Spacetime as a Membrane in Higher Dimensions

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Nucl. Phys. **B287** (1987) 717-742.

Received 29 September 1986
(Revised 5 December 1986)

Abstract

By means of a simple model we investigate the possibility that spacetime is a membrane embedded in higher dimensions. We present cosmological solutions of d-dimensional Einstein-Maxwell theory which compactify to two dimensions. These solutions are analytically continued to obtain dual solutions in which a $(d-2)$-dimensional Einstein spacetime "membrane" is embedded in d-dimensions. The membrane solutions generalise Melvin's 4-dimensional flux tube solution. The flat membrane is shown to be classically stable. It is shown that there are zero mode solutions of the d-dimensional Dirac equation which are confined to a neighbourhood of the membrane and move within it like massless chiral $(d-2)$-dimensional fermions. An investigation of the spectrum of scalar perturbations shows that a well-defined mass gap between the zero modes and massive modes can be obtained if there is a positive cosmological term in $(d-2)$ dimensions or a negative cosmological term in d dimensions.

[*Note added 2001:* In this early brane world model we sought to realise spacetime as a codimension two "thick brane" warped product submanifold of a higher–dimensional curved spacetime, as distinct from the previous models of K. Akama [Lect. Notes Phys. **176** (1982) 267 = hep-th/0001113] and Rubakov and Shaposhnikov [26], which involved field theoretic trappings of matter in a higher-dimensional flat spacetime, and Visser's model [9] in which the physical spacetime did not possess an exact Lorentz invariance.]

1. Introduction

The idea of regarding our 4-dimensional spacetime as a world sheet or membrane in some higher-dimensional spacetime is an old one. Most frequently one thinks of isometrically embedding spacetime into some higher-dimensional *flat* spacetime to aid the study of its geometry. A considerable amount is known about when this is possible and there are many examples of embeddings for familiar spacetimes [1-4]. Attempts have also been made to find an action principle governing the embedding variables which would give rise to the Einstein equations for the metric on the membrane. However, this approach encounters various difficulties [5].

In this paper we wish to examine a rather different idea. We will show in section 3 that there are solutions of d-dimensional Einstein-Maxwell theory which behave like a $(d-2)$-dimensional membrane embedded in the d-dimensional curved spacetime. Furthermore, in section 5 we find zero mode solutions of the Dirac equation in d dimensions which are confined to a neighbourhood of the membrane and move within it like massless $(d-2)$-dimensional fermions. Fermion chirality is preserved in the reduction from d to $(d-2)$ dimensions (for even d).

This has suggested to us an alternative solution to the problem that many currently attractive quantum gravity models seem only to be viable in higher dimensions. In contrast to the standard spontaneous compactification picture [6], in which the ground state is viewed as a product of 4-dimensional Minkowski spacetime with a compact internal space K, our idea is that we are confined to a 4-dimensional Minkowski membrane in a topologically trivial higher dimensional universe. An alternative description is to say that the "internal" space K is not compact but is topologically $\mathbb{R}^n$.

The idea that "internal symmetries" arise from higher dimensions and that some potential or force confines us to a 4-dimensional subspace is not new. Some rudimentary speculations along these lines may be found in [7] and no doubt there are earlier papers. More recently Rubakov and Shaposhnikov [8] and Visser [9] have discussed models of a nature similar to that of the membrane solutions studied here.

The main problem that all models with non-compact internal spaces face is to explain why one may ignore the massive states in lower dimensions which arise from harmonic expansions of the higher dimensional fields. These states may be ignored if they are separated from the zero modes by a well defined "mass gap". It does not seem essential that the spectrum of masses necessarily be discrete. We find that this condition is met automatically for the membrane solutions if the membrane has curvature, or if it is flat then provided that there is a negative higher-dimensional cosmological term.

2. Einstein-Maxwell Theory and Monopole Compactifications

In this paper we shall deal mainly with the standard Einstein-Maxwell theory in d dimensions. The case $d = 6$ is of most interest but we shall give solutions for arbitrary d. The action, including a possible cosmological term, is

$$S = \int \mathrm{d}^d x \sqrt{|\det g_{ab}|} \left[\frac{-1}{4\kappa^2}(R + 2\Lambda) - \frac{1}{4} F_{ab} F^{ab} \right] . \tag{2.1}$$

We use signature $(+ - - \cdots -)$ and conventions such that $R^a{}_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \ldots$, $R_{ab} = R^c{}_{acb}$ and $\kappa^2 = 4\pi G$. The usual approach to compactification is to assume that $\mathcal{M}^d = \mathcal{M}^{d-2} \times S^2$ and that there is a monopole or Freund-Rubin type field on the S^2 [10], i.e.

$$F_{ab} \mathrm{d}x^a \mathrm{d}x^b = \frac{Q}{4\pi} \varepsilon_{mn} \mathrm{d}y^m \wedge \mathrm{d}y^n , \tag{2.2}$$

where y^m, $m = 1, 2$, are coordinates on the 2-sphere and ε_{mn} is the alternating tensor. The Einstein field equations (without a cosmological term) now tell us that spacetime must be a $(d-2)$-dimensional anti-de Sitter space, $(AdS)_{d-2}$, with radius of curvature equal to that of the 2-sphere. The case $d = 4$ coincides with the Robinson-Bertotti metric in 4-dimensional Einstein-Maxwell theory.

To obtain a flat spacetime factor one can include a positive cosmological term Λ to "fine tune" the physical cosmological constant to be zero [11]. Another more appealing procedure is to replace the cosmological constant by a scalar field in the manner of Salam and Sezgin [12]. The potential acts as a positive cosmological term and one obtains a flat spacetime factor without special adjustment of the parameters.

These ground states have been found to be stable against small localised perturbations [13,14]. One should in addition check for cosmological stability. This has been done by Okada in the fine tuning case [15] and the cosmology of the Salam-Sezgin model has been studied by Maeda and Nishino [16], Lonsdale [17], Halliwell [18] and Gibbons and Townsend [19]. In the fine tuning case Okada found cosmological models with flat spatial sections in which the radius of the internal space oscillates in time. The spatially flat case is unstable amongst all Friedmann-Robertson-Walker universes. A more interesting case to consider is when the spatial sections have negative curvature. In that case one finds that if the universe expands the oscillations die out and the internal radius settles down to a constant value, the spacetime factor being asymptotic to an empty Milne universe. Similar remarks apply to the Salam-Sezgin model. Thus these models have the pleasing feature that the ground state acts as a cosmological attractor.

In addition to the monopole compactification in which there is a magnetic 2-form on the internal space solutions exist with an electric 2-form on a 2-dimensional spacetime factor and an internal space which can be any 4-dimensional Einstein space of positive curvature, of which four examples are known: S^4, $S^2\times S^2$, $\mathbb{C}P^2$ and $\mathbb{C}P^2\ \#\ \overline{\mathbb{C}P}^2$. If there is no cosmological term the spacetime factor is $(AdS)_2$. This may be fine tuned to 2-dimensional Minkowski spacetime by addition of a positive cosmological constant.

The reduction from d to two dimensions is not relevant to our world of course but the possibility is contained in the theory and if one really believes that higher dimensions played a dynamical role in the very early universe one should presumably explain why such a compactification did not occur, at least in our neck of the woods. Some insight into this can be gained by looking at the cosmological models. Let us assume that the metric takes the form

$$\mathrm{d}s^2=\mathrm{d}t^2-a(t)^2\mathrm{d}\chi^2-b(t)^2\bar{g}_{IJ}(y)\mathrm{d}y^I\mathrm{d}y^J\ , \tag{2.3}$$

where $\bar{g}_{IJ}(y)$, $I,J=2,\dots,d-1$ is a $(d-2)$-dimensional Einstein space,

$$\bar{R}_{IJ}=(d-3)\bar{\lambda}\bar{g}_{IJ}\ . \tag{2.4}$$

We are primarily interested in solutions with a compact internal space, i.e. $\bar{\lambda}>0$. We include the general case here, however, as it will be of interest in the extension to the membrane solutions. Solutions based on the pure gravity ansatz (2.3), (2.4) have been outlined by Ishihara et al. [20]. We will assume in addition that there is electric field given by

$$\mathbf{F}=\frac{Qa}{4\pi b^{d-2}}\mathrm{d}t\wedge\mathrm{d}\chi\ . \tag{2.5}$$

Maxwell's equations are trivially satisfied by (2.5). The remaining field equations obtained by variation of the action (2.1) are

$$\frac{\dot{a}}{a}+(d-2)\frac{\dot{b}}{b}=\frac{2\Lambda}{(d-2)}-\frac{(d-3)GQ^2}{2\pi(d-2)b^{2(d-2)}}\ , \tag{2.6a}$$

$$\frac{\dot{a}}{a}+(d-2)\frac{\dot{a}\dot{b}}{ab}=\frac{2\Lambda}{(d-2)}-\frac{(d-3)GQ^2}{2\pi(d-2)b^{2(d-2)}}\ , \tag{2.6b}$$

$$\frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{(d-3)}{b^2}\left(\bar{\lambda} + \dot{b}^2\right) = \frac{2\Lambda}{(d-2)} + \frac{GQ^2}{2\pi(d-2)b^{2(d-2)}} \ , \tag{2.6c}$$

where $\dot{}\equiv \mathrm{d}/\mathrm{d}t$. The difference of (2.6a) and (2.6b) may be immediately integrated to give the result

$$a = c\dot{b} \ , \tag{2.7}$$

where c is an arbitrary constant. If we regard b as an independent time coordinate (instead of t) and write $a(b) = \dot{b}(t) = \dot{b}(b(t)) = \dot{b}(b)$ (setting $c = 1$) then (2.6c) becomes a Bernoulli equation which may be integrated to give

$$a^2 \equiv \Delta = \frac{2GM}{b^{d-3}} - \frac{GQ^2}{2\pi(d-2)(d-3)b^{2(d-3)}} + \frac{2\Lambda b^2}{(d-1)(d-2)} - \bar{\lambda} \ . \tag{2.8}$$

The metric therefore has the form

$$\mathrm{d}s^2 = \frac{db^2}{\Delta} - \Delta \mathrm{d}\chi^2 - b^2 g_{IJ}\mathrm{d}y^I\mathrm{d}y^J \ . \tag{2.9}$$

Reality of $a \equiv \dot{b}$ requires $\Delta > 0$, i.e. $\dfrac{2Q^2}{(d-2)(d-3)} < \kappa^2 M^2$ if $\Lambda = 0$ and $\bar{\lambda} = 1$.

Consider the case $\Lambda = 0$, $\bar{\lambda} = 1$. The spacetime (2.9) is incomplete and if it is extended the cosmological region described by (2.9) corresponds to the intermediate region $b_- < b < b_+$ of a d-dimensional black hole, where

$$(b_\pm)^{d-3} = GM \pm \frac{\kappa}{4\pi}\left(\kappa^2 M^2 - \frac{2Q^2}{(d-2)(d-3)}\right)^{1/2} \ . \tag{2.10}$$

The exterior region of the black hole is given by (2.9) with $\Delta < 0$ (and $\Lambda = 0$). This metric is then a trivial generalization of the d-dimensional Reissner-Nordström metric written down by Tangherlini [21], who chose the "internal" space to be S^{d-2}. The global structure of the metric is best revealed by a Carter-Penrose diagram, as shown in Fig. 1, in which each point represents an internal space. The t-dependence of the radii b and a is illustrated in Fig. 2. (In the extreme case $\kappa^2 M^2 = \dfrac{2Q^2}{(d-2)(d-3)}$ we display the behaviour of $a(t)$ for the solution dual to the Freund-Rubin solution which may be obtained from (2.9) by a limiting procedure.) The main point is that the general oscillating solution corresponds to the interior of a black hole in d dimensions. This would presumably be a very inhospitable place to live and indeed one would expect the Cauchy horizons to be unstable and turn into real curvature singularities if the metric were perturbed, just as in the 4-dimensional case [22-24].

Another reason for believing that the reduction to two dimensions should be unstable is that the electric fields can presumably create charged particle-antiparticle pairs which will tend to reduce those fields. However, in a supersymmetric theory it is possible that this process would be suppressed (c.f. [25]).

It seems therefore that while the "electric" reduction from d to two dimensions might come about naturally by gravitational collapse to form a black hole in d dimensions it is as yet unclear what might bring about the "magnetic" reduction to $(d-2)$ dimensions. The global structure of the analogous cosmological solutions has not yet been investigated because exact solutions are not known. It is difficult to believe, however, that it can be any less complicated than the behaviour in the electric case. Some idea about the compactification process can be gleaned from a study of the membrane-like solutions, however, and it is to this that we now turn.

3. Membranes

In this section we shall describe some solutions of the d-dimensional Einstein-Maxwell theory in which a flat $(d-2)$-dimensional Minkowski spacetime is naturally picked out, without fine tuning, and the extra dimensions form a non-compact "internal" space D_2, which is topologically equivalent to $\mathbb{R}^2$.

The possibility of non-compact internal spaces as an alternative to the standard Kaluza-Klein theory has been considered in pure gravity models [26-28], in models with fermions coupled to gravity [29-31], in Einstein-Maxwell theory [32] and in non-linear sigma models coupled to gravity [33,34]. In these models the non-compact space D_2 was assumed to be of finite volume. Spaces with this property generally suffer from the problem that they are geodesically incomplete: there are timelike geodesics which reach a singularity at the boundary in a finite proper time. It is therefore possible that basic conservation laws could be violated since normally conserved quantities could leak out of the internal space at the boundary. In the case of the non-linear sigma models Gell-Mann and Zwiebach have shown that if certain boundary conditions are placed on small fluctuations of the fields then the fact that the internal space is geodesically incomplete is no longer a problem [34].

Here we will consider a model in which D_2 is geodesically complete but of infinite volume. The interpretation of the model is therefore very different from that of the standard

Kaluza-Klein picture or the models with non-compact internal spaces mentioned above. The extra dimensions should now be viewed as being very large while spacetime is a membrane embedded in a spacetime of higher dimension.

We consider a d-dimensional metric

$$\mathrm{d}s^2 = \beta^2 \bar{g}_{\mu\nu} \mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu - \gamma^2 \mathrm{d}\rho^2 - \frac{\rho^2 \mathrm{d}\phi^2}{\beta^{2(d-3)}} \ , \tag{3.1}$$

coupled to a cylindrically symmetric electromagnetic field

$$\mathbf{F} = B_0 \rho \gamma \beta^{-(2d-5)} \mathrm{d}\rho \wedge \mathrm{d}\phi \ , \tag{3.2}$$

where $\beta = \beta(\rho)$, $\gamma = \gamma(\rho)$, B_0 is a constant representing the magnetic field strength and $\bar{g}_{\mu\nu}(\bar{x}^\mu)$ is the metric of a $(d-2)$-dimensional Einstein spacetime:

$$\bar{R}_{\mu\nu} = -\bar{\lambda}(d-3)\bar{g}_{\mu\nu} \ . \tag{3.3}$$

Maxwell's equations are trivially satisfied by (3.2).

This model has already been discussed by Wetterich [32] in the case $d = 6$ (using different coordinates). However, Wetterich restricted his attention to the solutions for which D_2 has finite volume (cases (iv) and (v) below). We will study all the cases and will integrate the field equations completely. In fact, the solution of the Einstein-Maxwell-de Sitter equations with the ansatz (3.1), (3.2) and (3.3) is easily seen to be obtained by analytically continuing the solutions of section 2: $a \to \rho\beta^{-(d-3)}$, $b \to \beta$, $\frac{Q}{4\pi} \to iB_0$, $\mathrm{d}t \to -i\gamma(\rho)\mathrm{d}\rho$, $\chi \to \phi$, $y_2 \to i\bar{x}_0$ and $y_I \to \bar{x}_{I-2}$, $I = 3, \ldots, d-1$ etc. Integration of the field equations yields the results

$$\gamma(\rho) = C\rho^{-1}\beta^{d-3}\frac{\mathrm{d}\beta}{\mathrm{d}\rho} \ , \tag{3.4}$$

and

$$\frac{\rho^2}{C^2\beta^{2(d-3)}} \equiv \tilde{\Delta} = \frac{2Gk}{\beta^{d-3}} - \frac{8\pi G B_0^2}{(d-2)(d-3)\beta^{2(d-3)}} - \frac{2\Lambda\beta^2}{(d-1)(d-2)} + \bar{\lambda} \ , \tag{3.5}$$

which correspond to equations (2.7) and (2.8). Here C and k are arbitrary constants of dimension L^2 and L^{-4} respectively. The complete solution is therefore

$$\mathrm{d}s^2 = \beta^2 \bar{g}_{\mu\nu} \mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu - \frac{\mathrm{d}\beta^2}{\tilde{\Delta}} - \tilde{\Delta}\mathrm{d}\tilde{\phi}^2 \ , \tag{3.6}$$

where $\tilde{\phi} = C\phi$. A further constant may be removed by rescaling $\tilde{\Delta}$ thus leaving three independent parameters. Equation (3.5) can be solved analytically for $\beta(\rho)$ only if $\Lambda = 0$. The global properties of the general solutions, if they exist, are easily derived, however. There are a number of cases to consider.

(i) $\Lambda = 0, \bar{\lambda} = 0$

This case is the simplest. If we choose $k = \dfrac{4\pi B_0^2}{(d-2)(d-3)}$ and $C = \dfrac{(d-2)}{4\pi G B_0^2}$ the solution may be written

$$\mathrm{d}s^2 = \left(1 + \frac{\rho^2}{a^2}\right)^{2/(d-3)} \left(\bar{g}_{\mu\nu}\mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu - \mathrm{d}\rho^2\right) - \frac{\rho^2 \mathrm{d}\phi^2}{\left(1 + \frac{\rho^2}{a^2}\right)^2} , \tag{3.7a}$$

$$\mathbf{F} = \frac{B_0 \rho}{\left(1 + \frac{\rho^2}{a^2}\right)^2} \mathrm{d}\rho \wedge \mathrm{d}\phi , \tag{3.7b}$$

where

$$a^2 = \frac{(d-2)}{2\pi(d-3)G B_0^2} , \tag{3.7c}$$

and $\bar{g}_{\mu\nu}$ is the metric of a Ricci-flat spacetime, $\bar{R}_{\mu\nu} = 0$. In particular, we may choose $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. The metric (3.7a) is then a warped product of a $(d-2)$-dimensional Minkowski spacetime and a non-compact space D_2 with 2-metric

$$\mathrm{d}s_2^2 = -\left(1 + \frac{\rho^2}{a^2}\right)^{2/(d-3)} \mathrm{d}\rho^2 - \frac{\rho^2 \mathrm{d}\phi^2}{\left(1 + \frac{\rho^2}{a^2}\right)^2} . \tag{3.8}$$

If $d = 4$ the 2-dimensional submanifold is necessarily flat and (3.7) is Melvin's solution [35], whose geometry is described at length in [36-38]. The properties of the d-dimensional solution (3.7) are similar. The space D_2 is of infinite total volume and geodesically complete since the length of radial lines of constant ϕ tends to infinity as $\rho \to \infty$. The circumference of the circles $\rho =$ constant at first increases and then monotonically decreases to zero as $\rho \to \infty$. The effect of the magnetic field (3.7b) is thus to cause D_2 to almost pinch off and mimic the geometry of the monopole compactification. This is shown in Fig. 3a where we draw a cross-section of a surface of constant ρ, ϕ embedded in Euclidean 3-space. Classically a complete "pinch off" is not allowed if one starts from topologically trivial initial data. The metric (3.7a) thus represents a compromise between collapsing and maintaining non-singularity. More information about the $d = 4$ case is given in [39].

(ii) $\Lambda = 0,\ \bar{\lambda} > 0$

If the d-dimensional cosmological constant is set to zero we can obtain solutions in which the $(d-2)$-dimensional spacetime is a de Sitter space with arbitrary radius $\bar{\lambda}^{-1/2}$. The space D_2 is still non-compact and geodesically complete but instead of pinching off the circumference tends to a constant value at large β (or ρ), (see Fig. 3b). Since the membrane is time-dependent and time-symmetric one might view this solution as representing a membrane which shrinks and then expands again, the magnetic flux presumably preventing collapse.

(iii) $\Lambda < 0,\ \bar{\lambda}$ arbitrary

If the d-dimensional cosmological constant is negative there are two families of solutions for which D_2 is non-compact and geodesically complete. In these cases D_2 "opens out", that is the circumference $2\pi\tilde{\Delta}^{1/2} \to \infty$ as $\beta \to \infty$, (see Fig 3c). If the $(d-2)$-dimensional manifold is flat these metrics tend asymptotically to that of d-dimensional anti-de Sitter space expressed in horospherical coordinates, namely

$$\mathrm{d}s^2 = \rho^{2/(d-2)}\left(\mathrm{d}\bar{t}^2 - \mathrm{d}\bar{x}_1^2 - \ldots - \mathrm{d}\bar{x}_{d-3}^2 - \mathrm{d}\phi^2\right) - \frac{\mathrm{d}\rho^2}{(d-2)^2\rho^2}\ , \tag{3.9}$$

where $\bar{t}$, $\bar{x}_i$ and ϕ have been suitably rescaled.

(iv) $\Lambda > -4\pi G B_0^2(d-3)\left(\dfrac{16\pi B_0^2}{(d-1)(d-3)k}\right)^{2\left(\frac{d-2}{d-3}\right)}$, $\bar{\lambda} > \dfrac{-(d-3)(d-1)^2Gk^2}{32\pi B_0^2}$

For these values of Λ and $\bar{\lambda}$ there are solutions for which D_2 pinches off in a singular fashion at a finite value of β. It is therefore topologically equivalent to a two-sphere with a point removed. For given $\bar{\lambda} > \dfrac{-(d-3)(d-1)^2Gk^2}{32\pi B_0^2}$ there are either one or two families of such solutions. If

$$\frac{4\pi G B_0^2}{\beta_-^{2(d-2)}} + \frac{(d-2)(d-3)\bar{\lambda}}{2\beta_-^2} < \Lambda < \frac{4\pi G B_0^2}{\beta_+^{2(d-2)}} + \frac{(d-2)(d-3)\bar{\lambda}}{2\beta_+^2}\ , \tag{3.10}$$

where

$$\beta_\pm^{-(d-3)} = \frac{(d-1)(d-3)k \pm \sqrt{(d-1)^2(d-3)^2k^2 + 32\pi B_0^2(d-3)\bar{\lambda}/G}}{16\pi B_0^2}\ , \tag{3.11}$$

then there is one family. If

$$\bar{\lambda} > 0 \qquad \text{and} \qquad 0 < \Lambda < \frac{4\pi G B_0^2}{\beta_-^{2(d-2)}} + \frac{(d-2)(d-3)\bar{\lambda}}{2\beta_-^2} \ , \tag{3.12}$$

then there are two families. The region of the Λ,$\bar{\lambda}$ plane which contains the solutions (3.10) overlaps with the region containing the solutions of case (iii). The space D_2 is now non-compact but of finite volume. These are the solutions discussed by Wetterich [32]. They also closely resemble the "tear drop" solutions discussed by Gell-Mann and Zwiebach in the case of non-linear sigma models coupled to gravity [33,34]. Since D_2 is not geodesically complete, however, these solutions are not of interest to us here.

(v) Product solutions

Case (iv) possesses a degenerate limit in which D_2 pinches off regularly and is equivalent to S^2 (by a suitable redefinition of variables). This occurs when $\left.\frac{\mathrm{d}\tilde{\Delta}}{\mathrm{d}\beta}\right|_{\beta_\pm} = 0$ and

$P(\beta_\pm) \equiv \left.\frac{-1}{2}\frac{\mathrm{d}^2\tilde{\Delta}}{\mathrm{d}\beta_\pm^2}\right|_{\beta_\pm} = \frac{8\pi G B_0^2(d-3)}{(d-2)\beta_\pm^{2(d-2)}} + \frac{2\Lambda}{d-2} > 0$, that is when

$$\Lambda = \frac{4\pi G B_0^2}{\beta_+^{2(d-2)}} + \frac{(d-2)(d-3)\bar{\lambda}}{2\beta_+^2} \ , \qquad \bar{\lambda} > \frac{-(d-3)(d-1)^2 G k^2}{32\pi B_0^2} \ , \tag{3.13}$$

or when

$$\Lambda = \frac{4\pi G B_0^2}{\beta_-^{2(d-2)}} + \frac{(d-2)(d-3)\bar{\lambda}}{2\beta_-^2}, \qquad \bar{\lambda} > 0 \ , \tag{3.14}$$

where $\beta_\pm$ are given by (3.11). If we define z by $\beta = \beta_+(1+z)$ or $\beta = \beta_-(1+z)$ in the respective cases (3.13) and (3.14), take the limit of (3.6) as $z \to 0$, and then analytically continue to variables θ and ψ defined by $z = \sin\theta \mathrm{e}^{i\psi}$, $\beta_\pm P(\beta_\pm)\tilde{\phi} = i\cot\theta \mathrm{e}^{-i\psi}$, we obtain the familiar product metric

$$\mathrm{d}s^2 = \beta_\pm^2 \bar{g}_{\mu\nu}\mathrm{d}\bar{x}^\mu \ \mathrm{d}\bar{x}^\nu - \frac{1}{P(\beta_\pm)}\left(d\theta^2 + \sin^2\theta \mathrm{d}\psi^2\right) \ . \tag{3.15}$$

In particular, when the d-dimensional cosmological constant is zero the only regular solution with compact D_2 is the product solution $(AdS)_{d-2} \times S^2$.

In Fig. 4 we display the Λ,$\bar{\lambda}$ plane for fixed B_0^2/k in the case $d = 6$ to summarize the various cases we have discussed. In region I there are no solutions. At each point of region

II there are two independent solutions with D_2 non-compact and geodesically complete. At points in region III there are three solutions with D_2 non-compact, two being geodesically complete and one geodesically incomplete. At points in region IV there is one solution with D_2 non-compact and geodesically incomplete. At points in region V there are two such solutions. The point X, the open curve joining X with the origin, and the positive $\bar{\lambda}$ axis all belong to region II. At the origin there is a single solution with D_2 non-compact and geodesically complete. The dotted curves extending upwards from point X and the origin represent the product solutions.

4. Stability of Flat Membrane Solutions

The case $d = 4$, with $\Lambda = \bar{\lambda} = 0$, is known to be stable; Melvin has shown that the solution is stable against small radial (i.e ρ-dependent) perturbations [37] and Thorne has shown that stability is maintained for arbitrarily large radial perturbations [38]. The arguments of Melvin and Thorne may be generalized in a straight-forward manner to the case of arbitrary d in the case that $\bar{g}_{\mu\nu}$ is flat and $\Lambda = 0$. We present these arguments here.

Let us first consider small perturbations of the metric and electromagnetic field for the flat membranes. We will investigate ρ dependent perturbations. Thus we are interested in the most general metric which is homogeneous on the spatial section of the (flat) membrane and which admits 2-spaces orthogonal to the Killing vectors $\xi_{(i)} = \partial/\partial\bar{x}_i$, $i = 1, \ldots, d-3$ and $\xi_{(\phi)} = \partial/\partial\phi$. Provided that the source term gives rise to an energy-momentum tensor which satisfies $T_{\bar{t}\bar{t}} = T_{\rho\rho}$, which is true in our case, such a metric may without loss of generality be transformed to the canonical form

$$ds^2 = e^{-2U}\left(e^{2k}(d\bar{t}^2 - d\rho^2) - \rho^2 d\phi^2\right) - e^{2U/(d-3)}\left(d\bar{x}_1^2 + \ldots + d\bar{x}_{d-3}^2\right), \qquad (4.1)$$

where $U = U(\rho, \bar{t})$ and $k = k(\rho, \bar{t})$. The proof of this is a simple generalization of that given by Melvin in four dimensions. (If the coefficient of $d\phi^2$ in (4.1) is replaced by $-W^2 e^{-2U}$ then the condition $T_{\bar{t}\bar{t}} = T_{\rho\rho}$ ensures that W is harmonic and hence can be chosen to be ρ.) The metric (4.1) is a d-dimensional generalization of the time-dependent Weyl-type metric †. The Maxwell field ansatz is taken to be

$$\mathbf{F} = E(\rho, \bar{t})d\bar{t} \wedge d\phi + B(\rho, \bar{t})d\rho \wedge d\phi\ . \qquad (4.2)$$

† A d-dimensional generalization of the cylindrically symmetric Weyl solution [40,41]

(This is not the most general ansatz possible. One could also add terms such as a homogeneous electric field on the membrane. An electric field in the ρ-direction is ruled out, however, as it would lead to a singularity at $\rho=0$.) On account of the Maxwell equation $F_{[ab,c]}=0$ one may express E and B as derivatives of a single potential $A(\rho,\bar{t})\equiv A_\phi$

$$B=\partial_\rho A, \qquad E=\partial_{\bar{t}}A \; . \tag{4.3}$$

The field equations governing the above system are Maxwell's equation

$$\partial_\rho(\rho^{-1}\mathrm{e}^{2U}\partial_\rho A)-\partial_{\bar{t}}(\rho^{-1}\mathrm{e}^{2U}\partial_{\bar{t}}A)=0 \; , \tag{4.4}$$

and the three independent Einstein equations

$$\left(\frac{d-2}{d-3}\right)\nabla^2 U=2\kappa^2\rho^{-2}\mathrm{e}^{2U}\left[(\partial_\rho A)^2-(\partial_{\bar{t}}A)^2\right] \; , \tag{4.5a}$$

$$\frac{1}{\rho}\partial_{\bar{t}}k-\left(\frac{d-2}{d-3}\right)(\partial_{\bar{t}}U)(\partial_\rho U)=2\kappa^2\rho^{-2}\mathrm{e}^{2U}(\partial_{\bar{t}}A)(\partial_\rho A) \; , \tag{4.5b}$$

$$\frac{2}{\rho}\partial_\rho k-\left(\frac{d-2}{d-3}\right)\left((\partial_{\bar{t}}U)^2+(\partial_\rho U)^2\right)=2\kappa^2\rho^{-2}\mathrm{e}^{2U}\left((\partial_{\bar{t}}A)^2+(\partial_\rho A)^2\right), \tag{4.5c}$$

where $\nabla^2\equiv\dfrac{-\partial^2}{\partial\bar{t}^2}+\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}+\dfrac{\partial^2}{\partial\rho^2}$. The static membrane solution is given by

$$\bar{U}=\left(\frac{d-3}{d-2}\right)\bar{k}=\ln\left(1+\frac{\rho^2}{a^2}\right), \quad \bar{A}=\frac{-a^2B_0}{2}\left(1+\frac{\rho^2}{a^2}\right)^{-1} \; . \tag{4.6}$$

Instead of working with the equations (4.4) and (4.5) it is convenient to introduce the (dimensionless) quantities

$$\mathcal{W}(\rho,\bar{t})=a\rho^{-1}\mathrm{e}^U \; , \tag{4.7a}$$

$$\mathcal{G}(\rho,\bar{t})=k-\left(\frac{d-2}{d-3}\right)U+\ln\frac{\rho}{a} \; , \tag{4.7b}$$

$$\mathcal{A}(\rho,\bar{t})=-2B_0^{-1}a^{-1}\rho^{-1}\mathrm{e}^U A \; . \tag{4.7c}$$

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is similarly given by

$$\mathrm{d}s^2=\mathrm{e}^{2U}\mathrm{d}t^2-\mathrm{e}^{-2U/(d-3)}\left[\mathrm{e}^{2k}(\mathrm{d}\rho^2+\mathrm{d}z_1^2)+\mathrm{d}z_2^2+\ldots+\mathrm{d}z_{d-3}^2+\rho^2\mathrm{d}\phi^2\right] \; ,$$

where $U=U(\rho,z_1)$, $k=k(\rho,z_1)$.

In terms of $\mathcal{G}$ and $\mathcal{W}$ the metric (4.1) is given by

$$\mathrm{d}s^2 = \left(\mathcal{W}(\rho/a)^{-(d-4)}\right)^{2/(d-3)} \mathrm{e}^{2\mathcal{G}} \left(\mathrm{d}\bar{t}^2 - \mathrm{d}\rho^2\right) - \frac{a^2 \mathrm{d}\phi^2}{\mathcal{W}^2} - \left(\frac{\mathcal{W}\rho}{a}\right)^{2/(d-3)} \left(\mathrm{d}\bar{x}_1^2 + \ldots + \mathrm{d}\bar{x}_{d-3}^2\right) . \tag{4.8}$$

The field equations (4.4) and (4.5) now become

$$\frac{\nabla^2 \mathcal{A}}{\mathcal{A}} = \frac{\nabla^2 \mathcal{W}}{\mathcal{W}} , \tag{4.9}$$

$$\frac{\nabla^2 \mathcal{W}}{\mathcal{W}} + \frac{(\mathcal{W},_{\bar{t}}^2 - \mathcal{W},_\rho^2)}{\mathcal{W}^2} = \mathcal{A},_\rho^2 - \mathcal{A},_{\bar{t}}^2 + \frac{2\mathcal{A}}{\mathcal{W}} (\mathcal{A},_{\bar{t}} \mathcal{W},_{\bar{t}} - \mathcal{A},_\rho \mathcal{W},_\rho) + \frac{\mathcal{A}^2}{\mathcal{W}^2} (\mathcal{W},_\rho^2 - \mathcal{W},_{\bar{t}}^2) , \tag{4.10a}$$

$$\frac{1}{\rho}\mathcal{G},_{\bar{t}} - \left(\frac{d-2}{d-3}\right) \frac{\mathcal{W},_{\bar{t}} \mathcal{W},_\rho}{\mathcal{W}^2} = \left(\frac{d-2}{d-3}\right) \left[\mathcal{A},_{\bar{t}} \mathcal{A},_\rho - \frac{\mathcal{A}}{\mathcal{W}} (\mathcal{A},_{\bar{t}} \mathcal{W},_\rho + \mathcal{A},_\rho \mathcal{W},_{\bar{t}}) + \frac{\mathcal{A}^2}{\mathcal{W}^2} \mathcal{W},_{\bar{t}} \mathcal{W},_\rho\right] , \tag{4.10b}$$

$$\frac{2}{\rho}\mathcal{G},_\rho + \left(\frac{d-4}{d-3}\right) \frac{1}{\rho^2} - \left(\frac{d-2}{d-3}\right) \frac{(\mathcal{W},_{\bar{t}}^2 + \mathcal{W},_\rho^2)}{\mathcal{W}^2} = \left(\frac{d-2}{d-3}\right) \left[\mathcal{A},_{\bar{t}}^2 + \mathcal{A},_\rho^2 - \frac{2\mathcal{A}}{\mathcal{W}} (\mathcal{A},_{\bar{t}} \mathcal{W},_{\bar{t}} + \mathcal{A},_\rho \mathcal{W},_\rho) + \frac{\mathcal{A}^2}{\mathcal{W}^2} (\mathcal{W},_{\bar{t}}^2 + \mathcal{W},_\rho^2)\right] \tag{4.10c}$$

The static membrane solution (4.6) is now given by

$$\bar{\mathcal{W}} = \frac{a}{\rho}\left(1 + \frac{\rho^2}{a^2}\right), \quad \bar{\mathcal{G}} = \ln \frac{\rho}{a}, \quad \bar{\mathcal{A}} = \frac{a}{\rho} . \tag{4.11}$$

We now perturb the quantities $\mathcal{W}$,$\mathcal{G}$ and $\mathcal{A}$ about the static solution (4.11). We set

$$\begin{aligned} \mathcal{W} &= \bar{\mathcal{W}} + \varepsilon w + \mathrm{O}(\varepsilon^2) , \\ \mathcal{G} &= \bar{\mathcal{G}} + \varepsilon g + \mathrm{O}(\varepsilon^2) , \\ \mathcal{A} &= \bar{\mathcal{A}} + \varepsilon \alpha + \mathrm{O}(\varepsilon^2) . \end{aligned} \tag{4.12}$$

Substituting (4.12) in (4.9) and (4.10) we find after some calculation that the $\mathrm{O}(\varepsilon)$ terms give

$$\left(\nabla^2 - 1/\rho^2\right) \alpha = \left(1 + \rho^2/a^2\right)^{-1} \left(\nabla^2 - 1/\rho^2\right) w , \tag{4.13a}$$

$$g_{,\bar{t}} = \left(\frac{d-2}{d-3}\right)\frac{\rho}{a}\left(1+\rho^2/a^2\right)^{-1}\left[w_{,\bar{t}} - 2\alpha_{,\bar{t}}\right] \;, \tag{4.13b}$$

$$g_{,\rho} = \left(\frac{d-2}{d-3}\right)\frac{\rho}{a}\left(1+\rho^2/a^2\right)^{-1}\left[w_{,\rho} - 2\alpha_{,\rho} + \frac{1}{\rho}\left(\frac{1-\rho^2/a^2}{1+\rho^2/a^2}\right)(w-2\alpha)\right], \tag{4.13c}$$

$$\left(\nabla^2 - 1/\rho^2\right) w = 2\left(\frac{d-2}{d-3}\right)\frac{a}{\rho^2}\left(1+\rho^2/a^2\right) g_{,\rho} \;. \tag{4.13d}$$

Integrating (4.13b) and (4.13c) we obtain

$$g = \left(\frac{d-2}{d-3}\right)\frac{\rho}{a}\left(1+\rho^2/a^2\right)^{-1}(w-2\alpha) + \text{const.} \tag{4.14}$$

A useful dependent equation which may be derived from (4.14), (4.13a) and (4.13d) is

$$\nabla^2 g = 0. \tag{4.15}$$

We also have

$$\left(\nabla^2 - 1/\rho^2\right) w = \frac{2}{\rho}\left[\partial_\rho + \frac{1}{\rho}\left(\frac{1-\rho^2/a^2}{1+\rho^2/a^2}\right)\right](w-2\alpha) \;, \tag{4.16a}$$

and

$$\left(\nabla^2 - 1/\rho^2\right)\alpha = \frac{2}{\rho}\left(1+\rho^2/a^2\right)^{-1}\left[\partial_\rho + \frac{1}{\rho}\left(\frac{1-\rho^2/a^2}{1+\rho^2/a^2}\right)\right](w-2\alpha)\,, \tag{4.16b}$$

Since (4.16a) and (4.16b) are linear, each has as its solution the general solution to the corresponding homogeneous equation ($w^{(h)}$ and $\alpha^{(h)} \equiv h$) plus a particular integral ($w^{(part)}$ and $\alpha^{(part)}$):

$$w = w^{(h)} + w^{(part)}, \qquad \alpha = h + \alpha^{(part)} \;. \tag{4.17}$$

One may determine $w^{(part)}$ by taking $w^{(part)} = gf(\rho)$ as a trial function and determining f, and similarly for $\alpha^{(part)}$. After some calculation one finds that the general solutions are

$$w = 2h + \left(\frac{d-3}{d-2}\right)\left[\text{const.}\left(\frac{\rho}{a}+\frac{a}{\rho}\right) + g\left(\frac{\rho}{a}-\frac{a}{\rho}\right)\right] \;, \tag{4.18a}$$

and

$$\alpha = h - \left(\frac{d-3}{d-2}\right)\frac{ag}{\rho} \;. \tag{4.18b}$$

Thus we have only to determine the solutions of the equations

$$\nabla^2 g = 0, \qquad \nabla^2 h - \frac{h}{\rho^2} = 0 \; , \tag{4.19}$$

subject to appropriate boundary conditions, and the complete solution to the perturbation problem is immediately given by (4.18a,b).

The boundary conditions on g are imposed by the requirements that: (i) the metric is locally flat at $\rho = 0$; and (ii) at all times to $\mathrm{O}(\varepsilon)$ the static metric at $\rho = \infty$ is unaltered, i.e. the ratio of the added ε term to the static metric goes to zero as $\rho \to \infty$.

The requirement of local flatness at $\rho = 0$ gives rise to the following conditions: firstly, since infinitesimal circles in the ρ,ϕ plane must shrink to zero regularly as $\rho \to 0$ one finds that

$$\mathcal{W}^{d-2} \sim \frac{a\mathrm{e}^{-(d-3)\mathcal{G}}}{\rho} \qquad \text{as} \;\; \rho \to 0 \qquad \text{or} \qquad k(0,\bar{t}) = 0 \quad \forall \quad \bar{t} \; . \tag{4.20}$$

Secondly, since the coordinate velocity of light in any $\bar{x}_i$ direction is bounded it follows that

$$\mathcal{G}(\rho,\bar{t}) - \ln\rho < \infty \qquad \text{as} \;\; \rho \to 0 \qquad \text{or} \qquad -\left(\frac{d-2}{d-3}\right) U(0,\bar{t}) < \infty \quad \forall \quad \bar{t} \; . \tag{4.21}$$

For perturbations about the static membrane (4.20) and (4.21) lead to the boundary conditions

$$w \sim -\left(\frac{d-3}{d-2}\right)\frac{ga}{\rho} \qquad \text{as} \;\; \rho \to 0 \qquad \text{and} \qquad g(0,\bar{t}) < \infty \quad \forall \quad \bar{t} \; . \tag{4.22}$$

The requirement that the static metric at $\rho = \infty$ is unaltered at times first-order in ε gives the conditions

$$\frac{wa}{\rho} \to 0 \qquad \text{and} \qquad g \to 0 \qquad \text{as} \; \rho \to \infty \; . \tag{4.23}$$

From (4.18a) and (4.22) it follows that

$$h(\rho,\bar{t}) \to -\frac{(d-3)}{2(d-2)}\, \text{const.} \frac{1}{\rho} \; . \tag{4.24}$$

Further conditions on h result from regularity conditions on the electromagnetic source term. We require that (i) at $\rho = 0$ the physical magnetic field is finite (at least over a finite time interval) and the physical electric field is zero; and (ii) at $\rho = \infty$ the physical magnetic field and the physical electric field are zero at all times. The physical magnetic and electric fields are given by the frame components of **F**:

$$\begin{aligned}B_{phys} =& F_{\hat{\rho}\hat{\phi}} = B_0 \left(1+\frac{\rho^2}{a^2}\right)^{-\left(\frac{d-2}{d-3}\right)} + \frac{\varepsilon B_0}{2}\left(1+\frac{\rho^2}{a^2}\right)^{-\left(\frac{2d-5}{d-3}\right)} \\ &\times\left[-\text{const.}\left(1+\frac{\rho^2}{a^2}\right) - \left(\frac{5d-12}{d-2}\right)\frac{\rho^2 g}{a^2} + 2\left(\frac{d-3}{d-2}\right)\left(1+\frac{\rho^2}{a^2}\right)\rho g_{,\rho} \right. \\ &\left. + \left(1-\frac{\rho^4}{a^4}\right) a h_{,\rho} + \left(1-2\left(\frac{2d-5}{d-3}\right)\frac{\rho^2}{a^2}+\frac{\rho^4}{a^4}\right)\frac{ah}{\rho}\right], \end{aligned} \tag{4.25a}$$

and

$$E_{phys} = F_{\hat{\bar{t}}\hat{\phi}} = \frac{\varepsilon B_0}{2}\left(1+\frac{\rho^2}{a^2}\right)^{-\left(\frac{d-2}{d-3}\right)}\left[-2\left(\frac{d-3}{d-2}\right)\rho g_{,\bar{t}} + \left(1-\frac{\rho^2}{a^2}\right) a h_{,\bar{t}}\right]. \tag{4.25b}$$

From (4.25a) we see that the constant in (4.24) must be zero:

$$h(\rho,\bar{t}) \to 0 \qquad \text{as} \quad \rho \to 0 \ . \tag{4.26}$$

Equations (4.22), (4.23) and (4.26) form a complete set of boundary conditions for the perturbation problem†.

By separation of variables the solutions of (4.19) are

$$g(\rho,\bar{t}) = \mathbf{S}[u(\omega)\mathrm{e}^{i\omega\bar{t}} + v(\omega)\mathrm{e}^{-i\omega\bar{t}}]J_0(\omega\rho) \ , \tag{4.27a}$$

and

$$h(\rho,\bar{t}) = \mathbf{S}[\tilde{u}(\omega)\mathrm{e}^{i\omega\bar{t}} + \tilde{v}(\omega)\mathrm{e}^{-i\omega\bar{t}}]J_1(\omega\rho) \ , \tag{4.27b}$$

where J_0 and J_1 are modified Bessel function kernels and **S** denotes a summation or integration over all admissible eigenvalues ω. Additional terms with a ρ dependence on Bessel functions of the second kind are ruled out by the requirements that g is bounded and $h \to 0$ as $\rho \to 0$. Furthermore, since $g \to 0$ and $h\rho/a \to 0$ as $\rho \to \infty$ (by (4.23) and (4.18a)) the eigenvalues ω must be real because $\lim_{\rho\to\infty} J_0(\omega\rho)$ and $\lim \rho \to \infty J_1(\omega\rho)$ blow up exponentially for all eigenvalues except those which are purely real. All real eigenvalues are allowed, however.

† Note that the condition $g \to 0$ as $\rho \to \infty$ is stronger than that given by Melvin [37]. This small error of his does not alter his conclusions.

We therefore conclude that both the g-and h-mode solutions are purely oscillatory, i.e. the system is stable. Further properties of the modes are discussed by Melvin [37] and his remarks remain valid in the arbitrary dimensional case. The fact that *all* real eigenvalues are allowed, however, will have dire consequences for the spectrum of particles confined to the membrane, as we shall see in section 6.

The membrane solutions completely break supersymmetry and so Witten type arguments are not directly applicable. However, Thorne has given arguments in four dimensions [36,38] which may be readily generalised to d dimensions to give a fully non-linear proof of stability.

Let us define a vector P^a by

$$P^a = \frac{\varepsilon^{abc_1c_2\cdots c_{d-3}d}\mathrm{E}_{,b}\,\xi_{(1)c_1}\xi_{(2)c_2}\cdots\xi_{(d-3)c_{d-3}}\xi_{(\phi)d}}{\sqrt{|\det g_{ab}|}|\xi_(1)|^2|\xi_(2)|^2\cdots|\xi_{(d-3)}|^2|\xi_{(\phi)}|^2}\ , \tag{4.28a}$$

where

$$\mathrm{E} = \frac{-1}{8G}\ln\left\{\frac{g^{ab}\left(|\xi_{(1)}|\cdots|\xi_{(d-3)}||\xi_{(\phi)}|\right)_{,a}\left(|\xi_{(1)}|\cdots|\xi_{(d-3)}||\xi_{(\phi)}|\right)_{,b}}{4\pi^2|\xi_{(1)}|^2|\xi_{(2)}|^2\cdots|\xi_{(d-3)}|^2|\xi_{(\phi)}|^2}\right\}\ , \tag{4.28b}$$

and $\underline{\xi}_{(i)}$, $i=1,\ldots,d-3$, and $\underline{\xi}_{(\phi)}$ are the Killing vectors associated with the invariant translations and rotations of the system. In the coordinates (4.1), for example, the non-trivial components of P^a are

$$P^{\bar{t}} = \frac{\mathrm{e}^{2(U-k)}}{2\pi h_{(1)}h_{(2)}\cdots h_{(d-3)}\rho}\frac{\partial\mathrm{E}}{\partial\rho}\ , \tag{4.29a}$$

and

$$P^{\rho} = \frac{-\mathrm{e}^{2(U-k)}}{2\pi h_{(1)}h_{(2)}\cdots h_{(d-3)}\rho}\frac{\partial\mathrm{E}}{\partial\bar{t}}\ , \tag{4.29b}$$

where $h_{(i)}$ is the $\bar{x}_i$-coordinate interval associated with a translation of unit proper length at $\rho=0$ when there is no gravitational radiation there.

Now $P^a_{\ ;a}=0$ so that

$$\int_{\substack{\textit{closed}\\ \textit{surface}}} P^a\mathrm{d}\Sigma_a = 0\ . \tag{4.30}$$

Thus (4.30) defines a conserved quantity which Thorne calls the "C-energy" [36]. In the coordinates (4.1) the C-energy on a canonical $\bar{t}=$ const. hypersurface is given by

$$\int P^{\bar{t}}\sqrt{|\det g_{ab}|}\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\bar{x}^1\cdots\mathrm{d}\bar{x}^{d-3} = \frac{1}{\pi G\prod_{i=1}^{d-3}h_{(i)}}\int k_{,\rho}\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\bar{x}^1\ldots\mathrm{d}\bar{x}^{d-3}\ . \tag{4.31}$$

On account of (4.5c) the C-energy per unit $(d-3)$-volume is

$$\frac{1}{8G}\left(\frac{d-2}{d-3}\right)\int \mathrm{d}\rho\left[\rho(U_{,\bar{t}}^{\,2}+U_{,\rho}^{\,2})+\mathrm{e}^{2U}\rho^{-1}(\tilde{A}_{,\bar{t}}^{\,2}+\tilde{A}_{,\rho}^{\,2})\right] , \tag{4.32a}$$

where

$$\tilde{A}(\bar{t},\rho)=\frac{-2A(\bar{t},\rho)}{aB_0} . \tag{4.32b}$$

One may show that the integral (4.32a) is an absolute minimum for the flat membrane solution. Since (4.32a) differs form the 4-dimensional case only by an overall multiplicative factor the proof of this statement is almost identical to that given by Thorne [36]. Thorne's arguments for stability under arbitrarily large ρ-perturbations [38] therefore apply also to the flat membrane in arbitrary dimensions.

5. Fermion zero modes

Non-compact internal spaces provide one solution to the problem of obtaining chiral fermions when dimensionally reducing higher-dimensional gravity theories [29-31]. We will demonstrate this for our model by an explicit calculation. To obtain a low energy 4-dimensional world with chiral fermions it seems necessary to start with chiral fermions in higher dimensions [42]. In this section we will therefore assume that $d=4\mathrm{k}+2$, where k is a positive integer. We will look for solutions of the minimally coupled massless Dirac equation

$$i\gamma^a\left(D_a-ieA_a\right)\Psi=0 , \tag{5.1}$$

which effectively describe charged fermions, with $U(1)$ charge e, trapped in the $(d-2)$-dimensional membrane. This problem is very similar to that of the interaction of fermions with topologically non-trivial objects in four dimensions, e.g. the fermion-vortex system [43-45].

We first note that in order for (5.1) to be well-defined the total magnetic flux Φ threading the membrane must satisfy the Dirac quantisation condition

$$e\Phi=2\pi n\hbar, \quad n\in\mathbb{Z}. \tag{5.2}$$

Evaluating Φ using (3.2) and (3.4) we find that

$$\frac{eB_0\bar{a}^2}{2\hbar}=n , \tag{5.3a}$$

where

$$\bar{a}^2 = \frac{2C}{(d-3)}\left(1/\beta_{min}^{d-3} - 1/\beta_{max}^{d-3}\right) \ . \tag{5.3b}$$

If the spacetime metric is Ricci-flat, so that we are dealing with the solution (3.7), then $a = \bar{a}$. We will restrict our attention to this case for the remainder of this section. The magnetic field strength B_0 and membrane thickness a are then quantised according to

$$B_0 = \frac{(d-2)e}{4\pi(d-3)G\hbar}\frac{1}{n} \ , \tag{5.4a}$$

$$a = \frac{2\hbar}{e}\left(\frac{2\pi(d-3)G}{(d-2)}\right)^{1/2} n \ . \tag{5.4b}$$

We now look for solutions of (5.1) which will be assumed to obey the chirality condition

$$\gamma^{\hat{0}}\gamma^{\hat{1}}\ldots\gamma^{\widehat{d-3}}\gamma^{\hat{\rho}}\gamma^{\hat{\phi}}\Psi = +\Psi \ , \tag{5.5}$$

in $d = 4\mathrm{k}+2$ dimensions. Equation (5.1) is most easily solved by making the conformal transformation

$$g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}, \quad \gamma^a \rightarrow \tilde{\gamma}^a = \Omega^{-1}\gamma^a, \quad \Psi \rightarrow \tilde{\Psi} = \Omega^{-(d-1)/2}\Psi \ , \tag{5.6a}$$

where

$$\Omega = \left(1 + \frac{\rho^2}{a^2}\right)^{-1/(d-3)} \ . \tag{5.6b}$$

Since the Dirac equation is conformally invariant we can therefore find solutions of (5.1) of the form

$$\Psi = \left(1 + \frac{\rho^2}{a^2}\right)^{\frac{-(d-1)}{2(d-3)}} \tilde{\chi}(\rho)\mathrm{e}^{im\phi}\psi_{/\!/}(\bar{t}, \bar{x}^i) \ , \tag{5.7a}$$

where $\psi_{/\!/}(\bar{t}, \bar{x}^i)$ satisfies the uncharged massless Dirac equation in spacetime

$$i\gamma^\mu D_\mu \psi_{/\!/} = 0 \ , \tag{5.7b}$$

and $\tilde{\chi}(\rho)$ is a 2-component spinor. From (5.1) and (5.5) it follows that if

$$i\gamma^{\hat{0}}\gamma^{\hat{1}}\ldots\gamma^{\widehat{d-3}}\psi_{/\!/} = +\psi_{/\!/} \tag{5.8}$$

then $\tilde{\chi}(\rho)$ must satisfy

$$i\gamma^{\hat{\rho}}\gamma^{\hat{\phi}}\tilde{\chi} = -\tilde{\chi} \ , \tag{5.9a}$$

and

$$\left\{\partial_\rho + \frac{1}{2\rho}\left(1+\rho^2/a^2\right)^{-1}\left[1-\left(\frac{d-1}{d-3}\right)\rho^2/a^2\right] + \frac{1}{\rho}\left(1+\rho^2/a^2\right)^{1/d-3}\left[\frac{1}{2}ea^2B_0 + m\left(1+\rho^2/a^2\right)\right]\right\}\tilde{\chi} = 0. \tag{5.9b}$$

Independent solutions of opposite $(d-2)$-dimensional chirality can also be found if we change the sign of the right hand sides of (5.8) and (5.9a). The explicit form of these solutions is then otherwise the same except that we make the replacements $m \to -m$ and $e \to -e$ in (5.7a) and (5.9b).

One may integrate (5.9b) exactly by making use of the substitution $\beta = \left(1+\frac{\rho^2}{a^2}\right)^{1/(d-3)}$ where necessary. For $d=6$, for example, the complete solutions are

$$\begin{aligned}\Psi = \psi_{/\!/}\mathrm{e}^{im\phi}\left(\frac{a}{\rho}\right)^{\frac{1}{2}}\left(1+\frac{\rho^2}{a^2}\right)^{\frac{-1}{6}}\left[\left(1+\frac{\rho^2}{a^2}\right)^2+\left(1+\frac{\rho^2}{a^2}\right)+1\right]^{\frac{1}{4}(m+\frac{1}{2}ea^2B_0)} \\ \times\left[\left(1+\frac{\rho^2}{a^2}\right)^{\frac{1}{3}}-1\right]^{\frac{-1}{2}(m+\frac{1}{2}ea^2B_0)} \exp\left\{\frac{-3}{4}\left(1+\frac{\rho^2}{a^2}\right)^{\frac{1}{3}}\left(ea^2B_0+\frac{5}{2}m+\frac{m\rho^2}{2a^2}\right)\right. \\ \left.+\frac{\sqrt{3}}{2}\left(\frac{1}{2}ea^2B_0+m\right)\tan^{-1}\frac{1}{\sqrt{3}}\left(1+2\left(1+\frac{\rho^2}{a^2}\right)^{\frac{1}{3}}\right)\right\}.\end{aligned} \tag{5.10}$$

The requirement that the solutions are normalisable imposes the condition

$$0 < m < |n| + 1\ , \tag{5.11}$$

independently of d, where n is the integer occurring in the flux quantisation rule (5.2). Thus there are $|n|$ zero modes which is precisely what one would expect from the index theorem. These zero modes fall off at large values of ρ, so we may regard them as being confined in the vicinity of the membrane. Presumably quantum effects would cause the zero modes to interact among themselves but one hopes that their coupling to the massive modes would be small. In the case of a spacetime with non-vanishing cosmological constant there will, by the index theorem, still be the same number of zero modes.

6. Spectrum and mass gap

For the conventional Kaluza-Klein picture with a compact internal space K there are massive modes in addition to the massless modes which satisfy an effective low energy theory. These massive modes are separated by a finite "mass gap" from the massless zero modes and so, provided the gap is sufficiently large, the massive modes cannot be created by collisions of massless modes or appear in thermal ensembles with low temperature. This renders the effects of the extra dimensions comparatively unobservable.

If the extra dimensions are not compact then the kinetic operators which describe small fluctuations about the background geometry are not guaranteed to have a discrete spectrum with a finite mass gap. Particular models must be investigated individually. Nicolai and Wetterich [27] have discussed the properties of scalar fluctuations for some general pure gravity models with a vacuum $\mathcal{M}^4 \times D_2$ (without warp factor) and have established some criteria for the existence of a mass gap in these models.

In the membrane picture, however, it appears that a clean separation between the massless and massive modes does not always occur. For example, we have shown in section 4 that gravitational and electromagnetic modes travelling in the ρ direction can have any arbitrarily small energy. Similar results apply for the simple case of a neutral scalar field Φ minimally coupled to the higher dimensional geometry. In the case of Ricci-flat membranes (3.7) there are solutions of the form

$$\Phi = \Phi_{/\!/}(\bar{t}, \bar{x}^i) \mathrm{e}^{im\phi} R(\rho) \tag{6.1a}$$

where

$$(\Box + \mu^2)\Phi_{/\!/}(\bar{t}, \bar{x}^i) = 0 \ , \tag{6.1b}$$

and $R(\rho)$ satisfies

$$\frac{-1}{\rho}\partial_\rho(\rho\partial_\rho R) - \mu^2 R + \frac{m^2}{\rho^2}\left(1 + \frac{\rho^2}{a^2}\right)^{2\left(\frac{d-2}{d-3}\right)} R = 0 \ . \tag{6.1c}$$

In (6.1b) $\Box$ denotes the scalar Laplacian on the $(d-2)$-dimensional spacetime membrane. From (6.1c) it is easy to see that if $m \neq 0$ then the eigenvalue μ^2 is bounded below and the spectrum is discrete. However, if $m = 0$ then (6.1c) is simply a Bessel equation. If we require that Φ be bounded at $\rho = 0$ and $\Phi \to 0$ as $\rho \to \infty$ then the solutions are

$$R = J_0(\mu\rho) \ , \tag{6.2}$$

where μ^2 can take any real positive value. Therefore the conventional Kaluza-Klein mechanism will not work in this case.

The situation improves, however, if we allow cosmological terms in $(d-2)$- or d-dimensions. In particular, let us consider the solutions of cases (ii) and (iii) discussed in section 3. If we expand Φ in the background (3.6) as in (6.1a,b) but replace $R(\rho)$ by $R(\beta)$ then (6.1c) is replaced by the equation

$$\frac{-d}{\mathrm{d}\beta}\left(\beta^{d-2}\tilde{\Delta}\frac{dR}{\mathrm{d}\beta}\right)-\mu^2\beta^{d-4}R+m^2\beta^{d-2}\tilde{\Delta}^{-1}R=0. \tag{6.3}$$

To make the problem well-defined it is necessary to impose certain boundary conditions. We require that

$$\int_{D_2}\partial_\alpha\left(\beta^{d-2}\Phi^*\partial^\alpha\Phi\right)=0\ , \tag{6.4}$$

where α denotes indices on the "internal" space. This ensures that the modified scalar Laplacian on D_2 is Hermitian and any discrete eigenvalues μ^2 are positive. The boundary conditions

$$\lim_{\beta\to\beta_{min}}\tilde{\Delta}R\frac{dR}{\mathrm{d}\beta}=0\ , \tag{6.5a}$$

and

$$\lim_{\beta\to\infty}\beta^{d-2}\tilde{\Delta}R\frac{dR}{\mathrm{d}\beta}=0\ , \tag{6.5b}$$

guarantee that (6.4) is satisfied. The problem may be simplified if we define a new variable y, $0\le y\le\infty$, by

$$y=\int\frac{\mathrm{d}\beta}{\beta\tilde{\Delta}^{1/2}}\ , \tag{6.6a}$$

and a new β-dependent function P by

$$P=\left(\beta^{d-3}\tilde{\Delta}^{1/2}\right)^{1/2}R\ . \tag{6.6b}$$

Equation (6.3) becomes

$$\left[\frac{-\mathrm{d}^2}{\mathrm{d}y^2}+V(y)\right]P=\mu^2P\ , \tag{6.7a}$$

where the "effective potential" V is given by

$$V=\frac{1}{4}(d-3)^2\bar{\lambda}-\Lambda\beta^2+\frac{1}{\tilde{\Delta}}\left\{m^2\beta^2-\left[\frac{1}{2}(d-3)Gk\beta^{-(d-3)}-\Lambda\beta^2/(d-1)\right]^2\right\}. \tag{6.7b}$$

We now have a standard eigenvalue problem. The nature of the spectrum is determined by the behaviour of V as $y \to 0$ and as $y \to \infty$ [46]. As $y \to 0$,

$$V \sim \frac{4}{c_1^2 y^2} \left(m^2 \beta_{min}^2 - \frac{1}{16} \left[2(d-3) G k \beta_{min}^{-(d-3)} - 4\beta_{min}^2 \Lambda/(d-1) \right]^2 \right) , \tag{6.8a}$$

where

$$c_1 \equiv \beta_{min} \left[\frac{\mathrm{d}\tilde{\Delta}}{\mathrm{d}\beta} \right]_{\beta_{min}} . \tag{6.8b}$$

A short calculation shows that in all the cases of interest, namely cases (i)-(iii) of section 3, as $y \to 0$

$$V \geq \frac{1}{y^2} \left[4m^2 \beta_{min}^2 / c_1^2 - \frac{1}{4} \right] . \tag{6.9}$$

Therefore the nature of the spectrum is the same as if there were no singularity at $y = 0$ and is entirely determined by the behaviour of V as $y \to \infty$. If $\Lambda = 0$, $\bar{\lambda} > 0$ and the azimuthal quantum number $m = 0$ then $V \to \frac{1}{4}(d-3)^2 \bar{\lambda}$ as $y \to \infty$. Otherwise (i.e. for $\Lambda < 0$ or for $\Lambda = 0$, $\bar{\lambda} > 0$, $m \neq 0$) $V \to \infty$ as $y \to \infty$. This means that there are two possibilities for the spectrum:

(i) If $\Lambda = 0$, $\bar{\lambda} > 0$ and $m = 0$ then there is possibly a discrete spectrum for $0 \leq \mu^2 < \frac{1}{4}(d-3)^2 \bar{\lambda}$, while a continuous spectrum extends from $\mu^2 = \frac{1}{4}(d-3)^2 \bar{\lambda}$ upwards.

(ii) If either $\Lambda = 0$, $\bar{\lambda} > 0$ and $m \neq 0$; or $\Lambda < 0$ then there is a discrete spectrum of positive eigenvalues μ^2.

Therefore we do have a definite mass gap in both cases, provided the zero modes are allowed solutions. There are no tachyon instabilities. One may observe from (6.3) that if $m = 0$ the regular zero mode solution is given by $R = \text{const}$, which is not square integrable, however. In order to allow this solution a boundary condition of the form $\left. \frac{dR}{\mathrm{d}\beta} \right|_{\infty} = 0$ is required.

It would seem that non-compact "internal" spaces provide a perfectly consistent 4-dimensional interpretation provided there is a mass gap and provided the modes allow for the expansion of a sufficiently general set of perturbations around the "ground state". This appears to be so in the cases described above where there was a non-vanishing positive $(d-2)$-dimensional cosmological term or a non-vanishing negative d-dimensional cosmological term. The question of what modes should be allowed is one of boundary conditions and it seems that suitable boundary conditions will exist in the present case.

7. Conclusion

We have shown through investigation of a simple model that membrane-type solutions may provide a viable alternative to the standard spontaneous compactification scenario. A well-defined mass gap can be obtained despite the fact that the extra dimensions are non-compact; the only issue is the normalizability of the zero modes. A further point we have not studied is the stability of the generalised solutions with non-zero cosmological terms in $(d-2)$-or d-dimensions. The argument of section 4 cannot be directly generalised to these cases.

One sometimes sees the requirement that the internal space should have finite volume [27,47]. If there is a warp factor one might similarly demand that when an ansatz such as (3.1) is substituted into the higher-dimensional Einstein action then the internal space integral of the term linear in $\bar{R}$ (the lower dimensional curvature scalar) should converge, so as to yield the standard lower-dimensional Einstein action up to a finite multiplicative constant. This latter requirement is certainly not required classically and it is not obvious to us that it is required quantum mechanically either.

Even if the solutions discussed here (or more sophisticated generalisations) are not at all realistic they should still prove interesting for a deeper overall understanding of higher dimensional theories. If compactification to $\mathcal{M} \times K$ is achieved by some dynamical mechanism, as is the standard view, then that mechanism should at least explain why alternative classically stable solutions (similar to the $\Lambda = 0$, $\bar{\lambda} = 0$ membrane) are definitely ruled out.

Acknowledgement

We wish to thank P.J. Ruback for useful discussions.

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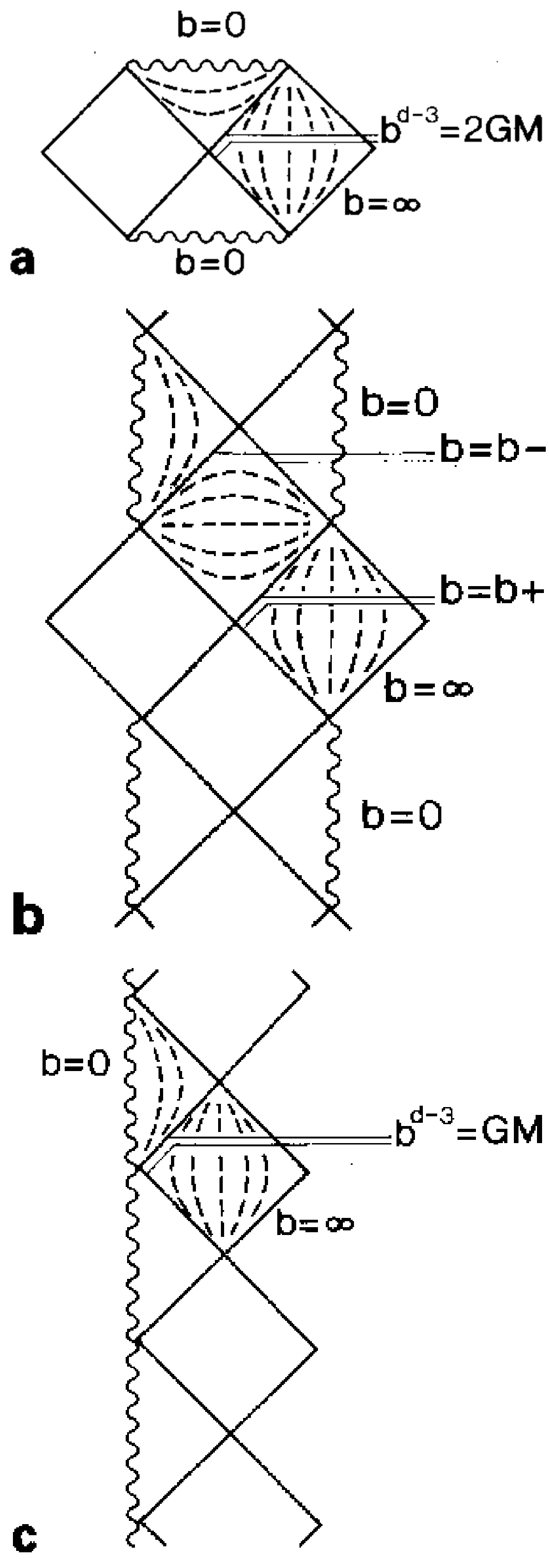

Fig. 1. Carter-Penrose diagrams for the case $\Lambda = 0$, $\bar{\lambda} = 1$: (a) $Q = 0$; (b) $\kappa^2 M^2 < \frac{2Q^2}{(d-2)(d-3)}$; (c) $\kappa^2 M^2 = \frac{2Q^2}{(d-2)(d-3)}$.

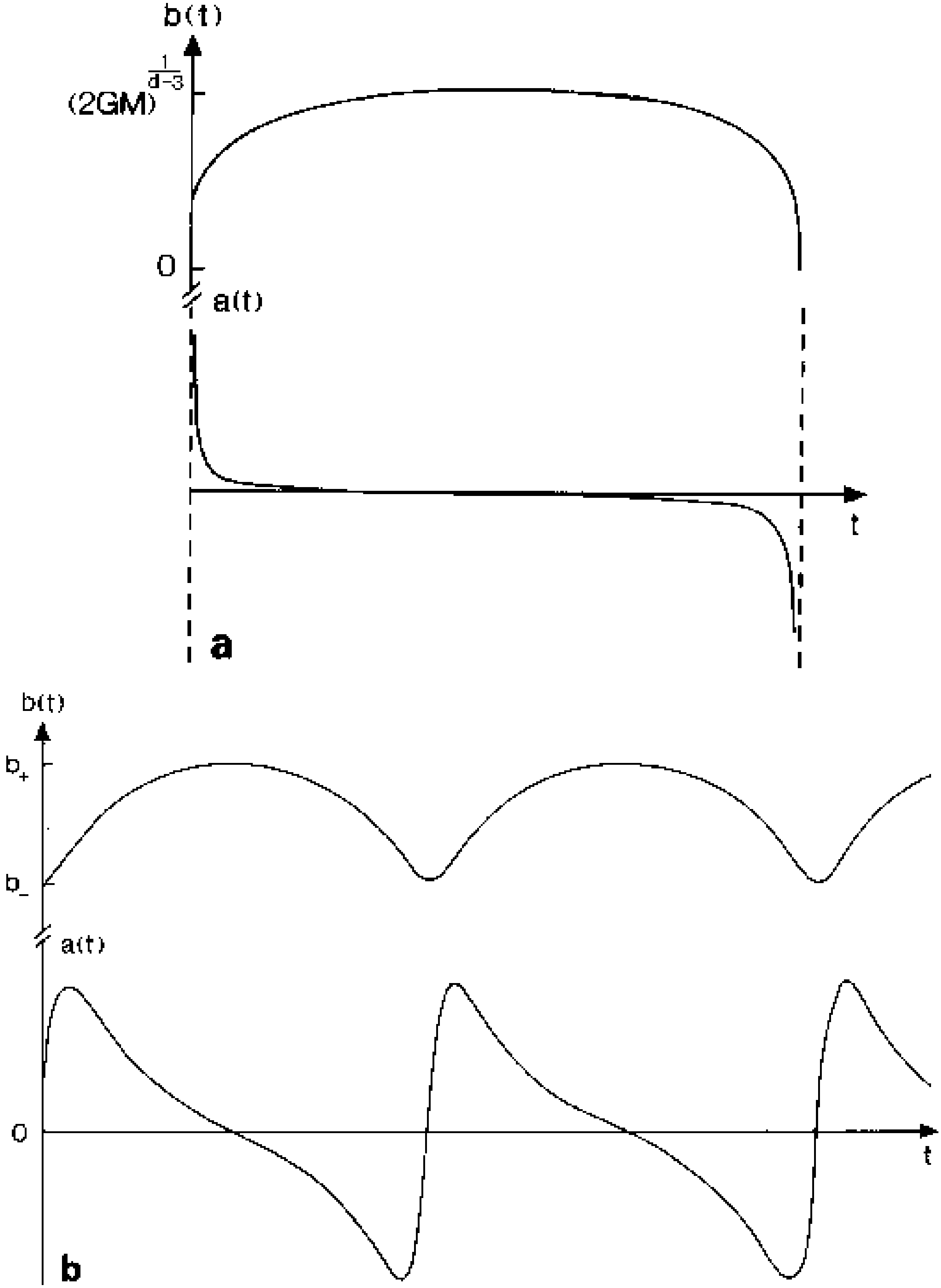

Fig. 2. Explicit t-dependence of the internal and external radii b and a for the case $\Lambda = 0$, $\bar{\lambda} = 1$: (a) $Q = 0$; (b) $\kappa^2 M^2 < \dfrac{2Q^2}{(d-2)(d-3)}$

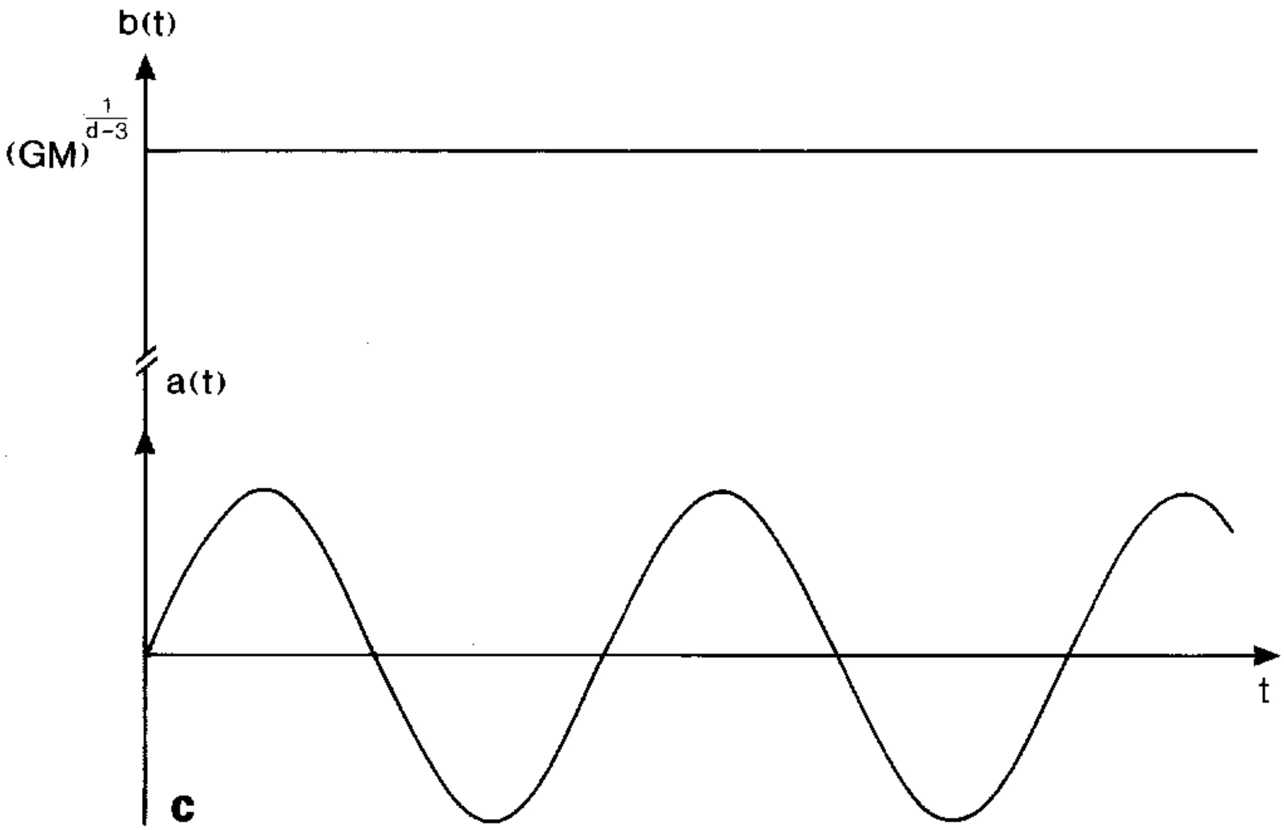

Fig. 2. Explicit t-dependence of the internal and external radii b and a for the case $\Lambda = 0$, $\bar{\lambda} = 1$: (c) $\kappa^2 M^2 = \dfrac{2Q^2}{(d-2)(d-3)}$.

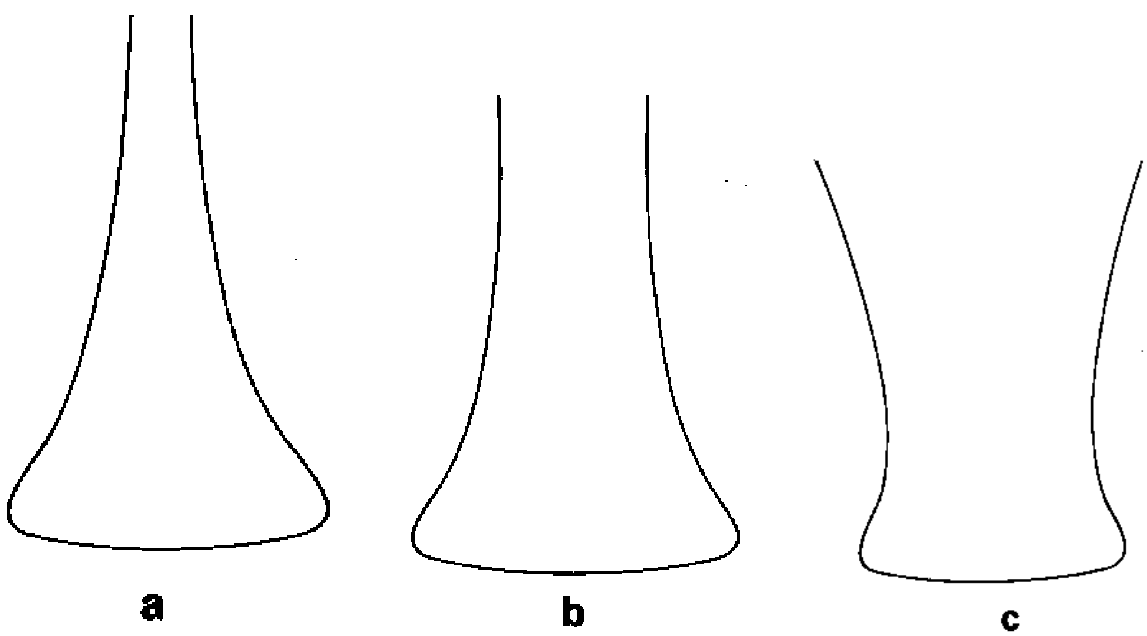

Fig. 3. Embedding diagrams for D_2 in the cases: (a) $\Lambda = 0$, $\bar{\lambda} = 0$; (b) $\Lambda = 0$, $\bar{\lambda} > 0$; (c) $\Lambda < 0$, $\bar{\lambda}$ arbitrary

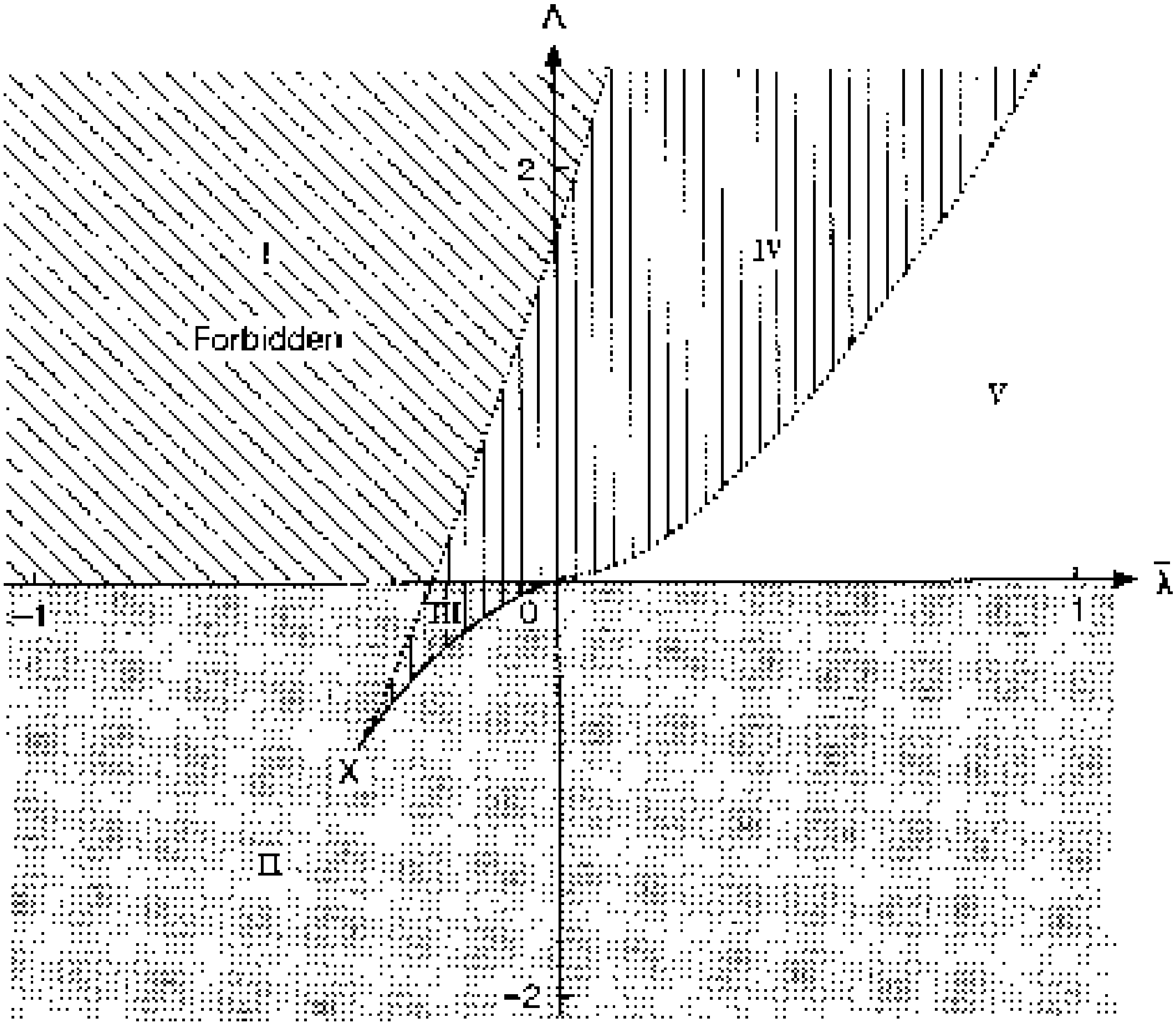

Fig. 4. Range of general membrane solutions for $B_0^2/k = 3/\pi$ (see text). Λ and $\bar{\lambda}$ are given in units of $2Gk$.