \in Z_1 - \mathrm{fcl}_k(T)$, and S_1 contains a subset S' such that no member of $\mathcal{F}$ contains S' and $|S' \cap Z_1| = |S' \cap Z_2| = k-1$, **then**

      Set $\tau_2 = (Z_1, [(S_1 \cap Z_2, T_1 \cap U)], V \cap Z_2)$.

    else

      Set $\tau_2 = (Z_1, [(U \cap Z_2)], V \cap Z_2)$.

    Let $\tau_2 = (Z_1, [(P_1, \ldots, P_p)], Q)$ with $p \in \{1, 2\}$, and $P = \bigcup_{i=1}^{p} P_i$.

    ▷ See if Z_1 breaks into three petals.

    if there exists a k-separation (S, T) such that S contains both $V - P$ and an element $s \in Z_1 - \mathrm{fcl}_k(V - P)$; and T contains P, an element $t \in Z_1 - (\mathrm{fcl}_k(P) \cup \{s\})$, and a k-element subset T' such that no member of $\mathcal{F}$ contains T', **then**

      ▷ (S, T) non-sequential, so corresponding flower irredundant.

      output $(V \cap Z_1, [(S \cap U, T \cap Z_1, P_1, \ldots, P_p)], Q)$.

    else if there exists a k-separation (S, T) such that S contains both $(Z_1 \cap U) \cup P_1$ and an element $s \in Z_1 - \mathrm{fcl}_k((Z_1 \cap U) \cup P_1)$; and T contains $Z_2 - P_1$, an element $t \in Z_1 - (\mathrm{fcl}_k(Z_2 - P_1) \cup \{s\})$, and a k-element subset T' such that no member of $\mathcal{F}$ contains T', **then**

      output $(T \cap Z_1, [(S \cap V, U \cap Z_1, P_1, \ldots, P_p)], Q)$.

▷ Algorithm continues on the next page.

else
output $(V \cap Z_1, [(U \cap Z_1, P_1, \ldots, P_p)], Q)$.
else ▷ No such (U, V) exists
output $(X_0 \cup Z_1, Z_2)$. ▷ This completes the $m = 2$ case.
else ▷ $m \geq 3$
Let $\tau_m = (X_0 \cup Z_1, Z_2, \ldots, Z_m)$.
if Z_{m-1} is k-separating, **then**
▷ See if Z_m breaks into at least two petals.
if there exists a k-separation (U, V) such that U contains Z_{m-1}, the set V contains Z^-_{m-1}, and $|U \cap Z_m|, |V \cap Z_m| \geq k-1$, **then**
▷ Ensure that the corresponding flower is irredundant.
if there exists a k-separation (U_1, V_1) such that U_1 contains both U and a k-element subset U', and V_1 contains a k-element subset V' and $|V_1 \cap Z_m| \geq k-1$, where no member of $\mathcal{F}$ contains U' or V', **then**
▷ See if Z_m breaks into three petals.
if there exists a k-separation (S, T) such that S contains both $U_1 - Z^-_{m-1}$ and an element $s \in Z_m - \mathrm{fcl}_k(U_1 - Z^-_{m-1})$, and T contains Z^-_{m-1} and $|T \cap Z_m| \geq k-1$, **then**
Set $\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1}, U_1 \cap Z_m, S \cap V_1 \cap Z_m)], T \cap Z_m\big)$.
else if there exists a k-separation (S, T) such that S contains both Z_{m-1} and a k-element subset S', and $|S \cap U_1 \cap Z_m| \geq k-1$, and T contains a k-element subset T' and $|T \cap U_1 \cap Z_m| \geq k-1$, where no member of $\mathcal{F}$ contains S' or T' **then**
Set $\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1}, S \cap U_1 \cap Z_m, T \cap U_1 \cap Z_m)], V_1 \cap Z_m\big)$.
else ▷ No such (S, T) exists
Set $\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1}, U_1 \cap Z_m)], V_1 \cap Z_m\big)$.
else ▷ No such non-sequential (U_1, V_1)
$\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1})], Z_m\big)$.
else ▷ No such (U, V) exists
Set $\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1})], Z_m\big)$.
else if $Z_{m-1} - \mathrm{fcl}_k(Z_m)$ is k-separating, **then**
$\tau_{m-1} = \big(\tau_m(Z^-_{m-1}), [(Z_{m-1} - \mathrm{fcl}_k(Z_m))], Z_{m-1} \cap \mathrm{fcl}_k(Z_m), Z_m\big)$.
else
Set $\tau_{m-1} = \tau_m$. ▷ Continued on the next page.

▷ Uncover flower structure in $Z_{m-2}, Z_{m-3}, \dots, Z_2$.
for each i from $m-2$ down to 2, **do**
  if Z_i is k-separating, **then**
    if $\tau_{i+1}(Z_i^+) = ([(P_1, \dots, P_p), (Q_1, \dots, Q_q)], \dots)$, where $p \geq 1$ and $q \geq 0$, **then**
      if $Z_i \cup P_1$ is k-separating, **then**
        Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_{i+1}(Z_i^{++})\big)$.
      else if $q \geq 1$ and $Z_i \cup Q_1$ is k-separating, **then**
        Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(P_1, \dots, P_p), (Z_i, Q_1, \dots, Q_q)], \tau_{i+1}(Z_i^{++})\big)$.
      else if $q = 0$ and $Z_i \cup \tau_{i+1}(Z_i^{++})$ is k-separating, **then**
        Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(P_1, \dots, P_p), (Z_i)], \tau_{i+1}(Z_i^{++})\big)$.
      else
        Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i)], [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_{i+1}(Z_i^{++})\big)$.
    else ▷ $\tau_{i+1}(Z_i^+) = (Z_{i+1}, \dots)$
      Set $\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i)], \tau_{i+1}(Z_i^+)\big)$.
  else ▷ Z_i is not k-separating
    if $Z_i - \mathrm{fcl}_k(Z_i^+)$ is k-separating, **then**
      $\tau_i = \big(\tau_{i+1}(Z_i^-), [(Z_i - \mathrm{fcl}_k(Z_i^+))], Z_i \cap \mathrm{fcl}_k(Z_i^+), \tau_{i+1}(Z_i^+)\big)$.
    else
      Set $\tau_i = \tau_{i+1}$. ▷ Continued on the next page.

▷ See if Z_1 breaks into at least two petals.

if X_0 is empty, **and** $\tau_2 = (Z_1, [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], \dots)$ for some $p \geq 1$ and $q \geq 0$, **and** there exists a k-separation (U, V) for which U contains P_1 and an element $u \in Z_1 - \mathrm{fcl}_k(E - Z_1)$, and V contains both $E - (Z_1 \cup P_1)$ and an element $v \in Z_1 - (\mathrm{fcl}_k(E - Z_1) \cup \{u\})$, **then**

  ▷ Ensure that the corresponding flower will be irredundant.

  if there exists a k-separation (U_1, V_1) such that U_1 contains both U and a k-element subset U', and V_1 contains a k-element subset V' and an element $v \in Z_1 - \mathrm{fcl}_k(E - Z_1)$, where no member of $\mathcal{F}$ contains U' or V', **then**

    ▷ See if Z_1 breaks into three petals.

    if there exists a k-separation (S, T) such that S contains both $U_1 \cap (Z_1 \cup P_1)$ and an element $s \in Z_1 - (\mathrm{fcl}_k(U_1 \cap (Z_1 \cup P_1)) \cup \mathrm{fcl}_k(E - Z_1))$, and T contains both $E - (Z_1 \cup P_1)$ and an element $t \in Z_1 - (\mathrm{fcl}_k(E - Z_1) \cup \{s\})$, **then**

      output $\big(T \cap Z_1, [(S \cap V_1 \cap Z_1, U_1 \cap Z_1, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_2(Z_1^{++})\big)$.

    else if there exists a k-separation (S, T) such that S contains both an element $s \in (U_1 \cap Z_1) - \mathrm{fcl}_k(E - Z_1)$ and a k-element subset S', and T contains both an element $t \in (U_1 \cap Z_1) - (\mathrm{fcl}_k(E - Z_1) \cup \{s\})$ and a k-element subset T', where no member of $\mathcal{F}$ contains S' or T', **then**

      output $\big(V_1 \cap Z_1, [(S \cap U_1 \cap Z_1, T \cap U_1 \cap Z_1, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_2(Z_1^{++})\big)$.

    else ▷ No such (S, T) exists

      output $\big(V_1 \cap Z_1, [(U_1 \cap Z_1, P_1, \dots, P_p), (Q_1, \dots, Q_q)], \tau_2(Z_1^{++})\big)$.

  else

    ▷ No non-sequential (U_1, V_1) where $U \subseteq U_1$ and $V \cap Z_1 \subseteq V_1$.

    output τ_2.

else ▷ Either X_0 non-empty, τ_2 not of the correct form, or no such (U, V) exists

  output τ_2.

We now give an example of a k-connected matroid M, its corresponding k-tree T, and a brief walk-through of the algorithm when applied to M. This example is inspired by the corresponding example of a 3-tree for a 3-connected matroid given by Oxley and Semple (2013).

The *Higgs lift* of a matroid N, denoted $L(N)$, is obtained by freely coextending N by a non-loop element e, and then deleting e. Note that $L(N) = (T(N^*))^*$. By the next lemma, which is a consequence of Lemma 4.4.1 and duality, we can obtain a $(k+1)$-connected matroid by performing the Higgs lift on an appropriate k-connected matroid.

Lemma 5.2.1. *Let M be a k-connected matroid with $r^*(M) > k$ and no k-cocircuits. Then $L(M)$ is $(k+1)$-connected.*

The Higgs lift turns k-flowers into $(k+1)$-flowers, due to the following result of Aikin (2009, Lemma 2.6.2).

Lemma 5.2.2. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ in a k-connected matroid M, with $n \geq 4$. If every petal of Φ is a dependent set, then Φ is a $(k+1)$-flower in $L(M)$.*

We start by constructing the matroid M'. Fix $j \geq k-1$, and let S be a free $(5, j)$-swirl (V_1, V_2, V_3, V_4, L), where each of V_1, V_2, V_3, V_4, and L is a line of S. Use L as the spine of a paddle to which we attach three free $(4, j)$-swirls (X_1, X_2, X_3, L), (Y_1, Y_2, Y_3, L), and (Z_1, Z_2, Z_3, L). The resulting matroid M' is 3-connected.

We now repeatedly perform the Higgs lift to obtain $L(M')$, $L^2(M'), \ldots, L^{k-3}(M')$, for some $k \geq 4$. It is easily verified that for $i \in \{0, 1, 2, \ldots, k-4\}$, the matroid $L^i(M')$ has corank greater than $i+3$ and has no $(i+3)$-cocircuits, so $L^{k-3}(M')$ is a k-connected matroid. Moreover, for each 3-flower Φ in M', every petal of Φ is dependent in $L(M'), L^2(M'), \ldots, L^{k-4}(M')$, so Φ is a k-flower in $L^{k-3}(M')$. A possible k-tree for this matroid, irrespective of the precise value of k, is given in Figure 5.1, where large open circles represent bag vertices.

Now suppose that k-Tree is applied to $M = L^{k-3}(M')$. Let $X = X_1 \cup X_2 \cup X_3$, let $Y = Y_1 \cup Y_2 \cup Y_3$, and let $Z = Z_1 \cup Z_2 \cup Z_3$. If $(V_2 \cup V_3 \cup V_4, V_1 \cup L \cup X \cup Y \cup Z)$ is the k-separation found in line 3 of k-Tree, then a possible k-path returned by the first call to ForwardSweep is

$$(V_2 \cup V_3, V_4, V_1 \cup L, X, Z, Y_1, Y_2 \cup Y_3).$$

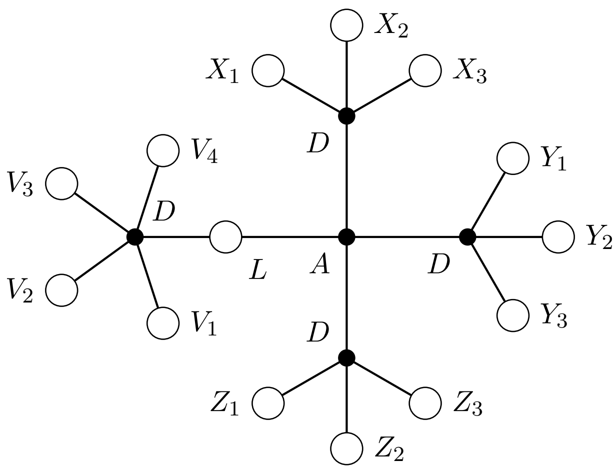


Figure 5.1: A k-tree for M.

Observe that the k-path is left-justified and maximal. With this k-path, a possible generalised k-path returned by the immediate subsequent call to BACKWARDSWEEP is

$$(V_3, [(V_2, V_1), (V_4)], L, [(X, Z)], [(Y_1, Y_2)], Y_3).$$

Comparing the k-path and the generalised k-path, both $V_2 \cup V_3$ and $Y_2 \cup Y_3$ are split parts. The splitting of $Y_2 \cup Y_3$ and $V_2 \cup V_3$ is the result of end moves performed due to k-separations being found as described in lines 25 and 63 of BACKWARDSWEEP, respectively. The path realization T_1 of this generalised k-path, produced in line 6 of k-TREE, is shown in Figure 5.2, where we note that X and Z are petals of an anemone. The algorithm now enters the loop in line 7 of k-TREE.

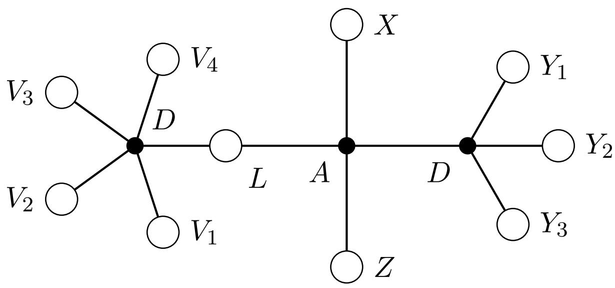


Figure 5.2: The path realization T_1.

Since all bag vertices in T_1 are unmarked, line 9 of k-TREE selects a bag vertex and, depending on whether it is a non-terminal or terminal bag, attempts to find a particular type of k-separation. If there is no such k-separation, such as when one of the bag vertices labelled V_1, V_2, V_3, V_4, L, Y_1, Y_2, or Y_3 is selected, the bag vertex is marked at line 19 of k-Tree. On the other hand, if there is such a k-separation, such as when one of the bag vertices labelled X or Z is selected, then lines 13–17 are invoked, so k-Tree calls ForwardSweep, BackwardSweep, and then updates the current π-labelled tree. For example, assume the bag vertex labelled X is selected before the bag vertex labelled Z. When this happens, k-Tree finds an appropriate k-separation in line 9, and then, in line 14, calls ForwardSweep using this k-separation. The subroutine BackwardSweep is subsequently called and a possible generalised k-path returned by this call is

$$\big(E(M) - X, [(X_1, X_2)], X_3\big).$$

A path realization of this generalised k-path is then merged with the current π-labelled tree, in this case T_1, in line 17 of k-Tree to produce the π-labelled tree T_2 shown in Figure 5.3. This process continues until all bag vertices are marked. The k-tree finally returned by k-Tree is as shown in Figure 5.1.

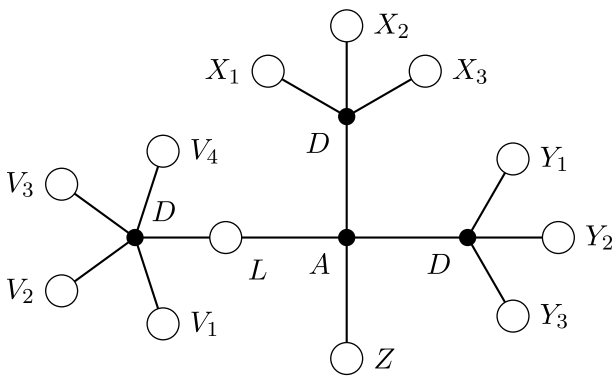


Figure 5.3: The π-labelled tree T_2.

5.3 An alternative approach

It was noted earlier that the proof of Theorem 4.0.1 (Clark and Whittle, 2013, Theorem 7.1) does not appear to yield an efficient algorithm for finding a k-tree for a k-connected matroid. We now describe the approach taken in this proof, and the difficulty in using this approach to obtain an algorithm for constructing a k-tree.

Let M be a k-connected matroid. A tight irredundant maximal k-flower is a partial k-tree T for M (Clark and Whittle, 2013, Lemma 5.10). If there exists a k-separation that is not equivalent to a k-separation displayed by T, we can modify T to obtain a partial k-tree T' where $T \preccurlyeq T'$ and T' displays a k-separation not displayed by T (Clark and Whittle, 2013, Lemma 6.3). Thus, we can eventually obtain a k-tree for M. The difficulty in using a similar approach to obtain an algorithm for constructing a k-tree lies in finding a tight irredundant maximal k-flower for M. Given a 3-separation (X, Y), it seems difficult to detect in polynomial time whether it can be refined to a 3-flower with at least three petals (Oxley and Semple, 2013, Section 7). Similarly, it is not clear whether a k-separation (X, Y) can be refined to a k-flower with at least three petals.

Chapter 6

Correctness of the algorithm

Let M be a k-connected matroid where $|E(M)| \geq 8k - 15$, and let T be the π-labelled tree returned by k-TREE when applied to M. In this chapter, we prove that T is a k-tree for M, and that k-TREE runs in time polynomial in $|E(M)|$. The crux is Lemma 6.1.4, where we prove that T is a conforming tree. Lemma 6.1.5 demonstrates that, additionally, each flower vertex of T corresponds to a tight irredundant flower. We prove these two lemmas in Section 6.1. Subsequently, for T to be a partial k-tree it remains to show that each flower vertex corresponds to a maximal flower, which we address in Section 6.2. Again, the situation is more complex for general k, but we prove, as Proposition 6.2.5, that T is indeed a partial k-tree. Finally, in Section 6.3, we prove Theorem 4.0.2 by showing that T is a k-tree and that the algorithm runs in polynomial time.

6.1 Conformance

The goals of this section are two-fold. First, we show that the tree returned by k-TREE is a conforming tree. Second, we prove that each flower vertex of this tree corresponds to a tight irredundant flower.

We begin by showing that FORWARDSWEEP outputs a left-justified maximal k-path. Lemmas 6.1.1 and 6.1.2 are straightforward generalisations of the case when $k = 3$ (Oxley and Semple, 2013, Lemmas 6.1 and 6.2), but we provide the proofs for completeness.

Lemma 6.1.1. *Let M be a k-connected matroid with $|E(M)| \geq 8k-15$. Let $(X_0 \cup X_1, X_2)$ be a k-path in M with $X_0 \cup X_1$ fully closed and let $\mathcal{F}$ be the set* *of maximal sequential k-separating sets of M. Let $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ be the output of* ForwardSweep *when applied to $(M, (X_0 \cup X_1, X_2), \mathcal{F})$. Then $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ is a left-justified maximal X_0-rooted k-path of M.*

Proof. By construction, $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ is a left-justified X_0-rooted k-path. Thus, if the lemma fails, then there is a partition (Y_j, Z_j) of X'_j for some j in $\{1, 2, \ldots, m\}$ such that $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1} \cup Y_j, Z_j \cup X'_{j+1} \cup \cdots \cup X'_m)$ is a non-sequential k-separation of M. We need to show that this k-separation is equivalent to $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1}, X'_j \cup \cdots \cup X'_m)$ or $(X_0 \cup X'_1 \cup \cdots \cup X'_j, X'_{j+1} \cup \cdots \cup X'_m)$.

If $j = m$, then the result follows immediately from lines 10–15 of ForwardSweep. Thus, in what follows we assume that $j < m$.

Suppose $X_0 = \emptyset$ and $j = 1$. Then, because $(X_0 \cup Y_1, Z_1 \cup X'_2 \cup \cdots \cup X'_m)$ is a non-sequential k-separation of M, there is a k-element subset Y'_1 of Y_1 that is not contained in any member of $\mathcal{F}$, by Lemma 5.1.3. Since $Z_1 \cup X'_2 \cup \cdots \cup X'_m$ contains $X'_2 \cup \cdots \cup X'_m$, line 5 of ForwardSweep implies that every element of Z_1 is in $\mathrm{fcl}_k(X'_2 \cup \cdots \cup X'_m)$, otherwise lines 13–15 will further refine the k-path. Hence every element of Z_1 is in $\mathrm{fcl}_k(Y_1)$ and $(X_0 \cup Y_1, Z_1 \cup X'_2 \cup \cdots \cup X'_m)$ is equivalent to $(X_0 \cup X'_1, X'_2 \cup \cdots \cup X'_m)$, by Corollary 4.3.7, as required.

We may now assume that either $X_0 \neq \emptyset$ or $j \geq 2$. Then, to prevent lines 13–15 of ForwardSweep from further refining the k-path, either every element of Y_j is in $\mathrm{fcl}_k(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1})$ or every element of Z_j is in $\mathrm{fcl}_k(X'_{j+1} \cup \cdots \cup X'_m)$. Hence $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1} \cup Y_j, Z_j \cup X'_{j+1} \cup \cdots \cup X'_m)$ is equivalent to $(X_0 \cup X'_1 \cup \cdots \cup X'_{j-1}, X'_j \cup \cdots \cup X'_m)$ or $(X_0 \cup X'_1 \cup \cdots \cup X'_j, X'_{j+1} \cup \cdots \cup X'_m)$, as required. □

Lemma 6.1.2. *Let M be a k-connected matroid with ground set E, where $|E| \geq 8k - 15$. Let T_i and T_{i+1} be π-labelled trees constructed by k-*Tree*(M) in line 6 or 17, where $i \geq 0$. Suppose that T_i is a conforming tree for M, and T_{i+1} satisfies* (F1)–(F4) *but is not a conforming tree for M. Let $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$ be the k-path returned when* ForwardSweep *is applied in line 5 or 14 of k-*Tree *depending on whether $i = 0$ or i is positive. Let (R, G) be a non-sequential k-separation in M that does not conform with T_{i+1} for which X_0 is monochromatic and no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts in $(X_0 \cup X'_1, X'_2, \ldots, X'_m)$. Then $X_0 \cup X'_1$ is monochromatic unless $i = 0$. In the exceptional case, either*

X_1' is monochromatic, or both $R \cap X_1'$ and $G \cap X_1'$ are sequential k-separating sets with $|R \cap X_1'|, |G \cap X_1'| \geq k-1$.

Proof. Assume that $X_0 \cup X_1'$ is bichromatic. First suppose that $i \geq 1$. Then X_0 is non-empty. As X_0 is monochromatic, we may assume that $X_0 \subseteq G$. Furthermore, as (R, G) does not conform with T_{i+1}, the set $X_2' \cup \cdots \cup X_m'$ contains at least one red element. Since $X_0 \cup X_1'$ is bichromatic, $|R \cap (X_2' \cup \cdots \cup X_m')| \geq k-1$ by Lemma 4.3.15. Thus, since G and $X_0 \cup X_1'$ are both k-separating, it follows, by uncrossing, that $G \cap (X_0 \cup X_1')$, which equals $X_0 \cup (G \cap X_1')$, is k-separating. Therefore $(X_0 \cup (G \cap X_1'), (R \cap X_1') \cup X_2' \cup \cdots \cup X_m')$ is a k-separation. If this k-separation is non-sequential, then, by Lemma 6.1.1, it is equivalent to $(X_0 \cup X_1', X_2' \cup \cdots \cup X_m')$. Thus, we can recolour all the elements in $R \cap X_1'$ green thereby reducing the number of bichromatic parts; a contradiction. Therefore, either $X_0 \cup (G \cap X_1')$ or $(R \cap X_1') \cup X_2' \cup \cdots \cup X_m'$ is k-sequential. By Lemma 4.3.2, the last set is non-sequential as $X_2' \cup X_3' \cup \cdots \cup X_m'$ is non-sequential. Thus $X_0 \cup (G \cap X_1')$ is sequential. But, as $i \geq 1$, the set X_0 contains at least one non-sequential k-separation, contradicting Lemma 4.3.2. Hence $X_0 \cup X_1'$ is monochromatic when $i \geq 1$.

Now suppose that $i = 0$. Then X_0 is empty. If $|R \cap X_1'| \leq k-2$, then $|R - X_1'| \geq k-1$, by Lemma 4.3.5, and so, as G and X_1' are both k-separating, by uncrossing, $G \cap X_1'$ is k-separating. Therefore, as X_1' is k-separating, it follows that $R \cap X_1' \subseteq \mathrm{fcl}_k(G \cap X_1')$. Thus we can recolour $R \cap X_1'$ green thereby reducing the number of bichromatic parts; a contradiction. Hence $|R \cap X_1'| \geq k-1$ and, by symmetry, $|G \cap X_1'| \geq k-1$. If $R - X_1'$ is empty, then, as $(X_0 \cup X_1', X_2', \ldots, X_m')$ is a maximal X_0-rooted k-path, (R, G) is equivalent to $(X_1', E - X_1')$. Hence $G \cap X_1' \subseteq \mathrm{fcl}_k(R)$ and so we can recolour the elements in $G \cap X_1'$ red, reducing the number of bichromatic parts; a contradiction. Thus $|R - X_1'| \geq 1$ and so, by Lemma 4.3.15, $|R - X_1'| \geq k-1$. Similarly, $|G - X_1'| \geq k-1$. It now follows by uncrossing that both $G \cap X_1'$ and $R \cap X_1'$ are k-separating.

Consider the k-separation $(G \cap X_1', E - (G \cap X_1'))$. If this k-separation is non-sequential, then, by Lemma 6.1.1, it is equivalent to $(X_1', E - X_1')$ and so $R \cap X_1' \subseteq \mathrm{fcl}_k(G \cap X_1') \subseteq \mathrm{fcl}_k(G)$. Thus we can recolour all the elements in $R \cap X_1'$ green thereby reducing the number of bichromatic parts; a contradiction. Hence either $G \cap X_1'$ or $E - (G \cap X_1')$ is sequential. As $E - (G \cap X_1')$ contains the non-sequential set $X_2' \cup X_3' \cup \cdots \cup X_m'$, it follows, by Lemma 4.3.2, that $G \cap X_1'$ is sequential. By symmetry, $R \cap X_1'$ is also sequential, and the lemma follows. □

In order to prove Lemma 6.1.4, we require one more lemma. We omit the proof, which follows directly from results of Clark and Whittle (2013, Lemmas 5.5 and 5.9).

Lemma 6.1.3. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight k-flower of order at least three in a k-connected matroid M. Let (R, G) be a non-sequential k-separation such that P_1 is bichromatic, P_2 is red, and no equivalent k-separation has fewer bichromatic petals. Then, there is a tight k-flower $(G \cap P_1, R \cap P_1, P_2, \ldots, P_n)$ that refines Φ.*

The next two lemmas collectively generalise a result by Oxley and Semple (2013, Lemma 6.3). As the proof of that result is sizeable, we present the generalisation as two lemmas. When proving the result for arbitrary k, the main difference is that we have to deal with the possibility of end parts breaking into three and not just two petals. In the proof of Lemma 6.1.4, these are the cases where (6.1.4.1)(ii) or (6.1.4.2)(ii) hold. In Lemma 6.1.5, the last two paragraphs of (6.1.5.1) handle this possibility. Recall that a k-flower $\Phi = (P_1, P_2, \ldots, P_n)$ is *irredundant* if Φ is a k-daisy and, for all $i \in \{1, 2, \ldots, n\}$, there is a non-sequential k-separation (X, Y) displayed by Φ with $P_i \subseteq X$ and $P_{i+1} \subseteq Y$; or Φ is a k-anemone and, for all distinct $i, j \in \{1, 2, \ldots, n\}$, there is a non-sequential k-separation (X, Y) displayed by Φ with $P_i \subseteq X$ and $P_j \subseteq Y$. As we are interested in the non-sequential k-separations of a matroid, it is most efficient for the tree to display irredundant flowers. Whereas every tight 3-flower is irredundant, the same cannot be said of tight k-flowers for arbitrary k. However, in (6.1.5.2) we show that every k-flower corresponding to a flower vertex of the tree returned by k-Tree is irredundant.

Lemma 6.1.4. *Let M be a k-connected matroid with $|E(M)| \geq 8k - 15$. The tree returned by k-Tree, when applied to M, is a conforming tree for M.*

Proof. Let E denote the ground set of M. We prove the lemma by showing that each of the π-labelled trees T_p constructed in lines 6 and 17 of k-Tree is a conforming tree for M. Since T_0 consists of a single bag vertex labelled E, the result holds trivially if $p = 0$. Now suppose that $p \geq 0$ and T_p is a conforming tree for M. We will eventually show that T_{p+1} is a conforming tree for M. The structure of the proof is as follows. First we show that T_{p+1} satisfies (F1)–(F4). Then, we suppose towards a contradiction that (R, G) is a non-sequential k-separation that does not conform with T_{p+1}. End moves require special attention: we show, as (6.1.4.1) and (6.1.4.2), that when one is performed, we can assume the end part breaks into two or three petals in a flower displayed by T_{p+1}, and these petals are monochromatic with respect to (R, G). To derive the contradiction, we handle the cases where $p \geq 1$ and $p = 0$ separately, as (6.1.4.3) and (6.1.4.4) respectively.

It follows by induction, Lemma 6.1.1, and the construction in BackwardSweep that T_{p+1} satisfies (F1) in the definition of a conforming tree. Furthermore, T_{p+1} trivially satisfies (F2) in this definition. To see that (F3) and (F4) hold for T_{p+1}, let $\Phi = (Q_1, Q_2, \ldots, Q_k)$ be a k-flower in M corresponding to a flower vertex v in the path realisation of the generalised k-path returned by BackwardSweep in the construction of T_{p+1} from T_p. By induction, to show that (F3) and (F4) hold for T_{p+1}, it suffices to show that v satisfies either (F3) or (F4) depending upon whether it is labelled A or D, respectively. Without loss of generality, we may assume that, relative to the generalised k-path, Q_1 is the entry petal. By construction, each petal of Φ is k-separating and, apart from at most one of $Q_1 \cup Q_2$ and $Q_1 \cup Q_k$, each pair of consecutive petals is k-separating. Thus, by symmetry, it suffices to check that $Q_1 \cup Q_2$ is k-separating. This check is done by induction by showing, for all i in $\{3, 4, \ldots, k\}$, that $Q_3 \cup Q_4 \cup \cdots \cup Q_i$ is k-separating. In particular, this will show that $Q_3 \cup Q_4 \cup \cdots \cup Q_k$ is k-separating, so $Q_1 \cup Q_2$ is k-separating. Clearly, Q_3 and $Q_3 \cup Q_4$ are k-separating. Now let $i \geq 5$ and assume that $Q_3 \cup Q_4 \cup \cdots \cup Q_{i-1}$ is k-separating. As $Q_{i-1} \cup Q_i$ is also k-separating, and Q_{i-1} contains at least $k - 1$ elements, it follows, by uncrossing, that $Q_3 \cup Q_4 \cup \cdots \cup Q_i$ is k-separating, as desired.

To complete the proof that T_{p+1} is a conforming tree for M, suppose there is a non-sequential k-separation (R', G') that does not conform with T_{p+1}. Because this k-separation does conform with T_p, it is equivalent to a k-separation (R, G) such that R or G is contained in a bag of T_p. Only one bag of T_p is affected in the construction of T_{p+1}, so we may assume that R or G is contained in this bag B. As $X_0 = E - \pi(B)$, which may be empty, we deduce that, with respect to (R, G), the set X_0 is monochromatic. Thus (R, G) is a non-sequential k-separation that does not conform with T_{p+1} and has X_0 monochromatic. From among the collection of choices for (R, G) satisfying these conditions, choose one such that no equivalent k-separation in which X_0 is monochromatic has fewer bichromatic parts with respect to the X_0-rooted k-path $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ returned by FORWARDSWEEP during the construction of T_{p+1} from T_p. By Lemma 6.1.1, the k-path is left-justified and maximal. By Lemma 6.1.2, we may further assume that if $p \geq 1$, then $X_0 \cup Z_1$ is monochromatic, and if $p = 0$, in which case X_0 is empty, then either Z_1 is monochromatic, or each of $R \cap Z_1$ and $G \cap Z_1$ is a sequential k-separating set consisting of at least $k - 1$ elements.

Shortly, we handle the case where $X_0 \cup Z_1$ is monochromatic, as (6.1.4.3). First, we show that when $m \geq 3$ and Z_m or Z_1 is bichromatic, then we can assume the generalised k-path returned by BACKWARDSWEEP during the construction of T_{p+1} from T_p breaks Z_m or Z_1, respectively, into monochromatic petals.

6.1.4.1. *Consider the call to* BACKWARDSWEEP *while constructing T_{p+1} from T_p. If Z_m and Z_m^- are bichromatic and Z_{m-1} is monochromatic, where $m \geq 3$, then, up to recolouring elements of Z_m to give a k-separation equivalent to (R, G), the generalised k-path τ_{m-1} is of the form*

(i) $(\ldots, [(Z_{m-1}, X)], Y)$*, where (X, Y) is a partition of Z_m such that X and Y are monochromatic, or*

(ii) $(\ldots, [(Z_{m-1}, A, B)], C)$*, where (A, B, C) is a partition of Z_m such that A, B, and C are monochromatic.*

As $|G \cap Z_m^-| \geq k - 1$, by Lemma 4.3.15, and both Z_m and R are k-separating, $R \cap Z_m$ is k-separating by uncrossing. Now, if $|G \cap Z_m| \leq k - 2$, then $G \cap Z_m \subseteq \mathrm{fcl}_k(R \cap Z_m)$, so we can recolour $G \cap Z_m$ red to obtain a k-separation equivalent to (R, G) with fewer bichromatic parts; a contradiction. Thus $|G \cap Z_m| \geq k - 1$. A similar argument shows that $|R \cap Z_m| \geq k - 1$.

We next show that line 25 of BACKWARDSWEEP is invoked. If $Z_{m-1} \subseteq R$, then, as R and $Z_{m-1} \cup Z_m$ are both k-separating and $|G \cap Z_{m-1}^-| \geq k - 1$, the set $R \cap (Z_{m-1} \cup Z_m)$ is k-separating by uncrossing. As $|G \cap Z_m| \geq k - 1$, it follows that Z_{m-1} is k-separating by uncrossing $R \cap (Z_{m-1} \cup Z_m)$ and Z_m^-. Using the fact that Z_m^- is bichromatic, the same argument shows that Z_{m-1} is k-separating when $Z_{m-1} \subseteq G$. Thus line 25 is invoked. Furthermore, as $Z_{m-1} \cup (R \cap Z_m)$ is k-separating if $Z_{m-1} \subseteq R$ and, similarly, $Z_{m-1} \cup (G \cap Z_m)$

is k-separating if $Z_{m-1} \subseteq G$, it follows that BackwardSweep finds a k-separation (U, V) as described in this line.

Suppose that both $U \cap Z_m$ and $V \cap Z_m$ are monochromatic in an (R, G)-equivalent k-separation obtained by recolouring elements of Z_m. Then, since (R, G) is non-sequential, BackwardSweep finds a k-separation (U_1, V_1) as described in line 27. It follows that τ_{m-1} is of the form $(\dots, [(Z_{m-1}, U \cap Z_m)], V \cap Z_m)$ or $(\dots, [(Z_{m-1}, A, B)], C)$, where either $(A, B \cup C) = (U \cap Z_m, V \cap Z_m)$ or $(A \cup B, C) = (U \cap Z_m, V \cap Z_m)$. Thus (i) or (ii) holds.

Now we may assume that no recolouring of elements in Z_m gives a k-separation equivalent to (R, G) such that both $U \cap Z_m$ and $V \cap Z_m$ are monochromatic. First, we show that BackwardSweep finds a non-sequential k-separation (U_1, V_1) as described in line 27. If U is non-sequential, then (U, V) is such a k-separation, so let U be k-sequential. Without loss of generality we may assume that Z_{m-1} is red. Suppose that no recolouring of elements in Z_m gives an (R, G)-equivalent k-separation such that $U \cap Z_m$ is monochromatic. Since Z_m^- is bichromatic, it follows that $|G \cap V| \geq k - 1$ by Lemma 4.3.15. By uncrossing and Lemma 4.3.2, $R \cap U$ and $U \cap Z_m$ are sequential k-separating sets. If $|R \cap U \cap Z_m| \leq k - 2$, then, since $R \cap U$ is k-separating, $R \cap U \cap Z_m \subseteq \mathrm{fcl}_k(Z_{m-1})$; a contradiction. It follows, by Lemma 4.4.8, that since no recolouring of elements of Z_m gives an (R, G)-equivalent k-separation where $U \cap Z_m$ is monochromatic, either $Z_{m-1} \subseteq \mathrm{fcl}_k(R \cap U \cap Z_m)$ or $R \cap U \cap Z_m \subseteq \mathrm{fcl}_k(Z_{m-1})$. But if the former holds, then $Z_{m-1} \subseteq \mathrm{fcl}_k(Z_m)$; a contradiction. If the latter holds, then $(Z_1, Z_2, \dots, Z_m)$ is not a left-justified k-path; a contradiction. Now we may assume that $U \cap Z_m$ is monochromatic. If U is monochromatic, then the non-sequential k-separation (R, G) satisfies the requirements of (U_1, V_1) in line 27, so we may assume that $U \cap Z_m$ is green. Recall that, as $Z_{m-1} \subseteq R$, the set $R \cap (Z_{m-1} \cup Z_m)$ is k-separating. Thus $U \cup (R \cap Z_m)$ is k-separating by uncrossing U and $R \cap (Z_{m-1} \cup Z_m)$. Suppose that $U \cup (R \cap Z_m)$ is k-sequential. Then $R \cap (Z_{m-1} \cup Z_m)$ and U are k-sequential by Lemma 4.3.2. Thus, we can apply Lemma 4.4.8. However, since $(Z_1, Z_2, \dots, Z_m)$ is a k-path, $Z_{m-1} \nsubseteq \mathrm{fcl}_k(R \cap Z_m)$ and $Z_{m-1} \nsubseteq \mathrm{fcl}_k(U \cap Z_m)$. Moreover, if either $R \cap Z_m \subseteq \mathrm{fcl}_k(Z_{m-1})$ or $U \cap Z_m \subseteq \mathrm{fcl}_k(Z_{m-1})$, then the k-path is not left-justified; a contradiction. We deduce that $U \cup (R \cap Z_m)$ is non-sequential, so a k-separation (U_1, V_1) is found as described in line 27.

By Lemma 4.3.20, $R \cap Z_m$ and $G \cap Z_m$ are sequential k-separating sets. If $V_1 \cap Z_m$ is non-sequential, then, as $(Z_1, Z_2, \ldots, Z_m)$ is a left-justified maximal k-path, $U_1 \cap Z_m \subseteq \mathrm{fcl}_k(V_1 \cap Z_m) \subseteq \mathrm{fcl}_k(V_1)$. But then, by Corollary 4.3.7(i), $U_1 \cap Z_m \subseteq \mathrm{fcl}_k(U_1 - Z_m)$; a contradiction. It follows that $V_1 \cap Z_m$ is k-sequential and, by a similar argument, $U_1 \cap Z_m$ is k-sequential. By Lemma 4.4.10, we may assume, by recolouring elements of Z_m if necessary, that one of $U_1 \cap Z_m$ and $V_1 \cap Z_m$ is monochromatic and the other is bichromatic.

Suppose, up to swapping R and G, that $U_1 \cap Z_m$ is red and $V_1 \cap Z_m$ is bichromatic. Since $|V_1 \cap Z^-_{m-1}| \geq k - 1$, as $V_1 \cap Z_m$ is k-sequential, $U_1 \cap (Z_{m-1} \cup Z_m)$ is k-separating, by uncrossing. By a further application of uncrossing, it follows that since $|U_1 \cap Z_m| \geq k-1$, the set $Z_{m-1} \cup (R \cap Z_m)$ is k-separating. Moreover, $R \cap Z_m$ has an element that is not in $\mathrm{fcl}_k(U_1 - Z^-_{m-1})$, by Lemma 4.4.7, since no (R, G)-equivalent recolouring of elements in Z_m has both $U \cap Z_m$ and $V \cap Z_m$ monochromatic. As $|G \cap Z_m| \geq k-1$, it follows that BackwardSweep finds a k-separation (S, T) as described in line 29.

We are almost ready to invoke Corollary 4.4.11 with (S, T) in the role of (R, G). First, we show that (S, T) is non-sequential. By Corollary 4.3.3, T is non-sequential as it contains Z^-_{m-1}. Suppose that S is k-sequential, and let $U_2 = U_1 - Z^-_{m-1}$. Then U_2 and $S \cap Z_m$ are also k-sequential by Lemma 4.3.2. Next, we apply Lemma 4.4.8. If $U_2 - Z_m \subseteq \mathrm{fcl}_k(U_2 \cap Z_m)$, then $U_2 - Z_m \subseteq \mathrm{fcl}_k(Z_m)$ where $U_2 - Z_m = Z_{m-1}$; a contradiction. By line 29 of BackwardSweep, $S - U_2 \not\subseteq \mathrm{fcl}_k(U_2 \cap Z_m)$. Since $(Z_1, Z_2, \ldots, Z_m)$ is a left-justified k-path, $U_2 \cap Z_m \not\subseteq \mathrm{fcl}_k(U_2 - Z_m)$. Moreover, if $U_2 \cap Z_m \subseteq \mathrm{fcl}_k(S - U_2)$, then $U_2 \cap Z_m \subseteq \mathrm{fcl}_k(V_2 \cap Z_m)$, so, by Corollary 4.3.7(i), $U_2 \cap Z_m \subseteq \mathrm{fcl}_k(Z^-_m)$; a contradiction. We deduce that S is also non-sequential.

By applying Lemma 4.3.20, but with (S, T) in the role of (R, G), we deduce that $S \cap Z_m$ and $T \cap Z_m$ are k-sequential sets. It follows, by Corollary 4.4.11, that $\Phi = (V_1 - Z_m, U_1 - Z_m, U_1 \cap Z_m, S \cap V_1 \cap Z_m, T \cap Z_m)$ is a tight k-flower. If possible, recolour elements of $V_1 \cap Z_m$ to give a k-separation equivalent to (R, G) such that Φ has fewer bichromatic petals. Now, if $S \cap V_1 \cap Z_m$ is bichromatic, then, by Lemma 6.1.3, there exists a tight refinement $\Phi' = (V_1 - Z_m, U_1 - Z_m, U_1 \cap Z_m, R \cap S \cap V_1 \cap Z_m, G \cap S \cap V_1 \cap Z_m, T \cap Z_m)$ of Φ. But $V_1 \cap Z_m$ is sequential, so Φ' has three consecutive petals whose union is a sequential set, contradicting Corollary 4.4.9. Thus $S \cap V_1 \cap Z_m$ is monochromatic and, by the same argument, $T \cap Z_m$ is monochromatic. We deduce, by line 30 of BackwardSweep, that (ii) holds.

Now suppose, up to swapping R and G, that $U_1 \cap Z_m$ is bichromatic and $V_1 \cap Z_m$ is green. By Corollary 4.4.11, and a reversal and cyclic shift of the petals, $\Phi = (V_1 - Z_m, U_1 - Z_m, R \cap U_1 \cap Z_m, G \cap U_1 \cap Z_m, V_1 \cap Z_m)$ is a tight k-flower. It follows, by Lemma 6.1.3, that if there is a k-separation as described in line 29 of BackwardSweep, then Φ has a tight refinement with three consecutive petals, $G \cap U_1 \cap Z_m$, $S \cap V_1 \cap Z_m$, and $T \cap V_1 \cap Z_m$, whose union is the sequential set $G \cap Z_m$; a contradiction. Therefore, the algorithm reaches line 31. If $Z_{m-1} \subseteq R$, then (R, G) is a k-separation that satisfies the requirements of this line, while if $Z_{m-1} \subseteq G$, then (G, R) is such a k-separation; so the algorithm finds a k-separation (S, T) as described. Suppose $S \cap Z_m$ is non-sequential. Since $(Z_1, Z_2, \ldots, Z_m)$ is a left-justified maximal k-path, $T \cap Z_m \subseteq \mathrm{fcl}_k(S \cap Z_m) \subseteq \mathrm{fcl}_k(S)$. It follows, by Corollary 4.3.7(i), that $T \cap Z_m \subseteq \mathrm{fcl}_k(T - Z_m)$; a contradiction. Thus $S \cap Z_m$ is non-sequential. By a similar argument, $T \cap Z_m$ is also non-sequential. If, up to recolouring elements of Z_m to give an (R, G)-equivalent k-separation, $S \cap Z_m$ and $T \cap Z_m$ are monochromatic, then (ii) holds, so assume otherwise. By applying Corollary 4.4.11 with (V_1, U_1) and (S, T) in the roles of (U, V) and (R, G) respectively, we deduce that $\Phi' = (U_1 - Z_m, V_1 - Z_m, V_1 \cap Z_m, S \cap U_1 \cap Z_m, T \cap U_1 \cap Z_m)$ is a tight k-flower. If possible, recolour elements of $U_1 \cap Z_m$ to give an (R, G)-equivalent k-separation such that Φ' has fewer bichromatic petals. Now, if $S \cap U_1 \cap Z_m$ is bichromatic, then, by Lemma 6.1.3, there exists a tight refinement of Φ' with three consecutive petals $G \cap S \cap U_1 \cap Z_m$, $R \cap S \cap U_1 \cap Z_m$, and $T \cap U_1 \cap Z_m$. But the union of these petals, $U_1 \cap Z_m$, is sequential, contradicting Corollary 4.4.9. So $S \cap U_1 \cap Z_m$ is monochromatic and, by a similar argument, $T \cap U_1 \cap Z_m$ is monochromatic. We deduce, by line 32 of BackwardSweep, that (ii) holds in this case, completing the proof of (6.1.4.1).

6.1.4.2. *Consider the call to* BackwardSweep *while constructing T_1 in line 6 of k-*Tree*. If Z_1 and $E - Z_1$ are bichromatic, $m \geq 3$, and τ_2 starts with $(Z_1, [(P_1, \ldots, P_s), (Q_1, \ldots, Q_t)], \ldots)$ where $s \geq 1$, $t \geq 0$, and P_1 is monochromatic, then, up to recolouring elements of Z_1 to give a k-separation equivalent to (R, G),* BackwardSweep *returns a generalised k-path that starts with either*

(i) $(X, [(Y, P_1, \dots, P_s), (Q_1, \dots, Q_t)], \dots)$, *where* (X, Y) *is a partition of* Z_1 *such that* X *and* Y *are monochromatic, or*

(ii) $(A, [(B, C, P_1, \dots, P_s), (Q_1, \dots, Q_t)], \dots)$, *where* (A, B, C) *is a partition of* Z_1 *such that* A, B *and* C *are monochromatic.*

As P_1 is monochromatic, and Z_1 and $E - Z_1$ are bichromatic, it follows, by uncrossing $Z_1 \cup P_1$ and either R or G, that the call to BackwardSweep reaches line 63 and finds a k-separation (U, V) as described in that line. If we can recolour elements of Z_1 to give an (R, G)-equivalent k-separation where both $U \cap Z_1$ and $V \cap Z_1$ are monochromatic, then, since (R, G) is non-sequential, a k-separation is found as described in line 65. It follows that the generalised k-path returned by BackwardSweep starts with $(V \cap Z_1, [(U \cap Z_1, P_1, \dots, P_s), (Q_1, \dots, Q_t)], \dots)$ or $(A, [(B, C, P_1, \dots, P_s), (Q_1, \dots, Q_t)], \dots)$, where $(A, B \cup C) = (V \cap Z_1, U \cap Z_1)$ or $(A \cup B, C) = (V \cap Z_1, U \cap Z_1)$, in which case either (i) or (ii) holds.

Now we may assume that there is no k-separation equivalent to (R, G) such that both $U \cap Z_1$ and $V \cap Z_1$ are monochromatic. First, we show that BackwardSweep finds a non-sequential k-separation (U_1, V_1) as described in line 65. If U is non-sequential, then (U, V) is such a k-separation, so let U be k-sequential. Without loss of generality we may assume that P_1 is red. Suppose that no recolouring of elements in Z_1 gives an (R, G)-equivalent k-separation such that $U \cap Z_1$ is monochromatic. By uncrossing and Lemma 4.3.2, $R \cap U$ and $U \cap Z_1$ are sequential k-separating sets. Towards a contradiction, suppose that $R \cap U \cap Z_1 \subseteq \mathrm{fcl}_k(P_1)$. Then, by the construction of U in line 63 of BackwardSweep, $G \cap U \cap Z_1 \not\subseteq \mathrm{fcl}_k(P_1)$ and, in particular, $|G \cap U \cap Z_1| \geq k-1$. If $|R \cap V \cap Z_1| \leq k-2$, then $R \cap Z_1 \subseteq \mathrm{fcl}_k(R - Z_1)$, so $R \cap Z_1 \subseteq \mathrm{fcl}_k(G)$ by Corollary 4.3.7(i); a contradiction. Hence, by uncrossing, $V \cup (R \cap Z_1)$ is k-separating. Thus $R \cap U \cap Z_1 \subseteq \mathrm{fcl}_k(U - (R \cap Z_1))$. By applying Lemma 4.4.7 with $(Z_1, E - Z_1)$ in the role of (R, G), we deduce that $R \cap U \cap Z_1 \subseteq \mathrm{fcl}_k(G \cap U \cap Z_1) \subseteq \mathrm{fcl}_k(G)$; a contradiction. So $R \cap U \cap Z_1 \not\subseteq \mathrm{fcl}_k(P_1)$. It follows that $|R \cap U \cap Z_1| \geq k - 1$. Now we can apply Lemma 4.4.8 with $R \cap U$ and $U \cap Z_1$ in the roles of A and B respectively. Since no (R, G)-equivalent k-separation has $U \cap Z_1$ monochromatic, it follows that $P_1 \subseteq \mathrm{fcl}_k(R \cap U \cap Z_1)$. Thus, $P_1 \subseteq \mathrm{fcl}_k(Z_1)$; a contradiction.

Now suppose that there is a recolouring of elements in Z_1 that results in an (R,G)-equivalent k-separation for which $U \cap Z_1$ is monochromatic. If U is monochromatic, then the non-sequential k-separation (R,G) satisfies the requirements of (U_1,V_1) in line 65, so we may assume that $U \cap Z_1$ is green. As P_1 is red, the set $P_1 \cup (R \cap Z_1)$ is k-separating by uncrossing $Z_1 \cup P_1$ and R. Thus, by uncrossing $P_1 \cup (R \cap Z_1)$ and U, we deduce that $U \cup (R \cap Z_1)$ is k-separating. Suppose that $U \cup (R \cap Z_1)$ is k-sequential. Then $P_1 \cup (R \cap Z_1)$ and U are k-sequential by Lemma 4.3.2. Thus, we can apply Lemma 4.4.8. However, since $(Z_1, Z_2, \dots, Z_m)$ is a left-justified k-path, $P_1 \not\subseteq \mathrm{fcl}_k(R \cap Z_1)$ and $P_1 \not\subseteq \mathrm{fcl}_k(U \cap Z_1)$, and, moreover, $U \cap Z_1 \not\subseteq \mathrm{fcl}_k(P_1)$ by the construction of U in line 63 of BackwardSweep. Therefore, $R \cap Z_1 \subseteq \mathrm{fcl}_k(P_1)$, in which case $R \cap Z_1 \subseteq \mathrm{fcl}_k(R - Z_1)$, so, by Corollary 4.3.7(i), we can recolour $R \cap Z_1$ green to give an (R,G)-equivalent k-separation where $U \cap Z_1$ and $V \cap Z_1$ are monochromatic; a contradiction. We deduce that $U \cup (R \cap Z_1)$ is non-sequential, so a k-separation (U_1,V_1) is found as described in line 65.

By Lemma 6.1.2, $R \cap Z_1$ and $G \cap Z_1$ are sequential k-separating sets. If $V_1 \cap Z_1$ is non-sequential, then, as $(Z_1, Z_2, \dots, Z_m)$ is a left-justified maximal k-path, $U_1 \cap Z_1 \subseteq \mathrm{fcl}_k(V_1 \cap Z_1) \subseteq \mathrm{fcl}_k(V_1)$. Thus, by Corollary 4.3.7(i), $U_1 \cap Z_1 \subseteq \mathrm{fcl}_k(U_1 - Z_1)$, contradicting the construction of U and U_1 in lines 63 and 65. Thus $V_1 \cap Z_1$ is k-sequential, and, by a similar argument, $U_1 \cap Z_1$ is k-sequential. Now we may assume, by Lemma 4.4.10, that, up to recolouring elements of Z_1 to give an (R,G)-equivalent k-separation, one of $U_1 \cap Z_1$ and $V_1 \cap Z_1$ is monochromatic and the other is bichromatic. Suppose, up to swapping R and G, that $U_1 \cap Z_1$ is red and $V_1 \cap Z_1$ is bichromatic. Since $|V_1 - (Z_1 \cup P_1)| \geq k-1$, as $V_1 \cap Z_1$ is k-sequential, $(U_1 \cap Z_1) \cup P_1$ is k-separating by uncrossing U_1 and $Z_1 \cup P_1$. By a further application of uncrossing, it follows that $P_1 \cup (R \cap Z_1)$ is k-separating. If $G \cap Z_1 \subseteq \mathrm{fcl}_k(E - Z_1)$, then, by Corollary 4.3.7(i), $G \cap Z_1$ can be recoloured red in an (R,G)-equivalent k-separation; a contradiction. Likewise, if $R \cap V_1 \cap Z_1 \subseteq \mathrm{fcl}_k(U_1 \cap (Z_1 \cup P_1))$, then, by Lemma 4.4.7, $R \cap V_1 \cap Z_1 \subseteq \mathrm{fcl}_k(R \cap U_1 \cap (Z_1 \cup P_1)) \subseteq \mathrm{fcl}_k(R - (V_1 \cap Z_1))$, so $R \cap V_1 \cap Z_1 \subseteq \mathrm{fcl}_k(G)$ by Corollary 4.3.7(i); a contradiction. Thus BackwardSweep finds a k-separation (S,T) as described in line 67.

We are almost ready to invoke Corollary 4.4.11 with (S,T) in the role of (R,G). First, we show that (S,T) is non-sequential. By Corollary 4.3.3, T is non-sequential as it contains Z_m. Suppose that S is k-sequential. Let $(U_2, V_2) = (U_1 \cap (Z_1 \cup P_1), V_1 \cup (E - (Z_1 \cup P_1)))$.

Then U_2 and $S \cap Z_1$ are also k-sequential by Lemma 4.3.2. By lines 63 and 67, $U_2 \cap Z_1 \not\subseteq \mathrm{fcl}_k(U_2 - Z_1)$ and $S - U_2 \not\subseteq \mathrm{fcl}_k(U_2 \cap Z_1)$, and, since $(Z_1, Z_2, \dots, Z_m)$ is a left-justified k-path, $U_2 - Z_1 \not\subseteq \mathrm{fcl}_k(U_2 \cap Z_1)$. Hence, by Lemma 4.4.8, $U_2 \cap Z_1 \subseteq \mathrm{fcl}_k(S - U_2) \subseteq \mathrm{fcl}_k(V_2 \cap Z_1)$. By Corollary 4.3.7(i), $U_2 \cap Z_1 \subseteq \mathrm{fcl}_k(E - Z_1)$. By an application of Lemma 4.4.7 with (U_2, V_2) in the role of (R, G), we deduce that $U_2 \cap Z_1 \subseteq \mathrm{fcl}_k(U_2 - Z_1)$; a contradiction. Hence S is also non-sequential.

Next we show that $S \cap Z_1$ and $T \cap Z_1$ are k-sequential. Suppose that $S \cap Z_1$ is non-sequential. Since $(Z_1, Z_2, \dots, Z_m)$ is maximal and left-justified, we deduce that $T \cap Z_1 \subseteq \mathrm{fcl}_k(S \cap Z_1)$, so $T \cap Z_1 \subseteq \mathrm{fcl}_k(S)$. As T is non-sequential, it follows, by Corollary 4.3.7(i), that $T \cap Z_1 \subseteq \mathrm{fcl}_k(T - Z_1)$, contradicting the construction of (S, T) in line 67. We deduce that $S \cap Z_1$ is k-sequential and, by a similar argument, $T \cap Z_1$ is also k-sequential. Thus, by Corollary 4.4.11, $\Phi = (T \cap Z_1, S \cap V_1 \cap Z_1, U_1 \cap Z_1, U_1 - Z_1, V_1 - Z_1)$ is a k-flower where the first three petals are tight, and thus Φ is tight. If possible, recolour elements of $V_1 \cap Z_1$ to give a k-separation equivalent to (R, G) such that Φ has fewer bichromatic petals. Now, if $S \cap V_1 \cap Z_1$ is bichromatic, then, by Lemma 6.1.3, there exists a refinement of Φ with consecutive tight petals $T \cap Z_1$, $G \cap S \cap V_1 \cap Z_1$ and $R \cap S \cap V_1 \cap Z_1$. The union of these three petals, $V_1 \cap Z_1$, is k-sequential, contradicting Corollary 4.4.9. So $S \cap V_1 \cap Z_1$ is monochromatic and, by a similar argument, $T \cap Z_1$ is monochromatic. We deduce, by line 68 of BackwardSweep, that (ii) holds.

Now suppose, up to swapping R and G, that $U_1 \cap Z_1$ is bichromatic and $V_1 \cap Z_1$ is green. By Corollary 4.4.11, $\Phi = (V_1 - Z_1, U_1 - Z_1, R \cap U_1 \cap Z_1, G \cap U_1 \cap Z_1, V_1 \cap Z_1)$ is a tight k-flower. It follows, by Lemma 6.1.3, that if there is a k-separation as described in line 67, then Φ has a tight refinement with three consecutive petals $G \cap U_1 \cap Z_1$, $S \cap V_1 \cap Z_1$ and $T \cap V_1 \cap Z_1$ whose union is $G \cap Z_1$, contradicting Corollary 4.4.9. Thus, the algorithm reaches line 69. If $P_1 \subseteq R$, then (R, G) is a non-sequential k-separation that satisfies the requirements of line 69, while if $P_1 \subseteq G$, then (G, R) is such a k-separation; so a k-separation (S, T) is found as described. If $S \cap Z_1$ is non-sequential, then $T \cap Z_1 \subseteq \mathrm{fcl}_k(S \cap Z_1)$, since $(Z_1, Z_2, \dots, Z_m)$ is a maximal k-path. But then $T \cap Z_1 \subseteq \mathrm{fcl}_k(S)$, so $T \cap Z_1 \subseteq \mathrm{fcl}_k(T - Z_1)$ by Corollary 4.3.7(i), contradicting the construction of (S, T) in line 69. We deduce that $S \cap Z_1$ and, similarly, $T \cap Z_1$ are k-sequential. If, up to recolouring elements of Z_1 to give an (R, G)-equivalent k-separation, $S \cap Z_1$ and $T \cap Z_1$ are monochromatic, then (ii) holds, by line 70, so assume otherwise. By applying Corollary 4.4.11, $\Phi' = (E-(Z_1\cup P_1), P_1, S\cap U_1\cap Z_1, T\cap U_1\cap Z_1, V_1\cap Z_1)$ is a tight k-flower. If possible, recolour elements of $U_1\cap Z_1$ to give an (R,G)-equivalent k-separation such that Φ' has fewer bichromatic petals. Now, if $T\cap U_1\cap Z_1$ is bichromatic, then, by Lemma 6.1.3, there exists a refinement of Φ' with three consecutive petals $S\cap U_1\cap Z_1$, $R\cap T\cap U_1\cap Z_1$ and $G\cap T\cap U_1\cap Z_1$. But the union of these petals, $U_1\cap Z_1$ is k-sequential, contradicting Corollary 4.4.9. So $T\cap U_1\cap Z_1$ is monochromatic and, by the same argument, $S\cap U_1\cap Z_1$ is monochromatic. Thus (6.1.4.2) holds.

6.1.4.3. *If $X_0\cup Z_1$ is monochromatic, then T_{p+1} displays (R,G).*

Suppose that $X_0\cup Z_1$ is monochromatic. Without loss of generality, we may assume that $X_0\cup Z_1\subseteq G$. Let b be the number of bichromatic parts amongst $Z_2,\dots,Z_m$. Assume that $b\geq 2$ and let Z_i be the bichromatic part with the smallest subscript. If $Z_i^-\cap R$ is non-empty, then, by Lemmas 4.3.14 and 4.3.15, Z_i is monochromatic; a contradiction. Therefore $Z_i^-\subseteq G$. But then, by Lemma 4.3.17, Z_i^+ is monochromatic; a contradiction. Thus $b\in\{0,1\}$.

Assume that $b=1$ and Z_i is bichromatic. We first consider $i\neq m$. If Z_i^+ is bichromatic, then, by Lemma 4.3.17, Z_i^- is bichromatic, and so, by Lemma 4.3.15, $|R\cap Z_i^-|, |G\cap Z_i^-|, |R\cap Z_i^+|, |G\cap Z_i^+|\geq k-1$. But then, by Lemma 4.3.14, Z_i is monochromatic; a contradiction. Thus we may assume that Z_i^+ is monochromatic.

Suppose that Z_i^- is monochromatic. As $X_0\cup Z_1\subseteq G$, we have $Z_i^-\subseteq G$. Then, by Lemma 4.3.17, $Z_i^+\subseteq G$, so $R\subseteq Z_i$. The only lines in BackwardSweep that do not leave Z_i intact are lines 40 and 59. As (R,G) does not conform with T_{p+1}, we may assume that one of these is invoked. Then, both $R\cap(Z_i-\text{fcl}_k(Z_i^+))$ and $R\cap(Z_i\cap\text{fcl}_k(Z_i^+))$ are non-empty. But, as $R\cap(Z_i\cap\text{fcl}_k(Z_i^+))\subseteq\text{fcl}_k(Z_i^+)$, it follows that $R\cap(Z_i\cap\text{fcl}_k(Z_i^+))\subseteq\text{fcl}_k(G)$. Therefore we can recolour all the elements in $R\cap(Z_i\cap\text{fcl}_k(Z_i^+))$ green thereby obtaining an equivalent k-separation in which all the red elements are in $Z_i-\text{fcl}_k(Z_i^+)$, a single bag of T_{p+1}. This contradiction implies that Z_i^- is bichromatic.

By Lemma 4.3.15, $|R\cap Z_i^-|, |G\cap Z_i^-|\geq k-1$. Without loss of generality, we may assume that $Z_i^+\subseteq R$. By Lemma 4.3.19, $R\cap Z_i\subseteq\text{fcl}_k(Z_i^+)$. Furthermore, by recolouring if necessary, we may assume that $R\cap Z_i=$

$Z_i \cap \mathrm{fcl}_k(Z_i^+)$. Since $|R \cap Z_i^-| \geq k-1$, it follows, by uncrossing G and $Z_i \cup Z_i^+$, that $G \cap Z_i$ is k-separating. Moreover, by Lemma 4.3.16, Z_i is not k-separating. Therefore the generalised k-path τ_i at the end of the iteration of BackwardSweep in which Z_i is considered is

$$\tau_i = \big(X_0 \cup Z_1, Z_2, \ldots, Z_{i-1}, [(Z_i - \mathrm{fcl}_k(Z_i^+))], Z_i \cap \mathrm{fcl}_k(Z_i^+), \tau_{i+1}(Z_i^+)\big).$$

Now $Z_i - \mathrm{fcl}_k(Z_i^+) \subseteq G$ and $(Z_i \cap \mathrm{fcl}_k(Z_i^+)) \cup Z_i^+ \subseteq R$. Let h be the smallest index for which $Z_h^- \subseteq G$, but $Z_h \subseteq R$. Since $X_0 \cup Z_1 \subseteq G$ and $|R \cap Z_i^-| \geq k-1$, we have $2 \leq h \leq i-1$. By applying Lemma 4.3.18 to the k-path $\big(Z_h^-, Z_h, Z_{h+1}, \ldots, Z_{i-1}, Z_i - \mathrm{fcl}_k(Z_i^+), (Z_i \cap \mathrm{fcl}_k(Z_i^+)) \cup Z_i^+\big)$, we deduce that M has a k-flower in which the parts of the k-path are petals of the flower. It now follows by Lemma 4.3.18 and the construction in BackwardSweep that T_{p+1} displays (R, G), so (6.1.4.3) is satisfied when $b = 1$ and $i \neq m$.

Now consider $i = m$. If Z_m^- is monochromatic, that is, $Z_m^- \subseteq G$, then either $(X_0 \cup Z_1, Z_2, \ldots, Z_m)$ is not left-justified or it is not maximal; a contradiction. Therefore Z_m^- is bichromatic, and so $m \geq 3$. Let h denote the smallest index for which $Z_h^- \subseteq G$, but $Z_h \subseteq R$. Then, by Lemma 4.3.18, M has a flower with petals $Z_h^-, Z_h, Z_{h+1}, \ldots, Z_{m-1}, Z_m', Z_m''$, where $\{Z_m', Z_m''\} = \{Z_m \cap R, Z_m \cap G\}$. Thus, by Lemma 4.3.18, (6.1.4.1), and the construction in BackwardSweep, T_{p+1} displays (R, G).

We may now assume that $b = 0$. Let h denote the smallest index for which $Z_h^- \subseteq G$, but $Z_h \subseteq R$. Say $Z_h \cup Z_h^+$ is bichromatic. Let h' denote the largest index for which $Z_{h'} \cup Z_{h'}^+$ is not monochromatic, but $Z_{h'}^+$ is monochromatic. Note that $h' \geq h$. Then it follows, by Lemma 4.3.18, that each of the sets $Z_h, Z_{h+1}, \ldots, Z_{h'}$ is k-separating and so, by the construction in BackwardSweep and Lemma 4.3.18, T_{p+1} displays (R, G) as the petals of a k-flower. Now say $Z_h \cup Z_h^+$ is monochromatic. It follows from the construction in BackwardSweep that if (R, G) does not conform with T_{p+1}, then $h \geq 3$ and line 59 of BackwardSweep is invoked when Z_{h-1} is considered. But then we can recolour all the elements in $Z_{h-1} \cap \mathrm{fcl}_k(Z_h \cup Z_h^+)$ red, resulting in a k-separation equivalent to (R, G); so T_{p+1} displays (R, G). This completes the proof of (6.1.4.3).

6.1.4.4. *If $p = 0$, then T_1 displays (R, G).*

Suppose that $p = 0$, in which case X_0 is empty. If Z_1 is monochromatic, then (6.1.4.4) holds by (6.1.4.3). Thus we may assume that Z_1 is bichromatic, in which case both $R \cap Z_1$ and $G \cap Z_1$ are sequential k-separating sets consisting of at least $k-1$ elements. Let b denote the number of bichromatic parts amongst $Z_1, \dots, Z_m$. By Lemmas 4.3.14 and 4.3.15, $b \in \{1, 2\}$.

First assume that $b = 2$, and let Z_i denote the bichromatic part with $i > 1$. Say $i \neq m$. By Lemmas 4.3.14 and 4.3.15, Z_i^+ is monochromatic. Without loss of generality, we may assume that $Z_i^+ \subseteq R$. By Lemma 4.3.16, Z_i is not k-separating. Furthermore, by Lemma 4.3.19, $R \cap Z_i \subseteq \mathrm{fcl}_k(Z_i^+)$. By recolouring elements of X_i, if necessary, we may assume that $R \cap Z_i = Z_i \cap \mathrm{fcl}_k(Z_i^+)$. Since $|R \cap Z_i^-| \geq k-1$, it follows, by uncrossing G and $Z_i \cup Z_i^+$, that $G \cap Z_i$, which equals $Z_i - \mathrm{fcl}_k(Z_i^+)$, is k-separating. Thus, by the construction in BackwardSweep, the generalised k-path τ_i at the end of the iteration in which Z_i is considered is

$$\tau_i = \big(Z_1, Z_2, \dots, Z_{i-1}, [(Z_i - \mathrm{fcl}_k(Z_i^+))], Z_i \cap \mathrm{fcl}_k(Z_i^+), \tau_{i+1}(Z_i^+)\big).$$

Now $Z_i - \mathrm{fcl}_k(Z_i^+) \subseteq G$ and $(Z_i \cap \mathrm{fcl}_k(Z_i^+)) \cup Z_i^+ \subseteq R$ and so, by Lemma 4.3.18, M has a flower with petals $R \cap Z_1, G \cap Z_1, Z_2, \dots, Z_{i-1}$, $Z_i - \mathrm{fcl}_k(Z_i^+), (Z_i \cap \mathrm{fcl}_k(Z_i^+)) \cup Z_i^+$. It follows, by the construction in BackwardSweep, that τ_2 is eventually constructed and is of the form

$$\tau_2 = \big(Z_1, [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], Z_i \cap \mathrm{fcl}_k(Z_i^+), \tau_{i+1}(Z_i^+)\big),$$

where $\{P_1, \dots, P_p, Q_1, \dots, Q_q\} = \{Z_2, \dots, Z_{i-1}, Z_i - \mathrm{fcl}_k(Z_i^+)\}$. Therefore, by Lemma 4.3.18, (6.1.4.2), and construction, (R, G) is displayed by T_{p+1}. So (6.1.4.4) holds when Z_1 and Z_i are bichromatic, for $i \in \{2, 3, \dots, m-1\}$.

Now say $i = m$. There are two cases depending upon whether $m = 2$ or $m \geq 3$. If $m \geq 3$, then Z_{m-1} is monochromatic. Lemma 4.3.18 implies that M has a flower with petals $R \cap Z_1, G \cap Z_1, Z_2, \dots, Z_{m-1}, R \cap Z_m, G \cap Z_m$. It follows, by the construction in BackwardSweep and (6.1.4.1), that eventually we construct τ_2 and it is of the form $(Z_1, [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], W)$, where either $\{P_1, \dots, P_p, Q_1, \dots, Q_q, W\} = \{Z_2, \dots, Z_{m-1}, X, Y\}$ or $\{P_1, \dots, P_p, Q_1, \dots, Q_q, W\} = \{Z_2, \dots, Z_{m-1}, A, B, C\}$, for some partition (X, Y) or (A, B, C), respectively, of Z_m with monochromatic parts. As P_1 is monochromatic, we can apply (6.1.4.2). It follows that Z_1 either breaks into two petals or three petals, each of which is monochromatic. Thus (R, G) is displayed by T_{p+1}.

Consider the case where $m = 2$. Since $|G \cap Z_1| \geq k-1$, it follows, by uncrossing, that $R \cap Z_2$ is k-separating. If $|G \cap Z_2| \leq k-2$, then $Z_2 \subseteq \mathrm{fcl}_k(R \cap Z_2)$, in which case we can recolour $G \cap Z_2$ red thereby obtaining an (R, G)-equivalent k-separation with fewer bichromatic parts; a contradiction. Hence $|G \cap Z_2| \geq k-1$ and, by symmetry, $|R \cap Z_2| \geq k-1$. As (R, G) is non-sequential, it follows, by Lemma 5.1.3, that BackwardSweep finds a k-separation (U, V) as described in line 2. If, up to a k-separation equivalent to (R, G), the sets $U \cap Z_1$, $V \cap Z_1$, $U \cap Z_2$, and $V \cap Z_2$ are monochromatic, then, as lines 2–18 output a refinement of $(V \cap Z_1, U \cap Z_1, U \cap Z_2, V \cap Z_2)$ up to a cyclic shift, (R, G) is displayed by T_{p+1}.

We may now assume that there is no k-separation equivalent to (R, G) such that both $U \cap Z_i$ and $V \cap Z_i$ are monochromatic for some $i \in \{1, 2\}$. By Lemma 4.4.10, we can assume, for such an i, that one of $U \cap Z_i$ and $V \cap Z_i$ is monochromatic and the other is bichromatic. Suppose that $U \cap Z_2$ is monochromatic; without loss of generality, we may assume that $U \cap Z_2$ is red. Recall that $R \cap Z_2$ is k-separating. If $R \cap V \cap Z_2 \subseteq \mathrm{fcl}_k(U \cap Z_2)$, then $R \cap V \cap Z_2 \subseteq \mathrm{fcl}_k(R - (V \cap Z_2))$, in which case, by Corollary 4.3.7(i), $R \cap V \cap Z_2 \subseteq \mathrm{fcl}_k(G)$; a contradiction. So $R \cap V \cap Z_2$ contains an element not in $\mathrm{fcl}_k(U \cap Z_2)$. Since (R, G) is non-sequential, BackwardSweep finds a k-separation as described in line 4. By Corollaries 4.4.9 and 4.4.11, it follows that, up to an equivalent recolouring of (R, G), the last three petals of the generalised k-path output by BackwardSweep are monochromatic. If $V \cap Z_2$ is monochromatic, a similar argument applies, where line 6 of BackwardSweep is invoked instead of line 4. Likewise, a similar argument applies when $V \cap Z_1$ or $U \cap Z_1$ is monochromatic and the other is bichromatic, where line 12 or 15 of BackwardSweep, respectively, is invoked in this case. As each of the petals in the generalised k-path returned by BackwardSweep is monochromatic, we deduce that (R, G) is displayed by T_{p+1}. So (6.1.4.4) holds when Z_1 and Z_m are bichromatic, and, more generally, when $b = 2$.

Now assume that $b = 1$, so Z_1 is the only bichromatic part. Since $R \cap Z_1$ and $G \cap Z_1$ are sequential k-separating sets and (R, G) is non-sequential, we deduce that Z_1^+ is bichromatic and $m \geq 3$. Let h denote the largest index for which $Z_h \cup Z_h^+$ is not monochromatic, but Z_h^+ is monochromatic. By Lemma 4.3.18, M has a flower with petals $R \cap Z_1, G \cap Z_1, Z_2, \ldots, Z_h, Z_h^+$. Therefore, by construction and Lemma 4.3.18, τ_2 is eventually constructed and begins with $\tau_2 = (Z_1, [(P_1, \dots, P_p), (Q_1, \dots, Q_q)], \dots)$, where $\{P_1, \dots, P_p, Q_1, \dots, Q_q\} = \{Z_2, \dots, Z_h\}$. Since P_1 is monochromatic, we can apply (6.1.4.2). Thus T_{p+1} displays (R, G), completing the proof of (6.1.4.4).

When $p \geq 1$, $X_0 \cup Z_1$ is monochromatic so, by (6.1.4.3), T_{p+1} displays (R, G); a contradiction. Otherwise, $p = 0$ and we can apply (6.1.4.4); again we derive the contradiction that T_{p+1} displays (R, G). Thus we deduce that T_{p+1} is a conforming tree for M. By induction, this completes the proof of the lemma. □

Lemma 6.1.5. *Let M be a k-connected matroid with $|E(M)| \geq 8k - 15$, and let T be the conforming tree returned by k-*Tree *when applied to M. If v is a flower vertex of T, then the flower corresponding to v is tight and irredundant.*

Proof. Let E denote the ground set of M. We prove the lemma by showing that each of the π-labelled trees T_p constructed in lines 6 and 17 of k-Tree has the property that for each flower vertex, the corresponding flower is tight and irredundant. Since T_0 consists of a single bag vertex labelled E, the result holds trivially if $p = 0$. Now suppose that $p \geq 0$ and T_p has the property that if v is a flower vertex of T_p, then the flower corresponding to v is tight and irredundant. We show, as (6.1.5.1) and (6.1.5.2), that the flower corresponding to each flower vertex of T_{p+1} is tight and irredundant, respectively.

6.1.5.1. *If v is a flower vertex of T_{p+1}, then the flower corresponding to v is tight.*

By induction, T_p has this property on its flower vertices. Therefore, by construction, it suffices to consider only the flower vertices in the path realisation T'_{p+1} of the generalised k-path returned by BackwardSweep in the construction of T_{p+1} from T_p, in line 16 of k-Tree. Let $(X_0 \cup X_1, X_2, \dots, X_m)$ be the left-justified maximal X_0-rooted k-path returned by ForwardSweep in the construction of T_{p+1} from T_p in k-Tree. Let v be a flower vertex of T'_{p+1} and let Φ be the flower corresponding to v. Suppose that Φ is not tight. By construction, we may assume that v has degree at least three. For clarity, we shall assume that line 59 in BackwardSweep is not invoked in the construction of Φ. The straightforward extension of the proof below to include the case when this line is invoked is omitted.

It follows from the description of BackwardSweep that if no end moves are performed, then, for some i and j with $1 \leq i \leq j \leq m$, the entry and exit petals of Φ are X_i^- and X_j^+ respectively, and the union of the set of clockwise petals and the set of anticlockwise petals of Φ is $\{X_i, X_{i+1}, \ldots, X_j\}$. Ignoring the possibility of end moves for now, if X_i^- is loose, then $X_i^- \subseteq \mathrm{fcl}_k(X_i \cup X_i^+)$, and so $(X_i^-, X_i \cup X_i^+)$ is sequential; a contradiction. Similarly, if X_j^+ is loose, then we deduce a contradiction. Assume that, for some $i \leq s \leq j$, the petal X_s is loose. Since the clockwise and anticlockwise petals are each subsequences of $\{X_i, X_{i+1}, \ldots, X_j\}$ that induce a partition of this set, there is a cyclic shift of the petals of Φ that results in a flower Φ' equivalent to Φ with a concatenation (X_s^-, X_s, X_s^+). Thus, by Lemma 4.3.12, either $X_s \subseteq \mathrm{fcl}_k(X_s^-)$ or $X_s \subseteq \mathrm{fcl}_k(X_s^+)$, contradicting the fact that $(X_0 \cup X_1, X_2, \ldots, X_m)$ is a k-path.

Now consider the possibility of end moves. First suppose that $m \geq 3$. If X_m breaks into two petals Y_m and Y_m' in BackwardSweep, then the algorithm finds a k-separation as described in line 25. It follows, by Lemma 4.3.20, that Y_m and Y_m' are both sequential. If $Y_m \subseteq \mathrm{fcl}_k(Y_m')$, then $Y_m \subseteq \mathrm{fcl}_k(E - X_m)$ by Corollary 4.3.7(i), so X_m is sequential; a contradiction. Thus, by Lemma 4.3.12, Y_m is tight and, by symmetry, Y_m' is also tight. Similarly, if X_1 breaks into two petals Y_1 and Y_1', then BackwardSweep finds a non-sequential k-separation (U_1, V_1) as described on line 65, where $\{U_1 \cap X_1, V_1 \cap X_1\} = \{Y_1, Y_1'\}$. If $U_1 \cap X_1$ is non-sequential, then, since $(X_1, X_2, \ldots, X_m)$ is a left-justified maximal k-path, $V_1 \cap X_1 \subseteq \mathrm{fcl}_k(U_1 \cap X_1) \subseteq \mathrm{fcl}_k(U_1)$. Thus, by Corollary 4.3.7(i), $V_1 \cap X_1 \subseteq \mathrm{fcl}_k(V_1 - X_1)$, contradicting the construction of V_1 in line 65. Thus $U_1 \cap X_1$ is k-sequential and, by a similar argument $V_1 \cap X_1$ is k-sequential. Since Y_1 and Y_1' are sequential, Y_1 and Y_1' are tight by the same argument as for Y_m and Y_m'. If X_m breaks into three petals, then line 29 or line 31 is invoked and a k-separation (S, T) is found as described on that line. It follows, by Corollary 4.4.11, that the three petals, whose union is X_m, are tight. The same argument applies if X_1 breaks into three petals, where, in this case, the k-separation (S, T) is found at line 67 or line 69 of BackwardSweep.

It remains to consider end moves when $m = 2$ and X_0 is empty. In this case, line 2 of BACKWARDSWEEP is invoked and a k-separation (U, V) is found as described in that line. It follows, by Lemma 4.3.21, that $U \cap X_1$, $V \cap X_1$, $U \cap X_2$ and $V \cap X_2$ are sequential. Since (X_1, X_2) is non-sequential, neither $U \cap X_2$ nor $V \cap X_2$ is a subset of $\mathrm{fcl}_k(X_1)$, and so, by Lemma 4.3.12, if $U \cap X_2$ and $V \cap X_2$ are petals of Φ, then they are tight. Similarly, if $U \cap X_1$ and $V \cap X_1$ are petals of Φ, then they are tight. We deduce that when line 9 is invoked, the last two petals of Φ are tight; and when line 18 is invoked, the first two petals of Φ are tight. If line 4 or 6 is invoked and the condition is satisfied, then the last three petals of Φ are tight by Corollary 4.4.11. Similarly, if line 12 or 15 is invoked and the condition is satisfied, then the first three petals of Φ are tight by Corollary 4.4.11. This completes the proof of (6.1.5.1).

6.1.5.2. *If v is a flower vertex of T_{p+1}, then the flower corresponding to v is irredundant.*

By induction, T_p has this property on its flower vertices. Hence, it suffices to consider only the flower vertices in the path realisation T'_{p+1} of the generalised k-path returned by BACKWARDSWEEP in the construction of T_{p+1} from T_p in line 16 of k-TREE. Let $(X_0 \cup X_1, X_2, \ldots, X_m)$ be the left-justified maximal X_0-rooted k-path returned by FORWARDSWEEP in the construction of T'_{p+1} in line 14 of k-TREE. Let v be a flower vertex of T'_{p+1} and let Φ be the flower corresponding to v.

First, assume that no end moves are performed in the construction of the generalised k-path. It follows from the description of BACKWARDSWEEP that if line 59 in BACKWARDSWEEP is not invoked, then, for some i and j with $1 \leq i \leq j \leq m$, the entry and exit petals of Φ are X_i^- and X_j^+, respectively, and the clockwise petals $(X_{a,1}, X_{a,2}, \ldots, X_{a,p})$ and anticlockwise petals $(X_{b,1}, X_{b,2}, \ldots, X_{b,q})$ of Φ are subsequences of $(X_i, X_{i+1}, \ldots, X_j)$ that induce a partition of $\{X_i, X_{i+1}, \ldots, X_j\}$. For any l such that $i-1 \leq l \leq j$, the non-sequential k-separation $\big(X_i^- \cup (\bigcup_{s=i}^{l} X_s), (\bigcup_{s=l+1}^{j} X_s) \cup X_j^+\big)$ is displayed by Φ. Since $\Phi = (X_i^-, X_{a,1}, X_{a,2}, \ldots, X_{a,p}, X_j^+, X_{b,1}, X_{b,2}, \ldots, X_{b,q})$, it follows that Φ is irredundant. When line 59 in BACKWARDSWEEP is invoked,

$$\Phi = \big(X_i^-, X_{a,1}, X_{a,2}, \ldots, X_{a,p}, (X_j \cap \mathrm{fcl}_k(X_j^+)) \cup X_j^+, X_{b,1}, X_{b,2}, \ldots, X_{b,q}\big)$$

where $(X_{a,1}, X_{a,2}, \ldots, X_{a,p})$ and $(X_{b,1}, X_{b,2}, \ldots, X_{b,q})$ are subsequences of

$(X_i, X_{i+1}, \ldots, X_{j-1}, X_j - \mathrm{fcl}_k(X_j^+))$. By the same argument, Φ is irredundant.

Now consider the possibility of end moves. First suppose that $m \geq 3$ and that X_m comprises at least two petals of Φ. Then the algorithm reaches line 25 of BackwardSweep, and finds both a k-separation (U, V) as described on that line, and a k-separation (U_1, V_1) as described on line 27. By Lemma 5.1.3, (U_1, V_1) is non-sequential. Let $\Phi = (P_1, P_2, \ldots, P_n)$. Since (X_m, X_m^-) is a non-sequential k-separation displayed by Φ, it suffices to show that for each pair of distinct petals A, B contained in X_m, there is a non-sequential k-separation (A', B') displayed by Φ such that $A \subseteq A'$ and $B \subseteq B'$. By construction, there exists an index $i \in \{n-2, n-1\}$ such that $P_i \subseteq U_1 \cap X_m \subseteq U_1$ and $P_{i+1} \subseteq V_1 \cap X_m \subseteq V_1$. If a k-separation (S, T) is found at line 29, then it follows that Φ has a concatenation $(X_{m-1}^-, X_{m-1}, U_1 \cap X_m, S \cap V_1 \cap X_m, T \cap X_m)$ that is tight, by (6.1.5.1). As T contains X_{m-1}^- and S contains $X_{m-1} \cup (U_1 \cap X_m)$, the k-separation (S, T) is non-sequential by Corollary 4.4.9. If, instead, line 31 of BackwardSweep is invoked and a k-separation (S, T) is found as described, then (S, T) is non-sequential by Lemma 5.1.3. Thus, for distinct petals A, B of Φ contained in X_m, there is a non-sequential k-separation (A', B') displayed by Φ such that $A \subseteq A'$ and $B \subseteq B'$.

We can argue in a similar fashion when X_1 comprises at least two petals of Φ. In this case, k-separations (U, V) and (U_1, V_1) are found as described in lines 63 and 65 of BackwardSweep, respectively. Furthermore, (U_1, V_1) and (X_1, X_1^+) are non-sequential. If line 67 is invoked and a k-separation (S, T) is found as described on that line, then (S, T) is non-sequential by (6.1.5.1) and Corollary 4.4.9. If, instead, line 69 of BackwardSweep is invoked and a k-separation (S, T) is found as described on that line, then (S, T) is non-sequential by Lemma 5.1.3. It now follows that when $m \geq 3$ and an end move, or end moves, is performed, the flower Φ is irredundant.

It remains to consider when $m = 2$ and, in particular, line 2 of BackwardSweep is invoked and a non-sequential k-separation (U, V) is found as described in that line. If the algorithm invokes lines 9 and 18 of BackwardSweep, so Φ has four petals, then Φ is irredundant. Otherwise, at least one of X_1 and X_2 breaks into three petals of Φ.

First we consider the case where X_2 breaks into three petals. Suppose line 4 is invoked, and k-separations (S, T) and (S_1, T_1) are found as described. Thus $\Phi = (\dots, P_{n-2}, P_{n-1}, P_n) = (\dots, U \cap X_2, S_1 \cap V, T_1 \cap X_2)$. Now, by construction, the non-sequential k-separation (U, V) is displayed by Φ with $P_{n-2} \subseteq U$ and $P_{n-1} \subseteq V$. Moreover, (S_1, T_1) is a k-separation with $P_{n-2} \cup P_{n-1} \subseteq S_1$ and $P_n \subseteq T_1$; we will show that (S_1, T_1) is a non-sequential k-separation displayed by Φ. By Corollary 4.4.11, $(X_1, U \cap X_2, S_1 \cap V \cap X_2, T_1 \cap X_2)$ is a tight flower. It follows, by Lemma 6.1.3, that $(T_1 \cap X_1, S_1 \cap X_1, U \cap X_2, S_1 \cap V \cap X_2, T_1 \cap X_2)$ is a tight flower where $U \cap X_2 \subseteq S_1$. Thus, by Corollary 4.4.9, the set S_1 is non-sequential. If T_1 is sequential, then, by Corollary 4.3.4, it is contained in a member F of $\mathcal{F}$. It follows that any subset T' of T_1 will also be contained in F, contradicting the construction of T_1 in line 4. So (S_1, T_1) is non-sequential. Since (S_1, T_1) conforms with Φ, by Lemma 6.1.4, either (S_1, T_1) is displayed by Φ or (S_1, T_1) is equivalent to a k-separation (S_2, T_2) where S_2 or T_2 is contained in a petal of Φ. Suppose the latter. Then such a petal is non-sequential by Corollary 4.3.3. But Φ is a refinement of $(V \cap X_1, U \cap X_1, U \cap X_2, V \cap X_2)$, where each part of this partition is sequential by Lemma 4.3.21, so we have a contradiction. We deduce that (S_1, T_1) conforms with Φ.

Suppose instead that line 6 is invoked and k-separations (S, T) and (S_1, T_1) are found as described, so $\Phi = (\dots, P_{n-2}, P_{n-1}, P_n) = (\dots, S_1 \cap X_2, T_1 \cap U, V \cap X_2)$. Then (U, V) is a non-sequential k-separation displayed by Φ such that $P_{n-1} \subseteq U$ and $P_n \subseteq V$, and, by a similar argument as in the previous paragraph, (S, T) is a non-sequential k-separation such that $P_{n-2} \subseteq S$ and $P_{n-1} \cup P_n \subseteq T$.

Now we consider the two cases where X_1 breaks into three petals. First we suppose that line 12 is invoked and a k-separation (S, T) is found as described, so $\Phi = (P_1, P_2, P_3, \dots) = (V \cap X_1, S \cap U, T \cap X_1, \dots)$. Since $T \cap X_1 \subseteq U$, the non-sequential k-separation (U, V) displayed by Φ has $P_1 \subseteq V$ and $P_2 \subseteq U$. Moreover, the k-separation (S, T) has $P_1 \cup P_2 \subseteq S$ and $P_3 \subseteq T$; we will show that this k-separation is non-sequential and is displayed by Φ. By Corollary 4.4.11 and Lemma 6.1.3, $(V \cap X_1, S \cap U, T \cap X_1, T \cap X_2, S \cap X_2)$ is a tight k-flower. Since $V \cap X_1 \subseteq S$, the set S is non-sequential by Corollary 4.4.9. If T is sequential, then, by Corollary 4.3.4, the subset T' of T is contained in a member of $\mathcal{F}$; a contradiction. Hence (S, T) is non-sequential and, since T_{p+1} is conforming by Lemma 6.1.4, is displayed by Φ. Suppose instead that line 15 is invoked and a k-separation (S, T) is found as described. Now $\Phi = (P_1, P_2, P_3, \dots) = (T \cap X_1, S \cap V, U \cap X_1, \dots)$. Then

(U, V) is a non-sequential k-separation displayed by Φ such that $P_2 \subseteq V$ and $P_3 \subseteq U$, and, by a similar argument as earlier in the paragraph, (S, T) is a non-sequential k-separation displayed by Φ such that $P_1 \subseteq T$ and $P_2 \cup P_3 \subseteq S$. Finally, since (X_1, X_2) is also a non-sequential k-separation, we deduce that Φ is irredundant when X_1 or X_2 is the union of three petals of Φ. So (6.1.5.2) holds, thus completing the proof of the lemma. □

6.2 Maximality

In this section, we show that each flower vertex of a conforming tree returned by k-Tree is maximal. In other words, the tree is a partial k-tree.

The next lemma is a straightforward consequence of the way in which flowers are constructed in k-Tree.

Lemma 6.2.1. *Let M be a k-connected matroid with $|E(M)| \geq 8k - 15$. The tree T returned by k-Tree(M) has the property that every k-flower corresponding to a flower vertex in T displays at least two inequivalent non-sequential k-separations.*

It now follows, by Lemmas 6.1.4, 6.1.5 and 6.2.1, that if T is a π-labelled tree returned by k-Tree, then T is conforming, and every flower Φ_v corresponding to a flower vertex v of T is tight, irredundant, and displays at least two inequivalent non-sequential k-separations. The following lemma, which is implicit in a result by Oxley and Semple (2013, Lemma 6.5), says that, when $k = 3$, these are sufficient conditions for each Φ_v to be a maximal flower.

Lemma 6.2.2. *Let M be a 3-connected matroid and let T be a conforming 3-tree for M. If, for every flower vertex v of T, the 3-flower corresponding to v is tight and displays at least two inequivalent non-sequential 3-separations, then T is a partial 3-tree for M.*

When $k \geq 4$, however, a conforming tree T, where every flower Φ_v corresponding to a flower vertex v of T is tight and displays at least two inequivalent non-sequential k-separations, is not necessarily a partial k-tree. This remains the case even if, additionally, each Φ_v is irredundant. The next example demonstrates this. In this example we construct a 4-flower in a similar manner to Example 4.4.4.

Example 6.2.3. Let Ψ be the free $(4,3)$-swirl with $x_i, y_i, z_i \in E(\Psi)$ such that $r(\{x_i, y_i, z_i\}) = 2$ and $r(\{x_i, y_i, z_i, x_{i+1}, y_{i+1}, z_{i+1}\}) = 3$, for all $i \in \{1,2,3,4\}$, where the subscripts are interpreted modulo four. Let Ψ' be the coextension of Ψ by an element e where $\{x_3, y_3, x_4, y_4\}$ is the only dependent flat not containing e in the coextension. Take the direct sum of $\Psi' \backslash e$ with a copy of $U_{2,2}$ having ground set $\{w_1, w_2\}$. Then, for each $i \in \{1,2\}$, freely add the elements s_i, t_i, u_i, and v_i, in turn, to the flat spanned by $\{w_i, x_i, y_i, z_i\}$. The resulting rank-7 matroid M is 4-connected, and $\Phi' = (Q_1, Q_2, Q_3, Q_4)$ is a swirl-like 4-flower, where $Q_i = \{x_i, y_i, z_i\}$ for $i \in \{3,4\}$, and $Q_i = \{s_i, t_i, \dots, z_i\}$ for $i \in \{1,2\}$. An illustration of M is given in Figure 6.1, where the elements in Q_1 and Q_2 are suppressed. Note that as $\{x_3, y_3, x_4, y_4\}$ is 4-separating in M, the set $Q_3 \cup Q_4$ is 4-sequential.

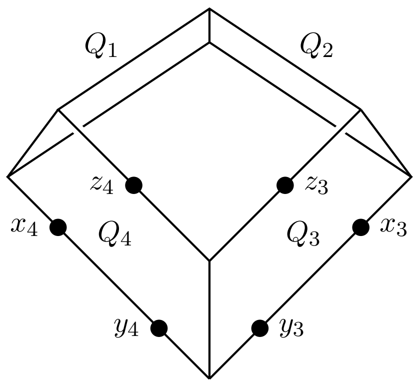


Figure 6.1: The 4-connected rank-7 matroid M.

Let T be a tree consisting of a single flower vertex, labelled D, with corresponding 4-flower $\Phi = (Q_1 \cup Q_4, Q_2, Q_3)$. Then T is a conforming 4-tree, and Φ is tight, irredundant, and displays the inequivalent non-sequential 4-separations $(Q_1 \cup Q_4, Q_2 \cup Q_3)$ and $(Q_2, E(M) - Q_2)$. However Φ is not maximal since Φ' is a 4-flower that displays all the non-sequential 4-separations displayed by Φ, as well as the non-sequential 4-separation $(Q_1, E(M) - Q_1)$.

Fortunately, all tight irredundant non-maximal flowers displaying at least two inequivalent non-sequential k-separations have the same predominant structure as the 4-flower Φ in Example 6.2.3. We make this more precise in the next lemma.

We say that a k-separation (X, Y) *crosses* a k-separation (U, V) if each of $X \cap U, X \cap V, Y \cap U, Y \cap V$ is non-empty.

Lemma 6.2.4. *Let M be a k-connected matroid with ground set E and let T be a conforming k-tree for M. Suppose that, for every flower vertex v of T, the k-flower corresponding to v is tight, irredundant, and displays at least two inequivalent non-sequential k-separations. Then, either*

(i) *T is a partial k-tree for M, or*

(ii) *there is a flower vertex of T for which the corresponding k-flower is k-equivalent to $(Q_1 \cup Q_4, Q_2, Q_3)$, but (Q_1, Q_2, Q_3, Q_4) is a maximal tight irredundant k-flower and the only non-sequential k-separations displayed by this maximal k-flower are $(Q_1, E - Q_1)$, $(Q_2, E - Q_2)$, and $(Q_1 \cup Q_4, Q_2 \cup Q_3)$.*

Proof. Let Φ be a k-flower corresponding to a flower vertex v of T. By hypothesis, Φ is tight, irredundant, and displays at least two inequivalent non-sequential k-separations. Assume that Φ is not maximal. We will show that v satisfies (ii). Since Φ is not maximal, there exists a tight irredundant maximal k-flower Φ' that displays, up to k-equivalence, all the non-sequential k-separations displayed by Φ, as well as at least one non-sequential k-separation (R, G) that, up to k-equivalence, is not displayed by Φ. In particular, for every union U of petals of Φ such that $(U, E - U)$ is a non-sequential k-separation in M, there is a union U' of petals of Φ' such that $(U, E - U)$ is k-equivalent to $(U', E - U')$.

We may assume that $\Phi' = (Q_1, Q_2, \ldots, Q_n)$, where $R = Q_1 \cup Q_2 \cup \cdots \cup Q_l$ for some $1 \leq l \leq n - 1$. Let $\Phi = (P_1, P_2, \ldots, P_m)$. As T is a conforming k-tree for M, there is an (R, G)-equivalent k-separation (R', G') that conforms with T and, without loss of generality, we may assume that R' is properly contained in some petal P_r of Φ. By Corollary 4.3.3, P_r is non-sequential. If $E - P_r$ is sequential, then it follows, by Lemma 4.3.2, that Φ displays no non-sequential k-separations; a contradiction. Hence $(P_r, E - P_r)$ is non-sequential and Φ' displays an equivalent k-separation $\big(\bigcup_{i \in I} Q_i, \bigcup_{j \in \{1,2,\ldots,n\} - I} Q_j\big)$ for some proper subset I of $\{1, 2, \ldots, n\}$, where $\mathrm{fcl}_k(P_r) = \mathrm{fcl}_k(\bigcup_{i \in I} Q_i)$.

6.2.4.1. *There are no non-sequential k-separations displayed by Φ' that cross $\big(\bigcup_{i \in I} Q_i, \bigcup_{j \in \{1,2,\ldots,n\} - I} Q_j\big)$.*

Suppose there is a non-sequential k-separation $(Q, E - Q)$ displayed by Φ' such that Q contains the petals Q_{i_1} and Q_{j_1}, and $E - Q$ contains the petals Q_{i_2} and Q_{j_2}, for some $i_1, i_2 \in I$ and $j_1, j_2 \in \{1, 2, \ldots, n\} - I$. Now $(Q, E - Q)$ is k-equivalent to a non-sequential k-separation $(Q', E - Q')$, where $\mathrm{fcl}_k(Q) = \mathrm{fcl}_k(Q')$, that conforms with T. Hence either

(I) $(Q', E - Q')$ is displayed by Φ, or

(II) Q' or $E - Q'$ is contained in a petal of Φ.

Recall that $\mathrm{fcl}_k(P_r) = \mathrm{fcl}_k(\bigcup_{i \in I} Q_i)$. Suppose that (I) holds. Then we may assume that $Q' = \bigcup_{i \in K} P_i$ for some proper subset K of $\{1, 2, \ldots, m\}$. Now $\mathrm{fcl}_k(Q')$ contains the petal Q_{i_1}, so $\mathrm{fcl}_k(E - Q')$ does not contain Q_{i_1} by Corollary 4.3.11. But $Q_{i_1} \subseteq \mathrm{fcl}_k(P_r)$, so $P_r \subseteq Q'$. Then $Q_{i_2} \subseteq \mathrm{fcl}_k(P_r) \subseteq \mathrm{fcl}_k(Q') = \mathrm{fcl}_k(Q)$. Since $Q_{i_2} \subseteq E - Q$, it follows, by Corollary 4.3.9, that Q_{i_2} is loose; a contradiction. Thus we deduce that (II) holds.

Without loss of generality, either $Q' \subseteq P_1$ or $E - Q' \subseteq P_1$. First assume that $Q' \subseteq P_1$. Then $Q_{j_1} \subseteq \mathrm{fcl}_k(Q) = \mathrm{fcl}_k(Q') \subseteq \mathrm{fcl}_k(P_1)$. But $Q_{j_1} \subseteq \mathrm{fcl}_k(E - P_r)$, so $Q_{j_1} \nsubseteq \mathrm{fcl}_k(P_r)$, by Corollary 4.3.11. Hence $P_r \neq P_1$. As $Q' \subseteq P_1$ and $R' \subseteq P_r \subseteq E - P_1$, it follows, by Corollary 4.3.3, that $(P_1, E - P_1)$ is non-sequential. Thus, there is a union $\bigcup_{w \in W} Q_w$ of petals of Φ' such that $(P_1, E - P_1)$ is equivalent to $\big(\bigcup_{w \in W} Q_w, \bigcup_{w \in \{1,2,\ldots,n\} - W} Q_w\big)$, where $\mathrm{fcl}_k(P_1) = \mathrm{fcl}_k(\bigcup_{w \in W} Q_w)$. Now $Q_{i_1} \subseteq \mathrm{fcl}_k(Q) = \mathrm{fcl}_k(Q') \subseteq \mathrm{fcl}_k(P_1) = \mathrm{fcl}_k(\bigcup_{w \in W} Q_w)$ and $Q_{i_1} \subseteq \mathrm{fcl}_k(P_r) \subseteq \mathrm{fcl}_k(E - P_1) \subseteq \mathrm{fcl}_k(\bigcup_{w \in \{1,2,\ldots,n\} - W} Q_w)$, contradicting Corollary 4.3.11.

Thus, we may assume that $E - Q' \subseteq P_1$. Suppose that $P_r \neq P_1$. Then $P_r \subseteq Q'$, so $Q_{i_2} \subseteq \mathrm{fcl}_k(P_r) \subseteq \mathrm{fcl}_k(Q') = \mathrm{fcl}_k(Q)$. Hence, by Corollary 4.3.9, Q_{i_2} is loose; a contradiction. We deduce that $P_r = P_1$. Thus $Q_{j_2} \subseteq \mathrm{fcl}_k(E - Q') \subseteq \mathrm{fcl}_k(P_r) = \mathrm{fcl}_k(\bigcup_{i \in I} Q_i)$, so, by Corollary 4.3.9 again, Q_{j_2} is loose; a contradiction. This completes the proof of (6.2.4.1).

6.2.4.2. $\Phi' = (Q_1, Q_2, Q_3, Q_4)$*, and the only non-sequential k-separations displayed by* Φ' *are* $(Q_1, E - Q_1)$, $(Q_2, E - Q_2)$ *and* $(Q_1 \cup Q_4, Q_2 \cup Q_3)$.

Suppose that $|I| = n - 1$. By assumption, Φ displays a non-sequential k-separation $(O, E - O)$ that is not equivalent to $(P_r, E - P_r)$. As P_r is a petal of Φ, it follows that $\mathrm{fcl}_k(P_r)$ is a proper subset of either $\mathrm{fcl}_k(O)$ or $\mathrm{fcl}_k(E - O)$. Let $(O', E - O')$ be the k-separation displayed by Φ' that is equivalent to $(O, E - O)$. Since Φ' has only one petal Q_j such that $j \notin I$, either O' or $E - O'$ is contained in $\bigcup_{i \in I} Q_i$. Hence $\mathrm{fcl}_k(\bigcup_{i \in I} Q_i)$ contains $\mathrm{fcl}_k(O')$ or

$\text{fcl}_k(E - O')$, so $\text{fcl}_k(P_r)$ contains $\text{fcl}_k(O)$ or $\text{fcl}_k(E - O)$; a contradiction. Thus $|I| \leq n - 2$.

Since $\text{fcl}_k(R) = \text{fcl}_k(Q_1 \cup Q_2 \cup \cdots \cup Q_l) = \text{fcl}_k(R') \subseteq \text{fcl}_k(\bigcup_{i \in I} Q_i)$ and Φ' is a tight flower, it follows, by Corollary 4.3.9, that $\{1, 2, \ldots, l\} \subseteq I$. Moreover, I contains at least one element in $\{l+1, l+2, \ldots, n\}$, since no k-separation equivalent to (R, G) is displayed by Φ. Thus we may assume that

$$I = \{n - s + 1, \ldots, n, 1, 2, \ldots, l, l + 1, \ldots, l + t\},$$

where $s \geq 1$ and $l + t \leq n - s - 2$, and hence $n \geq 4$.

Let $(Q, E-Q) = (Q_1 \cup Q_2 \cup \cdots \cup Q_{l+t+1}, Q_{l+t+2} \cup \cdots \cup Q_n)$. Since $\{1, n\} \subseteq I$ and $\{l+t+1, l+t+2\} \subseteq \{1, 2, \ldots, n\} - I$, the k-separation $(Q, E - Q)$ crosses $\left(\bigcup_{i \in I} Q_i, \bigcup_{j \in \{1,2,\ldots,n\}-I} Q_j\right)$. By (6.2.4.1), and since $\text{fcl}_k(Q)$ contains $\text{fcl}_k(R)$, the set $E - Q$ is k-sequential. Thus, by Corollary 4.4.9, we may assume that $l + t + 1 = n - 2$ and $Q_{n-1} \cup Q_n$ is k-sequential.

Since Φ' is irredundant, there exists a non-sequential k-separation $(Q', E - Q')$ displayed by Φ' where $Q_{l+t+1} = Q_{n-2} \subseteq Q'$ and $Q_{n-1} \subseteq E - Q'$. If $Q_n \subseteq Q'$, then we obtain a contradiction to (6.2.4.1) unless $Q_1 \cup Q_2 \cup \cdots \cup Q_{l+t} \subseteq Q'$, in which case Q_{n-1} is non-sequential. But then $Q_{n-1} \cup Q_n$ is non-sequential by Corollary 4.3.3; a contradiction. Thus we may assume that $Q_n \subseteq E - Q'$. But now the existence of $(Q', E - Q')$ contradicts (6.2.4.1) unless $Q_1 \cup Q_2 \cup \cdots \cup Q_{l+t} \subseteq E - Q'$, in which case Q_{n-2} is non-sequential. In the exceptional case, when $n \geq 5$, the k-separation $(Q_2 \cup \cdots \cup Q_{n-2}, Q_{n-1} \cup Q_n \cup Q_1)$ is non-sequential by Corollary 4.4.9, again contradicting (6.2.4.1). In the remaining case, $\Phi' = (Q_1, Q_2, Q_3, Q_4)$ and the k-separations $(Q_2, E - Q_2)$ and $(Q_1 \cup Q_4, Q_2 \cup Q_3)$ are non-sequential, but $Q_3 \cup Q_4$ is k-sequential. Since Φ' is irredundant, there exists a non-sequential k-separation (U, V) displayed by Φ' with $Q_1 \subseteq U$ and $Q_4 \subseteq V$. Since $Q_3 \cup Q_4$ is k-sequential, either $(U, V) = (Q_1 \cup Q_3, Q_2 \cup Q_4)$ or $(U, V) = (Q_1, E - Q_1)$. But if the former, then (U, V) crosses $\left(\bigcup_{i \in I} Q_i, \bigcup_{j \in \{1,2,\ldots,n\}-I} Q_j\right)$, contradicting (6.2.4.1). Thus $(Q_1, E - Q_1)$ is a non-sequential k-separation, and Φ displays no other non-sequential k-separations apart from $(Q_2, E - Q_2)$ and $(Q_1 \cup Q_4, Q_2 \cup Q_3)$. This completes the proof of (6.2.4.2).

Since T is a conforming tree and Φ displays at least two inequivalent non-sequential k-separations, the k-separation (R, G) displayed by Φ', but not Φ, is either $(Q_1, E - Q_1)$ or $(Q_2, E - Q_2)$. Thus, up to swapping Q_1 and Q_2, the flower Φ displays the same non-sequential k-separations as $(Q_1 \cup Q_4, Q_2, Q_3)$. Hence, when Φ is not maximal, (ii) holds. This completes the proof of the lemma. □

Proposition 6.2.5. *Let M be a k-connected matroid with $|E(M)| \geq 8k-15$. The tree returned by k-*TREE*(M) is a partial k-tree for M.*

Proof. By Lemma 6.1.4, the tree T returned by k-TREE(M) is a conforming tree for M and, by Lemmas 6.1.5 and 6.2.1, for each flower vertex u of T, the flower corresponding to u is tight, irredundant, and displays at least two inequivalent non-sequential k-separations. Suppose T is not a partial k-tree for M. Then, by Lemma 6.2.4, T has a flower vertex for which the corresponding k-flower Φ is $(Q_1 \cup Q_4, Q_2, Q_3)$. Furthermore, the non-sequential k-separations displayed by this k-flower are precisely $(Q_2, E-Q_2)$ and $(Q_1 \cup Q_4, Q_2 \cup Q_3)$, but $(Q_1, E-Q_1)$ is also a non-sequential k-separation.

By construction, the algorithm k-TREE at some stage invokes BACKWARDSWEEP, either in line 6 or line 15, at which point a generalised k-path τ is returned with a concatenation τ' that is, up to a reversal of the parts, one of $(Q_3, [(Q_1 \cup Q_4)], Q_2)$, $(Q_1 \cup Q_4, [(Q_2)], Q_3)$, and $(Q_2, [(Q_3)], Q_1 \cup Q_4)$. Since Q_3 is k-sequential and no other petal is k-sequential, it follows that Q_3 is not an entry or exit petal of Φ. Thus $\tau' = (Q_2, [(Q_3)], Q_1 \cup Q_4)$ or $\tau' = (Q_1 \cup Q_4, [(Q_3)], Q_2)$.

Let $(Z_0 \cup Z_1, Z_2, \ldots, Z_m)$ be the left-justified maximal k-path provided to the call to BACKWARDSWEEP. First, assume that $\tau' = (Q_2, [(Q_3)], Q_1 \cup Q_4)$. Since $(Q_1, E-Q_1)$ conforms with T, and Q_4 is k-sequential, it follows, by BACKWARDSWEEP, that, up to equivalence, τ is a refinement of $(Q_2, [(Q_3)], Q_4, Q_1)$. Suppose that $\tau = (\ldots, [(Q_3)], [(S_1, \ldots, S_s), (T_1, \ldots, T_t)], \ldots)$, where $s \geq 1$ and $t \geq 0$. Then $Q_3 = Z_j$ for some $j \in \{2, 3, \ldots, m-1\}$. By construction, $(Q_4 \cup Q_1) - S_1$ and $(Q_4 \cup Q_1) - T_1$ are k-separating and, up to equivalence, either S_1 or T_1 is a subset of Q_4. If S_1 is a subset of Q_4, then, by uncrossing $(Q_4 \cup Q_1) - S_1$ and $Q_1 \cup Q_2$, we deduce that $(Q_4 - S_1) \cup Q_1 \cup Q_2$ is k-separating, hence $Q_3 \cup S_1$ is k-separating. Then, line 48 of BACKWARDSWEEP is invoked when $i = j$, so τ is of the form $(\ldots, [(Q_3, S_1, \ldots, S_s), (T_1, \ldots, T_t)], \ldots)$; a contradiction. Otherwise, T_1 is a subset of Q_4, and, similarly, $Q_3 \cup T_1$ is k-separating, so line 50 is invoked; a contradiction. Now suppose $\tau = (\ldots, [(Q_3)], Z_{j+1}, \ldots)$. Then, up to equivalence, $Z_{j+1} \subseteq Q_4$. Hence line 61 of BACKWARDSWEEP is invoked when $i = j + 1$, so Z_{j+1} is not k-separating. But $Q_2 \cup Q_3 \cup Z_{j+1}$ is k-separating by construction, and it follows, by uncrossing $Q_2 \cup Q_3 \cup Z_{j+1}$ and Q_4, that Z_{j+1} is k-separating; a contradiction.

Now assume that $\tau' = (Q_1 \cup Q_4, [(Q_3)], Q_2)$. Since $(Q_1, E - Q_1)$ conforms with T, and Q_4 is k-sequential, τ is a refinement of $(Q_1, Q_4, [(Q_3)], Q_2)$, up to equivalence. Consider the construction of τ_i in BackwardSweep where $i \in \{2, 3, \ldots, m-2\}$ such that $\tau_{i+1}(Z_i^+) = ([(Q_3)], \ldots)$. The algorithm reaches line 45 of BackwardSweep and $Z_i \subseteq Q_4$. Since $Z_i \cup Q_3 \cup Q_2$ and Q_4 are k-separating, Z_i is also k-separating, by uncrossing. Moreover, by uncrossing $Z_i \cup Q_3 \cup Q_2$ and $Q_4 \cup Q_3$, we deduce that $Z_i \cup Q_3$ is k-separating. Hence line 48 is invoked, and τ_i is of the form $(\ldots, [(Z_i, Q_3)], \ldots)$; a contradiction. Thus T has no flower vertex as described in Lemma 6.2.4(ii), so T is a partial k-tree as required. □

6.3 The proof of correctness

The proof of Theorem 4.0.2 is a simple upgrade of the $k = 3$ case (Oxley and Semple, 2013, Theorem 2.2).

Proof of Theorem 4.0.2. To prove the theorem, we show that k-Tree is a polynomial-time algorithm for finding a k-tree for M. Let T be the tree returned by a call to k-Tree(M). Then every vertex of T is marked. Moreover, by Proposition 6.2.5, T is a partial k-tree for M. Now T is a k-tree for M unless there is a non-sequential k-separation of M with the property that no equivalent k-separation is displayed by T. Suppose there is such a k-separation (R, G). Since T is conforming, we may assume, by taking an equivalent k-separation if necessary, that G is contained in a bag B of T. If T consists of the single bag vertex B, then line 3 of k-Tree would have found a non-sequential k-separation (Y, Z) of M; a contradiction. But if T consists of at least two vertices, then line 9 of k-Tree would have found a non-sequential k-separation (Y, Z) of M with the property that $Z \subseteq \pi(B)$, contradicting the fact that B is marked. Hence T is a k-tree for M.

We next show that k-Tree runs in polynomial time in the size n of $E(M)$. By Lemma 5.1.1, the collection $\mathcal{F}$ of maximal sequential k-separating sets of M can be constructed in polynomial time in n, and, by Theorem 5.1.2, for fixed disjoint subsets Y' and Z' of $E(M)$, we can find a k-separation (Y, Z) with $Y' \subseteq Y$ and $Z' \subseteq Z$ in polynomial time in n, or determine that none exists. Thus, by Lemma 5.1.3, we can find a non-sequential k-separation by iterating over all k-element subsets of $E(M)$ not contained in a member of $\mathcal{F}$. As there are $O(n^k)$ such subsets, where k is fixed, this can be done in polynomial time in n. Extending this, whenever k-Tree, or one of the two subroutines, is called upon to find a k-separation where each part contains particular subsets, it either finds such a k-separation or correctly determines that there is no such k-separation in time polynomial in n. Therefore, as every k-path of M has length $O(n)$, it follows that each call to ForwardSweep takes time polynomial in n.

Now consider a call from k-Tree to the subroutine BackwardSweep. When $m \geq 3$, this subroutine considers each of the following subsets of $E(M)$ in turn: the subsets Z_m and Z_{m-1}, a subset Z_i where $i \in \{m-2, m-3, \ldots, 2\}$, and finally the subset $X_0 \cup Z_1$. For each of the subsets $Z_2, Z_3, \ldots, Z_{m-2}$, it is clear that their consideration takes polynomial time in n. Note that finding the full closure of a subset X of $E(M)$, as in line 58 of BackwardSweep, takes time $O(n^{k-1})$. For the subsets Z_m and $X_0 \cup Z_1$, BackwardSweep may, up to five times, attempt to find k-separations where each part contains particular subsets. As mentioned above, each call takes time polynomial in n, so the time taken for BackwardSweep to consider each of Z_m and $X_0 \cup Z_1$ is also polynomial in n. Since $m \leq n$, it follows that, when $m \geq 3$, BackwardSweep takes time polynomial in n. Similarly, the subroutine takes time polynomial in n when $m = 2$, so each call to BackwardSweep takes time polynomial in n.

At the completion of each call to BackwardSweep, the algorithm k-Tree extends the current π-labelled tree to a new π-labelled tree in polynomial time in n. This extension is non-trivial in that at least one new edge is created. Since the terminal bags of each such constructed π-labelled tree contain at least $k-1$ elements of $E(M)$ and there is no empty bag vertex of degree two, the number of edges of each constructed π-labelled tree is linear in n, and so there are $O(n)$ calls to ForwardSweep and BackwardSweep from k-Tree. As marked bags are never reconsidered, we deduce that k-Tree terminates in time polynomial in n. This completes the proof of the theorem. □

Bibliography

Aikin, J. M. (2009), The structure of 4-separations in 4-connected matroids, Ph.D. dissertation, Louisiana State University.

Aikin, J. and Oxley, J. (2008), 'The structure of crossing separations in matroids', *Advances in Applied Mathematics* **41**(1), 10–26.

Aikin, J. and Oxley, J. (2012), 'The structure of the 4-separations in 4-connected matroids', *Advances in Applied Mathematics* **48**(1), 1–24.

Beavers, B. D. (2006), Circuits and structure in matroids and graphs, Ph.D. dissertation, Louisiana State University.

Bixby, R. E. (1982), 'A simple theorem on 3-connectivity', *Linear Algebra and its Applications* **45**, 123–126.

Brettell, N. and Semple, C. (2014*a*), 'A Splitter Theorem relative to a fixed basis', *Annals of Combinatorics* **18**(1), 1–20.

Brettell, N. and Semple, C. (2014*b*), 'An algorithm for constructing a k-tree for a k-connected matroid', *Annals of Combinatorics*, to appear.

Brylawski, T. (1975), 'Modular constructions for combinatorial geometries', *Transactions of the American Mathematical Society* **203**, 1–44.

Chen, R. and Xiang, K. N. (2012), 'Decomposition of 3-connected representable matroids', *Journal of Combinatorial Theory, Series B* **102**(3), 647–670.

Clark, B. and Whittle, G. (2013), 'Tangles, trees, and flowers', *Journal of Combinatorial Theory, Series B* **103**(3), 385–407.

Cook, W. J., Cunningham, W. H., Pulleyblank, W. R. and Schrijver, A. (1998), Matroid Intersection, *in* 'Combinatorial Optimization', Wiley, New York, pp. 287–294.

Coullard, C. and Oxley, J. (1992), 'Extensions of Tutte's wheels-and-whirls theorem', *Journal of Combinatorial Theory, Series B* **56**(1), 130–140.

Cunningham, W. H. (1973), A combinatorial decomposition theory, Ph.D. dissertation, University of Waterloo.

Cunningham, W. H. and Edmonds, J. (1980), 'A combinatorial decomposition theory', *Canadian Journal of Mathematics* **32**(3), 734–765.

Edmonds, J. (1970), Submodular functions, matroids, and certain polyhedra, *in* 'Combinatorial Structures and Their Applications', Gordon and Breach, New York, pp. 69–87.

Geelen, J. F., Gerards, A. M. H. and Kapoor, A. (2000), 'The excluded minors for $GF(4)$-representable matroids', *Journal of Combinatorial Theory, Series B* **79**(2), 247–299.

Geelen, J. and Whittle, G. (2013), 'Inequivalent representations of matroids over prime fields', *Advances in Applied Mathematics* **51**(1), 1–175.

Hall, R., Oxley, J. and Semple, C. (2005), 'The structure of equivalent 3-separations in a 3-connected matroid', *Advances in Applied Mathematics* **35**(2), 123–181.

Kahn, J. (1988), 'On the uniqueness of matroid representations over $GF(4)$', *Bulletin of the London Mathematical Society* **20**, 5–10.

Oxley, J. (1996), 'Structure theory and connectivity for matroids', *Contemporary Mathematics* **197**, 129–170.

Oxley, J. (2011), *Matroid Theory*, Vol. 21 of *Oxford Graduate Texts in Mathematics*, second edition, Oxford University Press, New York.

Oxley, J. and Semple, C. (2013), 'Constructing a 3-tree for a 3-connected matroid', *Advances in Applied Mathematics* **50**(1), 176–200.

Oxley, J., Semple, C. and Whittle, G. (2004), 'The structure of the 3-separations of 3-connected matroids', *Journal of Combinatorial Theory, Series B* **92**(2), 257–293.

Oxley, J., Semple, C. and Whittle, G. (2008*a*), 'Maintaining 3-connectivity relative to a fixed basis', *Advances in Applied Mathematics* **41**(1), 1–9.

Oxley, J., Semple, C. and Whittle, G. (2008*b*), 'Wild triangles in 3-connected matroids', *Journal of Combinatorial Theory, Series B* **98**(2), 291–323.

Oxley, J., Semple, C. and Whittle, G. (2012), 'An upgraded Wheels-and-Whirls Theorem for 3-connected matroids', *Journal of Combinatorial Theory, Series B* **102**(3), 610–637.

Oxley, J., Vertigan, D. and Whittle, G. (1996), 'On inequivalent representations of matroids over finite fields', *Journal of Combinatorial Theory, Series B* **67**(2), 325–343.

Oxley, J. and Wu, H. (2000), 'On the structure of 3-connected matroids and graphs', *European Journal of Combinatorics* **21**(5), 667–688.

Seymour, P. D. (1980), 'Decomposition of regular matroids', *Journal of Combinatorial Theory, Series B* **28**(3), 305–359.

Seymour, P. D. (1995), Matroid Minors, *in* 'Handbook of combinatorics', R. Graham, M. Grötschel and L. Lovász, editors, MIT press, Cambridge, pp. 527–550.

Tan, J. J.-M. (1981), Matroid 3-connectivity, Ph.D. dissertation, Carleton University.

Tutte, W. T. (1966), 'Connectivity in matroids', *Canadian Journal of Mathematics* **18**, 1301–1324.

Whittle, G. (1999), 'Stabilizers of classes of representable matroids', *Journal of Combinatorial Theory, Series B* **77**(1), 39–72.

Whittle, G. and Williams, A. (2013), 'On preserving matroid 3-connectivity relative to a fixed basis', *European Journal of Combinatorics* **34**(6), 957–967.