GROUP THEORETICAL METHODS AND MULTIQUARK HADRONS

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PREFACE

This thesis has been divided into two sections. The first deals exclusively with multiquark hadrons and the application of useful group theoretic techniques. The second contains a discussion of the foundations of these techniques with emphasis on phase freedom and simplifying phase choices. It also provides an up-to-date account of Butler's method of calculating 3jm factors and 6j symbols. Those interested solely in multiquark hadrons can regard Part II as an overgrown appendix containing a few tables of 3jm factors and 6j symbols plus a rigorous justification of the phase choices made and comments on the methods used to calculate them while those interested only in the group theory will find Part II completely self-contained and may regard Part I merely as an application.

In Part I, my contribution to the subject has been to remove Jaffe's (1977a,b) approximation from the spherical cavity multiquark M.I.T. bag spectrum. (This approximation is troublesome in that it predicts degeneracies which in fact are not present in a correct evaluation of the colour-magnetic interaction.) The means for removing this approximation is presented in chapter 6 and the resulting spectrum is discussed in chapter 7. I have also tried to elucidate, in chapter 3, the group theoretic nature of dissociation calculations for multiquark systems — dissociations playing an important role in the phenomenology. Techniques are

described which allow more difficult dissociations to be handled than has been the case in the past. In both cases, the methods used are not new, being simply an extension of standard angular momentum theory, but for the first time in elementary particle physics this work takes into account the considerable advances on the generalized theory made fifteen years ago by Derome and Sharp (1965, Derome 1966) and more recently by Butler and King (1974, Butler 1975). These authors showed how to isolate general quantities in the algebra such as permutation and complex-conjugation phases and further, noted simplifying choices for these quantities. These canonical choices add power and elegance to the generalized theory.

In chapters 2 and 4 I have provided some review material on relevant theoretical aspects of multiquark hadrons. A pessimistic stance has been deliberately taken in an attempt to counter the optimism widespread in the literature. Several theoretical ideas, lately popular, rest on surprisingly shaky foundations and I have tried to point out these weaknesses; recent experimental results provide plenty of reason to be critical. I have also attempted, in chapter 5, a review of the P-matrix formalism of Jaffe and Low (1979), since there is little independent material available in the literature. With the once common air of euphoria rapidly fading from the multiquark scene this formalism provides an exciting alternative interpretation of theoretical predictions which I have adopted in discussing my results.

In Part II, my contribution is mainly the elucidation of certain aspects of phase freedom. I have approached the problem by considering the fundamental connection between coupling theory and basis transformations. This connection is seldom paid any attention so I have devoted some space in chapters 10 and 11 to developing it. An important role in this development is played by the concept of a general transformation factor. I have managed to rederive the Derome-Sharp lemma using antilinear transformations. This new derivation displays the general nature of the Derome-Sharp result and reveals the true identity of the Derome-Sharp A matrix. I have gone to some length in these chapters to accurately identify and precisely describe all the phase freedom that exists in the algebra. In chapter 12 it is shown how to determine phase freedom analytically, even for 6j symbols and 3jm factors, and how to go about choosing phases. I have felt it an opportune time to include a review of canonical choices. The result is a comprehensive account of the phase problem. This provides a sound foundation on which to base a discussion, in chapters 13 and 14, of Butler's method of calculating 3jm factors and 6j symbols. Some small contributions are also made here with regard to the use of the Biedenharn-Elliott identity and algorithms for calculating 3jm factors.

Throughout this thesis I have tried to present group theoretic results in a general manner rather than deriving hosts of analogous formulae for every specific group. I am convinced of the worth of this and have quite unashamedly

ignored some idiosyncratic conventions that exist for several groups e.g. the Condon and Shortley (1935) phase convention for SU_2.

I would like to express my gratitude to my supervisor, Professor B.G. Wybourne, for his continued support and guidance, particularly for demonstrating the power of the tensor operator technique to me. Also I would like to express thanks to Dr G.E. Stedman for his interest when he took over supervision during Professor Wybourne's term of leave and for several helpful discussions, particularly on antilinearity. Further, I am indebted to Dr P.H. Butler for numerous discussions on the Wigner-Racah algebra and for making available to me a copy of his book prior to publication, as well as providing some other unpublished tables. This thesis has benefited greatly from interactions with him and his students. I would also like to thank Dr W.R. Moreau for his encouragement and several useful conversations particularly on scattering theory and quantum field theory. Appreciation is also recorded of the many stimulating conversations with my fellow students and the titbits of advice they were able to offer. Lastly, I would like to thank my wife, Lois, for her perseverance and support and Janet Warburton for her excellent job of typing a very difficult manuscript.

Christchurch, Paul Bickerstaff

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ABSTRACT

Tensor operator techniques are used to evaluate the colour-spin matrix elements of multiquark hadrons in the static spherical cavity approximation to the M.I.T. bag model, thereby obviating the necessity for the Jaffe approximation, which creates isospin degeneracies. All $q^4\bar{q}$, $q^2\bar{q}^2$ and q^6 isospin multiplet masses are tabulated and are to be regarded as Jaffe-Low primitives. The dissociation of multiquark bag model eigenstates is shown to be related to a basis transformation and techniques for performing this transformation are described. Tables of 3jm factors and 6j symbols, adequate to calculate dissociations for all $q^2\bar{q}^2$ and $q^4\bar{q}$ primitives and q^6 primitives for strangeness $\leqslant -2$, are provided.

The generalized Wigner-Racah algebra is reviewed with emphasis on phase freedom. A method of choosing phases is described and the simplifications due to certain canonical choices are noted. This leads naturally onto a discussion of Butler's method for calculating 3jm factors and 6j symbols for arbitrary compact group chains. The 6j symbols and 3jm factors required for the multiquark calculations are used as examples.

GROUP THEORETICAL METHODS AND MULTIQUARK HADRONS

Part I

Multiquark hadrons
and their colour hyperfine structure
in the MIT bag model

CHAPTER 1

INTRODUCTION

Symmetry principles, in particular the theory of groups, have proved useful in many areas of physics. Especially notable in this respect has been elementary particle physics where symmetries have been not only a guiding light but at times one of the few tools available for probing the nature of these particles and their interactions. Even when theories are available, group theory can greatly simplify many spectroscopic calculations. It is this latter type of application of group theory which is studied herein.

The advances in our understanding of hadronic matter over the past twelve or so years have been considerable. We now know that hadrons are composite particles and their prime constituents are quarks. (See for example, Feynman 1972, Close 1979a, 1979b, Hendry and Lichtenberg 1978, cf. Gell-Mann and Ne'eman 1964). Quarks possess various quantum numbers such as spin and electric charge. Different varieties of quarks with different "masses" are distinguished by a "flavour" quantum number (which includes isospin and strangeness). To date five different types of quark (or alternatively five different flavour quantum numbers) are known to exist; we term them u,d,s,c and b. However, nearly all the hadronic matter in the universe is composed of just u and d quarks and we shall often refer to them both as ordinary, o. In addition to these quantum numbers the quarks possess a

further one called the "colour" charge. This colour charge gives rise to a colour field whose quanta are termed gluons and which are analogous to the photons of the electromagnetic field. (For an interesting history of gluons, see Ellis 1980). It is believed that the colour forces arising from gluon exchange are responsible for binding the quarks together to form a hadron. Thus hadrons are composed of (valence) quarks and gluons together with a few virtual quark-antiquark pairs, called sea quarks. The two major classes of hadrons have though a different composition in terms of valence quarks. Mesons are comprised of a quark and an antiquark, $q\bar{q}$, while baryons are made up of three quarks, q^3 (and antibaryons of three antiquarks, $\bar{q}^3$).

A fascinating feature of this scenario is that while it is a relatively simple matter to free an electron from an atom or knock a nucleon from an atomic nucleus, nobody has ever succeeded in removing a quark from a hadron. It does not appear that this is simply a matter of insufficient energy because one finds that a jet of new hadrons is produced in the direction expected for an ejected quark. Further, when one probes inside the hadron with high-energy leptons one finds that at short interquark separations the quarks behave as if they are only weakly interacting i.e. quasi-free. It appears as though the interquark forces actually grow with distance until the vacuum "breaks down" and quark-antiquark pairs are created from the energy stored in the colour field. These would allow the formation of new hadrons and are responsible for the observed jets. Thus we

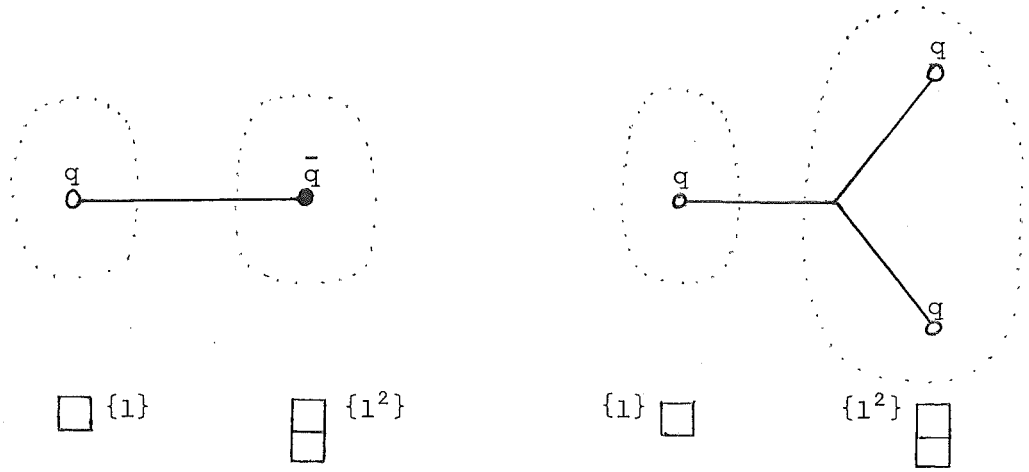
are led to the concept of quark confinement. Quarks are permanently imprisoned inside hadrons by the colour force and attempts to release them result not in free quarks but rather in new prisons with more inmates. It follows that if this confinement mechanism is absolute then the only observable hadrons will be those with a net neutral colour charge.

The colour charge comes in three different varieties, called red, green and blue (together with their "anticolours" cyan, magenta, and yellow). By assigning these to the defining representation of the Lie group SU_3 it is possible to construct a gauge theory of the colour interaction called quantum chromodynamics (QCD). (For reviews, see Abers and Lee (1973) on gauge theories and Marciano and Pagels (1978) on QCD.) This theory is analogous to the gauge theory, based on the group U_1, of the electromagnetic interaction commonly known as quantum electrodynamics (QED). The crucial difference between these theories is that SU_3 is a non-abelian group with the result that the gluons themselves possess a colour charge and are therefore self-interacting. QCD has been very successful in predicting the short-range behaviour of the colour force (Buras, 1980) but unfortunately, due to the breakdown of perturbation theory in the strong coupling regime, a long-range solution is so far non-existent. Nobody has succeeded in showing that QCD is a confining theory although theorists do have high hopes. (See for example, Bjorken, 1980.)

Within the framework of SU_3 it is quite straightforward to show (see chapter 3) that colour neutral states are of the form $q^{m-n}\bar{q}^{n}$ where m-n = 0 mod 3. This nicely accommodates $q\bar{q}$, q^3,$\bar{q}^3$ and rules out q,q^2, etc but what about $q^2\bar{q}^2$, $q^3\bar{q}^3$, $q^4\bar{q}$, q^6, q^9 etc? These are what we term multiquark hadrons and are the subject of Part I of this thesis. Of course the mere fact that group theory allows their existence does not mean that they do, although if they do not then we would like to know why. It is this basic question of existence that one ultimately hopes to answer.

For many years all observed hadrons could be accounted for by the configurations $q\bar{q}$, q^3 and $\bar{q}^3$. Recently there have emerged several candidates for multiquark states but they have a disconcerting habit of failing to make second stage appearances in higher statistics experiments. The present experimental situation is one of confusion. (A discussion of the status of the prime candidates can be found in the review by Montanet et al., 1980.)

Multiquark hadrons are of considerable interest to theorists because of their internal colour degree of freedom. In a baryon, any two quarks must possess the same net colour charge as the antiquark in a meson so as to produce an overall colour neutral state. This requirement is dictated by group theory and is depicted in the string picture in fig 1.1. (We use Young diagrams to denote the colour charge. The number of components of this charge is also indicated.) However, in a multiquark system there are many various possibilities depending on the actual

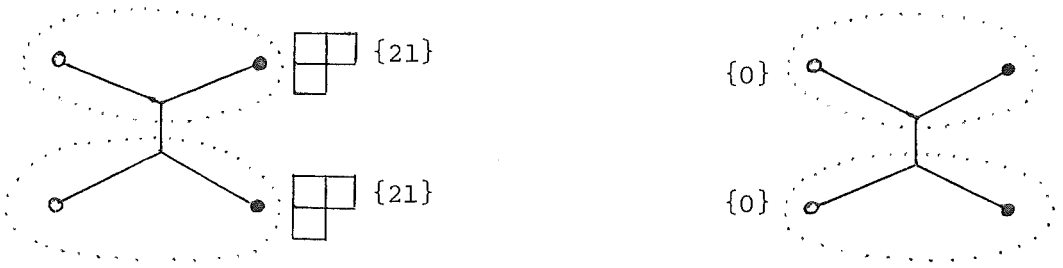


$$|\{1\}| = |\{1^2\}| = 3$$

Fig. 1.1: Colour content of mesons and baryons

$$|\{2\}| = |\{2^2\}| = 6$$

Fig 1.2a: Colour content of $q^2\bar{q}^2$ system: diquark-antiquark structure



$$|\{21\}| = 8 \qquad |\{0\}| = 1$$

Fig 1.2b: Colour content of $q^2\bar{q}^2$ system: $(q\bar{q})(q\bar{q})$ structure.

system. For example in $q^2\bar{q}^2$ the diquark-antiquark system can be either $\{1^2\}$ coupled with $\{1\}$ (as in mesons and baryons) or $\{2\}$ coupled with $\{2^2\}$, see fig. 1.2a. We could also though look at the $(q\bar{q})(q\bar{q})$ nature of the system where we find the two possibilities $\{21\}$ coupled with $\{21\}$ or $\{0\}$ coupled with $\{0\}$, see fig. 1.2b. The last possibility is just two (colour neutral) mesons coupled together. This sort of possibility is present in all multiquark systems and is the source of major theoretical difficulties! Meson, baryon and antibaryon configurations turn out to be the only overall colour neutral ones which cannot be divided into colour neutral components.

The above mentioned difficulties arise because of our poor understanding of the confinement mechanism. For mesons and baryons, considerable insight into the physical spectrum can be gained by assuming an arbitrary confining potential and calculating corrections due to the differing quark "masses" and colour hyperfine interactions (e.g. De Rújula et al. 1975). However similar procedures for multiquark systems are liable to give spurious results because colour neutral subsystems are not confined — as evinced by atomic nuclei which are apparently combinations of colour neutral systems — and there is a worrying possibility that the multiquark system will simply dissociate into such subsystems. Thus the possibility of dissociation invalidates any model of multiquark hadrons which incorporates artificial confinement, even though that model may work quite well for mesons and baryons. Like it or not, multiquark systems are fundamentally different.

All this notwithstanding, this thesis is mainly concerned with technical difficulties in evaluating the colour-hyperfine contributions to the multiquark spectrum in just such a model. That chosen is the M.I.T. bag model (an account of which may be found in chapter 4). Previous calculations using this model have all approximated the colour-hyperfine term when dealing with multiquark systems. Indeed quite extensive calculations have been performed. Jaffe (1977 a,b,c) has studied $q^2\bar{q}^2$ and q^6 systems while Strottman (1978, 1979) has studied $q^4\bar{q}$ and even some $q^5\bar{q}^2$ states. De Swart and his students (Aerts 1979, Mulders 1980, Mulders et al. 1979, Aerts et al. 1980, Mulders et al. 1980) have made extensive calculations for $q^2\bar{q}^2$, q^6 and $q^4\bar{q}$ using a slightly different approximation and including several orbital excitations. It is the purpose of this thesis to show that the colour-hyperfine term can be evaluated exactly, using group theoretical methods, without inordinate effort. The results are to a certain degree more general than the bag model because the form of the colour-hyperfine term is similar in most models of the multiquark spectrum in current use. The group theoretical techniques used are certainly more general. Indeed they had their origins long ago in atomic physics calculations.

To facilitate the calculations it is necessary to introduce a new basis for multiquark states. This means that the problem of dissociation must be considered as well because previous work on that problem cannot be readily utilized in this new scheme. Part of the reason for this

is the necessity for phase consistency. Wong and Liu (1980) have shown that Jaffe (1977a, who only considers a very simple case) has run into difficulties with phases. The approach used by So and Strottman (1979, Strottman 1979) is incompatible with this work because of the special phase choices which their methods dictate. (It is worth remarking that So and Strottman have also run into problems with phases. They use an incorrect phase prescription based on their failure to appreciate the difference between the groups U_6 and SU_6. Fortunately it has no effect on their most important results.) The method used in this thesis is the same in principle as that adopted by Matveev and Sorba (1978) but they only considered systems without strange quarks. Further, none of these authors make full use of symmetries.

Both the above mentioned problems are simplified by using the highly symmetric 3jm factors and 6j symbols rather than the coupling and recoupling coefficients which arise naturally. The calculation of these symbols and the associated phase difficulties is a problem in itself and the treatment of this aspect is reserved for Part II of this thesis.

To ensure that the results of Part I are viewed with proper perspective we shall spend a little time in the early chapters looking at the theoretical foundations for multiquark hadrons and some of the models for them. Of particular importance is the P-matrix formalism discussed in Chapter 5. This allows some meaning to be attributed to masses calculated in the M.I.T. bag model by relating

bag model eigenstates to poles in what is termed the P-matrix (Jaffe and Low, 1979) — a quantity which can be constructed from experimental phase shifts.

CHAPTER 2

QUESTIONS OF EXISTENCE

Recognition of the phenomenon of quark confinement and the necessity for colour neutral hadrons has greatly reduced the number of possible configurations admissible in a quark model. As has been mentioned, colour neutral hadrons are restricted by group theory to be of the form $q^{m-n}\bar{q}^{n}$ where $n-m = 0 \bmod 3$. However, group theory is not the beginning and end of physics and the mere fact that it admits such configurations does not mean that they exist. For the simplest configurations, $q\bar{q}$, q^3 and $\bar{q}^3$, experiment settles the matter but for multiquark configurations the situation is unclear. To decide one way or the other on theoretic grounds one needs to be able to calculate a spectrum and see whether or not there are any bound states. It is not difficult to see how bound states would arise in a confining potential but for multiquark systems the possibility of colour neutral components means that there is no guarantee that the potential is confining. It might even be repulsive! Unfortunately there are at present no reliable means of calculating a multiquark spectrum. (Some authors have claimed that multiquark hadrons are "predicted by QCD" — usually on the mere grounds of admissibility — but this is just not true. Nobody has ever been able to calculate a spectrum using QCD and so it is impossible to tell at this stage whether that theory predicts them or not.)

If one naively considers quarks as "building blocks" for hadronic matter then it is difficult to understand the apparent absence of multiquark hadrons. Even if multiquark hadrons do in fact exist then one thing at least is clear from experiment: they are not as readily produced as the ordinary mesons and baryons. One could rightfully claim that it is very mysterious for Nature not to take advantage of a degree of freedom available to it. However the emergence of quark confinement as the principal binding mechanism in hadrons and the possibility of dissociation for multiquark systems casts a different light on the matter. It is even conceivable that bound multiquark systems are a myth!

With such doubt in mind it is desirable to have some independent support for the concept of multiquark hadrons. An alternative argument does exist (Rosner 1968, Roy and Suzuki 1969) and the rest of this chapter is devoted to it. Its appreciation requires an understanding of scattering theory and the concept of duality. A brief explanation of crossed channels in scattering processes is given in appendix IA. Those wishing more background information, particularly on duality, should find the text by Novozhilov (1975) useful. More advanced treatments of duality may be found in the reviews by Fukugita and Igi (1977) and Rosner (1974).

(a) Duality and Baryonium

The scattering amplitude can be expanded in terms of partial wave amplitudes: one for each possible value of the

angular momentum, J. In the t-channel, these partial wave amplitudes are analytic functions $a_J(t)$ of the t-channel invariant, t. Further, J can be considered to be a complex variable and the partial waves can be continued into the region of complex J. The poles in J of this new function, play an important role in determining the scattering. These "Regge" poles depend on t and as the energy varies a pole describes an analytic "Regge" trajectory $\alpha(t_J)$ in the complex angular momentum plane. For $0 < t <$ elastic threshold (= square of masses of particles in initial state), $\alpha(t_J)$ is real and the physical values of J (positive integers for meson trajectories and half-(odd)integers for baryon trajectories) correspond to bound states, of the scattering particles, with mass $m_J = (t_J)^{\frac{1}{2}}$. For values of t above the elastic threshold, $\alpha(t_J)$ acquires a positive imaginary part and moves further from the real axis as t increases. Provided the trajectory is not too far from the real axis, physical values of Re $\alpha(t_J)$ correspond to resonances of mass $m_J = (t_J)^{\frac{1}{2}}$ and widths given by Im $\alpha(t_J)$. (N.B. There are actually two complementary trajectories, $\alpha^{\pm}(t_J)$ of even and odd signature, and the spins of neighbouring resonances on each differ by two.) For finite $t < 0$ the trajectory $\alpha(t)$ describes the asymptotic behaviour of the amplitude in the s-channel; in the s-channel, $t<0$ corresponds to a momentum transfer and the exchanged quantum numbers are those of the trajectory $\alpha(t)$ i.e. the quantum numbers of t-channel resonances. (We are referring to strong interaction processes and conserved quantum numbers. One should note

that the reverse exchange, with conjugate quantum numbers takes place at the same time so that conventions are important.) The scattering is said to take place via "Reggeon" exchange. Similar comments apply to the crossed channels.

As just indicated, the asymptotic behaviour of the scattering amplitude at high energies in the s-channel can be described in terms of Regge trajectories in the t-channel. At low energies, s-channel resonances govern the properties of the amplitude. However, the behaviour of the amplitude in these two regions must be connected. This connection can be expressed in terms of "finite energy sum rules", whose derivation is based on Cauchy's theorem (e.g. Novozhilov, 1975).

The finite energy sum rules show two things. Firstly, the average over the low energy region of the imaginary parts of the resonant amplitude (expressed in terms of $\nu = \frac{1}{2}(s-u)$) and the asymptotic form of this amplitude are equal. Secondly, they relate the imaginary part of this resonant amplitude, integrated over the low energy region of the s- and u-channels, to a sum over Regge trajectories $\alpha(t)$ in the t-channel (where $t \leqslant 0$). (The Pomeranchuk trajectory is, however, excluded from the summation because it corresponds to the diffractive part of the amplitude i.e. the non-resonant background. It has vacuum quantum numbers and, unlike all other known trajectories, no resonances in the physical region of the t-channel.) The terms in the summation involve the trajectories themselves and their

couplings to the scattering particles. The imaginary part of the resonant amplitude appearing in the integrand can be replaced by the contribution of resonances in the s- and u-channels using the resonance approximation in dispersion theory. We thus obtain a relationship between these resonances and t-channel resonances. One can consider the s- and u-channel resonances to "build" the trajectory in the t-channel and vice-versa. This leads to the concept of duality.

Of particular importance is that the finite energy sum rules imply (Roy and Suzuki, 1969) that if there are no resonances in the s- and u- channels then the contributions to the t-channel must vanish. Further, since there are actually two distinct sum rules relating the amplitudes symmetric and antisymmetric under the crossing interchange $s \leftrightarrow u$ to summations over the trajectories $\alpha^+(t)$ and $\alpha^-(t)$ respectively, the vector and tensor contributions must vanish separately. This means that either there are no t-channel resonances or the couplings are such so as to cause cancellation.

All this works quite well for meson-meson and meson-baryon scattering. It predicts for instance ρ,ω degeneracy and the Okubo (1963), Zweig (1964), Iiuzuka (1966) or OZI rule governing ϕ decays. (A good elementary discussion of the OZI rule is given by Hendry and Lichtenberg, 1978. The relationship with duality is described, for example by Roy, 1980. Note that ϕ is sometimes called strangeonium because of its $s\bar{s}$ quark content — mesons of this generic

type, with both quark and antiquark of the same flavour, are collectively referred to as quarkonium.) Consider, however, the following baryon-antibaryon scattering process (Lipkin, 1970)

			B	Q	I_z
s:	$\Delta^+ + (\bar{\Delta})^+$	$\rightarrow \Delta^{++} + (\bar{\Delta})^0$	0	+2	+2
t:	$\Delta^+ + (\bar{\Delta})^{--}$	$\rightarrow \Delta^- + (\bar{\Delta})^0$	0	-1	-1
u:	$\Delta^+ + \Delta^0$	$\rightarrow \Delta^{++} + \Delta^-$	2	+1	0

where B is the total baryon number, Q the charge and I_z the z-component of isospin. Both the s- and u-channels are exotic but the only vector and tensor exchanges in the t-channel are ρ and A_2 respectively. There is no possibility of cancellation and they cannot decouple as this would be inconsistent with the requirements imposed by meson-baryon amplitudes. If duality is to hold, there must be meson resonances in exotic baryon-antibaryon channels (which must decouple from meson-meson channels in order to maintain consistency with amplitudes for those processes) — a B=2 resonance would be inconsistent with other data, Roy (1980). The only means of obtaining a Q = +2, I = 2 meson via the quark model is within a multiquark configuration. The simplest possibility is $q^2\bar{q}^2$ so that we are led to the picture of $B\bar{B}$ scattering given in fig. 2.1 in terms of quark line diagrams (Harari 1969, Rosner 1969).

These new exotic states are referred to as baryonium. In $B\bar{B}$ scattering baryonium exchange is dual to normal meson resonances and vice versa, just as the ω,ρ resonances

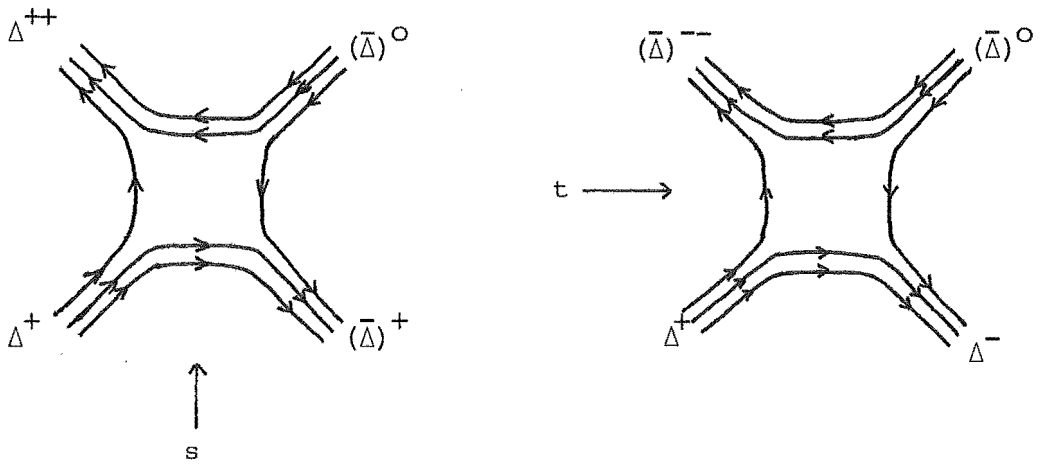
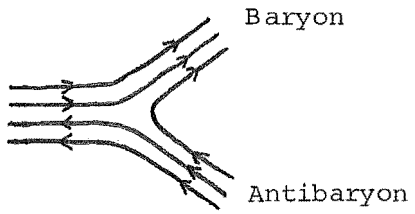
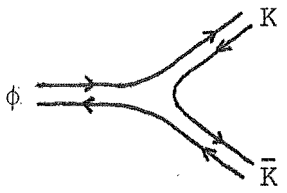


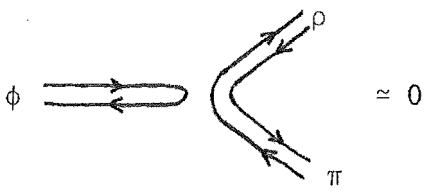
Fig. 2.1: Quark line diagrams for $\Delta\bar{\Delta}$ scattering.
Time proceeds in the direction of the arrow.



Baryonium → B + $\bar{B}$



$\phi \rightarrow K + \bar{K}$



$\phi \rightarrow \rho + \pi$ (OZI rule)

Fig. 2.2: Baryonium and strangeonium decays.
Time proceeds to the right.

are dual to strangeonium exchange and vice versa in $K\bar{K}$ scattering. Similarly baryonium couples to (or decays into) baryon-antibaryon channels rather than meson-meson, just as strangeonium couples to $K\bar{K}$ rather than, for instance, $\rho\pi$. (See fig. 2.2.)

(b) Higher Exotics

Applying the same hypotheses to baryonium-baryonium scattering and baryonium-baryon scattering etc implies still higher exotics (Roy and Suzuki, 1969). These would correspond to $q^3\bar{q}^3$ and $q^4\bar{q}$ multiquark systems and so on. It seems unlikely that any closed set of "meson" and "baryon" resonances would provide a self-consistent solution to the duality constraints.

These predictions of duality are however not beyond question. They involve an extrapolation of approximate techniques and for this reason alone we should be wary. Shapiro (1978) has criticised the predominantly qualitative nature of the predictions. Quantitative constraints on the masses and couplings of these exotics do exist though. Roy (1980) discusses some of these in relation to the experimental data for candidate states and indeed notes some serious discrepancies. It has been suggested that exotic non-resonant contributions would undermine the above predictions (e.g. Lipkin, 1970). Thus a $q^2\bar{q}^2$ continuum of meson pairs could be an alternative (cf. Rossi and Veneziano, 1977).

Thus it seems wise to proceed with some caution.

CHAPTER 3

CLASSIFICATION OF MULTIQUARK STATES

It is appropriate now to discuss the purely group theoretic problem of classifying multiquark states. First though, a word on notation: we shall use Schur functions — (S-functions) or equivalently Young diagrams — to label the irreducible representations of the general linear group in N dimensions, GL(N) and its unitary subgroups (see Wybourne 1970). The only exception will be SU_2 where we shall normally use either the "total spin quantum number" — more appropriate to the isomorphic three dimensional rotation group — or the spin multiplicity. For the other unitary groups, the labelling by representation dimensions leads to an ambiguous scheme and further, such a scheme is useless for performing calculations. The prevalent use of dimensional schemes in elementary particle physics is most unfortunate, particularly in relation to multiquark states where the number of representations arising is rather large. To aid those unaccustomed with the group theorist's notation, some reference will occasionally be made to the representation dimensions. It is a simple matter though to calculate the dimensions of a representation of GL(N) from the associated Young diagram. We shall only be dealing directly with covariant tensor representations for which the dimension of the representation $\{\lambda\}$ (associated with a partition (λ) into not more than N parts) is given by (Robinson, 1961)

$$|\lambda|_{GL(N)} = G_N^{\{\lambda\}}/H_{[\lambda]} \quad . \tag{3.1}$$

Here,

$$G_N^{\{\lambda\}} = \prod_{i,j} (N + i - j) \tag{3.2}$$

(i and j specify the column and row respectively of each cell in the Young diagram) and $H_{[\lambda]}$ is the product of hook lengths of the cells in the Young diagram. (The hook length of a cell is equal to $\alpha + \beta + 1$ where β is the number of cells directly below the given cell in the same column and α is the number of cells to the right of the given cell in the same row.) A useful expression is

$$H_{[\lambda]} = \frac{\ell_0!\ \ell_1! \cdots \ell_p!}{\prod_{i<j} (\ell_i - \ell_j)} \tag{3.3}$$

which involves only the hook lengths $\ell_0, \ell_1, \cdots \ell_p$ of cells in the first column. We shall also use S-functions to label irreducible representations of the symmetric group on n objects, S_n. The representations of S_n involve Young diagrams corresponding to all partitions of n. In S_n the dimension of the representation $[\lambda]$ is given by

$$|\lambda|_{S_n} = n!/H_{[\lambda]} \quad . \tag{3.4}$$

The relationship between S_n and GL(N) was greatly developed by Weyl (1939) and will prove useful in what follows.

Apart from the close correspondence with the symmetric group, the advantage of using S-function notation for GL(N), as opposed to other unambiguous schemes, is that product and branching rules can be formulated in a group independent manner. For both GL(N) and U_N the Kronecker product of representations is given independently of N by the Littlewood-Richardson rule; this involves combining the cells of two Young diagrams in a prescribed manner so as to give new diagrams (Littlewood 1950, p.94). As will become apparent in a moment, this rule holds with only minor modification for SU_N. The other branching rules required are as follows.

$GL(N) \rightarrow U_N$:

Irreducible representations of GL(N) remain irreducible on restriction to U_N, i.e.

$$\{\lambda\} \rightarrow \{\lambda\} \tag{3.5}$$

$U_{pq} \rightarrow U_p \times U_q$:

This branching rule is given independently of p and q by the S-function result (King 1975, Whippman 1965)

$$\{\lambda\} \rightarrow \sum_{\xi} (\{\lambda\} \circ \{\xi\})\{\xi\} \tag{3.6}$$

where the symbol $\circ$ denotes inner S-function multiplication (e.g. Wybourne 1970) and the sum is over all partitions (ξ) of the weight of λ (i.e. the number of cells) into not more than q parts. Of course partitions coming from $\{\lambda\} \circ \{\xi\}$ must not consist of more than p parts. For the special case

of the $\{1^n\}$ representations we have the simpler result

$$\{1^n\} \to \sum_{\xi} \{\tilde{\xi}\}\{\xi\} \tag{3.7}$$

where $(\tilde{\xi})$ is the partition conjugate to (ξ) i.e. the rows and columns in the Young diagram are interchanged.

$U_{p+q} \to U_p \times U_q$:

Again, this branching rule is given independently of p and q by an S-function result (King 1975, Whippman 1965)

$$\{\lambda\} \to \sum_{\xi} (\{\lambda\}/\{\xi\})\{\xi\} \tag{3.8}$$

where / denotes S-function division (Wybourne 1970). For the representations $\{1^n\}$ this becomes

$$\{1^n\} \to \sum_{x=t}^{u} \{1^{n-x}\}\{1^x\} \tag{3.9}$$

where t is the larger of 0 and n-p, and u is the smaller of n and q.

$U_N \to SU_N$:

Irreducible representations of U_N (isomorphic to $U_1 \times SU_N$) remain irreducible on restriction to SU_N but we have in SU_N the equivalences

$$\{\lambda_1, \lambda_2, \cdots, \lambda_N\} \equiv \{\lambda_1 - \lambda_N, \lambda_2 - \lambda_N, \cdots, 0\} \ . \tag{3.10}$$

Thus irreducible representations of SU_N can be described by partitions into not more than N-1 parts. In particular, for SU_2 we have the following correspondence between the S-function notation and the S (or I) quantum number:

$$S = \tfrac{1}{2}(\lambda_1 - \lambda_2) \quad . \tag{3.11}$$

It was stated above that we shall only be directly concerned with covariant tensor representations. GL(N) has though other, mixed tensor, representations; in particular the representation complex conjugate to a covariant tensor representation is a contravariant tensor representation. Of importance to us is that in SU_N all such representations are equivalent to covariant tensor representations. (This is just a generalization of the equivalence 3.10). The important result is

$$\{\lambda_1,\lambda_2, \cdots, \lambda_N\}^* \equiv \{\lambda_1-\lambda_N,\lambda_2 - \lambda_{N-1}, \cdots, 0\} \quad . \tag{3.12}$$

Armed with these results we now return to our problem of classification.

(a) Classification of Multiparticle States

A set of functions, or quantum states, spanning an N-dimensional representation of any group G will also span the defining representation {1} of the general linear group GL(N) (e.g. Wybourne 1970). Thus if a single particle has N quantum states then it will transform according to the N-dimensional representation {1} of GL(N). It follows that n

particles will transform according to the nth Kronecker power of this representation. Such a product can be labelled by the symmetric group, S_n (Weyl, 1939):

$$\{1\}^{\times n} = \sum_{\lambda} |\lambda|_{S_n} (\{1\} \otimes \{\lambda\}) \tag{3.13}$$

where the sum is over all representations $[\lambda]$ of S_n and the quantity in parentheses is the "$[\lambda]$-symmetrized" part of the nth Kronecker power. The symmetrized Kronecker powers are conveniently evaluated by Littlewood's (1950) algebra of plethysm and $\{1\} \otimes \{\lambda\}$ (read as "$\{1\}$ plethys $\{\lambda\}$") denotes the use of plethysm. We have the very simple general result

$$\{1\} \otimes \{\lambda\} = \{\lambda\} \quad . \tag{3.14}$$

These two equations can be viewed as expressing the decomposition of the representation $\{1\}^{\times n}$ of $GL(N)^{\times n}$ under restriction to $S_n \times GL(N)$. By reducing the representation $\{\lambda\}$ of $GL(N)$ to representations of the group G one obtains a complete classification of the n-particle states under transformations of $S_n \times [GL(N) \supset G]$.

The importance of the above procedure is that if the particle is a fermion then the spin-statistics theorem of quantum field theory requires the n-particle state to be a totally antisymmetric combination i.e. one which transforms as the representation $[1^n]$ of S_n and thus also as the $N!/[(N-n)!n!]$ dimensional representation $\{1^n\}$ of $GL(n)$.

Similarly an n-boson state must be totally symmetric i.e. transform as [n] under S_n and thus also as the $(N+n-1)!/[(N-1)!n!]$ dimensional representation {n} of GL(N).

(b) Bases for Multiquark States

A quark can exist in any one of at least five flavour states (u,d,s,c and b — it is generally believed that there is at least one more, t). The precise number is not very important and indeed we shall only consider the first three as hadrons with quarks in the c and b states are much more "massive" and harder to produce. The ordinary, o (or u and d) states are associated with two components of isospin, I and span the two-dimensional defining representation {1} (or I = ½) of SU_2^I while the s state is associated with the strangeness quantum number, S and transforms as the one-dimensional defining representation {1} (or S = -1) of U_1^S. These three states could be combined in the three-dimensional defining representation {1} of U_3^{fl} if desired. (Extension to N_f flavours would thus simply involve using $U_{N_f}^{fl}$.) Notice that we do not use SU_3^{fl}; this would require us to drop U_1^S and use instead the hypercharge quantum number, Y which transforms as a three-valued representation of U_1^Y and is less convenient — principally because the anti-particle does not have hypercharge -Y when the particle has hypercharge +Y.

A quark can also exist in any one of two spin states (up and down) which span the defining (spin, S = ½) representation {1} of SU_2^S. Further, a quark can exist in

any one of three colour states (r,g and b) which span the defining representation of SU_3^C.

Thus the quantum states of a quark span the eighteen dimensional mixed product representation $(\{1\}^I\{0\}^S + \{0\}^I\{1\}^S)\{1^3\}^C$ of the product group $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$. A quark can therefore be labelled by the defining representation of GL(18). In calculating the branching rules for the decomposition of irreducible representations of GL(18) on restriction to $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ it is useful to insert a chain of intermediate groups. Some possibilities are

$$GL(18) \supset U_{18} \supset (U_3^{fl} \supset SU_2^I \times U_1^S) \times (SU_6^{CS} \supset SU_2^S \times SU_3^C) \tag{3.15}$$

(c.f. Wybourne 1978a) or

$$GL(18) \supset U_{18} \supset (SU_{12} \supset SU_2^I \times SU_6^{CS}) \times (U_6 \supset U_1^S \times SU_6^{CS}) \supset SU_2^I \times U_1^S \times (SU_6^{CS} \supset SU_2^S \times SU_3^C) \tag{3.16}$$

or (Bickerstaff and Wybourne, 1980a,b)

$$GL(18) \supset U_{18} \supset [SU_{12} \supset SU_2^I \times (SU_6^{CS} \supset SU_2^S \times SU_3^C)] \times [U_6 \supset U_1^S \times (SU_6^{CS} \supset SU_2^S \times SU_3^C)] \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C. \tag{3.17}$$

Whichever scheme is used is largely arbitrary; however, some may have advantages over others. The first scheme involves U_3^{fl}, but in the multiquark sector there is strong mixing of flavour quantum numbers (as there is also for the mesons) and

it is a good idea to ignore the flavour group altogether. The second, SU_6^{CS} coupled, scheme provides a basis in which the M.I.T. bag model Hamiltonian (see chapter 4) is nearly diagonal but evaluation of that Hamiltonian (chapter 6) is simpler in the third, $SU_2^S \times SU_3^C$ coupled scheme. Therefore it is the third scheme which we shall use throughout the remainder of this thesis.

In the $SU_2^S \times SU_3^C$ coupled scheme the states formed by the o quarks are associated with the subgroup chain

$$SU_{12} \supset SU_2^I \times (SU_6^{CS} \supset SU_2^S \times SU_3^C) \tag{3.18}$$

while the states formed by the s quarks are associated with the subgroup chain

$$U_6 \supset U_1^S \times (SU_6^{CS} \supset SU_2^S \times SU_3^C) \ . \tag{3.19}$$

The spin and colour quantum numbers of the combined quark system are then found by restricting the outer product of the two $SU_2^S \times SU_3^C$ groups to the inner product group i.e.

$$(SU_2^S \times SU_3^C) \times (SU_2^S \times SU_3^C) \supset SU_2^S \times SU_3^C \ . \tag{3.20}$$

The branching rules for the last group chain are of course just the decomposition rules for the Kronecker product of representations of SU_2 and SU_3 considered separately. All the other branching rules required were given at the beginning of this chapter. For quick

reference, the specific cases of interest are listed in tables I1 to I4 . The isospin and spin quantum numbers are determined by (3.11) — we denote spin by writing the spin multiplicity 2S+1 as a left superscript — and the strangeness quantum number is given by

$$S = - \lambda_1 \quad . \tag{3.21}$$

(The minus sign in (3.21) is historical and comes from the arbitrary assignment of S = +1 and -1 to the K^0 and Λ respectively in the strangeness conserving reaction $\pi^- + p \rightarrow \Lambda + K^0$.)

A state of m quarks will transform under $GL(18)^{\times m} \supset GL(18)$. Because quarks are fermions we select only the totally antisymmetric part of the Kronecker power. Thus m quarks will transform under the representation $\{1^m\}$ of GL(18). Similarly n antiquarks will transform under $\{1^n\}^*$. A system of m quarks and n antiquarks could be described using mixed tensor representations of GL(18) and generalized Young diagrams (King, 1970) but it is simpler to separately reduce $\{1^m\}$ and $\{1^n\}^*$ and to couple the two systems at the $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ level. (The representations arising in the decomposition of $\{1^n\}^*$ are just the complex-conjugates of those arising in the decomposition of $\{1^n\}$. When an SU_N group is reached the equivalence (3.12) can be applied. For U_1^S, one requires the trivial relationship: $\{\lambda_1\}^*$ has strangeness, $S = +\lambda_1$ cf.(3.21).)

Because of the trivial branching rule for $GL(N) \to U_N$ the use of GL(18) is superfluous if we are going to employ U_{18}. For this reason we shall henceforth cease to make any reference to GL(18). A complete classification scheme based on the group chain (3.17) is accordingly given in fig. 3.1. (Additional quantum numbers arising from subgroups of SU_2 and SU_3 will not feature in our discussion.) Note that direct evaluation of the plethysm $(\{1\}\{0\}\{1\}\{1\}) \otimes \{1^n\}$ in $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ would fail to provide adequate classification of the terms arising.

(c) Colour-neutral States

According to the colour hypothesis, the only observable hadrons are those with a net neutral colour i.e. they transform as $\{0\}^C$ under SU_3^C. Because of the equivalence (3.10), this is the same as saying that these states are associated with any partition into three equal parts.

Consider a hadron composed only of quarks. For the sake of generality combine all quantum numbers other than colour into one unitary group U_p; then it follows from the branching rule (3.7) for $U_{3p} \to U_p \times SU_3^C$ that the only possibility of obtaining a representation $\{\xi\}$ of SU_3^C involving a partition into three equal parts is when the number of quarks is a multiple of three.

If antiquarks are present we must combine the colour representation of the quarks with the colour representation of the antiquarks to obtain $\{0\}^C$. The only way of doing this is when the colour representation of the antiquarks is

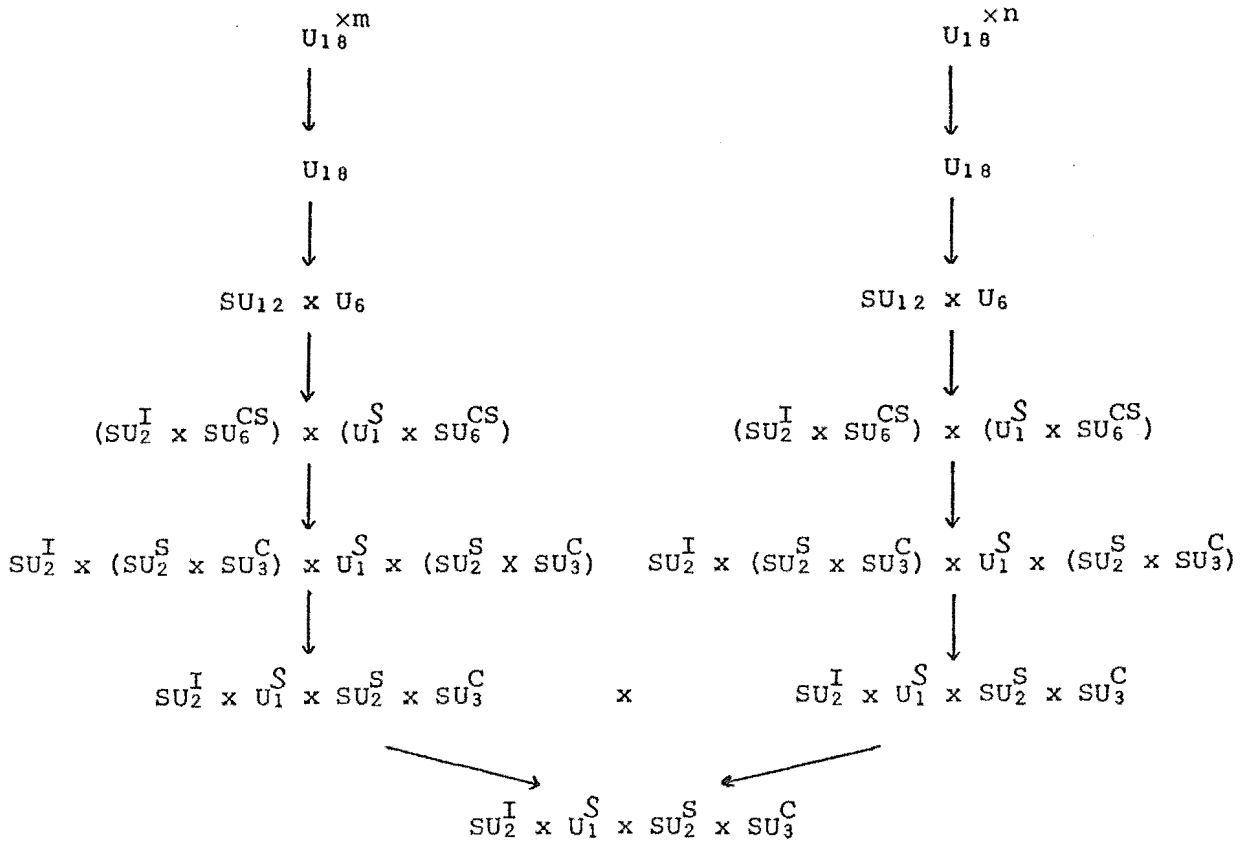


Fig. 3.1: Classification scheme for multiquark hadrons.

complex-conjugate to that of the quarks. Suppose that there are m quarks with colour $\{\xi\}$, then because of the equivalence (3.10) the colour $\{\xi\}^*$ can come not only from m antiquarks but from any number n that can give rise to a representation of the form $\{\xi_1 + i, \xi_2 + i, \xi_3 + i\}^*$ where i is a positive or negative integer. Clearly n-m = 3i. Thus we obtain the well-known result that a configuration $q^{m-n}\bar{q}^n$ can only be in a colour neutral state if

$$m - n = 0 \text{ mod } 3. \tag{3.22}$$

(It is interesting to note that this result would still have followed even if the quarks were bosons.)

It will be found helpful later on to have the colour neutral states readily available. Therefore, all possible q^3, $q\bar{q}$, q^6, $q^4\bar{q}$ and $q^2\bar{q}^2$ colour neutral states are listed in tables I5 - I9 respectively. (Included in these tables are some shorthand labels for the SU_6 x SU_6 content of the states. The significance of these will become apparent in chapter 6. Suffice it to say here that they are related to mixing of states by the bag model Hamiltonian.) Notice that the number of multiquark states is rather large!

It should be emphasized that the basis states listed in these tables need have no direct correspondence with observable hadrons, even if multiquark hadrons exist. For instance, in the ordinary meson sector the physical η and η' mesons are linear combinations of the states listed as η_o and η_s.

(d) Dissociation Transformations

While the basis states listed in tables I7 - I9 are convenient for the calculations to be performed in chapter 6 they are inappropriate for considering the dissociation of multiquark systems into colour-neutral components. Hadron-hadron combinations ought to be classified by a dissociated scheme based on the product group $(U_{18}^{\times p} \times U_{18}^{\times r}) \times (U_{18}^{\times q} \times U_{18}^{\times s})$ where both p-r and q-s equal 0 mod 3. In contrast, the standard scheme is based on $U_{18}^{\times(p+q)=m} \times U_{18}^{\times(r+s)=n}$ and it becomes necessary to transform between schemes. However, the nature of the Kronecker product ensures that the product groups at the head of the two classification schemes are identical i.e. $U_{18}^{\times(m+n)}$, and thus the required transformation is just a change of basis.

It would be straightforward to formulate this basis transformation in general terms but it is more instructive to consider some specific examples.

(i) q^6

This configuration is conceptually simple because of the absence of antiquarks. However, in practice one requires extensive tabulations of transformation coefficients.

The only allowed dissociation is into two baryons. Multiquark q^6 states are classified according to $U_{18}^{\times 6} \supset U_{18} \supset \cdots$ which, since we are only considering totally antisymmetric states, is uniquely related to $U_{18}^{\times 6} \equiv U_{18}^{\times 3} \times U_{18}^{\times 3} \supset U_{18} \times U_{18} \supset U_{18} \supset \cdots$. (In other words, the transformation factor $\langle (1^{\times 3} \cdot 1^{\times 3} \equiv 1^{\times 6})(1^3 \cdot 1^3)1^6 | 1^{\times 6} 1^6 \rangle$ is unity

— for clarity we drop the braces around representations in expressions of this sort.) Thus the required transformation reduces to one from the U_{18} coupled basis $U_{18} \times U_{18} \supset U_{18} \supset \ldots \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ to a basis coupled at the $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ level, namely $(U_{18} \supset \ldots \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C) \times (U_{18} \supset \ldots \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C) \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$. The transformation coefficients are just isoscalar factors for $U_{18} \supset \ldots \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$. Applying Racah's (1949) factorization lemma and using the factorization property of transformation coefficients for direct product groups (see Part II) we can write the transformation as

$$|(1^3, 1^3)1^6(1^{n_o}(I,\lambda_o^{CS}\ S_o\ \mu_o^C),\ 1^{n_s}(S,\ \lambda_s^{CS}\ S_s\ \mu_s^C))S\ 0^C\ i\rangle$$

$$= \sum |[1^3(1^{n_o'}(I',\lambda_o'\ S_o'\ \mu_o'),\ 1^{n_s'}(S',\lambda_s'\ S_s'\ \mu_s'))S'\mu'^C;$$

$$1^3(1^{n_o''}(I'',\lambda_o''\ S_o''\ \mu_o''),\ 1^{n_s''}(S'',\lambda_s''\ S_s''\ \mu_s''))S''\mu''^C]IS\ S0^C\ i\rangle$$

$$\times \langle(1^3(1^{n_o'}.1^{n_s'});\ 1^3(1^{n_o''}.1^{n_s''}))(1^{n_o}.1^{n_s})|(1^3,1^3)1^6(1^{n_o}.1^{n_s})\rangle$$

$$\times \langle(1^{n_o'}(I'\lambda_o');1^{n_o''}(I''\lambda_o''))(I\lambda_o)|(1^{n_o'},1^{n_o''})1^{n_o}(I\lambda_o)\rangle$$

$$\times \langle(1^{n_s'}(S'\lambda_s');1^{n_s''}(S''\lambda_s'')(S\lambda_s)|(1^{n_s'},1^{n_s''})1^{n_s}(S\lambda_s)\rangle$$

$$\times \langle(\lambda_o'\ S_o'\mu_o';\lambda_o''S_o''\mu_o'')S_o\mu_o|(\lambda_o',\lambda_o'')\lambda_o S_o\mu_o\rangle$$

$$x\ \langle(\lambda'_s S'_s \mu'_s; \lambda''_s S''_s \mu''_s) S_s \mu_s \,|\, (\lambda'_s, \lambda''_s) \lambda_s S_s \mu_s \rangle$$

$$x\ \langle((S'_o S'_s) S', (S''_o S''_s) S'') S \,|\, ((S'_o S''_o) S_o, (S'_s S''_s) S_s) S \rangle$$

$$x\ \langle((\mu'_o \mu'_s) \mu', (\mu''_o \mu''_s) \mu'') 0^C \,|\, ((\mu'_o \mu''_o) \mu_o, (\mu'_s \mu''_s) \mu_s) 0^C \rangle \ . \qquad (3.23)$$

Here, n_o is the number of ordinary quarks, n_s the number of strange quarks, I, S and S are respectively the isospin, strangeness and total spin quantum numbers, λ denotes representations of SU_6^{CS} and μ (0^C is the identity) representations of SU_3^C. Further subgroup labels complete the classification of the state and these are denoted collectively by $\mathcal{i}$. (Multiplicity does not arise in this example.) The summation is over all primed and double-primed labels. Note that the isoscalar factor for $G \times G \supset G$ is just a recoupling coefficient for four representations of G.

The proportion of baryon-baryon components in the q^6 state is obtained by considering the coefficients of the states with $\mu' = \mu'' = 0^C$. Of course, to obtain a complete dissociation one must completely decouple the states using the coupling coefficients for $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$. This means that different multiplet members will have different couplings to baryon-baryon channels, even in the degeneracy limit. It is also necessary to completely decouple the states in order to explicitly display the antisymmetry. (Note that a q^6 state totally antisymmetric under quark exchange will also be antisymmetric under baryon exchange.)

(ii) $q^2\bar{q}^2$

The only allowed dissociation is into two mesons. We can perform the desired transformation in two steps: first we separately decouple the q^2 and $\bar{q}^2$ parts of the system in a manner analogous to that described for q^6 — so that the standard basis is transformed into one described by $[(U_{18} \supset \ldots \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C) \times (U_{18} \supset \ldots \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C) \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C] \times [$ ... same again ...$] \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ — and then we perform a transformation within $(SU_2^I \times U_1^S \times SU_2^S \times SU_3^C)^{\times 4}$ which interchanges the order of coupling giving finally $(q\bar{q})(q\bar{q})$ states. This last step is just a recoupling of four $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ representations and is given by (suppressing parentage — i.e. higher group — labels for clarity)

$$|[(I_1S_1\tfrac{1}{2}1^C, I_2S_2\tfrac{1}{2}1^C)I_qS_qS_q\mu_q, (I_3S_3\tfrac{1}{2}1^{2C}, I_4S_4\tfrac{1}{2}1^{2C})I_{\bar{q}}S_{\bar{q}}S_{\bar{q}}\mu_{\bar{q}}]ISS0^C i\rangle$$

$$= \sum |[(I_1S_1\tfrac{1}{2}1^C, I_3S_3\tfrac{1}{2}1^{2C})I'S'S'\mu', (I_2S_2\tfrac{1}{2}1^C, I_4S_4\tfrac{1}{2}1^{2C})I"S"S"\mu"]ISS0^C i\rangle$$

$$\times \langle((I_1I_3)I', (I_2I_4)I")I|((I_1I_2)I_q, (I_3I_4)I_{\bar{q}})I\rangle$$

$$\times \langle((S_1S_3)S', (S_2S_4)S")S|((S_1S_2)S_q, (S_3S_4)S_{\bar{q}})S\rangle$$

$$\times \langle((\tfrac{1}{2}\tfrac{1}{2})S', (\tfrac{1}{2}\tfrac{1}{2})S")S|((\tfrac{1}{2}\tfrac{1}{2})S_q, (\tfrac{1}{2}\tfrac{1}{2})S_{\bar{q}})S\rangle$$

$$\times \langle((1\ 1^2)\mu', (1\ 1^2)\mu")0^C|((1\ 1)\mu_q, (1^2\ 1^2)\mu_{\bar{q}})0^C\rangle \quad . \qquad (3.24)$$

Most of the recoupling coefficients simplify as a consequence of the appearance of the identity.

The required proportion of meson-meson components is obtained by again considering the cases where $\mu' = \mu'' = 0^C$. (Note this time that a $q^2\bar{q}^2$ state totally antisymmetric under both quark and antiquark exchange will be symmetric under exchange of mesons.)

(iii) $q^4\bar{q}$

The only allowed dissociation is into a baryon plus a meson. Again the transformation is easiest understood when performed in two steps. First a single quark is decoupled from the q^4 system and then the resulting $[(U_{18}^{\times 3} \supset U_{18} \supset \ldots \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C) \times (U_{18} \supset \ldots \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C) \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C] \times (U_{18} \supset \ldots \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C) \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ states are recoupled at the $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ level using a recoupling coefficient for three representations. This last step is written explicitly as (where again we suppress parentage labels for clarity)

$$|[(I_3S_3S_3\mu_3, I_1S_1\tfrac{1}{2}1^C)I_qS_qS_q1^C; (I_{\bar{q}}S_{\bar{q}}\tfrac{1}{2}1^{2C})]ISS0^C i\rangle$$

$$= \sum |[I_3S_3S_3\mu_3, (I_1S_1\tfrac{1}{2}1^C, I_{\bar{q}}S_{\bar{q}}\tfrac{1}{2}1^{2C})I'S'S'\mu']ISS0^C i\rangle$$

$$\times \langle (I_3, (I_1I_{\bar{q}})I')I|((I_3I_1)I_q, I_{\bar{q}})I\rangle$$

$$\times \langle (S_3(S_1S_{\bar{q}})S')S|((S_3S_1)S_q, S_{\bar{q}})S\rangle$$

$$x \langle (S_3, (\tfrac{1}{2}\tfrac{1}{2})S')S | ((S_3\tfrac{1}{2})S_q, \tfrac{1}{2})S \rangle$$

$$x \langle (\mu_3, (1^C\ 1^{2C})\mu')0^C | ((\mu_3 1^C)1^C, 1^{2C})0^C \rangle \quad . \qquad (3.25)$$

This time the required proportion of baryon-meson states is obtained by setting $\mu_3 = \mu' = 0^C$.

(iv) $q^3\bar{q}^3$

This is the first configuration that has two available dissociation channels. However, the baryon-antibaryon channel is only available to a few states. Our classification scheme explicitly shows these. Perhaps of more importance is the three meson dissociation available to all states. This dissociation can be calculated by separately decoupling one quark from each of the q^3 and $\bar{q}^3$ systems to obtain $(q^2q)(\bar{q}^2\bar{q})$ labelled states and using the recoupling of four representations to give $(q^2\bar{q}^2)(q\bar{q})$ states. The $q^2\bar{q}^2$ states are then dissociated as previously described giving finally $((q\bar{q})(q\bar{q}))(q\bar{q})$ states. The reader is spared details.

These dissociation transformations are analogous to fractional parentage calculations in atomic and nuclear physics (Racah 1942a,b,1943,1949). For example in atomic physics one has an electron with two spin states and $2\ell+1$ orbital states transforming as $SU_2^S \times SO_3^{orbital}$ which can be embedded in $U_{4\ell+2}$ via either (e.g. Wybourne 1970)

$$U_{4\ell+2} \supset Sp_{4\ell+2} \supset SU_2^S \times (SO_{2\ell+1} \supset SO_3^{orbital})$$

or

$$U_{4\ell+2} \supset SU_2 \times SU_{2\ell+1} \supset SU_2^S \times (SO_{2\ell+1} \supset SO_3^{orbital}).$$

Thus a configuration of n electrons will transform as $\{1^n\}$ of $U_{4\ell+2} \subset U_{4\ell+2}^{\times n}$. The groups higher in the scheme than (the physically important) $SU_2^S \times SO_3^{orbital}$ provide what are called parentage labels. Under some circumstances (e.g. Judd 1963) it is desirable to express the n-electron wavefunction as a product of a wavefunction for n-1 electrons and a wavefunction for a single electron. The transformation coefficients are known as "coefficients of fractional parentage". Of course they are just isoscalar factors for whichever group chain is used in classifying the states, although this fact was not appreciated until some time after Racah's pioneering work.

It is particularly important to appreciate that although the required transformation simply involves decoupling one or more particles from a many particle state, it is impossible to perform this decoupling without taking proper account of the parentage of the system. One must apply the appropriate weighting factors (fractional parentage coefficients) to the different parents. Notice that a complete parentage classification takes full care of antisymmetry requirements, both in the original classification and during the fractional parentage calculation.

(e) 3jm Factors, 6j and 9j Symbols

Use of the transformations (3.23-25) is simplified by expressing the coefficients in terms of the more symmetric 3jm factors, 6j symbols and 9j symbols. These entities and

their evaluation are discussed in Part II. All the groups considered here are quasiambivalent and no nonsimple-phase representations arise in the calculations performed. The canonical choices of permutation and conjugation matrices (Derome 1966, Butler 1975) lead to

$$\langle(\lambda_1\lambda_2)r\lambda a\mu\,|\,(\lambda_1 a_1\mu_1,\lambda_2 a_2\mu_2)s\mu\rangle$$

$$= |\lambda|^{\frac{1}{2}}|\mu|^{-\frac{1}{2}}(\lambda)^{a\mu,a^*\mu^*}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda^* \\ a_1\mu_1 & a_2\mu_2 & a^*\mu^*\end{Bmatrix}_s^r . \qquad (3.26)$$

$$\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3)r\lambda\,|\,(\lambda_1,(\lambda_2\lambda_3)r_{23}\lambda_{23})s\lambda\rangle$$

$$= |\lambda_{12},\lambda_{23}|^{\frac{1}{2}}\{\lambda_2^*\}\{\lambda_1\lambda_{12}^*\lambda_2 r_{12}\}\{\lambda_3\lambda_{12}\lambda^* r\}$$

$$\begin{Bmatrix}\lambda_1 & \lambda_{23} & \lambda^* \\ \lambda_3^* & \lambda_{12} & \lambda_2\end{Bmatrix}r_{12}r_{23}rs \qquad (3.27)$$

$$\langle((\lambda_1\lambda_2)r_{12}\lambda_{12},(\lambda_3\lambda_4)r_{34}\lambda_{34})r\lambda\,|\,((\lambda_1\lambda_3)r_{13}\lambda_{13},(\lambda_2\lambda_4)r_{24}\lambda_{24})s\lambda\rangle$$

$$= |\lambda_{12},\lambda_{34},\lambda_{13},\lambda_{24}|^{\frac{1}{2}}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_{12}^* \\ \lambda_3 & \lambda_4 & \lambda_{34}^* \\ \lambda_{13}^* & \lambda_{24}^* & \lambda\end{Bmatrix}\begin{matrix}r_{12}\\ r_{34}\\ r\end{matrix}$$

$$r_{13}\quad r_{24}\quad s \qquad (3.28)$$

For SU_2, it is customary to insert some additional historical phases (Condon and Shortley, 1935) in equations (3.26) and (3.27) However, it will cause us less bother if we adopt the sensible phase relationship (Butler 1975) for all groups.

The trivial 9j symbol reduces to a 6j symbol

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_4 & \lambda_{12}^* \\ \lambda_{13} & \lambda_{13}^* & 0 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ 0 \end{matrix} = |\lambda_{12},\lambda_{13}|^{-\frac{1}{2}} \{\lambda_2\}\{\lambda_{13}\}\{\lambda_2\lambda_4\lambda_{13}^* r_4\}\{\lambda_3\lambda_4\lambda_{12}^* r_2\}$$

$$r_3 \quad r_4 \quad 0$$

$$\times \begin{Bmatrix} \lambda_1 & \lambda_3 & \lambda_{13} \\ \lambda_4 & \lambda_2^* & \lambda_{12} \end{Bmatrix} r_1 r_2 r_3 r_4 \tag{3.29}$$

and non-trivial 9j symbols can be expressed as a sum over products of three 6j symbols

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \lambda_9 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} = \sum_{\lambda t_1 t_2 t_3} |\lambda| \{\lambda\}$$

$$s_1 \quad s_2 \quad s_3$$

$$\times \begin{Bmatrix} \lambda_1 & \lambda_4 & \lambda_7 \\ \lambda_8^* & \lambda_9 & \lambda \end{Bmatrix} t_2 t_1 r_3 s_1 \begin{Bmatrix} \lambda_2 & \lambda_5 & \lambda_8 \\ \lambda_4 & \lambda & \lambda_6^* \end{Bmatrix} t_3 r_2 t_1 s_2 \begin{Bmatrix} \lambda_3 & \lambda_6 & \lambda_9 \\ \lambda & \lambda_1^* & \lambda_2 \end{Bmatrix} r_1 t_3 t_2 s_3 \ . \tag{3.30}$$

The use of 3jm factors and 6j and 9j symbols not only simplifies the calculation of the required coefficients (see Part II) but allows one to fully exploit symmetries when performing the desired transformations. As an example consider the q^6 state

$$|^3I(0^4S^2)> = |q^6(1^4(0,2^2\ ^31),1^2(-2,1^2\ ^31^2))^30> \ . \tag{3.31}$$

This is the $(I,S) = (0,-2)$ member of a flavour octet denoted as H* by Jaffe (1977c). Using the tables in Part II one finds that in the dissociation basis it takes the appearance (Bickerstaff and Wybourne, 1980b)

$$|{}^3I({}^4O{}^2S)\rangle =$$

$$\frac{1}{3}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^3(\tfrac{1}{2},21\ {}^421),0(0,0\ {}^10))^421;q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^421]0,-2,{}^30\rangle$$
$$+|[q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^421;q^3(1^3(\tfrac{1}{2},21^421),0(0,0\ {}^10))^421]0,-2,{}^30\rangle\}$$

$$-\sqrt{\frac{1}{45}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^3(\tfrac{1}{2},21\ {}^421),0(0,0\ {}^10))^421;q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^221]0,-2,{}^30\rangle$$
$$-|[q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^221;q^3(1^3(\tfrac{1}{2},21\ {}^421),0(0,0\ {}^10))^421]0,-2,{}^30\rangle\}$$

$$-\sqrt{\frac{4}{45}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^3(\tfrac{1}{2},21\ {}^221),0(0,0\ {}^10))^221;q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^221]0,-2,{}^30\rangle$$
$$+|[q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^221;q^3(1^3(\tfrac{1}{2},21\ {}^221),0(0,0\ {}^10))^221]0,-2,{}^30\rangle\}$$

$$-\sqrt{\frac{2}{45}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^3(\tfrac{1}{2},21\ {}^221),0(0,0\ {}^10))^221;q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^421]0,-2,{}^30\rangle$$
$$-|[q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^421;q^3(1^3(\tfrac{1}{2},21\ {}^221),0(0,0\ {}^10))^221]0,-2,{}^30\rangle\}$$

$$+\sqrt{\frac{4}{45}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^3(\tfrac{1}{2},21\ {}^20),0(0,0\ {}^10))^20;q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^20]0,-2,{}^30\rangle$$
$$+|[q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^20;q^3(1^3(\tfrac{1}{2},21\ {}^20),0(0,0\ {}^10))^20]0,-2,{}^30\rangle\}$$

$$+\sqrt{\frac{2}{45}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^3(\tfrac{1}{2},21\ {}^20),0(0,0\ {}^10))^20;q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2\ ))^40]0,-2,{}^30\rangle$$
$$-|[q^3(1(\tfrac{1}{2},1\ {}^21),1^2(-2,1^2\ {}^31^2))^40;q^3(1^3(\tfrac{1}{2},21\ {}^20),0(0,0\ {}^10))^20]0,-2,{}^30\rangle\}$$

$$+\sqrt{\frac{1}{5}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^2(0,2\ {}^32),1(-1,1\ {}^21))^221;\ q^3(1^2(0,2\ {}^11^2),1(-1,1\ {}^21))^221]0,-2,\ {}^30\rangle$$
$$-|[q^3(1^2(0,2\ {}^11^2),1(-1,1\ {}^21))^221;q^3(1^2(0,2\ {}^32),1(-1,1\ {}^21))^221]0,-2,{}^30\rangle\}$$

$$+\sqrt{\frac{1}{10}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^2(0,2\ {}^32),1(-1,1\ {}^21))^421;q^3(1^2(0,2\ {}^11^2),1(-1,1\ {}^21))^221]0,-2,{}^30\rangle$$
$$+|[q^3(1^2(0,2\ {}^11^2),1(-1,1\ {}^21))^221;q^3(1^2(0,2\ {}^32),1(-1,1\ {}^21))^421]0,-2,\ {}^30\rangle\}$$

$$+\sqrt{\frac{1}{15}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^221;q^3(1^2(1,1^2\ {}^12),1(-1,1\ {}^21))^221]0,-2,{}^30\rangle$$
$$-|[q^3(1^2(1,1^2\ {}^12),1(-1,1\ {}^21))^221;q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^221]0,-2,\ {}^30\rangle\}$$

$$+\sqrt{\frac{1}{30}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^421;q^3(1^2(1,1^2\ {}^11^2),1(-1,1\ {}^21))^221]0,-2,{}^30\rangle$$
$$+|[q^3(1^2(1,1^2\ {}^11^2),1(-1,1\ {}^21))^221;q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^421]0,-2,{}^30\rangle\}$$

$$-\sqrt{\frac{1}{15}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^20;q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^40]0,-2,\ {}^30\rangle$$
$$+|[q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^40;q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^20]0,-2,{}^30\rangle\}$$

$$-\sqrt{\frac{2}{15}}\cdot\frac{1}{\sqrt{2}}\{|[q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^221;q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^421]0,-2,{}^30\rangle$$
$$+|[q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^421;q^3(1^2(1,1^2\ {}^31^2),1(-1,1\ {}^21))^221]0,-2,{}^30\rangle\}$$
(3.32)

Comparing with table I5 we see that H* consists of 8.9% NΞ, 4.4% NΞ* and 6.7% ΣΣ* (the fifth, sixth and eleventh terms respectively) with colour octet-octet components comprising the remaining 80.0%. Notice that the state in the new basis is indeed antisymmetric under baryon exchange — this requirement was taken care of by the permutation symmetry of the 3jm factors. Consider for instance the NΞ term:

$$\sqrt{\frac{4}{45}}\cdot\frac{1}{\sqrt{2}}\{|(N\Xi)\rangle+|(\Xi N)\rangle\}\quad.$$

The states $|(N\Xi)\rangle$ and $|(\Xi N)\rangle$ are coupled at the $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ level. Under permutation of N and Ξ this coupling is even in colour, spin and strangeness but odd in isospin and therefore $|(N\Xi)0,-2,{}^30\rangle \leftrightarrow -|(\Xi N)0,-2,{}^30\rangle$. Hence, the NΞ term changes sign on exchange of baryons and is thus antisymmetric. In contrast $|(N\Xi^*)0,-2,{}^30\rangle \leftrightarrow +|(\Xi^*N)0,-2,{}^30\rangle$ under baryon exchange but the form of the NΞ* term is such that the NΞ* state is still an antisymmetric one. Permutation symmetry also rules out any ΣΣ or Σ*Σ* terms which would

necessarily be symmetric. (Specifically, one finds that an SU_2 9j symbol vanishes because of its odd permutational symmetry — the permutational symmetries of the general 9j symbol simply involve multiplying the transposed symbol by the 3j phases for the affected triads, see for example Butler, 1980b.) In addition, note that H* could not possibly couple to $\Lambda\Lambda$ because of the group theoretic selection rule which operates on the spin of the ordinary quark, cf. Jaffe (1977c).

Table I1: Branching rules for $U_{18} \rightarrow SU_{12} \times U_6$

$\lvert 1^n \rvert_{U_{18}}$	$\{1^n\}$	$\sum \{1^{n-x}\}\{1^x\}$
1	$\{0\}$	$\{0\}\{0\}$
18	$\{1\}$	$\{1\}\{0\} + \{0\}\{1\}$
153	$\{1^2\}$	$\{1^2\}\{0\} + \{1\}\{1\} + \{0\}\{1^2\}$
816	$\{1^3\}$	$\{1^3\}\{0\} + \{1^2\}\{1\} + \{1\}\{1^2\} + \{0\}\{1^3\}$
3060	$\{1^4\}$	$\{1^4\}\{0\} + \{1^3\}\{1\} + \{1^2\}\{1^2\} + \{1\}\{1^3\} + \{0\}\{1^4\}$
8568	$\{1^5\}$	$\{1^5\}\{0\} + \{1^4\}\{1\} + \{1^3\}\{1^2\} + \{1^2\}\{1^3\} + \{1\}\{1^4\} + \{0\}\{1^5\}$
18564	$\{1^6\}$	$\{1^6\}\{0\} + \{1^5\}\{1\} + \{1^4\}\{1^2\} + \{1^3\}\{1^3\} + \{1^2\}\{1^4\} + \{1\}\{1^5\} + \{0\}\{1^6\}$
31824	$\{1^7\}$	$\{1^7\}\{0\} + \{1^6\}\{1\} + \{1^5\}\{1^2\} + \{1^4\}\{1^3\} + \{1^3\}\{1^4\} + \{1^2\}\{1^5\} + \{1\}\{1^6\}$
43758	$\{1^8\}$	$\{1^8\}\{0\} + \{1^7\}\{1\} + \{1^6\}\{1^2\} + \{1^5\}\{1^3\} + \{1^4\}\{1^4\} + \{1^3\}\{1^5\} + \{1^2\}\{1^6\}$
48620	$\{1^9\}$	$\{1^9\}\{0\} + \{1^8\}\{1\} + \{1^7\}\{1^2\} + \{1^6\}\{1^3\} + \{1^5\}\{1^4\} + \{1^4\}\{1^5\} + \{1^3\}\{1^6\}$
43758	$\{1^{10}\}$	$\{1^{10}\}\{0\} + \{1^9\}\{1\} + \{1^8\}\{1^2\} + \{1^7\}\{1^3\} + \{1^6\}\{1^4\} + \{1^5\}\{1^5\} + \{1^4\}\{1^6\}$
31824	$\{1^{11}\}$	$\{1^{11}\}\{0\} + \{1^{10}\}\{1\} + \{1^9\}\{1^2\} + \{1^8\}\{1^3\} + \{1^7\}\{1^4\} + \{1^6\}\{1^5\} + \{1^5\}\{1^6\}$
18564	$\{1^{12}\}$	$\{0\}\{0\} + \{1^{11}\}\{1\} + \{1^{10}\}\{1^2\} + \{1^9\}\{1^3\} + \{1^8\}\{1^4\} + \{1^7\}\{1^5\} + \{1^6\}\{1^6\}$
8568	$\{1^{13}\}$	$+ \{0\}\{1\} + \{1^{11}\}\{1^2\} + \{1^{10}\}\{1^3\} + \{1^9\}\{1^4\} + \{1^8\}\{1^5\} + \{1^7\}\{1^6\}$
3060	$\{1^{14}\}$	$+ \{0\}\{1^2\} + \{1^{11}\}\{1^3\} + \{1^{10}\}\{1^4\} + \{1^9\}\{1^5\} + \{1^8\}\{1^6\}$
816	$\{1^{15}\}$	$\{0\}\{1^3\} + \{1^{11}\}\{1^4\} + \{1^{10}\}\{1^5\} + \{1^9\}\{1^6\}$
153	$\{1^{16}\}$	$\{0\}\{1^4\} + \{1^{11}\}\{1^5\} + \{1^{10}\}\{1^6\}$
18	$\{1^{17}\}$	$\{0\}\{1^5\} + \{1^{11}\}\{1^6\}$
1	$\{1^{18}\}$	$\{0\}\{1^6\}$

Table I2: Branching rules for $SU_{12} \rightarrow SU_2 \times SU_6$

$\lvert 1^n \rvert_{SU_{12}}$	$\{1^n\}$	$\sum \{\tilde{\xi}\}\{\xi\}$
1	{0 }	{0}{0 }
12	{1 }	{1}{1 }
66	$\{1^2\}$	{0}{2 } + {2}$\{1^2$ }
220	$\{1^3\}$	{1}{21 } + {3}$\{1^3$ }
495	$\{1^4\}$	{0}$\{2^2$ } + {2}$\{21^2$ } + {4}$\{1^4$ }
792	$\{1^5\}$	{1}$\{2^21\}$ + {3}$\{21^3$ } + {5}$\{1^5$ }
924	$\{1^6\}$	{0}$\{2^3$ } + {2}$\{2^21^2\}$ + {4}$\{21^4\}$ + {6}{0}

Table I3: Representation equivalences for $U_6 \cong U_1 \times SU_6$

$\lvert 1^n \rvert_{U_6}$	$\{1^n\}$	$\{n\}\{1^n\}$
1	{0 }	{0}{0}
6	{1 }	{1}{1}
15	$\{1^2\}$	{2}$\{1^2\}$
20	$\{1^3\}$	{3}$\{1^3\}$
15	$\{1^4\}$	{4}$\{1^4\}$
6	$\{1^5\}$	{5}$\{1^5\}$
1	$\{1^6\}$	{6}{0 }

Table I4: Branching rules for $SU_6 \rightarrow SU_2 \times SU_3$

$\lvert\lambda\rvert$	$\{\lambda\}$	$\sum \lvert S\rvert_{\{\mu\}}$
1	$\{0\}$	$^1\{0\}$
6	$\{1\}$	$^2\{1\}$
6	$\{1^5\}$	$^2\{1^2\}$
15	$\{1^2\}$	$^3\{1^2\} + {}^1\{2\}$
15	$\{1^4\}$	$^3\{1\} + {}^1\{2^2\}$
21	$\{2\}$	$^3\{2\} + {}^1\{1^2\}$
21	$\{2^5\}$	$^3\{2^2\} + {}^1\{1\}$
35	$\{21^4\}$	$^3[\{21\} + \{0\}] + {}^1\{21\}$
20	$\{1^3\}$	$^4\{0\} + {}^2\{21\}$
70	$\{21\}$	$^4\{21\} + {}^2[\{3\} + \{21\} + \{0\}]$
70	$\{2^4 1\}$	$^4\{21\} + {}^2[\{3^2\} + \{21\} + \{0\}]$
84	$\{21^3\}$	$^4[\{2\} + \{1^2\}] + {}^2[\{32\} + \{2\} + \{1^2\}]$
84	$\{2^2 1^3\}$	$^4[\{2^2\} + \{1\}] + {}^2[\{31\} + \{2^2\} + \{1\}]$
56	$\{3\}$	$^4\{3\} + {}^2\{21\}$
56	$\{3^5\}$	$^4\{3^2\} + {}^2\{21\}$
120	$\{31^4\}$	$^4[\{31\} + \{1\}] + {}^2[\{31\} + \{2^2\} + \{1\}]$
120	$\{32^4\}$	$^4[\{32\} + \{1^2\}] + {}^2[\{32\} + \{2\} + \{1^2\}]$
105	$\{21^2\}$	$^5\{1\} + {}^3[\{31\} + \{2^2\} + \{1\}] + {}^1[\{31\} + \{1\}]$
105	$\{2^3 1^2\}$	$^5\{1^2\} + {}^3[\{32\} + \{2\} + \{1^2\}] + {}^1[\{32\} + \{1^2\}]$
105	$\{2^2\}$	$^5\{2^2\} + {}^3[\{31\} + \{1\}] + {}^1[\{4\} + \{2^2\}]$
105	$\{2^4\}$	$^5\{2\} + {}^3[\{32\} + \{1^2\}] + {}^1[\{4^2\} + \{2\}]$
189	$\{2^2 1^2\}$	$^5[\{21\} + \{0\}] + {}^3[\{3\} + \{3^2\} + 2\{21\}]$ $+{}^1[\{42\} + \{21\} + \{0\}]$
210	$\{31\}$	$^5\{31\} + {}^3[\{4\} + \{31\} + \{2^2\} + \{1\}] + {}^1[\{31\} + \{1\}]$
210	$\{3^4 2\}$	$^5\{32\} + {}^3[\{4^2\} + \{32\} + \{2\} + \{1^2\}] + {}^1[\{32\} + \{1^2\}]$
280	$\{31^3\}$	$^5[\{3\} + \{21\}] + {}^3[\{42\} + \{3\} + 2\{21\} + \{0\}]$ $^1[\{3\} + \{3^2\} + \{21\}]$
280	$\{3^2 2^3\}$	$^5[\{3^2\} + \{21\}] + {}^3[\{42\} + \{3^2\} + 2\{21\} + \{0\}]$ $+{}^1[\{3^2\} + \{3\} + \{21\}]$

384 $\{321^3\}$ $^5[\{32\} + \{2\} + \{1^2\}] + {}^3[\{41\} + 2\{32\} + 2\{2\} + 2\{1^2\}]$
$+^1[\{41\} + \{32\} + \{2\} + \{1^2\}]$

384 $\{32^31\}$ $^5[\{31\} + \{2^2\} + \{1\}] + {}^3[\{42\} + 2\{31\} + 2\{2^2\} + 2\{1\}]$
$+^1[\{43\} + \{31\} + \{2^2\} + \{1\}]$

126 $\{4\}$ $^5\{4\} + {}^3\{31\} + {}^1\{2^2\}$

126 $\{4^5\}$ $^5\{4^2\} + {}^3\{32\} + {}^1\{2\}$

315 $\{41^4\}$ $^5[\{41\} + \{2\}] + {}^3[\{41\} + \{32\} + \{2\} + \{1^2\}]$
$+^1[\{32\} + \{2\}]$

315 $\{43^4\}$ $^5[\{43\} + \{2^2\}] + {}^3[\{43\} + \{31\} + \{2^2\} + \{1\}]$
$+^1[\{31\} + \{2^2\}]$

405 $\{42^4\}$ $^5[\{42\} + \{21\} + \{0\}] + {}^3[\{42\} + \{3\} + \{3^2\} + 2\{21\}]$
$+^1[\{42\} + \{21\} + \{0\}]$

210 $\{2^21\}$ $^6\{1^2\} + {}^4[\{32\} + \{2\} + \{1^2\}]$
$+^2[\{41\} + \{32\} + \{2\} + \{1^2\}]$

210 $\{2^31\}$ $^6\{1\} + {}^4[\{31\} + \{2^2\} + \{1\}]$
$+^2[\{43\} + \{31\} + \{2^2\} + \{1\}]$

540 $\{32^21^2\}$ $^6[\{21\} + \{0\}] + {}^4[\{42\} + \{3\} + \{3^2\} + 3\{21\} + \{0\}]$
$+^2[2\{42\} + \{3\} + \{3^2\} + 3\{21\} + \{0\}]$

175 $\{2^3\}$ $^7\{0\} + {}^5\{21\} + {}^3[\{42\} + \{21\} + \{0\}] + {}^1[\{3\} + \{3^2\}]$

896 $\{321\}$ $^7\{21\} + {}^5[\{42\} + \{3\} + \{3^2\} + 2\{21\} + \{0\}]$
$+^3[\{51\} + 2\{42\} + 2\{3\} + \{3^2\} + 3\{21\} + \{0\}]$
$+^1[\{51\} + \{42\} + \{3\} + 2\{21\}]$

896 $\{3^321\}$ $^7\{21\} + {}^5[\{42\} + \{3\} + \{3^2\} + 2\{21\} + \{0\}]$
$+^3[\{54\} + 2\{42\} + 2\{3\} + \{3^2\} + 3\{21\} + \{0\}]$
$^1[\{54\} + \{42\} + \{3^2\} + 2\{21\}]$

490 $\{3^2\}$ $^7\{3^2\} + {}^5[\{42\} + \{21\}] + {}^3[\{51\} + \{3\} + \{3^2\} + \{21\}]$
$+^1[\{6\} + \{42\} + \{0\}]$

490 $\{3^4\}$ $^7\{3\} + {}^5[\{42\} + \{21\}] + {}^3[\{54\} + \{3\} + \{3^2\} + \{21\}]$
$+^1[\{6^2\} + \{42\} + \{0\}]$

This table is complete up to power 4. (See Part II for a discussion of the concept of power.) The colour-singlet content of a few additional power 5 and 6 representations has been given by Wybourne (1978a).

Table I5 : Q^3 basis vectors

Name	$\|q^3(1^{n_o}(I,\lambda_o^{CS}\ \|s_o\|_{\mu_o}^{C}),1^{n_s}(S,\lambda_s^{CS}\ \|s_s\|_{\mu_s}^{C}))\ \|s\|_0^{C}\rangle$
N	$\|q^3(1^3(\frac{1}{2},21\ ^20),0(0,0\ ^10))^20\rangle$
Λ	$\|q^3(1^2(0,2\ ^11^2),1(-1,1\ ^21))^20\rangle$
Σ	$\|q^3(1^2(1,1^2\ ^31^2),1(-1,1\ ^21))^20\rangle$
Ξ	$\|q^3(1(\frac{1}{2},1\ ^21),1^2(-2,1^2\ ^31^2))^20\rangle$
Δ	$\|q^3(1^3(\frac{3}{2},1^3\ ^40),0(0,0\ ^10))^40\rangle$
Σ^*	$\|q^3(1^2(1,1^2\ ^31^2),1(-1,1\ ^21))^40\rangle$
Ξ^*	$\|q^3(1(\frac{1}{2},1\ ^21),1^2(-2,1^2\ ^31^2))^40\rangle$
Ω	$\|q^3(0(0,0\ ^10),1^3(-3,1^3\ ^40))^40\rangle$

Table I6 : $Q\bar{Q}$ basis vectors

Name	$\vert[q(1^{n_o}(I_q,\lambda_o^{CS}\ \vert s_o\vert_{\mu_o}^{C}),1^{n_s}(S_q,\lambda_s^{CS}\ \vert s_s\vert_{\mu_s}^{C}))\ \vert s_q\vert_{\mu_q}^{C};\bar{q}(1^{n_{\bar{o}}^{*}}(I_{\bar{q}},\lambda_{\bar{o}}^{CS}\ \vert s_{\bar{o}}\vert_{\mu_{\bar{o}}}^{C}),1^{n_{\bar{s}}^{*}}(S_{\bar{q}},\lambda_{\bar{s}}^{CS}\ \vert s_{\bar{s}}\vert_{\mu_{\bar{s}}}^{C}))\ \vert s_{\bar{q}}\vert_{\mu_{\bar{q}}}^{C}]I,S,\ \vert s\vert_0^C\rangle$
η_o	$\vert[q(1(\frac{1}{2},1\ ^2\!1),0(0,0\ ^1\!0))^2\!1;\bar{q}(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),0(0,0\ ^1\!0))^2\!1^2]0,0,\ ^1\!0\rangle$
π	$\vert[q(1(\frac{1}{2},1\ ^2\!1),0(0,0\ ^1\!0))^2\!1;\bar{q}(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),0(0,0\ ^1\!0))^2\!1^2]1,0,\ ^1\!0\rangle$
K	$\vert[q(1(\frac{1}{2},1\ ^2\!1),0(0,0\ ^1\!0))^2\!1;\bar{q}(0(0,0\ ^1\!0),1^*\ (1,1^5\ ^2\!1^2))^2\!1^2]\frac{1}{2},1,\ ^1\!0\rangle$
$\bar{K}$	$\vert[q(0(0,0\ ^1\!0),1(-1,1\ ^2\!1))^2\!1;\bar{q}(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),0(0,0\ ^1\!0))^2\!1^2]\frac{1}{2},-1,\ ^1\!0\rangle$
η_s	$\vert[q(0(0,0\ ^1\!0),1(-1,1\ ^2\!1))^2\!1;\bar{q}(0(0,0\ ^1\!0),1^*\ (1,1^5\ ^2\!1^2))^2\!1^2]0,0,\ ^1\!0\rangle$
ω_o	$\vert[q(1(\frac{1}{2},1\ ^2\!1),0(0,0\ ^1\!0))^2\!1;\bar{q}(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),0(0,0\ ^1\!0))^2\!1^2]0,0,\ ^3\!0\rangle$
ρ	$\vert[q(1(\frac{1}{2},1\ ^2\!1),0(0,0\ ^1\!0))^2\!1;\bar{q}(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),0(0,0\ ^1\!0))^2\!1^2]1,0,\ ^3\!0\rangle$
K*	$\vert[q(1(\frac{1}{2},1\ ^2\!1),0(0,0\ ^1\!0))^2\!1;\bar{q}(0(0,0\ ^1\!0),1^*\ (1,1^5\ ^2\!1^2))^2\!1^2]\frac{1}{2},1,\ ^3\!0\rangle$
$\bar{K}$*	$\vert[q(0(0,0\ ^1\!0),1(-1,1\ ^2\!1))^2\!1;\bar{q}(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),0(0,0\ ^1\!0))^2\!1^2]\frac{1}{2},-1,\ ^3\!0\rangle$
ϕ_s	$\vert[q(0(0,0\ ^1\!0),1(-1,1\ ^2\!1))^2\!1;\bar{q}(0(0,0\ ^1\!0),1^*\ (1,1^5\ ^2\!1^2))^2\!1^2]0,0,\ ^3\!0\rangle$

Table I7: Q^6 basis vectors

Reduced colour-spin identification	$\|q^6(1^{n_o}(I,\lambda_o^{CS}\ \|s_o\|_{\mu_o}^{C}),1^{n_s}(S,\lambda_s^{CS}\ \|s_s\|_{\mu_s}^{C}))\ \|s\|_0^C\rangle$
^1A	$\|q^6(1^6(1,2^21^2\ {}^10),0(0,0\ {}^10))^10\rangle$
^1B	$\|q^6(1^6(3,0\ {}^10),0(0,0\ {}^10))^10\rangle$
^1E	$\|q^6(1^5(\tfrac{1}{2},2^21\ {}^21^2),1(-1,1\ {}^21))^10\rangle$
^1F	$\|q^6(1^5(\tfrac{3}{2},21^3\ {}^21^2),1(-1,1\ {}^21))^10\rangle$
^1G	$\|q^6(1^5(\tfrac{5}{2},1^5\ {}^21^2),1(-1,1\ {}^21))^10\rangle$
$^1I_1$	$\|q^6(1^4(0,2^2\ {}^31),1^2(-2,1^2\ {}^31^2))^10\rangle$
$^1I_2$	$\|q^6(1^4(0,2^2\ {}^12^2),1^2(-2,1^2\ {}^12))^10\rangle$
^1H	$\|q^6(1^4(1,21^2\ {}^31),1^2(-2,1^2\ {}^31^2))^10\rangle$
$^1J_1$	$\|q^6(1^4(2,1^4\ {}^31),1^2(-2,1^2\ {}^31^2))^10\rangle$
$^1J_2$	$\|q^6(1^4(2,1^4\ {}^12^2),1^2(-2,1^2\ {}^12))^10\rangle$
^1L	$\|q^6(1^3(\tfrac{1}{2},21\ {}^221),1^3(-3,1^3\ {}^221))^10\rangle$
$^1M_1$	$\|q^6(1^3(\tfrac{3}{2},1^3\ {}^40),1^3(-3,1^3\ {}^40))^10\rangle$
$^1M_2$	$\|q^6(1^3(\tfrac{3}{2},1^3\ {}^221),1^3(-3,1^3\ {}^221))^10\rangle$
$^1J_1$	$\|q^6(1^2(1,1^2\ {}^31^2),1^4(-4,1^4\ {}^31))^10\rangle$
$^1J_2$	$\|q^6(1^2(1,1^2\ {}^12),1^4(-4,1^4\ {}^12^2))^10\rangle$
^1G	$\|q^6(1(\tfrac{1}{2},1\ {}^21),1^5(-5,1^5\ {}^21^2))^10\rangle$
^1B	$\|q^6(0(0,0\ {}^10),1^6(-6,0\ {}^10))^10\rangle$
^3C	$\|q^6(1^6(0,2^3\ {}^30),0(0,0\ {}^10))^30\rangle$
^3D	$\|q^6(1^6(2,21^4\ {}^30),0(0,0\ {}^10))^30\rangle$
$^3E_1$	$\|q^6(1^5(\tfrac{1}{2},2^21\ {}^21^2),1(-1,1\ {}^21))^30\rangle$
$^3E_2$	$\|q^6(1^5(\tfrac{1}{2},2^21\ {}^41^2),1(-1,1\ {}^21))^30\rangle$
$^3F_1$	$\|q^6(1^5(\tfrac{3}{2},21^3\ {}^21^2),1(-1,1\ {}^21))^30\rangle$
$^3F_2$	$\|q^6(1^5(\tfrac{3}{2},21^3\ {}^41^2),1(-1,1\ {}^21))^30\rangle$
^3G	$\|q^6(1^5(\tfrac{5}{2},1^5\ {}^21^2),1(-1,1\ {}^21))^30\rangle$
^3I	$\|q^6(1^4(0,2^2\ {}^31),1^2(-2,1^2\ {}^31^2))^30\rangle$
$^3H_1$	$\|q^6(1^4(1,21^2\ {}^51),1^2(-2,1^2\ {}^31^2))^30\rangle$
$^3H_2$	$\|q^6(1^4(1,21^2\ {}^31),1^2(-2,1^2\ {}^31^2))^30\rangle$
$^3H_3$	$\|q^6(1^4(1,21^2\ {}^11),1^2(-2,1^2\ {}^31^2))^30\rangle$
$^3H_4$	$\|q^6(1^4(1,21^2\ {}^32^2),1^2(-2,1^2\ {}^12))^30\rangle$
^3J	$\|q^6(1^4(2,1^4\ {}^31),1^2(-2,1^2\ {}^31^2))^30\rangle$
$^3L_1$	$\|q^6(1^3(\tfrac{1}{2},21\ {}^20),1^3(-3,1^3\ {}^40))^30\rangle$
$^3L_2$	$\|q^6(1^3(\tfrac{1}{2},21\ {}^221),1^3(-3(1^3\ {}^221))^30\rangle$
$^3L_3$	$\|q^6(1^3(\tfrac{1}{2},21\ {}^421),1^3(-3,1^3\ {}^221))^30\rangle$

$^3M_1$ $|q^6(1^3(\frac{3}{2},1^3\ ^4 0),1^3(-3,1^3\ ^4 0))^3 0\rangle$

$^3M_2$ $|q^6(1^3(\frac{3}{2},1^3\ ^2 21),1^3(-3,1^3\ ^2 21))^3 0\rangle$

$^3K_1$ $|q^6(1^2(0,2\ ^1 1^2),1^4(-4,1^4\ ^3 1))^3 0\rangle$

$^3K_2$ $|q^6(1^2(0,2\ ^3 2),1^4(-4,1^4\ ^1 2^2))^3 0\rangle$

^3J $|q^6(1^2(1,1^2\ ^3 1^2),1^4(-4,1^4\ ^3 1))^3 0\rangle$

^3G $|q^6(1(\frac{1}{2},1\ ^2 1),1^5(-5,1^5\ ^2 1^2)^3 0\rangle$

^5A $|q^6(1^6(1,2^2 1^2\ ^5 0),0(0,0\ ^1 0)^5 0\rangle$

$^5E_1$ $|q^6(1^5(\frac{1}{2},2^2 1\ ^4 1^2),1(-1,1\ ^2 1)^5 0\rangle$

$^5E_2$ $|q^6(1^5(\frac{1}{2},2^2 1\ ^6 1^2),1(-1,1\ ^2 1)^5 0\rangle$

^5F $|q^6(1^5(\frac{3}{2},21^3\ ^4 1^2),1(-1,1\ ^2 1)^5 0\rangle$

$^5I_1$ $|q^6(1^4(0,2^2\ ^3 1),1^2(-2,1^2\ ^3 1^2)^5 0\rangle$

$^5I_2$ $|q^6(1^4(0,2^2\ ^5 2^2),1^2(-2,1^2\ ^1 2)^5 0\rangle$

$^5H_1$ $|q^6(1^4(1,21^2\ ^5 1),1^2(-2,1^2\ ^3 1^2)^5 0\rangle$

$^5H_2$ $|q^6(1^4(1,21^2\ ^3 1),1^2(-2,1^2\ ^3 1^2)^5 0\rangle$

^5J $|q^6(1^4(2,1^4\ ^3 1),1^2(-2,1^2\ ^3 1^2)^5 0\rangle$

$^5L_1$ $|q^6(1^3(\frac{1}{2},21\ ^2 0),1^3(-3,1^3\ ^4 0)^5 0\rangle$

$^5L_2$ $|q^6(1^3(\frac{1}{2},21\ ^4 21),1^3(-3,1^3\ ^4 21)^5 0\rangle$

^5M $|q^6(1^3(\frac{3}{2},1^3\ ^4 0),1^3(-3,1^3\ ^4 0)^5 0\rangle$

^5J $|q^6(1^2(1,1^2\ ^3 1^2),1^4(-4,1^4\ ^3 1)^5 0\rangle$

^7C $|q^6(1^6(0,2^3\ ^7 0),0(0,0\ ^1 0)^7 0\rangle$

^7E $|q^6(1^5(\frac{1}{2},2^2 1\ ^6 1^2),1(-1,1\ ^2 1)^7 0\rangle$

^7H $|q^6(1^4(1,21^2\ ^5 1),1^2(-2,1^2\ ^3 1^2)^7 0\rangle$

^7M $|q^6(1^3(\frac{3}{2},1^3\ ^4 0),1^3(-3,1^3\ ^4 0)^7 0\rangle$

Table I8: $Q^4\bar{Q}$ basis vectors

Reduced colour-spin identification	$\lvert q^4(1^{n_o}(I_q,\lambda_o^{CS}\ \lvert s_o\rvert_{\mu_o^C}),1^{n_s}(S_q,\lambda_s^{CS}\ \lvert s_s\rvert_{\mu_s^C}))\ \lvert s_q\rvert_{\mu_q^C};\bar{q}(1^{n_{\bar{o}}^*}(I_{\bar{q}},\lambda_o^{CS}\ \lvert s_{\bar{o}}\rvert_{\mu_{\bar{o}}^C}),1^{n_{\bar{s}}^*}(S_{\bar{q}},\lambda_{\bar{s}}^{CS}\ \lvert s_{\bar{s}}\rvert_{\mu_{\bar{s}}^C}))\ \lvert s_{\bar{q}}\rvert_{\mu_{\bar{q}}^C}]I,S,\ \lvert s\rvert_0{}^C\rangle$
^2B	$\lvert[q^4(1^4(0,2^2\ {}^31),0(0,0\ {}^10))^31;\bar{q}(1^{11}(\tfrac{1}{2},1^5\ {}^21^2),0(0,0\ {}^10))^21^2]\tfrac{1}{2},0,{}^20\rangle$
$^2A_1$	$\lvert[q^4(1^4(1,21^2\ {}^31),0(0,0\ {}^10))^31;$ " " $]\tfrac{1}{2},0,{}^20\rangle$
$^2A_2$	$\lvert[$ " ^{11} " $^10\ {}^11;$ " " $]$ " $\rangle$
$^2A_1$	$\lvert[q^4(1^4(1,21^2\ {}^31),0(0,0\ {}^10))^31;$ " " $]\tfrac{3}{2},0,{}^20\rangle$
$^2A_2$	$\lvert[$ " ^{11} " $^10\ {}^11;$ " " $]$ " $\rangle$
^2C	$\lvert[q^4(1^4(2,1^4\ {}^31),0(0,0\ {}^10))^31;$ " " $]\tfrac{3}{2},0,{}^20\rangle$
^2C	$\lvert[q^4(1^4(2,1^4\ {}^31),0(0,0\ {}^10))^31;$ " " $]\tfrac{5}{2},0,{}^20\rangle$
$^2D_1$	$\lvert[q^4(1^3(\tfrac{1}{2},21\ {}^421),1(-1,1\ {}^21))^31;$ " " $]0,-1,{}^20\rangle$
$^2D_2$	$\lvert[$ " ^{221} " $^21\ {}^31;$ " " $]$ " $\rangle$
$^2D_3$	$\lvert[$ " ^{221} " $^21\ {}^11;$ " " $]$ " $\rangle$
$^2D_4$	$\lvert[$ " ^{20} " $^21\ {}^31,$ " " $]$ " $\rangle$
$^2D_5$	$\lvert[$ " ^{20} " $^21\ {}^11;$ " " $]$ " $\rangle$
$^2D_1$	$\lvert[q^4(1^3(\tfrac{1}{2},21\ {}^421),1(-1,1\ {}^21))^31;$ " " $]1,-1,{}^20\rangle$
$^2D_2$	$\lvert[$ " ^{221} " $^21\ {}^31;$ " " $]$ " $\rangle$
$^2D_3$	$\lvert[$ " ^{221} " $^21\ {}^11;$ " " $]$ " $\rangle$
$^2D_4$	$\lvert[$ " ^{20} " $^21\ {}^31;$ " " $]$ " $\rangle$
$^2D_5$	$\lvert[$ " ^{20} " $^21\ {}^11;$ " " $]$ " $\rangle$

$^2E_1$ | $[q^4(1^3(\frac{3}{2},1^3\ {}^4 0),\ 1(-1,\ 1\ {}^2 1))^3 1;\bar{q}(1^{11}(\frac{1}{2},1^5\ {}^2 1^2),0(0,0\ {}^1 0))^2 1^2]1,-1,{}^2 0>$

$^2E_2$ | [" $^2 21$ " $^2 1\ {}^3 1$; " "] " >

$^2E_3$ | [" $^2 21$ " $^2 1\ {}^1 1$; " "] " >

$^2\dot{E}_1$ | $[q^4(1^3(\frac{3}{2},1^3\ {}^4 0\ ),\ 1(-1,\ 1\ {}^2 1))^3 1$; " "]2,-1,$^2 0>$

$^2E_2$ | [" $^2 21$ " $^2 1\ {}^3 1$; " "] " >

$^2E_3$ | [" $^2 21$ " $^2 1\ {}^1 1$; " "] " >

$^2F_1$ | $[q^4(1^2(0,2\ {}^3 2),1^2(-2,\ 1^2\ {}^3 1^2))^3 1$; " "]$\frac{1}{2}$,-2,$^2 0>$

$^2F_2$ | [" $^3 2$ " $^3 1^2\ {}^1 1$; " "] " >

$^2F_3$ | [" $^1 1^2$ " $^3 1^2\ {}^3 1$; " "] " >

$^2F_4$ | [" $^1 1^2$ " $^1 2\ {}^1 1$; " "] " >

$^2G_1$ | $[q^4(1^2(1,1^2\ {}^3 1^2),1^2(-2,\ 1^2\ {}^3 1^2))^3 1$; " "]$\frac{1}{2}$,-2,$^2 0>$

$^2G_2$ | [" $^3 1^2$ " $^3 1^2\ {}^1 1$; " "] " >

$^2G_3$ | [" $^3 1^2$ " $^1 2\ {}^3 1$; " "] " >

$^2G_4$ | [" $^1 2$ " $^3 1^2\ {}^3 1$; " "] " >

$^2G_1$ | $[q^4(1^2(1,1^2\ {}^3 1^2),1^2(-2,\ 1^2\ {}^3 1^2))^3 1$; " "]$\frac{3}{2}$,-2,$^2 0>$

$^2G_2$ | [" $^3 1^2$ " $^3 1^2\ {}^1 1$; " "] " >

$^2G_3$ | [" $^3 1^2$ " $^1 2\ {}^3 1$; " "] " >

$^2G_4$ | [" $^1 2$ " $^3 1^2\ {}^3 1$; " "] " >

$^2E_1$ | $[q^4(1\ (\frac{1}{2},1\ \ ^2\!1),1^3(-3,1^3\ \ ^4\!0))\ ^3\!1;\bar{q}(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),0(0,0\ ^1\!0))^2\!1^2]0,-3,^2\!0>$

$^2E_2$ | [" $^2\!1$ " $^2\!21$ $^3\!1$; " "] " >

$^2E_3$ | [" $^2\!1$ " $^2\!21$ $^1\!1$; " "] " >

$^2E_1$ | $[q^4(1\ (\frac{1}{2},1\ \ ^2\!1),1^3(-3,1^3\ \ ^4\!0))\ ^3\!1;$ " " $]1,-3,^2\!0>$

$^2E_2$ | [" $^2\!1$ " $^2\!21$ $^3\!1$; " "] " >

$^2E_3$ | [" $^2\!1$ " $^2\!21$ $^1\!1$; " "] " >

^2C | $[q^4(0\ (0,0\ \ ^1\!0),1^4(-4,1^4\ \ ^3\!1))\ ^3\!1;$ " " $]\frac{1}{2},-4,^2\!0>$

^2B | $[q^4(1^4(0,2^2\ \ ^3\!1),\ 0(\ 0,0\ \ ^1\!0))\ ^3\!1;\bar{q}(0(0,0\ ^1\!0),1^*\ (1,1^5\ ^2\!1^2))^2\!1^2]0,\ 1,^2\!0>$

$^2A_1$ | $[q^4(1^4(1,21^2\ \ ^3\!1),\ 0(\ 0,0\ \ ^1\!0))\ ^3\!1;$ " " $]1,\ 1,^2\!0>$

$^2A_2$ | [" $^1\!1$ " $^1\!0$ $^1\!1$; " "] " >

^2C | $[q^4(1^4(2,1^4\ \ ^3\!1),\ 0(\ 0,0\ \ ^1\!0))\ ^3\!1;$ " " $]2,\ 1,^2\!0>$

$^2D_1$ | $[q^4(1^3(\frac{1}{2},21\ \ ^4\!21),\ 1(-1,1\ \ ^2\!1))\ ^3\!1;$ " " $]\frac{1}{2},\ 0,^2\!0>$

$^2D_2$ | [" $^2\!21$ " $^2\!1$ $^3\!1$; " "] " >

$^2D_3$ | [" $^2\!21$ " $^2\!1$ $^1\!1$; " "] " >

$^2D_4$ | [" $^2\!0$ " $^2\!1$ $^3\!1$; " "] " >

$^2D_5$ | [" $^2\!0$ " $^2\!1$ $^1\!1$; " "] " >

$^2E_1$ | $[q^4(1^3(\frac{3}{2},1^3\ \ ^4\!0),\ 1(-1,1\ \ ^2\!1))\ ^3\!1;$ " " $]\frac{3}{2},\ 0,^2\!0>$

$^2E_2$ | [" $^2\!21$ " $^2\!1$ $^3\!1$; " "] " >

$^2E_3$ | [" $^2\!21$ " $^2\!1$ $^1\!1$; " "] " >

$^2F_1$ | $[q^4(1^2(0,2\ {}^3)2),1^2(-2,1^2\ {}^3)1^2))^3)1;\bar{q}(0(0,0\ {}^1)0),1^*\ (1,1^5\ {}^2)1^2))^2)1^2]0,-1,^20>$

$^2F_2$ | [" ^{32} " $^31^2$ ^{11}; " "] " >

$^2F_3$ | [" $^11^2$ " $^31^2$ ^{31}; " "] " >

$^2F_4$ | [" $^11^2$ " ^{12} ^{11}; " "] " >

$^2G_1$ | $[q^4(1^2(1,1^2\ {}^3)1^2),1^2(-2,1^2\ {}^3)1^2))^3)1$; " "]1,-1,^{20}>

$^2G_2$ | [" $^31^2$ " $^31^2$ ^{11}; " "] " >

$^2G_3$ | [" $^31^2$ " ^{12} ^{31}; " "] " >

$^2G_4$ | [" ^{12} " $^31^2$ ^{31}; " "] " >

$^2E_1$ | $[q^4(1\ (½,1\ {}^4)0),1^3(-3,1^3\ {}^2)1))^3)1$; " "]½,-2,^{20}>

$^2E_2$ | [^{221} ^{21} ^{31}; " "] " >

$^2E_3$ | [" ^{221} ^{21} ^{11}; " "] " >

^2C | $[q^4(0\ (0,0\ {}^1)0),1^4(-4,1^4\ {}^3)1))^3)1$; " "]0,-3,^{20}>

$^4\mathcal{B}$ | $[q^4(1^4(0,2^2\ {}^31),\ 0(\ 0,0\ {}^10))\ {}^31;\bar{q}(1^{11}(\frac{1}{2},1^5\ {}^21^2),0(0,0\ {}^10))^21^2]\frac{1}{2},0,{}^40\rangle$

$^4A_1$ | $[q^4(1^4(1,21^2\ {}^31),\ 0(\ 0,0\ {}^10))\ {}^31;$ " " $]\frac{1}{2},0,{}^40\rangle$

$^4A_2$ | $[$ " ^{51} " $^10\ {}^51;$ " " $]$ " $\rangle$

$^4A_1$ | $[q^4(1^4(1,21^2\ {}^31),\ 0(\ 0,0\ {}^10))\ {}^31;$ " " $]\frac{3}{2},0,{}^40\rangle$

$^4A_2$ | $[$ " ^{51} " $^10\ {}^51;$ " " $]$ " $\rangle$

^4C | $[q^4(1^4(2,1^4\ {}^31),\ 0(\ 0,0\ {}^10))\ {}^31;$ " " $]\frac{3}{2},0,{}^40\rangle$

^4C | $[q^4(1^4(2,1^4\ {}^31),\ 0(\ 0,0\ {}^10))\ {}^31;$ " " $]\frac{5}{2},0,{}^40\rangle$

$^4\mathcal{D}_1$ | $[q^4(1^3(\frac{1}{2},21\ {}^421),\ 1(-1,1\ {}^21))\ {}^31;$ " " $]0,-1,{}^40\rangle$

$^4\mathcal{D}_2$ | $[$ " ^{421} " $^21\ {}^51;$ " " $]$ " $\rangle$

$^4\mathcal{D}_3$ | $[$ " ^{221} " $^21\ {}^31;$ " " $]$ " $\rangle$

$^4\mathcal{D}_4$ | $[$ " ^{20} " $^21\ {}^31;$ " " $]$ " $\rangle$

$^4\mathcal{D}_1$ | $[q^4(1^3(\frac{1}{2},21\ {}^421),\ 1(-1,1\ {}^21))\ {}^31;$ " " $]1,-1,{}^40\rangle$

$^4\mathcal{D}_2$ | $[$ " $^421),$ " $^21))\ {}^51;$ " " $]$ " $\rangle$

$^4\mathcal{D}_3$ | $[$ " ^{221} " $^21\ {}^31;$ " " $]$ " $\rangle$

$^4\mathcal{D}_4$ | $[$ " ^{20} " $^21\ {}^31;$ " " $]$ " $\rangle$

$^4E_1$ | $[q^4(1^3(\frac{3}{2},1^3\ {}^4 0),\ 1(-1,1\ {}^2 1))\ {}^3 1; \bar{q}(1^{11}(\frac{1}{2},1^5\ {}^2 1^2),0(0,0\ {}^1 0))\ {}^2 1^2]\ 1,-1,{}^4 0\rangle$

$^4E_2$ | [" $^2 21$ " $^2 1$ $^3 1$; " "] " >

$^4E_3$ | [" $^2 21$ " $^2 1$ $^1 1$; " "] " >

$^4E_1$ | $[q^4(1^3(\frac{3}{2},1^3\ {}^4 0),\ 1(-1,1\ {}^2 1))\ {}^3 1$; " "] $2,-1,{}^4 0\rangle$

$^4E_2$ | [" $^2 21$ " $^2 1$ $^3 1$; " "] " >

$^4E_3$ | [" $^2 21$ " $^2 1$ $^1 1$; " "] " >

$^4F_1$ | $[q^4(1^2(0,2\ {}^3 2),1^2(-2,1^2\ {}^3 1^2))\ {}^3 1$; " "] $\frac{1}{2},-2,{}^4 0\rangle$

$^4F_2$ | [" $^3 2$ " $^3 1^2$ $^5 1$; " "] " >

$^4F_3$ | [" $^1 1^2$ " $^3 1^2$ $^3 1$; " "] " >

$^4G_1$ | $[q^4(1^2(1,1^2\ {}^3 1^2),1^2(-2,1^2\ {}^3 1^2))\ {}^3 1$; " "] $\frac{1}{2},-2,{}^4 0\rangle$

$^4G_2$ | [" $^3 1^2$ " $^3 1^2$ $^5 1$; " "] " >

$^4G_3$ | [" $^3 1^2$ " $^1 2$ $^3 1$; " "] " >

$^4G_4$ | [" $^1 2$ " $^3 1^2$ $^3 1$; " "] " >

$^4G_1$ | $[q^4(1^2(1,1^2\ {}^3 1^2),1^2(-2,1^2\ {}^3 1^2))\ {}^3 1$; " "] $\frac{3}{2},-2,{}^4 0\rangle$

$^4G_2$ | [" $^3 1^2$ " $^3 1^2$ $^5 1$; " "] " >

$^4G_3$ | [" $^3 1^2$ " $^1 2$ $^3 1$; " "] " >

$^4G_4$ | [" $^1 2$ " $^3 1^2$ $^3 1$; " "] " >

$^4E_1$ | $[q^4(1\ (\frac{1}{2},1\ \ ^2\!1),1^3(-3,1^3\ \ ^4\!0))^5\!1;\bar{q}(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),0(0,0\ ^1\!0))^2\!1^2]0,-3,^4\!0>$

$^4E_2$ | $[\quad"\quad\ ^2\!1\quad"\quad\ ^4\!0\ \ ^3\!1;\quad"\quad\quad"\quad\ ]\quad"\quad>$

$^4E_3$ | $[\quad"\quad\ ^2\!1\quad\quad\ ^2\!21\ \ ^3\!1;\quad"\quad\quad"\quad\ ]\quad"\quad>$

$^4E_1$ | $[q^4(1\ (\frac{1}{2},1\ \ ^2\!1),1^3(-3,1^3\ \ ^4\!0))^5\!1;\quad"\quad\quad"\quad\ ]1,-3,^4\!0>$

$^4E_2$ | $[\quad"\quad\ ^2\!1\quad"\quad\ ^4\!0\ \ ^3\!1;\quad"\quad\quad"\quad\ ]\quad"\quad>$

$^4E_3$ | $[\quad"\quad\ ^2\!1\quad"\quad\ ^2\!21\ \ ^3\!1;\quad"\quad\quad"\quad\ ]\quad"\quad>$

^4C | $[q^4(0\ (0,0\ \ ^1\!0),1^4(-4,1^4\ \ ^3\!1))^3\!1;\quad"\quad\quad"\quad\ ]\frac{1}{2},-4,^4\!0>$

^4B | $[q^4(1^4(0,2^2\ \ ^3\!1),0\ (0\ ,0\ \ ^1\!0))^3\!1;\bar{q}(0(0,0\ ^1\!0),1^*\ (1,1^5\ ^2\!1^2))^2\!1^2]0,\ 1,^4\!0>$

$^4A_1$ | $[q^4(1^4(1,21^2\ \ ^3\!1),0\ (0,0\ \ ^1\!0))^3\!1;\quad"\quad\quad"\quad\ ]1,\ 1,^4\!0>$

$^4A_2$ | $[\quad"\quad\ ^5\!1\quad"\quad\ ^1\!0\ \ ^5\!1;\quad"\quad\quad"\quad\ ]\quad"\quad>$

^4C | $[q^4(1^4(2,1^4\ \ ^3\!1),0\ (0\ ,0\ \ ^1\!0))^3\!1;\quad"\quad\quad"\quad\ ]2,\ 1,^4\!0>$

$^4D_1$ | $[q^4(1^3(\frac{1}{2},21\ \ ^4\!21),1\ (-1,1\ \ ^2\!1))^3\!1;\quad"\quad\quad"\quad\ ]\frac{1}{2},\ 0,^4\!0>$

$^4D_2$ | $[\quad"\quad\ ^4\!21\quad"\quad\ ^2\!1\ \ ^5\!1;\quad"\quad\quad"\quad\ ]\quad"\quad>$

$^4D_3$ | $[\quad"\quad\ ^2\!21\quad"\quad\ ^2\!1\ \ ^3\!1;\quad"\quad\quad"\quad\ ]\quad"\quad>$

$^4D_4$ | $[\quad"\quad\ ^2\!0\quad"\quad\ ^2\!1\ \ ^3\!1;\quad"\quad\quad"\quad\ ]\quad"\quad>$

$^4E_1$ | $[q^4(1^3(\frac{3}{2},1^3\ {}^40),\ 1(-1,1\ {}^21))^51;\bar{q}(0(0,0\ {}^10),1^*\ (1,1^5\ {}^21^2))^21^2]^{\frac{3}{2}},0,{}^40\rangle$

$^4E_2$ | $[$ " ^{40} " $^21\ {}^31;$ " " $]$ " $\rangle$

$^4E_3$ | $[$ " ^{221} " $^21\ {}^31;$ " " $]$ " $\rangle$

$^4F_1$ | $[q^4(1^2(0,2\ {}^32),1^2(-2,1^2\ {}^31^2))^31;$ " " $]0,-1,{}^40\rangle$

$^4F_2$ | $[$ " ^{32} " $^31^2\ {}^51;$ " " $]$ " $\rangle$

$^4F_3$ | $[$ " $^11^2$ " $^31^2\ {}^31;$ " " $]$ " $\rangle$

$^4G_1$ | $[q^4(1^2(1,1^2\ {}^31^2),1^2(-2,1^2\ {}^31^2))^31;$ " " $]1,-1,{}^40\rangle$

$^4G_2$ | $[$ " $^31^2$ " $^31^2\ {}^51;$ " " $]$ " $\rangle$

$^4G_3$ | $[$ " $^31^2$ " $^12\ {}^31;$ " " $]$ " $\rangle$

$^4G_4$ | $[$ " ^{12} " $^31^2\ {}^31;$ " " $]$ " $\rangle$

$^4E_1$ | $[q^4(1(\frac{1}{2},1\ {}^40),1^3(-3,1^3\ {}^21\ ))^51;$ " " $]\frac{1}{2},-2,{}^40\rangle$

$^4E_2$ | $[$ " ^{40} " $^21\ {}^31;$ " " $]$ " $\rangle$

$^4E_3$ | $[$ " ^{221} " $^21\ {}^31;$ " " $]$ " $\rangle$

^4C | $[q^4(0\ (0,0\ {}^10),1^4(-4,1^4\ {}^31))\ {}^31;$ " " $]0,-3,{}^40\rangle$

^6A $|[q^4(1^4(1,21^2\ {}^5\!1),0(0,0\ {}^1\!0))^5\!1;\bar{q}(1^{11}(\tfrac{1}{2},1^5\ {}^2\!1^2),0(0,0\ {}^1\!0))^2\!1^2]\tfrac{1}{2},0,{}^6\!0\rangle$

^6A $|[q^4(1^4(1,21^2\ {}^5\!1),0(0,0\ {}^1\!0))^5\!1;\quad"\quad"\quad]\tfrac{3}{2},0,{}^6\!0\rangle$

^6D $|[q^4(1^3(\tfrac{1}{2},21\ {}^4\!21),1(-1,1\ {}^2\!1))^5\!1;\quad"\quad"\quad]0,-1,{}^6\!0\rangle$

^6D $|[q^4(1^3(\tfrac{1}{2},21\ {}^4\!21),1(-1,1\ {}^2\!1))^5\!1;\quad"\quad"\quad]1,-1,{}^6\!0\rangle$

^6E $|[q^4(1^3(\tfrac{3}{2},1^3\ {}^4\!0),1(-1,1\ {}^2\!1))^5\!1;\quad"\quad"\quad]1,-1,{}^6\!0\rangle$

^6E $|[q^4(1^3(\tfrac{3}{2},1^3\ {}^4\!0),1(-1,1\ {}^2\!1))^5\!1;\quad"\quad"\quad]2,-1,{}^6\!0\rangle$

^6F $|[q^4(1^2(0,2\ {}^3\!2),1^2(-2,1^2\ {}^3\!1^2))^5\!1;\quad"\quad"\quad]\tfrac{1}{2},-2,{}^6\!0\rangle$

^6G $|[q^4(1^2(1,1^2\ {}^3\!1^2),1^2(-2,1^2\ {}^3\!1^2))^5\!1;\quad"\quad"\quad]\tfrac{1}{2},-2,{}^6\!0\rangle$

^6G $|[q^4(1^2(1,1^2\ {}^3\!1^2),1^2(-2,1^2\ {}^3\!1^2))^5\!1;\quad"\quad"\quad]\tfrac{3}{2},-2,{}^6\!0\rangle$

^6E $|[q^4(1(\tfrac{1}{2},1\ {}^2\!1),1^3(-3,1^3\ {}^4\!0))^5\!1;\quad"\quad"\quad]0,-3,{}^6\!0\rangle$

^6E $|[q^4(1(\tfrac{1}{2},1\ {}^2\!1),1^3(-3,1^3\ {}^4\!0))^5\!1;\quad"\quad"\quad]1,-3,{}^6\!0\rangle$

^6A $|[q^4(1^4(1,21^2\ {}^5\!1),0(0,0\ {}^1\!0))^5\!1;\bar{q}(0(0,0\ {}^1\!0),1^*\ (1,1^5\ {}^2\!1^2))^2\!1^2\ ]1,1,{}^6\!0\rangle$

^6D $|[q^4(1^3(\tfrac{1}{2},21\ {}^4\!21),1(-1,1\ {}^2\!1))^5\!1;\quad"\quad"\quad]\tfrac{1}{2},0,{}^6\!0\rangle$

^6E $|[q^4(1^3(\tfrac{3}{2},1^3\ {}^4\!0),1(-1,1\ {}^2\!1))^5\!1;\quad"\quad"\quad]\tfrac{3}{2},0,{}^6\!0\rangle$

^6F $|[q^4(1^2(0,2\ {}^3\!2),1^2(-2,1^2\ {}^3\!1^2))^5\!1;\quad"\quad"\quad]0,-1,{}^6\!0\rangle$

^6G $|[q^4(1^2(1,1^2\ {}^3\!1^2),1^2(-2,1^2\ {}^3\!1^2))^5\!1;\quad"\quad"\quad]1,-1,{}^6\!0\rangle$

^6E $|[q^4(1(\tfrac{1}{2},1\ {}^2\!1),1^3(-3,1^3\ {}^4\!0))^5\!1;\quad"\quad"\quad]\tfrac{1}{2},-2,{}^6\!0\rangle$

Table I9: $Q^2\bar{Q}^2$ basis vectors

Reduced colour-spin identification	$\vert[q^2(1^{n_o}(I_q,\lambda_o^{CS}\vert s_o\vert\mu_o^C),1^{n_s}(S_q,\lambda_s^{CS}\vert s_s\vert\mu_s^C))\vert s_q\vert\mu_q^C;\bar{q}^2(1^{n_{\bar{o}}^*}(I_{\bar{q}},\lambda_{\bar{o}}^{CS}\vert s_{\bar{o}}\vert\mu_{\bar{o}}^C),1^{n_{\bar{s}}^*}(S_{\bar{q}},\lambda_{\bar{s}}^{CS}\vert s_{\bar{s}}\vert\mu_{\bar{s}}^C))\vert s_{\bar{q}}\vert\mu_{\bar{q}}^C]I,S,\vert s\vert 0^C\rangle$
$^1A_1$	$\vert[q^2(1^2(0,2\ ^32),0\ (0,0\ ^10))^32;\bar{q}^2(1^{10}(0,2^5\ ^32^2),0\ (0,0\ ^10))^32^2]0,0,^10\rangle$
$^1A_2$	$\vert[\quad"\quad ^11^2\quad"\quad ^10\quad ^11^2;\quad"\quad ^11\quad"\quad ^10\quad ^11\ ]\quad"\quad\rangle$
$^1B_1$	$\vert[q^2(1^2(1,1^2\ ^31^2),0\ (0,0\ ^10))^31^2;\bar{q}^2(1^{10}(1,1^4\ ^31),0\ (0,0\ ^10))^31\ ]0,0,^10\rangle$
$^1B_2$	$\vert[\quad"\quad ^12\quad"\quad ^10\quad ^12;\quad"\quad ^12^2\quad"\quad ^10\quad ^12^2]\quad"\quad\rangle$
$^1B_1$	$\vert[q^2(1^2(1,1^2\ ^31^2),0\ (0,0\ ^10))^31^2;\bar{q}^2(1^{10}(1,1^4\ ^31),0\ (0,0\ ^10))^31\ ]1,0,^10\rangle$
$^1B_2$	$\vert[\quad"\quad ^12\quad"\quad ^10\quad ^12;\quad"\quad ^12^2\quad"\quad ^10\quad ^12^2]\quad"\quad\rangle$
$^1B_1$	$\vert[q^2(1^2(1,1^2\ ^31^2),0\ (0,0\ ^10))^31^2;\bar{q}^2(1^{10}(1,1^4\ ^31),0\ (0,0\ ^10))^31\ ]2,0,^10\rangle$
$^1B_2$	$\vert[\quad"\quad ^12\quad"\quad ^10\quad ^12;\quad"\quad ^12^2\quad"\quad ^10\quad ^12^2]\quad"\quad\rangle$
$^1D_1$	$\vert[q^2(1\ (\tfrac{1}{2},1\ ^21),1\ (-1,1\ ^21))^32;\bar{q}^2(1^{10}(0,2^5\ ^32^2),0\ (0,0\ ^10))^32^2]\tfrac{1}{2},-1,^10\rangle$
$^1D_2$	$\vert[\quad"\quad ^21\quad"\quad ^21\quad ^11^2;\quad"\quad ^11\quad"\quad ^10\quad ^11\ ]\quad"\quad\rangle$
$^1E_1$	$\vert[q^2(1\ (\tfrac{1}{2},1\ ^21),1\ (-1,1\ ^21))^12;\bar{q}^2(1^{10}(1,1^4\ ^12^2),0\ (0,0\ ^10))^12^2]\tfrac{1}{2},-1,^10\rangle$
$^1E_2$	$\vert[\quad"\quad ^21\quad"\quad ^21\quad ^31^2;\quad"\quad ^31\quad"\quad ^10\quad ^31\ ]\quad"\quad\rangle$
$^1E_1$	$\vert[q^2(1\ (\tfrac{1}{2},1\ ^21),1\ (-1,1\ ^21))^12;\bar{q}^2(1^{10}(1,1^4\ ^12^2),0\ (0,0\ ^10))^12^2]\tfrac{3}{2},-1,^10\rangle$
$^1E_2$	$\vert[\quad"\quad ^21\quad"\quad ^21\quad ^31^2;\quad"\quad ^31\quad"\quad ^10\quad ^31\ ]\quad"\quad\rangle$
$^1B_1$	$\vert[q^2(0\ (0,0\ ^10),1^2(-2,1^2\ ^31^2))^31^2;\bar{q}^2(1^{10}(1,1^4\ ^31),0\ (0,0\ ^10))^31\ ]1,-2\ ^10\rangle$
$^1B_2$	$\vert[\quad"\quad ^10\quad"\quad ^12\quad ^12;\quad"\quad ^12^2\quad"\quad ^10\quad ^12^2]\quad"\quad\rangle$

$^1\bar{D}_1$ | $[q^2(1^2(0,2\ {}^3\!2\ ),0\ (0\ ,0\ {}^1\!0))^3\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^3\!2^2]\frac{1}{2},1,{}^1\!0\rangle$

$^1\bar{D}_2$ | [" $^1\!1^2$ " $^1\!0\ {}^1\!1^2$; " $^2\!1^2$ " $^2\!1^2\ {}^1\!1\ ]$ " $\rangle$

$^1\bar{E}_1$ | $[q^2(1^2(1,1^2\ {}^1\!2\ ),0\ (0\ ,0\ {}^1\!0))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^1\!2^2]\frac{1}{2},1,{}^1\!0\rangle$

$^1\bar{E}_2$ | [" $^3\!1^2$ " $^1\!0\ {}^3\!1^2$; " $^2\!1^2$ " $^2\!1^2\ {}^3\!1\ ]$ " $\rangle$

$^1\bar{E}_1$ | $[q^2(1^2(1,1^2\ {}^1\!2\ ),0\ (0\ ,0\ {}^1\!0))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^1\!2^2]\frac{3}{2},1,{}^1\!0\rangle$

$^1\bar{E}_2$ | [" $^3\!1^2$ " $^1\!0\ {}^3\!1^2$; " $^2\!1^2$ " $^2\!1^2\ {}^3\!1\ ]$ " $\rangle$

$^1F_1$ | $[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^1\!2^2]0,0,{}^1\!0\rangle$

$^1F_2$ | [" $^2\!1$ " $^2\!1\ {}^3\!2$; " $^2\!1^2$ " $^2\!1^2\ {}^3\!2^2]$ " $\rangle$

$^1F_3$ | [" $^2\!1$ " $^2\!1\ {}^1\!1^2$; " $^2\!1^2$ " $^2\!1^2\ {}^1\!1\ ]$ " $\rangle$

$^1F_4$ | [" $^2\!1$ " $^2\!1\ {}^3\!1^2$; " $^2\!1^2$ " $^2\!1^2\ {}^3\!1\ ]$ " $\rangle$

$^1F_1$ | $[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^1\!2^2]1,0,{}^1\!0\rangle$

$^1F_2$ | [" $^2\!1$ " $^2\!1\ {}^3\!2$; " $^2\!1^2$ " $^2\!1^2\ {}^3\!2^2]$ $\rangle$

$^1F_3$ | [" $^2\!1$ " $^2\!1\ {}^1\!1^2$; " $^2\!1^2$ " $^2\!1^2\ {}^1\!1\ ]$ " $\rangle$

$^1F_4$ | [" $^2\!1$ " $^2\!1\ {}^3\!1^2$; " $^2\!1^2$ " $^2\!1^2\ {}^3\!1\ ]$ " $\rangle$

$^1\bar{E}_1$ | $[q^2(0\ (0,0\ {}^1\!0\ ),1^2(-2,1^2\ {}^1\!2))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^1\!2^2]\frac{1}{2},-1,{}^1\!0\rangle$

$^1\bar{E}_2$ | [" $^1\!0$ " $^3\!1^2\ {}^3\!1^2$; " $^2\!1^2$ " $^2\!1^2\ {}^3\!1\ ]$ " $\rangle$

$^1B_1$ $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (0,0\ {}^1\!0))^3\!1^2;\bar{q}^2(0\ (0,0\ {}^1\!0),1^{2*}(2,1^4\ {}^3\!1))^3\!1]1,2,{}^1\!0\rangle$

$^1B_2$ $|[\quad"\quad{}^1\!2\quad"\quad{}^1\!0\quad{}^1\!2\ ;\quad"\quad{}^1\!0\quad"\quad{}^1\!2^2\quad{}^1\!2^2]\quad"\quad\rangle$

$^1E_1$ $|[q^2(1\ (\tfrac{1}{2},1\ {}^2\!1),1\ (-1,1\ {}^2\!1))^1\!2\ ;\bar{q}^2(0\ (0,0\ {}^1\!0),1^{2*}(2,1^4\ {}^1\!2^2))^1\!2^2]\tfrac{1}{2},1,{}^1\!0\rangle$

$^1E_2$ $|[\quad"\quad{}^2\!1\quad"\quad{}^2\!1\quad{}^3\!1^2;\quad"\quad{}^1\!0\quad"\quad{}^3\!1\quad{}^3\!1]\quad"\quad\rangle$

$^1B_1$ $|[q^2(0\ (0,0\ {}^1\!0),1^{10}(-2,1^2\ {}^3\!1^2))^3\!1^2;\bar{q}^2(0\ (0,0\ {}^1\!0),1^{2*}(2,1^4\ {}^3\!1))^3\!1]0,0,{}^1\!0\rangle$

$^1B_2$ $|[\quad"\quad{}^1\!0\quad"\quad{}^1\!2\quad{}^1\!2\quad"\quad{}^1\!0\quad"\quad{}^1\!2^2\quad{}^1\!2^2]\quad"\quad\rangle$

^3A $|[q^2(1^2(0,2\ {}^3\!2),0\ (0,0\ {}^1\!0))^3\!2\ ;\bar{q}^2(1^{10}(0,2^5\ {}^3\!2^2),0\ (0,0\ {}^1\!0))^3\!2^2]0,0,{}^3\!0\rangle$

$^3\bar{C}_1$ $|[q^2(1^2(1,1^2\ {}^1\!2),0\ (0,0\ {}^1\!0))^1\!2\ ;\bar{q}^2(1^{10}(0,2^5\ {}^3\!2^2),0\ (0,0\ {}^1\!0))^3\!2^2]1,0,{}^3\!0\rangle$

$^3\bar{C}_2$ $|[\quad"\quad{}^3\!1^2\quad"\quad{}^1\!0\quad{}^3\!1^2;\quad"\quad{}^1\!1\quad"\quad{}^1\!0\quad{}^1\!1]\quad"\quad\rangle$

$^3C_1$ $|[q^2(1^2(0,2\ {}^3\!2),0\ (0,0\ {}^1\!0))^3\!2\ ;\bar{q}^2(1^{10}(1,1^4\ {}^1\!2^2),0\ (0,0\ {}^1\!0))^1\!2^2]1,0,{}^3\!0\rangle$

$^3C_2$ $|[\quad"\quad{}^1\!1^2\quad"\quad{}^1\!0\quad{}^1\!1^2;\quad"\quad{}^3\!1\quad"\quad{}^1\!0\quad{}^3\!1]\quad"\quad\rangle$

^3B $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (0,0\ {}^1\!0))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1),0\ (0,0\ {}^1\!0))^3\!1]0,0,{}^3\!0\rangle$

^3B $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (0,0\ {}^1\!0))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1),0\ (0,0\ {}^1\!0))^3\!1]1,0,{}^3\!0\rangle$

^3B $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (0,0\ {}^1\!0))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1),0\ (0,0\ {}^1\!0))^3\!1]2,0,{}^3\!0\rangle$

$^3D_1$ $|[q^2(1^2(\tfrac{1}{2},1\ {}^2\!1),1\ (-1,1\ {}^2\!1))^1\!2\ ;\bar{q}^2(1^{10}(0,2^5\ {}^3\!2^2),0\ (0,0\ {}^1\!0))^3\!2^2]\tfrac{1}{2},-1,{}^3\!0\rangle$

$^3D_2$ $|[\quad"\quad{}^2\!1\quad"\quad{}^2\!1\quad{}^3\!2\ ;\quad"\quad{}^3\!2^2\quad"\quad{}^1\!0\quad{}^3\!2^2]\quad"\quad\rangle$

$^3D_3$ $|[\quad"\quad{}^2\!1\quad"\quad{}^2\!1\quad{}^3\!1^2;\quad"\quad{}^1\!1\quad"\quad{}^1\!0\quad{}^1\!1]\quad"\quad\rangle$

$^3E_1$ $|[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1\ ))^3\!2\ ;\bar{q}^2(1^{10}(1,1^4\ {}^1\!2^2),0\ (0,0\ {}^1\!0\ ))^1\!2^2]_{\frac{1}{2},-1,^3\!0}>$

$^3E_2$ $|[\quad"\quad{}^2\!1\quad"\quad{}^2\!1\quad{}^1\!1^2;\quad"\quad{}^3\!1\quad"\quad{}^1\!0\quad{}^3\!1\ ]\ "\quad>$

$^3E_3$ $|[\quad"\quad{}^2\!1\quad"\quad{}^2\!1\quad{}^3\!1^2;\quad"\quad{}^3\!1\quad"\quad{}^1\!0\quad{}^3\!1\ ]\ "\quad>$

$^3E_1$ $|[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1\ ))^3\!2\ ;\bar{q}^2(1^{10}(1,1^4\ {}^1\!2^2),0\ (0,0\ {}^1\!0\ ))^1\!2^2]_{\frac{3}{2},-1\ {}^3\!0}>$

$^3E_2$ $|[\quad"\quad{}^2\!1\quad"\quad{}^2\!1\quad{}^1\!1^2\quad"\quad{}^3\!1\quad"\quad{}^1\!0\quad{}^3\!1\ ]\ "\quad>$

$^3E_3$ $|[\quad"\quad{}^2\!1\quad"\quad{}^2\!1\quad{}^3\!1^2;\quad"\quad{}^3\!1\quad"\quad{}^1\!0\quad{}^3\!1\ ]\ "\quad>$

$^3\bar{C}_1$ $|[q^2(0\ (0,0\ {}^1\!0\ ),1^2\ (-2,1^2\ {}^1\!2\ ))^1\!2\ ;\bar{q}^2(1^{10}(0,2^5\ {}^3\!2^2),0\ (0,0\ {}^1\!0\ ))^3\!2^2]_{0,-2,^3\!0}>$

$^3\bar{C}_2$ $|[\quad"\quad{}^1\!0\quad"\quad{}^3\!1^2\quad{}^3\!1^2;\quad"\quad{}^1\!1\quad"\quad{}^1\!0\quad{}^1\!1\ ]\ "\quad>$

^3B $|[q^2(0\ (0,0\ {}^1\!0\ ),1^2\ (-2,1^2\ {}^3\!1^2))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1\ ),0\ (0,0\ {}^1\!0\ ))^3\!1\ ]_{1,-2,^3\!0}>$

$^3\bar{D}_1$ $|[q^2(1^2(0,2\ {}^3\!2\ ),0\ (\ 0,0\ {}^1\!0\ ))^3\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^1\!2^2]_{\frac{1}{2},1,\ ^3\!0}>$

$^3\bar{D}_2$ $|[\quad"\quad{}^3\!2\quad"\quad{}^1\!0\quad{}^3\!2\ ;\quad"\quad{}^2\!1^2\quad\quad{}^2\!1^2\quad{}^3\!2^2]\ "\quad>$

$^3\bar{D}_3$ $|[\quad"\quad{}^1\!1^2\quad"\quad{}^1\!0\quad{}^1\!1^2;\quad"\quad{}^2\!1^2\quad"\quad{}^2\!1^2\quad{}^3\!1\ ]\ "\quad>$

$^3\bar{E}_1$ $|[q^2(1^2(1,1^2\ {}^1\!2\ ),0\ (\ 0,0\ {}^1\!0\ ))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^3\!2^2]_{\frac{1}{2},1,^3\!0}>$

$^3\bar{E}_2$ $|[\quad"\quad{}^3\!1^2\quad"\quad{}^1\!0\quad{}^3\!1^2;\quad"\quad{}^2\!1^2\quad"\quad{}^2\!1^2\quad{}^1\!1\ ]\ "\quad>$

$^3\bar{E}_3$ $|[\quad"\quad{}^3\!1^2\quad"\quad{}^1\!0\quad{}^3\!1^2;\quad"\quad{}^2\!1^2\quad"\quad{}^2\!1^2\quad{}^3\!1\ ]\ "\quad>$

$^3\bar{E}_1$ $|[q^2(1^2(1,1^2\ {}^1\!2\ ),0\ (\ 0,0\ {}^1\!0\ ))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^3\!2^2]_{\frac{3}{2},1,^3\!0}>$

$^3\bar{E}_2$ $|[\quad"\quad{}^3\!1^2\quad"\quad{}^1\!0\quad{}^3\!1^2;\quad"\quad{}^2\!1^2\quad\quad{}^2\!1^2\quad{}^1\!1\ ]\ "\quad>$

$^3\bar{E}_3$ $|[\quad"\quad{}^3\!1^2\quad"\quad{}^1\!0\quad{}^3\!1^2;\quad"\quad{}^2\!1^2\quad"\quad{}^2\!1^2\quad{}^3\!1\ ]\ "\quad>$

$^3F_1$ $|[q^2(1\ (\frac{1}{2},1\ ^2\!1\ ),1\ (-1,1\ ^2\!1\ ))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),1^*\ (1,1^5\ ^2\!1^2))^3\!2^2]0,0,^3\!0>$

$^3F_2$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^3\!2\ ;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^1\!2^2]\quad "\quad >$

$^3F_3$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^3\!2\ ;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^3\!2^2]\quad "\quad >$

$^3F_4$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^1\!1^2;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^3\!1\ ]\quad "\quad >$

$^3F_5$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^3\!1^2;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^1\!1\ ]\quad "\quad >$

$^3F_6$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^3\!1^2;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^3\!1\ ]\quad "\quad >$

$^3F_1$ $|[q^2(1\ (\frac{1}{2},1\ ^2\!1\ ),1\ (-1,1\ ^2\!1\ ))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),1^*\ (1,1^5\ ^2\!1^2))^3\!2^2]1,0,^3\!0>$

$^3F_2$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^3\!2\ ;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^1\!2^2]\quad "\quad >$

$^3F_3$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^3\!2\ ;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^3\!2^2]\quad "\quad >$

$^3F_4$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^1\!1^2;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^3\!1\ ]\quad "\quad >$

$^3F_5$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^3\!1^2;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^1\!1\ ]\quad "\quad >$

$^3F_6$ $|[\quad "\quad ^2\!1\quad\quad "\quad ^2\!1\quad ^3\!1^2;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^3\!1\ ]\quad "\quad >$

$^3\bar{E}_1$ $|[q^2(0\ (0,0\ ^1\!0\ ),1^{10}(-2,1^2\ ^1\!2\ ))^1\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ ^2\!1^2),1^*\ (1,1^5\ ^2\!1^2))^3\!2^2]\frac{1}{2},-1,^3\!0>$

$^3\bar{E}_2$ $|[\quad "\quad ^1\!0\quad\quad "\quad ^3\!1^2\quad ^3\!1^2;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^1\!1\ ]\quad "\quad >$

$^3\bar{E}_3$ $|[\quad "\quad ^1\!0\quad\quad "\quad ^3\!1^2\quad ^3\!1^2;\quad\quad "\quad ^2\!1^2\quad\quad "\quad ^2\!1^2\quad ^3\!1\ ]\quad "\quad >$

$^3C_1$ $|[q^2(1^2(0,2\ ^3\!2\ ),0\ (\ 0,0\ ^1\!0\ ))^3\!2\ ;\bar{q}^2(0\ (0,0\ ^1\!0\ ),1^{2*}(2,1^4\ ^1\!2^2))^1\!2^2]0,2,^3\!0\ >$

$^3C_2$ $|[\quad "\quad ^1\!1^2\quad\quad "\quad ^1\!0\quad ^1\!1^2;\quad\quad "\quad ^1\!0\quad\quad "\quad ^3\!1\quad ^3\!1\ ]\quad "\quad >$

^3B $|[q^2(1^2(1,1^2\ ^3\!1^2),0\ (\ 0,0\ ^1\!0\ ))^3\!1^2;\bar{q}^2(0\ (0,0\ ^1\!0\ ),1^{2*}(2,1^4\ ^3\!1\ ))^3\!1\ ]1,2,^3\!0>$

$^3E_1$ $|[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1\ ))^3\!2\ ;\bar{q}^2(0\ \ (0,0\ {}^1\!0\ ),1^{2*}\ (2,1^4\ {}^1\!2^2))^1\!2^2]\frac{1}{2},1,{}^3\!0\rangle$

$^3E_2$ $|[\quad\ "\ {}^2\!1\quad\ "\ {}^2\!1\ {}^1\!1^2;\quad\ "\ {}^1\!0\quad\ "\ {}^3\!1\ {}^3\!1\ ]\ "\ \rangle$

$^3E_3$ $|[\quad\ "\ {}^2\!1\quad\ "\ {}^2\!1\ {}^3\!1^2;\quad\ "\ {}^1\!0\quad\ "\ {}^3\!1\ {}^3\!1\ ]\ "\ \rangle$

^3B $|[q^2(0\ (0,0\ {}^1\!0\ ),1^2\ (-2,1^2\ {}^3\!1^2))^3\!1^2;\bar{q}^2(0\ \ (0,0\ {}^1\!0\ ),1^{2*}\ (2,1^4\ {}^3\!1\ ))^3\!1\ ]0,0,{}^3\!0\rangle$

^5A $|[q^2(1^2(0,2\ {}^3\!2\ ),0\ (\ 0,0\ {}^1\!0\ ))^3\!2\ ;\bar{q}^2(1^{10}(0,2^5\ {}^3\!2^2),0\ \ (0,0\ {}^1\!0\ ))^3\!2^2]0,0,{}^5\!0\rangle$

^5B $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (\ 0,0\ {}^1\!0\ ))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1\ ),0\ \ (0,0\ {}^1\!0\ ))^3\!1\ ]0,0,{}^5\!0\rangle$

^5B $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (\ 0,0\ {}^1\!0\ ))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1\ ),0\ \ (0,0\ {}^1\!0\ ))^3\!1\ ]1,0,{}^5\!0\rangle$

^5B $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (\ 0,0\ {}^1\!0\ ))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1\ ),0\ \ (0,0\ {}^1\!0\ ))^3\!1\ ]2,0,{}^5\!0\rangle$

^5D $|[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1\ ))^3\!2\ ;\bar{q}^2(1^{10}(0,2^5\ {}^3\!2^2),0\ \ (0,0\ {}^1\!0\ ))^3\!2^2]\frac{1}{2},-1,{}^5\!0\rangle$

^5E $|[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1\ ))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1\ ),0\ \ (0,0\ {}^1\!0\ ))^3\!1\ ]\frac{1}{2},-1,{}^5\!0\rangle$

^5E $|[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1\ ))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1\ ),0\ \ (0,0\ {}^1\!0\ ))^3\!1\ ]\frac{3}{2},-1,{}^5\!0\rangle$

^5B $|[q^2(0\ (0,0\ {}^1\!0\ ),1^2\ (-2,1^2\ {}^3\!1^2))^3\!1^2;\bar{q}^2(1^{10}(1,1^4\ {}^3\!1\ ),0\ \ (0,0\ {}^1\!0\ ))^3\!1\ ]1,-2,{}^5\!0\rangle$

$^5\bar{D}$ $|[q^2(1^2(0,2\ {}^3\!2\ ),0\ (\ 0,0\ {}^1\!0\ ))^3\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^3\!2^2]\frac{1}{2},\ 1,{}^5\!0\rangle$

$^5\bar{E}$ $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (\ 0,0\ {}^1\!0\ ))^3\!1^2;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^3\!1\ ]\frac{1}{2},\ 1,{}^5\!0\rangle$

$^5\bar{E}$ $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (\ 0,0\ {}^1\!0\ ))^3\!1^2;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2),1^*\ (1,1^5\ {}^2\!1^2))^3\!1\ ]\frac{3}{2},\ 1,{}^5\!0\rangle$

$^5F_1$ $|[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1\ ))^3\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2)\ ,1^*\ (1,1^5\ {}^2\!1^2))^3\!2^2]0,0,{}^5\!0>$

$^5F_2$ $|[\quad\ "\quad {}^2\!1\quad\quad "\quad {}^2\!1\quad {}^3\!1^2;\quad\quad "\quad {}^2\!1^2\quad\quad "\quad {}^2\!1^2\ {}^3\!1\ ]\ "\quad >$

$^5F_1$ $|[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1\ ))^3\!2\ ;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2)\ ,1^*\ (1,1^5\ {}^2\!1^2))^3\!2^2]1,0,{}^5\!0>$

$^5F_2$ $|[\quad\ "\quad {}^2\!1\quad\quad "\quad {}^2\!1\quad {}^3\!1^2;\quad\quad "\quad {}^2\!1^2\quad\quad "\quad {}^2\!1^2\ {}^3\!1\ ]\ "\quad >$

$^5\bar{E}$ $|[q^2(0\ (0,0\ {}^1\!0\ ),1^2\ (-2,1^2\ {}^3\!1^2))^3\!1^2;\bar{q}^2(1^{11}(\frac{1}{2},1^5\ {}^2\!1^2)\ ,1^*\ (1,1^5\ {}^2\!1^2))^3\!1\ ]\frac{1}{2},-1,{}^5\!0>$

^5B $|[q^2(1^2(1,1^2\ {}^3\!1^2),0\ (\ 0,0\ {}^1\!0\ ))^3\!1^2;\bar{q}^2(0\ (0,0\ {}^1\!0\ )\ ,1^{2^*}(2,1^4\ {}^3\!1\ ))^3\!1\ ]1,2,{}^5\!0>$

^5E $|[q^2(1\ (\frac{1}{2},1\ {}^2\!1\ ),1\ (-1,1\ {}^2\!1\ ))^3\!1^2;\bar{q}^2(0\ (0,0\ {}^1\!0\ )\ ,1^{2^*}(2,1^4\ {}^3\!1\ ))^3\!1\ ]\frac{1}{2},1,{}^5\!0>$

^5B $|[q^2(0\ (0,0\ {}^1\!0\ ),1^2\ (-2,1^2\ {}^3\!1^2))^3\!1^2;\bar{q}^2(0\ (0,0\ {}^1\!0\ )\ ,1^{2^*}(2,1^4\ {}^3\!1\ ))^3\!1\ ]0,0,{}^5\!0>$

CHAPTER 4

THEORETICAL MODELS

In the absence of a non-perturbative solution to QCD, theorists have had to make do with what are often referred to as "QCD inspired" models. It is not the purpose of this chapter to review all of these. In particular we shall omit discussion of dual topological unitarization (e.g. Chew and Rosenzweig, 1978) and the $1/N_c$ expansion (N_c is the number of colours, 't Hooft 1974). The former in particular has an important place in multiquark theory being essentially a "QCD inspired" extension of duality concepts; a review can be found in Montanet et al. (1980). For a discussion of the $1/N_c$ expansion the reader is referred to Witten (1979); we shall merely note here that Witten suggests that this approximation (to QCD) implies that multiquark hadrons would be hard to produce. Of more interest to us are those models which purportedly permit direct spectroscopic calculations. Our discussion shall be restricted to a few salient features of these models and their major flaws.

Before beginning it is emphasized that "QCD inspired" should not be taken to mean that these models are in any way derived from QCD. Rather, they are "educated guesses".

(a) Non-relativistic Potential Models

These have a Hamiltonian of the general form

$$H = V_{conf} + K + H_{gluon} \tag{4.1}$$

where V_{conf} is an arbitrary confining potential, K is a simple kinetic energy term

$$K = \sum_i \underset{\sim}{p}_i^{\,2}/2m_i \tag{4.2}$$

and H_{gluon} is usually just the Fermi-Breit, non-relativistic reduction of one-gluon exchange contributions (De Rújula et al., 1975). The confining potential is usually taken to be either a harmonic oscillator potential or a linearly rising potential. It is the term H_{gluon} which is of immediate interest. This contains a colour magnetic dipole-dipole interaction which is the colour analogue of the ordinary magnetic dipole-dipole interaction responsible for hyperfine splittings in atomic spectra. For zero angular momentum states this takes the form of a Fermi contact term

$$H_{gluon} \propto -\alpha_c \sum_{i>j} \frac{\underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j \; \underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j}{m_i m_j} \delta^3(\underset{\sim}{r}) \tag{4.3}$$

where $\underset{\sim}{\sigma}_i$ and $\underset{\sim}{\lambda}_i$ are the spin and colour operators acting on the ith quark, of mass m_i, $\alpha_c = g^2/4\pi$ is the strong interaction fine structure constant and $\delta^3(\underset{\sim}{r})$ is a Dirac delta function expressing the contact nature of the interaction. The form of H_{gluon} is similar in many other models

and is thought to be responsible for the gross features of the meson and baryon spectrum (De Rújula et al., 1975). For instance it predicts the π-ρ, N-Δ and Λ-Σ splittings. Of some considerable importance is that the sign of the splitting between spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ baryons would be reversed in an abelian gauge theory (e.g. QED). However, (4.3) is not entirely successful by itself. It is thought that higher order gluon exchanges are necessary to raise the degeneracy of the π and η' mesons in this model (and similarly for the ω and ρ, De Rújula et al., 1975).

In applying such models to the multiquark sector one must be wary because a confining potential which works well for mesons and baryons may not be at all appropriate to multiquark hadrons. It becomes necessary to allow a variation in V_{conf} from case to case (e.g. Gavela et al. 1978, Anderson and Joshi 1979, Barbour and Ponting 1980).

It is also worth noting that a non-relativistic treatment of confined light quarks (o and s quarks come under this category) is inconsistent with the uncertainty principle (e.g. DeGrand et al. 1975).

(b) The M.I.T. Bag Model

It is this phenomenological model, named after the Massachusetts Institute of Technology where it was developed (Chodos et al. 1974a,b, DeGrand et al. 1975, DeGrand and Jaffe 1976) which is used for the calculations in this thesis.

Rather than trying to show that quarks and gluons are confined, the bag model attempts to describe the (apparent) observation that they are. Quarks and gluons are assigned only to the region of space inside the hadron. Inside this region — called the "bag" — colour electric fields can exist and quarks and gluons interact weakly (via lowest order QCD processes). By only allowing colour neutral combinations of quarks and gluons, Gauss's law can be invoked to exclude colour electric fields from the region of space outside the bag. In order to be consistent with relativity, the bag boundary is deformable and is maintained in equilibrium by a constant pressure B, exerted by the surrounding vacuum, which balances the radiation pressure of the quarks and gluons. The pressure B can also be thought of as the energy per unit volume which must be expended by the quarks and gluons in establishing a region of space in which they can exist. (This situation is reminiscent (e.g. Jaffe 1977d) of the Meissner effect in superconductivity where ordinary magnetic flux lines can only exist trapped in "flux tubes". The connection is however speculative.)

The bag model has been reviewed by Hasenfratz and Kuti (1978) and Squires (1979) — see also Johnson (1976) for a more elementary account and Jaffe (1979b). To begin, it is a relativistic model and quite generally formulated in terms of boundary conditions and equations of motion. However, it has only been solved in the static spherical cavity approximation (Chodos et al. 1974b, DeGrand et al. 1975) and the stringlike approximation (Johnson and Thorn,

1976). It is the former solution appropriate to zero angular momentum, or S-wave, states which concerns us most. In this approximation the bag Hamiltonian is given by

$$H = E_v + E_0 + E_k + E_m \tag{4.4}$$

where the various terms are as follows (DeGrand et al. 1975).

$$E_v = \frac{4}{3}\pi B R^3 \tag{4.5}$$

is a volume term depending on B and the bag volume, $V = \frac{4}{3}\pi R^3$ where R is the bag radius.

$$E_0 = - Z_0/R \tag{4.6}$$

is a zero-point energy term arising from the quantum fluctuations. Normally, the zero-point energy can be ignored in a model by redefining the zero of the energy scale but because of the varying radius in the bag model this term must be included. The zero-point energy can be estimated using a QCD argument (Milton 1980a,b) and it is found that the contributions from the quarks and gluons differ in sign and sum to give a value for the parameter Z_0 opposite in sign and of a different magnitude to that obtained in a phenomenological fit (DeGrand et al., 1975). The reason for this seems to be at least partly due to centre of mass (c.m.) corrections — necessary because the c.m. is not really confined. These are of similar form to

the zero-point energy term and cause large changes in the fitted value of Z_0 when taken into account (Wong and Liu 1980a, Wong 1980 cf. Rebbi, 1975).

Next there is a kinetic energy term

$$E_k = [n_o\omega(m_oR) + n_s\omega(m_sR)]/R \tag{4.7}$$

where n_o and n_s are the numbers of ordinary and strange quarks of mass m_o and m_s respectively and

$$\omega(m_iR)/R = [x^2 + (m_iR)^2]^{\frac{1}{2}}/R \tag{4.8}$$

is the frequency of the lowest quark eigenmode in the bag. In this last equation $x = x(m_iR)$ is the smallest positive root of

$$\tan x = x/\{1 - m_iR - [x^2 + (m_iR)^2]^{\frac{1}{2}}\} \quad . \tag{4.9}$$

In the approximation considered here, the o quarks are considered to be massless and the strange quark mass is fitted phenomenologically. This introduces breaking of flavour symmetry and is responsible for instance for the splitting between isospin multiplets in the spin $\frac{3}{2}$ baryon decuplet.

The last term

$$E_m = -(\alpha_c/R) \sum_{i>j} \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j \; \underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j \; M(m_iR, m_jR) \tag{4.10}$$

is a colour magnetic contribution from single gluon exchange, giving rise to colour hyperfine splittings (cf. 4.3). It is often referred to as the colour-spin interaction. $M(m_iR, m_jR)$ is a radial integral given by

$$M(m_iR,m_jR) = 3\mu(m_iR)\mu(m_jR)I(m_iR,m_jR)/R^2 \tag{4.11}$$

where $\mu(mR)$ is the magnetic moment, given in the lowest quark eigenmode by

$$\mu = \frac{R}{6}\frac{(4\omega + 2mR - 3)}{2\omega(\omega - 1) + mR} \tag{4.12}$$

and

$$I(m_iR,m_jR)$$

$$= 1 + \{-\frac{3}{2}y_iy_j - 2x_ix_j\sin^2x_i\sin^2x_j$$

$$+ \tfrac{1}{2}x_ix_j[2x_iSi(2x_i) + 2x_j\,Si(2x_j) - (x_i+x_j)Si[2(x_i+x_j)]$$

$$-(x_i-x_j)Si[2(x_i - x_j)]]\}/\{(x_i\sin^2x_i - \frac{3}{2}y_i)$$

$$\times (x_j\sin^2x_j - \frac{3}{2}y_j)\} \tag{4.13}$$

in which

$$y_i = x_i - \sin x_i \cos x_i \tag{4.14}$$

and

$$Si(x) = \int_0^x (\sin t/t)\,dt\ . \tag{4.15}$$

The complete evaluation of (4.10) is not too difficult for mesons and baryons but previous authors have approximated it when considering the multiquark sector. Proper evaluation of (4.10) is the primary problem tackled in this thesis and is considered in detail in chapter 6.

Also present is a colour electric term, also coming from single gluon exchange, which can give contributions of the order of 5 MeV (DeGrand et al., 1975). It is neglected here.

The mass of a state is found by minimizing the energy eigenvalue with respect to R. By using the experimental masses for the N,Δ,Ω and ω particles as input, DeGrand et al (1975) have obtained the following fit for the parameters (m_o is arbitrarily taken to be zero)

$$\begin{aligned} B^{\frac{1}{4}} &= 146 \text{ MeV} \\ Z_0 &= 1.84 \\ \alpha_c &= 0.55 \\ m_s &= 279 \text{ MeV} \quad . \end{aligned} \tag{4.16}$$

Using these parameters the masses of the other mesons and baryons are obtained with a fair degree of success. The worst cases are the η,η' and π mesons. The model predicts the η' to be degenerate with the π and the η to be a pure $s\bar{s}$ state, too high in mass. In fact the η' is much more massive (958 MeV) than the η (549 MeV). It is thought (DeGrand et al. 1975, De Rújula et al. 1975) that inclusion of higher order gluon effects, involving the

annihilation of S = 0, $\{0\}^{fl}$ quark-antiquark pairs into two gluons and their re-emergence as a (possibly) different flavour pair, can largely account for this discrepancy with experiment. The π appears to pose more of a problem; in the bag model it turns out to have a size smaller than its Compton wavelength! Problems with the pion occur in all quark models and are thought to be related to its identification in PCAC theory (for a review see Pagels, 1975) with the massless Goldstone boson associated with the spontaneous breaking of chiral $SU_2 \times SU_2$ symmetry. Attempts at rectifying the bag model of the pion have been made, for example, by Donoghue and Johnson (1980) and Goldman and Haymaker (1980).

The remarkable thing about the bag model is that all its parameters can be fitted in the meson and baryon sector. At first sight it seems as though it can be applied without modification to multiquark systems. Jaffe (1977a,b) calculated an approximate spectrum for $q^2\bar{q}^2$ (and also for q^6, Jaffe 1977c) and noticed several things. Firstly, the masses of multiquark hadrons turned out to be rather low — some less than 1 GeV — and well within experimentally accessible limits. Secondly, those with quantum numbers shared by mesons and baryons ("cryptoexotics") were lower in mass than those with exotic quantum numbers. This last facet is a consequence of the colour-spin interaction and thus the result is more general than the bag model. Jaffe (1977a, Jaffe and Johnson 1976) noted that it was possible that some multiquark states could have been misidentified with ordinary hadrons and suggested that the $J^{PC} = 0^{++}$

mesons $\varepsilon(700)$, $S^*(993)$, $\delta(976)$ and $K(800 - 1100)$, usually taken to be orbitally excited, $\ell = 1$ $q\bar{q}$ states, might in fact be $q^2\bar{q}^2$ cryptoexotics. This assignment has been both supported and disputed (e.g. Holmgren and Pennington 1978, Greenhut and Intemann 1979, Bramon and Massó 1980, Achasov et al. 1980; note that one must find alternative candidates for the $\ell = 1$ $q\bar{q}$ states and there is indeed some evidence for such states e.g. Martin 1978) but the bag model also predicts lots of exotics for which there is no evidence. (The resonant nature of the 0^{++} mesons is also suspect.)

It is well-known that a zero-width approximation will sometimes give a rather poor estimate of the mean energy of a state. The static spherical cavity approximation is such a zero-width approximation. However, the situation is even worse than this. As we have already seen, all multiquark systems can be divided into colour-neutral subsystems and will spend a fraction of their lifetime as such. However, colour neutral objects are not confined by Nature and yet the bag model does just that by its imposition of artificial boundary conditions. It has been argued at this point that the bag model will be classically unstable against fissioning and, provided that energy conservation allows it, will decay into separate hadrons. Note that such a process does not require the creation of quark-antiquark pairs as in common strong decays; it is a zeroth-order process in the strong interaction coupling constant and is often referred to as superallowed.

But is this really a decay? The term decay is appropriate to a quasi-stationary state. Do quasi-stationary states really

exist or are they simply artefacts of the artificial boundary condition? We shall return to this question in chapter 5.

(c) High-ℓ Stability?

It is appropriate at this stage to mention a mechanism for inhibiting dissociation processes which has received a great deal of attention.

Consider a $q^2\bar{q}^2$ system in which there exists a diquark cluster separated from an antidiquark cluster. Recall from fig. 1.2 that the diquark can possess either a triplet, $\{1^2\}$ or a sextet, $\{2\}$ colour charge. A pure colour triplet-triplet system can couple "strongly" to baryon-antibaryon channels by a pair creation process as in fig. 2.2. For a pure sextet-sextet state though, this process is colour inhibited because $\{2\}^C$ cannot couple with the colour of a quark, $\{1\}^C$ to produce a colour neutral hadron. Recalling duality and baryonium, pure colour triplet-triplet states are called true or T-baryonium and pure colour sextet-sextet states are called mock or M-baryonium (Chan and Hogaasen, 1977). (Some authors refer to these states as diquarkonium or simply diquonium to distinguish them from baryon-antibaryon resonances of a more general origin.) In S-wave states the colour-spin interaction mixes most (but not all!) of these states. However, the strength of this interaction fades with distance and as the separation of diquark clusters should increase with high angular momentum, ℓ this leads to pure T and M states. (Note that it is not stated what high is. It is common to

apply these ideas to ℓ = 1,2,3!) Similar notions can be applied to all sorts of clusters in any multiquark hadron and leads to the concept of "colour chemistry" (Chan and Hogaasen 1978a, Chan et al. 1978, Chan 1980). The cluster is treated as a "chromion", the colour flux between clusters as a "colour bond" and the multiquark hadron as the colour analogue of an ionic molecule!

At high ℓ the energy of a hadron of this sort is supposed to be contained mostly in the colour flux tube separating the clusters. In the stringlike approximation to the bag model (Johnson and Thorn, 1976), the energy density of this tube is proportional to $\sqrt{C_\mu}$ where C_μ is the value of the quadratic Casimir operator acting on the cluster with colour $\{\mu\}^C$; it is also independent of ℓ. (This picture is thought to at least partially explain the linearity and parallel slopes for the Regge trajectories of mesons and baryons.) Note that a key assumption in the application of this approximation to the bag model is that clusters must be "tightly knit" and located at opposite ends of the stringlike bag. Now it so happens that

$$\sqrt{C_\mu} < \sqrt{C_{\mu_1}} + \sqrt{C_{\mu_2}} \qquad (4.17)$$

whenever $\{\mu_1\} \times \{\mu_2\} \supset \{\mu\}$ and therefore it is not possible (Chan and Hogaasen 1978b, Chan 1980) for, for instance, a $q^2 - \bar{q}^2$ system to split along its length into two high-ℓ mesons (fig 4.1). (The length is assumed to stay the same otherwise an angular momentum barrier must be overcome; Chan

Fig. 4.1: Splitting of a high-ℓ diquonium into two high-ℓ mesons.

et al., 1978). The angular momentum barrier is supposed to prevent quarks and antiquarks from moving from one cluster to another and dissociating that way so high angular momentum states are argued to be stable against dissociation.

Thus although typical hadronic widths (~ 100 MeV) are expected for T-baryonium because of pair creation processes, M-baryonium is expected to be narrow (~ 10 MeV). The easiest decay mode for M-baryonium appears to be a cascade process to another M-baryonium state with smaller ℓ via the emission of a meson (Chan and Hogaasen, 1977). The cascade continues until ℓ is small enough for appreciable mixing to occur, thereby allowing a decay via pair creation in the T state.

Ideas such as these became very popular a few years ago when a narrow state at 2.95 GeV was seen in a $p\bar{p}$ reaction, decaying into states at 2.204 GeV and 2.020 GeV via π emission (Evangelista et al., 1977). Other "good" candidate states for baryonia were the S,T,U and V resonances, assumed to be T-baryonia, which fall on a straight line in a Chew-Frautschi plot, of J vs m^2. (See the Particle Data Group tables, 1980 for references.) Fukugita (1980) has listed several other baryonium candidates as well as some unusual baryon ($q^4\bar{q}$) candidates. Mulders et al. (1980) discuss some candidates for q^6 states. Unfortunately, perhaps, doubt has been cast on the authenticity of these states after some subsequent experiments have failed to find some of them; many others have only been observed once. An appraisal of the status of the prime baryonium candidates can be found in Montanet et al. (1980). Even the "textbook" baryonium S(1936) which had appeared in several experiments (see Particle Data Group, 1980) has now been placed under suspicion. A recent experiment (Hamilton et al. 1980, see also Kamae et al. 1980) found no more than a broad and gentle enhancement of the background which may or may not be a resonance. The current experimental situation does not rule out the existence of narrow baryonia but it certainly places them under a great deal of suspicion. Although Hogaasen and Sorba (1980) are optimistic about the correspondence between colour chemistry models and experimental states in the $q^4\bar{q}$ sector it is, at this early stage, wise to be suspicious of the interpretation of the data — not only in

respect of its resonant nature but also the identification of states with multiquark systems (e.g. Bowler et al. 1980).

With the experimental situation in doubt it is worthwhile taking a closer look at the theoretical reasoning behind these states. The entire argument hinges on the assumed existence of closely knit clusters with large separations between them. There is no firm evidence to support this conjecture! Further, not too much weight can be placed on the argument against splitting dissociations. In employing the bag model one is using a model which essentially assumes that such dissociations do not take place anyway. It is in this sense a rather empty prediction. Chan and Hogaasen (1978b) have attempted to show that the result is more general than the bag model. They argued that the volume energy density must be proportional to C_μ divided by the cross-sectional area, A of the colour flux tube. But it is precisely the confining bag boundary conditions together with the assumption of tightly knit clusters at the tube ends which give $A \propto \sqrt{C_\mu}$ (Johnson and Thorn, 1976) and hence the relationship for the linear density. Chan and Hogaasen (1978b) also noted that a string picture predicts that M-diquonium is unstable against dissociation. (In contrast to the bag model, the energy density in the string picture depends on the number of quarks at the ends of the flux tube, or string.) They rejected this model in favour of the bag picture on experimental grounds i.e. the narrow state at 2.95 GeV — which has since been discredited! It is also interesting to note that calculations using non-relativistic potential

models (Gavela et al. 1978, Barbour and Ponting 1980) also show that M-baryonium has a strong tendency to decay into mesons, at least for small values of ℓ.

It seems certain that early unsound experimental results have confounded theoretical thinking on multiquark hadrons. In this respect the talk by Pietrzyk (1980) makes interesting reading. The article by Hey (1980) further underlines the weaknesses in the once orthodox picture of multiquark hadrons. Bearing in mind that we are only just beginning to understand the production and decay mechanisms for ordinary mesons and baryons (due largely to the work of Isgur and Karl — see Hey, 1979 and Koniuk and Isgur, 1980 for a review and references) it is probably premature to consider high-ℓ states until the S-wave states are properly understood. Therefore, for the remainder of this thesis we shall only consider the less problematic S-wave states.

CHAPTER 5

P-MATRIX FORMALISM

After having virtually dismissed the M.I.T. bag model in the last chapter as being irrelevant we shall now go about restoring at least some confidence in it. In an intriguing paper, Jaffe and Low (1979) showed that even if bag model eigenstates and eigenenergies have little connection with observable states and their physical masses, the eigenenergies can still be related to measurable quantities. The purpose of this chapter is to provide a brief, but comprehensive, survey of their idea. More detailed information can be found in the original paper (Jaffe and Low, 1979) and the lectures by Low (1979). (See also Jaffe 1978, 1979a for some early accounts.)

(a) The "Square-well" Problem

Jaffe and Low (1979) presented the following pedagogic example which provides an easy way to grasp the gist of their idea. Consider non-relativistic S-wave scattering by a shallow spherical square-well, fig. 5.1. This problem can be tackled by a separation of variables

$$\psi_{\ell}^{m}(r,\theta,\phi) = Y_{\ell}^{m}(\theta,\phi)\ \chi_{\ell}(r) \qquad (5.1)$$

where the Y_ℓ^m are spherical harmonics and $\chi_\ell(r)$ is a radial wavefunction. The stationary states are then found by solving the radial Schrödinger equation both inside and outside the well, subject to boundary conditions at the origin and infinity, and matching the solutions and their logarithmic derivatives at r = b. (This problem is treated in nearly every elementary text on quantum mechanics e.g. Schiff 1968, Messiah 1961 vol. I.) The *S*-wave solutions are

$$\chi_0(r) = A\, j_0(qr) = A\,\frac{\sin qr}{qr} \qquad r < b \tag{5.2}$$

$$\chi_0(r) = B\, h_0^{(+)}(kr) = B\,\frac{e^{+ikr}}{kr} \qquad r > b, \quad E < 0 \tag{5.3}$$

$$\begin{aligned}\chi_0(r) &= C[\cos\delta_0\, j_0(kr) + \sin\delta_0\, n_0(kr)] \\ &= C[\cos\delta_0\left(\frac{\sin kr}{kr}\right) + \sin\delta_0\left(\frac{\cos kr}{kr}\right)] \\ &= C\,\frac{\sin(kr+\delta_0)}{kr} \qquad r > b,\ E > 0\end{aligned} \tag{5.4}$$

with logarithmic derivatives

$$\chi_0'(r)/\chi_0(r) = q\cot qr - \frac{1}{r} \tag{5.5}$$

$$\chi_0'(r)/\chi_0(r) = +ik - \frac{1}{r} \tag{5.6}$$

$$\chi_0'(r)/\chi_0(r) = k\cot(kr+\delta_0) - \frac{1}{r} \tag{5.7}$$

respectively. In these equations, A,B and C are arbitrary (complex) normalization constants,

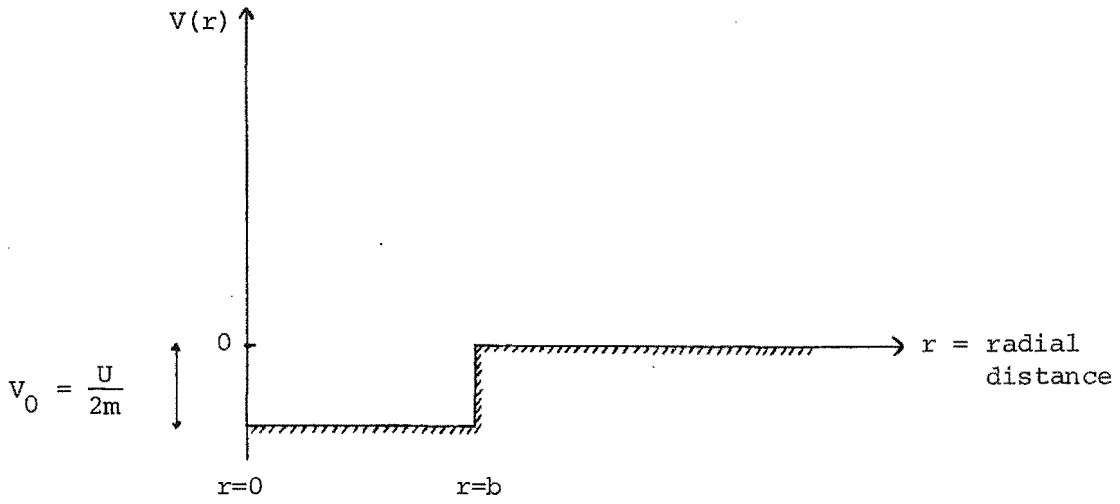


Fig. 5.1: Spherical square-well potential

$$k = + (2mE)^{\frac{1}{2}} \tag{5.8}$$

is the momentum of the scattering particle with mass m and kinetic energy E (note that for $E < 0$ we have chosen $\mathrm{Im}\, k > 0$ so that $h_0^{(+)}(kr)$ is an exponentially decreasing function),

$$q = (k^2 + U)^{\frac{1}{2}} \tag{5.9}$$

and δ_0 is the S-wave phase-shift. The phase-shift fixes the linear combination of spherical Bessel and spherical Neumann functions that forms an acceptable solution and is determined by equating the logarithmic derivatives at $r = b$ i.e. solving

$$q \cot qb = k \cot(kb + \delta_0) \quad . \tag{5.10}$$

Thus knowledge of the phase-shift determines the scattering states.

Suppose that we impose the artificial boundary condition that the internal wavefunction vanishes at r = b, — this is equivalent to approximating the potential by an infinite square well! — then we create an infinite set of internal states at

$$q_n b = n\pi \quad . \qquad (n = 1,2, \ldots) \tag{5.11}$$

Jaffe and Low term these artificial states "primitives". Clearly they have little relevance to the actual bound states of this problem — found by solving

$$q \cot qb = +ik = - Im(k) \quad . \tag{5.12}$$

Indeed there are no bound states at all unless (e.g. Schiff 1968)

$$V_0 > \pi^2/8mb^2 \quad . \tag{5.13}$$

(Note that there are primitives even when U = 0).

Nevertheless, as we increase the energy the scattering wave-function will still vanish at r = b for various energies. But by continuity these are precisely the energies at which the internal wavefunction vanishes i.e. the energies

of the primitives. Clearly one can identify the primitives through the scattering wavefunction by looking for poles in the quantity

$$P = k \cot (kb + \delta_0) \quad . \tag{5.14}$$

Of course, in a problem as simple as this one there is no need to consider the primitives at all; one can solve the phase-shift exactly. However, the spherical cavity approximation to the M.I.T. bag model is a covariant version of an infinite square-well potential — see the reference by DeGrand et al (1975) to Bogoliubov (1967) — and calculated eigenenergies can not be expected to have a direct correspondence with actual bound states or resonances, especially when dissociation can occur. Rather the eigenstates are primitives and should instead be identified with poles in a quantity like P in the dissociation (or, for ordinary hadrons, decay) channels.

(b) Low-energy Scattering and the P-matrix

Jaffe and Low (1979) assume that outside a relative separation r = b in the centre of mass (c.m.) frame the n-channel two-hadron system is free and that continuum channels are unimportant; the true nature of the interaction potential inside b is unknown. Considering S-wave scattering, they then parametrize the exterior radial scattering wavefunction by

$$r\chi_0(r_j)_i = \delta_{ij} \cos k_j(r_j - b) + \frac{P_{ji}}{k_j} \sin k_j(r_j - b) \qquad (5.15)$$

(where j labels the channel and i the scattering state — i, j = 1, ... , n). In this equation, k_j is the channel momentum which is given in a nonrelativistic system simply by $k_j = (2\mu_j E_j)^{\frac{1}{2}}$ where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass (which differs from channel to channel) and E is the total kinetic energy in the c.m. frame. However, for a relativistic system one must use equation IA.10. (Actually 5.15 is only an approximation for a relativistic system — see Roiesnel, 1979).

Equation (5.15) is to be taken as the defining relation for the S-wave P-matrix. One notes that when P has a pole, the second term dominates (5.15) and thus the pole corresponds to a state for which the exterior wavefunction vanishes at r = b. (To see this it is helpful to change the arbitrary normalization of (5.15) by a factor of P^{-1}, e.g. Low 1979.) Therefore the interior wavefunction also vanishes by continuity and we have a primitive. Jaffe and Low (1979) show that the P-matrix is related to the scattering S-matrix by

$$S \equiv e^{2i\delta} = - e^{-ikb} \cdot \frac{1 - \left(\frac{i}{\sqrt{k}}\right) P \left(\frac{1}{\sqrt{k}}\right)}{1 + \left(\frac{i}{\sqrt{k}}\right) P \left(\frac{1}{\sqrt{k}}\right)} \cdot e^{-ikb} \qquad (5.16)$$

and therefore can be constructed from scattering data.

In the one channel case, solving (5.16) yields

$$P = k \cot [kb + \delta(k)]$$

as in (5.14). It is interesting to note that when $\delta = 0$ (the no interaction case), P has a pole at

$$k_c = \pi/b \quad . \tag{5.17}$$

Jaffe and Low call the energy corresponding to k_c the "compensation" energy, E_c. If $\delta > 0$ (attractive hadron-hadron potential) then the first pole in P is at

$$k < k_c$$

whereas if $\delta < 0$ (repulsive potential) then the first pole is at

$$k > k_c \quad .$$

Hence, if one calculates the energy of a primitive to be below E_c this is tantamount to predicting a positive phase-shift and an attractive potential in the real problem. This turns out to be very useful.

It is possible in a two-channel problem to use a single-channel P-matrix below the second threshold. However, Jaffe and Low (1979, Low 1979) show that the effect of a nearby closed channel is to produce an effective open channel

P-matrix, $\tilde{P}$ with displaced poles. A (second) pole in $\tilde{P}$ below the second channel threshold will then be at a different energy from the true pole to be found above the threshold in P.

Jaffe and Low (1979) have given a derivation of the P-matrix for both the two-channel problem and higher partial waves. They emphasize however, the inability of this formalism to parametrize multibody channels and thus it is only useful at low energies before the proliferation of multibody thresholds. Consequently, it is also of no value in the three meson dissociations of the $q^3\bar{q}^3$ system.

Consider now the pole residues. These turn out to play an important role in the phenomenology. In the vicinity of a pole at $s = s_0(b)$, the P-matrix can be written in terms of its residue by

$$P_{ij}(b,s) = \frac{r(b)\ Q_{ij}}{s - s_0(b)} \quad . \tag{5.18}$$

It can be shown (Jaffe and Low 1979, Low 1979) that to leading order

$$r(b) = - \frac{ds_0}{db} \tag{5.19}$$

while $Q = Q^2$ is a projection operator, which in the absence of accidental degeneracy (of poles in P) factorizes:

$$Q_{ij} = \xi_i\ \xi_j \quad . \tag{5.20}$$

Here ξ_i is the projection of the scattering state at s_0 onto the physical channel space, labelled by i. It is therefore related to the coupling of the primitive to external channels.

If there is actually a physical barrier, at R_B, which creates the internal state then moving the artificial barrier at $b > R_B$ further outwards should cause negligible change in s_0. However, if the system is largely a creation of the artificial barrier then s_0 will be very sensitive to any changes in b — for any reasonable system, s_0 will decrease as b increases (Jaffe 1979a, Low 1979). Thus, according to (5.19), the residue should be a measure of the presence or absence of a physical barrier. Inserting (5.18) into (5.16) shows that the S-matrix will (in the one channel case) have a pole at

$$s = s_0 - i\, r(b)/k \ , \qquad (5.21)$$

which will be near the real axis if r(b) is small and therefore generate a resonance. It must be emphasized however (Jaffe and Shatz, 1980) that the condition $b > R_B$ can only be reasonably satisfied for a deeply bound system. For a loosely bound system, its spatial extent becomes larger as the binding energy decreases so that the condition $b > R_B$ becomes unreasonable. (One notes that if we could send b to infinity then we would be able to calculate the bound states directly!)

(c) Connection with the Bag Model

What then are the primitives of a two hadron system confined to a spherical region of space in their c.m. frame? The fact that at close distances quarks experience only relatively feeble forces means that for a small radius the internal degree of freedom should be that of a multiquark system — not just a two hadron system. We can reasonably expect that the static spherical cavity approximation to the bag model will give a good description of this circumstance; provided that the radius b at which the two-hadron wavefunction is required to vanish corresponds to the radius R_0 of the bag for which the energy of the primitive has a minimum. (One cannot expect the bag model to work for any R since the confining boundary has to be in equilibrium.) The question is: what is the relationship between b and R_0? (It should be clear that the spherical bag is not the same as the sphere of radius b.)

Jaffe and Low (1979) solve the problem by calculating the effective two-body density in the spherical bag — note that one must take account of c.m. motion — and the density for a two-hadron wavefunction with its first zero at r = b and then equating the root-mean-square (r.m.s.) values of the relative separation. The resulting relationship between b and R_0 depends on the hadrons involved and is given below (Jaffe and Low 1979, Roiesnel 1979, Jaffe and Shatz 1980 respectively).

$$q^2\bar{q}^2: \qquad b \simeq 1.4\ R_0 \tag{5.22}$$

$q^4\bar{q}$: $b \simeq 1.25\ R_0$ (5.23)

q^6 : $b \simeq 1.1\ R_0$ (5.24)

Thus one associates bag model eigenstates with hadron-hadron scattering states which vanish at a relative separation, in the c.m. frame, of $r = b$ where for $q^2\bar{q}^2$, $q^4\bar{q}$ and q^6, b is given by (5.22), (5.23) and (5.24) respectively.

In constructing the P-matrix one is faced with the problem that if it has two poles at different energies then since R_0 will be different so will b. One way of surmounting this is to express b in terms of the scattering energy. Jaffe and Low (1979) use a bag model virial theorem

$$R_0 = \left(\frac{3}{16\pi B}\right)^{\frac{1}{3}} s_0^{\frac{1}{6}} \quad \text{GeV}^{-1} \tag{5.25}$$

for s_0 in GeV^2, so that the required relationships are as follows.

$q^2\bar{q}^2$: $b = 7.0\ s^{\frac{1}{6}}$ (5.26)

$q^4\bar{q}$: $b = 6.4\ s^{\frac{1}{6}}$ (5.27)

q^6 : $b = 5.7\ s^{\frac{1}{6}}$ (5.28)

However, this theorem is only exactly true for massless quarks and will introduce errors as much as 10% or more for systems containing s quarks. The actual values of b will be somewhat less than those predicted by (5.26-28).

The compensation energy can be calculated by inserting (5.26-28) in (5.17) and solving (IA.10) for the energy

(in GeV). This amounts to finding the first root above threshold of one of the following polynomials.

$$q^2\bar{q}^2: E^4 - 2(m_1^2 + m_2^2)E^2 + (m_1^2 - m_2^2)^2 - 0.80\, E^{\frac{4}{3}} = 0 \qquad (5.29)$$

$$q^4\bar{q}: E^4 - 2(m_1^2 + m_2^2)E^2 + (m_1^2 - m_2^2)^2 - 0.96\, E^{\frac{4}{3}} = 0 \qquad (5.30)$$

$$q^6: E^4 - 2(m_1^2 + m_2^2)E^2 + (m_1^2 - m_2^2)^2 - 1.21\, E^{\frac{4}{3}} = 0 \qquad (5.31)$$

One can also extract from the data the residues of the poles and the couplings of the primitives to the scattering channels. These present a more difficult problem for the bag model. In order to calculate the residue, one has to ask: what would be the change in mass of the primitive if the constraining radius b was increased? As remarked earlier, the spherical cavity approximation is inherently incapable of determining the mass for arbitrary bag radii but Jaffe and Low (1979) made the following assumptions in order to obtain some sort of estimate. The bag model eigenstate will have projections onto confined and unconfined components. Some of the unconfined components will have channel thresholds above the primitive energy and so these dissociation channels are closed as well as all confining channels. Denote projections onto open channels by ζ_o and closed channels by ζ_c; assume that at R_0 these projections are a true indication of the actual content of the primitive. As the constraining radius is increased the multiquark system will begin to expand into its open channels. However, separation of the colour neutral components should not be significant until these can fit comfortably into the allowed

volume. Assume therefore that the multiquark description is still a good one and that the expansion can be taken into account simply by switching off the bag pressure in the open channels. Then the new Hamiltonian of the system, constrained to the radius $R > R_0$ (but not too large!), is given by

$$H = H_B(R) - \langle\Lambda\rangle \tfrac{4}{3} \pi B R^3 \tag{5.32}$$

where $H_B(R)$ is the spherical cavity bag Hamiltonian (4.4) and

$$\langle\Lambda\rangle = \sum_o \zeta_o^2 \tag{5.33}$$

is the probability that the primitive will be in an open channel at $R = R_0$. It is then a simple matter to show (Jaffe and Low, 1979) that

$$\left.\frac{ds_0}{db}\right|_{R = R_0} = -\frac{3}{2}\frac{s_0}{b} \langle\Lambda\rangle \quad . \tag{5.34}$$

Further, if one assumes that the relative strengths of the projections ζ_o do not change as the system expands into the open channels then one obtains the channel couplings

$$\xi_o = \zeta_o/(1 - \sum_c \zeta_c^2)^{\frac{1}{2}} \quad . \tag{5.35}$$

(Note that the projection ξ_o is only onto physical channels; hence we must change the normalizations.)

Of course, as Jaffe and Low emphasize, the approximations made here are gross. Roiesnel (1979) has suggested modifying (5.33) because some confined channels can become open by exchanging a gluon in a first order process. Thus a crude refinement of (5.33) would be

$$<\Lambda> = \sum_{o} \zeta_o^2 + \alpha_c \, O(1) \sum_{c'} \zeta_{c'}^2, \qquad (5.36)$$

where c' denotes those confined channels that can become open in first order. However, it is not at all clear that this provides a fair description either, although it does admit couplings of ordinary mesons and baryons to open channels. (Actually Roiesnel uses $\alpha_c{}^2$ but his reasoning is obscure and seems erroneous; one power of α_c is more natural.)

Thus, although we expect the bag model to give a good description of the pole positions we must regard predicted residues and even couplings as rather crude estimates.

(d) Experimental Pole Positions

Application of the P-matrix formalism is handicapped by lack of experimental data on phase shifts and the profusion of multibody channels. However, Jaffe and Low (1979), Roiesnel (1979) and Jaffe and Shatz (1980) have been able to determine a few S-wave pole parameters for $q^2\bar{q}^2$, $q^4\bar{q}$ and q^6 respectively. These are listed in table I10.

Consider the $q^2\bar{q}^2$ sector. Table I10 includes data on $\pi\pi$, πK and $K\bar{K}$ scattering, both in exotic and nonexotic channels.

The $\pi\pi$ compensation energy is (from 5.29) 0.95 GeV while the πK and $K\bar{K}$ compensation energies are 1.11 and 1.29 GeV respectively. Jaffe (1977a,b) has calculated the approximate masses of $q^2\bar{q}^2$ primitives. These bag model predictions include primitives at 1.15 and 1.35 GeV in the $\pi\pi$ I = 2 and πK I = $\frac{3}{2}$ exotic channels respectively. Both primitives are above the relevant compensation energies, signalling negative phase-shifts corresponding to repulsive potentials. Bearing in mind the approximations involved, the agreement with the observed phase-shifts and poles is quite remarkable. (The predicted residues are too small but we need not be perturbed by that.) Thus the problem of low-lying exotics predicted by the bag model is resolved; although the data show no exotic resonances they instead reveal the exotics by falling phase-shifts with P-matrix poles close to the values predicted by the bag model.

The nonexotic channels are just as revealing and bring us back to the cryptoexotic nonet ($\{0\} + \{21\}$ in SU_3^{fl}) referred to in the last chapter. Jaffe's (1977a,b) predictions put primitives at 0.65, 1.10 and 0.90 GeV in the $\pi\pi$ I = 0, $K\bar{K}$ I = 0 and πK I = ½ nonexotic channels respectively. All primitives are below the relevant compensation energies, signalling positive phase-shifts corresponding to attractive potentials. Again the agreement with the observed phase-shifts and pole positions is quite remarkable. The primitive in the $K\bar{K}$ channel, which is just above threshold, also couples to the $\pi\pi$ channel and induces a pole with a small residue in the

reduced $\pi\pi$ single-channel P-matrix. This pole is associated with the narrow S*(993) "resonance". (Jaffe and Low (1979) have noted a problem with the S*(993) coupling to $\pi\pi$ which is forbidden in the OZI limit with their identification, though they did suggest mechanisms for generating the coupling. We shall return to this question in chapter 7.) The poles at 0.69 and 0.96 GeV are associated with the broad enhancements known as the $\varepsilon(700)$ and $\kappa(800\text{-}1100)$. Lack of any $\pi\eta$ phase-shift analysis prevented Jaffe and Low from considering the I = 1 pole in this channel, presumably associated with the $\delta(980)$, and predicted to be degenerate with the S*(993). However, the identification of all these effects with $q^2\bar{q}^2$ primitives looks promising. Note that these enhancements are not to be regarded as resonances; they are generated by P-matrix poles associated with non-resonant phase shifts. (The narrowness of the S*(993) is to be associated with threshold effects.)

The poles found by Roiesnel (1979) in the $q^4\bar{q}$ sector also fit bag model predictions (Strottman 1979) fairly well. (A complication in meson-baryon scattering is the rapid onset of multibody channels with two pions in the final state.)

In nucleon-nucleon scattering, final state interactions cannot be ignored. Jaffe and Shatz (1980) have modified the P-matrix to take this into account. They find poles in pn channels, above E_c = 2.07 GeV, which are again in reasonable agreement with bag model predictions (Jaffe 1977c) although the observed isosinglet $^3S_1$ and isotriplet $^1S_0$ poles are

nearly degenerate whereas the bag model predicts a splitting ~ 100 MeV. They point out, however, that the model calculation does not include *S*-wave *D*-wave mixing which is known to be significant. Of some importance is that these primitives have no relation with the bound deuteron and virtual dinucleon states. As Jaffe and Shatz (1980) emphasize, the P-matrix formalism has no correspondence with loosely bound states.

It is worth mentioning that Jaffe and Low (1979) analysed the *P*-wave $\pi\pi$ P-matrix and found a pole at 788 MeV, with a residue 0.03 GeV^3, associated with the ρ meson S-matrix pole at 770 MeV. The ρ meson couples to $\pi\pi$ in a relative *P*-wave by a first order process involving creation of a quark-antiquark pair and there is a genuine physical barrier inhibiting this. Consequently the ρ meson is deeply bound — the *P*-wave compensation energy is 1.23 GeV (Jaffe and Low, 1979) — and the artificial confinement imposed by the spherical cavity approximation has little effect. Therefore, (5.21) correctly implies a resonance.

Thus the P-matrix concept looks a good one. It has caused a drastic change in thinking from the situation prevalent a few years ago when it was generally thought that superallowed processes would simply imply very broad resonances. Instead of interpreting spherical cavity eigenstates as resonances we now see that they must be regarded as P-matrix poles, seen in the continuum region of dissociation channels. This point of view will be adopted for the remainder of this thesis where we shall be concerned

with a refined evaluation of the spherical cavity eigenenergies. Further discussion of the poles in table I10 will come in chapter 7 after we have generated a more accurate spectrum.

Table I10: P-matrix pole parameters

$q^2\bar{q}^2$:

channel ($J^{PC} = 0^{++}$)	δ_o	pole location (GeV)	residue (GeV^3)	ξ_i
I = 0 (nonexotic) $\pi\pi$	+ve	$0.69, 0.98_{eff}$	$0.064, 0.009_{eff}$	0.8
$+K\bar{K}$	+ve	(1.04)	(0.10)	0.6
I = ½ (nonexotic) πK	+ve	0.96	0.079	-
I = 2 (exotic) $\pi\pi$	-ve	1.04	0.21	-
I = $\frac{3}{2}$ (exotic) πK	-ve	1.19	0.22	-

$q^4\bar{q}$:

channel ($J^P = \frac{1}{2}^-$)	δ_o	pole location (GeV)	residue (GeV^3)	ξ_i
I = 0 (nonexotic) $\pi\Sigma$		1.41_{eff}	0.006_{eff}	0.6
$+\bar{K}N$		1.45	0.052	0.8
I = 1 (nonexotic) $\bar{K}N$		(1.54)	?	-
I = ½ (nonexotic) πN	+ve	1.43	0.14	-
I = $\frac{3}{2}$ (nonexotic) πN	-ve	(1.56)	?	-
I = 0 (exotic) KN	-ve	1.705 ± 0.010	0.19 ± 0.01	-
I = 1 (exotic) KN	-ve	1.78_{eff}	0.27_{eff}	-
$+K^*N$		?	?	

q^6:

channel	δ_o	pole location (GeV)	residue (GeV^3)
I = 0, $^3S_1$ pn		2.10	0.39 ± 0.02
I = 1, $^1S_0$ pn		2.11	0.37 ± 0.05

CHAPTER 6

COLOUR-SPIN MATRIX ELEMENTS

We come now to the problem of evaluating the colour-magnetic term (4.10) in the bag model Hamiltonian. The mathematics is complicated by the presence of the radial integral $M(m_i R, m_j R)$: if all quarks in the hadron are of the same mass then this integral can be removed from the summation in (4.10) and the remaining sum can be expressed in terms of simple group theoretic operators but if the masses of the quarks differ then the radial integrals spoil this correspondence. To overcome this, Jaffe (1977 a,b) took an average value for $M(m_i R, m_j R)$,

$$\bar{M}(R) = M\left(\frac{n_s m_s}{N} R, \frac{n_s m_s}{N} R\right) \tag{6.1}$$

where n_s is the number of s quarks (or antiquarks) of mass m_s in a state for which the total number of quarks plus antiquarks is N. If $n_s = 0$ or N then this procedure is exact but in between it amounts to a linear interpolation. Wybourne (1978a) and Strottman (1978,79) for example have copied Jaffe's approximation but Mulders et al. (1979) use a different averaging procedure

$$\bar{M}(R) = \frac{\frac{1}{2}n_o(n_o-1)M(m_o R,m_o R) + \frac{1}{2}n_s(n_s-1)M(m_s R,m_s R) + n_o n_s M(m_o R,m_s R)}{\frac{1}{2}N(N-1)} \tag{6.2}$$

and further, calculate the energies at an estimated average radius (Aerts et al. 1978)

$$R_{av} \approx r_0 \, N^{\frac{1}{3}} \quad \text{where } r_o = 3.63 \text{ GeV}^{-1} \tag{6.3}$$

rather than conducting a minimization procedure.

Important spectroscopic structure can be concealed by using an averaging procedure such as (6.1) or (6.2). For example, in the ordinary baryon sector it fails to predict the Λ-Σ splitting. This kind of approximation can be confusing in the multiquark sector where it is not always easy to distinguish these artificially induced degeneracies of the colour-magnetic interaction. Further, even in the absence of this degeneracy, the masses thus obtained are only approximate. It is the purpose of this chapter to show how the colour magnetic interaction may be evaluated exactly.

To begin, we can for the sake of brevity write

$$M_{ab} \equiv M(m_a R, m_b R) \tag{6.4}$$

where a and b denote any quark species (i.e. flavour) and introduce the colour-spin operator

$$\Delta_g^{ab} = - \sum_{i<j}^{a,b} \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j \, \underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j \tag{6.5}$$

where the summation over i and j is understood to be over quarks of flavour a and b respectively. The colour-magnetic interaction then becomes

$$E_M = (\alpha_c/R) \sum_{a \leqslant b} \Delta_g^{ab} M_{ab} \qquad (6.6)$$

where the summation is over all pairs of quark flavours that can arise in a given multiquark configuration e.g. in the specific configuration $o^2 s^2 \bar{o}$ the summation includes the terms oo, os, $o\bar{o}$, ss and $s\bar{o}$. One understands that while M_{ab} does not depend on whether a or b represent quarks or antiquarks, the colour-spin operator (6.5) must be replaced in the presence of antiquarks by

$$\Delta_g^{a\bar{b}} = \Delta_g^{a\bar{b}} = - \Delta_g^{\bar{a}\bar{b}} = - \Delta_g^{ab} \ . \qquad (6.7)$$

The necessity for this goes right back to the derivation of the colour-magnetic term from the relevant Feynman diagram and is a consequence of the inclusion of a factor of (-1) for each antiparticle in the initial state. Note that here we interpret $\underset{\sim}{\sigma}_i$ and $\underset{\sim}{\lambda}_i$ as operators (namely the generators of SU_2^S and SU_3^C) acting on the ith quark (or antiquark) and no replacement for them is necessary.

The key problem now is to evaluate the colour-spin operator.

(a) <u>The Generators of $SU_6^{CS} \supset SU_2^S \times SU_3^C$ and Δ_g^{ab}</u>

The generators of SU_6^{CS} belong to the 35-dimensional adjoint representation $\{21^4\}^{CS}$. Under the restriction to $SU_2^S \times SU_3^C$ we have

$$\{21^4\} \to {}^3\{21\} + {}^3\{0\} + {}^1\{21\} \ . \qquad (6.8)$$

The three spin generators $\underset{\sim}{\sigma}$ transform as ${}^{3}\{0\}^{C}$ under $SU_2^S \times SU_3^C$ and generate the Lie algebra associated with the spin group SU_2^S while the eight colour operators $\underset{\sim}{\lambda}$ transform as ${}^{1}\{21\}^{C}$ and generate SU_3^C. In (6.5) they are normalized to $Tr\ \sigma^a\sigma_b = 2\delta^a{}_b$ and $Tr\ \lambda^a\lambda_b = 2\delta^a{}_b$ in the defining representations. Further, the twenty-four operators $\underset{\sim\sim}{\sigma\lambda}$ transform as ${}^{3}\{21\}^{C}$ under $SU_2^S \times SU_3^C$ and can, in conjunction with $\underset{\sim}{\sigma}$ and $\underset{\sim}{\lambda}$, be used to generate SU_6^{CS}. Jaffe (1977b) chooses the generators $\underset{\sim}{\alpha}$ of SU_6^{CS} to be normalized to $Tr\ \alpha^a\alpha_b = 4\delta^a{}_b$ in the defining representation so that the thirty-five generators are

$$\underset{\sim}{\alpha} = \{ \sqrt{\frac{2}{3}}\ \underset{\sim}{\sigma}\ 1_3,\ 1_2\underset{\sim}{\lambda},\ \underset{\sim\sim}{\sigma\lambda}\} \tag{6.9}$$

where 1_2 and 1_3 are the identity operators in SU_2 and SU_3 respectively and a direct product of operators is implied in each case.

It is just a scalar product of the twenty-four operators $\underset{\sim\sim}{\sigma\lambda}$ which appears in (6.5). If all quarks are of the same species then it becomes a simple matter to express Δ_g^{aa} in terms of the quadratic Casimir invariants of SU_2^{Sa}, SU_3^{Ca} and SU_6^{CSa} where these groups refer to the total spin, colour and colour-spin respectively of all the type a quarks. The quadratic Casimir invariants for any SU_N group may be defined in terms of the group generators $\underset{\sim}{\gamma}$ by

$$C_N = \underset{\sim}{\gamma}\cdot\underset{\sim}{\gamma} \tag{6.10}$$

If the generators are normalized to $Tr\ \gamma^a\gamma_b = \frac{1}{2}\ \delta^a{}_b$ in the

defining representation then it can be shown (e.g. Judd 1963) that this operator has eigenvalues

$$C_N(\{\lambda\}) = \tfrac{1}{2} \sum_{i=1}^{N} \lambda_i(\lambda_i + N + 1 - 2i) - \tfrac{1}{2}\cdot\frac{m^2}{N} \qquad (6.11)$$

where m is the weight of $\{\lambda\}$. Hence for SU_2, SU_3 and SU_6 with the "Jaffe" normalizations for the generators we have

$$C_6^J(\{\lambda\}) = 4 \sum_{i=1}^{6} \lambda_i(\lambda_i + 7 - 2i) - \frac{2}{3} m^2 \qquad (6.12)$$

$$C_3^J(\{\mu_1\mu_2\}) = \frac{4}{3} [\mu_1^2 + \mu_2^2 - \mu_1\mu_2 + 3\mu_1] \qquad (6.13)$$

$$C_2^J(S) = 4S(S+1) \quad . \qquad (6.14)$$

The generators for groups acting on a combined system of n_a particles can be taken as $\sum_{i=1}^{n_a} \chi_i$, where the χ_i act on particle i, and thus the Casimir invariant for the combined system is

$$C_N = (\sum_{i=1}^{n_a} \chi_i)\cdot(\sum_{i=1}^{n_a} \chi_i) = 2 \sum_{i<j} \chi_i\cdot\chi_j + \sum_i (\chi_i)^2 \quad . \qquad (6.15)$$

The term $\sum_i (\chi_i)^2$ is a single particle operator which is to be evaluated between two states of the q^{n_a} configuration. This can easily be done using standard fractional parentage techniques (e.g. Judd 1963, cf. chapter 3) by expressing the n_a-particle system in terms of (n_a-1)-particle and 1-particle systems

$$\langle q^n k| \sum_i (\chi_i)^2 |q^n k'\rangle$$

$$= \sum_{\ell m \ell' m'} \langle q^n k|qm;q^{n-1}\ell\rangle(\langle q^{n-1}\ell|\langle qm|\sum_i^n(\chi_i)^2|qm'\rangle|q^{n-1}\ell'\rangle)$$

$$\times \langle qm';q^{n-1}\ell'|q^n k'\rangle$$

$$= n \sum_{\ell m \ell' m'} \langle q^n k|qm;q^{n-1}\ell\rangle\langle q^{n-1}\ell|q^{n-1}\ell'\rangle\langle qm|\chi^2|qm'\rangle$$

$$\times \langle qm';q^{n-1}\ell'|q^n k'\rangle$$

$$= n \sum_{\ell m m'} \langle q^n k|qm;q^{n-1}\ell\rangle\langle qm|\chi_i^2|qm'\rangle\langle qm';q^{n-1}\ell|q^n k'\rangle \qquad (6.16)$$

where we have used the equivalence of the quarks in the second step. Here, the labels k, m and ℓ collectively denote parentage, $SU_N^{(C,S \text{ or } CS)}$, subgroup and other labels. In our case the "other" labels are (mostly) flavour quantum numbers but $(\chi_i)^2$ does not act on these and must therefore be diagonal in them. Further since the generators χ are group operators, $(\chi_i)^2$ will be diagonal in SU_N and parentage labels and because $(\chi_i)^2$ is an invariant of SU_N it will also be diagonal in subgroup labels. Thus $(\chi_i)^2$ is diagonal in all m and by the orthogonality of the fractional parentage coefficients we have simply

$$\langle q^{n_a} k| \sum_i (\chi_i)^2 |q^{n_a} k'\rangle = n_a \, C_N(\{1\}) \, \delta^k{}_{k'}, \qquad (6.17)$$

where {1} is the defining representation of SU_N, associated with a single quark.

It is now trivial to show that (Jaffe 1977c)

$$\Delta^{aa} = \frac{1}{2}\sum_i [(\underset{\sim}{\alpha}_i)^2 - \frac{2}{3}(\underset{\sim}{\sigma}_i)^2 - (\underset{\sim}{\lambda}_i)^2] - \tfrac{1}{2}(\sum \underset{\sim}{\alpha}_i)^2$$

$$+ \frac{1}{3}(\sum_i \underset{\sim}{\sigma}_i)^2 + \frac{1}{2}(\sum_i \underset{\sim}{\lambda}_i)^2$$

$$= 8n_a - \frac{1}{2} C_6^J + \frac{1}{3} C_2^J + \frac{1}{2} C_3^J \qquad (6.18)$$

where the action of the operators is on the combined state of the n_a quarks.

If antiquarks of the same flavour are present then the summation in (6.6) amounts to calculating

$$\Delta_g(a+\bar{a}) \equiv \Delta^{aa} + \Delta^{a\bar{a}} + \Delta^{\bar{a}\bar{a}}$$

$$= \Delta^{aa} - [\Delta^{tot} - \Delta^{aa} - \Delta^{\bar{a}\bar{a}}] + \Delta^{\bar{a}\bar{a}} \qquad (6.19)$$

where Δ^{tot} is that operator obtained by treating quarks and antiquarks the same (i.e. 6.18 acting on the combined quantum numbers of a and $\bar{a}$ quarks) and we have used (6.7). Thus we obtain the result (Jaffe 1977b)

$$\Delta_g(q+\bar{q}) = 8n + \frac{1}{2}C_6^J(q+\bar{q}) - \frac{1}{3}C_2^J(q+\bar{q}) - \frac{1}{2}C_3^J(q+\bar{q})$$

$$- C_6^J(q) + \frac{2}{3}C_2^J(q) + C_3^J(q)$$

$$- C_6^J(\bar{q}) + \frac{2}{3}C_2^J(\bar{q}) + C_3^J(\bar{q}) \; . \qquad (6.20)$$

When quarks and antiquarks of more than one flavour are present, the exact evaluation of (6.6) amounts to calculating (for o and s quarks)

$$(R/\alpha_c)E_m = \Delta(o+\bar{o})M_{oo} + \Delta(o,s)M_{os} + \Delta(s+\bar{s})M_{ss} \qquad (6.21)$$

where $\Delta(o+\bar{o})$ and $\Delta(s+\bar{s})$ are given by (6.20) and $\Delta(o,s)$ is easily shown to be

$$\begin{aligned}\Delta(o,s) \equiv\ & \Delta^{os} + \Delta^{o\bar{s}} + \Delta^{s\bar{o}} + \Delta^{\bar{o}\bar{s}} \\ =\ & -\tfrac{1}{2}C_6^J(o+s) + \tfrac{1}{3}C_2^J(o+s) + \tfrac{1}{2}C_3^J(o+s) \\ & + \tfrac{1}{2}C_6^J(o+\bar{s}) - \tfrac{1}{3}C_2^J(o+\bar{s}) - \tfrac{1}{3}C_3^J(o+\bar{s}) \\ & + \tfrac{1}{2}C_6^J(s+\bar{o}) - \tfrac{1}{3}C_2^J(s+\bar{o}) - \tfrac{1}{2}C_3^J(s+\bar{o}) \\ & - \tfrac{1}{2}C_6^J(\bar{o}+\bar{s}) + \tfrac{1}{3}C_2^J(\bar{o}+\bar{s}) + \tfrac{1}{2}C_3^J(\bar{o}+\bar{s}) \qquad (6.22)\end{aligned}$$

The averaging procedure (6.1) amounts to treating o and s quarks the same in (6.22) and the right hand side of (6.21) reduces to (6.20) multiplied by an $\bar{M}(R)$. Unfortunately neither (6.21) nor (6.20) is diagonal in our basis nor in the SU_6^{CS} coupled basis, nor indeed in any simple basis. The colour-magnetic term is however, diagonal in n_o, n_s, all flavour quantum numbers, $\{\lambda_o\}^{CS}$, $\{\lambda_{\bar{o}}\}^{CS}$, $\{\lambda_s\}^{CS}$, $\{\lambda_{\bar{s}}\}^{CS}$ and total spin and colour. Clearly its eigenstates are also eigenstates of the complete bag Hamiltonian.

One could attempt to evaluate (6.21) exactly by using various isoscalar factors and recoupling coefficients but the ones that arise are many and difficult to calculate. We shall find it more convenient to take a different approach using tensor operators. The formulae given here will though be useful for checking procedures, especially (6.18).

(b) Tensor Operators

The operators $\underset{\sim}{\sigma\lambda}$ are generators of SU_6^{CS} and therefore transform as $\{21^4\}^{CS}$. They are consequently tensor operators (e.g. Butler 1975 or Part II)

$$(\underset{\sim}{\sigma\lambda})^a_i = c\underset{\sim}{x}^a_i \equiv c(\underset{\sim}{x}^{21^4\ 3}21)^a_i \qquad (6.23)$$

where a denotes the species of quark the single particle operator $\underset{\sim}{x}_i$ acts upon and c is a proportionality constant which depends on the arbitrary normalization of the tensor operators — which is yet to be chosen.

The two-particle colour-spin operator involves the scalar product of $\underset{\sim}{\sigma\lambda}$ and therefore must transform as the identity $^1\{0\}^C$ under $SU_2^S \times SU_3^C$, where these groups refer to the total quantum numbers of the a and b quarks combined. This suggests that we express the colour-spin operator in terms of scalar coupled products of the single particle operators in (6.23):

$$[\underset{\sim}{x}_i^a \underset{\sim}{x}_j^b]^{10}_{0} = \sum_{k_1 k_2} (x^{21^4\ {}^321}_{k_1})^a_i (x^{21^4\ {}^321}_{k_2})^b_j \langle {}^321\ k_1; {}^321\ k_2 | {}^10\ 0\rangle$$

$$= -\frac{1}{2\sqrt{6}} \sum_{mq} (-1)^m (x^{21^4\ {}^321}_{mq})^a_i (x^{21^4\ {}^321}_{-mq^*})^b_i$$

$$= -\frac{1}{2\sqrt{6}}\ \underset{\sim}{x}_i^a \cdot \underset{\sim}{x}_j^b \qquad (6.24)$$

using our phase conventions.

We note that any two-particle operator may be rewritten as

$$\sum_{i<j} \underset{\sim}{x}_i^a \cdot \underset{\sim}{x}_j^b = \delta_{ab} \sum_{i>j}^{n_a} \underset{\sim}{x}_i^a \cdot \underset{\sim}{x}_j^a + (1-\delta_{ab}) \sum_{i}^{n_a} \sum_{j}^{n_b} \underset{\sim}{x}_i^a \cdot \underset{\sim}{x}_j^b$$

$$= \tfrac{1}{2}[(2-\delta_{ab})\underset{\sim}{X}^a \cdot \underset{\sim}{X}^b - \sum_{i=1}^{n_a} (\underset{\sim}{x}_i^a)^2\ \delta_{ab}] \qquad (6.25)$$

where

$$\underset{\sim}{X}^a \equiv (\underset{\sim}{X}^{21^4\ {}^321})^a = \sum_{i=1}^{n_a} \underset{\sim}{x}_i^a \quad . \qquad (6.26)$$

As in (6.24) we can define the operator

$$\underset{\sim}{X}^{ab} \equiv [\underset{\sim}{X}^{21^4\ {}^321}_a\ \underset{\sim}{X}^{21^4\ {}^321}_b]^0_0$$

$$= \frac{-1}{2\sqrt{6}}\ \underset{\sim}{X}^a \cdot \underset{\sim}{X}^b \qquad (6.27)$$

This may be used, together with (6.24), to evaluate (6.25).

Let us recall some properties of scalar coupled products of tensor operators which are generalizations (e.g. Butler 1980b) of the familiar SU_2 angular momentum tensor operator results (Judd 1963) to arbitrary compact groups. The matrix

elements of such a tensor are simply

$$\langle x_1\lambda_1 i_1 | [P^K Q^{K*}]^0_0 | x_2\lambda_2 i_2\rangle$$

$$= \delta_{i_1 i_2}\, \delta_{\lambda_1\lambda_2}\, |\lambda_1|^{-1} |\kappa|^{-\frac{1}{2}} \sum_{s x_3 \lambda_3} \{\lambda_1\}\{\lambda_1^* \kappa \lambda_3 s\}$$

$$\langle x_1\lambda_1 \| P^K \| x_3\lambda_3\rangle_s \langle x_3\lambda_3 \| Q^{K*} \| x_2\lambda_2\rangle_s \qquad (6.28)$$

where s is a product multiplicity label. (The reader is reminded that throughout, we are employing canonical phase choices — see Part II.) A further useful result is for when the scalar coupled tensor is acting on coupled vectors:

$$\langle (\lambda_1\lambda_2) r_1 \lambda i | [P^K Q^{K*}]^0_0 | (\mu_1\mu_2) r_2 \mu j\rangle$$

$$= \delta_{ij}\, \delta_{\lambda\mu} |\kappa|^{-\frac{1}{2}} \sum_{s_1 s_2} \{\lambda_2\}\{\lambda_2 \kappa \mu_2^* s_2\}\{\lambda_1\lambda_2\lambda^* r_1\} \begin{Bmatrix} \mu_1 & \mu_2 & \lambda^* \\ \lambda_2^* & \lambda_1 & \kappa \end{Bmatrix}_{s_1 s_2 r_1 r_2}$$

$$\langle \lambda_1 \| P^K \| \mu_1\rangle_{s_1} \langle \lambda_2 \| Q^{K*} \| \mu_2\rangle_{s_2} \,. \qquad (6.29)$$

The enumeration of essential results is completed by considering the reduced matrix elements for operators that act only on one part of a coupled vector. Thus if $\underset{\sim}{P}^{K1}$ acts only on part 1 of a system then

$$\langle (\lambda_1\lambda_2) r_1 \lambda \| P^{K1} \| (\mu_1\mu_2) r_2 \mu\rangle_s$$

$$= \delta_{\lambda_2\mu_2} |\lambda|^{\frac{1}{2}} |\mu|^{\frac{1}{2}} \sum_{s_1} \{\lambda_1\}\{\lambda_1\lambda_2\lambda^* r_1\}\{\lambda_1^* \kappa_1 \mu_1 s_1\} \begin{Bmatrix} \lambda^* & \kappa_1 & \mu \\ \mu_1 & \lambda_2^* & \lambda_1 \end{Bmatrix}_{r_1 s_1 r_2 s}$$

$$\langle \lambda_1 \| P^{K1} \| \mu_1\rangle_{s_1} \qquad (6.30)$$

while if Q^{K2} acts only on part 2 of a system then

$$<(\lambda_1\lambda_2)r_1\lambda\|Q^{K2}\|(\mu_1\mu_2)r_2\mu>_s$$

$$= \delta_{\lambda_1\mu_1}|\lambda|^{\frac{1}{2}}|\mu|^{\frac{1}{2}}\sum_{s_2}\{\mu_2\}\{\mu_1\mu_2\mu^*r_2\}\{\lambda_2^*\kappa_2\mu_2 s_2\}\begin{Bmatrix}\lambda^* & \kappa_2 & \mu\\ \mu_2 & \lambda_1^* & \lambda_2\end{Bmatrix}_{r_1s_2r_2s}$$

$$<\lambda_2\|Q^{K2}\|\mu_2>_{s_2} \quad . \tag{6.31}$$

(c) Reduced Matrix Elements

The formulae of the last section require knowledge of reduced matrix elements. A standard procedure for calculating these is to compare the matrix elements of the tensor with those of a known operator (e.g. Butler et al. 1979). We note that the spin operator S_z must transform as a scalar component $T^{21^4\ {}^30}_{00}$ of a tensor with the same SU_6^{CS} transformation properties as $\underset{\sim}{x}^{21^4\ {}^321}$ and hence

$$<\lambda^{CS}\ |s|_\mu C\ S_z I^C Y^C I_z^C|x^{21^4\ {}^321}_{0,000}|\ \lambda^{CS}\ |s|_\mu C S_z I^C Y^C I_z^C> = kS_z \tag{6.32}$$

where k is a proportionality constant associated with the normalization of the tensor operators, $\underset{\sim}{x}$. Explicit use of the Wigner-Eckart theorem (see Butler 1975 or Part II) leads to the result (Bickerstaff and Wybourne 1980a)

$$\sum_r \begin{pmatrix} \lambda^* & 21^4 & \lambda \\ |S|_{\mu^*} & {}^30 & |S|_\mu \end{pmatrix}^r <\lambda\|\underset{\sim}{x}^{21^4}\|\lambda>_r$$

$$= k[|\mu|S(S+1)(2S+1)]^{\frac{1}{2}} \tag{6.33}$$

where r is a product multiplicity index. Use of the known values of the $SU_6 \supset SU_2 \times SU_3$ 3jm factors (Part II) leads to equations in the reduced matrix elements. Choosing $\langle 1 \| \underset{\sim}{x}^{21^4} \| 1 \rangle = 1$ requires

$$k = (2/105)^{\frac{1}{2}} \tag{6.34}$$

leading to the results given in table I11. (It should be clear that $\underset{\sim}{x}$ and $\underset{\sim}{X}$ both have the same reduced matrix elements.)

The $SU_2 \times SU_3$ dependence of the matrix elements may be obtained by noting that

$$\begin{aligned} &\langle \lambda \ |s|_{\mu} \| \underset{\sim}{x}^{21^4} \ {}^321 \| \lambda \ |s'|_{\mu'} \rangle_{\delta} \\ &= \sum_r \begin{pmatrix} \lambda^* & 21^4 & \lambda \\ |s|_{\mu^*} & {}^321 & |s'|_{\mu'} \end{pmatrix}_{\delta}^{r} \langle \lambda \| \underset{\sim}{x}^{21^4} \| \lambda \rangle_r \end{aligned} \tag{6.35}$$

where δ is an SU_3 product multiplicity index. Because $\underset{\sim}{x}$ is constructed from SU_6 generators it is diagonal in SU_6 representations and parentage but it is not included amongst $SU_2 \times SU_3$ generators and is not diagonal in representations of those groups. Most of the necessary reduced matrix elements are given in table I12. We do not though have sufficient 3jm factors to be able to calculate those for the representations $\{21^3\}^{CS}$ and $\{2^21\}^{CS}$ which arise when five o quarks are present. These could be obtained, but at this stage it would be hardly worth the effort.

Note that in evaluating (6.28 - 31), the $SU_6 \supset SU_2 \times SU_3$ 3jm factors arise only in the calculation of the reduced matrix elements; after that we only require a few SU_2 and SU_3 6j symbols. This situation is much simpler than that arising in the direct evaluation of (6.21) via (6.22) and (6.20).

It is possible to derive (from 6.33-35 and the properties of 3jm factors) several useful formulae for the reduced matrix element symmetries which augment the scope of tables I11 and I12. We find for the SU_6 reduced matrix elements

$$\langle \lambda^{CS} \| \underset{\sim}{x}^{21^4} \| \lambda^{CS} \rangle_r^* = (-1)^r \langle \lambda^{CS} \| \underset{\sim}{x}^{21^4} \| \lambda^{CS} \rangle_r \tag{6.36}$$

$$\langle \lambda^{*CS} \| \underset{\sim}{x}^{21^4} \| \lambda^{*CS} \rangle_r = \langle \lambda^{CS} \| \underset{\sim}{x}^{21^4} \| \lambda^{CS} \rangle_r^* \tag{6.37}$$

and for the $SU_2 \times SU_3$ doubly-reduced matrix elements

$$\langle \lambda^{*CS} \; |\mu|_{\mu}{}^{*C} \| \underset{\sim}{x}^{21^4 \; {}^3 21} \| \lambda^{*CS} \; |s'|_{\mu'}{}^{*C} \rangle_{\delta}$$

$$= \langle \lambda^{CS} \; |s|_{\mu}{}^{C} \| \underset{\sim}{x}^{21^4 \; {}^3 21} \| \lambda^{CS} \; |s'|_{\mu'}{}^{C} \rangle_{\delta}^* \tag{6.38}$$

$$\langle \lambda^{CS} \; |s'|_{\mu'}{}^{C} \| \underset{\sim}{x}^{21^4 \; {}^3 21} \| \lambda^{CS} \; |s|_{\mu}{}^{C} \rangle_{\delta}$$

$$= (-1)^{2j_\lambda + j_\mu + j_{\mu'} + s + s' + \delta} \langle \lambda^{CS} \; |s|_{\mu}{}^{C} \| \underset{\sim}{x}^{21^4 \; {}^3 21} \| \lambda^{CS} \; |s'|_{\mu'}{}^{C} \rangle_{\delta}^* \tag{6.39}$$

The permutational sign change in (6.39) is included in table I12. It needs to be emphasized that both these

formulae and the reduced matrix elements appearing in tables I11-12 are dependent on our phase choices and multiplicity resolutions.

(d) Evaluation of Matrix Elements

We are now able to determine the constant c in (6.23). A simple way of doing this (cf. Bickerstaff and Wybourne 1980a) is to note that

$$\sum_i (\underset{\sim}{x}_i^a)^2 = (1/c^2) \sum_i [(\underset{\sim}{\sigma}\underset{\sim}{\lambda})_i^a]^2 \tag{6.40}$$

is proportional to the number operator. Since the right hand side of (6.40) is known to be given by (cf. 6.18)

$$\sum_i \underset{\sim}{\sigma}_i^2 \underset{\sim}{\lambda}_i^2 = 16\ n \tag{6.41}$$

it only remains to evaluate

$$\langle 1\ ^2 1\ \mathcal{L}\ |(\underset{\sim}{x}_i^a)^2| 1\ ^2 1\ \mathcal{L}\rangle$$

using (6.24) and (6.28). We find

$$c^2 = 140 \tag{6.42}$$

and thus

$$\begin{aligned} \Delta_g^{ab} &= -\ c^2 \sum_{i<j} \underset{\sim}{x}_i^a \cdot \underset{\sim}{x}_j^b \\ &= 140\sqrt{6}\ (2 - \delta_{ab})\underset{\sim}{X}^{ab} + 8n_a\delta_{ab}\ . \end{aligned} \tag{6.43}$$

It is now possible to calculate the required matrix elements. We note that Δ_g^{ab} does not depend on the flavour content of a multiquark state and we need only consider its $SU_6^{CS} \supset SU_2^S \times SU_3^C$ content. Δ_g^{ab} will mix states having the same (internal and external) flavour quantum numbers, the same $\{\lambda_o\}^{CS}$, $\{\lambda_s\}^{CS}$, $\{\lambda_{\bar{o}}\}^{CS}$, $\{\lambda_{\bar{s}}\}^{CS}$ and the same total spin. Thus for each flavour configuration there will be a colour-spin matrix determined by the total spin and SU_6^{CS} (internal) quantum numbers but which is independent of flavour. This means that several specific configurations e.g. $q_o^4\bar{q}_o$, $q_o^4\bar{q}_s$, $q_s^4\bar{q}_o$ and $q_s^4\bar{q}_s$ can have the same matrix elements; a fact which greatly reduces the necessary work. Consider for instance the $(I,S,\ |s|0^C) = (\tfrac{1}{2},0,{}^20^C)\ q_o^4\bar{q}_o$ state appearing at the head of table I8. Its matrix elements depend only on its $SU_6^{CS} \supset SU_2^S \times SU_3^C$ content

$$|(2^2\ {}^31, 1^5\ {}^21^2)^20\rangle \tag{6.44}$$

which is precisely the same as that of the $q_o^4\bar{q}_s$ state with quantum numbers $(0,1,{}^20)$. For this reason it is useful to represent sets of configurations by a generic configuration; within $q^4\bar{q}$ it suffices to consider the three generic configurations $q_a^4\bar{q}_b$, $q_a^3q_b\bar{q}_c$ and $q_a^2q_b^2\bar{q}_c$. (Actually there is a small complication in treating configurations such as $q_a^3q_b\bar{q}_c$ and $q_aq_b^3\bar{q}_c$ the same but we shall ignore this for the present and return to it later.) We also introduce a shorthand labelling scheme for the matrices; each possible SU_6^{CS} content is given an arbitrary label (a script letter) and the total spin is written as a left superscript. Thus

we denote the 1 x 1 matrix for the state in (6.44) by $^2\mathcal{B}$; an arbitrary subscript completes the classification of the $SU_6^{CS} \supset SU_2^S \times SU_3^C$ content of the state when the dimension of the matrix is greater than one. This reduced colour-spin identification is shown in tables I7-9. (A full classification of the states can be regained by adding the specific configuration and the total isospin.)

It is a simple matter to pick the reduced states out of the tables. The relevant contributions Δ_g^{ab} to the colour-magnetic term are now evaluated. First, we note that for a group of n_a quarks q_a of the same flavour, use of (6.28) to evaluate Δ^{aa} generally involves a summation over intermediate states in the reduced matrix elements; in this case it is easier to use (6.18). Consider now a general configuration $q_a^m q_b^{n-p} \bar{q}_c^p \bar{q}_d^q$. The term Δ^{ab} is independent of, and diagonal in, parts c and d of the system. It is immediately evaluated using (6.43) and (6.29); similarly for $\Delta^{\bar{c}\bar{d}}$. The terms $\Delta^{a\bar{c}}$, $\Delta^{a\bar{d}}$, $\Delta^{b\bar{c}}$, and $\Delta^{b\bar{d}}$ are all very similar. Consider for example $\Delta^{a\bar{c}}$. Again, it can be evaluated using (6.29): the appearance of the identity representation in the SU_3 6j symbol leads to considerable simplifications and, noting that for SU_2^S we have the 2j phase

$$\{S\} = (-1)^{2S} \tag{6.45}$$

and the 3j phase

$$\{S_1 S_2 S_3\} = (-1)^{S_1+S_2+S_3} \tag{6.46}$$

while for SU_3 the 2j phases may all be taken as unity (Butler and King 1979), we obtain

$$
\begin{aligned}
&\langle((\lambda_a S_a \mu_a, \lambda_b S_b \mu_b) r_{ab} S_{ab} \mu_{ab}, (\lambda_{\bar{c}} S_{\bar{c}} \mu_{\bar{c}}, \lambda_{\bar{d}} S_{\bar{d}} \mu_{\bar{d}}) r_{\bar{c}\bar{d}} S_{\bar{c}\bar{d}} \mu_{\bar{c}\bar{d}}) S0^{C} i| \\
&\times \Delta^{a\bar{c}} |((\lambda_a S'_a \mu'_a, \lambda_b S'_b \mu'_b) r'_{ab} S'_{ab} \mu'_{ab}, (\lambda_{\bar{c}} S'_{\bar{c}} \mu'_{\bar{c}}, \lambda_{\bar{d}} S'_{\bar{d}} \mu'_{\bar{d}}) r'_{\bar{c}\bar{d}} S'_{\bar{c}\bar{d}} \mu'_{\bar{c}\bar{d}}) S0^{C} i\rangle \\
&= 140\, \delta^{\mu_{\bar{c}\bar{d}}}_{\mu^*_{ab}} \delta^{\mu'_{\bar{c}\bar{d}}}_{\mu'^*_{ab}} (-1)^{S'_{ab}+S_{\bar{c}\bar{d}}+S} |\mu_{ab}|^{-\frac{1}{2}} |\mu'_{ab}|^{-\frac{1}{2}} \begin{Bmatrix} S'_{ab} & S'_{\bar{c}\bar{d}} & S \\ S_{\bar{c}\bar{d}} & S_{ab} & 1 \end{Bmatrix} \\
&\times \sum_{\delta} \langle(\lambda_a S_a \mu_a, \lambda_b S_b \mu_b) r_{ab} S_{ab} \mu_{ab} \| \underset{\sim}{x}^{21^4_a\ {}^321} \| (\lambda_a S'_a \mu'_a, \lambda_b S'_b \mu'_b) r'_{ab} S'_{ab} \mu'_{ab} \rangle_{\delta} \\
&\times \langle(\lambda_{\bar{c}} S_{\bar{c}} \mu_{\bar{c}}, \lambda_{\bar{d}} S_{\bar{d}} \mu_{\bar{d}}) r_{\bar{c}\bar{d}} S_{\bar{c}\bar{d}} \mu_{\bar{c}\bar{d}} \| \underset{\sim}{x}^{21^4_c\ {}^321} \| (\lambda_{\bar{c}} S'_{\bar{c}} \mu'_{\bar{c}}, \lambda_{\bar{d}} S'_{\bar{d}} \mu'_{\bar{d}}) r'_{\bar{c}\bar{d}} S'_{\bar{c}\bar{d}} \mu'_{\bar{c}\bar{d}} \rangle_{\delta} .
\end{aligned}
\tag{6.47}
$$

The equations for $\Delta^{a\bar{d}}$ $\Delta^{b\bar{c}}$ and $\Delta^{b\bar{d}}$ differ only in the reduced matrix elements that appear. These may be evaluated in terms of the reduced matrix elements appearing in table I12 using either (6.30) or (6.31) as appropriate. For example

$$
\begin{aligned}
&\langle(\lambda_a S_a \mu_a, \lambda_b S_b \mu_b) r_{ab} S_{ab} \mu_{ab} \| \underset{\sim}{x}^{21^4_a\ {}^321} \| (\lambda_a S'_a \mu'_a, \lambda_b S'_b \mu'_b) r'_{ab} S'_{ab} \mu'_{ab} \rangle_{s} \\
&= \delta^{S_b}_{S'_b} \delta^{\mu_b}_{\mu'_b} \sqrt{(2S_{ab}+1)(2S'_{ab}+1)}\ |\mu_{ab}|^{\frac{1}{2}} |\mu'_{ab}|^{\frac{1}{2}} \\
&\times (-1)^{S_b+S_{ab}+1+S'_a} \begin{Bmatrix} S_{ab} & 1 & S'_{ab} \\ S'_a & S_b & S_a \end{Bmatrix} \{\mu_a \mu_b \mu^*_{ab} r_{ab}\} \\
&\times \sum_{s_1} \{\mu^*_a\ 21\ \mu'_a\ s_1\} \begin{Bmatrix} \mu^*_{ab} & 21 & \mu'_{ab} \\ \mu'_a & \mu^*_b & \mu_a \end{Bmatrix} r_{ab} s_1\ r'_{ab} s \\
&\qquad \times \langle \lambda_a S_a \mu_a \| \underset{\sim}{x}^{21^4_a\ {}^321} \| \lambda_a S'_a \mu'_a \rangle_{s_1} .
\end{aligned}
\tag{6.48}
$$

Proceeding in this manner it is straightforward to derive all the relevant formulae. We point out though that in some cases there may exist further simplifications due to the absence of one of the species b or $\bar{d}$, or special symmetries such as between the matrix elements of $\Delta^{a\bar{c}}$ and $\Delta^{b\bar{c}}$ in the configuration $q_a q_b \bar{q}_c^2$. The calculations are further simplified by noting that many matrix elements only differ by a phase and the ratio of two 6j symbols, such as when the only difference is the total spin of the states.

The required matrix elements were all computed by hand and are tabulated in tables I13-17 for q^3, $q\bar{q}$, $q^4\bar{q}$, $q^2\bar{q}^2$ and q^6 respectively. Only the elements in the upper triangle are given as the matrices are symmetric; the elements are listed in the order M_{11}, M_{12}, M_{22}, M_{13}, M_{23}, ... We have used the SU_2 6j symbols tabulated by Rotenberg et al. (1959) and the SU_3 3j phases and 6j symbols of (Butler et al. 1979, Butler and Haase 1979(1977), see also Part II). The matrix elements for q^3 and $q\bar{q}$ are easily obtained by the Casimir invariant techniques described in the first section and provide an important check on the form of the colour-spin operator given in (6.43). There is some ambiguity remaining in three 2 x 2 matrices in the $S = -1$ part of the q^6 sector because we do not have the reduced matrix elements for the representations $\{21^3\}^{CS}$ and $\{2^21\}^{CS}$. The one dimensional matrices ${}^1E$, ${}^1F$, ${}^5F$, ${}^7E$ (and all $S = 0$ matrices) can be calculated using (6.18) and (6.22). This enables us to deduce

$$\langle 21^3\ {}^2 1^2 \| \underset{\sim}{x}^{21^4\ {}^3 21} \| 21^3\ {}^2 1^2 \rangle = -\sqrt{210}/70$$

$$\langle 21^3\ {}^4 1^2 \| \quad " \quad \| 21^3\ {}^4 1^2 \rangle = 0$$

$$\langle 2^2 1\ {}^2 1^2 \| \quad " \quad \| 2^2 1\ {}^2 1^2 \rangle = +\sqrt{210}/35$$

$$\langle 2^2 1\ {}^6 1^2 \| \quad " \quad \| 2^2 1\ {}^6 1^2 \rangle = -2\sqrt{6}/5 \quad .$$

The trace of ${}^3F$ when compared with the checking eigenvalues (see below) implies

$$\langle 2^2 1\ {}^4 1^2 \| \underset{\sim}{x}^{21^4\ {}^3 21} \| 2^2 1\ {}^4 1^2 \rangle = -6\sqrt{21}/35$$

and now all diagonal elements can be determined but the phase of the off-diagonal terms (whose magnitudes were determined by the sums of squares of the checking eigenvalues) can not be pinned down — it depends on a phase choice in the Wigner-Racah algebra. However, the magnitude alone is sufficient to calculate the eigenvalues and hence the energies. The ambiguity in sign reflects itself in the eigenvectors though and it is not possible to treat dissociations until this sign has been determined (in relationship to the iso-scalar factors that appear in the dissociation calculation).

The full colour-magnetic matrix is easily extracted from these tables simply by multiplying the terms Δ^{aa}, Δ^{ab}, ... by M_{oo}, M_{ss} or M_{os} as appropriate. For example the first matrix ${}^2A$ in table I15 yields the matrix

$$\begin{pmatrix} \frac{16}{3} M_{oo} & -8\ M_{oo} \\ -8\ M_{oo} & 0 \end{pmatrix}$$

for the $o^4\bar{o}$ $(I,S) = (\frac{1}{2},0)$ and $(\frac{3}{2},0)$ states in table I8 whereas for the $o^4\bar{s}$ $(I,S) = (1,1)$ states it yields

$$\begin{pmatrix} \frac{8}{3} M_{oo} + \frac{8}{3} M_{os} & -8 M_{os} \\ -8 M_{os} & 0 \end{pmatrix} .$$

There is one other matter, alluded to earlier, which must now be considered. In actual fact the matrix elements for a configuration such as $q_o q_s^3 \bar{q}_o$ are not the same as those for $q_o^3 q_s \bar{q}_o$ because of a problem with permutational symmetry. Our basis states have been given with o quark quantum numbers preceding s quark quantum numbers and one cannot arbitrarily permute them. It can be shown that

$$\begin{aligned} &\langle((\lambda_b S_b \mu_b, \lambda_a S_a \mu_a) r_{ab} S_{ab} \mu_{ab}, \lambda_c S_c \mu_c) S0^C i | \\ &\times \Delta^{a\bar{c}} | ((\lambda_b S_b' \mu_b', \lambda_a S_a' \mu_a') r_{ab}' S_{ab}' \mu_{ab}', \lambda_c S_c' \mu_c') S0^C i \\ &= (-1)^{S_a + S_b + S_{ab}} (-1)^{S_a' + S_b' + S_{ab}'} \{\mu_a \mu_b \mu_{ab}^* r_{ab}\} \{\mu_a' \mu_b' \mu_{ab}'^* r_{ab}'\} \\ &\times \langle((\lambda_a S_a \mu_a, \lambda_b S_b \mu_b) r_{ab} S_{ab} \mu_{ab}, \lambda_c S_c \mu_c) S0^C i | \\ &\times \Delta^{a\bar{c}} | ((\lambda_a S_a' \mu_a', \lambda_b S_b' \mu_b') r_{ab}' S_{ab}' \mu_{ab}', \lambda_c S_c' \mu_c') S0^C i \rangle \end{aligned} \qquad (6.49)$$

and precisely the same phase change occurs for each of the terms $\Delta^{b\bar{c}}$ and Δ^{ab}; clearly there is no phase change for the diagonal elements — note that terms such as Δ^{aa} can only

occur on the diagonal. Obviously the correct matrix is obtained using a transformation by a diagonal matrix of phases and this doesn't alter the eigenvalues. Hence all one has to do is use the same matrix all the time but change the sign of the coefficients of the basis vectors in the eigenstates as need be i.e. we simply make the substitution

$$|((\lambda_b S_b \mu_b, \lambda_a S_a \mu_a) r_{ab} S_{ab} \mu_{ab}, \lambda_c S_c \mu_c) S0^C i\rangle$$

$$\rightarrow (-1)^{S_a+S_b+S_{ab}} \{\mu_a\ \mu_b\ \mu^*_{ab}\ r_{ab}\}$$

$$\times |((\lambda_a S_a \mu_a, \lambda_b S_b \mu_b) r_{ab} S_{ab} \mu_{ab}, \lambda_c S_c \mu_c) S0^C i\rangle \qquad (6.50)$$

In actual practice, such substitutions are fairly rare but they must be made or else the dissociations will come out wrong.

(e) Checking Procedures

It is essential to be able to verify that the matrices are correct. To do this we note that if all quark flavours are treated identically by setting all radial integrals equal to unity and summing terms then the eigenvalues of the colour-magnetic matrix must be given by the eigenvalues of (6.20). However this poses problems if the matrix-dimension is greater than one so it is easier to treat not only all flavours the same but quarks and antiquarks as well by setting the radial integrals M_{ab}, $M_{\bar{a}\bar{b}}$ equal to +1 and $M_{a\bar{b}}$ equal to -1.

The eigenvalues of this checking matrix are then given by (6.18) where the Casimir invariants act on the total quantum numbers of the combined system. We must therefore determine the associated $\{\lambda_{tot}\}^{CS}$ representations. This is accomplished simply by forming the Kronecker product $\{\lambda_o\}^{CS} \times \{\lambda_s\}^{CS} \times \{\lambda_{\bar{o}}\}^{CS} \times \{\lambda_{\bar{s}}\}^{CS}$ and reducing the resulting $\{\lambda_{tot}\}^{CS}$ representations to $SU_2^S \times SU_3^C$ to see which ones contain colour singlets, $\{0\}^C$ with the necessary total spin value. For example, consider the $q^2\bar{q}^2$ matrix ^1A. We have

$$\{2\} \times \{0\} \times \{2^5\} \times \{0\} = \{0\} + \{21^4\} + \{42^4\}$$

but only $\{0\}^{CS}$ and $\{42^4\}^{CS}$ contain a $^1\{0\}^C$ term. Acting on these two states with (6.18) gives the checking eigenvalues +32 and -24 respectively. The associated $\{\lambda_{tot}\}^{CS}$ representations and checking eigenvalues are given for all matrices in tables I13-17. This checking procedure provides a powerful constraint on the matrices although it is still possible for a sign error to occur. Where feasible, a few random checks have also been made using (6.20) and (6.21). One should be entitled to a reasonable degree of confidence in the tables.

Notice that Δ^{check} is diagonal in a basis coupled at the SU_6^{CS} level to $SU_6^{CS(tot)}$. Thus the transformation which diagonalizes the checking matrix also transforms the states into such a basis. Consider the q^6 matrix ^1I associated with Jaffe's (1977c) $S = -2$ dihyperon. From table I17 we have

$$ {}^1I(O^2S^2) = \begin{pmatrix} -\frac{16}{3}M_{oo} - \frac{40}{3}M_{os} + \frac{8}{3}M_{ss} & 8\sqrt{3}\,M_{os} \\ 8\sqrt{3}\,M_{os} & -4M_{oo} + 4M_{ss} \end{pmatrix} $$

which goes into the following checking matrix:

$$ \begin{pmatrix} -16 & 8\sqrt{3} \\ 8\sqrt{3} & 0 \end{pmatrix} . $$

The checking matrix has eigenvalues -24 and +8 associated with $\{3^2\}^{CS}$ and $\{2^2 1^2\}^{CS}$ respectively and eigenvectors:

$$ |(2^2,1^2)3^{2CS}\ {}^1O^C\rangle = \frac{\sqrt{3}}{2}\,|{}^1I_1^{(O^4S^2)}\rangle - \frac{1}{2}|{}^1I_2^{(O^4S^2)}\rangle $$

$$ |(2^2,1^2)2^2 1^{2CS}\ {}^1O^C\rangle = \frac{1}{2}|{}^1I_1^{(O^4S^2)}\rangle + \frac{\sqrt{3}}{2}|{}^1I_2^{(O^4S^2)}\rangle . $$

The coefficients are of course just $SU_6 \supset SU_2 \times SU_3$ isoscalar factors — with random phases — and can be used to see what the colour-spin matrix looks like in the SU_6^{CS} coupled basis. We find

$$ \begin{pmatrix} -5M_{oo} - 22M_{os} + 3M_{ss} & -\frac{\sqrt{3}}{3}M_{oo} + \frac{2\sqrt{3}}{3}M_{os} - \frac{\sqrt{3}}{3}M_{ss} \\ -\frac{\sqrt{3}}{3}M_{oo} + \frac{2\sqrt{3}}{3}M_{os} - \frac{\sqrt{3}}{3}M_{ss} & -\frac{13}{3}M_{oo} + \frac{26}{3}M_{os} + \frac{11}{3}M_{ss} \end{pmatrix} . $$

One notices how small the off-diagonal element is (since $M_{oo}+M_{ss} \simeq 2M_{os}$). The reason for this can be related to the dominance of the Casimir invariants of SU_6^{CS} in (6.18),(6.20)

and (6.22) — cf. Jaffe (1977b). Thus the SU_6^{CS} coupled basis would be a good one to use if it were not for the fact that direct calculations within it are very difficult. (It perhaps should be remarked that not all cases are as extreme as the example considered here and mixing in the SU_6^{CS} coupled basis must be taken into account in general. The only reason that Jaffe (1977c) was able to calculate the energy of his dihyperon, H is that one is able to place lower and upper bounds on the mass by treating some unknown isoscalar factors as either 0 or 1. In this case the two bounds differ by the order of 1 MeV — see though Jaffe's paper for the uncertainty in some other energies!) It is essentially because Δ_g^{ab} is a scalar under the combined $SU_2 \times SU_3$ group that our $SU_2 \times SU_3$ coupled basis proves to be so convenient.

Table III: SU_6 reduced matrix elements

$$T \equiv \underset{\sim}{X}^{21^4}$$

$\langle 1 \| T \| 1 \rangle = 1$

$\langle 1^2 \| T \| 1^2 \rangle = 2$

$\langle 2 \| T \| 2 \rangle = -2\sqrt{2}$

$\langle 1^3 \| T \| 1^3 \rangle = -\sqrt{6}$

$\langle 21 \| T \| 21 \rangle_0 = -13/3$ $\qquad$ $\langle 21 \| T \| 21 \rangle_1 = -8i\sqrt{2}/3$

$\langle 2^2 \| T \| 2^2 \rangle = -8$

$\langle 21^2 \| T \| 21^2 \rangle_0 = 5$ $\qquad$ $\langle 21^2 \| T \| 21^2 \rangle_1 = -3i\sqrt{3}$

Table I12: $SU_6 \supset SU_2 \times SU_3$ reduced matrix elements

$$T \equiv \underset{\sim}{X}^{21^4\ {}^321}$$

$$\langle 1 \quad {}^21 \| T \| 1 \quad {}^21 \rangle^+ = 2\sqrt{210}/35$$

$$\langle 1^2 \quad {}^31^2 \| T \| 1^2 \quad {}^31^2 \rangle^+ = -2\sqrt{210}/35$$

$$\langle 1^2 \quad {}^31^2 \| T \| 1^2 \quad {}^12 \rangle^+ = 6\sqrt{35}/35$$

$$\langle 2 \quad {}^32 \| T \| 2 \quad {}^32 \rangle^+ = 2\sqrt{42}/7$$

$$\langle 2 \quad {}^32 \| T \| 2 \quad {}^11^2 \rangle^+ = -6\sqrt{35}/35$$

$$\langle 1^3 \quad {}^40 \| T \| 1^3 \quad {}^221 \rangle^- = -4\sqrt{70}/35$$

$$\langle 1^3 \quad {}^221 \| T \| 1^3 \quad {}^221 \rangle_0^+ = -4\sqrt{7}/7 \qquad \langle 1^3 \quad {}^221 \| T \| 1^3 \quad {}^221 \rangle_1^- = 0$$

$$\langle 21 \quad {}^421 \| T \| 21 \quad {}^421 \rangle_0^+ = 0 \qquad \langle 21 \quad {}^421 \| T \| 21 \quad {}^421 \rangle_1^- = 4i\sqrt{14}/7$$

$$\langle 21 \quad {}^421 \| T \| 21 \quad {}^221 \rangle_0^- = -4\sqrt{7}/7 \qquad \langle 21 \quad {}^421 \| T \| 21 \quad {}^221 \rangle_1^+ = -4i\sqrt{35}/35$$

$$\langle 21 \quad {}^421 \| T \| 21 \quad {}^20 \rangle^- = -4\sqrt{35}/35$$

$$\langle 21 \quad {}^221 \| T \| 21 \quad {}^20 \rangle^+ = -4\sqrt{35}/35$$

$$\langle 21 \quad {}^23 \| T \| 21 \quad {}^421 \rangle^+ = -4\sqrt{7}/7$$

$$\langle 21 \quad {}^23 \| T \| 21 \quad {}^23 \rangle^+ = 2\sqrt{14}/7$$

$$\langle 21 \quad {}^23 \| T \| 21 \quad {}^221 \rangle^- = -4\sqrt{7}/7$$

$$\langle 21 \quad {}^221 \| T \| 21 \quad {}^221 \rangle_0^+ = 0 \qquad \langle 21 \quad {}^221 \| T \| 21 \quad {}^221 \rangle_1^- = +4i\sqrt{35}/35$$

$$\langle 2^2 \quad {}^31 \| T \| 2^2 \quad {}^31 \rangle^+ = \sqrt{210}/14$$

$$\langle 2^2 \quad {}^31 \| T \| 2^2 \quad {}^12^2 \rangle^+ = 3\sqrt{70}/35$$

$$\langle 2^2 \quad {}^31 \| T \| 2^2 \quad {}^52^2 \rangle^+ = -3\sqrt{7}/7$$

$$\langle 21^2 \quad {}^51 \| T \| 21^2 \quad {}^51 \rangle^+ = \sqrt{42}/7$$

$$\langle 21^2 \quad {}^51 \| T \| 21^2 \quad {}^31 \rangle^- = -3\sqrt{7}/7$$

$$\langle 21^2 \quad {}^31 \| T \| 21^2 \quad {}^31 \rangle^+ = -\sqrt{210}/70$$

$$\langle 21^2 \quad {}^31 \| T \| 21^2 \quad {}^11 \rangle^- = -3\sqrt{35}/35$$

$$\langle 21^2 \quad {}^51 \| T \| 21^2 \quad {}^32^2 \rangle^+ = -\sqrt{42}/7$$

$$\langle 21^2 \quad {}^32^2 \| T \| 21^2 \quad {}^11 \rangle^+ = -2\sqrt{210}/35$$

$$\langle 21^2 \quad {}^32^2 \| T \| 21^2 \quad {}^31 \rangle^- = -3\sqrt{35}/35$$

N.B. These reduced matrix elements have the permutational symmetry

$$\langle \lambda \quad |s'|\ \mu' \| T \| \lambda \quad |s|\ \mu \rangle_\delta^\sigma = \sigma \langle \lambda \quad |s|\ \mu \| T \| \lambda \quad |s'|\ \mu' \rangle_\delta^{\sigma *}$$

Table I13: Q^3 colour-spin matrix elements

q_a^3:

Identification	Associated SU_6^{CS} irrep	Total spin	Checking eigenvalue	Δ^{aa}
N	{21}	$\frac{1}{2}$	-8	-8
Δ,Ω	$\{1^3\}$	$\frac{3}{2}$	+8	+8

$q_a^2 q_b$:

Identification	Associated SU_6^{CS} irrep	Total spin	Checking eigenvalue	Δ^{aa}	Δ^{ab}
Λ	{21}	$\frac{1}{2}$	-8	-8	0
Σ,Ξ	{21}	$\frac{1}{2}$	-8	+8/3	-32/3
Σ^*,Ξ^*	$\{1^3\}$	$\frac{3}{2}$	+8	+8/3	+16/3

Table I14: $Q\bar{Q}$ colour-spin matrix elements

$q_a\bar{q}_b$:

Identification	Associated SU_6^{CS} irrep	Total spin	Checking eigenvalue	$\Delta^{a\bar{b}}$
pseudoscalar mesons	{0}	0	+16	-16
vector mesons	$\{21^4\}$	1	-16/3	+16/3

Table I15: $Q^4\bar{Q}$ colour-spin matrix elements[†]

$q_a^4\bar{q}_b$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{aa}	$\Delta^{a\bar{b}}$
^2A	$\{21\}$	+8	+8/3	+8/3
	$\{32^21^2\}$	-8	0	-8
			0	0
^2B	$\{21\}$	+8	-16/3	-40/3
^2C	$\{2^41\}$	+8	+56/3	+32/3
^4A	$\{1^3\}$	+24	+8/3	-4/3
	$\{32^21^2\}$	-4	0	$-4\sqrt{10}$
			+8	-8
^4B	$\{3^21^3\}$	-12	-16/3	+20/3
^4C	$\{1^3\}$	+24	+56/3	-16/3
^6A	$\{32^21^2\}$	+8/3	+8	+16/3

$q_a^3 q_b \bar{q}_c$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{aa}	Δ^{ab}	$\Delta^{a\bar{c}}$	$\Delta^{b\bar{c}}$
$^2\mathcal{D}$	{21}	+8	+2	-10	-10	-2/3
	{21}	+8	0	+8/3	+16/3	0
			-2	+2	-4	+4/3
	{21}	+8	0	0	$-16\sqrt{3}/3$	0
			0	0	$-2\sqrt{3}$	$-2\sqrt{3}/3$
			-2	-6	0	0
	$\{32^21^2\}$	-8	0	+16/3	+8/3	0
			0	+8/3	+16/3	0
			0	0	$+8\sqrt{3}/3$	0
			-8	0	0	-32/3
	$\{421^3\}$	-24	0	0	$-8\sqrt{3}/3$	0
			0	0	$+8\sqrt{3}/3$	0
			0	-8	0	0
			0	0	0	$+16\sqrt{3}/3$
			-8	0	0	0

^2E	$\{21\}$	+8	+8	0	0	+16/3
	$\{2^41\}$	+8	0	$+16\sqrt{2}/3$	$+8\sqrt{2}/3$	0
			+10	+10/3	+20/3	+4/3
	$\{32^21^2\}$	-8	0	0	$-8\sqrt{6}/3$	0
			0	0	$+10\sqrt{3}/3$	$-2\sqrt{3}/3$
			+10	-10	0	0
^4D	$\{1^3\}$	+24	+2	-10	+5	+1/3
	$\{32^21^2\}$	-4	0	0	$-\sqrt{15}$	$-\sqrt{15}/3$
			+2	+6	-9	+1
	$\{3^21^3\}$	-12	0	+8/3	-8/3	0
			0	0	$-8\sqrt{15}/3$	0
			-2	+2	+2	-2/3
	$\{421^3\}$	-20	0	+16/3	-4/3	0
			0	0	$-4\sqrt{15}/3$	0
			0	+8/3	-8/3	0
			-8	0	0	+16/3

^4E	$\{1^3\}$	+24	+8	0	0	-8
	$\{1^3\}$	+24	0	0	0	$+8\sqrt{15}/3$
			+8	0	0	-8/3
	$\{32^21^2\}$	-4	0	0	$-4\sqrt{30}/3$	0
			0	$+16\sqrt{2}/3$	$-4\sqrt{2}/3$	0
			+10	+10/3	-10/3	-2/3
^6D	$\{32^21^2\}$	+8/3	+2	+6	+6	-2/3
^6E	$\{32^21^2\}$	+8/3	+8	0	0	+16/3

$q_a^2q_b^2\bar{q}_c$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{aa}	Δ^{ab}	$\Delta^{a\bar{c}}$	Δ^{bb}	$\Delta^{b\bar{c}}$
^2F	$\{21\}$	+8	-4/3	-20/3	-20/3	+8/3	+4/3
	$\{21\}$	+8	0	0	$-20\sqrt{2}/3$	0	$-4\sqrt{2}/3$
			-4/3	-40/3	0	+8/3	0
	$\{32^21^2\}$	-8	0	-8	-8	0	0
			0	0	$-4\sqrt{2}$	0	0
			-8	0	0	+8/3	-16/3

	$\{421^3\}$	−24	0	0	0	0	0
			0	$-4\sqrt{3}$	0	0	0
			0	0	0	0	$-4\sqrt{6}$
			−8	0	0	+4	0
^2G	$\{21\}$	+8	+8/3	−8/3	−8/3	+8/3	−8/3
	$\{21\}$	+8	0	0	$-8\sqrt{2}/3$	0	$+8\sqrt{2}/3$
			+8/3	−16/3	0	+8/3	0
	$\{2^41\}$	+8	0	−8	0	0	−8
			0	0	0	0	$+4\sqrt{2}$
			+8/3	0	+8/3	+4	0
	$\{32^21^2\}$	−8	0	−8	−8	0	0
			0	0	$-4\sqrt{2}$	0	0
			0	+4	0	0	0
			+4	0	0	+8/3	+8/3
^4F	$\{1^3\}$	+24	−4/3	−20/3	+10/3	+8/3	−2/3
	$\{32^21^2\}$	−4	0	0	$-10\sqrt{5}/3$	0	$-2\sqrt{5}/3$
			−4/3	+20/3	−10	+8/3	+2
	$\{421^3\}$	−20	0	−8	+4	0	0
			0	0	$+4\sqrt{5}$	0	0
			−8	0	0	+8/3	+8/3

^4G	$\{1^3\}$	+24	+8/3	−8/3	+4/3	+8/3	+4/3
	$\{1^3\}$	+24	0	0	$-4\sqrt{5}/3$	0	$+4\sqrt{5}/3$
			+8/3	+8/3	−4	+8/3	−4
	$\{32^21^2\}$	−4	0	−8	0	0	+4
			0	0	0	0	$-4\sqrt{5}$
			+8/3	0	−4/3	+4	0
	$\{3^21^3\}$	−12	0	−8	+4	0	0
			0	0	$+4\sqrt{5}$	0	0
			0	+4	0	0	0
			+4	0	0	+8/3	−4/3
^6F	$\{32^21^2\}$	+8/3	−4/3	+20/3	+20/3	+8/3	−4/3
^6G	$\{32^21^2\}$	+8/3	+8/3	+8/3	+8/3	+8/3	+8/3

† The matrix elements are listed vertically in the order M_{11}, M_{12}, M_{22}, M_{13}, M_{23}, ...

Table I16: $Q^2\bar{Q}^2$ colour-spin matrix elements†

$q_a^2\bar{q}_b^2$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{aa}	$\Delta^{a\bar{b}}$	$\Delta^{\bar{b}\bar{b}}$
^1A	$\{0\}$	+32	-4/3	-80/3	-4/3
	$\{42^4\}$	-24	0	$-8\sqrt{6}$	0
			-8	0	-8
^1B	$\{0\}$	+32	+8/3	-32/3	+8/3
	$\{2^21^2\}$	-8	0	$-8\sqrt{6}$	0
			+4	0	+4
^3A	$\{21^4\}$	+32/3	-4/3	-40/3	-4/3
^3C	$\{21^4\}$	+32/3	-4/3	0	+4
	$\{31^3\}$	-40/3	0	$+8\sqrt{2}$	0
			-8	0	+8/3
$^3\bar{C}$	$\{21^4\}$	+32/3	+4	0	-4/3
	$\{3^22^3\}$	-40/3	0	$+8\sqrt{2}$	0
			+8/3	0	-8
^3B	$\{21^4\}$	+32/3	+8/3	-16/3	+8/3
^5A	$\{42^4\}$	-16	-4/3	+40/3	-4/3
^5B	$\{2^21^2\}$	0	+8/3	+16/3	+8/3

$q_a q_b \bar{q}_c^2$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{ab}	$\Delta^{a\bar{c}}$	$\Delta^{b\bar{c}}$	$\Delta^{\bar{c}\bar{c}}$
$^1\mathcal{D}$	$\{0\}$	+32	-4/3	-40/3	-40/3	-4/3
	$\{42^4\}$	-24	0	$-4\sqrt{6}$	$-4\sqrt{6}$	0
			-8	0	0	-8
^1E	$\{0\}$	+32	+4	0	0	+4
	$\{2^21^2\}$	-8	0	$-4\sqrt{6}$	$-4\sqrt{6}$	0
			+8/3	-16/3	-16/3	+8/3
$^3\mathcal{D}$	$\{21^4\}$	+32/3	+4	0	0	-4/3
	$\{21^4\}$	+32/3	0	$-20\sqrt{2}/3$	$+20\sqrt{2}/3$	0
			-4/3	-20/3	-20/3	-4/3
	$\{3^22^3\}$	-40/3	0	$+4\sqrt{2}$	$+4\sqrt{2}$	0
			0	+8	-8	0
			+8/3	0	0	-8
^3E	$\{21^4\}$	+32/3	-4/3	0	0	+4
	$\{21^4\}$	+32/3	0	$+4\sqrt{2}$	$+4\sqrt{2}$	0
			-8	0	0	+8/3
	$\{31^3\}$	-40/3	0	+8	-8	0
			0	$-8\sqrt{2}/3$	$+8\sqrt{2}/3$	0
			+8/3	-8/3	-8/3	+8/3

$^5\mathcal{D}$	$\{42^4\}$	-16	$-4/3$	$+20/3$	$+20/3$	$-4/3$
^5E	$\{2^21^2\}$	0	$+8/3$	$+8/3$	$+8/3$	$+8/3$

$q_a^2\bar{q}_b\bar{q}_c$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{aa}	$\Delta^{a\bar{b}}$	$\Delta^{a\bar{c}}$	$\Delta^{\bar{b}\bar{c}}$
$^1\bar{\mathcal{D}}$	$\{0\}$	$+32$	$-4/3$	$-40/3$	$-40/3$	$-4/3$
	$\{42^4\}$	-24	0	$-4\sqrt{6}$	$-4\sqrt{6}$	0
			-8	0	0	-8
$^1\bar{E}$	$\{0\}$	$+32$	$+4$	0	0	$+4$
	$\{2^21^2\}$	-8	0	$-4\sqrt{6}$	$-4\sqrt{6}$	0
			$+8/3$	$-16/3$	$-16/3$	$+8/3$
$^3\bar{\mathcal{D}}$	$\{21^4\}$	$+32/3$	$-4/3$	0	0	$+4$
	$\{21^4\}$	$+32/3$	0	$+20\sqrt{2}/3$	$-20\sqrt{2}/3$	0
			$-4/3$	$-20/3$	$-20/3$	$-4/3$
	$\{31^3\}$	$-40/3$	0	$+4\sqrt{2}$	$+4\sqrt{2}$	0
			0	-8	$+8$	0
			-8	0	0	$+8/3$

$^3\bar{E}$	$\{21^4\}$	+32/3	+4	0	0	-4/3
	$\{21^4\}$	+32/3	0	$+4\sqrt{2}$	$+4\sqrt{2}$	0
			+8/3	0	0	-8
	$\{3^22^3\}$	-40/3	0	-8	+8	0
			0	$+8\sqrt{2}/3$	$-8\sqrt{2}/3$	0
			+8/3	-8/3	-8/3	+8/3
$^5\bar{D}$	$\{42^4\}$	-16	-4/3	+20/3	+20/3	-4/3
$^5\bar{E}$	$\{2^21^2\}$	0	+8/3	+8/3	+8/3	+8/3

$q_aq_b\bar{q}_c\bar{q}_d$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{ab}	$\Delta^{a\bar{c}}$	$\Delta^{a\bar{d}}$	$\Delta^{b\bar{c}}$	$\Delta^{b\bar{d}}$	$\Delta^{\bar{c}\bar{d}}$
^1F	$\{0\}$	+32	+4	0	0	0	0	+4
	$\{0\}$	+32	0	$-10\sqrt{3}/3$	$+10\sqrt{3}/3$	$+10\sqrt{3}/3$	$-10\sqrt{3}/3$	0
			-4/3	-20/3	-20/3	-20/3	-20/3	-4/3
	$\{2^21^2\}$	-8	0	0	0	0	0	0
			0	$-2\sqrt{6}$	$-2\sqrt{6}$	$-2\sqrt{6}$	$-2\sqrt{6}$	0
			-8	0	0	0	0	-8
	$\{42^4\}$	-24	0	$-2\sqrt{6}$	$-2\sqrt{6}$	$-2\sqrt{6}$	$-2\sqrt{6}$	0
			0	$-4\sqrt{2}$	$+4\sqrt{2}$	$+4\sqrt{2}$	$-4\sqrt{2}$	0
			0	$-4\sqrt{3}/3$	$+4\sqrt{3}/3$	$+4\sqrt{3}/3$	$-4\sqrt{3}/3$	0
			+8/3	-8/3	-8/3	-8/3	-8/3	+8/3

^3F	$\{21^4\}$	+32/3	+4	0	0	0	0	-4/3
	$\{21^4\}$	+32/3	0	-10/3	+10/3	+10/3	-10/3	0
			-4/3	0	0	0	0	+4
	$\{21^4\}$	+32/3	0	$+10\sqrt{2}/3$	$+10\sqrt{2}/3$	$-10\sqrt{2}/3$	$-10\sqrt{2}/3$	0
			0	$-10\sqrt{2}/3$	$+10\sqrt{2}/3$	$-10\sqrt{2}/3$	$+10\sqrt{2}/3$	0
			-4/3	-10/3	-10/3	-10/3	-10/3	-4/3
	$\{21^4\}$	+32/3	0	0	0	0	0	0
			0	$+2\sqrt{2}$	$+2\sqrt{2}$	$+2\sqrt{2}$	$+2\sqrt{2}$	0
			0	+4	-4	+4	-4	0
			-8	0	0	0	0	+8/3
	$\{31^3\}$	-40/3	0	$+2\sqrt{2}$	$+2\sqrt{2}$	$+2\sqrt{2}$	$+2\sqrt{2}$	0
			0	0	0	0	0	0
			0	-4	-4	+4	+4	0
			0	+4/3	-4/3	-4/3	+4/3	0
			+8/3	0	0	0	0	-8
	$\{3^2 2^3\}$	-40/3	0	+4	-4	+4	-4	0
			0	-4	-4	+4	+4	0
			0	$-2\sqrt{2}$	$+2\sqrt{2}$	$+2\sqrt{2}$	$-2\sqrt{2}$	0
			0	$+4\sqrt{2}/3$	$+4\sqrt{2}/3$	$-4\sqrt{2}/3$	$-4\sqrt{2}/3$	0
			0	$-4\sqrt{2}/3$	$+4\sqrt{2}/3$	$-4\sqrt{2}/3$	$+4\sqrt{2}/3$	0
			+8/3	-4/3	-4/3	-4/3	-4/3	+8/3

^5F	$\{2^2 1^2\}$	0	-4/3	+10/3	+10/3	+10/3	+10/3	-4/3
	$\{42^4\}$	-16	0	$+2\sqrt{2}$	$-2\sqrt{2}$	$-2\sqrt{2}$	$+2\sqrt{2}$	0
			+8/3	+4/3	+4/3	+4/3	+4/3	+8/3

† The matrix elements are listed vertically in the order M_{11}, M_{12}, M_{22}, M_{13}, M_{23}, ...

Table I17: Q^6 colour-spin matrix elements†

q_a^6:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{aa}
^1A	$\{2^2 1^2\}$	+8	+8
^1B	$\{0\}$	+48	+48
^3C	$\{2^3\}$	+8/3	+8/3
^3D	$\{21^4\}$	+80/3	+80/3
^5A	$\{2^2 1^2\}$	+16	+16
^7C	$\{2^3\}$	+16	+16

$q_a^5 q_b$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{aa}	Δ^{ab}
^1E	$\{2^2 1^2\}$	+8	0	+8
^1F	$\{2^2 1^2\}$	+8	+12	-4
^1G	$\{0\}$	+48	+32	+16
^3E	$\{2^3\}$	+8/3	0	-8/3
	$\{321\}$	-28/3	0	$\pm 8\sqrt{5}/3$
			+4	-8
^3F	$\{21^4\}$	+80/3	+12	+4/3
	$\{31^3\}$	+8/3	0	$\pm 16\sqrt{5}/3$
			+16	0
^3G	$\{21^4\}$	+80/3	+32	-16/3
^5E	$\{2^2 1^2\}$	+16	+4	+24/5
	$\{321\}$	-4	0	$\pm 48/5$
			+32/3	-112/15
^5F	$\{2^2 1^2\}$	+16	+16	0
^7E	$\{2^3\}$	+16	+32/3	+16/3

$q_a^4 q_b^2$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{aa}	Δ^{ab}	Δ^{bb}
^1H	$\{2^2 1^2\}$	+8	+8/3	+8/3	+8/3
^1I	$\{2^2 1^2\}$	+8	-16/3	-40/3	+8/3
	$\{3^2\}$	-24	0	$+8\sqrt{3}$	0
			-4	0	+4
^1J	$\{0\}$	+48	+56/3	+32/3	+8/3
	$\{2^2 1^2\}$	+8	0	$+8\sqrt{6}$	0
			+20	0	+4
^3H	$\{21^4\}$	+80/3	+8	-8	+8/3
	$\{2^3\}$	+8/3	0	$-4\sqrt{30}/3$	0
			+8/3	+4/3	+8/3
	$\{31^3\}$	+8/3	0	0	0
			0	$-8\sqrt{6}/3$	0
			0	0	+8/3
	$\{321\}$	-28/3	0	$-8\sqrt{15}/3$	0
			0	$+4\sqrt{2}$	0
			0	$-16\sqrt{3}/3$	0
			+20/3	0	+4
^3I	$\{321\}$	-28/3	-16/3	-20/3	+8/3
^3J	$\{21^4\}$	+80/3	+56/3	+16/3	+8/3
^3K	$\{21^4\}$	+80/3	+56/3	0	-8
	$\{31^3\}$	+8/3	0	$-8\sqrt{2}$	0
			+20	0	-4/3
^5H	$\{2^2 1^2\}$	+16	+8	-8/3	+8/3
	$\{321\}$	-4	0	$-4\sqrt{6}$	0
			+8/3	-4/3	+8/3
^5I	$\{2^2 1^2\}$	+16	-16/3	+20/3	+8/3
	$\{321\}$	-4	0	$-4\sqrt{6}$	0
			+4	0	+4
^5J	$\{2^2 1^2\}$	+16	+56/3	-16/3	+8/3
^7H	$\{2^3\}$	+16	+8	+16/3	+8/3

$q_a^3 q_b^3$:

Matrix identification	Associated SU_6^{CS} irreps	Checking eigenvalues	Δ^{aa}	Δ^{ab}	Δ^{bb}
^1L	$\{2^2 1^2\}$	+8	-2	0	+10
^1M	$\{0\}$	+48	+8	0	+8
	$\{2^2 1^2\}$	+8	0	+16	0
			+10	+20	+10
^3L	$\{21^4\}$	+80/3	-8	0	+8
	$\{31^3\}$	+8/3	0	+32/3	0
			-2	0	+10
	$\{321\}$	-28/3	0	-8/3	0
			0	-40/3	0
			+2	0	+10
^3M	$\{21^4\}$	+80/3	+8	0	+8
	$\{2^3\}$	+8/3	0	$-16\sqrt{5}/3$	0
			+10	-20/3	+10
^5L	$\{2^2 1^2\}$	+16	-8	0	+8
	$\{321\}$	-4	0	-8	0
			+2	0	+10
^5M	$\{2^2 1^2\}$	+16	+8	0	+8
^7M	$\{2^3\}$	+16	+8	0	+8

† The matrix elements are listed vertically in the order M_{11}, M_{12}, M_{22}, M_{13}, M_{23}, ...

CHAPTER 7

THE SPHERICAL CAVITY BAG SPECTRUM

Having obtained the colour-spin matrices we can now proceed to calculate the eigenenergies of the spherical cavity Hamiltonian (4.4). As remarked earlier, the eigenstates of the colour-spin matrix are also eigenstates of the full Hamiltonian and the other terms can simply be added to the eigenvalues of the colour-magnetic interaction after the colour-spin matrix has been diagonalized. Each eigenvalue can then be minimized separately; note that the eigenenergies will not all have minima at the same bag radius and this means that the eigenstates will not quite be orthogonal. From chapter 5 it is clear that the resulting eigenstates are primitives and the eigenenergies are to be interpreted as the energies for which the P-matrix has poles.

The energies of all q^3, $q\bar{q}$, $q^2\bar{q}^2$, $q^4\bar{q}$ and q^6 primitives have been calculated and are listed in tables I18-22 respectively. Also displayed in these tables are the equilibrium values of the dimensionless quantity $m_s R$ and the breakdown of the energies of the primitives into contributions from each of the terms (4.5), (4.6), (4.7) and (4.10) in the Hamiltonian. For $q^2\bar{q}^2$, $q^4\bar{q}$ and q^6 the bag eigenstates are also given using the reduced colour-spin identification and the specific configuration — the total isospin suffices to locate these states in tables I9, I8 and I7 respectively. (In tables I20-22 the primitives have been listed in order

of increasing S, decreasing S, increasing I and increasing mass.) The computational procedures used in obtaining these energies are given in appendix IB. Considerable care was taken to ensure that round-off errors did not affect the answers (which are given to the nearest MeV) and so the only errors are those inherent in the model. However, this accuracy is only of academic interest. In comparing the current results for q^3 and $q\bar{q}$ with those of DeGrand et al. (1975) — who were able to evaluate the colour-spin matrices for these cases exactly — one notes a small discrepancy. This discrepancy appears even for states where there is no colour electric contribution — those for which all quarks are of the same species — and this is taken to mean that the fit (4.16) obtained by DeGrand et al. is only good to a few MeV. We could have refitted the parameters but with the present state of the bag model such action would not really be warranted. Thus, when combined with the neglected colour electric term, this implies an error in our quoted energies of approximately 5-10 MeV. Of course there are even larger errors in some cases, such as with the η and η' mesons, due to our neglect of various mixing processes.

We discuss each configuration in turn, concentrating on those primitives which have a direct bearing on the P-matrix poles listed in table I10.

(a) $Q^2\bar{Q}^2$ Primitives

We consider the S=0 primitives and begin with exotic channels. Table I10 includes a P-matrix pole in the

$(I,S) = (\frac{3}{2},+1)$ πK channel. From table I20 the lowest mass primitive with these quantum numbers is the state

$$|{}^1\bar{E}^{(o^2\bar{o}\bar{s})}\ E(1322),\ \tfrac{3}{2},+1\rangle$$

$$= 0.582|{}^1\bar{E}_1^{(o^2\bar{o}\bar{s})},\tfrac{3}{2}\rangle + 0.813|{}^1\bar{E}_2^{(o^2\bar{o}\bar{s})},\tfrac{3}{2}\rangle \qquad (7.1)$$

at 1.322 GeV. (The mass in MeV, I and S quantum numbers and the fact that the primitive is exotic have been included in (7.1) along with the specific colour-spin matrix.) The basis states in (7.1) can be dissociated as described in chapter 3 to give

$$|{}^1\bar{E}_1^{(o^2\bar{o}\bar{s})},\tfrac{3}{2}\rangle = + \sqrt{\tfrac{1}{6}} \cdot \frac{1}{\sqrt{2}} \{|(\pi K)\rangle - |(K\pi)\rangle\}$$

$$+ \sqrt{\tfrac{1}{2}} \cdot \frac{1}{\sqrt{2}} \{|(\rho K^*)\rangle - |(K^*\rho)\rangle\}$$

$$+ \sqrt{\tfrac{1}{3}} \{\text{colour octet}\} \qquad (7.2)$$

and

$$|{}^1\bar{E}_2^{(o^2\bar{o}\bar{s})},\tfrac{3}{2}\rangle = + \sqrt{\tfrac{1}{4}} \cdot \frac{1}{\sqrt{2}} \{|(\pi K)\rangle - |(K\pi)\rangle\}$$

$$- \sqrt{\tfrac{1}{12}} \cdot \frac{1}{\sqrt{2}} \{|(\rho K^*)\rangle - |(K^*\rho)\rangle\}$$

$$- \sqrt{\tfrac{2}{3}} \{\text{colour octet}\} \qquad (7.3)$$

and thus

$$|{}^1\bar{E}^{(o^2\bar{o}\bar{s})}\ E(1322),\tfrac{3}{2},\ +1\rangle = 0.644 \cdot \frac{1}{\sqrt{2}} \{|(\pi K)\rangle - |(K\pi)\rangle\}$$

$$+ 0.177 \cdot \frac{1}{\sqrt{2}} \{|(\rho K^*)\rangle - |(K^*\rho)\rangle\}$$

$$+ 0.744\ \{\text{colour octet}\} \qquad (7.4)$$

i.e. this state comprises 41.5% πK components, 3.1% ρK^* and 55.4% colour octet-octet combinations. Note that we cannot simply add together the colour octet contributions in (7.2) and (7.3) to get the total in (7.4) because the colour octet components in (7.2) are the colour excitations

$$\sqrt{\frac{1}{12}} \cdot \frac{1}{\sqrt{2}} \{ |(\underset{\sim}{\pi} \cdot \underset{\sim}{K})> - |(\underset{\sim}{K} \cdot \underset{\sim}{\pi})> \} + \sqrt{\frac{1}{4}} \cdot \frac{1}{\sqrt{2}} \{ |(\underset{\sim}{\rho} \cdot \underset{\sim}{K}^*)> - |(\underset{\sim}{K}^* \cdot \underset{\sim}{\rho})> \}$$

whereas in (7.3) they are

$$-\sqrt{\frac{1}{2}} \cdot \frac{1}{\sqrt{2}} \{ |(\underset{\sim}{\pi} \cdot \underset{\sim}{K})> - |(\underset{\sim}{K} \cdot \underset{\sim}{\pi})> \} + \sqrt{\frac{1}{6}} \cdot \frac{1}{\sqrt{2}} \{ |(\underset{\sim}{\rho} \cdot \underset{\sim}{K}^*)> - |(\underset{\sim}{K}^* \cdot \underset{\sim}{\rho})> \}.$$

(To obtain the combined contribution we take the square-root of the sum of the two probabilities.) The coefficients of the colour contributions in (7.2) and 7.3) are colour recoupling coefficients and the specific colour octet components can be easily obtained from the colour neutral components by multiplying the coefficients of the latter by, in the case of (7.2), $\left(\sqrt{\frac{2}{3}} \right)^{-1} \cdot \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{2}}$, and in the case of (7.3), $\left(\sqrt{\frac{1}{3}} \right)^{-1} \cdot - \sqrt{\frac{2}{3}} = -\sqrt{2}$. This can be seen by considering (3.24).

The primitive is above the πK threshold at 635 MeV but well below the ρK^* threshold at 1662 MeV and should therefore appear as a pole in the πK $I = \frac{3}{2}$ P-matrix. (Note that $\xi_{\pi K} = 1$ for this primitive.) Its residue, given by (5.19), (5.34), (5.22) and (5.33) is 0.15, in agreement with Jaffe and Low (1979). The compensation energy is at 1.11 GeV, below the primitive mass, and therefore the phase-shift

should be negative indicating repulsion. The agreement with table I10 is quite good and there is little difference between the current prediction and that of Jaffe (1977a).

The other exotic channel included in table I10 is the $(I,S) = (2,0)$ $\pi\pi$ channel. From table I20 the lowest mass primitive with the right quantum numbers is the state

$$|{}^1\mathcal{B}^{(Q^2\bar{Q}^2)}\ E(1122),2,0\rangle$$

$$= 0.813|{}^1\mathcal{B}_1^{(Q^2\bar{Q}^2)},2\rangle + 0.582|{}^1\mathcal{B}_2^{(Q^2\bar{Q}^2)},2\rangle \qquad (7.5)$$

at 1.122 GeV, little different from Jaffe (1977a). The basis states in (7.5) can be dissociated to give

$$|{}^1\mathcal{B}_1^{(Q^2\bar{Q}^2)},2\rangle = \sqrt{\frac{1}{4}}\,|(\pi\pi)\rangle - \sqrt{\frac{1}{12}}\,|(\rho\rho)\rangle$$

$$- \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \qquad (7.6)$$

and

$$|{}^1\mathcal{B}_2^{(Q^2\bar{Q}^2)},2\rangle = \sqrt{\frac{1}{6}}\,|(\pi\pi)\rangle + \sqrt{\frac{1}{2}}\,|(\rho\rho)\rangle$$

$$+ \sqrt{\frac{1}{3}}\,\{\text{colour octet}\}\ . \qquad (7.7)$$

Thus

$$|{}^1\mathcal{B}^{(Q^2\bar{Q}^2)}\ E(1122),2,0\rangle$$

$$= 0.644|(\pi\pi)\rangle + 0.177|(\rho\rho)\rangle + 0.744\{\text{colour-octet}\} \qquad (7.8)$$

i.e. this state comprises 41.5% $\pi\pi$ components, 3.1% $\rho\rho$ and 55.4% colour octet-octet combinations. The projection onto open channels is $\xi_{\pi\pi} = 1$ and we expect to see a pole

in the $\pi\pi$ I = 2 P-matrix at 1.122 GeV, above the compensation energy at 0.95 GeV. The pole residue is predicted to be 0.11, again in agreement with Jaffe and Low (1979).

In the cryptoexotic $(I,S) = (\frac{1}{2}, +1)$ channel, table I20 has at state at only 0.882 GeV:

$$|{}^1\bar{\mathcal{D}}^{(o^2\bar{o}\bar{s})}C(882),\tfrac{1}{2},1\rangle$$

$$= 0.813|{}^1\bar{\mathcal{D}}_1^{(o^2\bar{o}\bar{s})},\tfrac{1}{2}\rangle + 0.582|{}^1\bar{\mathcal{D}}_2^{(o^2\bar{o}\bar{s})},\tfrac{1}{2}\rangle \quad . \qquad (7.9)$$

The basis states dissociate as:

$$\begin{aligned}
|{}^1\bar{\mathcal{D}}_1^{(o^2\bar{o}\bar{s})},\tfrac{1}{2}\rangle = {} & + \sqrt{\tfrac{1}{8}} \cdot \frac{1}{\sqrt{2}} \{|(\eta_o K)\rangle - |(K\eta_o)\rangle\} \\
& + \sqrt{\tfrac{3}{8}} \cdot \frac{1}{\sqrt{2}} \{|(\pi K)\rangle + |(K\pi)\rangle\} \\
& - \sqrt{\tfrac{1}{24}} \cdot \frac{1}{\sqrt{2}} \{|(\omega_o K^*)\rangle - |(K^*\omega_o)\rangle\} \\
& - \sqrt{\tfrac{1}{8}} \cdot \frac{1}{\sqrt{2}} \{|(\rho K^*)\rangle + |(K^*\rho)\rangle\} \\
& + \sqrt{\tfrac{1}{3}} \cdot \{\text{colour octet}\} \qquad (7.10)
\end{aligned}$$

$$\begin{aligned}
|{}^1\bar{\mathcal{D}}_2^{(o^2\bar{o}\bar{s})},\tfrac{1}{2}\rangle = {} & + \sqrt{\tfrac{1}{48}} \cdot \frac{1}{\sqrt{2}} \{|(\eta_o K)\rangle - |(K\eta_o)\rangle\} \\
& + \sqrt{\tfrac{1}{16}} \cdot \frac{1}{\sqrt{2}} \{|(\pi K)\rangle + |(K\pi)\rangle\} \\
& + \sqrt{\tfrac{1}{16}} \cdot \frac{1}{\sqrt{2}} \{|(\omega_o K^*)\rangle - |(K^*\omega_o)\rangle\}
\end{aligned}$$

$$+ \sqrt{\frac{3}{16}} \cdot \frac{1}{\sqrt{2}} \{ |(\rho K^*)> + |(K^* \rho)> \}$$

$$- \sqrt{\frac{2}{3}} \cdot \{\text{colour octet}\} \quad . \qquad (7.11)$$

It follows that this state comprises 41.4% πK, 13.8% $\eta_o K$, 0.1% ρK^*, 44.6% colour octet-octet components and a very small amount of $\omega_o K^*$. The $\eta_o K$ content poses a problem because η_o is not a physical state and mixing effects will arise here. We shall assume that we in fact have the open channel projection $\xi_{\pi K} = 1$ since the ηK and $\eta' K$ thresholds are above 0.882 GeV. This time the primitive is below the compensation energy at 1.11 GeV. The predicted residue this time is 0.08. Again the correspondence with table I10 is quite good and there is little difference between these predictions and those of Jaffe and Low (1979, Jaffe 1977a). The effect in this channel is associated with the broad $\kappa(800 - 1100)$, however no particle resonance is claimed and the phase shift is certainly not clearly resonant.

The other nonexotic channels included in table I10 are the $(I,S) = (0,0)$ $\pi\pi$ and $K\bar{K}$ channels. From table I20 we see that the bag model predicts a primitive as low as 0.642 GeV:

$$|{}^1A^{(o^2\bar{o}^2)} \; C(642),0,0>$$

$$= 0.813|{}^1A_1^{(o^2\bar{o}^2)},0> + 0.582|{}^1A_2^{(o^2\bar{o}^2)},0> \quad . \qquad (7.12)$$

The basis states here dissociate as

$$|{}^1A_1^{(o^2\bar{o}^2)},0> = + \sqrt{\frac{1}{8}}\,|(\eta_o\eta_o)> + \sqrt{\frac{3}{8}}\,|(\pi\pi)> - \sqrt{\frac{1}{24}}|(\omega_o\omega_o)>$$

$$- \sqrt{\frac{1}{8}}\,|(\rho\rho)> + \sqrt{\frac{1}{3}}\,\{\text{colour octet}\} \qquad (7.13)$$

and

$$|{}^1A_2^{(o^2\bar{o}^2)},0> = + \sqrt{\frac{1}{48}}\,|(\eta_o\eta_o)> + \sqrt{\frac{1}{16}}\,|(\pi\pi)> + \sqrt{\frac{1}{16}}\,|(\omega_o\omega_o)>$$

$$+ \sqrt{\frac{3}{16}}\,|(\rho\rho)> - \sqrt{\frac{2}{3}}\,\{\text{colour octet}\}. \qquad (7.14)$$

Thus this primitive is comprised of 41.4% $\pi\pi$, 13.8% $\eta_o\eta_o$, a very small amount of $\omega_o\omega_o$, 0.1% $\rho\rho$ and 44.6% colour octet-octet components. The open channel coupling is taken to be $\xi_{\pi\pi} = 1$ and the predicted residue is 0.04 in agreement with Jaffe and Low (1979). This primitive is associated with the $\varepsilon(700)$. Again, no claim is made that this is a particle resonance.

Also in the $(I,S) = (0,0)$ channel is a primitive at 1.115 GeV. Jaffe and Low (1979) have shown that this is associated with the narrow S*(993) — the narrow effect being produced by the proximity of the $K\bar{K}$ threshold. They noted that this state does not couple to $\pi\pi$ in the OZI limit but S*(993) does. However, looking at table I20 we see, in addition to the primitive at 1.115 GeV, another primitive at 1.122 GeV. This is only 7 MeV higher in mass — which is of the order of accuracy of our calculation — and appreciable mixing can be expected. Let us examine these states more closely. We have the dissociations

$$|{}^1F_1^{(OS\bar{O}\bar{S})},0\rangle = + \sqrt{\frac{1}{12}} \cdot \frac{1}{\sqrt{2}} \{|(\eta_O\eta_S)\rangle + |(\eta_S\eta_O)\rangle\}$$

$$+ \sqrt{\frac{1}{4}} \cdot \frac{1}{\sqrt{2}} \{|\omega_O\phi_S)\rangle + |(\phi_S\omega_O)\rangle\}$$

$$- \sqrt{\frac{1}{12}} \cdot \frac{1}{\sqrt{2}} \{|(K\bar{K})\rangle - |(\bar{K}K)\rangle\}$$

$$- \sqrt{\frac{1}{4}} \cdot \frac{1}{\sqrt{2}} \{|(K^*\bar{K}^*)\rangle - |(\bar{K}^*K^*)\rangle\}$$

$$+ \sqrt{\frac{1}{3}} \{\text{colour octet}\} \qquad (7.15)$$

$$|{}^1F_2^{(OS\bar{O}\bar{S})},0\rangle = + \sqrt{\frac{1}{4}} \cdot \frac{1}{\sqrt{2}} \{|(\eta_O\eta_S)\rangle + |(\eta_S\eta_O)\rangle\}$$

$$- \sqrt{\frac{1}{12}} \cdot \frac{1}{\sqrt{2}} \{|\omega_O\phi_S)\rangle + |(\phi_S\omega_O)\rangle\}$$

$$+ \sqrt{\frac{1}{4}} \cdot \frac{1}{\sqrt{2}} \{|(K\bar{K})\rangle - |(\bar{K}K)\rangle\}$$

$$- \sqrt{\frac{1}{12}} \cdot \frac{1}{\sqrt{2}} \{|(K^*\bar{K}^*)\rangle - |(\bar{K}^*K^*)\rangle\}$$

$$+ \sqrt{\frac{1}{3}} \{\text{colour octet}\} \qquad (7.16)$$

$$|{}^1F_3^{(OS\bar{O}\bar{S})},0\rangle = + \sqrt{\frac{1}{24}} \cdot \frac{1}{\sqrt{2}} \{|(\eta_O\eta_S)\rangle + |(\eta_S\eta_O)\rangle\}$$

$$+ \sqrt{\frac{1}{8}} \cdot \frac{1}{\sqrt{2}} \{|(\omega_O\phi_S)\rangle + |(\phi_S\omega_O)\rangle\}$$

$$+ \sqrt{\frac{1}{24}} \cdot \frac{1}{\sqrt{2}} \{|(K\bar{K})\rangle - |(\bar{K}K)\rangle\}$$

$$+ \sqrt{\frac{1}{8}} \cdot \frac{1}{\sqrt{2}} \{|(K^*\bar{K}^*)\rangle - |(\bar{K}^*K^*)\rangle\}$$

$$- \sqrt{\frac{2}{3}} \{\text{colour octet}\} \qquad (7.17)$$

$$|{}^1F_4^{(OS\bar{O}\bar{S})},0> = + \sqrt{\frac{1}{8}} \cdot \frac{1}{\sqrt{2}} \{|(\eta_o\eta_s)> + |(\eta_s\eta_o)>\}$$

$$- \sqrt{\frac{1}{24}} \cdot \frac{1}{\sqrt{2}} \{|(\omega_o\phi_s)> + |(\phi_s\omega_o)>\}$$

$$- \sqrt{\frac{1}{8}} \cdot \frac{1}{\sqrt{2}} \{|(K\bar{K})> - |(\bar{K}K)>\}$$

$$+ \sqrt{\frac{1}{24}} \cdot \frac{1}{\sqrt{2}} \{|(K^*\bar{K}^*)> - |(\bar{K}^*K^*)>\}$$

$$- \sqrt{\frac{2}{3}} \{\text{colour octet}\} \qquad (7.18)$$

$$|{}^1B_1^{(O^2\bar{O}^2)},0> = \sqrt{\frac{3}{16}} |(\eta_o\eta_o)> - \sqrt{\frac{1}{16}} |(\pi\pi)> - \sqrt{\frac{1}{16}} |(\omega_o\omega_o)>$$

$$+ \sqrt{\frac{1}{48}} |(\rho\rho)> - \sqrt{\frac{2}{3}} \{\text{colour octet}\} \qquad (7.19)$$

$$|{}^1B_2^{(O^2\bar{O}^2)},0> = \sqrt{\frac{1}{8}} |(\eta_o\eta_o)> - \sqrt{\frac{1}{24}} |(\pi\pi)> + \sqrt{\frac{3}{8}} |(\omega_o\omega_o)>$$

$$- \sqrt{\frac{1}{8}} |(\rho\rho)> + \sqrt{\frac{1}{3}} \{\text{colour octet}\} \qquad (7.20)$$

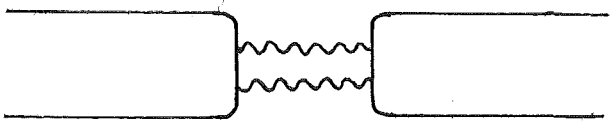
Thus $|{}^1F^{(OS\bar{O}\bar{S})}$ C(1115),0,0> comprises 27.0% $K\bar{K}$, 28.2% $\eta_o\eta_s$, 0.1% $\omega_o\phi_s$, 0.1% $K^*\bar{K}^*$ and 44.6% colour octet components. In contrast $|{}^1B^{(O^2\bar{O}^2)}$ C(1122),0,0> comprises 10.4% $\pi\pi$, 31.1% $\eta_o\eta_o$, 2.3% $\omega_o\omega_o$, 0.8% $\rho\rho$ and 55.4% colour octet components.

Mixing can occur between these states via annihilation processes into virtual gluons. This sort of mixing takes place between the η_o and η_s, 0^+ mesons and to a lesser degree for the ω_o and ϕ_s, 1^- mesons. However, in the $q^2\bar{q}^2$ system mixing can occur at first order (Jaffe, 1977a) since the annihilating pair can be both in a flavour singlet, $\{0\}^{fl}$

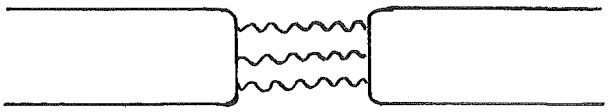
and $a_{\Lambda}^{\text{spin 1}}$ colour octet, $^{3}\{21\}^{C}$ state. The relevant processes are shown in fig. 7.1. However, although the mixing can occur here at first order, the fraction of the time spent by the annihilating pair in a $\{0\}^{fl}$ $\{21\}^{C}$ state will be rather small and this will decrease the mixing by a, perhaps large, numerical factor. Nevertheless the proximity of these primitives is good reason to believe the mixing to be substantial. Therefore we conclude that the coupling of the S*(993) to $\pi\pi$, rather than creating difficulties for this interpretation, is encouraging.

This pair of primitives is degenerate with another pair in the (I,S) = (1,0) channel. It is easily seen from the basis vectors that the only difference between these states is the total isospin and therefore this degeneracy is of the same nature as the $\pi\eta_o$ and $\rho\omega_o$ degeneracies encountered for ordinary mesons. This new pair also will mix and is to be associated with the $\delta(980)$ effect. The dissociations of the $^{1}F^{(OS\bar{O}\bar{S})}$ I = 1 states can be obtained from (7.15) to (7.18) simply by substituting $\eta_o \to \pi$ and $\omega_o \to \rho$. Thus $|^{1}F^{(OS\bar{O}\bar{S})}{}_{\Lambda}^{C(1115)}, 1, 0\rangle$ comprises 27.0% $K\bar{K}$, 28.2% $\pi\eta_s$, 0.1% $\rho\phi_s$, 0.1% $K^*\bar{K}^*$ and 44.6% colour octet components. The state $|^{1}B^{(O^2\bar{O}^2)}C(1122), 1, 0\rangle$ is also degenerate with the state in (7.8). It is comprised of 41.5% $\eta_o\pi$, 3.1% $\omega_o\rho$ and 55.4% colour octet components. (Note that this state does not couple to $|(\pi\pi)\rangle$ because an isospin 9j symbol is zero by permutation symmetry. This is related to conservation of G-parity.) Mixing between this pair of primitives will not be quite the same as for the I = 0 pair because their

(a)



(b)



(c)

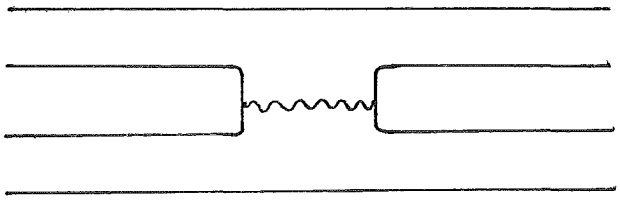


Fig. 7.1: Lowest-order virtual gluon annihilation processes inducing mixing, (a) in the 0^+ mesons, (b) in the 1^- mesons and (c) in the $q^2\bar{q}^2$ system.

dissociation is different. However, we do expect to see a P-matrix pole in the $\pi\rho$ channel approximately degenerate with the S*(993). There is little doubt that this is associated with the $\delta(980)$ effect.

(b) $Q^4\bar{Q}$ Primitives

Before embarking on a discussion of our results for this configuration we remark that the treatment by Hogaasen and Sorba (1978) of this problem in the flavour symmetry limit contains a spurious state and isoscalar factors with the wrong magnitudes.

Consider the $\frac{1}{2}^-$ primitives and let us begin once more with exotic channels. From table I21 we see that there is a state with quantum numbers $(I,S) = (0,1)$ at 1.717 GeV. Its dissociation may be calculated as described in chapter 3. Note that in (3.25) both the strangeness and colour recoupling coefficients are always unity. We find

$$|{}^2\mathcal{B}^{(Q^4\bar{S})},0\rangle = -\sqrt{\frac{1}{4}}\,|(NK)\rangle - \sqrt{\frac{1}{12}}\,|(NK^*)\rangle + \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \qquad (7.21)$$

easily discovering that $|{}^2\mathcal{B}^{(Q^4\bar{S})}$ E(1717),0,1> comprises 25.0% NK, 8.3% NK* and 66.7% colour octet-octet components. The open channel projection is $\xi_{NK} = 1$ and we expect a P-matrix pole in this channel at 1.717 GeV above the compensation energy, 1.67 GeV, signalling a negative phase-shift and a repulsive potential. The predicted residue is 0.15.

This is little different from the predictions of Strottman (1979) and Roiesnel (1979). The agreement with the observed pole position in table I10 is good.

The lowest state in the I = ½ KN channel is found from table I21 to be at 1.905 GeV. We have the dissociations

$$|{}^2A_1^{(0^4\bar{S})},1\rangle = -\sqrt{\frac{1}{12}}\,|(NK)\rangle - \sqrt{\frac{1}{36}}\,|(NK^*)\rangle + \sqrt{\frac{2}{9}}\,|(\Delta K^*)\rangle - \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \tag{7.22}$$

$$|{}^2A_2^{(0^4\bar{S})},1\rangle = -\sqrt{\frac{1}{12}}\,|(NK)\rangle + \sqrt{\frac{1}{4}}\,|(NK^*)\rangle + \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \tag{7.23}$$

and thus $|{}^2A^{(0^4\bar{S})}$ E(1905),1,1> comprises 16.1% NK, 10.1% NK*, 7.1% ΔK* and 66.7% colour octet components. This state is above both the NK threshold at 1.434 GeV and the NK* threshold at 1.831 GeV (but below the ΔK* threshold at 2.128 GeV) and thus this time there are two open channels. The open channel projections are $\xi_{NK} = 0.8$ and $\xi_{NK^*} = 0.6$ and the residue is 0.19. However the effect of the K*N threshold is to produce an "effective" pole in the reduced KN P-matrix below the K*N threshold with a different "effective" residue. Once again though the pole is above the KN compensation energy and the phase-shift is expected to be negative.

Note that the coefficients of the colour octet contributions in the $q^4\bar{q}$ dissociations do not have the same significance as they did in the $q^2\bar{q}^2$ dissociations. Indeed among the $q^4\bar{q}$ colour octet-octet components there occurs an N* state i.e. one with the flavour quantum numbers of the nucleon but with spin $\frac{3}{2}$.

In the nonexotic sector, table I21 includes a state with quantum numbers $(I,S) = (\frac{1}{2},0)$ at 1.502 GeV. We have the dissociation

$$|{}^2B^{(o^4\bar{o})},\tfrac{1}{2}> = -\sqrt{\frac{1}{16}}\,|(N\eta_o)> + \sqrt{\frac{3}{16}}\,|(N\pi)> - \sqrt{\frac{1}{48}}\,|(N\omega_o)>$$
$$+ \sqrt{\frac{1}{16}}\,|(N\rho)> + \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \qquad (7.23)$$

and thus $|{}^2B^{(o^4\bar{o})}C(1502),\frac{1}{2},0>$ comprises 18.8% $N\pi$, 6.3% $N\eta_o$, 2.1% $N\omega_o$, 6.3% $N\rho$ and 66.7% colour octet contributions. Both the $N\pi$ and $N\eta$ channels are open and an effective pole in the $N\pi$ channel can be expected. The $N\pi$ compensation energy is 1.49 GeV and the phase-shift is correctly predicted to be positive.

In the $I = \frac{3}{2}$ πN channel, the lowest mass primitive in table I21 is at 1.713 GeV. The appropriate dissociations are

$$|{}^2A_1^{(o^4\bar{o})},\tfrac{3}{2}> = -\sqrt{\frac{1}{12}}\,|(N\pi)> - \sqrt{\frac{1}{36}}\,|(N\rho)> + \sqrt{\frac{1}{12}}\,|(\Delta\omega)>$$
$$- \sqrt{\frac{5}{36}}\,|(\Delta\rho)> + \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \qquad (7.24)$$

and

$$|{}^2A_2^{(o^4\bar{o})},\tfrac{3}{2}> = -\sqrt{\frac{1}{12}}\,|(N\pi)> + \sqrt{\frac{1}{4}}\,|(N\rho)> + \sqrt{\frac{2}{3}}\,\{\text{colour octet}\}. \qquad (7.25)$$

Thus $|{}^2A^{(o^4\bar{o})}C(1713),\frac{3}{2},0>$ comprises 16.2% $N\pi$, 9.5% $N\rho$, 2.9% $\Delta\omega$, 4.8% $\Delta\rho$ and 66.7% colour octet components. Both the $N\pi$ and $N\rho$ channels are open and we again expect an effective pole in the $N\pi$ channel, below the $N\rho$ threshold.

The (I,S) = (0,-1) and (1,-1) primitives have caused some confusion in the literature. From table I21 we see that there are degenerate states at 1.432 GeV. Both Strottman (1979) and Roiesnel (1979) insinuate that this degeneracy is due to the Jaffe approximation but table I21 does not use this approximation. The degeneracies can be attributed to the same cause as the $\eta_o\pi$ and $\omega\rho$ degeneracies in the meson sector. It should be clear that one does not have to calculate the colour-spin matrices to see this; the fact is obvious from an examination of the basis states which only differ in their total isospin quantum numbers and we know that the energy is independent of these. States degenerate in the Jaffe approximation are those for which the colour-spin matrix would be the same in the flavour symmetry limit. An example from table I15 would be the matrices $^6\mathcal{D}$ and ^6E which have different coefficients of the radial integrals in the specific configuration $o^3s\bar{s}$.

We have for the (0,-1) primitive the dissociations

$$|^2\mathcal{D}_1^{(o^3s\bar{o})},0> = + \sqrt{\frac{2}{9}}\,|(\Sigma\pi)> + \sqrt{\frac{2}{27}}\,|(\Sigma\rho)> - \sqrt{\frac{1}{27}}\,|(\Sigma^*\rho)>$$

$$+ \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \qquad (7.26)$$

$$|^2\mathcal{D}_2^{(o^3s\bar{o})},0> = + \sqrt{\frac{1}{8}}\,|(\Lambda\eta_o)> + \sqrt{\frac{1}{24}}\,|(\Lambda\omega_o)> - \sqrt{\frac{1}{72}}\,|(\Sigma\pi)>$$

$$- \sqrt{\frac{1}{216}}\,|(\Sigma\rho)> - \sqrt{\frac{4}{27}}\,|(\Sigma^*\rho)>$$

$$+ \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \qquad (7.27)$$

$$|{}^2\mathcal{D}_3^{(o^3s\bar{o})},0\rangle = + \sqrt{\frac{1}{24}}\,|(\Lambda\eta_o)\rangle - \sqrt{\frac{1}{8}}\,|(\Lambda\omega_o)\rangle + \sqrt{\frac{1}{24}}\,|(\Sigma\pi)\rangle$$
$$- \sqrt{\frac{1}{8}}\,|(\Sigma\rho)\rangle + \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \qquad (7.28)$$

$$|{}^2\mathcal{D}_4^{(o^3s\bar{o})},0\rangle = - \sqrt{\frac{3}{16}}\,|(N\bar{K})\rangle - \sqrt{\frac{1}{16}}\,|(N\bar{K}^*)\rangle - \sqrt{\frac{1}{32}}\,|(\Lambda\eta_o)\rangle$$
$$- \sqrt{\frac{1}{96}}\,|(\Lambda\omega_o)\rangle - \sqrt{\frac{1}{288}}\,|(\Sigma\pi)\rangle - \sqrt{\frac{1}{864}}\,|(\Sigma\rho)\rangle$$
$$- \sqrt{\frac{1}{27}}\,|(\Sigma^*\rho)\rangle + \sqrt{\frac{2}{3}}\,\{\text{colour octet}\} \qquad (7.29)$$

$$|{}^2\mathcal{D}_5^{(o^3s\bar{o})},0\rangle = + \sqrt{\frac{1}{16}}\,|(N\bar{K})\rangle - \sqrt{\frac{3}{16}}\,|(N\bar{K}^*)\rangle - \sqrt{\frac{1}{96}}\,|(\Lambda\eta_o)\rangle$$
$$+ \sqrt{\frac{1}{32}}\,|(\Lambda\omega_o)\rangle + \sqrt{\frac{1}{96}}\,|(\Sigma\pi)\rangle - \sqrt{\frac{1}{32}}\,|(\Sigma\rho)\rangle$$
$$+ \sqrt{\frac{2}{3}}\,\{\text{colour octet}\}\ . \qquad (7.30)$$

Thus $|{}^2\mathcal{D}^{(o^3s\bar{o})}$ C(1432),0,-1> is comprised of 19.6% $\Sigma\pi$, 11.5% $N\bar{K}$, 1.9% $\Lambda\eta_o$, 0.1% $N\bar{K}^*$, 0.1% $\Lambda\omega_o$, 0.1% $\Sigma\rho$, a negligible amount of $\Sigma^*\rho$ and 66.7% colour octet components. This primitive lies almost on top of the $N\bar{K}$ threshold at 1.434 GeV. The P-matrix analysis of Roiesnel (1979) shows a pole at 1.45 GeV, just above this threshold, which induces a very narrow effective pole at 1.41 GeV, below threshold, in the reduced $\Sigma\pi$ P-matrix. (The $\Sigma\pi$ compensation energy is 1.69 GeV.) It is associated with the $\Lambda(1405)$ effect, as was also suggested by Strottman (1979) although his interpretation is inconsistent with the view taken here. Dalitz and McGinley (1980) have recently discussed the $\Lambda(1405)$ in some detail. Their conclusion that it is an "unstable $\bar{K}N$ bound state" is not too far removed from the P-matrix interpretation. The

$\Lambda(1405)$ is certainly not a normal particle resonance.

For the (1,-1) primitive we have the dissociations

$$|{}^2\mathcal{D}_1^{(o^3s\bar{o})},1\rangle = -\sqrt{\frac{2}{27}}\,|(\Sigma\eta_o)\rangle - \sqrt{\frac{2}{81}}\,|(\Sigma\omega_o)\rangle + \sqrt{\frac{4}{27}}\,|(\Sigma\pi)\rangle$$
$$+ \sqrt{\frac{4}{81}}\,|(\Sigma\rho)\rangle + \sqrt{\frac{1}{81}}\,|(\Sigma^*\omega_o)\rangle - \sqrt{\frac{2}{81}}\,|(\Sigma^*\rho)\rangle$$
$$+ \sqrt{\frac{2}{3}}\ \{\text{colour octet}\} \qquad (7.31)$$

$$|{}^2\mathcal{D}_2^{(o^3s\bar{o})},1\rangle = +\sqrt{\frac{1}{8}}\,|(\Lambda\pi)\rangle + \sqrt{\frac{1}{24}}\,|(\Lambda\rho)\rangle + \sqrt{\frac{1}{216}}\,|(\Sigma\eta_o)\rangle$$
$$+ \sqrt{\frac{1}{648}}\,|(\Sigma\omega_o)\rangle - \sqrt{\frac{1}{108}}\,|(\Sigma\pi)\rangle - \sqrt{\frac{1}{324}}\,|(\Sigma\rho)\rangle$$
$$+ \sqrt{\frac{4}{81}}\,|(\Sigma^*\omega_o)\rangle - \sqrt{\frac{8}{81}}\,|(\Sigma^*\rho)\rangle$$
$$+ \sqrt{\frac{2}{3}}\ \{\text{colour octet}\} \qquad (7.32)$$

$$|{}^2\mathcal{D}_3^{(o^3s\bar{o})},1\rangle = +\sqrt{\frac{1}{24}}\,|(\Lambda\pi)\rangle - \sqrt{\frac{1}{8}}\,|(\Lambda\rho)\rangle - \sqrt{\frac{1}{72}}\,|(\Sigma\eta_o)\rangle$$
$$+ \sqrt{\frac{1}{24}}\,|(\Sigma\omega_o)\rangle + \sqrt{\frac{1}{36}}\,|(\Sigma\pi)\rangle - \sqrt{\frac{1}{12}}\,|(\Sigma\rho)\rangle$$
$$+ \sqrt{\frac{2}{3}}\ \{\text{colour octet}\} \qquad (7.33)$$

$$|{}^2\mathcal{D}_4^{(o^2s\bar{o})},1\rangle = -\sqrt{\frac{3}{16}}\,|(N\bar{K})\rangle - \sqrt{\frac{1}{16}}\,|(N\bar{K}^*)\rangle - \sqrt{\frac{1}{32}}\,|(\Lambda\pi)\rangle$$
$$- \sqrt{\frac{1}{96}}\,|(\Lambda\rho)\rangle + \sqrt{\frac{1}{864}}\,|(\Sigma\eta_o)\rangle + \sqrt{\frac{1}{2592}}\,|(\Sigma\omega_o)\rangle$$
$$- \sqrt{\frac{1}{432}}\,|(\Sigma\pi)\rangle - \sqrt{\frac{1}{1296}}\,|(\Sigma\rho)\rangle + \sqrt{\frac{1}{81}}\,|(\Sigma^*\omega_o)\rangle$$

$$- \sqrt{\frac{2}{81}} \,|(\Sigma^*\rho)>$$

$$+ \sqrt{\frac{2}{3}} \,\{\text{colour octet}\} \qquad (7.34)$$

$$|{}^2\mathcal{D}_5^{(0^3S\bar{0})},1> = + \sqrt{\frac{1}{16}} \,|(N\bar{K})> - \sqrt{\frac{3}{16}} \,|(N\bar{K}^*)> - \sqrt{\frac{1}{96}} \,|(\Lambda\pi)>$$

$$+ \sqrt{\frac{1}{32}} \,|(\Lambda\rho)> - \sqrt{\frac{1}{288}} \,|(\Sigma\eta_o)> + \sqrt{\frac{1}{96}} \,|(\Sigma\omega_o)>$$

$$+ \sqrt{\frac{1}{144}} \,|(\Sigma\pi)> - \sqrt{\frac{1}{48}} \,|(\Sigma\rho)>$$

$$+ \sqrt{\frac{2}{3}} \,\{\text{colour octet}\} \quad . \qquad (7.35)$$

Thus $|{}^2\mathcal{D}^{(0^3S\bar{0})}$ C(1432),1,-1> is comprised of 1.9% $\Lambda\pi$, 13.1% $\Sigma\pi$, 11.5% $N\bar{K}$, 6.5% $\Sigma\eta_o$, 0.1% $N\bar{K}^*$, 0.1% $\Lambda\rho$, 0.1% $\Sigma\rho$, a negligible amount of $\Sigma\omega_o$, $\Sigma^*\rho$ and $\Sigma^*\omega_o$ and 66.7% colour octet components. Both the $\Lambda\pi$ and $\Sigma\pi$ channels are open and again the $N\bar{K}$ channel has its threshold very close to the primitive energy. Roiesnel (1979) attempted a three channel P-matrix analysis for this primitive but had to extrapolate the amplitudes till 1.54 GeV to find a pole.

The presence of η_o and ω_o in the dissociations of these two states indicates that some sort of mixing with other states must occur but there are no obvious candidates here like those met in the $q^2\bar{q}^2$ system.

Overall the agreement between bag model primitives in the $q^4\bar{q}$ sector and the P-matrix poles found by Roiesnel is quite good. The main difference between the primitives determined here and those of Strottman (1979), which were used by Roiesnel,

lies in the dissociations where there is some small qualitative difference. For example, equation (4.12) of Roiesnel indicates that the (I,S) = (0,-1) primitive considered here contains a substantial fraction of $N\bar{K}^*$ and $\Lambda\omega_o$ in contrast to our results.

(c) Q^6 Primitives

Table I10 contains two P-matrix poles found by Jaffe and Shatz (1980) in pn channels. These were discussed in chapter 5. The energies of the (I,S,S) = (1,0,0) and (0,0,1) primitives are given as 2.243 GeV and 2.165 GeV respectively in table I22. The current results are not significantly different from those of Jaffe (1977c) because they only involve one dimensional colour-spin matrices which are easily obtained exactly. In addition to the remarks made in chapter 5 we note only that both primitives are above the pn compensation energy, 2.07 GeV. This implies that the potential is repulsive. However, we know that the (0,0,1) channel contains a bound state, namely the deuteron. Presumably as we increase the constraining radius the energy of this primitive will drop below the compensation energy and this is possibly an indication of the well-known repulsive core in the n-p interaction.

Of some interest is the primitive

$$|{}^1I^{(O^4S^2)}\ E(2154),0,-2\rangle$$

$$= 0.867|{}^1I_1^{(O^4S^2)},0\rangle - 0.499|{}^1I_2^{(O^4S^2)},0\rangle \qquad (7.36)$$

predicted by Jaffe (1977c) to be stable against dissociation and named by him the H dihyperon. The relevant dissociations are

$$|{}^1I_1^{(O^4S^2)},0\rangle = \sqrt{\frac{2}{15}} \cdot \frac{1}{\sqrt{2}} \{|(N\Xi)\rangle - |(\Xi N)\rangle\} + \sqrt{\frac{2}{45}} |(\Sigma\Sigma)\rangle - \sqrt{\frac{1}{45}} |(\Sigma^*\Sigma^*)\rangle + \sqrt{\frac{4}{5}} \{\text{colour octet}\} \quad (7.37)$$

and

$$|{}^1I_2^{(O^4S^2)},0\rangle = + \sqrt{\frac{1}{10}} |(\Lambda\Lambda)\rangle - \sqrt{\frac{1}{30}} |(\Sigma\Sigma)\rangle - \sqrt{\frac{1}{15}} (\Sigma^*\Sigma^*)\rangle + \sqrt{\frac{4}{5}} \{\text{colour octet}\} . \quad (7.38)$$

Thus H is comprised of 2.5% $\Lambda\Lambda$, 10.0% $N\Xi$, ~~0.8~~ 7.5% $\Sigma\Sigma$, ~~6.7%~~ a negligible amount of $\Sigma^*\Sigma^*$ and 80.0% colour octet components. (~~Jaffe appears to have overlooked the $\Sigma^*\Sigma^*$ component~~. In the absence of flavour mixing the coupling to $\Xi^*\Xi^*$ would be nil.) The energy of the primitive is approximately 80 MeV below the $\Lambda\Lambda$ threshold. If the artificial confining boundary condition is realistic this would mean that a state exists so low in mass that it would have to decay weakly. Carroll et al. (1978) have searched for the H in the missing mass spectrum of the reaction $pp \to K^+K^+X$ but have failed to find any narrow structure. The presence of the H cannot be ruled out though, particularly as there was no strong signal for continuum $\Lambda\Lambda$ production (Jaffe, 1979b). Nevertheless, if a resonance does exist it would be most unlikely to possess an 80% colour content as the spherical cavity bag suggests.

Correlations between quarks, which could favour $\Lambda\Lambda$ clusters have been ignored in this model. The fact that the predicted (single-channel) residues have been consistently too small suggests that correlations may be important (e.g. Jaffe and Shatz, 1980). Also, if a $\Lambda\Lambda$ hypernuclear state exists then there would be no room amongst the allowed states for an H resonance as well but then, according to our discussion in chapter 5, the primitive would not have any direct correspondence with such a hypernuclear state either. Clearly, further theoretical and experimental work is required.

In the q^6 sector one knows that nuclear and hypernuclear states exist. These are multiquark states and a proper analysis of multiquark systems must produce them. The need for continuity between nuclear physics and elementary particle dynamics has been recently emphasized by, for example, Brodsky (1980). A first attempt in this direction using the bag model has been taken by DeTar (1978,1979) who considered deformations of the surface leading to fissioning. (See also the talk by Myhrer, 1979.) It is not altogether clear that nuclear states are pure baryon-baryon combinations. The multiquark picture suggests that a small colour contribution may remain. Matveev and Sorba (1977) and Hogaasen et al. (1980) have claimed some evidence for such a component. (See also Hogaasen, 1979 and Tsai 1980.) However, it needs to be stressed that the interpretation placed by these authors on the six quark spherical bag states is inconsistent with the interpretation taken in this thesis:

these states are primitives which are to be "identified with ordinary states in the continuum at the energies where the P-matrix has a pole" (Jaffe and Shatz, 1980). All the same though, it should be remembered that a complete theory of quark and gluon processes must include nuclear states in the multiquark sector.

We conclude this chapter by noting that in the Jaffe approximation, the colour-magnetic interaction is diagonal in the q^6 sector and its eigenvalues are just the checking eigenvalues in table I17. Thus for example, the primitives $|{}^5I^{(0^4S^2)}E(2633),0,-2>$, $|{}^5H^{(0^4S^2)}E(2656),1,-2>$ and $|{}^5J^{(0^4S^2)}E(2703),2,-2>$ are all degenerate in the Jaffe approximation.

Table I18: Masses of Q^3 primitives

Primitive	mR_o	E_{total} (MeV)	E_{volume} (MeV)	$E_{zeropoint}$ (MeV)	$E_{kinetic}$ (MeV)	E_{colour} (MeV)
N	1.389	940	235	-370	1231	-156
Λ	1.377	1101	229	-373	1403	-158
Σ	1.377	1142	229	-373	1403	-118
Ξ	1.364	1285	222	-376	1576	-137
Δ	1.523	1238	309	-337	1123	143
Σ^*	1.511	1382	303	-340	1295	124
Ξ^*	1.499	1528	295	-342	1468	107
Ω	1.486	1676	288	-345	1641	93

Table I19: Masses of $Q\bar{Q}$ primitives

Primitive	mR_o	E_{total} (MeV)	E_{volume} (MeV)	$E_{zeropoint}$ (MeV)	$E_{kinetic}$ (MeV)	E_{colour} (MeV)
η_o, π	0.924	277	69	-555	1233	-470
$K, \bar{K}$	0.903	490	64	-569	1416	-421
η_s	0.891	694	62	-576	1583	-376
ω_o, ρ	1.309	786	196	-392	871	111
$K^*, \bar{K}^*$	1.295	926	190	-396	1040	92
ϕ_s	1.280	1070	184	-401	1210	77

Table I20: Masses of $Q^2\bar{Q}^2$ primitives

(I,S,S)	mR_o	E_{total} (MeV)	E_{volume} (MeV)	$E_{zeropoint}$ (MeV)	$E_{kinetic}$ (MeV)	E_{colour} (MeV)	eigenstate
(1, 2,0)	1.450	1524	267	-354	1898	-287	$0.814\|{}^1B_1^{(O^2\bar{S}^2)}\rangle + 0.582\|{}^1B_2^{(O^2\bar{S}^2)}\rangle$
(")	1.711	2077	439	-300	1667	271	$0.581\|{}^1B_1^{(O^2\bar{S}^2)}\rangle - 0.814\|{}^1B_2^{(O^2\bar{S}^2)}\rangle$
($\frac{1}{2}$, 1,0)	1.209	882	155	-424	2043	-892	$0.813\|{}^1\bar{D}_1^{(O^2\bar{O}\bar{S})}\rangle + 0.582\|{}^1\bar{D}_2^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.463	1322	274	-351	1722	-323	$0.582\|{}^1\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle + 0.813\|{}^1\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.587	1599	351	-323	1601	-30	$0.582\|{}^1\bar{D}_1^{(O^2\bar{O}\bar{S})}\rangle - 0.813\|{}^1\bar{D}_2^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.444	1712	264	-356	2067	-263	$0.582\|{}^1E_1^{(OS\bar{S}^2)}\rangle + 0.813\|{}^1E_2^{(OS\bar{S}^2)}\rangle$
(")	1.722	1947	447	-298	1492	306	$0.813\|{}^1\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle - 0.582\|{}^1\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.699	2211	430	-302	1843	240	$0.813\|{}^1E_1^{(OS\bar{S}^2)}\rangle - 0.582\|{}^1E_2^{(OS\bar{S}^2)}\rangle$
($\frac{3}{2}$, 1,0)	1.463	1322	274	-351	1722	-323	$0.582\|{}^1\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle + 0.813\|{}^1\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.722	1947	447	-298	1492	306	$0.813\|{}^1\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle - 0.582\|{}^1\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle$
(0, 0,0)	1.223	642	160	-420	1864	-963	$0.813\|{}^1A_1^{(O^2\bar{O}^2)}\rangle + 0.582\|{}^1A_2^{(O^2\bar{O}^2)}\rangle$
(")	1.200	1115	151	-428	2217	-825	$0.007\|{}^1F_1^{(OS\bar{O}\bar{S})}\rangle + 0.814\|{}^1F_2^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.581\|{}^1F_3^{(OS\bar{O}\bar{S})}\rangle + 0.010\|{}^1F_4^{(OS\bar{O}\bar{S})}\rangle$
(")	1.474	1122	281	-348	1547	-357	$0.813\|{}^1B_1^{(O^2\bar{O}^2)}\rangle + 0.582\|{}^1B_2^{(O^2\bar{O}^2)}\rangle$
(")	1.598	1430	357	-321	1427	-33	$0.582\|{}^1A_1^{(O^2\bar{O}^2)}\rangle - 0.813\|{}^1A_2^{(O^2\bar{O}^2)}\rangle$

(0, 0,0)	1.454	1516	269	-353	1894	-294	$0.582\|{}^1F_1^{(OS\bar{O}\bar{S})}\rangle - 0.009\|{}^1F_2^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.018\|{}^1F_3^{(OS\bar{O}\bar{S})}\rangle + 0.813\|{}^1F_4^{(OS\bar{O}\bar{S})}\rangle$
(")	1.577	1770	344	-326	1776	-24	$0.002\|{}^1F_1^{(OS\bar{O}\bar{S})}\rangle - 0.580\|{}^1F_2^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.814\|{}^1F_3^{(OS\bar{O}\bar{S})}\rangle + 0.013\|{}^1F_4^{(OS\bar{O}\bar{S})}\rangle$
(")	1.732	1821	455	-296	1316	346	$0.582\|{}^1B_1^{(O^2\bar{O}^2)}\rangle - 0.813\|{}^1B_2^{(O^2\bar{O}^2)}\rangle$
(")	1.436	1903	260	-357	2238	-237	$0.813\|{}^1B_1^{(S^2\bar{S}^2)}\rangle + 0.582\|{}^1B_2^{(S^2\bar{S}^2)}\rangle$
(")	1.711	2077	439	-300	1667	271	$0.813\|{}^1F_1^{(OS\bar{O}\bar{S})}\rangle - 0.004\|{}^1F_2^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.005\|{}^1F_3^{(OS\bar{O}\bar{S})}\rangle - 0.583\|{}^1F_4^{(OS\bar{O}\bar{S})}\rangle$
(")	1.686	2347	420	-304	2020	212	$0.582\|{}^1B_1^{(S^2\bar{S}^2)}\rangle - 0.813\|{}^1B_2^{(S^2\bar{S}^2)}\rangle$
(1, 0,0)	1.200	1115	151	-428	2217	-825	$0.007\|{}^1F_1^{(OS\bar{O}\bar{S})}\rangle + 0.814\|{}^1F_2^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.581\|{}^1F_3^{(OS\bar{O}\bar{S})}\rangle + 0.010\|{}^1F_4^{(OS\bar{O}\bar{S})}\rangle$
(")	1.474	1122	281	-348	1547	-357	$0.813\|{}^1B_1^{(O^2\bar{O}^2)}\rangle + 0.582\|{}^1B_2^{(O^2\bar{O}^2)}\rangle$
(")	1.454	1516	269	-353	1894	-294	$0.582\|{}^1F_1^{(OS\bar{O}\bar{S})}\rangle - 0.009\|{}^1F_2^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.018\|{}^1F_3^{(OS\bar{O}\bar{S})}\rangle + 0.813\|{}^1F_4^{(OS\bar{O}\bar{S})}\rangle$
(")	1.577	1770	344	-326	1776	-24	$0.002\|{}^1F_1^{(OS\bar{O}\bar{S})}\rangle - 0.580\|{}^1F_2^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.814\|{}^1F_3^{(OS\bar{O}\bar{S})}\rangle + 0.013\|{}^1F_4^{(OS\bar{O}\bar{S})}\rangle$
(")	1.732	1821	455	-296	1316	346	$0.582\|{}^1B_1^{(O^2\bar{O}^2)}\rangle - 0.813\|{}^1B_2^{(O^2\bar{O}^2)}\rangle$
(")	1.711	2077	439	-300	1667	271	$0.813\|{}^1F_1^{(OS\bar{O}\bar{S})}\rangle - 0.004\|{}^1F_2^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.005\|{}^1F_3^{(OS\bar{O}\bar{S})}\rangle - 0.583\|{}^1F_4^{(OS\bar{O}\bar{S})}\rangle$

(2, 0,0)	1.474	1122	281	-348	1547	-357	$0.813\|{}^1B_1^{(O^2\bar{O}^2)}\rangle + 0.582\|{}^1B_2^{(O^2\bar{O}^2)}\rangle$
(")	1.732	1821	455	-296	1316	346	$0.582\|{}^1B_1^{(O^2\bar{O}^2)}\rangle - 0.813\|{}^1B_2^{(O^2\bar{O}^2)}\rangle$
(½,-1,0)	1.209	882	155	-424	2043	-892	$0.813\|{}^1D_1^{(OS\bar{O}^2)}\rangle + 0.582\|{}^1D_2^{(OS\bar{O}^2)}\rangle$
(")	1.463	1322	274	-351	1722	-323	$0.582\|{}^1E_1^{(OS\bar{O}^2)}\rangle + 0.813\|{}^1E_2^{(OS\bar{O}^2)}\rangle$
(")	1.587	1599	351	-323	1601	-30	$0.582\|{}^1D_1^{(OS\bar{O}^2)}\rangle - 0.813\|{}^1D_2^{(OS\bar{O}^2)}\rangle$
(")	1.444	1712	264	-356	2067	-263	$0.582\|{}^1\bar{E}_1^{(S^2\bar{O}\bar{S})}\rangle + 0.813\|{}^1\bar{E}_2^{(S^2\bar{O}\bar{S})}\rangle$
(")	1.722	1947	447	-298	1492	306	$0.813\|{}^1E_1^{(OS\bar{O}^2)}\rangle - 0.582\|{}^1E_2^{(OS\bar{O}^2)}\rangle$
(")	1.699	2211	430	-302	1843	240	$0.813\|{}^1\bar{E}_1^{(S^2\bar{O}\bar{S})}\rangle - 0.582\|{}^1\bar{E}_2^{(S^2\bar{O}\bar{S})}\rangle$
(⅜,-1,0)	1.463	1322	274	-351	1722	-323	$0.582\|{}^1E_1^{(OS\bar{O}^2)}\rangle + 0.813\|{}^1E_2^{(OS\bar{O}^2)}\rangle$
(")	1.722	1947	447	-298	1492	306	$0.813\|{}^1E_1^{(OS\bar{O}^2)}\rangle - 0.582\|{}^1E_2^{(OS\bar{O}^2)}\rangle$
(1,-2,0)	1.450	1524	267	-354	1898	-287	$0.814\|{}^1B_1^{(S^2\bar{O}^2)}\rangle + 0.582\|{}^1B_2^{(S^2\bar{O}^2)}\rangle$
(")	1.711	2077	439	-300	1667	271	$0.581\|{}^1B_1^{(S^2\bar{O}^2)}\rangle - 0.814\|{}^1B_2^{(S^2\bar{O}^2)}\rangle$
(0, 2,1)	1.498	1579	295	-343	1849	-223	$0.554\|{}^3C_1^{(O^2\bar{S}^2)}\rangle - 0.833\|{}^3C_2^{(O^2\bar{S}^2)}\rangle$
(")	1.651	1910	394	-311	1714	113	$0.835\|{}^3C_1^{(O^2\bar{S}^2)}\rangle + 0.551\|{}^3C_2^{(O^2\bar{S}^2)}\rangle$
(1, 2,1)	1.589	1796	352	-323	1765	2	$\|{}^3B^{(O^2\bar{S}^2)}\rangle$
(½, 1,1)	1.496	1344	293	-343	1688	-294	$0.356\|{}^3\bar{D}_1^{(O^2\bar{O}\bar{S})}\rangle - 0.803\|{}^3\bar{D}_2^{(O^2\bar{O}\bar{S})}\rangle$
							$- 0.479\|{}^3\bar{D}_3^{(O^2\bar{O}\bar{S})}\rangle$

($\frac{1}{2}$, 1,1)	1.512	1433	303	-340	1672	-202	$0.586\|{}^3\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle - 0.798\|{}^3\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle + 0.143\|{}^3\bar{E}_3^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.503	1437	298	-341	1680	-199	$0.439\|{}^3\bar{D}_1^{(O^2\bar{O}\bar{S})}\rangle + 0.597\|{}^3\bar{D}_2^{(O^2\bar{O}\bar{S})}\rangle - 0.672\|{}^3\bar{D}_3^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.598	1631	358	-321	1592	3	$0.015\|{}^3\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle - 0.176\|{}^3\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle - 0.984\|{}^3\bar{E}_3^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.662	1773	402	-309	1538	142	$0.826\|{}^3\bar{D}_1^{(O^2\bar{O}\bar{S})}\rangle + 0.033\|{}^3\bar{D}_2^{(O^2\bar{O}\bar{S})}\rangle + 0.563\|{}^3\bar{D}_3^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.489	1777	290	-345	2021	-189	$0.568\|{}^3E_1^{(O\bar{S}\bar{S}^2)}\rangle - 0.815\|{}^3E_2^{(O\bar{S}\bar{S}^2)}\rangle - 0.116\|{}^3E_3^{(O\bar{S}\bar{S}^2)}\rangle$
(")	1.662	1801	402	-309	1538	169	$0.810\|{}^3\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle + 0.579\|{}^3\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle - 0.095\|{}^3\bar{E}_3^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.579	1960	345	-325	1939	1	$0.018\|{}^3E_1^{(O\bar{S}\bar{S}^2)}\rangle + 0.161\|{}^3E_2^{(O\bar{S}\bar{S}^2)}\rangle - 0.987\|{}^3E_3^{(O\bar{S}\bar{S}^2)}\rangle$
(")	1.639	2076	386	-313	1889	113	$0.823\|{}^3E_1^{(O\bar{S}\bar{S}^2)}\rangle + 0.556\|{}^3E_2^{(O\bar{S}\bar{S}^2)}\rangle + 0.111\|{}^3E_3^{(O\bar{S}\bar{S}^2)}\rangle$
($\frac{3}{2}$, 1,1)	1.512	1433	303	-340	1672	-202	$0.586\|{}^3\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle - 0.798\|{}^3\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle + 0.143\|{}^3\bar{E}_3^{(O^2\bar{O}\bar{S})}\rangle$

$(\frac{3}{2}, 1,1)$	1.598	1631	358	-321	1592	3	$0.015\|{}^3\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle$ $-0.176\|{}^3\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle$ $-0.984\|{}^3\bar{E}_3^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.662	1801	402	-309	1538	169	$0.810\|{}^3\bar{E}_1^{(O^2\bar{O}\bar{S})}\rangle$ $+0.579\|{}^3\bar{E}_2^{(O^2\bar{O}\bar{S})}\rangle$ $-0.095\|{}^3\bar{E}_3^{(O^2\bar{O}\bar{S})}\rangle$
(0, 0,1)	1.500	1184	296	-342	1520	-290	$\|{}^3A^{(O^2\bar{O}^2)}\rangle$
(")	1.610	1463	366	-319	1416	0	$\|{}^3B^{(O^2\bar{O}^2)}\rangle$
(")	1.484	1526	286	-346	1863	-278	$0.258\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle$ $-0.258\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$ $-0.802\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle$ $+0.329\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$ $-0.329\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle$ $-0.086\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.500	1607	296	-342	1847	-194	$0.413\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle$ $+0.413\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$ $-0.000\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle$ $-0.574\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$ $-0.574\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle$ $-0.000\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.498	1642	295	-343	1849	-159	$0.309\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle$ $-0.309\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$ $+0.597\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle$ $+0.461\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$ $-0.461\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle$ $-0.168\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.589	1797	351	-323	1766	3	$0.001\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle$ $-0.001\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$ $-0.029\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle$ $-0.171\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$ $+0.171\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle$ $-0.970\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$

(0, 0,1)	1.652	1934	395	-311	1713	136	$0.573\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle + 0.573\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.000\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle + 0.414\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.414\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle - 0.000\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.650	1941	394	-311	1715	144	$0.582\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle - 0.582\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.043\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle - 0.388\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.388\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle + 0.143\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.569	2124	339	-327	2113	0	$\|{}^3B^{(S^2\bar{S}^2)}\rangle$
(1, 0,1)	1.520	1232	308	-338	1500	-238	$0.816\|{}^3\bar{C}_1^{(O^2\bar{O}^2)}\rangle - 0.577\|{}^3\bar{C}_2^{(O^2\bar{O}^2)}\rangle$
(")	1.520	1232	308	-338	1500	-238	$0.577\|{}^3C_1^{(O^2\bar{O}^2)}\rangle - 0.816\|{}^3C_2^{(O^2\bar{O}^2)}\rangle$
(")	1.610	1463	366	-319	1416	0	$\|{}^3B^{(O^2\bar{O}^2)}\rangle$
(")	1.484	1526	286	-346	1863	-278	$0.258\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle - 0.258\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.802\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle + 0.329\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.329\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle - 0.086\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.500	1607	296	-342	1847	-194	$0.413\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle + 0.413\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.000\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle - 0.574\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.574\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle - 0.000\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.672	1639	410	-307	1363	173	$0.577\|{}^3\bar{C}_1^{(O^2\bar{O}^2)}\rangle + 0.816\|{}^3\bar{C}_2^{(O^2\bar{O}^2)}\rangle$
(")	1.672	1639	410	-307	1363	173	$0.816\|{}^3C_1^{(O^2\bar{O}^2)}\rangle + 0.577\|{}^3C_2^{(O^2\bar{O}^2)}\rangle$

(1, 0,1)	1.498	1642	295	-343	1849	-159	$0.309\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle - 0.309\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.597\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle + 0.461\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.461\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle - 0.168\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.589	1797	351	-323	1766	3	$0.001\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle - 0.001\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.029\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle - 0.171\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.171\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle - 0.970\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.652	1934	395	-311	1713	136	$0.573\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle + 0.573\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$
							$- 0.000\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle + 0.414\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.414\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle - 0.000\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(")	1.650	1941	394	-311	1715	144	$0.582\|{}^3F_1^{(OS\bar{O}\bar{S})}\rangle - 0.582\|{}^3F_2^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.043\|{}^3F_3^{(OS\bar{O}\bar{S})}\rangle - 0.388\|{}^3F_4^{(OS\bar{O}\bar{S})}\rangle$
							$+ 0.388\|{}^3F_5^{(OS\bar{O}\bar{S})}\rangle + 0.143\|{}^3F_6^{(OS\bar{O}\bar{S})}\rangle$
(2, 0,1)	1.610	1463	366	-319	1416	0	$\|{}^3B^{(O^2\bar{O}^2)}\rangle$
(½,-1,1)	1.496	1344	293	-343	1688	-294	$0.356\|{}^3D_1^{(OS\bar{O}^2)}\rangle + 0.803\|{}^3D_2^{(OS\bar{O}^2)}\rangle$
							$- 0.479\|{}^3D_3^{(OS\bar{O}^2)}\rangle$
(")	1.512	1433	303	-340	1672	-202	$0.586\|{}^3E_1^{(OS\bar{O}^2)}\rangle - 0.798\|{}^3E_2^{(OS\bar{O}^2)}\rangle$
							$- 0.143\|{}^3E_3^{(OS\bar{O}^2)}\rangle$
(")	1.503	1437	298	-341	1680	-199	$0.439\|{}^3D_1^{(OS\bar{O}^2)}\rangle - 0.597\|{}^3D_2^{(OS\bar{O}^2)}\rangle$
							$- 0.672\|{}^3D_3^{(OS\bar{O}^2)}\rangle$

$(\frac{1}{2},-1,1)$	1.598	1631	358	-321	1592	3	$0.015\|{}^3E_1^{(OS\bar{O}^2)}\rangle - 0.176\|{}^3E_2^{(OS\bar{O}^2)}\rangle + 0.984\|{}^3E_3^{(OS\bar{O}^2)}\rangle$
(")	1.662	1773	402	-309	1538	142	$0.826\|{}^3D_1^{(OS\bar{O}^2)}\rangle - 0.033\|{}^3D_2^{(OS\bar{O}^2)}\rangle + 0.563\|{}^3D_3^{(OS\bar{O}^2)}\rangle$
(")	1.489	1777	290	-345	2021	-189	$0.568\|{}^3\bar{E}_1^{(S^2\bar{O}\bar{S})}\rangle - 0.815\|{}^3\bar{E}_2^{(S^2\bar{O}\bar{S})}\rangle + 0.116\|{}^3\bar{E}_3^{(S^2\bar{O}\bar{S})}\rangle$
(")	1.662	1801	402	-309	1538	169	$0.810\|{}^3E_1^{(OS\bar{O}^2)}\rangle + 0.579\|{}^3E_2^{(OS\bar{O}^2)}\rangle + 0.095\|{}^3E_3^{(OS\bar{O}^2)}\rangle$
(")	1.579	1960	345	-325	1939	1	$0.018\|{}^3\bar{E}_1^{(S^2\bar{O}\bar{S})}\rangle + 0.161\|{}^3\bar{E}_2^{(S^2\bar{O}\bar{S})}\rangle + 0.987\|{}^3\bar{E}_3^{(S^2\bar{O}\bar{S})}\rangle$
(")	1.639	2076	386	-313	1889	113	$0.823\|{}^3\bar{E}_1^{(S^2\bar{O}\bar{S})}\rangle + 0.556\|{}^3\bar{E}_2^{(S^2\bar{O}\bar{S})}\rangle - 0.111\|{}^3\bar{E}_3^{(S^2\bar{O}\bar{S})}\rangle$
$(\frac{3}{2},-1,1)$	1.512	1433	303	-340	1672	-202	$0.586\|{}^3E_1^{(OS\bar{O}^2)}\rangle - 0.798\|{}^3E_2^{(OS\bar{O}^2)}\rangle - 0.143\|{}^3E_3^{(OS\bar{O}^2)}\rangle$
(")	1.598	1631	358	-321	1592	3	$0.015\|{}^3E_1^{(OS\bar{O}^2)}\rangle - 0.176\|{}^3E_2^{(OS\bar{O}^2)}\rangle + 0.984\|{}^3E_3^{(OS\bar{O}^2)}\rangle$
(")	1.662	1801	402	-309	1538	169	$0.810\|{}^3E_1^{(OS\bar{O}^2)}\rangle + 0.579\|{}^3E_2^{(OS\bar{O}^2)}\rangle + 0.095\|{}^3E_3^{(OS\bar{O}^2)}\rangle$
(0,-2,1)	1.498	1579	295	-343	1849	-223	$0.833\|{}^3\bar{C}_1^{(S^2\bar{O}^2)}\rangle - 0.554\|{}^3\bar{C}_2^{(S^2\bar{O}^2)}\rangle$

(0,-2,1)	1.651	1910	394	-311	1714	113	$0.551\|{}^3\bar{C}_1^{(S^2\bar{O}^2)}\rangle + 0.835\|{}^3\bar{C}_2^{(S^2\bar{O}^2)}\rangle$
(1,-2,1)	1.589	1796	352	-323	1765	2	$\|{}^3B^{(S^2\bar{O}^2)}\rangle$
(1, 2,2)	1.651	1935	395	-311	1713	138	$\|{}^5B^{(O^2\bar{S}^2)}\rangle$
(½, 1,2)	1.662	1786	402	-309	1538	154	$\|{}^5\bar{D}^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.662	1786	402	-309	1538	154	$\|{}^5\bar{E}^{(O^2\bar{O}\bar{S})}\rangle$
(")	1.641	2084	387	-313	1888	122	$\|{}^5E^{(OS\bar{S}^2)}\rangle$
(3/2, 1,2)	1.662	1786	402	-309	1538	154	$\|{}^5\bar{E}^{(O^2\bar{O}\bar{S})}\rangle$
(0, 0,2)	1.672	1639	410	-307	1363	173	$\|{}^5A^{(O^2\bar{O}^2)}\rangle$
(")	1.672	1639	410	-307	1363	173	$\|{}^5B^{(O^2\bar{O}^2)}\rangle$
(")	1.652	1932	395	-311	1713	135	$0.577\|{}^5F_1^{(OS\bar{O}\bar{S})}\rangle - 0.816\|{}^5F_2^{(OS\bar{O}\bar{S})}\rangle$
(")	1.651	1938	394	-311	1714	141	$0.816\|{}^5F_1^{(OS\bar{O}\bar{S})}\rangle + 0.577\|{}^5F_2^{(OS\bar{O}\bar{S})}\rangle$
(")	1.629	2235	379	-315	2063	108	$\|{}^5B^{(S^2\bar{S}^2)}\rangle$
(1, 0,2)	1.672	1639	410	-307	1363	173	$\|{}^5B^{(O^2\bar{O}^2)}\rangle$
(")	1.652	1932	395	-311	1713	135	$0.577\|{}^5F_1^{(OS\bar{O}\bar{S})}\rangle - 0.816\|{}^5F_2^{(OS\bar{O}\bar{S})}\rangle$
(")	1.651	1938	394	-311	1714	141	$0.816\|{}^5F_1^{(OS\bar{O}\bar{S})}\rangle + 0.577\|{}^5F_2^{(OS\bar{O}\bar{S})}\rangle$
(2, 0,2)	1.672	1639	410	-307	1363	173	$\|{}^5B^{(O^2\bar{O}^2)}\rangle$
(½,-1,2)	1.662	1786	402	-309	1538	154	$\|{}^5D^{(OS\bar{O}^2)}\rangle$
(")	1.662	1786	402	-309	1538	154	$\|{}^5E^{(OS\bar{O}^2)}\rangle$
(")	1.641	2084	387	-313	1888	122	$\|{}^5\bar{E}^{(S^2\bar{O}\bar{S})}\rangle$

$(\frac{3}{2},-1,2)$	1.662	1786	402	-309	1538	154	$\|^5E^{(OS\bar{O}^2)}>$
(1,-2,2)	1.651	1935	395	-311	1713	138	$\|^5B^{(S^2\bar{O}^2)}>$

Table I21: Masses of $Q^4\bar{Q}$ primitives

(I,S,S)	mR_o	E_{total} (MeV)	E_{volume} (MeV)	$E_{zeropoint}$ (MeV)	$E_{kinetic}$ (MeV)	E_{colour} (MeV)	eigenstate
$(0, 1, \frac{1}{2})$	1.615	1717	369	-318	1930	-264	$\lvert {}^2B^{(O^4\bar{S})}\rangle$
$(1, 1, \frac{1}{2})$	1.688	1905	421	-304	1855	-68	$0.566\lvert {}^2A_1^{(O^4\bar{S})}\rangle + 0.824\lvert {}^2A_2^{(O^4\bar{S})}\rangle$
(")	1.770	2110	486	-290	1778	136	$0.825\lvert {}^2A_1^{(O^4\bar{S})}\rangle - 0.565\lvert {}^2A_2^{(O^4\bar{S})}\rangle$
$(2, 1, \frac{1}{2})$	1.849	2379	554	-278	1711	391	$\lvert {}^2C^{(O^4\bar{S})}\rangle$
$(\frac{1}{2}, 0, \frac{1}{2})$	1.624	1502	375	-316	1755	-312	$\lvert {}^2B^{(O^4\bar{O})}\rangle$
(")	1.489	1671	289	-345	2241	-515	$0.587\lvert {}^2D_1^{(O^3S\bar{S})}\rangle - 0.007\lvert {}^2D_2^{(O^3S\bar{S})}\rangle + 0.416\lvert {}^2D_3^{(O^3S\bar{S})}\rangle - 0.565\lvert {}^2D_4^{(O^3S\bar{S})}\rangle + 0.403\lvert {}^2D_5^{(O^3S\bar{S})}\rangle$
(")	1.697	1713	428	-303	1679	-92	$0.585\lvert {}^2A_1^{(O^4\bar{O})}\rangle + 0.811\lvert {}^2A_2^{(O^4\bar{O})}\rangle$
(")	1.607	1901	364	-319	2105	-248	$0.510\lvert {}^2D_1^{(O^3S\bar{S})}\rangle - 0.607\lvert {}^2D_2^{(O^3S\bar{S})}\rangle - 0.000\lvert {}^2D_3^{(O^3S\bar{S})}\rangle + 0.602\lvert {}^2D_4^{(O^3S\bar{S})}\rangle + 0.091\lvert {}^2D_5^{(O^3S\bar{S})}\rangle$
(")	1.780	1976	494	-288	1601	169	$0.811\lvert {}^2A_1^{(O^4\bar{O})}\rangle - 0.585\lvert {}^2A_2^{(O^4\bar{O})}\rangle$
(")	1.673	2034	410	-307	2037	-107	$0.498\lvert {}^2D_1^{(O^3S\bar{S})}\rangle + 0.025\lvert {}^2D_2^{(O^3S\bar{S})}\rangle - 0.348\lvert {}^2D_3^{(O^3S\bar{S})}\rangle - 0.274\lvert {}^2D_4^{(O^3S\bar{S})}\rangle - 0.745\lvert {}^2D_5^{(O^3S\bar{S})}\rangle$

$(\frac{1}{2}, 0, \frac{1}{2})$	1.679	2066	415	-306	2031	-74	$0.117\|{}^2\mathcal{D}_1^{(O^3S\bar{S})}\rangle + 0.413\|{}^2\mathcal{D}_2^{(O^3S\bar{S})}\rangle + 0.725\|{}^2\mathcal{D}_3^{(O^3S\bar{S})}\rangle + 0.378\|{}^2\mathcal{D}_4^{(O^3S\bar{S})}\rangle - 0.384\|{}^2\mathcal{D}_5^{(O^3S\bar{S})}\rangle$
(")	1.760	2254	478	-292	1956	112	$0.373\|{}^2\mathcal{D}_1^{(O^3S\bar{S})}\rangle + 0.678\|{}^2\mathcal{D}_2^{(O^3S\bar{S})}\rangle - 0.424\|{}^2\mathcal{D}_3^{(O^3S\bar{S})}\rangle + 0.315\|{}^2\mathcal{D}_4^{(O^3S\bar{S})}\rangle + 0.349\|{}^2\mathcal{D}_5^{(O^3S\bar{S})}\rangle$
$(\frac{3}{2}, 0, \frac{1}{2})$	1.697	1713	428	-303	1679	-92	$0.585\|{}^2A_1^{(O^4\bar{O})}\rangle + 0.811\|{}^2A_2^{(O^4\bar{O})}\rangle$
(")	1.780	1976	494	-288	1601	169	$0.811\|{}^2A_1^{(O^4\bar{O})}\rangle - 0.585\|{}^2A_2^{(O^4\bar{O})}\rangle$
(")	1.678	2098	414	-306	2032	-42	$0.508\|{}^2E_1^{(O^3S\bar{S})}\rangle - 0.350\|{}^2E_2^{(O^3S\bar{S})}\rangle + 0.787\|{}^2E_3^{(O^3S\bar{S})}\rangle$
(")	1.858	2248	562	-276	1534	429	$\|{}^2C^{(O^4\bar{O})}\rangle$
(")	1.761	2300	479	-291	1955	158	$0.664\|{}^2E_1^{(O^3S\bar{S})}\rangle - 0.421\|{}^2E_2^{(O^3S\bar{S})}\rangle - 0.618\|{}^2E_3^{(O^3S\bar{S})}\rangle$
(")	1.838	2507	544	-279	1889	352	$0.546\|{}^2E_1^{(O^3S\bar{S})}\rangle + 0.838\|{}^2E_2^{(O^3S\bar{S})}\rangle + 0.019\|{}^2E_3^{(O^3S\bar{S})}\rangle$
$(\frac{5}{2}, 0, \frac{1}{2})$	1.858	2248	526	-276	1534	429	$\|{}^2C^{(O^4\bar{O})}\rangle$
$(0, -1, \frac{1}{2})$	1.497	1432	294	-343	2068	-587	$0.603\|{}^2\mathcal{D}_1^{(O^3S\bar{O})}\rangle - 0.012\|{}^2\mathcal{D}_2^{(O^3S\bar{O})}\rangle + 0.414\|{}^2\mathcal{D}_3^{(O^3S\bar{O})}\rangle - 0.558\|{}^2\mathcal{D}_4^{(O^3S\bar{O})}\rangle + 0.392\|{}^2\mathcal{D}_5^{(O^3S\bar{O})}\rangle$

$(0,-1,\frac{1}{2})$	1.615	1688	369	-318	1930	-293	$0.507\|^2D_1^{(O^3S\bar{O})}\rangle - 0.616\|^2D_2^{(O^3S\bar{O})}\rangle - 0.027\|^2D_3^{(O^3S\bar{O})}\rangle + 0.598\|^2D_4^{(O^3S\bar{O})}\rangle + 0.076\|^2D_5^{(O^3S\bar{O})}\rangle$
(")	1.475	1817	281	-348	2422	-538	$0.591\|^2F_1^{(O^2S^2\bar{S})}\rangle + 0.488\|^2F_2^{(O^2S^2\bar{S})}\rangle + 0.584\|^2F_3^{(O^2S^2\bar{S})}\rangle + 0.267\|^2F_4^{(O^2S^2\bar{S})}\rangle$
(")	1.686	1871	420	-304	1857	-101	$0.405\|^2D_1^{(O^3S\bar{O})}\rangle + 0.346\|^2D_2^{(O^3S\bar{O})}\rangle + 0.325\|^2D_3^{(O^3S\bar{O})}\rangle + 0.135\|^2D_4^{(O^3S\bar{O})}\rangle - 0.770\|^2D_5^{(O^3S\bar{O})}\rangle$
(")	1.685	1903	419	-305	1858	-69	$0.285\|^2D_1^{(O^3S\bar{O})}\rangle - 0.233\|^2D_2^{(O^3S\bar{O})}\rangle - 0.725\|^2D_3^{(O^3S\bar{O})}\rangle - 0.470\|^2D_4^{(O^3S\bar{O})}\rangle - 0.343\|^2D_5^{(O^3S\bar{O})}\rangle$
(")	1.770	2118	486	-290	1778	144	$0.370\|^2D_1^{(O^3S\bar{O})}\rangle + 0.666\|^2D_2^{(O^3S\bar{O})}\rangle - 0.443\|^2D_3^{(O^3S\bar{O})}\rangle + 0.308\|^2D_4^{(O^3S\bar{O})}\rangle + 0.358\|^2D_5^{(O^3S\bar{O})}\rangle$
(")	1.663	2189	403	-309	2213	-119	$0.449\|^2F_1^{(O^2S^2\bar{S})}\rangle - 0.520\|^2F_2^{(O^2S^2\bar{S})}\rangle + 0.283\|^2F_3^{(O^2S^2\bar{S})}\rangle - 0.669\|^2F_4^{(O^2S^2\bar{S})}\rangle$
(")	1.669	2227	408	-308	2207	-80	$0.230\|^2F_1^{(O^2S^2\bar{S})}\rangle + 0.627\|^2F_2^{(O^2S^2\bar{S})}\rangle - 0.507\|^2F_3^{(O^2S^2\bar{S})}\rangle - 0.545\|^2F_4^{(O^2S^2\bar{S})}\rangle$
(")	1.749	2401	469	-294	2134	92	$0.632\|^2F_1^{(O^2S^2\bar{S})}\rangle - 0.318\|^2F_2^{(O^2S^2\bar{S})}\rangle - 0.566\|^2F_3^{(O^2S^2\bar{S})}\rangle + 0.424\|^2F_4^{(O^2S^2\bar{S})}\rangle$

$(1,-1,\frac{1}{2})$	1.497	1432	294	-343	2068	-587	$0.603\|{}^2D_1^{(O^3S\bar{O})}\rangle - 0.012\|{}^2D_2^{(O^3S\bar{O})}\rangle + 0.414\|{}^2D_3^{(O^3S\bar{O})}\rangle - 0.558\|{}^2D_4^{(O^3S\bar{O})}\rangle + 0.392\|{}^2D_5^{(O^3S\bar{O})}\rangle$
(")	1.615	1688	369	-318	1930	-293	$0.507\|{}^2D_1^{(O^3S\bar{O})}\rangle - 0.616\|{}^2D_2^{(O^3S\bar{O})}\rangle - 0.027\|{}^2D_3^{(O^3S\bar{O})}\rangle + 0.598\|{}^2D_4^{(O^3S\bar{O})}\rangle + 0.076\|{}^2D_5^{(O^3S\bar{O})}\rangle$
(")	1.686	1871	420	-304	1857	-101	$0.405\|{}^2D_1^{(O^3S\bar{O})}\rangle + 0.346\|{}^2D_2^{(O^3S\bar{O})}\rangle + 0.325\|{}^2D_3^{(O^3S\bar{O})}\rangle + 0.135\|{}^2D_4^{(O^3S\bar{O})}\rangle - 0.770\|{}^2D_5^{(O^3S\bar{O})}\rangle$
(")	1.685	1903	419	-305	1858	-69	$0.285\|{}^2D_1^{(O^3S\bar{O})}\rangle - 0.233\|{}^2D_2^{(O^3S\bar{O})}\rangle - 0.725\|{}^2D_3^{(O^3S\bar{O})}\rangle - 0.470\|{}^2D_4^{(O^3S\bar{O})}\rangle - 0.343\|{}^2D_5^{(O^3S\bar{O})}\rangle$
(")	1.688	1904	422	-304	1855	-69	$0.518\|{}^2E_1^{(O^3S\bar{O})}\rangle - 0.357\|{}^2E_2^{(O^3S\bar{O})}\rangle + 0.777\|{}^2E_3^{(O^3S\bar{O})}\rangle$
(")	1.597	2088	357	-321	2280	-228	$0.811\|{}^2G_1^{(O^2S^2\bar{S})}\rangle + 0.095\|{}^2G_2^{(O^2S^2\bar{S})}\rangle + 0.354\|{}^2G_3^{(O^2S^2\bar{S})}\rangle + 0.456\|{}^2G_4^{(O^2S^2\bar{S})}\rangle$
(")	1.770	2118	486	-290	1778	144	$0.370\|{}^2D_1^{(O^3S\bar{O})}\rangle + 0.666\|{}^2D_2^{(O^3S\bar{O})}\rangle - 0.443\|{}^2D_3^{(O^3S\bar{O})}\rangle + 0.308\|{}^2D_4^{(O^3S\bar{O})}\rangle + 0.358\|{}^2D_5^{(O^3S\bar{O})}\rangle$

$(1,-1,\frac{1}{2})$	1.771	2165	487	-290	1777	191	$0.660\|^2E_1^{(O^3S\bar{O})}\rangle - 0.408\|^2E_2^{(O^3S\bar{O})}\rangle - 0.631\|^2E_3^{(O^3S\bar{O})}\rangle$
(")	1.669	2250	407	-308	2208	-58	$0.097\|^2G_1^{(O^2S^2\bar{S})}\rangle - 0.812\|^2G_2^{(O^2S^2\bar{S})}\rangle + 0.459\|^2G_3^{(O^2S^2\bar{S})}\rangle - 0.349\|^2G_4^{(O^2S^2\bar{S})}\rangle$
(")	1.848	2374	553	-278	1712	387	$0.540\|^2E_1^{(O^3S\bar{O})}\rangle + 0.841\|^2E_2^{(O^3S\bar{O})}\rangle + 0.025\|^2E_3^{(O^3S\bar{O})}\rangle$
(")	1.750	2430	470	-293	2133	121	$0.022\|^2G_1^{(O^2S^2\bar{S})}\rangle - 0.575\|^2G_2^{(O^2S^2\bar{S})}\rangle - 0.605\|^2G_3^{(O^2S^2\bar{S})}\rangle + 0.550\|^2G_4^{(O^2S^2\bar{S})}\rangle$
(")	1.828	2638	535	-281	2067	316	$0.577\|^2G_1^{(O^2S^2\bar{S})}\rangle + 0.020\|^2G_2^{(O^2S^2\bar{S})}\rangle - 0.547\|^2G_3^{(O^2S^2\bar{S})}\rangle - 0.607\|^2G_4^{(O^2S^2\bar{S})}\rangle$
$(2,-1,\frac{1}{2})$	1.688	1904	422	-304	1855	-69	$0.518\|^2E_1^{(O^3S\bar{O})}\rangle - 0.357\|^2E_2^{(O^3S\bar{O})}\rangle + 0.777\|^2E_3^{(O^3S\bar{O})}\rangle$
(")	1.771	2165	487	-290	1777	191	$0.660\|^2E_1^{(O^3S\bar{O})}\rangle - 0.408\|^2E_2^{(O^3S\bar{O})}\rangle - 0.631\|^2E_3^{(O^3S\bar{O})}\rangle$
(")	1.848	2374	553	-278	1712	387	$0.540\|^2E_1^{(O^3S\bar{O})}\rangle + 0.841\|^2E_2^{(O^3S\bar{O})}\rangle + 0.025\|^2E_3^{(O^3S\bar{O})}\rangle$
$(\frac{1}{2},-2,\frac{1}{2})$	1.484	1576	286	-346	2248	-612	$0.598\|^2F_1^{(O^2S^2\bar{O})}\rangle + 0.485\|^2F_2^{(O^2S^2\bar{O})}\rangle + 0.583\|^2F_3^{(O^2S^2\bar{O})}\rangle + 0.258\|^2F_4^{(O^2S^2\bar{O})}\rangle$
(")	1.606	1877	363	-320	2106	-272	$0.809\|^2G_1^{(O^2S^2\bar{O})}\rangle + 0.111\|^2G_2^{(O^2S^2\bar{O})}\rangle + 0.341\|^2G_3^{(O^2S^2\bar{O})}\rangle + 0.466\|^2G_4^{(O^2S^2\bar{O})}\rangle$

$(\frac{1}{2},-2,\frac{1}{2})$	1.676	2031	413	-306	2034	-109	$0.491\|{}^2F_1^{(O^2S^2\bar{O})}\rangle + 0.012\|{}^2F_2^{(O^2S^2\bar{O})}\rangle - 0.139\|{}^2F_3^{(O^2S^2\bar{O})}\rangle - 0.860\|{}^2F_4^{(O^2S^2\bar{O})}\rangle$
(")	1.675	2063	412	-307	2035	-77	$0.111\|{}^2F_1^{(O^2S^2\bar{O})}\rangle - 0.809\|{}^2F_2^{(O^2S^2\bar{O})}\rangle + 0.576\|{}^2F_3^{(O^2S^2\bar{O})}\rangle - 0.041\|{}^2F_4^{(O^2S^2\bar{O})}\rangle$
(")	1.676	2063	412	-306	2034	-78	$0.117\|{}^2G_1^{(O^2S^2\bar{O})}\rangle - 0.798\|{}^2G_2^{(O^2S^2\bar{O})}\rangle + 0.478\|{}^2G_3^{(O^2S^2\bar{O})}\rangle - 0.348\|{}^2G_4^{(O^2S^2\bar{O})}\rangle$
(")	1.760	2262	477	-292	1956	121	$0.625\|{}^2F_1^{(O^2S^2\bar{O})}\rangle - 0.336\|{}^2F_2^{(O^2S^2\bar{O})}\rangle - 0.554\|{}^2F_3^{(O^2S^2\bar{O})}\rangle + 0.436\|{}^2F_4^{(O^2S^2\bar{O})}\rangle$
(")	1.761	2292	479	-292	1955	150	$0.026\|{}^2G_1^{(O^2S^2\bar{O})}\rangle - 0.591\|{}^2G_2^{(O^2S^2\bar{O})}\rangle - 0.603\|{}^2G_3^{(O^2S^2\bar{O})}\rangle + 0.535\|{}^2G_4^{(O^2S^2\bar{O})}\rangle$
(")	1.659	2399	400	-309	2384	-76	$0.430\|{}^2E_1^{(OS^3\bar{S})}\rangle + 0.310\|{}^2E_2^{(OS^3\bar{S})}\rangle + 0.848\|{}^2E_3^{(OS^3\bar{S})}\rangle$
(")	1.838	2503	544	-279	1890	348	$0.576\|{}^2G_1^{(O^2S^2\bar{O})}\rangle + 0.027\|{}^2G_2^{(O^2S^2\bar{O})}\rangle - 0.540\|{}^2G_3^{(O^2S^2\bar{O})}\rangle - 0.612\|{}^2G_4^{(O^2S^2\bar{O})}\rangle$
(")	1.738	2563	460	-295	2311	87	$0.672\|{}^2E_1^{(OS^3\bar{S})}\rangle + 0.520\|{}^2E_2^{(OS^3\bar{S})}\rangle - 0.527\|{}^2E_3^{(OS^3\bar{S})}\rangle$
(")	1.816	2772	525	-283	2246	283	$0.606\|{}^2E_1^{(OS^3\bar{S})}\rangle - 0.795\|{}^2E_2^{(OS^3\bar{S})}\rangle - 0.016\|{}^2E_3^{(OS^3\bar{S})}\rangle$

$(\frac{3}{2},-2,\frac{1}{2})$	1.606	1877	363	-320	2106	-272	$0.809\|^2G_1^{(O^2S^2\bar{O})}\rangle + 0.111\|^2G_2^{(O^2S^2\bar{O})}\rangle + 0.341\|^2G_3^{(O^2S^2\bar{O})}\rangle + 0.466\|^2G_4^{(O^2S^2\bar{O})}\rangle$
(")	1.676	2063	412	-306	2034	-78	$0.117\|^2G_1^{(O^2S^2\bar{O})}\rangle - 0.798\|^2G_2^{(O^2S^2\bar{O})}\rangle + 0.478\|^2G_3^{(O^2S^2\bar{O})}\rangle - 0.348\|^2G_4^{(O^2S^2\bar{O})}\rangle$
(")	1.761	2292	479	-292	1955	150	$0.026\|^2G_1^{(O^2S^2\bar{O})}\rangle - 0.591\|^2G_2^{(O^2S^2\bar{O})}\rangle - 0.603\|^2G_3^{(O^2S^2\bar{O})}\rangle + 0.535\|^2G_4^{(O^2S^2\bar{O})}\rangle$
(")	1.838	2503	544	-279	1890	348	$0.576\|^2G_1^{(O^2S^2\bar{O})}\rangle + 0.027\|^2G_2^{(O^2S^2\bar{O})}\rangle -0.540\|^2G_3^{(O^2S^2\bar{O})}\rangle - 0.612\|^2G_4^{(O^2S^2\bar{O})}\rangle$
$(0,-3,\frac{1}{2})$	1.666	2216	405	-308	2211	-92	$0.444\|^2E_1^{(OS^3\bar{O})}\rangle + 0.321\|^2E_2^{(OS^3\bar{O})}\rangle + 0.837\|^2E_3^{(OS^3\bar{O})}\rangle$
(")	1.750	2422	469	-293	2133	114	$0.657\|^2E_1^{(OS^3\bar{O})}\rangle + 0.521\|^2E_2^{(OS^3\bar{O})}\rangle - 0.545\|^2E_3^{(OS^3\bar{O})}\rangle$
(")	1.827	2634	535	-281	2068	312	$0.612\|^2E_1^{(OS^3\bar{O})}\rangle - 0.790\|^2E_2^{(OS^3\bar{O})}\rangle - 0.021\|^2E_3^{(OS^3\bar{O})}\rangle$
(")	1.804	2908	515	-285	2424	253	$\|^2C^{(S^4\bar{S})}\rangle$
$(1,-3,\frac{1}{2})$	1.666	2216	405	-308	2211	-92	$0.444\|^2E_1^{(OS^3\bar{O})}\rangle + 0.321\|^2E_2^{(OS^3\bar{O})}\rangle + 0.837\|^2E_3^{(OS^3\bar{O})}\rangle$
(")	1.750	2422	469	-293	2133	114	$0.657\|^2E_1^{(OS^3\bar{O})}\rangle + 0.521\|^2E_2^{(OS^3\bar{O})}\rangle - 0.545\|^2E_3^{(OS^3\bar{O})}\rangle$

$(1,-3,\frac{1}{2})$	1.827	2634	535	-281	2068	312	$0.612\|{}^2E_1^{(OS^3\bar{O})}\rangle - 0.790\|{}^2E_2^{(OS^3\bar{O})}\rangle - 0.021\|{}^2E_3^{(OS^3\bar{O})}\rangle$
$(\frac{1}{2},-4,\frac{1}{2})$	1.816	2768	525	-283	2246	280	$\|{}^2C^{(S^4\bar{O})}\rangle$
$(O, 1,\frac{3}{2})$	1.724	1968	449	-298	1821	-4	$\|{}^4B^{(O^4\bar{S})}\rangle$
$(1, 1,\frac{3}{2})$	1.654	1843	397	-310	1889	-132	$0.710\|{}^4A_1^{(O^4\bar{S})}\rangle + 0.704\|{}^4A_2^{(O^4\bar{S})}\rangle$
(")	1.780	2149	495	-288	1769	173	$0.702\|{}^4A_1^{(O^4\bar{S})}\rangle - 0.712\|{}^4A_2^{(O^4\bar{S})}\rangle$
$(2, 1,\frac{3}{2})$	1.781	2199	495	-288	1768	223	$\|{}^4C^{(O^4\bar{S})}\rangle$
$(\frac{1}{2}, 0,\frac{3}{2})$	1.663	1612	403	-309	1714	-196	$0.688\|{}^4A_1^{(O^4\bar{O})}\rangle + 0.725\|{}^4A_2^{(O^4\bar{O})}\rangle$
(")	1.733	1825	456	-296	1644	21	$\|{}^4B^{(O^4\bar{O})}\rangle$
(")	1.646	1985	391	-312	2064	-157	$0.138\|{}^4D_1^{(O^3S\bar{S})}\rangle + 0.682\|{}^4D_2^{(O^3S\bar{S})}\rangle + 0.532\|{}^4D_3^{(O^3S\bar{S})}\rangle + 0.482\|{}^4D_4^{(O^3S\bar{S})}\rangle$
(")	1.790	2010	503	-287	1592	202	$0.725\|{}^4A_1^{(O^4\bar{O})}\rangle - 0.688\|{}^4A_2^{(O^4\bar{O})}\rangle$
(")	1.672	2037	410	-307	2038	-104	$0.629\|{}^4D_1^{(O^3S\bar{S})}\rangle + 0.205\|{}^4D_2^{(O^3S\bar{S})}\rangle + 0.232\|{}^4D_3^{(O^3S\bar{S})}\rangle - 0.713\|{}^4D_4^{(O^3S\bar{S})}\rangle$
(")	1.713	2145	440	-300	1999	6	$0.711\|{}^4D_1^{(O^3S\bar{S})}\rangle - 0.022\|{}^4D_2^{(O^3S\bar{S})}\rangle - 0.552\|{}^4D_3^{(O^3S\bar{S})}\rangle + 0.435\|{}^4D_4^{(O^3S\bar{S})}\rangle$
(")	1.771	2294	487	-290	1946	151	$0.292\|{}^4D_1^{(O^3S\bar{S})}\rangle - 0.704\|{}^4D_2^{(O^3S\bar{S})}\rangle + 0.598\|{}^4D_3^{(O^3S\bar{S})}\rangle + 0.247\|{}^4D_4^{(O^3S\bar{S})}\rangle$

$(\frac{3}{2}, 0, \frac{3}{2})$	1.663	1612	403	-309	1714	-196	$0.688\|{}^4A_1^{(O^4\bar{O})}\rangle + 0.725\|{}^4A_2^{(O^4\bar{O})}\rangle$
(")	1.790	2010	503	-287	1592	202	$0.725\|{}^4A_1^{(O^4\bar{O})}\rangle - 0.688\|{}^4A_2^{(O^4\bar{O})}\rangle$
(")	1.790	2010	503	-287	1592	202	$\|{}^4C^{(O^4\bar{O})}\rangle$
(")	1.646	2035	391	-312	2063	-108	$0.710\|{}^4E_1^{(O^3S\bar{S})}\rangle - 0.576\|{}^4E_2^{(O^3S\bar{S})}\rangle + 0.405\|{}^4E_3^{(O^3S\bar{S})}\rangle$
(")	1.772	2336	488	-290	1945	193	$0.078\|{}^4E_1^{(O^3S\bar{S})}\rangle + 0.635\|{}^4E_2^{(O^3S\bar{S})}\rangle + 0.768\|{}^4E_3^{(O^3S\bar{S})}\rangle$
(")	1.773	2342	489	-289	1944	199	$0.701\|{}^4E_1^{(O^3S\bar{S})}\rangle + 0.513\|{}^4E_2^{(O^3S\bar{S})}\rangle - 0.496\|{}^4E_3^{(O^3S\bar{S})}\rangle$
$(\frac{5}{2}, 0, \frac{3}{2})$	1.790	2010	503	-287	1592	202	$\|{}^4C^{(O^4\bar{O})}\rangle$
$(0, -1, \frac{3}{2})$	1.653	1755	396	-311	1890	-220	$0.220\|{}^4D_1^{(O^3S\bar{O})}\rangle + 0.724\|{}^4D_2^{(O^3S\bar{O})}\rangle + 0.543\|{}^4D_3^{(O^3S\bar{O})}\rangle + 0.363\|{}^4D_4^{(O^3S\bar{O})}\rangle$
(")	1.685	1890	419	-305	1859	-83	$0.587\|{}^4D_1^{(O^3S\bar{O})}\rangle + 0.104\|{}^4D_2^{(O^3S\bar{O})}\rangle + 0.153\|{}^4D_3^{(O^3S\bar{O})}\rangle - 0.788\|{}^4D_4^{(O^3S\bar{O})}\rangle$
(")	1.722	2001	448	-298	1822	29	$0.724\|{}^4D_1^{(O^3S\bar{O})}\rangle - 0.025\|{}^4D_2^{(O^3S\bar{O})}\rangle - 0.544\|{}^4D_3^{(O^3S\bar{O})}\rangle + 0.424\|{}^4D_4^{(O^3S\bar{O})}\rangle$
(")	1.634	2133	383	-314	2242	-177	$0.333\|{}^4F_1^{(O^2S^2\bar{S})}\rangle + 0.694\|{}^4F_2^{(O^2S^2\bar{S})}\rangle - 0.639\|{}^4F_3^{(O^2S^2\bar{S})}\rangle$

$(0,-1,\frac{3}{2})$	1.780	2154	495	-288	1769	179	$0.296\|{}^4D_1^{(O^3S\bar{O})}\rangle - 0.681\|{}^4D_2^{(O^3S\bar{O})}\rangle + 0.622\|{}^4D_3^{(O^3S\bar{O})}\rangle + 0.249\|{}^4D_4^{(O^3S\bar{O})}\rangle$
(")	1.661	2199	402	-309	2215	-109	$0.778\|{}^4F_1^{(O^2S^2\bar{S})}\rangle + 0.185\|{}^4F_2^{(O^2S^2\bar{S})}\rangle + 0.600\|{}^4F_3^{(O^2S^2\bar{S})}\rangle$
(")	1.761	2441	478	-292	2123	131	$0.536\|{}^4F_1^{(O^2S^2\bar{S})}\rangle - 0.698\|{}^4F_2^{(O^2S^2\bar{S})}\rangle - 0.475\|{}^4F_3^{(O^2S^2\bar{S})}\rangle$
$(1,-1,\frac{3}{2})$	1.653	1755	396	-311	1890	-220	$0.220\|{}^4D_1^{(O^3S\bar{O})}\rangle + 0.724\|{}^4D_2^{(O^3S\bar{O})}\rangle + 0.543\|{}^4D_3^{(O^3S\bar{O})}\rangle + 0.363\|{}^4D_4^{(O^3S\bar{O})}\rangle$
(")	1.654	1808	396	-310	1890	-167	$0.725\|{}^4E_1^{(O^3S\bar{O})}\rangle - 0.562\|{}^4E_2^{(O^3S\bar{O})}\rangle + 0.397\|{}^4E_3^{(O^3S\bar{O})}\rangle$
(")	1.685	1890	419	-305	1859	-83	$0.587\|{}^4D_1^{(O^3S\bar{O})}\rangle + 0.104\|{}^4D_2^{(O^3S\bar{O})}\rangle + 0.153\|{}^4D_3^{(O^3S\bar{O})}\rangle - 0.788\|{}^4D_4^{(O^3S\bar{O})}\rangle$
(")	1.722	2001	448	-298	1822	29	$0.724\|{}^4D_1^{(O^3S\bar{O})}\rangle - 0.025\|{}^4D_2^{(O^3S\bar{O})}\rangle - 0.544\|{}^4D_3^{(O^3S\bar{O})}\rangle + 0.424\|{}^4D_4^{(O^3S\bar{O})}\rangle$
(")	1.781	2149	495	-288	1769	174	$0.000\|{}^4E_1^{(O^3S\bar{O})}\rangle - 0.577\|{}^4E_2^{(O^3S\bar{O})}\rangle - 0.816\|{}^4E_3^{(O^3S\bar{O})}\rangle$
(")	1.780	2154	495	-288	1769	179	$0.296\|{}^4D_1^{(O^3S\bar{O})}\rangle - 0.681\|{}^4D_2^{(O^3S\bar{O})}\rangle + 0.622\|{}^4D_3^{(O^3S\bar{O})}\rangle + 0.249\|{}^4D_4^{(O^3S\bar{O})}\rangle$

$(1,-1,\frac{3}{2})$	1.637	2182	384	-314	2240	-128	$0.052\|{}^4G_1^{(O^2S^2\bar{S})}\rangle + 0.717\|{}^4G_2^{(O^2S^2\bar{S})}\rangle + 0.480\|{}^4G_3^{(O^2S^2\bar{S})}\rangle - 0.503\|{}^4G_4^{(O^2S^2\bar{S})}\rangle$
(")	1.782	2209	496	-288	1768	233	$0.688\|{}^4E_1^{(O^3S\bar{O})}\rangle + 0.592\|{}^4E_2^{(O^3S\bar{O})}\rangle - 0.419\|{}^4E_3^{(O^3S\bar{O})}\rangle$
(")	1.704	2316	433	-301	2175	9	$0.811\|{}^4G_1^{(O^2S^2\bar{S})}\rangle - 0.070\|{}^4G_2^{(O^2S^2\bar{S})}\rangle + 0.427\|{}^4G_3^{(O^2S^2\bar{S})}\rangle + 0.394\|{}^4G_4^{(O^2S^2\bar{S})}\rangle$
(")	1.760	2452	478	-292	2124	142	$0.320\|{}^4G_1^{(O^2S^2\bar{S})}\rangle + 0.597\|{}^4G_2^{(O^2S^2\bar{S})}\rangle - 0.703\|{}^4G_3^{(O^2S^2\bar{S})}\rangle + 0.216\|{}^4G_4^{(O^2S^2\bar{S})}\rangle$
(")	1.763	2484	480	-291	2121	174	$0.488\|{}^4G_1^{(O^2S^2\bar{S})}\rangle - 0.353\|{}^4G_2^{(O^2S^2\bar{S})}\rangle - 0.305\|{}^4G_3^{(O^2S^2\bar{S})}\rangle - 0.738\|{}^4G_4^{(O^2S^2\bar{S})}\rangle$
$(2,-1,\frac{3}{2})$	1.654	1808	396	-310	1890	-167	$0.725\|{}^4E_1^{(O^3S\bar{O})}\rangle - 0.562\|{}^4E_2^{(O^3S\bar{O})}\rangle + 0.397\|{}^4E_3^{(O^3S\bar{O})}\rangle$
(")	1.781	2149	495	-288	1769	174	$0.000\|{}^4E_1^{(O^3S\bar{O})}\rangle - 0.577\|{}^4E_2^{(O^3S\bar{O})}\rangle - 0.816\|{}^4E_3^{(O^3S\bar{O})}\rangle$
(")	1.782	2209	496	-288	1768	233	$0.688\|{}^4E_1^{(O^3S\bar{O})}\rangle + 0.592\|{}^4E_2^{(O^3S\bar{O})}\rangle - 0.419\|{}^4E_3^{(O^3S\bar{O})}\rangle$
$(\frac{1}{2},-2,\frac{3}{2})$	1.642	1903	388	-313	2068	-241	$0.406\|{}^4F_1^{(O^2S^2\bar{O})}\rangle + 0.729\|{}^4F_2^{(O^2S^2\bar{O})}\rangle - 0.551\|{}^4F_3^{(O^2S^2\bar{O})}\rangle$

$(\frac{1}{2},-2,\frac{3}{2})$	1.643	1959	389	-312	2067	-185	$0.051\|^4G_1^{(o^2s^2\bar{o})}\rangle + 0.735\|^4G_2^{(o^2s^2\bar{o})}\rangle + 0.455\|^4G_3^{(o^2s^2\bar{o})}\rangle - 0.500\|^4G_4^{(o^2s^2\bar{o})}\rangle$
(")	1.673	2055	410	-307	2037	-86	$0.730\|^4F_1^{(o^2s^2\bar{o})}\rangle + 0.105\|^4F_2^{(o^2s^2\bar{o})}\rangle + 0.676\|^4F_3^{(o^2s^2\bar{o})}\rangle$
(")	1.714	2170	442	-299	1997	30	$0.806\|^4G_1^{(o^2s^2\bar{o})}\rangle - 0.074\|^4G_2^{(o^2s^2\bar{o})}\rangle + 0.446\|^4G_3^{(o^2s^2\bar{o})}\rangle + 0.382\|^4G_4^{(o^2s^2\bar{o})}\rangle$
(")	1.770	2284	486	-290	1947	141	$0.555\|^4G_1^{(o^2s^2\bar{o})}\rangle + 0.262\|^4G_2^{(o^2s^2\bar{o})}\rangle - 0.754\|^4G_3^{(o^2s^2\bar{o})}\rangle - 0.234\|^4G_4^{(o^2s^2\bar{o})}\rangle$
(")	1.770	2301	486	-290	1946	158	$0.553\|^4F_1^{(o^2s^2\bar{o})}\rangle - 0.675\|^4F_2^{(o^2s^2\bar{o})}\rangle - 0.488\|^4F_3^{(o^2s^2\bar{o})}\rangle$
(")	1.771	2326	487	-290	1946	183	$0.201\|^4G_1^{(o^2s^2\bar{o})}\rangle - 0.620\|^4G_2^{(o^2s^2\bar{o})}\rangle + 0.163\|^4G_3^{(o^2s^2\bar{o})}\rangle - 0.741\|^4G_4^{(o^2s^2\bar{o})}\rangle$
(")	1.626	2333	377	-316	2416	-144	$0.725\|^4E_1^{(os^3\bar{s})}\rangle + 0.562\|^4E_2^{(os^3\bar{s})}\rangle + 0.397\|^4E_3^{(os^3\bar{s})}\rangle$
(")	1.747	2576	467	-294	2303	99	$0.688\|^4E_1^{(os^3\bar{s})}\rangle - 0.592\|^4E_2^{(os^3\bar{s})}\rangle - 0.419\|^4E_3^{(os^3\bar{s})}\rangle$
(")	1.752	2620	471	-293	2299	143	$0.000\|^4E_1^{(os^3\bar{s})}\rangle + 0.577\|^4E_2^{(os^3\bar{s})}\rangle - 0.816\|^4E_3^{(os^3\bar{s})}\rangle$

$(\frac{3}{2},-2,\frac{3}{2})$	1.643	1959	389	-312	2067	-185	$0.051\|{}^4G_1^{(O^2S^2\bar{O})}\rangle + 0.735\|{}^4G_2^{(O^2S^2\bar{O})}\rangle + 0.455\|{}^4G_3^{(O^2S^2\bar{O})}\rangle - 0.500\|{}^4G_4^{(O^2S^2\bar{O})}\rangle$
(")	1.714	2170	442	-299	1997	30	$0.806\|{}^4G_1^{(O^2S^2\bar{O})}\rangle - 0.074\|{}^4G_2^{(O^2S^2\bar{O})}\rangle + 0.446\|{}^4G_3^{(O^2S^2\bar{O})}\rangle + 0.382\|{}^4G_4^{(O^2S^2\bar{O})}\rangle$
(")	1.770	2284	486	-290	1947	141	$0.555\|{}^4G_1^{(O^2S^2\bar{O})}\rangle + 0.262\|{}^4G_2^{(O^2S^2\bar{O})}\rangle - 0.754\|{}^4G_3^{(O^2S^2\bar{O})}\rangle + 0.234\|{}^4G_4^{(O^2S^2\bar{O})}\rangle$
(")	1.771	2326	487	-290	1946	183	$0.201\|{}^4G_1^{(O^2S^2\bar{O})}\rangle - 0.620\|{}^4G_2^{(O^2S^2\bar{O})}\rangle + 0.163\|{}^4G_3^{(O^2S^2\bar{O})}\rangle - 0.741\|{}^4G_4^{(O^2S^2\bar{O})}\rangle$
$(0,-3,\frac{3}{2})$	1.632	2113	381	-315	2245	-198	$0.747\|{}^4E_1^{(OS^3\bar{O})}\rangle + 0.544\|{}^4E_2^{(OS^3\bar{O})}\rangle + 0.383\|{}^4E_3^{(OS^3\bar{O})}\rangle$
(")	1.760	2426	477	-292	2124	116	$0.661\|{}^4E_1^{(OS^3\bar{O})}\rangle - 0.650\|{}^4E_2^{(OS^3\bar{O})}\rangle - 0.374\|{}^4E_3^{(OS^3\bar{O})}\rangle$
(")	1.760	2436	477	-292	2124	126	$0.045\|{}^4E_1^{(OS^3\bar{O})}\rangle + 0.532\|{}^4E_2^{(OS^3\bar{O})}\rangle - 0.845\|{}^4E_3^{(OS^3\bar{O})}\rangle$
(")	1.741	2766	462	-295	2477	122	$\|{}^4C^{(S^4\bar{S})}\rangle$
$(1,-3,\frac{3}{2})$	1.632	2113	381	-315	2245	-198	$0.747\|{}^4E_1^{(OS^3\bar{O})}\rangle + 0.544\|{}^4E_2^{(OS^3\bar{O})}\rangle + 0.383\|{}^4E_3^{(OS^3\bar{O})}\rangle$
(")	1.760	2426	477	-292	2124	116	$0.661\|{}^4E_1^{(OS^3\bar{O})}\rangle - 0.650\|{}^4E_2^{(OS^3\bar{O})}\rangle - 0.374\|{}^4E_3^{(OS^3\bar{O})}\rangle$

$(1,-3,\frac{3}{2})$	1.760	2436	477	-292	2124	126	$0.045\|{}^4E_1^{(OS^3\bar{O})}\rangle + 0.532\|{}^4E_2^{(OS^3\bar{O})}\rangle - 0.845\|{}^4E_3^{(OS^3\bar{O})}\rangle$
$(\frac{1}{2},-4,\frac{3}{2})$	1.748	2583	468	-294	2303	107	$\|{}^4C^{(S^4\bar{O})}\rangle$
$(1, 1,\frac{5}{2})$	1.781	2159	495	-288	1769	184	$\|{}^6A^{(O^4\bar{S})}\rangle$
$(\frac{1}{2}, 0,\frac{5}{2})$	1.790	2010	503	-287	1592	202	$\|{}^6A^{(O^4\bar{O})}\rangle$
(")	1.772	2307	487	-290	1945	164	$\|{}^6D^{(O^3S\bar{S})}\rangle$
$(\frac{3}{2}, 0,\frac{5}{2})$	1.790	2010	503	-287	1592	202	$\|{}^6A^{(O^4\bar{O})}\rangle$
(")	1.770	2313	486	-290	1947	170	$\|{}^6E^{(O^3S\bar{S})}\rangle$
$(0,-1,\frac{5}{2})$	1.781	2159	495	-288	1769	184	$\|{}^6D^{(O^3S\bar{O})}\rangle$
(")	1.762	2458	479	-291	2122	147	$\|{}^6F^{(O^2S^2\bar{S})}\rangle$
$(1,-1,\frac{5}{2})$	1.781	2159	495	-288	1769	184	$\|{}^6D^{(O^3S\bar{O})}\rangle$
(")	1.781	2159	495	-288	1769	184	$\|{}^6E^{(O^3S\bar{O})}\rangle$
(")	1.761	2462	479	-292	2123	151	$\|{}^6G^{(O^2S^2\bar{S})}\rangle$
$(2,-1,\frac{5}{2})$	1.781	2159	495	-288	1769	184	$\|{}^6E^{(O^3S\bar{O})}\rangle$
$(\frac{1}{2},-2,\frac{5}{2})$	1.771	2310	487	-290	1946	168	$\|{}^6F^{(O^2S^2\bar{O})}\rangle$
(")	1.771	2310	487	-290	1946	168	$\|{}^6G^{(O^2S^2\bar{O})}\rangle$
(")	1.751	2613	471	-293	2300	135	$\|{}^6E^{(OS^3\bar{S})}\rangle$

$(\frac{3}{2},-2,\frac{5}{2})$	1.771	2310	487	-290	1946	168	$\|^6G^{(O^2S^2\bar{O})}>$
$(0,-3,\frac{5}{2})$	1.760	2464	478	-292	2124	154	$\|^6E^{(OS^3\bar{O})}>$
$(1,-3,\frac{5}{2})$	1.760	2464	478	-292	2124	154	$\|^6E^{(OS^3\bar{O})}>$

Table I22: Masses of Q^6 primitives

(I,S,S)	mR_o	E_{total} (MeV)	E_{volume} (MeV)	$E_{zeropoint}$ (MeV)	$E_{kinetic}$ (MeV)	E_{colour} (MeV)	eigenstate
(1,0,0)	1.857	2243	561	-277	1842	117	$\|{}^1A^{(O^6)}\rangle$
(3,0,0)	2.000	2806	702	-257	1709	652	$\|{}^1B^{(O^6)}\rangle$
$(\frac{1}{2},-1,0)$	1.847	2383	553	-278	2021	88	$\|{}^1E^{(O^5S)}\rangle$
$(\frac{3}{2},-1,0)$	1.848	2428	554	-278	2020	132	$\|{}^1F^{(O^5S)}\rangle$
$(\frac{5}{2},-1,0)$	1.991	2919	692	-258	1889	596	$\|{}^1G^{(O^5S)}\rangle$
(0,-2,0)	1.693	2154	425	-303	2355	-323	$0.867\|{}^1I_1^{(O^4S^2)}\rangle - 0.499\|{}^1I_2^{(O^4S^2)}\rangle$
(")	1.837	2527	544	-279	2200	63	$0.499\|{}^1I_1^{(O^4S^2)}\rangle + 0.867\|{}^1I_2^{(O^4S^2)}\rangle$
(1,-2,0)	1.838	2556	544	-279	2199	91	$\|{}^1H^{(O^4S^2)}\rangle$
(2,-2,0)	1.840	2616	546	-279	2198	151	$0.634\|{}^1J_1^{(O^4S^2)}\rangle - 0.774\|{}^1J_2^{(O^4S^2)}\rangle$
(")	1.982	3036	682	-259	2069	543	$0.773\|{}^1J_1^{(O^4S^2)}\rangle + 0.634\|{}^1J_2^{(O^4S^2)}\rangle$
$(\frac{1}{2},-3,0)$	1.827	2688	535	-281	2380	55	$\|{}^1L^{(O^3S^3)}\rangle$
$(\frac{3}{2},-3,0)$	1.829	2731	536	-281	2378	98	$0.895\|{}^1M_1^{(O^3S^3)}\rangle - 0.446\|{}^1M_2^{(O^3S^3)}\rangle$
(")	1.972	3156	672	-260	2249	494	$0.445\|{}^1M_1^{(O^3S^3)}\rangle + 0.895\|{}^1M_2^{(O^3S^3)}\rangle$
(1,-4,0)	1.816	2852	525	-283	2560	50	$0.634\|{}^1J_1^{(O^2S^4)}\rangle - 0.774\|{}^1J_2^{(O^2S^4)}\rangle$
(")	1.962	3279	662	-262	2429	449	$0.773\|{}^1J_1^{(O^2S^4)}\rangle + 0.634\|{}^1J_2^{(O^2S^4)}\rangle$

$(\frac{1}{2},-5,0)$	1.951	3405	651	-263	2610	407	$\vert {}^1G^{(OS^5)}\rangle$
(0,-6,0)	1.939	3534	639	-265	2791	369	$\vert {}^1B^{(S^6)}\rangle$
(0,0,1)	1.835	2165	541	-280	1864	39	$\vert {}^3C^{(O^6)}\rangle$
(2,0,1)	1.928	2511	628	-266	1774	376	$\vert {}^3D^{(O^6)}\rangle$
$(\frac{1}{2},-1,1)$	1.774	2196	489	-289	2096	-100	$0.706\vert {}^3E_1^{(O^5S)}\rangle \mp 0.709\vert {}^3E_2^{(O^5S)}\rangle$
(")	1.826	2331	533	-281	2042	37	$0.707\vert {}^3E_1^{(O^5S)}\rangle \pm 0.707\vert {}^3E_2^{(O^5S)}\rangle$
$(\frac{3}{2},-1,1)$	1.827	2375	535	-281	2041	81	$0.763\vert {}^3F_1^{(O^5S)}\rangle \mp 0.646\vert {}^3F_2^{(O^5S)}\rangle$
(")	1.919	2636	619	-268	1953	332	$0.645\vert {}^3F_1^{(O^5S)}\rangle \pm 0.764\vert {}^3F_2^{(O^5S)}\rangle$
$(\frac{5}{2},-1,1)$	1.920	2702	620	-267	1952	397	$\vert {}^3G^{(O^5S)}\rangle$
(0,-2,1)	1.764	2329	481	-291	2275	-136	$\vert {}^3I^{(O^4S^2)}\rangle$
(1,-2,1)	1.765	2365	482	-291	2274	-100	$0.549\vert {}^3H_1^{(O^4S^2)}\rangle + 0.349\vert {}^3H_2^{(O^4S^2)}\rangle$ $+ 0.608\vert {}^3H_3^{(O^4S^2)}\rangle + 0.455\vert {}^3H_4^{(O^4S^2)}\rangle$
(")	1.817	2500	525	-283	2221	36	$0.363\vert {}^3H_1^{(O^4S^2)}\rangle + 0.712\vert {}^3H_2^{(O^4S^2)}\rangle$ $- 0.396\vert {}^3H_3^{(O^4S^2)}\rangle - 0.452\vert {}^3H_4^{(O^4S^2)}\rangle$
(")	1.816	2501	525	-283	2221	37	$0.605\vert {}^3H_1^{(O^4S^2)}\rangle - 0.402\vert {}^3H_2^{(O^4S^2)}\rangle$ $- 0.583\vert {}^3H_3^{(O^4S^2)}\rangle + 0.364\vert {}^3H_4^{(O^4S^2)}\rangle$
(")	1.909	2765	610	-269	2133	291	$0.452\vert {}^3H_1^{(O^4S^2)}\rangle - 0.457\vert {}^3H_2^{(O^4S^2)}\rangle$ $+ 0.360\vert {}^3H_3^{(O^4S^2)}\rangle - 0.676\vert {}^3H_4^{(O^4S^2)}\rangle$

(2,-2,1)	1.910	2816	611	-269	2132	343	$\lvert {}^3J^{(O^4S^2)}\rangle$
($\frac{1}{2}$,-3,1)	1.753	2483	472	-293	2455	-151	$0.715\lvert {}^3L_1^{(O^3S^3)}\rangle - 0.639\lvert {}^3L_2^{(O^3S^3)}\rangle$
							$- 0.285\lvert {}^3L_3^{(O^3S^3)}\rangle$
(")	1.805	2628	516	-284	2401	-5	$0.626\lvert {}^3L_1^{(O^3S^3)}\rangle + 0.408\lvert {}^3L_2^{(O^3S^3)}\rangle$
							$+ 0.665\lvert {}^3L_3^{(O^3S^3)}\rangle$
(")	1.899	2896	600	-270	2313	254	$0.307\lvert {}^3L_1^{(O^3S^3)}\rangle + 0.653\lvert {}^3L_2^{(O^3S^3)}\rangle$
							$- 0.693\lvert {}^3L_3^{(O^3S^3)}\rangle$
($\frac{3}{2}$,-3,1)	1.807	2672	517	-284	2400	39	$0.669\lvert {}^3M_1^{(O^3S^3)}\rangle + 0.743\lvert {}^3M_2^{(O^3S^3)}\rangle$
(")	1.900	2934	601	-270	2312	292	$0.743\lvert {}^3M_1^{(O^3S^3)}\rangle - 0.670\lvert {}^3M_2^{(O^3S^3)}\rangle$
(0,-4,1)	1.794	2758	506	-286	2581	-43	$0.836\lvert {}^3K_1^{(O^2S^4)}\rangle + 0.548\lvert {}^3K_2^{(O^2S^4)}\rangle$
(")	1.887	3031	589	-272	2494	220	$0.546\lvert {}^3K_1^{(O^2S^4)}\rangle - 0.838\lvert {}^3K_2^{(O^2S^4)}\rangle$
(1,-4,1)	1.888	3056	590	-272	2493	245	$\lvert {}^3J^{(O^2S^4)}\rangle$
($\frac{1}{2}$,-5,1)	1.876	3181	578	-274	2674	202	$\lvert {}^3G^{(OS^5)}\rangle$
(1,0,2)	1.888	2359	590	-272	1811	230	$\lvert {}^5A^{(O^6)}\rangle$
($\frac{1}{2}$,-1,2)	1.799	2279	510	-285	2070	-15	$0.642\lvert {}^5E_1^{(O^5S)}\rangle \mp 0.767\lvert {}^5E_2^{(O^5S)}\rangle$
(")	1.879	2495	581	-273	1990	196	$0.765\lvert {}^5E_1^{(O^5S)}\rangle \pm 0.644\lvert {}^5E_2^{(O^5S)}\rangle$
($\frac{3}{2}$,-1,2)	1.880	2530	582	-273	1989	231	$\lvert {}^5F^{(O^5S)}\rangle$

(0,-2,2)	1.788	2402	501	-287	2250	-61	$0.814\|{}^5I_1^{(O^4S^2)}\rangle + 0.581\|{}^5I_2^{(O^4S^2)}\rangle$
(")	1.869	2633	572	-275	2170	166	$0.579\|{}^5I_1^{(O^4S^2)}\rangle - 0.816\|{}^5I_2^{(O^4S^2)}\rangle$
(1,-2,2)	1.788	2428	501	-287	2249	-36	$0.600\|{}^5H_1^{(O^4S^2)}\rangle + 0.800\|{}^5H_2^{(O^4S^2)}\rangle$
(")	1.870	2656	573	-275	2169	188	$0.802\|{}^5H_1^{(O^4S^2)}\rangle - 0.598\|{}^5H_2^{(O^4S^2)}\rangle$
(2,-2,2)	1.871	2703	574	-274	2168	235	$\|{}^5J^{(O^4S^2)}\rangle$
(½,-3,2)	1.777	2541	492	-289	2430	-92	$0.916\|{}^5L_1^{(O^3S^3)}\rangle + 0.401\|{}^5L_2^{(O^3S^3)}\rangle$
(")	1.859	2786	563	-276	2350	150	$0.399\|{}^5L_1^{(O^3S^3)}\rangle - 0.917\|{}^5L_2^{(O^3S^3)}\rangle$
(3/2,-3,2)	1.860	2819	564	-276	2348	183	$\|{}^5M^{(O^3S^3)}\rangle$
(1,-4,2)	1.848	2940	553	-278	2529	136	$\|{}^5J^{(O^2S^4)}\rangle$
(0,0,3)	1.888	2359	590	-272	1811	230	$\|{}^7C^{(O^6)}\rangle$
(½,-1,3)	1.879	2510	582	-273	1990	211	$\|{}^7E^{(O^5S)}\rangle$
(1,-2,3)	1.870	2663	573	-275	2169	196	$\|{}^7H^{(O^4S^2)}\rangle$
(3/2,-3,3)	1.860	2819	564	-276	2348	183	$\|{}^7M^{(O^3S^3)}\rangle$

CHAPTER 8

CONCLUSIONS

In Part I we have described the group theoretical classification of multiquark states and have shown how dissociation calculations may be performed. We have also shown how tensor operator techniques may be used to completely evaluate the colour-magnetic interaction arising from single gluon exchange and have calculated matrix elements of this interaction for $q^2\bar{q}^2$, $q^4\bar{q}$ and q^6 multiquark systems. The M.I.T. bag model has been used to calculate the energies of all S-wave multiquark states in these configurations.

We have seen that the multiquark eigenstates of the spherical cavity bag Hamiltonian cannot be interpreted as particle resonances. The possibility that multiquark systems can dissociate, completely invalidates the bag model in the static spherical cavity approximation. Artificial confinement of colour-neutral subsystems by this model results in spurious states which have no correspondence with physical eigenstates. However, we have also seen that the P-matrix description (Jaffe and Low, 1979) of low-energy scattering allows these spurious states, or primitives, to be related to scattering data. They appear as ordinary scattering states in the continuum at energies for which the P-matrix has a pole.

The results given here confirm the earlier indications of Jaffe and Low (1979), Roiesnel (1979) and Jaffe and Shatz (1980) that the correspondence between the primitives as calculated in the spherical cavity bag model and the observed P-matrix poles is remarkably good. All earlier calculations of these primitives have approximated the colour-magnetic interaction. Here, for the first time we have seen what the actual splittings due to this term look like. We have been able to clarify some uncertainty surrounding degeneracies in the multiquark sector which have sometimes previously been attributed to approximations made in calculating the colour-magnetic interaction. In the main, the energies found here are not substantially different from those approximate ones determined by earlier authors but there are some qualitative features such as relative splittings and dissociation projections where these earlier calculations can be misleading.

Of some importance is that we have found that there are two almost degenerate primitives which must be associated with the S*(993) phase shift. This almost certainly explains why the S*(993) couples with the $\pi\pi$ channel, something which was not altogether easy to account for in the early identification made by Jaffe and Low (1979). Here and elsewhere in the multiquark sector, mixing via gluon annihilation processes is important, as it apparently is also for the η and η' mesons and a more sophisticated calculation of the primitives should take this into account. It is expected that this mixing will cause a substantial change in some

parts of the spectrum and it is probably the most pressing problem to be tackled next.

While the agreement with pole positions is quite good the predicted pole residues are consistently too small. This implies that the physical primitives are more strongly coupled to open channels than the bag model suggests. The exchange of gluons between colour octet subsystems would increase this coupling, as noted by Roiesnel, and would also have the effect of causing some correlations inside the bag since the reverse exchange becomes less likely as the colour-neutral subsystems separate.

We conclude, in agreement with Jaffe and Low (1979), that when properly interpreted there is substantial evidence in scattering data for multiquark S-wave states.

APPENDIX IA

CROSSED CHANNELS AND SCATTERING KINEMATICS

Of interest are scattering processes with a total of four particles in the initial and final states. Call these particles simply 1,2,3 and 4 and denote their 4-momenta by p_r, r = 1,2,3,4. (Throughout this thesis we use natural units in which $\hbar = c = \varepsilon_0 = 1$.) These momenta are involved in the kinematics of one of the following three reactions depending on the sign of the time-component p_r^0 (i.e. the energy).

s-channel:	$1 + 2 \to 3 + 4$	all $p_r^0 > 0$
t-channel:	$1 + \bar{3} \to \bar{2} + 4$	$p_1^0, p_4^0 > 0$; $p_2^0, p_3^0 < 0$
u-channel:	$1 + \bar{4} \to 3 + \bar{2}$	$p_1^0, p_3^0 > 0$; $p_2^0, p_4^0 < 0$

The above are called crossed reactions of one another. When passing to the crossed reaction the 4-momentum p_r of a particle r in the initial state is changed to the 4-momentum $-p_r$ of the antiparticle $\bar{r}$ in the final state. Thus all the particles in all three reactions have positive energies. (Note: it is conventional in reactions with baryons to designate the channels so that the t-channel contains a baryon and an antibaryon in the initial and final states.)

It is usual to introduce three invariants

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{IA.1}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \tag{IA.2}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \tag{IA.3}$$

related by

$$s + t + u = \sum_r m_r^2 . \tag{IA.4}$$

These are sometimes referred to as the Mandelstam variables. The quantity s is the square of the energy in the centre of mass (c.m.) system for the s-channel and similarly for t and u.

Consider the c.m. system for the s-channel and write

$$p_1 = (E_1, \underset{\sim}{k}) \tag{IA.5}$$

$$p_2 = (E_2, -\underset{\sim}{k}) \tag{IA.6}$$

so that

$$s = (E_1 + E_2)^2 \tag{IA.7}$$

From the invariants

$$E_1^2 - k^2 = m_1^2 \tag{IA.8}$$

$$E_2^2 - k^2 = m_2^2 \tag{IA.9}$$

it is not difficult to show that

$$4sk^2 = [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2] \qquad \text{(IA.10a)}$$

$$= s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2 \qquad \text{(IA.10b)}$$

$$= (s - m_1^2 - m_2^2)^2 - 4\, m_1^2 m_2^2 \ . \qquad \text{(IA.10c)}$$

Analogous equations hold in the t- and u-channels.

The scattering amplitude for any one of the three crossed processes can be shown to be an analytic function of momenta in a connected region of complex momentum space which includes all physical regions of the crossed processes. Thus if it is known in the physical region of one channel it may be analytically continued into the negative energy region corresponding to a crossed channel. This leads to the idea of crossing symmetry.

APPENDIX IB

COMPUTATIONAL PROCEDURES

The energies of the primitives were calculated using a computer program, written in B6700 ALGOL. Because of small discrepancies between our energies and those of DeGrand et al. (1975) and to allay any suppositions that mistakes were made here, this program is included. It is designed to be run either in batch mode using a card-reader or interactively via a terminal.

The program begins by taking a generic configuration such as $q_a^3 q_b \bar{q}_c$ and then setting up the general colour-spin matrices of tables I13-17, all of which must be read in as data. A checking option is included. After being informed of the particular quark flavours involved the program then proceeds to multiply the colour-spin matrix elements by the appropriate radial integrals to obtain a specific colour-spin matrix which is then diagonalized for values of $MR(\equiv m_s R) = 0.1, 0.2, \ldots, 3.0$ and the energies obtained. (The radial integrals and quark eigenfrequencies have been previously calculated for these values of MR and are contained in the value array FUNCTIONSOFMR.) The location of the value of MR for a minimum is then found for each energy eigenvalue separately and the energy at the minimum recomputed. In the absence of strange quarks the minimum can be calculated exactly; when strange quarks are present a curve fitting procedure is used. This procedure for obtaining the minimum may not be all that efficient but it is reliable in that it provides plenty of opportunity for checking.

The included files "NUMERALS/SYMEIGENVECTORS" and "NUMERALS/LINEQN" are procedures from the Burroughs Numerals package which respectively diagonalize real symmetric matrices and solve a set of linear equations. The parameters of these procedures are as follows:

SYMEIGENVECTORS(N,A,VALUES, VECTORS)

N — (input) order of matrix

A[0,0] — (input) the lower triangle of the matrix to be diagonalized is stored in A[1:N,1:N]

VALUES[0] — (output) eigenvalues of A in ascending order stored in VALUES[1:N]

VECTORS[0,0] — (output) eigenvectors of A stored by columns in VECTORS[1:N,1:N]

LINEQN(N,A,A,B,X) (solves AX = B)

N — (input) dimension of coefficient matrix

A[0,0] — (input) coefficient matrix stored in A[1:N,1:N]. This gets overwritten by its triangular decomposition.

B[0] — (input) right-hand side stored in B[1:N]

X[0] — (output) solution vector stored in X[1:N] .

Note that all arrays are declared with a lower bound of zero but that this row or column is not used.

```
M   M U   U L     TTTTT IIIII  QQQ  U   U  AAA  RRRR  K   K
MM MM U   U L       T     I   Q   Q U   U A   A R   R K  K
M M M U   U L       T     I   Q   Q U   U A   A R   R K K
M M M U   U L       T     I   Q   Q U   U AAAAA RRRR  KK
M   M U   U L       T     I   Q Q Q U   U A   A R R   K K
M   M U   U L       T     I   Q  QQ U   U A   A R  R  K  K
M   M  UUUU LLLLL   T   IIIII  QQQQ  UUUU A   A R   R K   K
```

663 RECORDS, CREATED 14/08/80

```
  100 BEGIN
  200    COMMENT: THIS PROGRAM IS DESIGNED TO CALCULATE THE ENERGIES OF
  300    MULTIQUARK HADRONS IN THE M.I.T. BAG MODEL AS FUNCTIONS OF MR,
  400    WHERE M IS THE STRANGE QUARK MASS AND R IS THE BAG RADIUS. THE
  500    MINIMUM ENERGIES ARE THEN DETERMINED BY A CURVE FITTING PROCEDURE.
  600    IT IS ASSUMED THAT THERE ARE ONLY 3 QUARK FLAVOURS AND THAT
  700    ISOSPIN SYMMETRY HOLDS;
  800    %                    %%%%%%
  900    FILE CARD(KIND=READER),LINEPRINT(KIND=PRINTER),
 1000    SCREEN(KIND=REMOTE),REMOTEIN(KIND=REMOTE);
 1100    BOOLEAN FINISHING,INTERACTIVE,OK,DEBUG;
 1200    LABEL FINISH;
 1300    ALPHA ARRAY LINEBUFF[0:30],DATA[0:11];
 1400    POINTER MESSAGE,INSTRUCTION;
 1500    DEFINE EOL="     " 48"9C"#,COMMA=,#,MARGIN=3#,
 1600    INFORM(TEXT)=BEGIN INTEGER I;
 1700       REPLACE MESSAGE BY TEXT,EOL;
 1800       SCAN MESSAGE FOR I:115 UNTIL =48"9C";
 1900       WRITE(LINEPRINT,(I:=(115-I) DIV 6) + MARGIN,LINEBUFF);
 2000       IF INTERACTIVE THEN WRITE(SCREEN,I,MESSAGE);
 2100       REPLACE LINEBUFF BY "      " FOR MARGIN WORDS;END#;

 2200    PROCEDURE RECEIVEINSTRUCTIONS;
 2300    BEGIN
 2400       IF INTERACTIVE THEN READ(REMOTEIN,12,DATA)
 2500       ELSE READ(CARD,12,DATA)[FINISH];
 2600       INSTRUCTION:=POINTER(DATA);
 2700       REPLACE MESSAGE BY "-> ",INSTRUCTION FOR 72;
 2800       WRITE(LINEPRINT,12+MARGIN,LINEBUFF);
 2900    END;
 3000
 3100    % THE FOLLOWING THREE PROCEDURES, X,OMEGA AND M CALCULATE THE QUARK
 3200    % EIGENFREQUENCIES AND RADIAL INTEGRALS THAT ARISE IN THE BAG MODEL

 3300    REAL PROCEDURE X(MR);
 3400    REAL MR;
 3500    % THIS PROCEDURE SOLVES THE TRANSCENDENTAL EQUATION
 3600    %       TAN(X)*(1-MR-SQRT(X**2+MR**2))-X=0
 3700    % VIA THE NEWTON-RAPHSON METHOD. THIS INVOLVES GUESSING
 3800    % AN APPROXIMATE VALUE, X0; AN IMPROVED VALUE IS THEN
 3900    % GIVEN BY
 4000    %       X=X0 + DELTA(X)
 4100    % WHERE
 4200    %       DELTA(X)=-FN(X0)/FN'(X0).
 4300    BEGIN
 4400       REAL THISGUESS,LASTGUESS,DERIVATIVE;
 4500       DEFINE REQDACCURACY=0.0000001#;
 4600       THISGUESS:=2.00; % FIRSTGUESS
 4700       LASTGUESS:=0;
 4800       WHILE ABS(THISGUESS-LASTGUESS)>=REQDACCURACY DO
 4900       BEGIN
 5000          IF ABS(THISGUESS-LASTGUESS)>=0.01 THEN
 5100          DERIVATIVE:=(1-MR-SQRT(THISGUESS**2+MR**2))/COS(THISGUESS)**2
 5200          -THISGUESS*TAN(THISGUESS)/SQRT(THISGUESS**2+MR**2)-1;
 5300          LASTGUESS:=THISGUESS;
 5400          THISGUESS:=THISGUESS-(TAN(THISGUESS)*(1-MR-SQRT(THISGUESS**2+MR**2
 5500          ))-THISGUESS)/DERIVATIVE;
 5600       END;
 5700       X:=THISGUESS;
 5800    END OF X;

 5900    REAL PROCEDURE OMEGA(MR);
 6000    REAL MR;
 6100    BEGIN
 6200       OMEGA:=SQRT(X(MR)**2+MR**2);
 6300    END OF OMEGA;

 6400    REAL PROCEDURE M(MIR,MJR);
 6500    REAL MIR,MJR;
 6600    % M(MIR,MJR)=3*MU(MIR)*MU(MJR)*I(MIR,MJR)/R**2
 6700    % WHERE
 6800    %       MU(MR)=(R/6)*(4*OMEGA(MR)+2*MR-3)/
 6900    %                    (2*OMEGA(MR)*(OMEGA(MR)-1)+MR)
 7000    % AND I(MIR,MJR) IS GIVEN BY
 7100    % EQUATION(2.24) OF DEGRAND ET AL PHYS. REV. D12,2060(1975).
 7200    BEGIN
 7300       REAL XI,XJ,YI,YJ,W;
```

```
 7400        REAL PROCEDURE MU(MR);
 7500        REAL MR;
 7600        BEGIN
 7700           W:=OMEGA(MR);
 7800           MU:=(4*W+2*MR-3)/(6*(2*W*(W-1)+MR));
 7900           % THIS IS THE MAGNETIC MOMENT DIVIDED BY THE BAG RADIUS.
 8000        END OF MU;

 8100        REAL PROCEDURE I(MIR,MJR);
 8200        REAL MIR,MJR;
 8300        BEGIN

 8400           REAL PROCEDURE SI(X);
 8500           REAL X;
 8600           % SI(X)=X - X**3/(3*3]) + X**5/(5*5]) - X**7/(7*7]) +...
 8700           BEGIN
 8800              DEFINE REQDACCURACY=0.000001#;
 8900              INTEGER I,SGN;
 9000              REAL S,SS,POWERX,FACTORIAL;
 9100              FACTORIAL:=I:=SGN:=1;
 9200              POWERX:=SS:=X;
 9300              S:=0;
 9400              WHILE ABS(SS-S)>=REQDACCURACY DO BEGIN
 9500                 SGN:=-SGN;
 9600                 POWERX:=POWERX*X**2;
 9700                 FACTORIAL:=FACTORIAL*(I+1)*(I+2);
 9800                 I:=I+2;
 9900                 %S:=SS;
10000                 %SS:=SS +SGN*POWERX/(I*FACTORIAL);
10100                 S:=READLOCK(SS +SGN*POWERX/(I*FACTORIAL),SS);
10200              END;
10300              SI:=SS;
10400           END OF SI;
10500
10600           XI:=X(MIR);
10700           YI:=XI-SIN(XI)*COS(XI);
10800           IF MIR=MJR THEN BEGIN
10900              % XJ=XI
11000              % YJ=YI
11100              I:=1+(-3*YI**2/2-2*XI**2*SIN(XI)**4+XI**2*(
11200              4*XI*SI(2*XI)-2*XI*SI(4*XI))/2)/(XI*SIN(XI)**2-3*YI/2)**2;
11300           END ELSE BEGIN
11400              XJ:=X(MJR);
11500              YJ:=XJ-SIN(XJ)*COS(XJ);
11600              I:=1+(-3*YI*YJ/2-2*XI*XJ*(SIN(XI)*SIN(XJ))**2
11700              +XI*XJ*(2*XI*SI(2*XI)+2*XJ*SI(2*XJ)
11800              -(XI+XJ)*SI(2*(XI+XJ))-(XI-XJ)*SI(2*(XI-XJ)) )/2)
11900              /((XI*SIN(XI)**2-3*YI/2)*(XJ*SIN(XJ)**2-3*YJ/2));
12000           END;
12100        END OF I;
12200        M:=3*MU(MIR)*MU(MJR)*I(MIR,MJR);
12300        % THE DIVISION BY R**2 HAS BEEN INCORPORATED IN THE PROCEDURE MU.
12400     END OF RADIAL INTEGRAL M;

12500     PROCEDURE GETENERGIES;
12600     BEGIN
12700        DEFINE B1QUARTER=146#,Z0=1.84#,ALPHAC=0.55#,MS=279#,MO=0#,
12800        FOURPIBOVER3MSCUBED=87.637#; % FOURPIBOVER3MSCUBED=4*PI*B/(3*MS**3)
12900        % THESE ARE THE M.I.T. BAG MODEL PARAMETERS
13000        REAL VALUE ARRAY FUNCTIONSOFMR(2.0428,2.0915,2.1418,2.1936,2.2469,
13100        2.3017,2.3579,2.4155,2.4745,2.5349,2.5966,2.6595,2.7237,2.7891,2.8556,
13200        2.9233,2.9920,3.0618,3.1326,3.2044,3.2771,3.3508,3.4253,3.5006,3.5767,
13300        3.6537,3.7313,3.8097,3.8888,3.9685,4.0489,0.1770,0.1723,0.1676,0.1629,
13400        0.1583,0.1537,0.1492,0.1447,0.1404,0.1360,0.1318,0.1277,0.1237,0.1198,
13500        0.1159,0.1122,0.1086,0.1051,0.1017,0.0985,0.0953,0.0922,0.0892,0.0864,
13600        0.0836,0.0810,0.0784,0.0759,0.0735,0.0712,0.0690,0.1770,0.1746,0.1722,
13700        0.1698,0.1673,0.1649,0.1624,0.1599,0.1574,0.1549,0.1524,0.1500,0.1475,
13800        0.1451,0.1427,0.1403,0.1380,0.1357,0.1334,0.1312,0.1290,0.1269,0.1247,
13900        0.1227,0.1206,0.1186,0.1167,0.1148,0.1129,0.1111,0.1093);
14000        % OMEGA(MR) IS STORED IN THE FIRST 31 ELEMENTS,
14100        % MSS(MR) IN THE NEXT 31 AND MOS(MR) IN THE LAST 31.
14200        % IF MO=0 THEN MOO(MR)=MSS(0).
14300        %  THESE RADIAL INTEGRALS HAVE PREVIOUSLY BEEN EVALUATED
14400        % FOR MR:=0.0,.............,3.0 USING THE PROCEDURES X,OMEGA
14500        % AND M LISTED ABOVE.
```

```
14600       DEFINE NUMPOSSIBLEQTYPES=4#; % O,-O,S,-S
14700       INTEGER ARRAY CONFIGURATION[0:NUMPOSSIBLEQTYPES-1];
14800       INTEGER NUM2PARTINTERACTIONS,NUMSPECIES,MATRIXDIMENSION,
14900       NUMBERINLOWERTRIANGLE,NUMOQUARKS,NUMSQUARKS;
15000       BOOLEAN CONFIGKNOWN,MATRIXKNOWN;
15100       DEFINE READMATRIXDIMENSION=BEGIN INTEGER I,J;POINTER P;
15200          RECEIVEINSTRUCTIONS;
15300          P:=POINTER(DATA);
15400          I:=72;
15500          SCAN P:P FOR I:I WHILE =" ";
15600          SCAN P   FOR J:I WHILE>="0";
15700          MATRIXDIMENSION:=INTEGER(P,I-J);END#;
15800       FORMAT HEADING(" MR    TOTAL  VOLUME  ZEROPT KINETIC  COLOUR"),
15900       FSMIN(F5.3,X1,I6,X2,4(I6,X2)),
16000       F1(F3.1,X3,I6,X2,4(I6,X2)),
16100       F2(X6     ,I6,X2,X24,I6,X2),
16200       FCHKV(14(F8.6,X1)),
16300       FLMIN(F5.3,X1,I6,X2,4(I6,X2),X2,14(F5.3,X1)),
16400       F3(F3.1,X3,I6,X2,4(I6,X2),X2,14(F5.3,X1)),
16500       F4(X6     ,I6,X2,X24,I6,X2,X2,14(F5.3,X1));
16600       SWITCH FORMAT NEATSCREEN:=F1,F2;
16700       SWITCH FORMAT NEATLINE  :=F3,F4;

16800       PROCEDURE NEWCONFIGURATION;
16900       BEGIN
17000          INTEGER I,J,K;
17100          POINTER P;
17200          BOOLEAN NEG;
17300          DEFINE READCONFIGURATION=RECEIVEINSTRUCTIONS#;
17400          NUM2PARTINTERACTIONS:=NUMSPECIES:=0;
17500          FILL CONFIGURATION[*] WITH NUMPOSSIBLEQTYPES(0);
17600          INFORM("WHAT IS THE CONFIGURATION?");
17700          % IF THE CONFIGURATION IS, FOR EXAMPLE, 2 QUARKS
17800          % OF TYPE A + 3 QUARKS OF TYPE B + 1 ANTIQUARK
17900          % OF TYPE C + 1 ANTIQUARK OF TYPE D THEN THIS
18000          % REQUEST SHOULD BE FOLLOWED UP BY INPUTTING THE
18100          % FREE FIELD DATA: 2,3,-1,-1
18200          READCONFIGURATION;
18300          % THE CONFIGURATION IS READ INTO THE ALPHA ARRAY DATA.
18400          P:=POINTER(DATA);
18500          I:=72;K:=0;
18600          SCAN P:P FOR I:I WHILE=" ";
18700          WHILE K<NUMPOSSIBLEQTYPES DO BEGIN
18800             IF I>0 THEN BEGIN
18900                IF P="+" OR NEG:=P="-" THEN
19000                SCAN P:P+1 FOR I:I-1 WHILE =" ";
19100                SCAN P FOR J:I WHILE >="0";
19200                CONFIGURATION[K]:=IF NEG THEN -INTEGER(P,I-J)
19300                ELSE INTEGER(P,I-J);
19400                SCAN P:P FOR I:I WHILE >="0";
19500                SCAN P:P FOR I:I WHILE  =" ";
19600                IF I>0 THEN IF P="," THEN SCAN P:P+1 FOR I:I-1 WHILE =" ";
19700             END;
19800             K:=K+1;
19900          END;
20000          IF DEBUG THEN WRITE(LINEPRINT,/,CONFIGURATION[*]);
20100          FOR I:=0 STEP 1 UNTIL NUMPOSSIBLEQTYPES-1 DO
20200          IF CONFIGURATION[I]^=0 THEN
20300          BEGIN
20400             NUMSPECIES:=*+1;
20500             IF ABS(CONFIGURATION[I])>1 THEN NUM2PARTINTERACTIONS:=*+1;
20600             FOR J:=I+1 STEP 1 UNTIL NUMPOSSIBLEQTYPES-1 DO
20700             IF CONFIGURATION[J]^=0 THEN
20800             NUM2PARTINTERACTIONS:=*+1;END;
20900          IF DEBUG THEN BEGIN
21000             INFORM("THE NUMBER OF PARTICLE SPECIES IS " COMMA
21100             NUMSPECIES FOR * DIGITS);
21200             INFORM("THE NUMBER OF 2 PARTICLE INTERACTIONS IS " COMMA
21300             NUM2PARTINTERACTIONS FOR * DIGITS);
21400          END;
21500       END OF NEWCONFIGURATION;

21600       PROCEDURE ADJUST(ROW,COLUMN);
21700       INTEGER ROW,COLUMN;
21800       BEGIN
21900          IF COLUMN >= ROW THEN BEGIN
22000             ROW:=ROW+1;
22100             COLUMN:=1;END
22200          ELSE COLUMN:=COLUMN+1;
22300       END OF ADJUST;
22400
22500       $INCLUDE "NUMERALS/SYMEIGENVECTORS"
```

```
22600       PROCEDURE SETUPMATRIX(NUMLT,MAT,CHKMATRIX,CHKVALUES,CHKVECTORS);
22700       INTEGER NUMLT;
22800       REAL ARRAY MAT,CHKMATRIX,CHKVECTORS[0,0],CHKVALUES[0];
22900       BEGIN
23000          INTEGER MATROW,MATCOL,I,J,NUM,DEN,SURD;
23100          BOOLEAN CHECK,NEG;
23200          POINTER P;
23300          DEFINE READMATRIXELEMENT=RECEIVEINSTRUCTIONS#;
23400          INFORM("FEED IN THE MATRIX ELEMENTS");
23500          %N.B.   THE MATRIX ELEMENTS MUST BE SUPPLIED IN A PARTICULAR
23600          %   ORDER ELSE THE PROGRAM WILL GIVE ERRONEOUS RESULTS. FOR
23700          %   EXAMPLE, IF THE MATRIX TO BE READ IN IS 3 BY 3 THEN THE
23800          %   ELEMENTS SHOWN BELOW MUST BE SUPPLIED IN THE ORDER
23900          %   INDICATED. (THE PROGRAM ASSUMES THAT THE MATRIX IS
24000          %   REAL AND SYMMETRIC.)
24100          %                          1 . .
24200          %                          2 3 .
24300          %                          4 5 6
24400          %
24500          %   FURTHER, EACH ELEMENT MUST BE SUPPLIED IN A PARTICULAR
24600          %   WAY. SUPPOSE THAT THE CONFIGURATION WAS
24700          %                        2  2
24800          %                       Q  Q  Q̄
24900          %                        A  B  C
25000          %
25100          %  THEN THE TERMS MUST BE SUPPLIED AS RATIONALISED FRACTIONS
25200          %  IN THE ORDER  AA,AB,AC,BB,BC. THUS IF THE SECOND ELEMENT
25300          %  WAS
25400          %             -2*SQRT(5)/3 BC̄ - 10*SQRT(5)/3 AC̄
25500          %
25600          %  THEN ONE WOULD NEED TO FEED IN
25700          %
25800          %             0,0,-10#5/3,0,-2#5/3
25900          %
26000          %  (IF TRAILING NOUGHTS ARE NOT GIVEN THEN THE DATA SHOULD
26100          %  NOT END IN A COMMA.)
26200          %
26300          FOR MATCOL:=0 STEP 1 UNTIL NUMLT-1 DO
26400          BEGIN
26500             READMATRIXELEMENT;
26600             % TERMS ARE READ INTO THE ALPHA ARRAY DATA.
26700             P:=POINTER(DATA);
26800             I:=72;
26900             SCAN P:P FOR I:I WHILE=" ";
27000             FOR MATROW:=0 STEP 1 UNTIL NUM2PARTINTERACTIONS-1 DO
27100             % MAXIMUM POSSIBLE NUMBER OF 2-PARTICLE INTERACTIONS=10
27200             IF I>0 THEN
27300             BEGIN
27400                IF P="," THEN SCAN P:P+1 FOR I:I-1 WHILE=" ";
27500                IF P="+" OR NEG:=P="-" THEN
27600                SCAN P:P+1 FOR I:I-1 WHILE=" ";
27700                IF P="#" THEN IF NEG THEN NUM:=-1 ELSE NUM:=1
27800                ELSE BEGIN
27900                   SCAN P FOR J:I WHILE >="0";
28000                   NUM:=IF NEG THEN -INTEGER(P,I-J) ELSE INTEGER(P,I-J);
28100                   SCAN P:P FOR I:I WHILE >="0";
28200                   SCAN P:P FOR I:I WHILE=" ";
28300                END;
28400                IF I>0 THEN IF P="#" THEN
28500                BEGIN
28600                   SCAN P:P+1 FOR I:I-1 WHILE=" ";
28700                   SCAN P FOR J:I WHILE >="0";
28800                   SURD:=INTEGER(P,I-J);
28900                   SCAN P:P FOR I:I WHILE >="0";
29000                   SCAN P:P FOR I:I WHILE=" ";
29100                END ELSE SURD:=1 ELSE SURD:=1;
29200                IF I>0 THEN IF P="/" THEN
29300                BEGIN
29400                   SCAN P:P+1 FOR I:I-1 WHILE=" ";
29500                   SCAN P FOR J:I WHILE >="0";
29600                   DEN:=INTEGER(P,I-J);
29700                   SCAN P:P FOR I:I WHILE >="0";
29800                   SCAN P:P FOR I:I WHILE=" ";
29900                END ELSE DEN:=1 ELSE DEN:=1;
30000                MAT[MATROW,MATCOL]:=NUM*SQRT(SURD)/DEN;
30100             END;
30200          END OF READING IN MATRIX ELEMENT;
30300          IF DEBUG THEN FOR MATCOL:=0 STEP 1 UNTIL NUMLT-1 DO
30400          WRITE(LINEPRINT,/,FOR MATROW:=0 STEP 1 UNTIL
30500          NUM2PARTINTERACTIONS-1 DO MAT[MATROW,MATCOL]);
30600          INFORM("DO YOU WANT THIS MATRIX CHECKED?");
30700          RECEIVEINSTRUCTIONS;
30800          OK:=CHECK:=TRUE;
30900          IF INSTRUCTION="NO" FOR 2 THEN CHECK:=FALSE ELSE
31000          IF INSTRUCTION="YES" FOR 3 OR INSTRUCTION="CHECK" FOR 5 THEN ELSE
31100          INFORM("INSTRUCTION NOT UNDERSTOOD. CHECKING BY DEFAULT");
```

```
31200          IF CHECK THEN
31300          BEGIN
31400             INTEGER CHKROW,CHKCOL;
31500             MATROW:=0;
31600             FOR CHKROW:=0 STEP 1 UNTIL MATRIXDIMENSION DO
31700             REPLACE CHKMATRIX[CHKROW,*] BY 0 FOR MATRIXDIMENSION+1 WORDS;
31800             FOR I:=0 STEP 1 UNTIL NUMSPECIES-1 DO
31900             BEGIN
32000                IF ABS(CONFIGURATION[I])>1 THEN BEGIN
32100                   CHKROW:=CHKCOL:=1;
32200                   FOR MATCOL:=0 STEP 1 UNTIL NUMLT-1 DO BEGIN
32300                      CHKMATRIX[CHKROW,CHKCOL]:=* + MAT[MATROW,MATCOL];
32400                      ADJUST(CHKROW,CHKCOL); END;
32500                   MATROW:=MATROW+1;
32600                END;
32700                FOR J:=I+1 STEP 1 UNTIL NUMSPECIES-1 DO
32800                BEGIN
32900                   CHKROW:=CHKCOL:=1;
33000                   IF SIGN(CONFIGURATION[I])*SIGN(CONFIGURATION[J])=-1 THEN
33100                   FOR MATCOL:=0 STEP 1 UNTIL NUMLT-1 DO BEGIN
33200                      CHKMATRIX[CHKROW,CHKCOL]:=* - MAT[MATROW,MATCOL];
33300                      ADJUST(CHKROW,CHKCOL); END
33400                   ELSE FOR MATCOL:=0 STEP 1 UNTIL NUMLT-1 DO BEGIN
33500                      CHKMATRIX[CHKROW,CHKCOL]:=* + MAT[MATROW,MATCOL];
33600                      ADJUST(CHKROW,CHKCOL); END;
33700                   MATROW:=MATROW+1;
33800                END;
33900             END;
34000             WRITE(LINEPRINT,<"THE CHECKING MATRIX IS">);
34100             FOR CHKROW:=1 STEP 1 UNTIL MATRIXDIMENSION DO
34200             WRITE(LINEPRINT,/,FOR I:=1 STEP 1 UNTIL MATRIXDIMENSION DO
34300             CHKMATRIX[CHKROW,I]);
34400             SYMEIGENVECTORS(MATRIXDIMENSION,CHKMATRIX,CHKVALUES,CHKVECTORS);
34500             IF INTERACTIVE THEN
34600             WRITE(SCREEN,<"THE CHECKING EIGENVALUES ARE">);
34700             WRITE(LINEPRINT,<"THE CHECKING EIGENVALUES ARE">);
34800             IF INTERACTIVE THEN WRITE(SCREEN,/,FOR I:=1 STEP 1 UNTIL
34900             MATRIXDIMENSION DO CHKVALUES[I]);
35000             WRITE(LINEPRINT,/,FOR I:=1 STEP 1 UNTIL MATRIXDIMENSION DO
35100             CHKVALUES[I]);
35200             WRITE(LINEPRINT,<"THE CHECKING EIGENVECTORS ARE THE ROW VECTORS">);
35300             FOR I:=1 STEP 1 UNTIL MATRIXDIMENSION DO
35400             WRITE(LINEPRINT,FCHKV,FOR J:=1 STEP 1 UNTIL MATRIXDIMENSION DO
35500             CHKVECTORS[J,I]);
35600             INFORM("DO YOU WANT TO PROCEED?");
35700             RECEIVEINSTRUCTIONS;
35800             IF INSTRUCTION="NO" FOR 2 THEN OK:=FALSE ELSE
35900             IF INSTRUCTION="YES" FOR 3 OR INSTRUCTION="PROCEED" FOR 7 THEN ELSE
36000             IF INSTRUCTION="FINISH" FOR 6 THEN BEGIN
36100                OK:=FALSE;FINISHING:=TRUE;END ELSE
36200             INFORM("INSTRUCTION NOT UNDERSTOOD. CONTINUING BY DEFAULT");
36300          END OF CHECKING;
36400       END OF SETUPMATRIX;

36500       PROCEDURE SPECIFY(MAT);
36600       REAL ARRAY MAT[0,0];
36700       BEGIN
36800          INTEGER I,J,K,L;
36900          ALPHA ARRAY SPECIES[0:(NUMSPECIES-1)DIV 6];
37000          POINTER PS;
37100          TRUTHSET QUARKTYPES("OS");
37200          DEFINE READSPECIES=BEGIN
37300             RECEIVEINSTRUCTIONS;
37400             REPLACE SPECIES[*] BY INSTRUCTION FOR
37500             (NUMSPECIES-1)DIV 6 + 1 WORDS;END#;
37600          INFORM("WHAT ARE THE SPECIES?");
37700          % FOR EACH QUARK TYPE, INPUT O OR S CONSECUTIVELY E.G.
37800          %    OSOS
37900          READSPECIES;
38000          OK:=TRUE;
38100          SCAN PS:POINTER(SPECIES) FOR I:NUMSPECIES WHILE IN QUARKTYPES;
38200          IF I^=0 THEN BEGIN
38300             INFORM("SPECIES UNKNOWN");
38400             OK:=FALSE;END
```

```
38500          ELSE BEGIN
38600             NUMOQUARKS:=NUMSQUARKS:=L:=0;
38700             PS:=POINTER(SPECIES);
38800             FOR K:=NUM2PARTINTERACTIONS STEP 1 UNTIL NUM2PARTINTERACTIONS+2 DO
38900             REPLACE MAT[K,*] BY 0 FOR NUMBERINLOWERTRIANGLE WORDS;
39000             FOR I:=0 STEP 1 UNTIL NUMSPECIES-1 DO
39100             BEGIN
39200                IF PS+I="O" THEN NUMOQUARKS:=*+ABS(CONFIGURATION[I]) ELSE
39300                IF PS+I="S" THEN NUMSQUARKS:=*+ABS(CONFIGURATION[I]);
39400                IF ABS(CONFIGURATION[I])>1 THEN
39500                BEGIN
39600                   IF PS+I="O" THEN
39700                   FOR K:=0 STEP 1 UNTIL NUMBERINLOWERTRIANGLE-1 DO
39800                   MAT[NUM2PARTINTERACTIONS,K]:=* + MAT[L,K]
39900                   ELSE IF PS+I="S" THEN
40000                   FOR K:=0 STEP 1 UNTIL NUMBERINLOWERTRIANGLE-1 DO
40100                   MAT[NUM2PARTINTERACTIONS+2,K]:=* + MAT[L,K]
40200                   ELSE OK:=FALSE;
40300                   L:=L+1;
40400                END;
40500                FOR J:=I+1 STEP 1 UNTIL NUMSPECIES-1 DO
40600                BEGIN
40700                   IF PS+I="O" AND PS+J="O" THEN
40800                   FOR K:=0 STEP 1 UNTIL NUMBERINLOWERTRIANGLE-1 DO
40900                   MAT[NUM2PARTINTERACTIONS,K]:=* + MAT[L,K]
41000                   ELSE IF PS+I="O" AND PS+J="S" OR
41100                   PS+I="S" AND PS+J="O" THEN
41200                   FOR K:=0 STEP 1 UNTIL NUMBERINLOWERTRIANGLE-1 DO
41300                   MAT[NUM2PARTINTERACTIONS+1,K]:=* + MAT[L,K]
41400                   ELSE IF PS+I="S" AND PS+J="S" THEN
41500                   FOR K:=0 STEP 1 UNTIL NUMBERINLOWERTRIANGLE-1 DO
41600                   MAT[NUM2PARTINTERACTIONS+2,K]:=* + MAT[L,K]
41700                   ELSE OK:=FALSE;
41800                   L:=L+1;
41900                END;
42000             END;
42100             INFORM("THE NUMBER OF ORDINARY QUARKS IS " COMMA
42200             NUMOQUARKS FOR * DIGITS);
42300             INFORM("THE NUMBER OF STRANGE QUARKS IS " COMMA
42400             NUMSQUARKS FOR * DIGITS);
42500          END;
42600       END OF SPECIFY;
42700
42800       $INCLUDE "NUMERALS/LINEQN"

42900       REAL PROCEDURE MRFORMIN(ENERGIES);
43000       REAL ARRAY ENERGIES[0];
43100       BEGIN
43200          INTEGER MRT;
43300          REAL MR;
43400          % THE BAG MODEL ENERGIES ARE GIVEN BY THE FORMULA
43500          %   E(I)[MR]=FOURPIBOVER3MSCUBED*MR**3 + P(I)/MR.
43600          % (HERE , I LABELS THE EIGENVALUE.)
43700          % IF THERE ARE NO STRANGE QUARKS THEN P(I) IS A CONSTANT.
43800          % IF STRANGE QUARKS ARE PRESENT THEN WE SHALL FIT P(I)
43900          % TO A CUBIC IN THE REGION OF THE MINIMUM ENERGY.
44000          %     THIS PROCEDURE HANDLES ONE EIGENVALUE AT A TIME. THE
44100          % EIGENENERGY AS A FUNCTION OF MR IS PASSED IN VIA THE
44200          % ARRAY ENERGIES.

44300          PROCEDURE NOMIN;
44400          BEGIN
44500             INFORM("NO MINIMUM");
44600             MRFORMIN:=0;
44700          END;
44800
44900          IF NUMSQUARKS=0 THEN BEGIN
45000             REAL P;
45100             % THE MINIMUM ENERGY OCCURS FOR THE VALUE OF MR GIVEN
45200             % BY THE DERIVATIVE
45300             %   E'(MR)=3*FOURPIBOVER3MSCUBED*MR**2 - P/MR**2
45400             %         =0
45500             % THUS
45600             %   MRFORMIN=SQRT(SQRT(P/(3*FOURPIBOVER3MSCUBED) ))
45700             MRT:=1;
45800             WHILE ENERGIES[MRT]>ENERGIES[MRT+1] AND MRT<29
45900             DO MRT:=MRT+1;
46000             % LOCATES APPROXIMATE MINIMUM
46100             IF MRT=1 OR (MRT=29 AND ENERGIES[MRT]>ENERGIES[MRT+1])
46200             THEN NOMIN
46300             ELSE BEGIN
46400                MR:=MRT/10;
46500                P:=(ENERGIES[MRT] - FOURPIBOVER3MSCUBED*MR**3) * MR;
46600                MRFORMIN:=SQRT(SQRT(P/(3*FOURPIBOVER3MSCUBED) ));
46700             END;
```

```
46800          END ELSE BEGIN
46900             REAL ARRAY P,C[0:4],POWERSOFMR[0:4,0:4];
47000             % WE TAKE FOUR VALUES OF P AND USE THEM TO FIT P TO A CUBIC
47100             % WITH COEFFICIENTS C[J].
47200             %   P=C[1] + C[2]*MR + C[3]*MR**2 + C[4]*MR**3
47300             % THE COEFFICIENTS C[J] OF THIS CUBIC POLYNOMIAL CAN BE
47400             % FOUND BY SOLVING FOUR SIMULTANEOUS EQUATIONS,GIVEN IN
47500             % MATRIX FORM BY
47600             %    POWERSOFMR[*,*] * C[*]=P[*].
47700             % THE MINIMUM ENERGY OCCURS FOR THE VALUE OF MR GIVEN
47800             % BY THE DERIVATIVE
47900             %  E'(MR)=3*FOURPIBOVER3MSCUBED*MR**2 + 2*C[4]*MR + C[3]
48000             %                - C[1]/MR**2
48100             %        =0
48200             % THUS WE MUST FIND THE CORRECT ROOT OF THE POLYNOMIAL
48300             %  3*FOURPIBOVER3MSCUBED*MR**4 + 2*C[4]*MR**3 + C[3]*MR**2
48400             %     - C[1]  =0
48500             MRT:=1;
48600             WHILE ENERGIES[MRT]>ENERGIES[MRT+4] AND MRT<26
48700             DO MRT:=MRT+1;
48800             % LOCATES 4 SMALLEST ENERGIES
48900             IF MRT=1 OR (MRT=26 AND ENERGIES[MRT]>ENERGIES[MRT+4])
49000             THEN NOMIN
49100             ELSE BEGIN
49200                INTEGER I,J;
49300                REAL THISGUESS,LASTGUESS,DERIVATIVE;
49400                DEFINE REQDACCURACY=0.00001#;
49500                FOR I:=0 STEP 1 UNTIL 3 DO BEGIN
49600                   MR:=(MRT + I)/10;
49700                   P[I+1]:=(ENERGIES[MRT+I]-FOURPIBOVER3MSCUBED*MR**3) * MR;
49800                   FOR J:=0 STEP 1 UNTIL 3 DO POWERSOFMR[I+1,J+1]:=MR**J;END;
49900                % AFTER THE ITERATION I=4.
50000                LINEQN(I,POWERSOFMR,POWERSOFMR,P,C);
50100                % FINDS COEFFICIENTS C[J] OF THE POLYNOMIAL
50200                IF ENERGIES[MRT+1]<ENERGIES[MRT+2]
50300                THEN MR:=(MRT+1)/10 ELSE MR:=(MRT+2)/10;
50400                % THIS IS AN APPROXIMATE VALUE OF MR AT THE MINIMUM.
50500                % WE NOW USE THE NEWTON-RAPHSON METHOD TO FIND THE PRECISE
50600                % ROOT OF THE POLYNOMIAL. IF MR0 IS AN APPROXIMATE VALUE
50700                % THEN AN IMPROVED VALUE IS GIVEN BY
50800                %       MRIMPRVD=MR0 + DELTA(MR0)
50900                % WHERE
51000                %       DELTA(MR0)= -POLYN(MR0)/POLYN'(MR0).
51100                THISGUESS:=MR;
51200                LASTGUESS:=0;
51300                WHILE ABS(THISGUESS-LASTGUESS)>=REQDACCURACY DO BEGIN
51400                   IF   ABS(THISGUESS-LASTGUESS)>=0.0001 THEN
51500                   DERIVATIVE:=(12*FOURPIBOVER3MSCUBED*THISGUESS**3
51600                   + 6*C[4]*THISGUESS**2 + 2*C[3]*THISGUESS);
51700                   LASTGUESS:=THISGUESS;
51800                   THISGUESS:=THISGUESS - (3*FOURPIBOVER3MSCUBED*THISGUESS**4
51900                   + 2*C[4]*THISGUESS**3 + C[3]*THISGUESS**2 -C[1])/DERIVATIVE;
52000                END;
52100                MRFORMIN:=THISGUESS;
52200             END;
52300          END;
52400       END OF MRFORMIN;
52500
52600       CONFIGKNOWN:=MATRIXKNOWN:=FALSE;
52700       WHILE ^FINISHING DO BEGIN
52800          BEGIN
52900             NEWCONFIGURATION;
53000             CONFIGKNOWN:=TRUE;
53100          END;
53200          WHILE CONFIGKNOWN AND ^FINISHING DO BEGIN
53300             INFORM("WHAT IS THE DIMENSION OF THE COLOUR-SPIN MATRIX?");
53400             % IF FOR EXAMPLE THE MATRIX IS 5 BY 5 THEN INPUT 5
53500             READMATRIXDIMENSION;
53600             IF DEBUG THEN WRITE(LINEPRINT,/,MATRIXDIMENSION);
53700             NUMBERINLOWERTRIANGLE:=(MATRIXDIMENSION+1)*MATRIXDIMENSION/2;
53800             IF DEBUG THEN
53900             INFORM("THE NUMBER OF ELEMENTS IN THE LOWER TRIANGLE IS " COMMA
54000             NUMBERINLOWERTRIANGLE FOR * DIGITS);
```

```
54100     BEGIN
54200        REAL MR,VOLENERGY,ZEROPTENERGY,QKINETICENERGY,MOO,MOS,MSS;
54300        INTEGER I,J,MRT,CSMROW,CSMCOL;
54400        REAL ARRAY GENMATRIX[0:NUM2PARTINTERACTIONS+2,
54500        0:NUMBERINLOWERTRIANGLE-1],
54600        % THE LAST 3 ROWS ARE FOR STORING THE SUMS OF THE
54700        % COEFFICIENTS OF THE RADIAL INTEGRALS MOO,MOS,MSS.
54800        CSMATRIX,CSVECTORS[0:MATRIXDIMENSION,0:MATRIXDIMENSION],
54900        CSVALUES[0:MATRIXDIMENSION],TOTENERGY[0:MATRIXDIMENSION,0:30];
55000        LIST L1(MR,TOTENERGY[I,MRT],
55100        VOLENERGY,ZEROPTENERGY,QKINETICENERGY,CSVALUES[I]),
55200        L2(TOTENERGY[I,MRT],CSVALUES[I]),
55300        L3(MR,TOTENERGY[I,MRT],
55400        VOLENERGY,ZEROPTENERGY,QKINETICENERGY,CSVALUES[I],
55500        FOR J:=1 STEP 1 UNTIL MATRIXDIMENSION DO CSVECTORS[J,I]),
55600        L4(TOTENERGY[I,MRT],CSVALUES[I],
55700        FOR J:=1 STEP 1 UNTIL MATRIXDIMENSION DO CSVECTORS[J,I]);
55800        SWITCH LIST ENERGIES:=L1,L2;
55900        SWITCH LIST ENRGYANDVECTOR:=L3,L4;
56000
56100        SETUPMATRIX(NUMBERINLOWERTRIANGLE,GENMATRIX,CSMATRIX,CSVALUES
56200        ,CSVECTORS);
56300        MATRIXKNOWN:=OK;
56400        WHILE CONFIGKNOWN AND MATRIXKNOWN AND ^FINISHING DO
56500        BEGIN
56600           SPECIFY(GENMATRIX);
56700           IF ^OK THEN INFORM("SPECIFICATION HAS FAILED. TO TRY " COMMA
56800           "AGAIN, INSTRUCT: SPECIES") ELSE
56900           BEGIN
57000              INFORM("AS A FUNCTION OF MR THE ENERGIES IN MEV ARE");
57100              IF INTERACTIVE THEN IF DEBUG THEN
57200              WRITE(SCREEN    ,HEADING) ELSE
57300              WRITE(SCREEN,<"BEING PRINTED">);
57400              WRITE(LINEPRINT,HEADING);
57500              FOR MRT:=1 STEP 1 UNTIL 30 DO
57600              BEGIN
57700                 MR:=MRT/10;
57800                 VOLENERGY:=FOURPIBOVER3MSCUBED*MR**3;
57900                 ZEROPTENERGY:= -Z0*MS/MR;
58000                 QKINETICENERGY:=(NUMOQUARKS*FUNCTIONSOFMR[0] +
58100                 NUMSQUARKS*FUNCTIONSOFMR[MRT])*MS/MR;
58200                 CSMROW:=CSMCOL:=1;
58300                 MOO:=FUNCTIONSOFMR[31];
58400                 MOS:=FUNCTIONSOFMR[62+MRT];
58500                 MSS:=FUNCTIONSOFMR[31+MRT];
58600                 FOR I:=0 STEP 1 UNTIL NUMBERINLOWERTRIANGLE-1 DO BEGIN
58700                    CSMATRIX[CSMROW,CSMCOL]:=
58800                    ( MOO*GENMATRIX[NUM2PARTINTERACTIONS  ,I]
58900                    + MOS*GENMATRIX[NUM2PARTINTERACTIONS+1,I]
59000                    + MSS*GENMATRIX[NUM2PARTINTERACTIONS+2,I]
59100                    )*ALPHAC*MS/MR;
59200                    ADJUST(CSMROW,CSMCOL);END;
59300                 SYMEIGENVECTORS(MATRIXDIMENSION,CSMATRIX,CSVALUES,
59400                 CSVECTORS);
59500                 FOR I:=1 STEP 1 UNTIL MATRIXDIMENSION DO BEGIN
59600                    TOTENERGY[I,MRT]:=VOLENERGY+ZEROPTENERGY+QKINETICENERGY+CSVALUES[I];
59700                    IF INTERACTIVE THEN IF DEBUG THEN
59800                    WRITE(SCREEN,NEATSCREEN[IF I=1 THEN 0 ELSE 1],
59900                    ENERGIES[IF I=1 THEN 0 ELSE 1]);
60000                    WRITE(LINEPRINT,NEATLINE[IF I=1 THEN 0 ELSE 1],
60100                    ENRGYANDVECTOR  [IF I=1 THEN 0 ELSE 1]);
60200                 END;
60300                 WRITE(LINEPRINT[SPACE 1]);
60400              END;
```

```
60500                     IF MATRIXDIMENSION>1 THEN
60600                     INFORM("THE MINIMA ARE ") ELSE
60700                     INFORM("THE MINIMUM IS ");
60800                     IF INTERACTIVE THEN
60900                     WRITE(SCREEN    ,HEADING);
61000                     WRITE(LINEPRINT,HEADING);
61100                     FOR I:=1 STEP 1 UNTIL MATRIXDIMENSION DO
61200                     IF (MR:=MRFORMIN(TOTENERGY[I,*]) )^=0 THEN BEGIN
61300                        IF DEBUG THEN WRITE(LINEPRINT,/,MR);
61400                        VOLENERGY:=FOURPIBOVER3MSCUBED*MR**3;
61500                        ZEROPTENERGY:=-Z0*MS/MR;
61600                        QKINETICENERGY:=(NUMOQUARKS*FUNCTIONSOFMR[0]
61700                        +NUMSQUARKS*OMEGA(MR) )*MS/MR;
61800                        CSMROW:=CSMCOL:=1;
61900                        MOO:=FUNCTIONSOFMR[31];
62000                        MOS:=M(0,MR);
62100                        MSS:=M(MR,MR);
62200                        FOR J:=0 STEP 1 UNTIL NUMBERINLOWERTRIANGLE-1 DO BEGIN
62300                           CSMATRIX[CSMROW,CSMCOL]:=
62400                           ( MOO*GENMATRIX[NUM2PARTINTERACTIONS  ,J]
62500                           + MOS*GENMATRIX[NUM2PARTINTERACTIONS+1,J]
62600                           + MSS*GENMATRIX[NUM2PARTINTERACTIONS+2,J]
62700                           )*ALPHAC*MS/MR;
62800                           ADJUST(CSMROW,CSMCOL);END;
62900                        SYMEIGENVECTORS(MATRIXDIMENSION,CSMATRIX,CSVALUES
63000                        ,CSVECTORS);
63100                        MRT:=0;
63200                        TOTENERGY[I,MRT]:=VOLENERGY+ZEROPTENERGY+QKINETICENERGY
63300                        +CSVALUES[I];
63400                        IF INTERACTIVE THEN
63500                        WRITE(SCREEN,FSMIN,ENERGIES[0]);
63600                        WRITE(LINEPRINT,FLMIN,ENRGYANDVECTOR[0]);
63700                     END;
63800                  END;
63900                  WRITE(LINEPRINT[SPACE 2]);
64000                  INFORM("AWAITING INSTRUCTIONS");
64100                  RECEIVEINSTRUCTIONS;
64200                  IF INSTRUCTION="FINISH" FOR 6 THEN FINISHING:=TRUE ELSE
64300                  IF INSTRUCTION="CONFIG" FOR 6 THEN
64400                  MATRIXKNOWN:=CONFIGKNOWN:=FALSE ELSE
64500                  IF INSTRUCTION="MAT" FOR 3 THEN MATRIXKNOWN:=FALSE ELSE
64600                  IF INSTRUCTION="SPECIES" FOR 4 THEN ELSE
64700                  IF INSTRUCTION="CONTINUE" FOR 8 THEN ELSE
64800                  INFORM("INSTRUCTION NOT UNDERSTOOD. CONTINUING BY DEFAULT.");
64900               END OF FIXED MATRIX LOOP;
65000            END;
65100         END OF FIXED CONFIGURATION LOOP;
65200      END;
65300   END OF GETENERGIES;
65400
65500   INTERACTIVE:=MYSELF.STATION^=0;
65600   FINISHING:=FALSE;DEBUG:=FALSE;
65700   MESSAGE:=POINTER(LINEBUFF[MARGIN]);
65800   REPLACE LINEBUFF BY "      " FOR MARGIN WORDS;
65900   INFORM("MULTIQUARK-HADRON ENERGY CALCULATOR READY AND " COMMA
66000   "REARING TO GO.");
66100   GETENERGIES;
66200   FINISH:INFORM("FINISHING , GOODBYE.");
66300 END OF PROGRAM.
```

GROUP THEORETICAL METHODS AND MULTIQUARK HADRONS

Part II

Phase freedom in the Wigner-Racah algebra and the calculation of 3jm factors and 6j symbols

CHAPTER 9

INTRODUCTION

It was recognized long ago (e.g. Wigner 1931) that the quantum theory of angular momentum was intimately related to the three-dimensional rotation group. Further it was early appreciated that many aspects of this theory could be generalized to other groups which were also of importance to physics. Indeed Racah's (1949) work on fractional parentage coefficients showed that even groups with no direct manifestation in a physical problem could be very useful. Calculations in angular momentum theory assume a terse and elegant form when the problem is expressed in terms of the highly symmetric $n$jm factors and nj symbols (Wigner, 1940 — the terminology is Butler's, 1975) and tensor operators which are symmetrized according to the group representations (Racah 1942b, see also for example Fano and Racah 1959, Judd 1963). Successful application of these concepts requires knowledge of the values of the $n$jm factors and nj symbols. Nowadays, most of those required can be found in the tabulation by Rotenberg et al (1959). In the generalized theory though it is common for authors to calculate and use the less symmetric isoscalar factors and recoupling coefficients e.g. Griffith (1961) for finite subgroups of SO_3 and de Swart (1963) for SU_3. Part of the reason for this has been lack of knowledge of the symmetry properties of the generalized $n$jm factors and nj

symbols. In an important paper, Derome and Sharp (1965) remedied this situation but still there has been a reluctance (e.g. Akiyama and Draayer 1973a,1973b, So and Strottman, 1979) to adopt the generalized theory (Wigner-Racah algebra) in its most powerful form — see though Griffith (1962). The usual policy seems to be to calculate some numbers and then determine their symmetry properties rather than beginning by demanding maximal symmetry and then calculating a set of numbers which satisfies the requirements. Of course the situation is compounded by the lack of a closed formula as exists for both the 3jm factors and 6j symbols of SO_3.

In part I of this thesis we met a problem in connection with multiquark hadrons whose solution required knowledge of 3jm factors and 6j symbols for the group chain (3.17) which involves the group-subgroup systems: $U_{18} \supset SU_{12} \times U_6$, $SU_{12} \supset SU_2 \times SU_6$, $U_6 \supset U_1 \times SU_6$, $SU_6 \supset SU_2 \times SU_3$ and $(SU_2 \times SU_3) \times (SU_2 \times SU_3) \supset SU_2 \times SU_3$. The ultimate aim of Part II is to calculate these. There are several methods which could be used to do so but we have selected one which has the considerable attraction of being applicable to any compact group, whether finite or continuous, of Lie type or otherwise. It has the further advantage of being peculiarly suited to obtaining coefficients subject to maximal symmetry requirements.

The method we use is mainly due to Butler (Butler and Wybourne 1976a, Butler 1976 — see also Fano and Racah 1959, appendix I — , Butler and Wybourne 1976b, Butler et al.

1978, 1979, Butler and Reid 1979, Reid and Butler 1980, Butler 1979, 1980a, 1980b). Consequently we shall refer to it as Butler's method. Basically, Butler has claimed that the generalized $n$jm and nj symbols can all be calculated solely from a knowledge of their general properties and the properties of the representations of the relevant groups; at least that is, up to a phase or multiplicity separation which is free to be chosen. It is therefore a uniqueness method. The key to success in performing Butler's calculations lies in understanding the phase choices that exist within the algebra. These phase choices also exist in angular momentum theory and the different choices of various authors have sometimes led to confusion. However, in the general theory it is not simply a question of establishing a universally accepted choice of phases but rather it is the need to obtain a choice — any choice — which yields a consistent Wigner-Racah algebra. Other methods usually conceal at least some of these phase choices, by for instance starting with an explicit matrix representation of the group and then constructing the required coefficients. It is however just this freedom which one requires to construct $n$jm factors and nj symbols with simple symmetries.

Derome (1966) showed when it was possible to have simple permutation symmetries for the 3jm factors but it was not until an often-overlooked paper by Butler and King (1974) that it was shown that symbols with contravariant and covariant multiplicity indices could be rendered equivalent — at least in most physically interesting cases. This last

fact considerably simplifies complex-conjugation symmetries and further simplifications with respect to the subgroup indices were noted by Butler and Wybourne (1976a). These simplifications amount to special choices of the (3j) permutation and (2jm and A) conjugation matrices. Naturally, the choices made here affect the later phase choices to be made when calculating 3jm factors and 6j symbols. It is necessary therefore to have a thorough understanding of these early choices and their implications. There are in fact several consistency conditions overlooked by Butler (1975) in his review of these matters and there are also some apparently unrecognized choices associated with some of the definitions in the algebra. For these reasons alone it will be worth our while undertaking a comprehensive review of these canonical choices. The experience we gain here will also be beneficial when we apply our techniques to the more difficult choices that exist for the 3jm factors and 6j symbols.

Butler's method has been in a continual state of development over the recent years. In fact its completeness is still a conjecture although Butler (1976) (see also Fano and Racah 1959, Appendix I) has shown it to be complete for $SO_3 \supset SO_2$. Butler and Wybourne (1976b) and Butler (1980b) have obtained complete tables for finite subgroups of SO_3 although the methods required are more general than those described by Butler and Wybourne (1976a). For instance, even in the $T \supset C_3$ calculation of Butler and Wybourne (1976b) one needs to make a phase choice which was not justified by

those authors. There are indeed six free phases in the $T \supset C_3$ 3jm algebra, remaining after the 6j symbols have been calculated, but only five of them can be directly attributed to the $T \rightarrow C_3$ branchings. This is because the phases are relative ones only and therefore we are not free to choose the phase of one of the six branchings. The rules of Butler and Wybourne (1976a) are again deficient in that they implied that there was no free phase associated with "primitive" couplings in a 6j calculation. However, Butler and Haase (private communication, 1978) have "discovered" (cf. So and Strottman 1979 for example) that for several groups there is apparently a free phase associated with some such couplings. Furthermore, Butler and Wybourne (1976a) assumed that no phase choice exists for products involving the identity representation but no inkling of why this should be so was given. It would therefore seem that claims by these authors that "the problems of phase specification are fully understood" (Butler et al. 1978) were a little premature. Not until the paper by Reid and Butler (1980) and Butler's (1980b) book was justification forthcoming.

The bulk of Part II constitutes a thorough and rigorous, independent, attempt to patch the early deficiences in Butler's phase fixing procedures. When read in conjunction with the paper by Reid and Butler (1980) it may at last be possible to claim that the problems of phase specification are understood. Several factors have influenced the approach taken here. Firstly, rigour is of paramount importance.

We will require to know exactly what phase freedom occurs; it will be found that certain phases cancel and we must be sure of when they do — a set of semi-empirical rules must be regarded as totally unsatisfactory. Secondly, it is highly desirable to use an argument as simple, as cogent and as general as possible. A disadvantage of Butler and Wybourne's approach is that their arguments are peculiar to coupling theory and not directly applicable to other types of transformation, where phase freedom also exists. A more general approach should be possible and would have wider application. Thirdly, Butler and Wybourne (1976a,b) and the earlier work of Butler (1975) make liberal and unashamed use of special phase choices. This is perfectly all right for the applications they have in mind but there are circumstances when one needs to know precisely what is the most general freedom available without any restrictions whatsoever imposed by special choices. An important example of this occurs in the algebra for the unitary groups U_N. Here it is found that certain classes of 6j symbols for U_N and U_M ($M \neq N$) are related by a simple dimensionality factor and a phase (e.g. chapter 14). If this phase can be chosen consistently then formulae can be obtained for these 6j symbols by working down to a unitary group where the appropriate symbol is known.

Unfortunately the main authorities on the foundations of a generalized Wigner-Racah algebra (Derome and Sharp 1965, Butler 1975) are inadequate for a rigorous discussion of phase freedom. Derome and Sharp, purposely, do not relate

their work to isoscalar factors and overlap integrals which makes a general treatment difficult and Butler, because of his special phase choices, largely ignores matters of covariance and contravariance of indices and often employs equations that are not independent of phase choice. Further, both authorities use Wigner's (1940) definition of the 1j symbol (which we shall hereafter refer to as the 2jm symbol in accordance with Butler's (1975) rationalization of such terms) but this involves making a particular phase choice! Therefore it is regrettably necessary to discuss some of the foundations again here. Only by starting from square one will we be able to rid ourselves of other people's phase conventions.

We shall begin in chapter 10 by reviewing some basic transformation theory. At first sight much of what is said may seem novel but for the larger part it is only the emphasis which is new. It is for instance well-known that coupling (or Clebsch-Gordan) coefficients are just special cases of transformation coefficients but the point is rarely emphasized and certainly it has never been extensively enunciated. We shall find that this relationship provides a key to solving the phase problem. By beginning with the general transformation coefficients one only has to specialize to coupling coefficients and thence to 3jm factors and 6j symbols. We find it necessary in the course of this development to conduct a careful examination of the relationship between complex-conjugate representations. This leads not only to the phase freedom in a 2jm factor and an explanation for the apparent lack of freedom for products involving

the identity representation but also to an important generalization of the Derome-Sharp lemma. In turn this provides a more profound view of the complex conjugation symmetries of 3jm factors and 6j symbols.

In chapter 11 the origin of phase freedom is described. Actually, when multiplicities are present in the Wigner-Racah algebra the freedom comprises an arbitrariness in the magnitude as well as the phase. Nevertheless we shall continue to use the popular term "phase freedom" (which originated for multiplicity-free groups) to collectively describe all cases. Some trivial phase choices for direct product groups are then discussed. Following this the 3jm factors, 3j permutation matrices and 6j symbols are introduced and their precise freedoms derived from the earlier results.

In chapter 12 it is shown how to systematically go about choosing phases. We review the work of Derome (1966) and Butler and King (1974) as a first step and then we proceed to discuss phase-fixing procedures required for the application of Butler's method.

Chapter 13 contains a discussion of Butler's method. Several improvements can be made to Butler and Wybourne's (1976a) algorithm. Butler et al. (1978,1979) found that they had to solve non-linear equations to obtain answers. This turns out to be unnecessary and was a consequence of them failing to use all the available equations. It is also shown how most calculations can be performed power by power via the solution of simultaneous equations. Some comments

on the completeness of the method are included.

Finally, in chapter 14, the use of the techniques developed is illustrated by calculating 3jm factors and 6j symbols for the group chain (3.17) used in Part I.

The notation adopted is essentially the Wigner style notation used by Butler (1975) and Butler and Wybourne (1976a) with only a few minor modifications to highlight certain points. In particular raised and lowered indices are used to distinguish contravariant and covariant labels in matrix elements. The reader is referred to Wigner (1940) or Derome and Sharp (1965) for a discussion of this.

CHAPTER 10

BASIC TRANSFORMATION THEORY

A representation λ of a group G consists of a representation vector space $V_{x\lambda}$ of dimension $|\lambda|$ with a chosen basis, together with a homomorphism, h_λ from the abstract group elements R,S, ... into the set of linear operators O^λ_R, O^λ_S, ... which map $V_{x\lambda}$ onto itself. The label x is a parentage label to distinguish different representation vector spaces of λ. If G is compact then all the representation vector spaces of G will be finite. The linear algebra which follows assumes this. Further we shall always assume that the representation is irreducible in the usual sense. We write (letting $|x\lambda i\rangle$ and $|x\lambda i'\rangle$ denote different basis vectors in the same basis for $V_{x\lambda}$)

$$O^\lambda_R|x\lambda i\rangle = \sum_{i'} |x\lambda i'\rangle\langle x\lambda i'|O^\lambda_R|x\lambda i\rangle$$

$$= \sum_{i'} |x\lambda i'\rangle\lambda(R)^{i'}{}_{i} \ . \qquad (10.1)$$

The coefficients $\langle x\lambda i'|O^\lambda_R|x\lambda i\rangle$ define a $|\lambda| \times |\lambda|$ invertible matrix $\lambda(R)$ with respect to the chosen basis, with elements $\lambda(R)^{i'}{}_{i}$ as indicated above. Note that by definition of a representation, O^λ_R is diagonal in the parentage label; in addition though the action of O^λ_R is defined independently of x so that spaces with different parentage do not give rise to separate representations. The justification for this

procedure rests with our ability to consider G independently of any larger group (cf. equation 10.5). On the other hand it is important to recognize that $\lambda(R)$ is basis dependent whereas O_R^λ is not. Strictly, there is a different representation for each choice of basis for $V_{x\lambda}$ but since the homomorphism, h_λ as defined here is independent of basis, all such representations can be considered equivalent and it is conventional to call them all by the vector space label, λ. Normally this does not lead to any confusion but in the context of this work one needs to be very careful as changes of basis play a principal role.

A (linear) transformation of basis in $V_{x\lambda}$ can be written as an operator mapping

$$|x\lambda\hat{i}\rangle = U|x\lambda i\rangle = \sum_{i'} |x\lambda i'\rangle U^{i'}{}_{i} \qquad (10.2a)$$

or alternatively by using the scalar product

$$|x\lambda\hat{i}\rangle = \sum_{i'} |x\lambda i'\rangle\langle\lambda i'|\lambda\hat{i}\rangle \qquad (10.2b)$$

where a circumflex has been used to denote the vectors in the new basis. Note that a basis transformation is performed independently of the parentage label. This statement will be justified in full in chapter 11. Assuming that the basis vectors are orthonormal, the transformation coefficients $\langle\lambda i'|\lambda\hat{i}\rangle$ are elements, $U^{i'}{}_{i}$ of a unitary matrix. In the transformed basis we must use the transformed representation matrices given by

$$\lambda(R)^{\hat{i}'}{}_{\hat{i}} = \sum_{i''i'''} \langle\lambda\hat{i}'|\lambda i''\rangle \lambda(R)^{i''}{}_{i'''} \langle\lambda i'''|\lambda\hat{i}\rangle \ . \qquad (10.3)$$

This follows because O_R^λ is defined independently of basis. If $i,\hat{i}$ and $\hat{\hat{i}}$ denote three different bases for $V_{x\lambda}$ then it is trivial to show that

$$\langle\lambda i|\lambda\hat{\hat{i}}''\rangle = \sum_{i'} \langle\lambda i|\lambda\hat{i}'\rangle\langle\lambda\hat{i}'|\lambda\hat{\hat{i}}''\rangle \ . \qquad (10.4)$$

Often it is helpful to choose a basis for $V_{x\lambda}$ in which the (basis) vectors $|x\lambda i\rangle$ are distinguished by their transformation properties under a subgroup H of G. For such a basis we replace the arbitrary label i by the labels $a\mu j$ where μ labels a subspace of $V_{x\lambda}$ which transforms as a representation space $V_{x\lambda a\mu}$ of the representation μ of H, a is a branching multiplicity label to distinguish different subspaces of $V_{x\lambda}$ with the same transformation properties under H and j labels the basis used for $V_{x\lambda a\mu}$. Note that in using the transformation properties (10.1) we would now require that

$$\lambda(R)^{a'\mu'j'}{}_{a\mu j} = \delta^{a'}{}_{a}\ \delta^{\mu'}{}_{\mu}\ \mu(R)^{j'}{}_{j} \qquad (10.5)$$

if $R \in H$.

The Wigner-Racah algebra is concerned with the matrices which perform changes of basis in the product spaces formed by tensor products of various representation vector spaces. Schur's lemmas (Schur, 1905) and the simple concepts presented here are all that are required to develop this algebra.

Indeed many of the more important equations in the quantum theory of angular momentum are just special cases of the formulae given here. Further, the techniques described here form the groundwork for handling the phase problem.

Let us now consider some consequences of Schur's lemmas, after which we shall examine the role of antilinear transformations. We shall then be in a position to tackle the phase problem itself.

(a) Transformation Factors

Suppose that we have a basis for $V_{x\lambda}$ consisting of the vectors $|x\lambda a\mu i\rangle$, then we can perform a unitary transformation in the branching multiplicity index and obtain a new basis, consisting of the vectors

$$|x\lambda\hat{a}\mu i\rangle = \sum_{a'\mu'i'} |x\lambda a'\mu'i'\rangle\langle\lambda a'\mu'i'|\lambda\hat{a}\mu i\rangle \ ,$$

in which we retain the same bases for the representation subspaces $V_{x\lambda\hat{a}\mu}$ as previously used for the $V_{x\lambda a\mu}$. This requirement has immediate and important consequences. Operating on both sides of the transformation with O^{λ}_{R}, where $R \in H$, it is simple to show that

$$\sum_{i'} \mu'(R)^{i''}{}_{i'} \langle\lambda a'\mu'i'|\lambda\hat{a}\mu i\rangle$$

$$= \sum_{i'''} \langle\lambda a'\mu'i''|\lambda\hat{a}\mu i'''\rangle \mu(R)^{i'''}{}_{i}$$

which means that the transformation commutes with the representation matrices $\mu(R)$. Since μ is irreducible, application of Schur's lemmas tells us that $\langle \lambda a' \mu' i' | \lambda \hat{a} \mu i \rangle$ is zero unless $\mu = \mu'$ and $i = i'$ and further that the coefficient is independent of i. Therefore we rewrite the transformation as

$$|x\lambda\hat{a}\mu i\rangle = \sum_{a'} |x\lambda a'\mu i\rangle\langle\lambda a'\mu|\lambda\hat{a}\mu\rangle \ . \qquad (10.6)$$

The use of Schur's lemma to show that $\langle \lambda a' \mu i | \lambda \hat{a} \mu i \rangle$ is independent of i warrants some comment since some proofs of this lemma (e.g. Hammermesh, 1962) only apply to transformations which map a representation space onto itself whereas this transformation maps the space $V_{x\lambda a\mu}$ onto $V_{x\lambda\hat{a}\mu}$. (Note that the matrix of coefficients in (10.6) is not a multiple of the identity.) However, it is not difficult to prove that Schur's lemma does indeed cover the more general case. Application of the Great Orthogonality theorem (e.g. chapter 11) immediately yields the more general result and furthermore reveals that it holds for the simple reason that the representation matrices are independent of parentage.

It is interesting to note an alternative and more direct derivation of (10.6) which avoids explicit use of Schur's lemma. That $\langle \lambda a' \mu' i' | \lambda \hat{a} \mu i \rangle$ is diagonal in μ and i follows directly from the definition of the dual vector space (the irreducibility of μ is crucial here) and then it is immediately apparent that

$$\langle \lambda a' \mu i | \lambda \hat{a} \mu i \rangle = \langle \lambda a' \mu i' | \lambda \hat{a} \mu i' \rangle$$

unless $\mu(R)^{i'}_{i} = 0$ for all elements R of H. Even then the result will still follow by transitivity unless for all rows i" for which $\mu(R)^{i"}_{i}$ is non-zero, $\mu(R)^{i'}_{i"}$ and $\mu(R)^{i"}_{i'}$ are both zero for all elements R of H. Continuing in this manner we must sooner or later obtain the desired result unless μ is completely reducible — but μ is assumed irreducible. This proof clearly exhibits the power of the scalar product notation. Of course this is equivalent to some proofs of Schur's lemma.

Having firmly established (10.6) we can now perform a basis transformation in $V_{x\lambda\hat{a}\mu}$ for which

$$|x\lambda\hat{a}\mu\hat{i}\rangle = \sum_{i'} |x\lambda\hat{a}\mu i'\rangle\langle\mu i'|\mu\hat{i}\rangle \tag{10.7}$$

as in (10.2b). Combining (10.6) and (10.7) yields

$$|x\lambda\hat{a}\mu\hat{i}\rangle = \sum_{a'i'} |x\lambda a'\mu i'\rangle\langle\lambda a'\mu|\lambda\hat{a}\mu\rangle\langle\mu i'|\mu\hat{i}\rangle \ . \tag{10.8}$$

It follows that we can always write

$$\langle\lambda a'\mu' i'|\lambda\hat{a}\mu\hat{i}\rangle = \langle\lambda a'\mu|\lambda\hat{a}\mu\rangle\ \delta^{\mu'}_{\mu}\langle\mu i'|\mu\hat{i}\rangle \ . \tag{10.9}$$

Any set of labels associated with any chain of subgroups can be substituted for any of the multiplicity labels and these results still hold. For instance in the transformation between $G \supset H \supset K \supset \ldots$ and $G \supset H \supset L \supset \ldots$ we can simply consider the transformation to be in the representation spaces of H and therefore the transformation would be

described by an equation of the form of (10.7). Similarly the transformation between $G \supset H \supset K \supset \ldots$ and $G \supset H' \supset K \supset \ldots$ can be performed, as in (10.6), by

$$|x\lambda a\mu b\xi i\rangle = \sum_{c\nu d} |x\lambda c\nu d\xi i\rangle\langle\lambda c\nu d\xi|\lambda a\mu b\xi\rangle \quad . \tag{10.10}$$

In this thesis we will, for emphasis, call transformation coefficients which depend only on certain portions of a group-subgroup chain, such as $\langle\lambda c\nu d\xi|\lambda a\mu b\xi\rangle$, transformation factors. They have obvious unitary properties; for instance

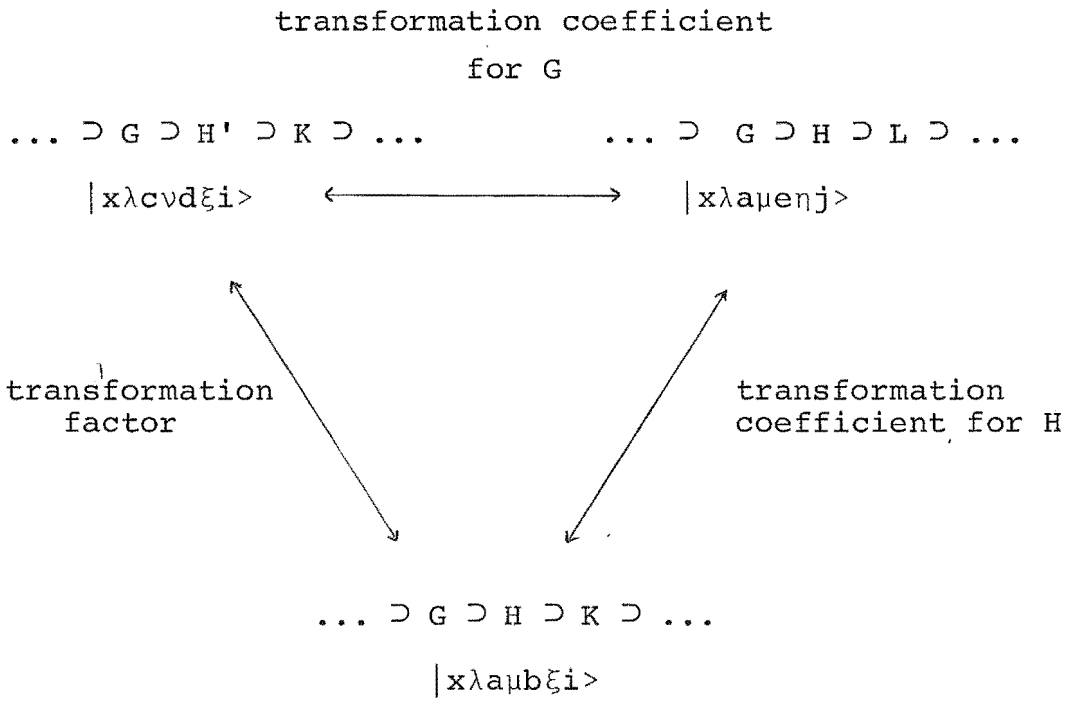
$$\sum_{a\mu b} \langle\lambda c\nu d\xi|\lambda a\mu b\xi\rangle\langle\lambda a\mu b\xi|\lambda c'\nu'd'\xi\rangle = \delta^{c}{}_{c'}\delta^{\nu}{}_{\nu'}\delta^{d}{}_{d'} \quad . \tag{10.11}$$

The transformation coefficient for G between the chains $G \supset H' \supset K \supset \ldots$ and $G \supset H \supset L \supset \ldots$ can be written in terms of the transformation coefficients for H between the chains $G \supset H \supset K \supset \ldots$ and $G \supset H \supset L \supset \ldots$ and the transformation factors between the chains $G \supset H \supset K \supset \ldots$ and $G \supset H' \supset K \supset \ldots$ by an argument analogous to that leading to equation (10.9). Letting λ label representations of G, μ representations of H, ν representations of H', ξ representations of K, η representations of L and Latin letters the remaining multiplicities we have

$$\langle\lambda c\nu d\xi i|\lambda a\mu e\eta j\rangle = \sum_{b} \langle\lambda c\nu d\xi|\lambda a\mu b\xi\rangle\langle\mu b\xi i|\mu e\eta j\rangle \quad . \tag{10.12}$$

This trivial — but important — equation is a generalization of the Racah (1949) factorization lemma for coupling

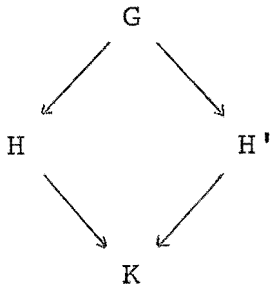
coefficients. It should be compared with equation (10.4) which can be regarded as the prototypal factorization. The special importance of (10.12) derives from the presence of a transformation factor rather than simply an ordinary transformation coefficient. The three group schemes involved can be represented schematically:



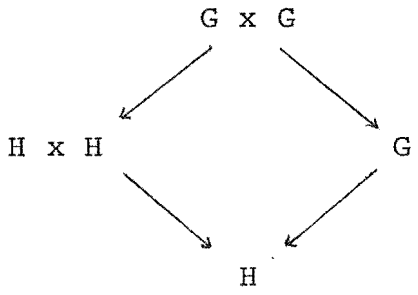
The particular group structure involved in (10.12) should be stressed; it is not an ordinary coefficient which appears on the left-hand side (cf. Butler and Reid 1979).
It is also crucial to realize that where we have required groups to be identical they must be just that; simple isomorphism is not enough. For instance if the two K groups above were only isomorphic then their representation spaces would not be the same and there would be no reason why the transformation (10.10) would be possible. Indeed

the representations ξ' occurring in one chain would not necessarily be the same as the representations ξ occurring in the other. A necessary and sufficient condition for the same subgroup K to be contained in both H and H' is that the set of elements of K be contained in the intersection of the set of elements of H and the set of elements of H'.

Our interest in transformation factors is mainly due to two special cases of the transformation described by equation (10.10). Schematically we have the two subgroup chains:



Consider the special case



The transformation factor could be written as

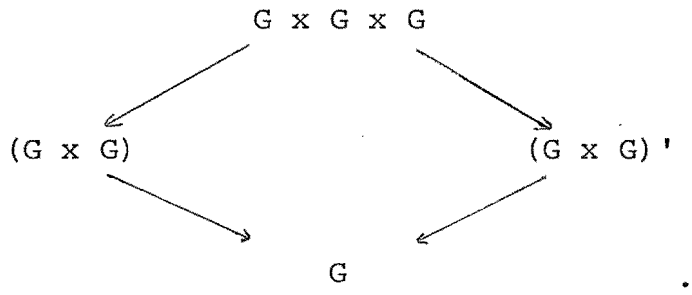
$$<(\lambda_1\lambda_2)r\lambda a\mu|(\lambda_1\lambda_2)(a_1a_2)(\mu_1\mu_2)s\mu>$$

$$\equiv<(\lambda_1\lambda_2)r\lambda a\mu|(\lambda_1 a_1\mu_1;\lambda_2 a_2\mu_2)s\mu>$$

$$\equiv <r\lambda a\mu s|\lambda_1 a_1\mu_1;\lambda_2 a_2\mu_2> \qquad (10.13)$$

and of course is just the well-known isoscalar (or coupling) factor for $G \supset H$. Of the three notations the third is that used by Butler (1975) but the second is more appropriate when discussing phases and will be adopted here. The first is only included for heuristic reasons. It is easy to see now that the well-known Racah factorization lemma for coupling coefficients is just a special case of equation (10.12).

Another special transformation factor is associated with the group scheme:



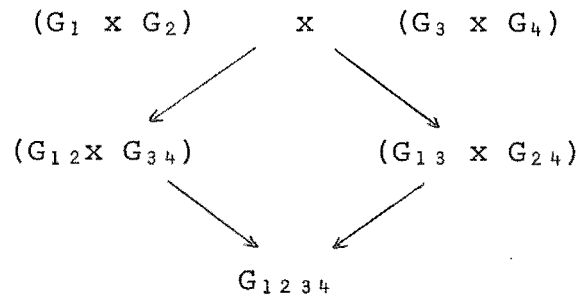
Here the two groups (G x G) and (G x G)' are isomorphic but not equal. The relevant transformation factor is obviously the overlap integral for G which we can write as

$$\langle(\lambda_1\lambda_2\lambda_3)r_{12}(\lambda_{12}\lambda_3)r_2\lambda|(\lambda_1\lambda_2\lambda_3)r_{23}(\lambda_1\lambda_{23})r_1\lambda\rangle$$

$$\equiv \langle(\lambda_1\lambda_2)r_{12}\lambda_{12},\lambda_3,r_2\lambda|\lambda_1(\lambda_2\lambda_3)r_{23}\lambda_{23},r_1\lambda\rangle. \qquad (10.14)$$

The latter is the more usual notation (Butler 1975) but we shall find advantage in the former. Although not generally recognized as such, Racah's factorization lemma in this case is again well-known, it being implicit in Butler's (1975) equation (9.12). Other various relatives of (10.4) are concealed in similar equations.

We finish by considering one last case; the isoscalar factor for $G \times G \supset G$ has the following group structure:



Clearly, this transformation factor is also a recoupling coefficient for four representations of G.

In these examples certain branchings are associated with the decomposition of Kronecker products and reflect the fact that product multiplicities are just special cases of branching multiplicities.

(b) Antilinear Transformations and the Complex-conjugate Representation

So far, our discussion has been limited to linear unitary transformations of basis. We have noted that although such a transformation induces a change in the representation matrices, given by (10.3), the new representation is regarded as being equivalent to the old one. However, quantum mechanics also admits antiunitary transformations (i.e. ones which are antilinear and unitary) and for these the situation is quite different; in general an antiunitary transformation maps $V_{x\lambda}$ onto a representation space with different group transformation properties. It is necessary for us to consider carefully the relationship between these new spaces and the old ones.

Recall that an operator L_λ is said to be linear if

$$L_\lambda(c_1|x\lambda 1\rangle + c_2|x\lambda 2\rangle) = c_1(L_\lambda|x\lambda 1\rangle) + c_2(L_\lambda|x\lambda 2\rangle) \qquad (10.15)$$

whereas an operator A_λ is said to be antilinear if

$$A_\lambda(c_1|x\lambda 1\rangle + c_2|x\lambda 2\rangle) = c_1^*(A_\lambda|x\lambda 1\rangle) + c_2^*(A_\lambda|x\lambda 2\rangle). \qquad (10.16)$$

Obviously the product of an even number of antilinear operators is a linear operator and the product of a linear operator and an antilinear operator is antilinear. Recall also that the linear form ($\langle v|L_\lambda$) is defined by requiring it to give the same result when acting on the vector $|u_\lambda\rangle$

as the linear form $\langle v|$ acting on $(L_\lambda|u_\lambda\rangle)$. Hence

$$(\langle v|L_\lambda)|u_\lambda\rangle = \langle v|(L_\lambda|u_\lambda\rangle) = \langle v|L_\lambda|u_\lambda\rangle \ . \tag{10.17}$$

However, if we tried to do the same thing with A_λ then $(\langle v|A_\lambda)$ would not be a linear form. Instead we define the linear form $(\langle v|A_\lambda)$ by (e.g. Messiah 1961, vol. II)

$$(\langle v|A_\lambda)|u_\lambda\rangle = [\langle v|(A_\lambda|u_\lambda\rangle)]^* \ . \tag{10.18}$$

The results in this section depend heavily on this equation.

Suppose that A_λ is an antiunitary operator acting on $V_{x\lambda}$. Let us denote by $AV_{x\lambda}$ the space spanned by the vectors $(A_\lambda|x\lambda i\rangle)$ and let us assume that the mapping $|x\lambda i\rangle \to (A_\lambda|x\lambda i\rangle)$ between the spaces is one-to-one and onto. This new space, $AV_{x\lambda}$ need not necessarily be the same as $V_{x\lambda}$ and therefore we define a new set of linear operators

$$O_R^{\lambda *} = K_\lambda \, O_R^\lambda \, K_\lambda^\dagger \qquad \text{for each } R \in G \ , \tag{10.19}$$

which map $AV_{x\lambda}$ onto itself. (K_λ is some particular antiunitary operator yet to be chosen.) These operators clearly have a well-defined action on $AV_{x\lambda}$ since $K_\lambda^\dagger A_\lambda$ is linear. It is trivial to show that the obvious mapping $h_{\lambda *}$ from R,S, ... into $O_R^{\lambda *}$, $O_S^{\lambda *}$, ... is a homomorphism i.e.

$$O_R^{\lambda *} \, O_S^{\lambda *} = O_{RS}^{\lambda *} \ . \tag{10.20}$$

Thus we have a (possibly) new representation.

The matrix elements of $O_R^{\lambda *}$ in the space K $V_{x\lambda}$, spanned by $(K_\lambda|x\lambda i>)$, are

$$(<x\lambda i'|K_\lambda^\dagger)O_R^{\lambda *}(K_\lambda|x\lambda i>)$$

$$= [(<x\lambda i'|K_\lambda^\dagger K_\lambda)O_R^{\lambda}(K_\lambda^\dagger K_\lambda|x\lambda i>)]^*$$

$$= <x\lambda i'|O_R^{\lambda}|x\lambda i>^*$$

$$= \lambda(R)_{i'}{}^{i} \ . \qquad (10.21)$$

It follows easily that for general A_λ the representation matrices of $O_R^{\lambda *}$ in $AV_{x\lambda}$ are related to $\lambda(R)^*$ by a unitary transformation and thus the various $AV_{x\lambda}$ differ only in the choice of basis i.e. all the representations obtained in this way are equivalent. Let us denote this (possibly) new class of representations by λ^* and denote its representation vector space by $V_{\bar{x}\lambda^*}$. (Note that in general we must use a parentage label different from x. Also note that it is not a foregone conclusion that all antilinear operators will give rise to the same parentage $\bar{x}$. Usually it will not matter what $\bar{x}$ is, just as it does not usually matter what x is; however, where confusion can arise, the simplest convention is to stipulate that $\bar{x}$ is that parentage obtained by the action of K_λ on $V_{x\lambda}$.) We call λ^* the representation complex-conjugate to λ. Clearly the characters are related by

$$\chi^{\lambda*}(R) = \chi^{\lambda}(R)^* \quad . \tag{10.22}$$

It follows immediately from the character criterion for irreducibility that λ^* is irreducible if and only if λ is. Obviously λ and λ^* are of the same dimension. Finally we note that λ and λ^* are equivalent if and only if $\chi^{\lambda}(R)$ is real for all R. If λ and λ^* are equivalent we call λ real, otherwise we call it complex.

Of course the theory of the complex-conjugate representation is well documented and understood. However, the way in which it has been introduced here is novel and some interesting and very useful information can be extracted from this approach; so let us proceed. Since the product of two antiunitary transformations is linear we must stipulate that

$$O_R^{\lambda} = K_{\lambda*} K_{\lambda} O_R^{\lambda} K_{\lambda}^{\dagger} K_{\lambda*}^{\dagger} \tag{10.23}$$

otherwise the theory would not make sense. (We also stipulate that $K_{\lambda*}K_{\lambda}$ does not affect the parentage.) It then follows by Schur's lemma that

$$K_{\lambda*}K_{\lambda} = \phi_{\lambda} I \tag{10.24}$$

where I is the identity operator and ϕ_{λ} a constant. In addition though the unitarity of $K_{\lambda*}$ and K_{λ} means that $K_{\lambda*}K_{\lambda}$ is norm-preserving and thus

$$\phi_\lambda^* \phi_\lambda = +1 \tag{10.25}$$

so that ϕ_λ is simply a phase factor. Further, it follows immediately from (10.24) that

$$K_{\lambda *} = \phi_\lambda K_\lambda^\dagger \quad . \tag{10.26}$$

On writing the corresponding equation for K_λ and substituting (10.26) for $K_{\lambda *}$ we find that we must have

$$\phi_{\lambda *} \phi_\lambda = +1. \tag{10.27}$$

(Remember that because of the antilinearity of K_λ, the adjoint of $K_{\lambda *}$ is equal to $\phi_\lambda K_\lambda$ rather than $\phi_\lambda^* K_\lambda$.) Comparing this with (10.25) we immediately deduce that ϕ_λ and $\phi_{\lambda *}$ are related by

$$\phi_{\lambda *} = \phi_\lambda^* \quad . \tag{10.28}$$

These relationships are restrictive but the operator K_λ is still not uniquely defined. It is most convenient to (arbitrarily) complete its specification later in conjunction with certain phase choices. For the time being we need simply regard it as being fixed but unknown.

The operator K_λ can be used to relate the basis vectors of $V_{\bar{x}\lambda *}$ to the basis vectors in $V_{x\lambda}$. (In fact this is one of the main reasons for introducing it.) Let us choose a standard basis for $V_{\bar{x}\lambda *}$ and for the time being let us denote

our standard basis vectors by $|\bar{x}\lambda^{*}\bar{i}>$. (Note that if λ is real we have no such choice; $|\bar{x}\lambda^{*}\bar{i}> \equiv |\bar{x}\lambda i>$.) We have the following general transformation between the standard basis vectors and the basis vectors $(K_\lambda|x\lambda i>)$

$$|\bar{x}\lambda^{*}\bar{i}> = \sum_{i'} (K_\lambda|x\lambda i'>)(<\lambda i'|K_\lambda^{\dagger})|\lambda^{*}\bar{i}>$$

$$= \sum_{i'} (K_\lambda|x\lambda i'><\lambda^{*}>_{i'\bar{i}} \qquad (10.29)$$

where the shorthand notation $<\lambda^{*}>_{i'\bar{i}}$ has been introduced for the coefficients $(<\lambda i'|K_\lambda^{\dagger})|\lambda^{*}\bar{i}>$ which perform a (linear) unitary transformation of basis in $V_{\bar{x}\lambda^{*}}$. These coefficients can also be regarded as the "matrix elements" of the operator K_λ evaluated between the standard bases. Later we shall be able to use this interpretation to complete the specification of K_λ by arbitrarily choosing these coefficients subject to certain restraints. These restraints derive from the properties of the coefficients: their obvious unitarity

$$\sum_{i} <\lambda^{*}>^{i\bar{i}'}<\lambda^{*}>_{i\bar{i}"} = \delta^{\bar{i}'}{}_{\bar{i}"}$$

$$\sum_{i} <\lambda^{*}>^{i'\bar{i}}<\lambda^{*}>_{i"\bar{i}} = \delta^{i'}{}_{i"} \qquad (10.30)$$

and their symmetry

$$\begin{aligned}
\langle\lambda^*\rangle_{i',\bar{i}} &= (\langle\lambda i'|K_\lambda^\dagger)|\lambda^*\bar{i}\rangle \\
&= [\langle\lambda^*\bar{i}|(K_\lambda|\lambda i'\rangle)]^* \\
&= (\langle\lambda^*\bar{i}|K_\lambda)|\lambda i'\rangle \\
&= \phi_\lambda(\langle\lambda^*\bar{i}|K_{\lambda^*}^\dagger)|\lambda i'\rangle \\
&= \phi_\lambda\langle\lambda\rangle_{\bar{i}i'} \quad . \qquad (10.31)
\end{aligned}$$

It follows from (10.19) and (10.29) that the representation matrices $\lambda^*(R)$ are given in terms of $\lambda(R)$ by (cf. 10.3)

$$\lambda^*(R)^{\bar{i}'}{}_{\bar{i}} = \sum_{i''i'''} \langle\lambda^*\rangle^{i''\bar{i}'} \lambda(R)_{i''}{}^{i'''} \langle\lambda^*\rangle_{i'''\bar{i}} \quad . \qquad (10.32)$$

It can be seen that the coefficients $\langle\lambda^*\rangle_{i',\bar{i}}$ play a role somewhat akin to that of a metric tensor in raising and lowering indices. Evidently these coefficients are related by a phase factor to the 2jm symbols $(\lambda)_{i',\bar{i}}$ (see Butler 1975, Derome and Sharp 1965). For the time being however, we shall preserve the distinction and persevere with the different notation; $\langle\lambda^*\rangle_{i',\bar{i}}$ can be conveniently called a conjugation coefficient. The reasons for doing this have already been alluded to and will become clear later on.

It is obvious from (10.28) that for real representations $\phi_\lambda = \pm 1$ while for complex ones the only restraint is its norm (cf. 11.8). Wigner (1959) has shown that for real

representations whose representation matrices can be transformed into real form, $\phi_\lambda = +1$ (we call these representations orthogonal) while for all other real representations $\phi_\lambda = -1$ (and we call these representations symplectic). Thus for real representations ϕ_λ is the same as the Frobenius-Schur (1906) invariant.

This approach to complex-conjugate representations has some advantages over the standard one and these are amply evident when we consider the properties of transformation coefficients. The coefficient $\langle\lambda^*\bar{i}'|\lambda^*\bar{\hat{i}}\rangle$ can be expressed in terms of transformation coefficients in $V_{x\lambda}$ by applying (10.29). We have

$$\langle\lambda^*\bar{i}'|\lambda^*\bar{\hat{i}}\rangle$$

$$= \sum_{i''i'''} \langle\lambda^*\rangle^{i''\bar{i}'} \left(\langle\lambda i''|K_\lambda^\dagger\right)\left(K_\lambda|\lambda\hat{i}'''\rangle\right)\langle\lambda^*\rangle_{\hat{i}'''\bar{\hat{i}}}$$

$$= \sum_{i''i'''} \langle\lambda^*\rangle^{i''\bar{i}'} \left[\langle\lambda i''|\left(K_\lambda^\dagger K_\lambda\right)|\lambda\hat{i}'''\rangle\right]^*\langle\lambda^*\rangle_{\hat{i}'''\bar{\hat{i}}}$$

$$= \sum_{i''i'''} \langle\lambda^*\rangle^{i''\bar{i}'} \langle\lambda i''|\lambda\hat{i}'''\rangle^*\langle\lambda^*\rangle_{\hat{i}'''\bar{\hat{i}}} \quad . \tag{10.33}$$

Using (10.30) to invert this we obtain a generalization of the celebrated Derome-Sharp lemma (cf. Stedman 1976)

$$\langle\lambda i'|\lambda\hat{i}\rangle^* = \sum_{i''i'''} \langle\lambda^*\rangle_{i'\bar{i}''}\langle\lambda^*\bar{i}''|\lambda^*\bar{\hat{i}}'''\rangle\langle\lambda^*\rangle^{\hat{i}\bar{\hat{i}}'''} \quad . \tag{10.34}$$

It is important to appreciate the notation here; if the bases are only related via (10.32) then an additional

phase can enter. A rearrangement of this equation gives the transformation properties of the conjugation coefficients under a change of basis:

$$\langle\lambda^*\rangle_{\hat{i}',\bar{\hat{i}}} = \sum_{i''i'''} \langle\lambda i''|\lambda\hat{i}'\rangle \langle\lambda^*\rangle_{i''\bar{i}'''}\langle\lambda^*\bar{i}'''|\lambda^*\bar{\hat{i}}\rangle \quad . \tag{10.35}$$

Consider now what happens when we choose a subgroup basis for $V_{x\lambda}$. Since the conjugation coefficient $\langle\lambda^*\rangle_{a'\mu'i',\bar{a}\ \bar{\mu}\ \bar{i}}$ is simply a transformation coefficient in $V_{\bar{x}\lambda^*}$ it is possible to immediately define a transformation factor by analogy with (10.9). However, it is more useful to define a slightly different factor by exploiting the relationship between K_λ and K_μ. The operator K_μ chosen in accordance with (10.19) must map $V_{x\lambda a\mu}$ onto the space $V_{\bar{x}\lambda^*\bar{a}\mu^*}$ where $\bar{x}$ and $\bar{a}$ are as previously defined. However, K_λ may map $V_{x\lambda a\mu}$ onto a different space $K_\lambda V_{x\lambda a\mu} \equiv V_{\bar{x}\lambda^* a^K\lambda\mu^*}$. Note that we require K_μ to map $V_{x\lambda}$ onto the same space $V_{\bar{x}\lambda^*}$ as does K_λ. (This is similar to our requirement that $K_{\lambda^*}K_\lambda$ does not affect the parentage.) Now the group operators $O_R^{\mu^*}$, where $R \in H$, may be defined either by (10.19) or as a restriction of $O_R^{\lambda^*}$ in keeping with (10.5):

$$O_R^{\lambda^*}|\bar{x}\lambda^*\bar{a}\mu^*\bar{i}\rangle = O_R^{\mu^*}|\bar{x}\lambda^*\bar{a}\mu^*\bar{i}\rangle \quad .$$

These two definitions must of course be consistent and the consequences of this can be seen by applying (10.19) to both sides of the above equation. Firstly the left hand side becomes

$$(K_\lambda O_R^\lambda K_\lambda^\dagger)\,|\bar{x}\lambda^*\bar{a}\mu^*\bar{i}>$$

$$= \sum_{a'\mu'i'} (K_\lambda O_R^\lambda |x\lambda a'\mu'i'>)(<\lambda a'\mu'i'|K_\lambda^\dagger)\,|\lambda^*\bar{a}\mu^*\bar{i}>$$

$$= \sum_{a'\mu'i'} (K_\lambda O_R^{\mu'} |x\lambda a'\mu'i'>)(<\lambda a'\mu'i'|K_\lambda^\dagger)\,|\lambda^*\bar{a}\mu^*\bar{i}>$$

$$= \sum_{\substack{a'\mu'i''' \\ a''\mu''^*i''}} |\bar{x}\lambda^*\bar{a}''\mu''^*\bar{i}''><\lambda^*\bar{a}''\mu''^*\bar{i}''|\,(K_\lambda\,|\lambda a'\mu'i'''>) \times \mu'(R)_{i'''}^{\ i'}\,(<\lambda a'\mu'i'|K_\lambda^\dagger)\,|\lambda^*\bar{a}\mu^*\bar{i}>$$

whereas the right hand side reduces to

$$(K_\mu\ O_R^\mu\ K_\mu^\dagger)\,|\bar{x}\lambda^*\bar{a}\mu^*\bar{i}>$$

$$= \sum_{i'i'''i''} |\bar{x}\lambda^*\bar{a}\mu^*\bar{i}''><\lambda^*\bar{a}\mu^*\bar{i}''|\,(K_\mu\,|\lambda a\mu i'''>) \times \mu(R)_{i'''}^{\ i'}\,(<\lambda a\mu i'|K_\mu^\dagger)\ |\lambda^*\bar{a}\mu^*\bar{i}>\ .$$

Equating coefficients of $|\bar{x}\lambda^*\bar{a}''\mu''^*\bar{i}''>$ we can deduce that

$$\sum_{i''} \mu'(R)^{i'}_{\ i''}\ <\lambda a'\mu'i''|\,(K_\lambda^\dagger K_\mu)\,|\lambda a\mu i>$$

$$= \sum_{i'''} <\lambda a'\mu'i'|\,(K_\lambda^\dagger K_\mu)\,|\lambda a\mu i'''>\mu(R)^{i'''}_{\ i}$$

and therefore by Schur's lemmas the operator $(K_\lambda^\dagger K_\mu)$ is diagonal in μ and i and its "diagonal" elements are independent of i. Hence we may conveniently write its matrix elements as

$$\langle \lambda a'\mu'i' | (K_\lambda^\dagger K_\mu) | \lambda a\mu i \rangle = \delta^{\mu'}{}_{\mu}\, \delta^{i'}{}_{i} \langle \lambda a'\mu | (K_\lambda^\dagger K_\mu) | \lambda a\mu \rangle \ . \quad (10.36)$$

Note that $K_\lambda^\dagger K_\mu$ need not be diagonal in the branching multiplicity label a and the arguments following equation (10.6) apply here also.

These factors may now be used to relate the conjugation coefficients $\langle \lambda^* \rangle_{a'\mu'i',\bar{a}\bar{\mu}\bar{i}}$ to the conjugation coefficients $\langle \mu^* \rangle_{i'\bar{i}}$ (from the discussion following (10.21) it is clear that $\bar{\mu} \equiv \mu^*$): we simply insert the identity in $(\langle \lambda a'\mu'i' | K_\lambda^\dagger) | \lambda^* \bar{a}\mu^* \bar{i} \rangle$ and easily show that

$$(\langle \lambda a'\mu'i' | K_\lambda^\dagger K_\mu K_\mu^\dagger) | \lambda^* \bar{a}\mu^* \bar{i} \rangle$$

$$= \sum_{a''} \langle \lambda a''\mu' | (K_\mu^\dagger K_\lambda) | \lambda a'\mu' \rangle (\langle \lambda a''\mu'i' | K_\mu^\dagger) | \lambda^* \bar{a}\mu^* \bar{i} \rangle \ .$$

But by definition

$$(\langle \lambda a''\mu'i' | K_\mu^\dagger) | \lambda^* \bar{a}\mu^* \bar{i} \rangle = \delta_{\bar{a}''}{}^{a}\, \delta_{\mu'}{}^{\mu} \langle \mu^* \rangle_{i'\bar{i}} \ .$$

Recalling that $\langle \lambda a\mu | (K_\mu^\dagger K_\lambda) | \lambda a'\mu \rangle$ is a transformation factor in $V_{\bar{x}\lambda^*}$ we introduce the shorthand notation

$$\langle \lambda^* \rangle_{a'\mu,\bar{a}\mu^*} \equiv \langle \lambda a\mu | (K_\mu^\dagger K_\lambda) | \lambda a'\mu \rangle \quad (10.37)$$

and thus arrive at the important factorization

$$\langle \lambda^* \rangle_{a'\mu'i',\bar{a}\bar{\mu}\bar{i}} = \langle \lambda^* \rangle_{a'\mu,\bar{a}\mu^*}\, \delta_{\mu'}{}^{\mu} \langle \mu^* \rangle_{i'\bar{i}} \ . \quad (10.38)$$

We call $<\lambda^*>_{a'\mu,\bar{a}\mu^*}$ a conjugation factor. (In case the reader is concerned that $\bar{a}$ does not appear explicitly in the right hand side of (10.37) he is reminded that it is implicit in the action of K_μ.) These factors obey the unitary conditions

$$\sum_a <\lambda^*>^{a\mu,\bar{a}'\mu^*} <\lambda^*>_{a\mu,\bar{a}''\mu^*} = \delta^{\bar{a}'}{}_{\bar{a}''}$$

$$\sum_a <\lambda^*>^{a'\mu,\bar{a}\mu^*} <\lambda^*>_{a''\mu,\bar{a}\mu^*} = \delta^{a'}{}_{a''} \qquad (10.39)$$

and possess the symmetry

$$<\lambda^*>_{a'\mu,\bar{a}\mu^*} = \phi_\lambda \, \phi_{\mu^*} <\lambda>_{\bar{a}\mu^*,a'\mu} \, . \qquad (10.40)$$

This last property is easily deduced from (10.38) and (10.31) but of course it must also follow directly from the definition. That the symmetry property does correctly follow from (10.37) can be demonstrated by using (10.26) as follows:

$$<\lambda a\mu|(K_\mu^\dagger K_\lambda)|\lambda a'\mu> = <\lambda a\mu i|(K_\mu^\dagger K_\lambda)|\lambda a'\mu i>$$

$$= \sum_{i'i''} <\mu>^{\bar{i}''i} (<\lambda^*\bar{a}\mu^*\bar{i}''|K_{\mu^*}^\dagger)(K_\mu^\dagger K_\lambda)(K_{\mu^*}|\lambda^*\bar{a}'\mu^*\bar{i}'>) <\mu>_{\bar{i}'i}$$

$$= \sum_{i'i''} <\mu>^{\bar{i}''i} \phi_\lambda \phi_{\mu^*} <\lambda^*\bar{a}'\mu^*\bar{i}'|(K_{\mu^*}^\dagger K_{\lambda^*})|\lambda^*\bar{a}\mu^*\bar{i}''> <\mu>_{\bar{i}'i}$$

$$= \phi_\lambda \phi_{\mu^*} <\lambda^*\bar{a}'\mu^*|(K_{\mu^*}^\dagger K_{\lambda^*})|\lambda^*\bar{a}\mu^*> \sum_{i'} <\mu>^{\bar{i}'i} <\mu>_{\bar{i}'i}$$

The summation is of course unity and the result is proven, thus providing an important consistency check.

The conjugation factor can be used to obtain a version of the generalized Derome-Sharp lemma for a transformation factor. For instance, remembering that $\langle \lambda c \nu d \xi | \lambda a \mu b \xi \rangle \equiv \langle \lambda c \nu d \xi i | \lambda a \mu b \xi i \rangle$ we readily deduce that

$$\langle \lambda c \nu d \xi | \lambda a \mu b \xi \rangle^*$$

$$= \sum_{a'b'c'd'} \langle \lambda^* \rangle_{c\nu,\bar{c}'\nu^*} \langle \nu^* \rangle_{d\xi,\bar{d}'\xi^*}$$

$$\times \langle \lambda^* \bar{c}' \nu^* \bar{d}' \xi^* | \lambda^* \bar{a}' \mu^* \bar{b}' \xi^* \rangle \langle \lambda^* \rangle^{a\mu,\bar{a}'\mu^*} \langle \mu^* \rangle^{b\xi,\bar{b}'\xi^*} . \qquad (10.41)$$

These results, while very important in their own right, enable us to simplify and slightly extend the complex-conjugation properties of the 3jm and 6j symbols discussed in the next section.

Before moving on, it perhaps ought to be stated that the representation conjugation operator K_λ is not the usual complex-conjugation operator K_o since that operator is always defined as having an invariant action on the basis vectors (e.g. Messiah 1961 vol. II) and cannot be generalized to complex representations; furthermore, $K_\lambda^2 \neq I$. Failure to fully appreciate the inherent property of K_λ in transforming vectors in $V_{x\lambda}$ to vectors in $V_{\bar{x}\lambda^*}$ (combined probably with an aversion to such operators), and the consequent properties (10.24) to (10.27), is perhaps the main reason why previous

authors have not obtained these results — the antilinear relationship between $V_{x\lambda}$ and $V_{\bar{x}\lambda^*}$ being widely known and commented upon. It should also be made quite clear that K_λ is not the time-reversal operator, though for some representation spaces it may mimic that operator (cf. Stedman and Butler, 1980).

CHAPTER 11

PHASE FREEDOM IN TRANSFORMATION COEFFICIENTS

The transformation coefficients which transform one basis into another specify the relationship between the different bases. Obviously, if both bases are fully specified then the transformation coefficients are fixed. However, even if the bases are not fully specified, provided that the relationship between the bases is known, the transformation coefficients are still uniquely determined. This may seem to be a very trivial statement but it has far-reaching practical consequences since it means that we may calculate transformation coefficients while possessing only a limited knowledge of the bases. At present though we are interested in cases where our knowledge of the bases is so poor that we do not even know the relationship between them. In such instances the value of the transformation coefficient is unknown. We may perhaps have enough information to be able to fix its norm; perhaps not. It is the purpose of this chapter to examine the transformation coefficients, and related entities which arise in the Wigner-Racah algebra and to accurately describe any indeterminateness from an elementary point of view. It is this indeterminateness which we refer to as phase freedom. The possibility that this indeterminateness can amount to more than a phase may irk the purist but little harm will be done by using the special term in the general case.

The question we must now address is: to what extent does a subgroup labelling scheme specify the representation space vectors and what effect does any indeterminateness have on the transformation coefficients? First let us consider the parentage label. We have stated earlier that the transformation coefficients are independent of parentage. This claim requires justification and the reasoning is as follows. We begin with a transformation of basis in $V_{x\lambda}$ given by

$$|x\lambda\hat{i}\rangle = \sum_{i'} |x\lambda i'\rangle\langle x\lambda i'|x\lambda\hat{i}\rangle \quad .$$

Clearly the transformation coefficients are uniquely determined once the two sets of basis vectors are specified; there is no room here for any choice. Now consider a set of basis vectors $\{|y\lambda i\rangle\}$ in $V_{y\lambda}$, related by one-to-one correspondence to the set $\{|x\lambda i\rangle\}$, which give rise to the same representation matrices as the set $\{|x\lambda i\rangle\}$. Consider also another set of basis vectors $\{|y\lambda\hat{i}\rangle\}$ which give rise to the same representation matrices as the set $\{|x\lambda\hat{i}\rangle\}$. Of course this criterion does not uniquely determine the vectors $|y\lambda\hat{i}\rangle$ since any overall phase change will lead to the same representation matrices. However, consider for the moment one such set. The transformation in $V_{y\lambda}$ between these two sets, corresponding to the above transformation in $V_{x\lambda}$, gives rise to the coefficients $\langle y\lambda i'|y\lambda\hat{i}\rangle$. Schur's lemma, combined with the unitarity of the transformations, implies that the coefficients $\langle x\lambda i'|x\lambda\hat{i}\rangle$ and $\langle y\lambda i'|y\lambda\hat{i}\rangle$ can differ by at most a phase which must be independent of i' and $\hat{i}$, i.e.

$$\langle x\lambda i'|x\hat{\lambda i}\rangle = e^{i\Theta}\,\langle y\lambda i'|y\hat{\lambda i}\rangle \ .$$

However, if we multiply each of the vectors $|y\hat{\lambda i}\rangle$ by $e^{-i\Theta}$ we obtain a new basis (note that this is properly regarded as a basis transformation and not a change in parentage) with the same representation matrices as before. For this basis the transformation coefficients linking with the old basis $\{|y\lambda i\rangle\}$ are the same as those in $V_{x\lambda}$. It is these new basis vectors which we should denote by $|y\hat{\lambda i}\rangle$. (Note that only the relative phase between the bases is important.) This explains why a transformation of basis is said to be performed independently of parentage; the statement merely reflects what we mean when we use the same notation for bases in spaces with different parentage.

We now move onto the representation label. Whether a vector belongs to a representation space of λ_1 or of λ_2 can be determined by standard projection techniques. The representations themselves are uniquely specified by their characters for each class of the group operations. Admittedly, it is often the case that the distinction between classes is arbitrary but once a convention has been decided upon the specification of the representations can be uniquely carried out. Note that projection techniques tell us nothing about the parentage.

We come now to the basis labels for $V_{x\lambda}$. There will automatically be a gap in our knowledge of the basis vectors when we choose a subgroup labelling scheme. The representation labels of subgroups merely specify the existence of a subspace

of $V_{x\lambda}$ with certain subgroup transformation properties. Because representation matrices are independent of parentage, subgroup representation labels do not provide any information on how this subspace is embedded in $V_{x\lambda}$ and are simply not sufficient. It is the branching multiplicity label which completes the specification of this subspace yet there is hardly ever any reason, and certainly never any necessity, to choose one resolution of branching multiplicity over another. Therefore the branching multiplicity must be resolved in an entirely arbitrary manner. However, for orthonormal bases the arbitrariness is restricted to a unitary transformation in the branching multiplicity index of the form (cf. 10.6, the derivation of which fully verifies the claim)

$$|x\lambda\hat{a}\mu i\rangle = \sum_{a'} |x\lambda a'\mu i\rangle\langle\lambda a'\mu|\lambda\hat{a}\mu\rangle$$

$$= \sum_{a'} |x\lambda a'\mu i\rangle U(\lambda,\mu)^{a'}{}_{a} \quad . \tag{11.1}$$

Clearly, until some resolution of branching multiplicity is made, the transformation coefficients between bases will also display this arbitrariness. At this point it should be emphasized that the branching multiplicity index, as used here, is more than just an integer for labelling the multiplicity of μ in the decomposition of λ. Even when this multiplicity is one it is obvious from the unitary properties of the transformation matrix $U(\lambda,\mu)$ that there still remains a complete phase ambiguity $e^{i\theta(\lambda,\mu)}$ in the

basis vector, which must be removed by specifying a particular embedding.

Finally we come to the arbitrariness in the basis for $V_{x\lambda a\mu}$. Clearly this is described by the transformation coefficients $\langle \mu i' | \hat{\mu i} \rangle$. However, the label i is usually replaced by further subgroup labels until a one-dimensional representation space is reached. For each branching in the group-subgroup chain there will exist an arbitrariness of the form (11.1). Note though that even when the one-dimensional subspace is reached there is still a phase specification required to determine its basis. Once all of these specifications have been carried out the basis vectors $|x\lambda a\mu i\rangle$ are unique.

Let us now consider the consequent phase freedom in various special transformation coefficients.

(a) Transformation Factors

The phase freedom in a transformation factor, such as that occurring in (10.10), is easily deduced. Suppose that we make the following changes in the basis vectors

$$|x\lambda \hat{a}\mu \hat{b}\xi i\rangle = \sum_{a'b'} |x\lambda a'\mu b'\xi i\rangle U(\lambda,\mu)^{a'}{}_{a} U(\mu,\xi)^{b'}{}_{b}$$

and

$$|x\lambda \hat{c}\nu \hat{d}\xi i\rangle = \sum_{c'd'} |x\lambda c'\nu d'\xi i\rangle U(\lambda,\nu)^{c'}{}_{c} U(\nu,\xi)^{d'}{}_{d}$$

then

$$\langle \lambda \hat{e} \nu \hat{d} \xi | \lambda \hat{a} \mu \hat{b} \xi \rangle$$

$$= \sum_{a'b'c'd'} U(\lambda,\nu)_{c'}{}^{c} U(\nu,\xi)_{d'}{}^{d} \langle \lambda c' \nu d' \xi | \lambda a' \mu b' \xi \rangle$$

$$\times U(\lambda,\mu)^{a'}{}_{a} U(\mu,\xi)^{b'}{}_{b} \qquad (11.2)$$

So the freedom in a transformation factor of this type is a product of coefficients of four unitary matrices, each associated with one of the branching multiplicities. (Note that matrices of the form $U(\lambda,\xi)$ are too general.) To fix the value of this transformation factor it is necessary to fix the relative resolutions of these four multiplicities. It should be clear that in general all of the multiplicity resolutions will be fixed relative to one another once a certain number of transformation factors are known; the remaining transformation factors must then be unique.

The situation can be slightly more complicated in an extended problem where more than one set of transformation factors is present. The freedom in a single factor is still given by (11.2) but one must be careful to check whether or not the relative multiplicity separation is fixed by other different transformation coefficients. For instance the product multiplicities that arise in the overlap integrals for G also arise in the coupling coefficients for G and one must be careful to ensure that the coupling coefficients are consistent with the relative resolutions of multiplicity specified by the overlap integrals. Such considerations are very elementary but because of the difficulty in keeping track of which resolutions are related and which are not it pays to spell them out.

(b) Conjugation Factors

The phase freedom in a conjugation coefficient is given by (10.35). Of more direct interest to us are the conjugation factors. It can be readily deduced from (10.35) that

$$<\lambda^*>_{\hat{a}'\mu,\bar{\hat{a}}\mu^*} = \sum_{a''a'''} U(\lambda,\mu)^{a''}{}_{a'} <\lambda^*>_{a''\mu,\bar{a}'''\mu^*} U(\lambda^*,\mu^*)^{\bar{a}'''}{}_{\bar{a}} \quad . \tag{11.3}$$

Note that the generalized Derome-Sharp lemma does not imply any relationship whatsoever between $U(\lambda,\mu)$ and $U(\lambda^*,\mu^*)$ until the conjugation factors have been fixed.

Later we shall find it advantageous to have (11.3) rewritten in matrix form. To accomplish this let $A(\lambda^*,\mu^*)$ be the unitary matrix of conjugation factors such that

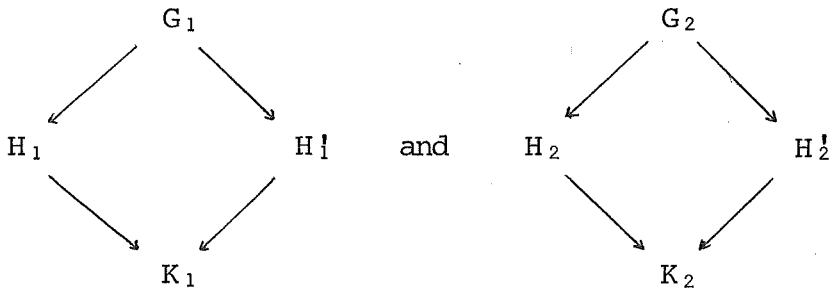
$$<\lambda^*>_{a'\mu,\bar{a}\mu^*} \equiv A(\lambda^*,\mu^*)^{a'^{K_\lambda}}{}_{a^{K_\lambda}} \quad . \tag{11.4}$$

(The K_λ appended to the indices a' and a is not really necessary here but it serves to denote the basis in which the corresponding unitary operator is evaluated, cf. 10.2a and 10.29, and thus plays a role analogous to $\hat{}$ and $\bar{}$.) The matrix form of (11.3) can then be written as

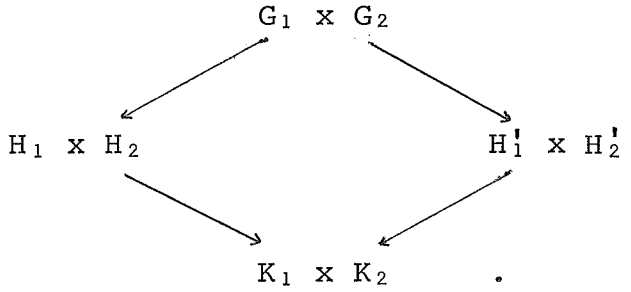
$$\hat{A}(\lambda^*,\mu^*) = U(\lambda,\mu)^{T} A(\lambda^*,\mu^*) U(\lambda^*,\mu^*) \quad . \tag{11.5}$$

(c) Direct-product Groups

Direct-product groups arise frequently in physical problems. We would like to know therefore what relationship exists between the transformation factors for the group schemes



and the transformation factors for the scheme



(G_1 and G_2 etc. need not necessarily be the same groups.) In general this relationship would not be simple but the freedom in the multiplicity separations for the direct-product scheme always allows us to turn the relationship into a trivial one. Consider the tensor product of the spaces $V_{x_1\lambda_1}$ and $V_{x_2\lambda_2}$. We can write

$$|x_1\lambda_1 a_1\mu_1 b_1\xi_1 i_1\rangle|x_2\lambda_2 a_2\mu_2 b_2\xi_2 i_2\rangle$$

$$\equiv |x_1\lambda_1 a_1\mu_1 b_1\xi_1 i_1; x_2\lambda_2 a_2\mu_2 b_2\xi_2 i_2\rangle$$

$$\equiv |(x_1x_2)(\lambda_1\lambda_2)(a_1a_2)(\mu_1\mu_2)(b_1b_2)(\xi_1\xi_2)(i_1i_2)\rangle$$

and similarly

$$|x_1\lambda_1 c_1\nu_1 d_1\xi_1 i_1\rangle|x_2\lambda_2 c_2\nu_2 d_2\xi_2 i_2\rangle$$

$$\equiv |x_1\lambda_1 c_1\nu_1 d_1\xi_1 i_1; x_2\lambda_2 c_2\nu_2 d_2\xi_2 i_2\rangle$$

$$\equiv |(x_1x_2)(\lambda_1\lambda_2)(c_1c_2)(\nu_1\nu_2)(d_1d_2)(\xi_1\xi_2)(i_1i_2)\rangle \quad .$$

Now these product states form bases for the product space $V_{(x_1\lambda_1;x_2\lambda_2)} \equiv V_{(x_1x_2)(\lambda_1\lambda_2)}$. Such bases may be termed outer-product bases. Clearly, we can always choose an outer-product basis for $V_{(x_1x_2)(\lambda_1\lambda_2)}$. In particular though we can always choose the multiplicity separations so that

$$|(x_1x_2)(\lambda_1\lambda_2)a(\mu_1\mu_2)b(\xi_1\xi_2)i\rangle$$

$$= |(x_1x_2)(\lambda_1\lambda_2)(a_1a_2)(\mu_1\mu_2)(b_1b_2)(\xi_1\xi_2)(i_1i_2)\rangle$$

and

$$|(x_1x_2)(\lambda_1\lambda_2)\ c\ (\nu_1\nu_2)d(\xi_1\xi_2)i\rangle$$

$$= |(x_1x_2)(\lambda_1\lambda_2)(c_1c_2)(\nu_1\nu_2)(d_1d_2)(\xi_1\xi_2)(i_1i_2)\rangle \quad .$$

(This justifies some of the statements made in chapter 10 regarding isoscalar factors.) If we do not then there is a transformation in the multiplicity indices which will bring us to such bases. With these basis choices it is trivial to show that

$$<(\lambda_1\lambda_2)(a_1a_2)(\mu_1\mu_2)(b_1b_2)(\xi_1\xi_2)\,|\,(\lambda_1\lambda_2)(c_1c_2)(\nu_1\nu_2)(d_1d_2)(\xi_1\xi_2)>$$

$$= <\lambda_1 a_1\mu_1 b_1\xi_1\,|\,\lambda_1 c_1\nu_1 d_1\xi_1><\lambda_2 a_2\mu_2 b_2\xi_2\,|\,\lambda_2 c_2\nu_2 d_2\xi_2> \ . \qquad (11.6)$$

In addition it is clear that such a choice of basis yields

$$<\lambda_1\lambda_2>(\bar{a}_1'\bar{a}_2')(\mu_1^*\mu_2^*),(a_1a_2)(\mu_1\mu_2) = <\lambda_1>\bar{a}_1'\mu_1^*,a_1\mu_1\,<\lambda_2>\bar{a}_2'\mu_2^*,a_2\mu_2 \qquad (11.7)$$

and consequently

$$\phi_{\lambda_1\lambda_2} = \phi_{\lambda_1}\phi_{\lambda_2} \qquad (11.8)$$

— the last equation clearly being true regardless of basis. (The reader may be puzzled that (11.8) seems to imply that $\phi_{\lambda\lambda^*} = +1$ even though for $\lambda \neq \lambda^*$, $\lambda \times \lambda^*$ is a complex representation in the outer product group. However, ϕ_λ does not have to be chosen the same in both G_1 and G_2, even though such a choice would be sensible.) Because of these simple relationships it makes obvious sense to always demand a product resolution of the multiplicities. The freedom associated with the multiplicity separations then (and only then) becomes, for instance

$$U(\lambda_1\lambda_2,\mu_1\mu_2)^{a_1'a_2'}{}_{a_1a_2} = U(\lambda_1,\mu_1)^{a_1'}{}_{a_1}\,U(\lambda_2,\mu_2)^{a_2'}{}_{a_2}. \qquad (11.9)$$

It is important to realize that in demanding a multiplicity resolution of this type one is using some of the available freedom. Therefore the freedom allowed by (11.9) may

actually be less than one would usually expect.

(d) 3jm Symbols and Factors

Although the isoscalar factors and overlap integrals are the fundamental entities appearing in the Wigner-Racah algebra it is useful to define some new (related) quantities, called 3jm factors and 6j symbols, which have higher symmetry. First let us consider the definition of a 3jm symbol. This leads us to the definition of a 3jm factor which is the analogue of an isoscalar factor in the Racah factorization of the 3jm symbol. There are several approaches which may be followed in defining such a quantity. Our aim is to define a 3jm factor in terms of the isoscalar factor and thereby obtain its phase freedom in terms of the known freedom in an isoscalar factor. (Actually a conjugation factor will also arise in the definition but its freedom is known too.) In order to achieve this aim we shall adopt an approach along the lines taken by Wigner (1959).

In the direct-product group G x G, with elements RS (R belonging to the first group and S to the second), we can choose an outer-product basis so that the representation matrices for the irreducible (outer-product) representation $\lambda_1 \times \lambda_2$ are given by

$$\lambda_1\lambda_2(RS)^{i_1'i_2'}_{\quad i_1 i_2} = \lambda_1(R)^{i_1'}_{\quad i_1}\ \lambda_2(S)^{i_2'}_{\quad i_2}. \qquad (11.10)$$

In cases such as this though where the groups are the same it is often more helpful to use a basis symmetrized according

to the subgroup chain $G \times G \supset G$ i.e. an inner-product group basis, in which case the representation matrices are given by (10.5) as

$$\lambda_1\lambda_2(RR)^{r'\lambda'i'}{}_{r\lambda i} = \delta^{r'}{}_{r}\,\delta^{\lambda'}{}_{\lambda}\,\lambda(R)^{i'}{}_{i} \ . \tag{11.11}$$

(In this equation, RR has been replaced by R on the right hand side — this notation is not intended to imply that $R \times R = R$!) The transformation (10.3) between $\lambda_1\lambda_2(RR)^{i_1'i_2'}{}_{i_1i_2}$ and $\lambda_1\lambda_2(RR)^{r'\lambda'i'}{}_{r\lambda i}$ then reads

$$\lambda_1(R)^{i_1'}{}_{i_1}\ \lambda_2(R)^{i_2'}{}_{i_2}$$

$$= \sum_{r\lambda i'i} \langle\lambda_1 i_1'; \lambda_2 i_2' | (\lambda_1\lambda_2) r\lambda i'\rangle \lambda(R)^{i'}{}_{i} \langle(\lambda_1\lambda_2) r\lambda i | \lambda_1 i_1; \lambda_2 i_2\rangle \ . \tag{11.12}$$

The transformation coefficients between the outer-product and inner-product bases are commonly known as coupling coefficients. (Note that an outer-product resolution of basis is implicit in their definition and does not have to be chosen.)

The Great Orthogonality theorem

$$|\lambda| \int_G \lambda(R)^{i'}{}_{i}\ \lambda'(R)_{j'}{}^{j}\ dR = \delta_{\lambda}{}^{\lambda'}\ \delta^{i'}{}_{j'}\ \delta_{i}{}^{j} \tag{11.13}$$

(where dR is the Haar measure) can be used to move $\lambda(R)^{i'}{}_{i}$ to the left of (11.12) but a more symmetric expression is obtained if first we substitute $\sum_{i''i'''} \langle\lambda\rangle^{\bar{i}''i'}\ \lambda^*(R)_{\bar{i}''}{}^{\bar{i}'''} \langle\lambda\rangle_{\bar{i}'''\,i}$

for $\lambda(R)^{i'}{}_{i}$. Then defining a 3jm symbol by

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^* \\ i_1 & i_2 & \bar{i}' \end{pmatrix}^r = \sum_i |\lambda|^{-\frac{1}{2}} \langle\lambda\rangle_{\bar{i}',i} \langle(\lambda_1\lambda_2)r\lambda i|\lambda_1 i_1;\lambda_2 i_2\rangle \qquad (11.14)$$

we can write

$$\lambda_1(R)^{i_1'}{}_{i_1} \lambda_2(R)^{i_2'}{}_{i_2}$$

$$= \sum_{r\lambda_3 i_3 i_3'} |\lambda_3| \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1' & i_2' & i_3' \end{pmatrix}^{r^*} \lambda_3(R)_{i_3'}{}^{i_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^r \qquad (11.15)$$

leading to

$$\int_G \lambda_1(R)^{i_1'}{}_{i_1} \lambda_2(R)^{i_2'}{}_{i_2} \lambda_3(R)^{i_3'}{}_{i_3} \, dR$$

$$= \sum_r \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1' & i_2' & i_3' \end{pmatrix}^{r^*} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^r \qquad (11.16)$$

The definition of the 3jm symbol given here is largely arbitrary. Any unitary transformation in the product multiplicity index on the right hand side of (11.14) would have led to an equally satisfactory 3jm symbol in that (11.15) and (11.16) would have again followed. Many authors (particularly in angular momentum theory) include such a transformation in their definition. (In angular momentum theory this takes the form of a phase which is independent of i_1, i_2 and i' cf. Butler 1975.) It is obvious though that the simple relationship between the indices on the right and left of (11.14) is the most convenient for a discussion of phases. This is essentially Butler's (1975)

"sensible" definition. Also there is no strong reason to order the columns in any particular way. The definition employed here uses the same ordering as Butler (1975) so as to facilitate correspondence with that author's work. It is pointed out however that the ordering $(\lambda^*\lambda_1\lambda_2)$ leads in a more natural fashion to a symmetric form for the Wigner-Eckart theorem (see appendix IIA). It is to be noted that the 3jm symbol defined here has a contravariant product multiplicity label. Butler (1975) erroneously writes it as being covariant (although he later makes special phase choices which render the distinction trivial). We could, if we liked, define a fully covariant 3jm symbol by including a conjugation factor $<\lambda_1\lambda_2>_{\bar{r}'\lambda^*,r\lambda}$ in the definition. Derome and Sharp (1965) do use a fully covariant symbol but since we shall find that one of the two product multiplicity indices in the 3jm factor must be contravariant, it will turn out to be simpler to define a 3jm symbol with a contravariant index. At any rate, if another definition of the 3jm symbol is preferred it is not difficult to obtain the desired symbol by a transformation of the one given here. Finally we note that definitions of the 3jm symbol using (11.15) are not unique. The definition (11.14) is unique even though it incorporates the arbitrariness in the resolution of multiplicities. For this reason it is an ideal starting point for a discussion on phase freedom.

Before defining a 3jm factor and deriving its phase freedom let us consider the complex-conjugation properties of 3jm symbols in the light of our discussion in chapter 10.

Using the generalized Derome-Sharp lemma (10.34) on the coupling coefficients we readily deduce that

$$\langle(\lambda_1\lambda_2)r\lambda i|\lambda_1 i_1;\lambda_2 i_2\rangle^*$$

$$= \sum_{r'i'\bar{i}_1'\bar{i}_2'} \langle\lambda_1\lambda_2\rangle_{\bar{r}'\lambda^*,r\lambda} \langle\lambda\rangle_{\bar{i}'i} \langle\lambda_1\rangle^{\bar{i}_1'i_1} \langle\lambda_2\rangle^{\bar{i}_2'i_2}$$

$$\times \langle(\lambda_1^*\lambda_2^*)\bar{r}'\lambda^*\bar{i}'|\lambda_1^*\bar{i}_1';\lambda_2^*\bar{i}_2'\rangle \qquad (11.17)$$

from which it is a simple step to prove that

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ i_1 & i_2 & i_3\end{pmatrix}\bar{r}\ *$$

$$= \sum_{r'\bar{i}_1'\bar{i}_2'\bar{i}_3'} \langle\lambda_1\lambda_2\rangle_{r'\lambda_3,\bar{r}\lambda_3^*} \langle\lambda_1\rangle^{\bar{i}_1'i_1} \langle\lambda_2\rangle^{\bar{i}_2'i_2} \langle\lambda_3\rangle^{\bar{i}_3'i_3}$$

$$\times \begin{pmatrix}\lambda_1^* & \lambda_2^* & \lambda_3^*\\ \bar{i}_1' & \bar{i}_2' & \bar{i}_3\end{pmatrix}r' \qquad (11.18)$$

This equation is recognized as being the ordinary Derome-Sharp lemma except that in the usual derivation (via equations 10.32 and 11.16 — see Derome and Sharp, 1965) it is necessary to introduce a matrix $A(\lambda_1\lambda_2\lambda_3)$ into the algebra. The derivation given here shows that the A matrix is not a fundamentally new quantity; it is simply a conjugation factor i.e.

$$A(\lambda_1\lambda_2\lambda_3)_{rr'} = \langle\lambda_1\lambda_2\rangle_{r'\lambda_3,\bar{r}\lambda_3^*} \equiv A(\lambda_1\lambda_2,\lambda_3^*)^{r'}{}_r \qquad (11.19)$$

although we note that it may incorporate an additional phase if the bases are only related by (10.32).

If we define a 3jm factor by

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^* \\ a_1\mu_1 & a_2\mu_2 & \bar{a}'\mu^* \end{pmatrix}^{r}_{s} = \sum_{a} |\lambda|^{-\frac{1}{2}} |\mu|^{\frac{1}{2}} \langle\lambda\rangle^{\bar{a}'\mu^*,a\mu} \times \langle(\lambda_1\lambda_2)r\lambda a\mu|(\lambda_1 a_1\mu_1, \lambda_2 a_2\mu_2)s\mu\rangle \tag{11.20}$$

then the Racah factorization lemma can be written in the form (e.g. Butler 1975)

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 i_1 & a_2\mu_2 i_2 & a_3\mu_3 i_3 \end{pmatrix}^{\bar{r}} = \sum_{s} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}^{\bar{r}}_{\bar{s}} \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^{\bar{s}} . \tag{11.21}$$

Because of this factorization property the 3jm factor is a more useful quantity to tabulate than the 3jm symbol. If we use the generalized Derome-Sharp lemma in the form (10.41) on the isoscalar factor in the definition (11.20) it is easy to prove that

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}^{\bar{r}}_{\bar{s}}{}^{*} = \sum_{r'a_1'a_2'a_3's'} \langle\lambda_1\lambda_2\rangle^{r'\lambda_3,\bar{r}\lambda_3^*} \langle\lambda_1\rangle^{\bar{a}_1'\mu_1^*,a_1\mu_1} \langle\lambda_2\rangle^{\bar{a}_2'\mu_2^*,a_2\mu_2} \langle\lambda_3\rangle^{\bar{a}_3'\mu_3^*,a_3\mu_3} \times \langle\mu_1\mu_2\rangle^{s'\mu_3,\bar{s}\mu_3^*} \begin{pmatrix} \lambda_1^* & \lambda_2^* & \lambda_3^* \\ \bar{a}_1'\mu_1^* & \bar{a}_2'\mu_2^* & \bar{a}_3'\mu_3^* \end{pmatrix}^{r'}_{s'} . \tag{11.22}$$

This important equation can also be derived from (11.18) and (11.21) and is the generalization of the symmetry given by Butler and Wybourne (1976a).

We now return to the phase problem. We shall have occasion to use the freedom in a 3jm symbol. In this case a change in basis is described by

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \hat{i}_1 & \hat{i}_2 & \hat{i}_3 \end{pmatrix}^{\hat{\bar{r}}}$$

$$= \sum_{r'i_1'i_2'i_3'} U(\lambda_1\lambda_2,\lambda_3^*)_{\bar{r}',}^{\bar{r}} U(\lambda_1)^{i_1'}{}_{i_1} U(\lambda_2)^{i_2'}{}_{i_2} U(\lambda_3)^{i_3'}{}_{i_3}$$

$$\times \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1' & i_2' & i_3' \end{pmatrix}^{\bar{r}'} . \qquad (11.23)$$

The freedom in a 3jm factor follows directly from its definition by use of (11.2),(11.3) and (11.9). We obtain the result

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \hat{a}_1\mu_1 & \hat{a}_2\mu_2 & \hat{a}_3\mu_3 \end{pmatrix}^{\hat{\bar{r}}}_{\hat{\bar{s}}}$$

$$= \sum_{r'a_1'a_2'a_3's'} U(\lambda_1,\mu_1)^{a_1'}{}_{a_1} U(\lambda_2,\mu_2)^{a_2'}{}_{a_2} U(\lambda_3,\mu_3)^{a_3'}{}_{a_3}$$

$$\times U(\lambda_1\lambda_2,\lambda_3^*)_{\bar{r}',}^{\bar{r}} U(\mu_1\mu_2,\mu_3^*)^{\bar{s}'}_{\bar{s}} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1'\mu_1 & a_2'\mu_2 & a_3'\mu_3 \end{pmatrix}^{\bar{r}'}_{\bar{s}'} . \qquad (11.24)$$

(e) 3j Permutation Matrices

There is one more entity whose phase freedom we must find before defining a 6j symbol. Since the left-hand side of (11.16) is independent of the order of λ_1,λ_2 and λ_3

it follows (Derome and Sharp, 1965) that there exists a unitary permutation matrix $M\{\pi,\lambda_1\lambda_2\lambda_3\}$ such that

$$\begin{pmatrix}\lambda_a & \lambda_b & \lambda_c\\ i_a & i_b & i_c\end{pmatrix}^{\bar{r}} = \sum_{r'} M\{\pi,\lambda_1\lambda_2\lambda_3\}^{\bar{r}}_{\bar{r}'} \begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ i_1 & i_2 & i_3\end{pmatrix}^{\bar{r}'} \tag{11.25}$$

where abc is the permutation π of 123. (Evidently Schur's lemma is again at work here — see Butler, 1979.) Not all of the thirty-six possible permutation matrices are independent. It is easy to show that they can all be expressed in terms of five fundamental transposition matrices (Derome, 1966). The permutation properties of the 3jm factors are readily deduced using the Racah factorization lemma (Butler, 1975).

A change in the product multiplicity in the 3jm symbol on the right of (11.25) given by

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ i_1 & i_2 & i_3\end{pmatrix}^{\hat{\bar{r}}'} = \sum_{r''} U(\lambda_1\lambda_2,\lambda_3^*)_{\bar{r}''}^{\bar{r}'} \begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ i_1 & i_2 & i_3\end{pmatrix}^{\bar{r}''}$$

induces a similar change

$$\begin{pmatrix}\lambda_a & \lambda_b & \lambda_c\\ i_a & i_b & i_c\end{pmatrix}^{\hat{\bar{r}}} = \sum_{r'''} U(\lambda_a\lambda_b,\lambda_c^*)_{\bar{r}'''}^{\bar{r}} \begin{pmatrix}\lambda_a & \lambda_b & \lambda_c\\ i_a & i_b & i_c\end{pmatrix}^{\bar{r}'''}$$

in the 3jm symbol on the left. It follows that the dependence of the 3j permutation matrix on the multiplicity separation is given by

$$M\{\pi,\lambda_1\lambda_2\lambda_3\}^{\hat{\bar{r}}}_{\hat{\bar{r}}'} = \sum_{r''r'''} U(\lambda_a\lambda_b,\lambda_c^*)_{\bar{r}'''}^{\bar{r}} M\{\pi,\lambda_1\lambda_2\lambda_3\}^{\bar{r}'''}_{\bar{r}''} U(\lambda_1\lambda_2,\lambda_3^*)^{\bar{r}''}_{\bar{r}'} . \tag{11.26}$$

This can be rewritten in matrix form as (note that Derome 1966

uses a slightly different notation)

$$\hat{M}\{\pi,\lambda_1\lambda_2\lambda_3\} = U(\lambda_a\lambda_b,\lambda_c^*)^\dagger M\{\pi,\lambda_1\lambda_2\lambda_3\}U(\lambda_1\lambda_2,\lambda_3^*) \ . \qquad (11.27)$$

Consistency between (11.18) and (11.25) demands that

$$A(\lambda_a\lambda_b,\lambda_c^*) = M\{\pi^{-1},\lambda_a^*\ \lambda_b^*,\lambda_c^*\}^T A(\lambda_1\lambda_2,\lambda_3^*)M\{\pi,\lambda_1\lambda_2\lambda_3\}^\dagger . \qquad (11.28)$$

One easily checks that both sides of this equation exhibit the same degree of freedom. This relationship will prove to be a key to consistent calculation of the conjugation factors and permutation matrices.

(f) 6j Symbols

A 6j symbol can be defined by "contracting the indices" of four 3jm symbols. We adopt Butler's (1975) definition

$$\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \lambda_4 & \lambda_5 & \lambda_6\end{Bmatrix}^{r_1r_2r_3r_4}$$

$$= \sum_{\substack{i_1i_2i_3i_1'i_2'i_3'\\ i_4i_5i_6i_4'i_5'i_6'}} <\lambda_4>^{\bar{i}_4'i_4}\ <\lambda_5>^{\bar{i}_5'i_5}\ <\lambda_6>^{\bar{i}_6'i_6}\ <\lambda_1>^{\bar{i}_1'i_1}\ <\lambda_2>^{\bar{i}_2'i_2}\ <\lambda_3>^{\bar{i}_3'i_3}$$

$$\times \begin{pmatrix}\lambda_1 & \lambda_5^* & \lambda_6\\ i_1 & \bar{i}_5' & i_6\end{pmatrix}^{r_1}\begin{pmatrix}\lambda_4 & \lambda_2 & \lambda_6^*\\ i_4 & i_2 & \bar{i}_6'\end{pmatrix}^{r_2}\begin{pmatrix}\lambda_4^* & \lambda_5 & \lambda_3\\ \bar{i}_4' & i_5 & i_3\end{pmatrix}^{r_3}\begin{pmatrix}\lambda_1^* & \lambda_2^* & \lambda_3^*\\ \bar{i}_1' & \bar{i}_2' & \bar{i}_3'\end{pmatrix}^{r_4} . \qquad (11.29)$$

Notice that this 6j symbol has contravariant indices. It is lengthy but quite straightforward to show that the overlap integral and the 6j symbol are related by

$$<(\lambda_1\lambda_2\lambda_3)r_{12}(\lambda_{12}\lambda_3)r\lambda\,|\,(\lambda_1\lambda_2\lambda_3)r_{23}(\lambda_1\lambda_{23})s\lambda>$$

$$= \sum_{s'r'_{23}\bar{r}''_{23}r'_{12}r'} |\lambda_{12},\lambda_{23}|^{\frac{1}{2}}\ \phi_{\lambda_2^*}\ <\lambda_1\ \lambda_{23}>^{\bar{s}'\lambda^*,s\lambda}\ <\lambda_2^*\lambda_3^*>_{r_{23}\lambda_{23},\bar{r}'_{23}\lambda_{23}^*}$$

$$\times M\{(132),\lambda_3^*\lambda_{23}\lambda_2^*\}^{\bar{r}'_{23}}{}_{\bar{r}''_{23}}\ M\{(23),\lambda_1\lambda_{12}^*\lambda_2\}^{r_{12}}{}_{r'_{12}}\ M\{(12),\lambda_3\lambda_{12}\lambda^*\}^{r}{}_{r'}$$

$$\times \begin{Bmatrix} \lambda_1 & \lambda_{23} & \lambda^* \\ \lambda_3^* & \lambda_{12} & \lambda_2 \end{Bmatrix}^{r'_{12}\bar{r}''_{23}r'\bar{s}'} \qquad . \qquad (11.30)$$

The phase freedom in the 6j symbol is most simply derived from the definition. We find that

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}^{\hat{r}_1\hat{r}_2\hat{r}_3\hat{r}_4}$$

$$= \sum_{r'_1r'_2r'_3r'_4} U(\lambda_1\lambda_5^*\lambda_6^*)_{r'_1}^{\ r_1}\ U(\lambda_4\lambda_2,\lambda_6)_{r'_2}^{\ r_2}\ U(\lambda_4^*\lambda_5,\lambda_3^*)_{r'_3}^{\ r_3}$$

$$\times U(\lambda_1^*\lambda_2^*,\lambda_3)_{r'_4}^{\ r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}^{r'_1r'_2r'_3r'_4} \qquad . \qquad (11.31)$$

At this stage the ordering of parameters and positioning of complex-conjugates in the phase freedom matrices is highly significant (cf. Butler and Wybourne 1976a). It is an important check on our work that use of the freedoms described by (11.2), (11.3) and (11.26) in equation (11.30) yields the same result.

CHAPTER 12

PHASE CHOICES

In the last section we were preoccupied with determining what freedom exists in the Wigner-Racah algebra. The treatment was exhaustive and shows that once we have defined the algebra and settled on a convention for the representations, the only source of freedom is the arbitrariness allowed by Schur's lemmas in the resolution of branching multiplicities. We come now to the problem of making consistent resolutions of these multiplicities. Our aim is to choose the permutation and conjugation matrices so that the symmetries of the 3jm factors and 6j symbols simplify. In this we follow the canonical choices of Derome (1966) and Butler and King (1974, Butler 1975). Much of what we have to say constitutes a review but there are several aspects (mostly relating to consistency) which have been largely overlooked in the past and which deserve some attention, even if they only assume real significance in rare cases or for obscure phase choices. One aspect which has certainly not been properly treated in the past is the choice of trivial 3jm factors and 6j symbols; we shall find some rather surprising consistency conditions here. Throughout these cases we will be endeavouring to determine exactly what freedom remains for later choices in the algebra. These remaining choices are the key to Butler's method for determining non-trivial 6j symbols and 3jm factors. The experience gained in handling permutation and conjugation

matrices will prove invaluable when we consider these later choices. It should be remarked that in the approach we take here the representation matrices are calculated last.

One problem which has worried earlier workers is whether or not it is possible to choose the phases so that all 3jm factors and 6j symbols are real. It is known that this is the case for angular momentum theory but Butler (1980a) has provided a beautiful counterexample using the $T \supset D_2$ 3jm factors which must have both real and imaginary parts. This counterexample is completely independent of any phase choices and rests only on the orthogonality properties of the 3jm factors. The situation with 6j symbols is not so clear-cut. It is plain that if the Derome-Sharp A matrix is chosen +1 as Butler (1975) recommends then complex 6j symbols must occur (Butler and Wybourne, 1976b). However these can be rendered real by choosing A = -1. This is still a fairly simple choice as far as complex-conjugation symmetries go but we regard real 6j symbols as far less appealing if a distinction has to be made between covariant and contravariant symbols. Therefore we shall not pursue the question of reality further and be quite content to deal with complex numbers.

(a) Choice of Permutation Matrices

The choice of 3j permutation matrices is nowadays well understood, thanks to the work of Derome (1965,1966). His arguments are easily transliterated and there is no need to repeat everything here. We shall just review the results

for completeness and restrict ourselves to some remarks on the salient features.

Given any permutation matrix $M\{\pi,\lambda_1\lambda_2\lambda_3\}$ we attempt to choose a transformation of the form (11.27) which will yield new permutation matrices of the desired form. For obvious reasons a diagonal permutation matrix is desirable. Three distinct cases arise.

Case (i): $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

There are six independent transformation matrices and five fundamental transposition matrices $M\{(12),\lambda_1\lambda_2\lambda_3\}$, $M\{(12),\lambda_1\lambda_3\lambda_2\}$, $M\{(12),\lambda_2\lambda_3\lambda_1\}$, $M\{(23),\lambda_1\lambda_2\lambda_3\}$ and $M\{(23),\lambda_2\lambda_1\lambda_3\}$. All of these five can be chosen to be the same arbitrary unitary matrix D. For instance, by choosing

$$U(\lambda_1\ \lambda_2,\lambda_3^*) = D \text{ and } U(\lambda_2\lambda_1,\lambda_3^*) = M\{(12),\lambda_1\lambda_2\lambda_3\}$$

we obtain

$$\hat{M}\{(12),\lambda_1\lambda_2\lambda_3\} = D\ .$$

If we chose D a Hermitian matrix (Derome, 1966), it follows that all eighteen transposition matrices will also be equal to D and all eighteen cyclic permutation matrices will be equal to unity. Of course D could be chosen to be the unit matrix but it is more usual to take a diagonal matrix of phases

$$D^r{}_{r'} = \{\lambda_1\lambda_2\lambda_3 r\}\delta^r{}_{r'}\ . \qquad (12.1)$$

The application of this symmetry is simplified if for each representation λ one chooses an integer (or if one is careful, perhaps a half-integer) $j(\lambda) = j(\lambda^*)$, termed a j-value (Butler and Wybourne, 1976a) such that the 3j transposition phase is given by

$$\{\lambda_1\lambda_2\lambda_3 r\} = (-1)^{j(\lambda_1) + j(\lambda_2) + j(\lambda_3) + r} \qquad (12.2)$$

where $r = 0,1,2, \ldots$ beginning with 0 for the first occurrence of λ_3^* in $\lambda_1 \times \lambda_2$. Although this is always possible in principle, later desired simplifications may make some choices of j-value intolerable. What does lead to later simplifications though is the choice of D as a Hermitian matrix. In addition to simplifying the permutation matrices this choice implies reality of the transposition phase and this is one way of satisfying equation (12.19).

Case (ii): $\lambda_1 = \lambda_2 \neq \lambda_3$.

There are now only three independent transformation matrices and three fundamental transposition matrices $M\{(12),\lambda_1\lambda_1\lambda_3\}$, $M\{(12),\lambda_1\lambda_3\lambda_1\}$ and $M\{(23),\lambda_1\lambda_1\lambda_3\}$. The first of these is its own inverse and therefore must be Hermitian. Its freedom is described by the unitary transformation

$$\hat{M}\{(12),\lambda_1\lambda_1\lambda_3\} = U(\lambda_1\lambda_1,\lambda_3^*)^\dagger \, M\{(12),\lambda_1\lambda_1\lambda_3\} \, U(\lambda_1\lambda_1,\lambda_3^*)$$

and it can be chosen diagonal by using a suitable choice of

$U(\lambda_1\lambda_1,\lambda_3^*)$. However, in general it cannot be chosen to be the unit matrix since clearly it is possible for it to have both +1 and -1 as its eigenvalues. It can be easily shown (Derome, 1965) that the degeneracies S and A of the eigenvalues +1 and -1 are the same as the multiplicities of λ_3^* in the symmetric and antisymmetric squares of λ_1, respectively. If as well we choose

$$U(\lambda_1\lambda_3,\lambda_1^*) = M\{(23),\lambda_1\lambda_1\lambda_3\}\, M\{(12),\lambda_1\lambda_1\lambda_3\}\, U(\lambda_1\lambda_1,\lambda_3^*)$$

and

$$U(\lambda_3\lambda_1,\lambda_1^*) = M\{(12),\lambda_1\lambda_3\lambda_1\}\, M\{(23),\lambda_1\lambda_1\lambda_3\}\, U(\lambda_1\lambda_1,\lambda_3^*)$$

we obtain (Derome, 1966)

$$\hat{M}\{(23),\lambda_1\lambda_1\lambda_3\} = \hat{M}\{(12),\lambda_1\lambda_3\lambda_1\} = \hat{M}\{(12),\lambda_1\lambda_1\lambda_3\} = \begin{pmatrix} 1_S & 0 \\ 0 & -1_A \end{pmatrix} \qquad (12.3)$$

where 1_S and 1_A are unit matrices of dimension S and A. It follows that all transposition matrices are equal to $\hat{M}\{(12),\lambda_1\lambda_1\lambda_3\}$ and all cyclic permutation matrices are again unity. Thus the diagonal choice of equation (12.1) is again possible but the transposition phase $\{\lambda_1\lambda_1\lambda_3 r\}$ is uniquely determined rather than being arbitrary.

Case (iii): $\lambda_1 = \lambda_2 = \lambda_3$

There is only one independent transformation matrix but two fundamental transposition matrices $M\{(12),\lambda_1\lambda_1\lambda_1\}$

and $M\{(23),\lambda_1\lambda_1\lambda_1\}$. In general these cannot be diagonalized simultaneously. However, one notes that the six matrices $M\{\pi,\lambda_1\lambda_1\lambda_1\}$ form a matrix representation of the symmetric group, S_3. In general this representation will be reducible but group theory then tells us that for all permutations π, $M\{\pi,\lambda_1\lambda_1\lambda_1\}$ can be completely reduced by the same unitary matrix. Setting $U(\lambda_1\lambda_1,\lambda_1^*)$ equal to this matrix we obtain

$$\hat{M}\{\pi,\lambda_1\lambda_1\lambda_1\} = \begin{pmatrix} [3](\pi) & & & \\ & \ddots & & \\ & & [21](\pi) & \\ & & & \ddots \\ & & & & [1^3](\pi) \\ & & & & & \ddots \end{pmatrix} \tag{12.4}$$

where each of the diagonal blocks is an irreducible representation matrix of S_3, usually taken to be in some standard form (Butler, 1975). It is not difficult to show that the degeneracy of $[3](\pi)$ and $[1^3](\pi)$ is the same as the multiplicity of ~~λ_1~~ the identity in the symmetric and antisymmetric Kronecker cubes of λ_1 respectively while the number of times $[21](\pi)$ occurs is one-half the multiplicity of ~~λ_1~~ the identity in the mixed-symmetry Kronecker cube. It is because the representation [21] of S_3 is two dimensional that M cannot always be chosen diagonal. If the mixed-symmetry term does not arise then λ_1 is said to be simple-phase (van Zanten and

de Vries, 1973) and it is easily shown that the choice (12.4) becomes equivalent to that of (12.3). Some important groups contain only simple-phase representations e.g. SU(2) and SU(3) (Derome 1967, Butler and King 1974), but in general nonsimple-phase representations do arise (Derome 1966). Note that both (12.3) and (12.4) resolve the multiplicities according to symmetry type.

This concludes the choice of permutation matrices but there is an important point that still needs to be made. In choosing the permutation matrices we have used the phase freedom matrices $U(\lambda_1\lambda_2,\lambda_3^*)$ but there is nothing to preclude us making new phase choices involving new matrices $U(\lambda_1\lambda_2,\lambda_3^*)$ provided that these new choices do not change the permutation matrices. This requirement is expressed by the restriction

$$U(\lambda_a\lambda_b,\lambda_c^*) = M\{\pi,\lambda_1\lambda_2\lambda_3\}U(\lambda_1\lambda_2,\lambda_3^*)\ M\{\pi,\lambda_1\lambda_2\lambda_3\}^\dagger \qquad (12.5)$$

which means that fixing the permutation matrix $M\{\pi,\lambda_1\lambda_2\lambda_3\}$ specifies the relationship between the resolutions of multiplicity in the embeddings of λ_3^* in $\lambda_1 \times \lambda_2$ and λ_c^* in $\lambda_a \times \lambda_b$. If a constant matrix was chosen for M then this restriction simplifies to

$$U(\lambda_a\lambda_b,\lambda_c^*) = U(\lambda_1\lambda_2,\lambda_3^*) \quad . \qquad (12.6a)$$

However, apart from cyclic permutations and multiplicity-free products a constant matrix is not a very common choice. A little manipulation of (12.5) readily shows that the choice (12.1) leads to (cf. Butler and Wybourne 1976a)

$$U(\lambda_a\lambda_b,\lambda_c^*)^r{}_{r'} = \{\lambda_1\lambda_2\lambda_3 r'\}^*\{\lambda_1\lambda_2\lambda_3 r\}U(\lambda_1\lambda_2,\lambda_3^*)^r{}_{r'} \qquad (12.6b)$$

while for $\lambda_1 = \lambda_2$ the choice (12.3) restricts $U(\lambda_1\lambda_1,\lambda_3^*)$ to being block diagonal with respect to symmetry type. If $\lambda_1 = \lambda_2 = \lambda_3$ then (12.5) says that $U(\lambda_1\lambda_1,\lambda_1^*)$ must commute with $M\{\pi,\lambda_1\lambda_1\lambda_1\}$ for all π. It follows from the choice (12.4) that $U(\lambda_1\lambda_1,\lambda_1^*)$ must also be block diagonal with respect to symmetry type and further, it cannot mix alternate rows or columns in the mixed symmetry sector.

(b) Choice of Conjugation Factors

Choosing the conjugation factors completes the specification of the representation conjugation operators in terms of the group-subgroup basis. Although some aspects of the selection process have been discussed by previous authors (Butler and King 1974, Butler 1975, Butler and Wybourne 1976a) a thorough account has never before been given and what has been done has not been entirely general. This and the next two sections are aimed at correcting this small deficiency.

As with the permutation matrices we attempt to choose a transformation, this time of the form (11.5), which will turn an arbitrary and unknown conjugation factor into a new one of the desired form. Again it is clear that a diagonal choice leads to simplifications. Two separate cases arise depending on whether or not the pairs (λ,μ) and (λ^*,μ^*) are distinct.

Case (i): Either λ or μ complex

In this case $U(\lambda,\mu)$ and $U(\lambda^*,\mu^*)$ are distinct matrices and can be chosen independently. Choosing $U(\lambda^*,\mu^*) = A(\lambda^*,\mu^*)^\dagger$ we obtain

$$\hat{A}(\lambda^*,\mu^*) = U(\lambda,\mu)^T$$

which clearly shows that the matrix of conjugation factors can be chosen to be any unitary matrix. In particular the unit matrix is possible. If the unit matrix is not chosen then summations on indices can still be rendered trivial by choosing the elements of $\hat{A}(\lambda^*,\mu^*)$ to be either of modulus one or zero e.g.

$$<\lambda^*>_{a\mu,\bar{a}'\mu^*} \equiv A(\lambda^*,\mu^*)^{a^{K_\lambda}}_{\quad a',K_\lambda} = \pm\,\delta_{a'a^*} \qquad (12.7)$$

where a^* is some multiplicity index arbitrarily related to a (Butler and Wybourne 1976a). (The choice $a^* = a$ would yield a diagonal matrix.) The conjugation factor $<\lambda>_{\bar{a}^*\mu^*,a\mu}$ is now determined by the symmetry (10.40). Usually, at least one of the phases ϕ_λ or ϕ_{μ^*} is at the moment arbitrary but we shall see at the end of this section that certain desired forms for the multiplicity metric tensor will severely restrict the tolerable values.

Case (ii): Both λ and μ are real

This time $U(\lambda,\mu)$ and $U(\lambda^*,\mu^*)$ are the same. The phase freedom therefore reduces to

$$\hat{A}(\lambda,\mu) = U(\lambda,\mu)^T A(\lambda,\mu)U(\lambda,\mu) \quad . \tag{12.8}$$

It is necessary to consider two possibilities separately.

(a) λ and μ are either both orthogonal or both symplectic.

In this case $A(\lambda,\mu)$ is a symmetric unitary matrix and can therefore be written in the form (see Gantmacher 1959, lemma 2 vol.II p.4 cf. Butler 1975 equation 8.17)

$$A(\lambda,\mu) = \pm e^{iS} \quad \text{where} \quad S = S^* = S^T \ . \tag{12.9}$$

Choosing

$$U(\lambda,\mu) = e^{-\frac{1}{2}iS}$$

then gives

$$\hat{A}(\lambda,\mu) = \pm I \tag{12.10a}$$

i.e.

$$\langle\lambda\rangle_{a\mu,a'\mu} = \pm \delta_{a'a^*} \quad \text{where } a^* = a \quad . \tag{12.10b}$$

(b) Either λ or μ is orthogonal and the other is symplectic.

In this case $A(\lambda,\mu)$ is an antisymmetric unitary matrix. It is well-known though that antisymmetric matrices of odd dimension are singular and therefore an antisymmetric unitary matrix cannot exist unless it is of even dimension. Since the dimension of $A(\lambda,\mu)$ is equal to the branching multiplicity this means that the multiplicity in this case is always even

i.e. μ must occur in pairs. This result is implicit in Malcev's (1944) theorems (see also Dynkin, 1952). A trivial example of this case is the restriction of a group to the trivial group consisting solely of the identity operation; then every representation λ decomposes to the single orthogonal identity representation with a multiplicity equal to the dimension of λ. We immediately obtain the well-known result that symplectic representations are even dimensional. However, apart from this simple example the case in hand is very rare. Nevertheless it must be considered. Clearly $A(\lambda,\mu)$ cannot be chosen diagonal but we find that it is possible to choose the elements of $A(\lambda,\mu)$ to be of modulus one or zero. The proof follows. It is trivial to show that any unitary skew-symmetric matrix, in particular $A(\lambda,\mu)$, can be written in the form

$$A(\lambda,\mu) = U + iV$$

where U and V are real skew-symmetric normal matrices. Then since

$$A(\lambda,\mu)A(\lambda,\mu)^* = -I$$

we also have

$$U^2 + V^2 = -I$$

and

$$UV = VU \ .$$

Therefore (Gantmacher 1959, vol. I p.292 Theorem 12' together with the note) U and V can be written as

$$U = Q\left\{\begin{pmatrix} 0 & \nu_1 \\ -\nu_1 & 0 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} 0 & \nu_q \\ -\nu_q & 0 \end{pmatrix}\right\} Q^{-1}$$

$$V = Q\left\{\begin{pmatrix} 0 & \nu_1' \\ -\nu_1' & 0 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} 0 & \nu_q' \\ -\nu_q' & 0 \end{pmatrix}\right\} Q^{-1}$$

where Q is a real orthogonal matrix. But from the above it also follows that

$$\nu_k^2 + \nu_k'^2 = 1$$

leading to the parametrization

$$\nu_k = \cos \phi_k \quad \text{and} \quad \nu_k' = \sin \phi_k$$

and the general result

$$A(\lambda,\mu) = Q\left\{\begin{pmatrix} 0 & e^{i\phi_1} \\ -e^{i\phi_1} & 0 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} 0 & e^{i\phi_q} \\ -e^{i\phi_q} & 0 \end{pmatrix}\right\} Q^{-1} .$$

We can go one step further and define a diagonal matrix

$$Y = \pm\left\{\begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_1} \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} e^{i\phi_q} & 0 \\ 0 & e^{i\phi_q} \end{pmatrix}\right\}$$

which commutes with the real orthogonal matrix

$$Z = \pm\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

so that $A(\lambda,\mu)$ can be written in the general form

$$A(\lambda,\mu) = QYZ\ Q^{-1} \ . \tag{12.11}$$

(This derivation is analogous to Gantmacher's proof of (12.9).) Choosing

$$U(\lambda,\mu) = QY^{-\frac{1}{2}}$$

we obtain

$$\hat{A}(\lambda,\mu) = Z \ . \tag{12.12a}$$

Thus the conjugation factors are of the form

$$\langle\lambda\rangle_{a\mu,a'\mu} = \pm\delta_{a'a^*} \quad \text{where} \quad a^* \neq a \ . \tag{12.12b}$$

In choosing the conjugation factors we have used the phase freedom matrices $U(\lambda,\mu)$ but there is nothing to preclude us making new phase choices involving new matrices $U(\lambda,\mu)$. However, in order to retain the same values for the conjugation factors we must restrict ourselves when making such changes. The situation is entirely analogous to that encountered for permutation matrices. This time the restrictions we must impose are in all cases (whatever the choice of conjugation factors may have been) described by

$$U(\lambda^*,\mu^*) = A(\lambda^*,\mu^*)^{\dagger}\ U(\lambda,\mu)^*\ A(\lambda^*,\mu^*) \ . \tag{12.13}$$

It is of particular interest to note that if a constant matrix was chosen we must restrict our future choices by

$$U(\lambda^*,\mu^*) = U(\lambda,\mu)^* \qquad (12.14)$$

which means of course that for case (iia) where this can occur, future choices would be restricted to a real orthogonal matrix. It should be emphasized though that (12.14) is not generally true, cf. Butler and Wybourne (1976a).

The choice of conjugation factors for the embedding of the inner product group in the outer product group i.e. Derome and Sharp's (1965) A matrix or multiplicity metric tensor, warrants special attention. In principle it is no different from the general case but we must contend with the restriction (12.5). This is no problem unless $(\lambda_1^*\lambda_2^*\lambda_3^*)$ is a permutation of $(\lambda_1\lambda_2\lambda_3)$ or itself. In the latter case i.e. when two or more of $\lambda_1,\lambda_2,\lambda_3$ are equal, we have seen that $U(\lambda_1\lambda_1,\lambda_3^*)$ and $U(\lambda_1\lambda_1,\lambda_1^*)$ are restricted to being block diagonal with respect to symmetry type. However, the choices (12.3) and (12.4) imply via (11.28) that $A(\lambda_1\lambda_1,\lambda_3^*)$ and $A(\lambda_1\lambda_1,\lambda_1^*)$ are also block diagonal (Butler, 1975). (This result follows even for λ_1 and λ_3 complex because the multiplicity types arising in $\lambda_1^* \times \lambda_1^*$ are the same as those in $\lambda_1 \times \lambda_1$ so that the choices (12.3) and (12.4) render $M\{(12),\lambda_1^*\lambda_1^*\lambda_3^*\} = M\{(12),\lambda_1\lambda_1\lambda_3\}$ and $M\{\pi,\lambda_1^*\lambda_1^*\lambda_1^*\} = M\{\pi,\lambda_1\lambda_1\lambda_1\}$.) Thus we simply choose the blocks independently. If $(\lambda_1^*\lambda_2^*\lambda_3^*)$ is a permutation of $(\lambda_1\lambda_2\lambda_3)$ there are only two possibilities to consider.

Case (i): λ_1,λ_2 and λ_3 are all real.

Then $U(\lambda_1^*\lambda_2^*,\lambda_3) = U(\lambda_1\lambda_2,\lambda_3^*)$ anyway and the general procedure for reality applies without modification, except

for the minor changes necessary for block diagonal matrices which may surface here.

Case (ii): One of $\lambda_1,\lambda_2,\lambda_3$ is real and the other two are a complex conjugate pair.

Consider without loss of generality $\lambda_2 = \lambda_1^*$ and $\lambda_3^* = \lambda_3$. The restriction (12.5) applied to the freedom in $A(\lambda_1^*\lambda_1,\lambda_3)$ leads to

$$M\{(12),\lambda_1^*\lambda_1\lambda_3\}^T \hat{A}(\lambda_1^*\lambda_1,\lambda_3)$$

$$= U(\lambda_1^*\lambda_1,\lambda_3)^T M\{(12),\lambda_1^*\lambda_1\lambda_3\}^T A(\lambda_1^*\lambda_1,\lambda_3)\ U(\lambda_1^*\lambda_1,\lambda_3). \tag{12.15}$$

To obtain a solution to this we note firstly that $\phi_{\lambda_1}{}^*\phi_{\lambda_1} = +1$ (even though $\lambda_1 \times \lambda_1^*$ is complex in the outer product group) so that the symmetry (10.40) reads

$$A(\lambda_1^*\lambda_1,\lambda_3) = \phi_{\lambda_3} A(\lambda_1\lambda_1^*,\lambda_3)^T . \tag{12.16}$$

Employing this together with (11.28) we then show that

$$[\,M\{(12),\lambda_1^*\lambda_1\lambda_3\}^T A(\lambda_1^*\lambda_1,\lambda_3)\,]^T$$

$$= \phi_{\lambda_3}\ M\{(12),\lambda_1^*\lambda_1\lambda_3\}^T A(\lambda_1^*\lambda_1,\lambda_3) \tag{12.17}$$

so that the product matrix is symmetric or skew-symmetric depending on whether λ_3 is orthogonal or symplectic. Thus a suitable choice of $U(\lambda_1^*\lambda_1,\lambda_3)$ will cast the right hand

side of (12.15) into one or the other of the canonical forms $\pm$ I or Z, leading to

$$\hat{A}(\lambda_1^*\lambda_1,\lambda_3) = \begin{cases} \pm M\{(12),\lambda_1^*\lambda_1\lambda_3\}^* & \text{for } \phi_{\lambda_3} = +1 \\ M\{(12),\lambda_1^*\lambda_1\lambda_3\}^* Z & \text{for } \phi_{\lambda_3} = -1 \ . \end{cases} \quad (12.18)$$

These choices are always possible but unless $M\{(12),\lambda_1^*\lambda_1\lambda_3\}$ was chosen to be the unit matrix they are not as simple as we would have liked. However, the right hand side of (12.15) could have been chosen to be any symmetric unitary matrix instead of $\pm$ I or any skew-symmetric unitary matrix instead of Z. One easily shows that for $\phi_{\lambda_3} = +1$ we can choose $\hat{A}(\lambda_1^*\lambda_1,\lambda_3) = I$ if and only if $M\{(12),\lambda_1^*\lambda_1\lambda_3\}$ was chosen symmetric while for $\phi_{\lambda_3} = -1$ we can choose $\hat{A}(\lambda_1^*\lambda_1,\lambda_3) = Z$ if and only if $ZM\{(12),\lambda_1^*\lambda_1\lambda_3\} = M\{(12),\lambda_1^*\lambda_1\lambda_3\}^T Z$. Actually, for λ_3 symplectic it is possible to choose $\hat{A}(\lambda_1^*\lambda_1,\lambda_3) = I$, but only if $M\{(12),\lambda_1^*\lambda_1\lambda_3\}$ was chosen skew-symmetric. Similarly for λ_3 orthogonal it is possible to choose $\hat{A}(\lambda_1^*\lambda_1,\lambda_3) = Z$ but only if $ZM\{(12),\lambda_1^*\lambda_1\lambda_3\} = -M\{(12),\lambda_1^*\lambda_2\lambda_3\}^T Z$. Obviously it pays to be careful when choosing the permutation matrices.

Having chosen $A(\lambda_1\lambda_2,\lambda_3^*)$ it follows from the restrictions (12.5) and (12.13) that there is no freedom remaining in $A(\lambda_a\lambda_b,\lambda_c^*)$ where again (abc) is a permutation of (123). However, equation (11.28) once more comes to our rescue by uniquely fixing $A(\lambda_a\lambda_b,\lambda_c^*)$ in terms of $A(\lambda_1\lambda_2,\lambda_3^*)$ and the appropriate permutation matrices. We note that if $A(\lambda_1\lambda_2,\lambda_3^*)$ was chosen a constant matrix then the condition for $A(\lambda_a\lambda_b,\lambda_c^*)$ to be the same constant matrix is that our earlier choice of

permutation matrices satisfies

$$M\{\pi,\lambda_1^*\lambda_2^*\lambda_3^*\} = M\{\pi,\lambda_1\lambda_2\lambda_3\}^*. \tag{12.19}$$

Since it was possible to choose all permutation matrices real, the desired simplification can be guaranteed.

Butler and King (1974) have shown that for all the classical Lie groups and for most finite groups, in particular the symmetric and alternating groups, the skew-symmetric case for the multiplicity metric tensor never arises. Further they show for these groups that even for complex λ, ϕ_λ can be chosen so that

$$\phi_{\lambda_1}\ \phi_{\lambda_2}\ \phi_{\lambda_3} = 1 \tag{12.20}$$

whenever $\lambda_1 \times \lambda_2 \supset \lambda_3^*$ and a table of the possible real values for ϕ_λ is given. (Note that this choice of ϕ_λ has nothing to do with phase choices but rather reflects a freedom in the definition of the algebra.) Hence these conjugation factors can always be chosen to be the same constant matrix — usually the identity, which greatly simplifies complex conjugation symmetries. Groups for which (12.20) can be satisfied with real ϕ_λ are termed quasi-ambivalent (Butler and Wybourne, 1976a) and the complex representations are termed quasi-orthogonal or quasi-symplectic depending on whether ϕ_λ is +1 or -1. Butler (1980b) writes

$$\phi_\lambda = \phi_\lambda^* \equiv \{\lambda\}$$

and this notation was adopted in Part I.

Having chosen the conjugation factors for a chain of groups the conjugation coefficients for any group in the chain follow from the factorization (10.38). This involves a conjugation coefficient for the lowest group in the chain. The process of choosing this last coefficient is precisely the same as for the conjugation factors.

(c) Trivial 3jm Factors

It is time now to reconcile Wigner's definition of the 2jm symbol with the more general conjugation coefficients introduced in chapter 10. The relationship between conjugation factors and 2jm factors then follows in a natural fashion.

If we set $\lambda_1 = 0$ (the identity representation) and $\lambda_2 = \lambda^*$ in equation (11.15) then we obtain

$$\lambda^*(R)^{\bar{i}'}{}_{\bar{i}}$$

$$= \sum_{i''i'''} \left\{ |\lambda|^{\frac{1}{2}} \begin{pmatrix} 0 & \lambda^* & \lambda \\ 0 & \bar{i}' & i'' \end{pmatrix} 0^* \right\} \lambda(R)_{i''}{}^{i'''} \left\{ |\lambda|^{\frac{1}{2}} \begin{pmatrix} 0 & \lambda^* & \lambda \\ 0 & \bar{i} & i''' \end{pmatrix} 0 \right\} . \tag{12.21}$$

This should be compared with (10.32). It follows from Schur's lemma, combined with the unitarity of the conjugation coefficients and coupling coefficients, that

$$|\lambda|^{\frac{1}{2}} \begin{pmatrix} 0 & \lambda^* & \lambda \\ 0 & \bar{i} & i' \end{pmatrix} 0 = e^{i\phi(\lambda)} \langle \lambda^* \rangle_{i'\bar{i}} \tag{12.22}$$

where $e^{i\phi(\lambda)}$ is some phase. The consistency of this result with the permutation symmetries of the conjugation coefficient and the trivial 3jm symbol leads to

$$\phi_\lambda \; e^{-i\phi(\lambda^*)} = e^{-i\phi(\lambda)} \; M\{(23),0\lambda\lambda^*\}^0{}_0 \; . \tag{12.23}$$

Also when we combine (12.22) with the Derome-Sharp lemma (11.18) we find that

$$<0\lambda^*>_{0\lambda,0\lambda^*} \; <0>^{00} = e^{-i\phi(\lambda)} \; e^{-i\phi(\lambda^*)} \; . \tag{12.24}$$

In choosing the trivial 3jm symbols we fix the phases $e^{i\phi(\lambda)}$ and $e^{i\phi(\lambda^*)}$. The freedom, described by (11.23), in the trivial 3jm can be written as

$$\begin{pmatrix} 0 & \lambda^* & \lambda \\ \hat{0} & \hat{\bar{i}} & \hat{i}' \end{pmatrix}^{\hat{0}}$$

$$= e^{-i\theta(0\lambda^*\lambda)} \; e^{i\theta(0)} \sum_{i''i'''} <\lambda i''|\lambda\hat{i}'> \begin{pmatrix} 0 & \lambda^* & \lambda \\ 0 & \bar{i}''' & i'' \end{pmatrix}^0 <\lambda^*\bar{i}'''|\lambda^*\hat{\bar{i}}> \tag{12.25}$$

which is greater than the freedom (10.35) in the conjugation coefficient. We assume now that all conjugation coefficients have been chosen as described in the last section so that (12.25) reduces to

$$\begin{pmatrix} 0 & \lambda^* & \lambda \\ \hat{0} & \bar{i} & i' \end{pmatrix}^{\hat{0}} = e^{-i\theta(0\lambda^*\lambda)} \; e^{i\theta(0)} . e^{i\phi(\lambda)} |\lambda|^{-\frac{1}{2}} <\lambda^*>_{i'\bar{i}} \; . \tag{12.26}$$

The choice of $<0>_{00}$ restricts $e^{i\theta(0)}$ to being a sign and the combination of (12.6) and (12.14) restricts $e^{-i\theta(0\lambda^*\lambda)}$

to being a sign also. Hence the relative free phase $e^{-i\theta(0\lambda^*\lambda)}e^{i\theta(0)}$ is only a sign. Evidently it is not possible at this late stage to choose phases so that $e^{i\phi(\lambda)}$ is whatever we want! In fact (12.23) and (12.24) fix $e^{i\phi(\lambda)}$ up to a sign

$$e^{i\phi(\lambda)} = \pm\,[<0\lambda^*>^{0\lambda,0\lambda^*}<0>_{00}\ \phi_{\lambda^*}\ M\{(23),0\lambda\lambda^*\}^0{}_0]^{\frac{1}{2}}. \tag{12.27}$$

The freedom in (12.26) merely allows us to choose this sign. We note that $e^{i\phi(\lambda^*)}$ is then uniquely determined by either (12.23) or (12.24).

However, the situation is not quite as messy as it may at first seem. We ask ourselves: if the result (12.27) was anticipated, could we have chosen the permutation matrices and conjugation factors so that the product of the factors appearing on the right of (12.27) turns out to be some simple number, say one? To answer this we begin by noting that some simplification occurs if $e^{i\phi(\lambda^*)} = e^{i\phi(\lambda)}$. This will be true if and only if

$$M\{(23),0\lambda\lambda^*\}^0{}_0 = \phi_\lambda \quad . \tag{12.28}$$

For λ real this is true anyway while for complex λ it merely amounts to having chosen the transposition phase to be equal to the value of ϕ_λ determined in accordance with (12.20). Clearly some care is required in choosing j-values if (12.28) is to be satisfied. In fact they are restricted to being integer or half-integer depending on whether $\phi_\lambda = +1$ or -1.

With (12.28) satisfied, we see that $e^{i\phi(\lambda)}$ can be chosen to be +1 if and only if

$$<0\lambda^*>_{0\lambda,0\lambda^*} = <0>_{00} \ . \tag{12.29}$$

This is a remarkable result since the right hand side is independent of λ and therefore

$$<0\lambda_1^*>_{0\lambda_1,0\lambda_1^*} = <0\lambda_2^*>_{0\lambda_2,0\lambda_2^*} \ . \tag{12.30}$$

Although stringent, these two conditions can be satisfied when we choose the conjugation factors and thus it is possible to arrange for

$$<\lambda^*>_{i'\bar{i}} = |\lambda|^{\frac{1}{2}} \begin{pmatrix} 0 & \lambda^* & \lambda \\ \hat{0} & \bar{i} & i' \end{pmatrix}^{\hat{}} \ . \tag{12.31}$$

It does not seem to have been recognized before that (12.29) and (12.30) are essential to the general validity of (12.31).

In order to retain this phase relationship we must restrict future choices by

$$e^{-i\theta(0\lambda^*\lambda)} \ e^{i\theta(0)} = 1 \ . \tag{12.32}$$

An important deduction from this restriction is that we must also have

$$e^{i\theta(0\lambda_1^*\lambda_1)} = e^{i\theta(0\lambda_2^* \ \lambda_2)} \tag{12.33}$$

since $e^{i\theta(0)}$ is independent of λ. This new restriction is certainly consistent with (12.6) and (12.14). It explains why Butler and Wybourne (1976a) found no free phase

for products involving the identity representation.

With these phase choices the conjugation coefficients have become equivalent to the 2jm symbols as defined by Butler (1975). The explicit relationship is

$$\langle\lambda^*\rangle_{i'\bar{i}} = (\lambda^*)_{\bar{i}i'} \ . \tag{12.34}$$

The ordering of the indices in a 2jm symbol is chosen so that contracted indices are adjacent when premultiplying a 3jm symbol by a 2jm symbol with the complex conjugate argument (see Butler, 1975). This is the same as the ordering of the columns in the trivial 3jm in (12.31). The ordering of indices in the conjugation coefficient arose naturally in (10.29). It is unfortunate that there is a difference but perhaps this will serve to emphasize that a definition of the 2jm symbol based on (12.31) incorporates a phase choice.

With this phase choice being made for every group in a chain, the relationship between a conjugation factor and a trivial 3jm factor follows immediately from the factorization properties of conjugation coefficients and 3jm symbols. We obtain

$$\langle\lambda^*\rangle_{a'\mu,\bar{a}\mu^*} = |\lambda|^{\frac{1}{2}}|\mu|^{-\frac{1}{2}}\begin{pmatrix} 0 & \lambda^* & \lambda \\ 0 & 0 & \bar{a}\mu^* \quad a'\mu \end{pmatrix}0 \ . \tag{12.35}$$

The restriction (12.32) on future phase choices for both groups implies the combined restriction

$$e^{-i\theta(0\lambda^*\lambda)}\ e^{i\theta(0,0)}\ e^{i\theta(0\mu^*\mu)} = 1 \tag{12.36}$$

and similarly (12.29) implies the relation

$$<0\lambda^*>_{0\lambda,0\lambda^*}<0\mu^*>^{0\mu,0\mu^*} = <0>_{00,00} \tag{12.37}$$

It is interesting to combine the choice (12.35) with the definition (11.20). The cyclic permutation matrix $M\{(132)\lambda^* \lambda 0\}$ would normally be chosen +1 as would the multiplicity metric tensors so that by (12.37), $<0>_{00,00}$ is also +1 and hence it follows that

$$<\lambda^*>_{a'\mu,\bar{a}\mu^*} = |\lambda|^{\frac{1}{2}}|\mu|^{-\frac{1}{2}} <000|\lambda^*\bar{a}\mu^*;\lambda a'\mu> \ . \tag{12.38}$$

This is the relation used by Butler and Wybourne (1976a) to define the 2jm factor. We emphasize the restrictions this sort of definition imposes.

(d) The Trivial 6j Symbol

All trivial 6j symbols can be cast, via their symmetries, into the form

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2^* & \lambda_1 & 0 \end{Bmatrix}^{00\bar{r}_3 r_4}$$

$$= \sum_{r_3'} M\{(123),0\lambda_1\lambda_1^*\}^0{}_0 M\{(123),0\lambda_2^*\lambda_2\}^0{}_0 M\{(12),\lambda_1\lambda_2\lambda_3\}^{\bar{r}_3}{}_{\bar{r}_3'}$$

$$<\lambda_1\lambda_2>^{r_4\lambda_3,\bar{r}_3'\lambda_3^*} <0>^{00} e^{i\phi(\lambda_2)} e^{i\phi(\lambda_1^*)} |\lambda_1,\lambda_2|^{-\frac{1}{2}} \ . \tag{12.39}$$

With all the phase choices made to date it is easy to see

that the freedom described by (11.31) vanishes for this symbol. However, all the quantities on the right of (12.39) have been determined. Taking the simplest possible choices we have

$$e^{i\phi(\lambda_2)} = e^{i\phi(\lambda_1^*)} = \langle 0\rangle^{00} = M\{(123),0\lambda_1\lambda_1^*\}^0_0$$

$$= M\{(123),0\lambda_2^*\lambda_2\}^0_0 = 1$$

and

$$M\{(12),\lambda_1\lambda_2\lambda_3\}^{\bar{r}_3}_{\bar{r}_3'} = \{\lambda_1\lambda_2\lambda_3,\bar{r}_3\}\delta^{r_3}_{r_3'}$$

(which is always possible, Butler 1975) so that

$$\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \lambda_2^* & \lambda_1 & 0\end{Bmatrix}^{00\bar{r}'r}$$

$$= |\lambda_1,\lambda_2|^{-\frac{1}{2}}\{\lambda_1\lambda_2\lambda_3,\bar{r}'\}\langle\lambda_1\lambda_2\rangle^{r\lambda_3,\bar{r}'\lambda_3^*} . \qquad (12.40)$$

In most cases one could also have chosen the multiplicity metric tensor to be

$$\langle\lambda_1\lambda_2\rangle^{r\lambda_3,\bar{r}'\lambda_3^*} = \delta^{rr'}$$

so that (12.40) simplifies further (Butler and Wybourne 1976a). However, it needs to be emphasized that these equations are not identically true.

(e) Non-trivial 6j Symbols

The 6j symbol has the freedom described by (11.31). The idea is to use this freedom subject to the restrictions

(12.5) and (12.13) — which ensure that the predetermined symmetries of the 6j symbol, given by the permutation and conjugation matrices, are not altered — to choose the value of the 6j symbols. Various restrictions on the phase freedom matrices may however completely eliminate the supposed freedom in (11.31). In fact, in practice only a small percentage of all 6j symbols can be chosen in this way. The calculation of the remaining symbols is discussed briefly in the next chapter. Here we shall be concerned only with those cases where freedom exists. Because of the restriction (12.5) it is helpful to write

$$U(\lambda_1\lambda_2,\lambda_3^*) \equiv U(\lambda_1\lambda_2\lambda_3) \qquad (12.41)$$

where $(\lambda_1\lambda_2\lambda_3)$ is termed a triad and retains its form under permutations.

We shall follow Butler and Wybourne (1976a) in choosing a faithful representation ε, termed the primitive, so that all representations λ are contained in some Kronecker power of ε or ε^*. The smallest value of p for which

$$(\varepsilon + \varepsilon^*)^p \supset \lambda$$

is termed the power, $p(\lambda)$ of λ (Butler 1980a, cf. Butler and Wybourne 1976a). The concept of a primitive and its powers allows us to consider 6j symbols involving representations of low power first and then to progress to the others via a building-up principle.

We note that the freedom in (11.31) is a relative freedom so that one only chooses relative separations of

multiplicity. However, every representation which occurs in the argument of a phase freedom matrix occurs as its complex conjugate in the argument of another phase freedom matrix e.g. λ_1 occurs in both the triads $(\lambda_1^*\lambda_2^*\lambda_3^*)$ and $(\lambda_1\lambda_5^*\lambda_6)$. Hence the first time a representation arises in the building up procedure it does so in association with at least two phase-freedom matrices. At least one of these matrices must also involve the primitive representation. It may well be that the two matrices concerned cancel each other, if their arguments are related, but if they do not it follows that we can regard the overall freedom as being the freedom of one of these products — a primitive one — relative to the other three products. Thus it is clear that for each complex-conjugate pair of representations (other than perhaps the primitive) there is at least one triad involving the primitive whose multiplicity can never be resolved in the 6j calculation. We term such triads antecedents as the other products are resolved relative to them. If a representation occurs for the first time in either of several primitive products then we may arbitrarily choose one of them as the antecedent. It seems highly unlikely that there can be more than one antecedent for the same representation but a proof is difficult to formulate. However, the number of antecedents is not particularly important since they only refer to the fact that it is impossible to fix resolutions of antecedents relative to each other by choosing 6j symbols. There is no suggestion that elsewhere a resolution of the associated multiplicity cannot

be made. It is however clear that in general the number of antecedents is less than the number of distinct primitive triads. As far as phase choices are concerned there is nothing special about primitive triads; those that are not antecedents are treated in precisely the same way as non-primitives. (The reason why this was not noted by Butler (1976) or Butler and Wybourne (1976a,b) seems to be simply that the groups SO_3 and T studied by them are too simple to reveal the general possibility.)

Consider now the available freedom. In the presence of multiplicity each multiplicity value of the 6j can be considered as a component of a vector whose overall norm is specified by orthogonality restrictions but the unitary transformation freedom in the multiplicity index has the effect of allowing us to choose the magnitude of each component. The allowed range is of course

$$0 \leqslant \left| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}^{000r'} \right| \leqslant \left(\sum_r \left| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}^{000r} \right|^2 \right)^{\frac{1}{2}} . \tag{12.42}$$

Naturally block-diagonal restrictions on the phase-freedom matrix may restrict the choice of magnitude to be within indices of the same symmetry type. For example, in a multiplicity-two case (for $\lambda_1 = \lambda_2$) with one symmetric and the other antisymmetric there is no freedom in the magnitude at all. Restrictions such as (12.6b) complicate the general proof that the magnitude can be chosen but the validity of the claim can be demonstrated in each case studied. (See for example chapter 14.)

The question of which possible value of the magnitude to choose is an interesting problem. Butler and Ford (1979, Butler 1980b) have found that certain resolutions of multiplicity cause simplifications elsewhere, in that some moderately large prime numbers can be eliminated from the tables of 6j symbols. However, our main interest is merely in having 6j symbols with simple symmetries and we find it convenient to choose as many 6j symbols as possible to be zero. This facilitates the subsequent solution of simultaneous equations in the calculation of other symbols. Thus if the multiplicity is R we choose all but the Rth symbol to be zero. The next time we come across this triad the magnitude of the Rth symbol will be fixed but we choose all others except the (R-1)th to be zero and so on until the phase-freedom matrix is restricted to a diagonal matrix of phases. One then proceeds as for the multiplicity-free case.

The multiplicity-free case is particularly straightforward. One first evaluates the freedom (11.31) subject to the restrictions; note that outright cancellation of the four phases can occur. If freedom still remains then clearly one can choose the phase although the freedom may only be a sign; in which case it must be determined whether the symbol is real or imaginary. Normally the symmetries of the symbol suffice to determine this but there is no guarantee that this will always be so. In fact cases do exist (Haase, private communication 1980) where one must determine the reality or otherwise by considering other equations in

the algebra (cf. the case for 3jm factors; see chapter 14). After the phase has been chosen, future choices must not alter it and we must impose the additional restriction

$$U(\lambda_1\lambda_5^*\lambda_6)_{r_1}{}^{r_1}\, U(\lambda_4\lambda_2\lambda_6^*)_{r_2}{}^{r_2}\, U(\lambda_4^*\lambda_5\lambda_3)_{r_3}{}^{r_3}\, U(\lambda_1^*\lambda_2^*\lambda_3^*)_{r_4}{}^{r_4} = 1\ . \tag{12.43}$$

Note that a zero 6j cannot impose any phase restriction.

The easiest practical way of handling these restrictions is to express one phase matrix relative to the other three. Thus all phase-freedom matrices will be expressed in terms of antecedents. When a phase-freedom matrix appears in (11.31) we substitute its expression in terms of antecedents (after first making the obvious cancellations). In most cases, complete cancellation will occur after doing this and therefore there is no freedom; the 6j symbol is completely fixed. We can state though that for every new non-antecedent triad there will be at least the freedom associated with its multiplicity resolution. It is emphasized that the freedom is a relative one, but it is relative to the antecedents. Only after the freedom expressed by (11.31) is cast in terms of antecedents can we know exactly what choice there is for a 6j symbol. In all cases known to the author, a sign choice for a triad ends up being fixed relative to antecedents with sign choices, so that overall there is only a sign choice, but it may be that cases exist where a sign freedom is relative to antecedents with a complete phase freedom (cf. the case for 3jm factors, to be discussed below).

(f) Non-trivial 3jm Factors

Non-trivial 3jm factors are calculated after the 6j symbols have been determined for both group and subgroup. The specification of phase is similar to that for 6j symbols: one first determines the phase-freedom described by (11.24), expressing in the process all product freedoms in terms of antecedents or (for the subgroup, possibly) trivials. If freedom remains, one makes a choice subject to any restrictions. This then imposes a new restriction on future choices

$$U\begin{pmatrix}\lambda_1\\ \mu_1\end{pmatrix}^{a_1}_{\ a_1}\ U\begin{pmatrix}\lambda_2\\ \mu_2\end{pmatrix}^{a_2}_{\ a_2}\ U\begin{pmatrix}\lambda_3\\ \mu_3\end{pmatrix}^{a_3}_{\ a_3} U(\lambda_1\lambda_2\lambda_3)^{*r}_{\ r}\ U(\mu_1\mu_2\mu_3)^{s}_{\ s} = 1 \tag{12.44}$$

where we have written

$$U\begin{pmatrix}\lambda\\ \mu\end{pmatrix}^{a}_{\ a'} \equiv U(\lambda,\mu)^{a}_{\ a'} \tag{12.45}$$

for the sake of convenience. This can be considered as fixing the freedom in one branching relative to the other two and the product antecedents. (Actually, sometimes the phases of the other two branchings cancel.) By choosing primitive 3jms first, all branching freedoms end up being fixed relative to the primitive branchings and the product antecedents (and sometimes subgroup trivials). These expressions in terms of antecedents should be inserted in (11.24), in place of the particular branching, when considering the freedom in subsequent 3jm factors. Complete cancellation may then occur; if it does not then a choice can be made. Clearly, for each new branching there is at

least the freedom available in that branching. The total freedom is however a relative one and it is possible to fix a free sign relative to complex antecedents, so that there is an overall complete phase freedom. This then results in restrictions among the antecedents themselves. Normally when the overall freedom is only a sign, the symmetry of the 3jm factor determines whether it is real or imaginary but this is not always the case (e.g. see chapter 14). Also it is possible for freedom to result from choosing branchings relative to different antecedents and again this implies a restriction among the antecedents. Such a choice is required to complete the $T \supset C_3$ calculation studied by Butler and Wybourne (1976b). Clearly it is imperative to cast the freedom (11.24) in terms of antecedents before one can be sure of the available freedom.

All the freedom exhibited in (11.24) is a consequence of the arbitrariness allowed by Schur's lemmas. However, it can be shown that using the absence of any prior restriction among antecedents to obtain freedom is equivalent to choosing different orientations of axes (cf. Reid and Butler, 1980).

(g) Basis Functions and Representation Matrices

By this stage there is little freedom left. All phases have been chosen relative to a few product antecedents and the primitive and trivial branchings. Thus, after the 6j symbols and 3jm factors have been calculated for an entire chain of groups, all we have to do is specify the primitive vectors (and the identity vector). If "orientation" choices

have been made then even the phase of the primitive vector is restricted. Consider for example $D_3 \supset C_3 \supset C_1$. After making all the possible choices we find that the combined phase freedom in the primitive branchings for $D_3 \supset C_3$, and $C_3 \supset C_1$ is

$$U\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \sqrt[6]{1} \tag{12.46}$$

(cf. Reid and Butler, 1980).

Similarly all the representation matrices are fixed except those for the primitive. It is helpful to use the Wigner-Eckart theorem if possible to calculate these (e.g. Butler et al. 1979). Once the generators are known then all other primitive matrices follow from the group product rules. Note that the reduced matrix elements in the Wigner-Eckart theorem incorporate the freedom remaining in a product antecedent (see appendix IIA).

The matrix elements for other representations are calculated by inverting (11.12) to obtain

$$\lambda(R)^{i'}{}_{i} = \sum_{i_1 i_2 i_1' i_2'} \langle r\lambda i' | \lambda_1 i_1'; \lambda_2 i_2' \rangle \lambda_1(R)^{i_1'}{}_{i_1} \lambda_2(R)^{i_2'}{}_{i_2} \times \langle \lambda_1 i_1; \lambda_2 i_2 | r\lambda i \rangle \quad . \tag{12.47}$$

CHAPTER 13

BUTLER'S METHOD

Butler's method consists in essence of phase choices, as described for example in the last chapter, together with the hypothesis that the general properties of the 6j symbols and 3jm factors suffice to determine them when no freedom remains. (Naturally one must also be able to determine the magnitude of these quantities when a phase freedom exists and also their reality or otherwise when only a sign freedom exists.) In this chapter we shall briefly examine the completeness of this method and algorithms. We shall assume that all the phase choices have been made and that the choices are canonical. In addition we shall not concern ourselves much with nonsimple-phase representations. These certainly arise in physical problems, e.g. in SU_6 the representation $\{432^21\}$ (see Butler and King, 1974) occurs in the SU_6^{CS} coupled basis (see Part I) for $q^3\bar{q}^3$ multiquark hadrons (Wybourne 1978[a]), however they only cause a problem when they occur three times in a single triad and such triads are very unlikely in physical problems.

A key concept in Butler's method is the power of a representation which was introduced in the last chapter. We have the obvious property

$$p(\lambda) = p(\lambda^*) \tag{13.1}$$

and the very useful relation,

$$|p(\lambda_1) - p(\lambda_2)| \leqslant p(\lambda_3) \leqslant p(\lambda_1) + p(\lambda_2) \qquad (13.2)$$

whenever $\lambda_1 \times \lambda_2 \supset \lambda_3$. The proof of this last relation is a straightforward use of the cyclic properties of the Kronecker product (Butler, private communication 1978). In what follows we shall find it helpful to explicitly denote the power of a representation where it is known. If λ is of power p we shall write $\lambda^{(p)}$; the primitive is then $\lambda^{(1)}$ and the identity $\lambda^{(0)}$.

The algorithms used by Butler (Butler and Wybourne 1976a) for SO_3 and its finite point groups are not sufficiently general for all groups and even where they do work, they can be improved on. In particular Butler avoids solving simultaneous equations by using orthogonality to leave only the unknown symbol on the left of his equations (see Butler and Wybourne 1976a, equations 27 and 41). The disadvantage of this is that it often involves an awkward summation which in general ranges over symbols of higher power than those in question. In most cases it is more direct to solve the simultaneous equations and in the majority of instances this avoids going to higher power.

Let us consider then some aspects of the calculation of 6j symbols and 3jm factors.

(a) 6j Symbols

Primitive 6j symbols can apparently be calculated power by power but it seems that in some cases one needs to consider higher power symbols in order to obtain non-primitives. The procedure is: for each power

(i) determine canonical permutation and A matrices, bearing in mind the choices for trivial 3jm symbols, and note the phase restrictions

(ii) calculate the trivial 6j symbols

(iii) make admissible phase choices for non-trivial primitive 6j symbols, noting phase restrictions, and calculate those for which there is no choice

(iv) calculate non-primitive 6j symbols, if possible.

(Butler and Wybourne (1976a) have proved that non-primitive 6j symbols can be obtained once all primitive 6j symbols are known but we note that this proof does not apply to a power-by-power calculation.)

The properties of the 6j symbols one requires in these calculations, in addition to their symmetries (e.g. Butler and Wybourne, 1976a), are their orthogonality

$$\sum_{\lambda_6 r_1 r_2} |\lambda_3,\lambda_6| \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}^{*}_{r_1 r_2 r_3 r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3' \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}_{r_1 r_2 r_3' r_4'}$$

$$= \delta_{\lambda_3 \lambda_3'} \delta_{r_3 r_3'} \delta_{r_4 r_4'} \qquad (13.3)$$

the Racah backcoupling relation

$$\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \lambda_4 & \lambda_5 & \lambda_6\end{Bmatrix}_{r_1r_2r_3r_4} = \sum_{\lambda rr'} |\lambda| \{\lambda_5\}\{\lambda_4\lambda_2\lambda_6^{*}r_2\}\{\lambda_1\lambda_2\lambda_3r_4\}$$

$$\times \{\lambda_1\lambda_4\lambda^{*}r\} \begin{Bmatrix}\lambda_2 & \lambda_1 & \lambda_3\\ \lambda_4 & \lambda_5 & \lambda\end{Bmatrix}_{r'rr_3r_4} \begin{Bmatrix}\lambda_1 & \lambda_4 & \lambda^{*}\\ \lambda_2 & \lambda_5 & \lambda_6\end{Bmatrix}_{r_1r_2r'r} \tag{13.4}$$

and the Biedenharn-Elliott sum rule

$$\sum_r \begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3r} \begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \nu_1 & \nu_2 & \nu_3\end{Bmatrix}^{*}_{s_1s_2s_3r}$$

$$= \sum_{\lambda t_1t_2t_3} |\lambda| \{\lambda_1\}\{\nu_1\}\{\lambda_1\mu_2^{*}\mu_3r_1\}\{\mu_1\lambda_2\mu_3^{*}r_2\}\{\mu_1^{*}\mu_2\lambda_3r_3\}$$

$$\times \{\lambda\mu_1^{*}\nu_1t_1\}\{\lambda\mu_2^{*}\nu_2t_2\}\{\lambda\mu_3^{*}\nu_3t_3\}$$

$$\times \begin{Bmatrix}\nu_2 & \mu_2^{*} & \lambda\\ \mu_3 & \nu_3 & \lambda_1^{*}\end{Bmatrix}_{s_1r_1t_3t_2} \begin{Bmatrix}\nu_3 & \mu_3^{*} & \lambda\\ \mu_1 & \nu_1 & \lambda_2^{*}\end{Bmatrix}_{s_2r_2t_1t_3} \begin{Bmatrix}\nu_1 & \mu_1^{*} & \lambda\\ \mu_2 & \nu_2 & \lambda_3^{*}\end{Bmatrix}_{s_3r_3t_2t_1} \tag{13.5}$$

the latter of which requires generalization for nonsimple-phase triads (e.g. Butler and Wybourne 1976a, Derome and Sharp 1965).

It is known that these are the only independent equations in the 6j algebra from an analogy which exists between them and the postulates of projective geometry (Fano and Racah 1959, Robinson 1970, 1972): the Biedenharn-Elliott identity corresponds to Desargues' theorem which does not follow from preceding postulates but must be accepted as a new, and in this case last, postulate. It is almost certain that this also implies that the method is complete for 6j symbols but the role of phase-freedom, complex representations and

multiplicity in this analogy has yet to be clarified. There certainly does not exist as yet a "first-strike" algorithm for obtaining unknown symbols. One must search the equations using different parameters (and orderings thereof) until an equation is found that works.

Butler (1976 , 1980b) has found it possible to calculate primitive 6j symbols for SO_3 and the finite point groups using only orthogonality and the Racah backcoupling rule but for other groups this does not work. Butler et al. (1978, 1979) found for E_7, SU_6 and SU_3 that this leads to nonlinear equations. They were able to examine the various roots and show that a wrong choice led to subsequent contradictions. However, it is clear from the analogy with projective geometry that there is no reason why the Biedenharn-Elliott identity should not also be used for primitives. One soon finds that on doing this the required symbols are readily obtained.

In the presence of multiplicity use of the Biedenharn-Elliott identity in the form (13.5) requires the solution of linear simultaneous equations. We can show that there are sufficient equations as follows. If there is no freedom in the 6j symbol

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r}$$

then, unless $(\lambda_1 \lambda_2 \lambda_3)$ is an antecedent, there must exist another (primitive) 6j symbol in which the multiplicity in

$(\lambda_1\lambda_2\lambda_3)$ was resolved. Let the multiplicity be R and consider the case $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$; then there would have been no block-diagonal restrictions on $U(\lambda_1\lambda_2\lambda_3)$ and we could have chosen all but the Rth 6j to be zero. However, this would still have left (R-1) phases to be fixed. Since all phases have been chosen there must exist (R-1) other (primitive) 6j symbols in which the choices were made. Thus there are R known 6j symbols which can be placed with the unknown symbol on the left of (13.5) leading to the simultaneous equations

$$\begin{aligned} \{?\}_R\{1\}_R &= \ldots \\ \{?\}_{R-1}\{2\}_{R-1} + \{?\}_R\{2\}_R &= \ldots \\ &\vdots \\ \{?\}_1\{R\}_1 + \{?\}_2\{R\}_2 + \ldots + \{?\}_{R-1}\{R\}_{R-1} + \{?\}_R\{R\}_R &= \ldots \end{aligned} \tag{13.6}$$

Clearly these equations are independent and this independence must hold for any different resolution of the multiplicities. If $\lambda_1 = \lambda_2 \neq \lambda_3$ then symmetry would have restricted the choice of magnitude to be within the symmetric and anti-symmetric parts and therefore phases could have been completely specified using less than R 6j symbols. However, consider the symmetry of such a symbol: interchanging columns one and two leads to a sign change dependent on the multiplicity index. Indeed, the use of j-values and the choice (12.2) leads to

$$\begin{Bmatrix} \lambda_1 & \lambda_1 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r} = (-1)^{r_1+r_2+r_3+r} \begin{Bmatrix} \lambda_1 & \lambda_1 & \lambda_3 \\ * & * & * \\ \mu_2 & \mu_1 & \mu_3 \end{Bmatrix}_{r_2 r_1 r_3 r} \tag{13.7}$$

where r is even for symmetric parts and odd for antisymmetric. If these two symbols are identical then either the symmetric or antisymmetric cases must give a zero 6j and therefore there must exist other 6j symbols in which the phase specification was completed. On the other hand, if the symbols are distinct then we have a new symbol which can be used. Either way, there are sufficient independent equations. The case $\lambda_1 = \lambda_2 = \lambda_3$ follows in an entirely analogous manner. Even in the nonsimple-phase case the argument still holds good because the permutation matrix for the (12) interchange can always be chosen to be of the form (Butler 1975)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Of course one still has to show that all symbols appearing on the right hand side of (13.6) are known. In general they will not be and one must either enter into a recursion procedure or use another equation.

We conclude this section by noting that if there is outright cancellation of phase in a 6j symbol then the Racah backcoupling relation always gives an expression in terms of just magnitudes of other 6j symbols. This can be a useful result. Its general proof though requires use of special symmetries such as exist between

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda & \lambda & \lambda \end{Bmatrix} \text{ and } \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda^* & \lambda^* & \lambda^* \end{Bmatrix}$$

and indicates some of the peculiar difficulties one encounters when trying to formulate a proof of completeness for Butler's method.

(b) 3jm Factors

In this method 3jm factors are calculated after the primitive 6j symbols have been determined for both group and subgroup. Again, it is more expedient to indulge in a power-by-power calculation as far as possible. The procedure recommended here is: for each power

(i) determine the trivial 3jm factors

(ii) determine those primitive 3jm factors for which the group triad is an antecedent; this will involve making phase choices as described in chapter 12 (and all restrictions should be noted) but there will remain some 3jm factors which must be determined by other means

(iii) calculate all other primitive 3jm factors

(iv) calculate non-primitive 3jm factors, if possible.

In addition to the permutation and conjugation symmetries of the 3jm factors (see chapter 11) one requires their orthogonality

$$\sum_{r\lambda_3 a_3} \frac{|\lambda_3|}{|\mu_3|} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}^{r}_{s}{}^{*} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1'\mu_1 & a_2'\mu_2 & a_3\mu_3 \end{pmatrix}^{r}_{s'}$$

$$= \delta_{a_1 a_1'} \delta_{a_2 a_2'} \delta_{\mu_1 \mu_1'} \delta_{\mu_2 \mu_2'} \delta_{ss'} \qquad (13.8)$$

$$\sum_{a_1\mu_1 a_2\mu_2 s} \frac{|\lambda_3|}{|\mu_3|} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}_{s}^{r} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3' \\ a_1\mu_1 & a_2\mu_2 & a_3'\mu_3 \end{pmatrix}_{s}^{r'}$$

$$= \delta_{a_3 a_3'} \delta_{\lambda_3 \lambda_3'} \delta_{rr'} \qquad (13.9)$$

and their property

$$\sum_{r_4} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}_{s_4}^{r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$$

$$= \sum_{a_4\mu_4 a_5\mu_5 a_6\mu_6 s_1 s_2 s_3} (\lambda_4)_{a_4\mu_4, a_4^*\mu_4^*} (\lambda_5)_{a_5\mu_5, a_5^*\mu_5^*} (\lambda_6)_{a_6\mu_6, a_6^*\mu_6^*}$$

$$\times \begin{pmatrix} \lambda_1 & \lambda_5^* & \lambda_6 \\ a_1\mu_1 & a_5^*\mu_5^* & a_6\mu_6 \end{pmatrix}_{s_1}^{r_1} \begin{pmatrix} \lambda_4 & \lambda_2 & \lambda_6^* \\ a_4\mu_4 & a_2\mu_2 & a_6^*\mu_6^* \end{pmatrix}_{s_2}^{r_2} \begin{pmatrix} \lambda_4^* & \lambda_5 & \lambda_3 \\ a_4^*\mu_4^* & a_5\mu_5 & a_3\mu_3 \end{pmatrix}_{s_3}^{r_3}$$

$$\times \begin{Bmatrix} \mu_1 & \mu_2 & \mu_3 \\ \mu_4 & \mu_5 & \mu_6 \end{Bmatrix}_{s_1 s_2 s_3 s_4} \qquad (13.10)$$

This last equation is related to the Racah factorization lemma (see chapter 10). Butler has dubbed it the Wigner relation. Sometimes it is helpful to use orthogonality to place two 3jms on the left of (13.10) (Butler et al. 1979) but this contains no new information; it merely presents the information contained in the orthogonality and Wigner relations in a different form.

Butler and Wybourne (1976a) have shown that once all primitive 3jms have been obtained then all non-primitive 3jms can be obtained by recursion. However, their proof does not apply to a power-by-power calculation. They also suggested that orthogonality was sufficient to calculate all primitive 3jms but examples considered in the next

chapter provide a counterexample. In fact it has been found more direct to use (13.10) as much as possible, reserving orthogonality to determine the norm where a phase choice occurs or as a last resort in the fixed-phase case. The proof that there exist R linearly independent equations in the presence of multiplicity follows exactly as for the Biedenharn-Elliott identity.

Assume that all 3jm factors with an antecedent group triad have been calculated and consider those remaining primitive 3jms for which the group triad is not an antecedent. These can usually all be calculated using (13.10) in the following straightforward fashion. Let the antecedent triad be $(\lambda_1^{(p)}\ \lambda_5^{*(p-1)}\ \lambda^{(1)})$ and the fixed primitive be $(\lambda_1^{(p)}\ \lambda_2^{(p')}\ \lambda^{(1)})$ or $(\lambda_1^{(p)}\ \lambda_2^{(p')}\ \lambda^{*(1)})$ where p' = p or p-1, then there will exist a 6j symbol in which the phase of the fixed primitive was chosen. Mostly, it seems, it will be of the form

$$\begin{Bmatrix} \lambda_1^{(p)} & \lambda_2^{(p')} & \lambda^{(1)} \\ \lambda_4^{(p'-1)} & \lambda_5^{(p-1)} & \lambda^{(1)} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \lambda_1^{(p)} & \lambda_2^{(p')} & \lambda^{*(1)} \\ \lambda_4^{(p'-1)} & \lambda_5^{(p-1)} & \lambda^{(1)} \end{Bmatrix} .$$

(The absence of a proof that such a symbol always exists was why we could not be more precise in our discussion of antecedents in chapter 12. Note that multiplicity rarely, or never, arises in primitive triads.) Inserting this 6j alongside the unknown 3jms in (3.10) gives the entire class of such factors, since the expansion of

$$\begin{pmatrix} \lambda_1^{(p)} & \lambda_2^{(p')} & \lambda^{(1)} \\ . & . & . \end{pmatrix} \begin{Bmatrix} \lambda_1^{(p)} & \lambda_2^{(p')} & \lambda^{(1)} \\ \lambda_4^{(p'-1)} & \lambda_5^{(p-1)} & \lambda^{(1)} \end{Bmatrix} = \ldots \qquad (13.11)$$

is entirely known if p' = p-1 or can be obtained by recursion if p' = p.

Assume now that all primitives are known and consider non-primitive 3jm factors. Let the group triad be $(\lambda_1^{(p_1)}\ \lambda_2^{(p_2)}\ \lambda_3^{(p_3)})$ with $p_1 \geqslant p_2 \geqslant p_3$. Since the phase has been fixed there will exist a primitive 6j in which this phase was chosen. (We have shown that multiplicity can easily be dealt with so we can ignore it here for clarity.) Let this 6j be

$$\begin{Bmatrix} \lambda_1^{(p_1)} & \lambda_2^{(p_2)} & \lambda_3^{(p_3)} \\ \lambda_4^{(p_4)} & \lambda_5^{(p_5)} & \lambda^{(1)} \end{Bmatrix} .$$

If $p_4 = p_2-1$ or p_2 and $p_5 = p_1-1$ or p_1 then the required non-primitive 3jms can be calculated without going to higher power. Insert this 6j alongside the unknown non-primitive 3jm and expand using (3.10)

$$\begin{pmatrix} \lambda_1^{(p_1)} & \lambda_2^{(p_2)} & \lambda_3^{(p_3)} \\ . & . & . \end{pmatrix} \begin{Bmatrix} \lambda_1^{(p_1)} & \lambda_2^{(p_2)} & \lambda_3^{(p_3)} \\ \lambda_4^{(p_4)} & \lambda_5^{(p_5)} & \lambda^{(1)} \end{Bmatrix} = \ldots \qquad (13.12)$$

The right-hand side contains at least two primitive 3jms and since the number of representations of any power is finite it follows that if the third is non-primitive, a recursion procedure will eventually reduce it down to primitives.

It remains to consider those primitives involving a group antecedent for which the phase is fixed. In these cases one tries

$$\begin{pmatrix} \lambda_1^{(p)} & \lambda_2^{(p-1)} & \lambda^{(1)} \\ a_1\mu_1 & a_2\mu_2 & a\mu \end{pmatrix} \begin{Bmatrix} \lambda_1^{(p)} & \lambda_2^{(p-1)} & \lambda^{(1)} \\ \lambda_4^{(p-2)} & \lambda_2^{*(p-1)} & \lambda^{(1)} \end{Bmatrix} = \ldots \qquad (13.13)$$

At least one 6j of the required form must exist; we only have to consider the antecedent triad $(\lambda_2^{(p-1)}\ \lambda_4^{(p-2)}\ \lambda^{*(1)})$. The relative phase of the branching $\lambda_1 \to \mu_1$ will have been fixed in the previous 3jm. If for this branching there are only two subgroup triads then the summation on the right hand side of (13.10) will contain both the unknown 3jm and the known 3jm (for which the phase was fixed) together with primitives of lower power. Thus we can solve for the unknown. If there are two or more unknowns for the same branching (i.e. for which there is no remaining freedom) then one must find other equations. Sometimes non-antecedent primitive triads will supply 6js for these; sometimes one can also resort to orthogonality relations to find extra equations. However, it is useless trying to discover a general proof that there are sufficient equations because a counterexample exists!

Consider $T \supset D_2$. (We use Butler's 1980b notation, see also Reid and Butler 1980). There are 7 distinct non-trivial primitive 3jms including

$$\begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \tilde{0} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

each of which has magnitude $(1/3)^{\frac{1}{2}}$. The phase of the first can be chosen real and positive but the methods of chapter 12 can then be used to prove that after this and other previous choices the freedom in the other two vanishes. The best we can do for the other two is to solve a quadratic equation which yields the solutions

$$\begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} = \frac{-1}{2\sqrt{3}} \pm \frac{i}{2}$$

$$\begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} = -\frac{1}{2\sqrt{3}} \mp \frac{i}{2} \qquad (13.14)$$

(It was these three factors which Butler, 1980a used to demonstrate that some 3jms must be general complex.) There are only three possibilities which may exist. Either

(i) there are more equations

(ii) there is an additional source of freedom

(iii) the method fails.

The first can easily be ruled out because Butler's method uses only the general properties of the 3jm factors and the properties of the representations. It happens that in D_2 and the $T \to D_2$ branching rules, $\tilde{0}$,1 and -1 are indistinguishable by their general properties; they have the same sets of characters but differ in the classes to which they are assigned. The only way the algebra can yield different values for the three factors concerned is by different phase choices but this has not happened here. Therefore it is impossible for the algebra to provide equations which could distinguish either of the pairs of values in (13.14).

It turns out that the second explanation is correct. Throughout we have assumed that the representations can be uniquely identified but this is clearly not true here. In many cases this indeterminateness simply results in a symmetry for the 6j symbols and 3jm factors containing similar

representations but here we are forced into making a distinction. Consider the representation matrices for T. The tetrahedral group can be generated by a 2-fold and a 3-fold rotation. Such matrices for the primitive $\frac{1}{2}$ representation are not difficult to set up for the C_2 class with character, $\chi^{\frac{1}{2}}(C_2) = 0$ but for the 3-fold rotations we must make an arbitrary distinction between the classes C_3 and $\bar{C}_3^{-1}$ both of which have character +1. A possible pair of generators is

$$\underset{\sim}{\frac{1}{2}}(C_{2y}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \underset{\sim}{\frac{1}{2}}(C_3) = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \\ -\frac{1}{2} - \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \end{pmatrix} .$$

(Note that $\frac{1}{2}$ is a faithful representation and therefore the second matrix cannot be in both C_3 and $\bar{C}_3^{-1}$. Also remember that we are dealing with a spinor representation of a double-valued group.) If one now proceeds to construct other representation matrices using the building up procedure in (12.47) one finds no problem with the 2-fold operation and also all product rules hold for both sets of 3jm factors in (13.14). However, for the 3-fold operation, when we get to the representation $\frac{3}{2}$ we find that one of the solutions in (13.14) yields a representation matrix with character $\omega^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ whereas the other solution yields a representation matrix with character $\omega = -\frac{1}{2} - \frac{\sqrt{3}}{2} i \ (= e^{-i2\pi/3})$. Only the second is in the class C_3, the other is in the class $\bar{C}_3^{-1}$ and thus the choice in (13.14) reflects the arbitrary choice that must be made between classes. Another way of viewing this is that the choice in (13.14) reflects

the arbitrary distinction that must be made between the representations $\frac{3}{2}$ and $-\frac{3}{2}$ in T. (This distinction is equivalent to the distinction that must be made between the classes.)

Once this choice is made all remaining 3jm factors can be obtained. Thus Butler's method still appears to work, provided that we can recognize such freedom and make an appropriate choice. It is rather disconcerting though that the group character table depends on the choice made.

Reid and Butler (1980) have shown that this choice corresponds to two distinct ways of orienting a tetrahedron about D_2 axes. Such orientations are in fact well known, especially within the context of the icosahedral group (e.g. Judd 1957), however their effect on the 3jm factors is only now becoming understood.

CHAPTER 14

EXAMPLES

In this chapter we shall use the group chain (3.17),

$$U_{18} \supset [SU_{12} \supset SU_2^I \times (SU_6^{CS} \supset SU_2^S \times SU_3^C)]$$

$$\times [U_6 \supset U_1^S \times (SU_6^{CS} \supset SU_2^S \times SU_3^C)] \supset SU_2^I \times U_1^S \times SU_2^S \times SU_3^C \ ,$$

from Part I to illustrate the application of Butler's method and the procedures for making phase choices. Thus we shall demonstrate how the 6j symbols and 3jm factors required in Part I may be obtained. It is not proposed to tabulate all the coefficients that may be required though. Firstly, such a tabulation would be lengthy and secondly a more systematic calculation in the future — incorporating different phase choices, based perhaps on the duality between S_n and U_N — would render it obsolete. Instead we shall concentrate on presenting those coefficients required for the dissociation calculations in Part I. (Since we have tabulated in Part I the matrix elements of the colour-spin operator there is little point in also tabulating the 3jm factors and 6j symbols used to calculate them. It should more than suffice to list a few symbols indicating the phase choices used.)

(a) 6j Symbols

We require 6j symbols for U_1, SU_2, SU_3, SU_6, U_6, SU_{12} and U_{18} — the last four groups being included as 6j symbols for them will be required when we calculate the necessary 3jm factors. For SU_2 we can use the tabulation of Rotenberg et al. (1959). With U_1 it is possible to choose phases so that all 6j symbols are +1. For the other groups we enter upon a calculation using Butler's method as described in chapters 12 and 13. We note that U_N is isomorphic to $U_1 \times SU_N$ and 6j symbols for U_N can be factorized into U_1 and SU_N 6js. It follows that U_N 6j symbols can be chosen to be equivalent to SU_N 6js. Note that separations of this sort depend on a factorizable phase choice (chapter 11) and while always possible, and recommended, it must be stressed that non-factorizable choices also exist.

The most extensive tabulations required are for SU_3 and SU_6. Butler et al. (1979) give j-values for representations of these groups up to power four (but not complete for this power). To these we can add, for SU_6, $j(2^2) = 0$, $j(31) = 1$, $j(4) = 0$. One chooses these j-values by attempting to describe as many fixed transposition phases as possible in terms of them. In general it is not always possible to do this but in the examples considered here one shows by explicit checking that it can be done. The main advantage of using j-values is that the column interchange symmetry of a 6j is given by (13.7), where one has also labelled the multiplicity indices alternately symmetric and antisymmetric.

The representations of a given power are all found by explicitly evaluating that power of the primitives

({1} + {1}*) and removing lower power terms. It is useful though to have a formula to determine the power of a given representation. Wybourne (1978b) has shown that in the defining representation

$$p(\lambda) = \sum_{i=1}^{N} \lambda_i - \sum_{i=N+1}^{2N-1} \lambda_i \qquad \text{for } SU_{2N} \tag{14.1}$$

$$p(\lambda) = \sum_{i=1}^{N} \lambda_i - \sum_{i=N+2}^{2N} \lambda_i \qquad \text{for } SU_{2N+1} \; . \tag{14.2}$$

Some useful, but incomplete, tabulations of SU_3 and SU_6 6j symbols have been computer generated by Butler and Haase (1979, 1978). In the examples below we follow their choices where the symbols under consideration appear.

Consider the SU_6 6j symbols. One begins by calculating trivials, easily obtaining by (12.40)

$$\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} = +1 \quad , \quad \begin{Bmatrix} 1^5 & 1 & 0 \\ 0 & 0 & 0 \end{Bmatrix} = \frac{-1}{\sqrt{2.3}} \quad , \quad \begin{Bmatrix} 1^5 & 1 & 0 \\ 1^5 & 1^5 & 0 \end{Bmatrix} = \frac{-1}{2.3}$$

etc. (14.3)

The first non-trivial symbol encountered is

$$\begin{Bmatrix} 21^4 & 1^5 & 1 \\ 1^2 & 1 & 1 \end{Bmatrix}$$

of maximum power two and phase freedom

$$U(21^4\ 1^5\ 1)\ U(1^2\ 1^5\ 1^5)\ U(1^4\ 1\ 1)\ U(21^4\ 1\ 1^5) = +1;$$

the freedom completely cancelling because of the restrictions (12.6a) and (12.14). Similarly the freedom in

$$\begin{Bmatrix} 21^4 & 1^5 & 1 \\ 2 & 1 & 1 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} 21^4 & 1^5 & 1 \\ 21^4 & 1^5 & 1^5 \end{Bmatrix}$$

cancels outright. Consider the last of these. Using the Racah backcoupling rule (13.4) one finds

$$\begin{Bmatrix} 1 & 1^5 & 21^4 \\ 1 & 1 & 21^4 \end{Bmatrix} = 15 \cdot \left| \begin{Bmatrix} 21^4 & 1^5 & 1 \\ 1^2 & 1 & 1 \end{Bmatrix} \right|^2 - 21 \cdot \left| \begin{Bmatrix} 21^4 & 1^5 & 1 \\ 2 & 1 & 1 \end{Bmatrix} \right|^2 .$$

(Note how we permute the columns to obtain a tractable equation.) These magnitudes are readily obtained. For example, orthogonality (13.3) gives

$$35 \cdot \left| \begin{Bmatrix} 1^5 & 1^5 & 2 \\ 1 & 1^5 & 21^4 \end{Bmatrix} \right|^2 + 1 \cdot \left| \begin{Bmatrix} 1^5 & 1^5 & 2 \\ 1 & 1^5 & 0 \end{Bmatrix} \right|^2 = \frac{1}{21}$$

$$\Rightarrow \left| \begin{Bmatrix} 21^4 & 1^5 & 1 \\ 2 & 1 & 1 \end{Bmatrix} \right| = \frac{1}{2.3.7} .$$

Thus

$$\begin{Bmatrix} 21^4 & 1^5 & 1 \\ 21^4 & 1^5 & 1^5 \end{Bmatrix} = \frac{+1}{2.3.5.7} . \qquad (14.4)$$

Now using the Racah backcoupling rule again we find (using 14.4 and known trivials)

$$\begin{Bmatrix} 1^5 & 1 & 21^4 \\ 1 & 1 & 1^2 \end{Bmatrix} = \frac{+1}{2.3.5} \qquad (14.5)$$

and similarly

$$\begin{Bmatrix} 1^5 & 1 & 21^4 \\ 1 & 1 & 2 \end{Bmatrix} = \frac{+1}{2.3.7} . \qquad (14.6)$$

Next we come to

$$\begin{Bmatrix} 21^4 & 1^4 & 1^2 \\ 1 & 1^5 & 1^5 \end{Bmatrix}$$

with freedom

$$U(21^4\ 1\ 1^5)\ U(1\ 1^4\ 1)\ U(1^5\ 1^5\ 1^2)\ U(21^4\ 1^2\ 1^4)$$

$$= U(21^4\ 1\ 1^5)U(21^4\ 1^2\ 1^4)$$

which allows only a sign choice. However, permutation and conjugation symmetry tells us that

$$\begin{Bmatrix} 21^4 & 1^4 & 1^2 \\ 1 & 1^5 & 1^5 \end{Bmatrix} = + \begin{Bmatrix} 21^4 & 1^2 & 1^4 \\ 1^5 & 1 & 1 \end{Bmatrix} = \begin{Bmatrix} 21^4 & 1^4 & 1^2 \\ 1 & 1^5 & 1^5 \end{Bmatrix}^*$$

and therefore the symbol is real. Its magnitude is easily found from orthogonality and thus we can choose

$$\begin{Bmatrix} 21^4 & 1^4 & 1^2 \\ 1 & 1^5 & 1^5 \end{Bmatrix} = \frac{+1}{3.5} \quad . \tag{14.7}$$

This implies the restriction

$$U(21^4\ \ 1^4\ \ 1^2) = U(21^4\ \ 1^5\ \ 1) \tag{14.8}$$

on future choices. Similarly we choose the sign in

$$\begin{Bmatrix} 21^4 & 2^5 & 2 \\ 1 & 1^5 & 1^5 \end{Bmatrix} = \frac{\sqrt{2}}{3.7} \tag{14.9}$$

which implies the phase restriction

$$U(21^4\ \ 2^5\ \ 2) = U(21^4\ \ 1^5\ \ 1) \ . \tag{14.10}$$

The 6j symbol

$$\begin{Bmatrix} 21^4 & 2 & 1^4 \\ 1^5 & 1 & 1 \end{Bmatrix}$$

has the freedom

$$U(21^4\ \ 1^5\ \ 1)\ U(1^5\ \ 2\ \ 1^5)\ U(1\ \ 1\ \ 1^4)\ U(21^4\ \ 2^5\ \ 1^2)$$

which is not restricted. Hence we choose the symbol to be real and positive

$$\begin{Bmatrix} 21^4 & 2 & 1^4 \\ 1^5 & 1 & 1 \end{Bmatrix} = \frac{+1}{\sqrt{2.3.5.7}}\ , \qquad (14.11)$$

where once again the magnitude is easily obtained from orthogonality. This leads to the restriction on future choices

$$U(21^4\ \ 2\ \ 1^4) = U(21^4\ \ 1^5\ \ 1)\ U(2\ \ 1^5\ \ 1^5)\ U(1^4\ \ 1\ \ 1). \qquad (14.12)$$

Notice how the restrictions (14.8), (14.10) and (14.12) fix the freedom in a non-primitive triad in terms of the freedom in antecedents.

Consider next the pair of symbols

$$\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0000} \text{ and } \begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0001}.$$

$U(21^4\ \ 21^4\ \ 21^4)$ is restricted to being a diagonal matrix of signs and so the magnitudes are fixed. The overall

freedom in either symbol is also only a sign but by symmetry the first 6j is real and the second imaginary. Orthogonality gives

$$\left|\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0000}\right|^2 + \left|\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0001}\right|^2 = \frac{17}{3.35.35}$$

while the Biedenharn-Elliott identity (13.5) gives

$$\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0000} \begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1^5 & 1^5 & 1^5 \end{Bmatrix}^*_{0000}$$

$$+ \begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0001} \begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1^5 & 1^5 & 1^5 \end{Bmatrix}^*_{0001}$$

$$= \left|\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0000}\right|^2 - \left|\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0001}\right|^2$$

$$= \frac{1}{3.35.35} \quad .$$

Thus we can choose

$$\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0000} = \frac{\sqrt{3}}{5.7} \tag{14.13}$$

$$\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0001} = \frac{+2i\sqrt{2}}{5.7\sqrt{3}} \tag{14.14}$$

which imposes the restrictions

$$U(21^4 \quad 21^4 \quad 21^4)^0_0 = U(21^4 \quad 1^5 \quad 1) \tag{14.15}$$

$$U(21^4 \quad 21^4 \quad 21^4)^1_1 = U(21^4 \quad 1^5 \quad 1) \tag{14.16}$$

on future choices.

We now know all power-two primitives. The power-two non-primitives present difficulties to a power-by-power calculation so let us proceed to power three primitives. Of some interest is the symbol

$$\begin{Bmatrix} 21 & 2^5 & 1^5 \\ 1 & 1^2 & 1^5 \end{Bmatrix}$$

with freedom given by

$$U(21\ \ 1^4\ \ 1^5)\ U(1\ \ 2^5\ \ 1)\ U(1^5\ \ 1^2\ \ 1^5)\ U(2^4 1\ \ 2\ \ 1)\ .$$

Although all triads are primitives there exists complete phase freedom; orthogonality fixes the magnitude and we choose

$$\begin{Bmatrix} 21 & 2^5 & 1^5 \\ 1 & 1^2 & 1^5 \end{Bmatrix} = \frac{+1}{2\sqrt{3.5.7}}\ . \tag{14.17}$$

This implies the phase restriction

$$U(21\ \ 2^5\ \ 1^5) = U(21\ \ 1^4\ \ 1^5)\ U(2^5\ \ 1\ \ 1)\ U(1^2\ \ 1^5\ \ 1^5) \tag{14.18}$$

where we have selected $(21\ \ 1^4\ \ 1^5)$ as the antecedent triad. The symbol

$$\begin{Bmatrix} 21 & 1^3 & 21^4 \\ 1 & 1 & 1^4 \end{Bmatrix}$$

has an obvious freedom since the triad $(2^4 1\ \ 1^3\ \ 21^4)$ occurs for the first time and by itself. The magnitude can be easily

obtained from orthogonality (by summing over representations in the position occupied by $\{21^4\}$) and we can choose the phase giving

$$\begin{Bmatrix} 21 & 1^3 & 21^4 \\ 1 & 1 & 1^4 \end{Bmatrix} = \frac{+1}{5\sqrt{3.7}} \qquad (14.19)$$

This implies the phase restriction

$$U(21\ \ 1^3\ \ 21^4) = U(21\ \ 1^4\ \ 1^5)\ U(1^3\ \ 1^2\ \ 1)\ U(21^4\ \ 1^5\ \ 1). \qquad (14.20)$$

Consider now the symbol

$$\begin{Bmatrix} 21 & 1^3 & 21^4 \\ 1^2 & 2 & 1^5 \end{Bmatrix}$$

which has the freedom expressed by

$$U(21\ \ 2^5\ \ 1^5)\ U(1^2\ \ 1^3\ \ 1)\ U(1^4\ \ 2\ \ 21^4)\ U(2^4 1\ \ 1^3\ \ 21^4)\ .$$

Inserting the restrictions (14.18), (14.12) and (14.20) shows that this freedom equals +1, i.e. no choice is possible; the symbol is completely fixed by previous choices. It can be found by using the Biedenharn-Elliott identity in the form

$$\begin{Bmatrix} 21 & 1^3 & 21^4 \\ 1^2 & 2 & 1^5 \end{Bmatrix} \begin{Bmatrix} 21 & 1^3 & 21^4 \\ 1 & 1 & 1^4 \end{Bmatrix}^* = \ldots$$

on the right hand side of which appears the primitive symbols (14.17), (14.11) and

$$\begin{Bmatrix} 1^3 & 1^2 & 1 \\ 1^5 & 1^4 & 1 \end{Bmatrix} = \frac{+1}{3.5}$$

whose freedom cancels outright and which could have been previously determined by the Racah backcoupling rule. We obtain

$$\begin{Bmatrix} 21 & 1^3 & 21^4 \\ 1^2 & 2 & 1^5 \end{Bmatrix} = \frac{-1}{5\sqrt{2.3.7}} \quad . \tag{14.21}$$

Note that if a choice had not been made in (14.19) the phase of (14.21) could have been chosen and this would have led to precisely the same restriction as (14.20), since we always express phase restrictions in terms of antecedents.

It is instructive to examine an example of multiplicity separation. The power-four symbols

$$\begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 1^3 & 1 \end{Bmatrix}_{0000} , \quad \begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 1^3 & 1 \end{Bmatrix}_{0001}$$

and

$$\begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 2^4 1 & 1 \end{Bmatrix}_{0000} , \quad \begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 2^4 1 & 1 \end{Bmatrix}_{0001}$$

both have the freedom

$$U(1 \;\; 21^4 \;\; 1^5) \; U(21^2 \;\; 21^4 \;\; 2^3 1^2)$$

where $U(1 \;\; 21^4 \;\; 1^5)$ is restricted to ± 1 and $U(21^2 \;\; 21^4 \;\; 2^3 1^2)$ is a two-by-two unitary matrix, restricted by (12.6b) and (12.14) to the form

$$\begin{pmatrix} r_1 & i_1 \\ i_2 & r_2 \end{pmatrix}$$

where r_1 and r_2 are real and i_1 and i_2 are imaginary.

Clearly we could use the freedom to choose either pair of symbols. Consider one pair and denote the multiplicity, r = 0 symbol by x and the r = 1 symbol by y. Symmetry tells us that x is real and y imaginary — since we have chosen all 3j transposition phases to be given by (12.2) and the permutation symmetry is then given by (13.7). For any x and y, let

$$U(21^2 \quad 21^4 \quad 2^3 1^2) = \frac{1}{\sqrt{|x|^2 + |y|^2}} \begin{pmatrix} \pm x & \pm y^* \\ y & -x \end{pmatrix} \tag{14.22}$$

then a new pair of 6j symbols is given by

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = U(21^2 \quad 21^4 \quad 2^3 1^2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \pm \sqrt{|x|^2 + |y|^2} \\ 0 \end{pmatrix} . \tag{14.23}$$

An alternative allowed form for U is

$$U(21^2 \quad 21^4 \quad 2^3 1^2) = \frac{i}{\sqrt{|x|^2 + |y|^2}} \begin{pmatrix} y & -x \\ \pm x^* & \pm y^* \end{pmatrix} \tag{14.24}$$

leading to

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \pm i\sqrt{|x|^2 + |y|^2} \end{pmatrix} . \tag{14.25}$$

Thus the freedom in $U(21^2 \quad 21^4 \quad 2^3 1^2)$ is still enough to allow us to choose the magnitudes of x and y, but after choosing the magnitudes we are restricted to a sign choice. By orthogonality (summing over the position occupied by $\{21^4\}$):

$$\sum_r \left| \begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 1^3 & 1 \end{Bmatrix}_{000r} \right|^2 = \frac{61}{4.9.25.49}$$

and

$$\sum_r \left| \begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 2^4 1 & 1 \end{Bmatrix}_{000r} \right|^2 = \frac{4}{9.25.49} \quad .$$

We choose

$$\begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 2^4 1 & 1 \end{Bmatrix}_{0000} = \frac{+2}{3.5.7} \tag{14.26}$$

and

$$\begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 2^4 1 & 1 \end{Bmatrix}_{0001} = 0 \quad . \tag{14.27}$$

Now $U(21^2 \quad 21^4 \quad 2^3 1^2)$ is restricted to a diagonal matrix where

$$U(21^2 \quad 21^4 \quad 2^3 1^2)^0_0 = U(21^4 \quad 1^5 \quad 1) \tag{14.28}$$

but (14.27) does not impose any restriction on $U(21^2\, 21^4\, 2^3 1^2)^1_1$ and it is still an independent real phase. To find the other pair of symbols we begin by using the Biedenharn-Elliott identity

$$\begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 1^3 & 1 \end{Bmatrix}_{0000} \begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 2^4 1 & 1 \end{Bmatrix}^*_{0000} + \begin{Bmatrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 1^3 & 1 \end{Bmatrix}_{0001} \cdot 0$$

$$= -35 \begin{Bmatrix} 1 & 1^5 & 21^4 \\ 1 & 1 & 21^4 \end{Bmatrix} \left| \begin{Bmatrix} 21 & 1 & 2^3 1^2 \\ 1^5 & 1^3 & 21^4 \end{Bmatrix} \right|^2 \quad .$$

The required magnitude is simply obtained and the other symbol is known, (14.4) so that

$$\left\{\begin{matrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 1^3 & 1 \end{matrix}\right\}_{0000} = \frac{-1}{4.3.5.7} \quad . \tag{14.29}$$

It is now trivial to obtain the magnitude of the remaining symbol and make the sign choice

$$\left\{\begin{matrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 1^3 & 1 \end{matrix}\right\}_{0001} = \frac{-i3\sqrt{3}}{4.5.7} \tag{14.30}$$

resulting in the final restriction

$$U(21^2 \;\; 21^4 \;\; 2^3 1^2)^1_1 = U(21^4 \;\; 1^5 \;\; 1) \; . \tag{14.31}$$

These examples cover most of the problems that arise in a primitive 6j calculation and certainly should suffice to give the reader familiarity with the techniques presented in earlier chapters. We list in tables II1 and II2 some of the phase choices made and resulting restrictions for SU_3 and SU_6 6j symbols respectively.

The 6j symbols for U_{18}, SU_{12} and U_6 can be obtained in the same manner but it is interesting to consider them together.

We require 6j symbols of the form

$$\left\{\begin{matrix} 1^{n*} & 1^{n-m} & 1^{m} \\ 1^{\ell *} & 1^{m+\ell *} & 1^{n-m-\ell} \end{matrix}\right\}$$ where m=0, ... , n/2 for n even or (n-1)/2 for n odd and ℓ = 0, ..., n-2m.

From orthogonality the modulus of all such 6js is $|1^{n-m}, 1^{m+\ell}|^{-\frac{1}{2}}$. (The required dimensions can be found in tables I1-3 of part I.) For all SU_N with N even, the

canonical 2j phases are given by $\phi_{1^n} \equiv \{1^n\} = (-1)^n$ (Butler and King, 1974). Further, all 3j interchange phases (both fixed and arbitrary) of the form $\{1^{n*}\ 1^{n-m}\ 1^m\}$ can be given via j-values. With such choices, none of these 6j symbols change sign under interchange of columns. A valid choice of j-value is

$$j(1^n) = j(1^{n*}) = \begin{cases} n & n \text{ even} \\ n/2 & n \text{ odd} \end{cases} . \qquad (14.32)$$

Most of the 6j symbols of the stated form are trivial. Note that the phases of these depend, via (12.40), only on a 3j interchange phase which is independent of the group. The non-trivial 6j symbols can be calculated power by power in the defining representation. There are only thirteen such symbols up to power six. Interestingly, it is possible to choose phases so that the phase of all 6j symbols is group independent. (This means that the recoupling coefficients can be chosen group independently.) The phases are listed in table II3, together with comments on the method of calculation. Note that while there are four free phases for SU_{18} and SU_{12}, three disappear in SU_6 where they are fixed either by symmetry or the triviality condition. To maintain group independence, the SU_{18} and SU_{12} phases must be chosen the same as their SU_6 counterparts. The one free choice remaining was chosen so as to coincide with the already mentioned computer calculation of SU_6 6js (Butler and Haase, 1978) used when calculating colour-spin matrix elements in Part I. Unfortunately, that arbitrary

choice does not coincide with the corresponding SU_4 6j which is fixed. Notice that in several 6js there is an outright cancellation of phase freedom and the Racah backcoupling rule gives the group independent formula

$$\begin{Bmatrix} 1^{n*} & 1^{n-m} & 1^{m} \\ 1^{n-2m*} & 1^{n-m*} & 1^{m} \end{Bmatrix}$$

$$= \{1^{m}\}\{1^{n-m*}\ 1^{n-2m}\ 1^{m}\}\{1^{n*}\ 1^{n-m}\ 1^{m}\}\{1^{n*}\ 1^{n-2m}\ 1^{2m}\}|1^{n-m}|^{-1} . \quad (14.33)$$

The remaining three symbols depend on phase choices made elsewhere and are fixed by the Biedenharn-Elliott identity. It is clear that their phases will be group independent if the previous choices were. Readers interested in this group independence are referred to Butler (1980a, and references therein) and the recent papers by Sullivan (1980) and Chen (1980a,b). These approach the problem via the duality between S_n and U_N, relating U_N overlap integrals to transformation coefficients between symmetric group chains, which are independent of N. However, it is clear from the development given here that the essential feature of this phenomenon is the group independence of the Kronecker products and symmetrized powers. Thus similar results must also hold for classes of groups such as SO_N and Sp_N; a fact not apparent from duality arguments.

(b) 3jm Factors

We require 3jm factors for $U_{18} \supset SU_{12} \times U_6$, $SU_{12} \supset SU_2 \times SU_6$, $U_6 \supset U_1 \times SU_6$, $SU_6 \supset SU_2 \times SU_3$ and

$(SU_2 \times SU_3) \times (SU_2 \times SU_3) \supset SU_2 \times SU_3$. The 3jm factors for the last case are proportional to 9j symbols (see chapter 10 and Part I, chapter 3) which can be factorized. For the case $U_6 \supset U_1 \times SU_6$, the 3jm factors are trivial because the subgroup is isomorphic to the group itself. With our factorized 6js for U_6, the 3jm factors are all +1. The 3jm factors for the other group-subgroup embeddings can be calculated using Butler's method as described in the last two chapters. We shall illustrate the application of these techniques using $SU_6 \supset SU_2 \times SU_3$.

In the decomposition of a representation $\{\lambda\}$ of SU_6 to irreducible representations of $SU_2 \times SU_3$, the SU_2 representations appearing correspond to integer values of the spin if the weight of $\{\lambda\}$ is even and half-(odd-)integer values if the weight is odd (e.g. see Part I, chapter 3). Since the canonical 2j-phases are +1 for SU_3, $(-1)^{2S}$ for SU_2 and $(-1)^{\ell}$ (where ℓ = weight of $\{\lambda\}$) for SU_6, this means that the permutational symmetry of the 2jm factors, given by (10.40) and (11.8), is even (Butler et al. 1979). Thus we can choose all 2jm factors to be +1 which imposes the restriction, (12.14)

$$U\begin{pmatrix} \lambda^* \\ |S|_{\mu^*} \end{pmatrix} = U\begin{pmatrix} \lambda \\ |S|_{\mu} \end{pmatrix}^* \quad .$$

The trivial 3jm factors are then immediately given by the choice (12.35) e.g.

$$\begin{pmatrix} 0 & 0 & 0 \\ {}^{1}0 & {}^{1}0 & {}^{1}0 \end{pmatrix}^{+} = 1, \quad \begin{pmatrix} 1^5 & 1 & 0 \\ {}^{2}1^{2} & {}^{2}1 & {}^{1}0 \end{pmatrix}^{+} = 1, \begin{pmatrix} 1^4 & 1^2 & 0 \\ {}^{3}1 & {}^{3}1^{2} & {}^{1}0 \end{pmatrix}^{+} = (2/5)^{\frac{1}{2}}$$

etc. (In all 3jm factors we shall, for convenience, indicate the column interchange symmetry — given by the product of 3j transposition phases for group and subgroup triads (Butler, 1975) — by a + or - superscript.)

For all power-two non-trivial primitive 3jms a choice exists. For example

$$\begin{pmatrix} 1^4 & 1 & 1 \\ {}^1 2^2 & {}^2 1 & {}^2 1 \end{pmatrix}^+$$

has the freedom (11.24)

$$U\begin{pmatrix} 1^4 \\ {}^1 2^2 \end{pmatrix} U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}^2 U_6(1^4\ 1\ 1)^*\, U_2(0\ \tfrac{1}{2}\ \tfrac{1}{2})\, U_3(2^2\ 1\ 1)$$

which allows a complete phase choice. The magnitude is easily found from orthogonality (13.9) and we choose

$$\begin{pmatrix} 1^4 & 1 & 1 \\ {}^1 2^2 & {}^2 1 & {}^2 1 \end{pmatrix}^+ = +(2/5)^{\frac{1}{2}} \qquad (14.34)$$

resulting in the phase restriction

$$U\begin{pmatrix} 1^2 \\ {}^1 2 \end{pmatrix} = U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}^2 U_6(1^4\ 1\ 1)^*\, U_2(0\ \tfrac{1}{2}\ \tfrac{1}{2})\, U_3(2^2\ 1\ 1)\ . \qquad (14.35)$$

Similarly we can choose the phases of

$$\begin{pmatrix} 1^4 & 1 & 1 \\ {}^3 1 & {}^2 1 & {}^2 1 \end{pmatrix}^+ , \begin{pmatrix} 2^5 & 1 & 1 \\ {}^1 1 & {}^2 1 & {}^2 1 \end{pmatrix}^+ , \begin{pmatrix} 2^5 & 1 & 1 \\ {}^3 2^2 & {}^2 1 & {}^2 1 \end{pmatrix}^+$$

resulting in the restrictions

$$U\begin{pmatrix} 1^2 \\ {}^3 1^2 \end{pmatrix} = U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}^2 U_6(1^4\ 1\ 1)^*\ U_2(1\ \tfrac{1}{2}\ \tfrac{1}{2})\ U_3(1\ 1\ 1) \qquad (14.36)$$

$$U\begin{pmatrix} 2 \\ {}^1 1^2 \end{pmatrix} = U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}^2 U_6(2^5\ 1\ 1)^*\ U_2(0\ \tfrac{1}{2}\ \tfrac{1}{2})\ U_3(1\ 1\ 1) \qquad (14.37)$$

$$U\begin{pmatrix} 2 \\ {}^3 2 \end{pmatrix} = U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}^2 U_6(2^5\ 1\ 1)^*\ U_2(1\ \tfrac{1}{2}\ \tfrac{1}{2})\ U_3(2^2\ 1\ 1)\ . \qquad (14.38)$$

The 3jm factor

$$\begin{pmatrix} 21^4 & 1^5 & 1 \\ {}^3 0 & {}^2 1^2 & {}^2 1 \end{pmatrix}^+$$

has the freedom

$$U\begin{pmatrix} 21^4 \\ {}^3 0 \end{pmatrix}\ U\begin{pmatrix} 1^5 \\ {}^2 1^2 \end{pmatrix}\ U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}\ U_6(21^4\ 1^5\ 1)^*\ U_2(1\ \tfrac{1}{2}\ \tfrac{1}{2})\ U_3(0\ 1^2\ 1)$$

$$= U\begin{pmatrix} 21^4 \\ {}^3 0 \end{pmatrix} U_6(21^4\ 1^5\ 1)\ U_2(1\ \tfrac{1}{2}\ \tfrac{1}{2})\ U_3(0\ 1^2\ 1) = \pm 1$$

and only a sign choice is allowed. However, the factor is real by symmetry and we choose

$$\begin{pmatrix} 1^5 & 21^4 & 1 \\ {}^2 1^2 & {}^3 0 & {}^2 1 \end{pmatrix}^+ = +(3/5.7)^{\frac{1}{2}} \qquad (14.39)$$

resulting in the restriction

$$U\begin{pmatrix} 21^4 \\ {}^3 0 \end{pmatrix} = U_6(21^4\ 1^5\ 1)\ U_2(1\ \tfrac{1}{2}\ \tfrac{1}{2})\ U_3(0\ 1^2\ 1). \qquad (14.40)$$

Similarly a sign choice leads to

$$\begin{pmatrix} 1^5 & 21^4 & 1 \\ {}^2 1^2 & {}^3 21 & {}^2 1 \end{pmatrix}^{+} = + (8.3/5.7)^{\frac{1}{2}} \tag{14.41}$$

$$\begin{pmatrix} 1^5 & 21^4 & 1 \\ {}^2 1^2 & {}^1 21 & {}^2 1 \end{pmatrix}^{-} = + i(8/5.7)^{\frac{1}{2}} \tag{14.42}$$

with the accompanying restrictions

$$U\begin{pmatrix} 21^4 \\ {}^3 21 \end{pmatrix} = U_6(21^4\ 1^5\ 1)\ U_2(1\ \tfrac{1}{2}\ \tfrac{1}{2})\ U_3(21\ 1^2\ 1) \tag{14.43}$$

$$U\begin{pmatrix} 21^4 \\ {}^1 21 \end{pmatrix} = U_6(21^4\ 1^5\ 1)\ U_2(0\ \tfrac{1}{2}\ \tfrac{1}{2})\ U_3(21\ 1^2\ 1)\ . \tag{14.44}$$

The power-two non-primitives include the SU_6 triads $(1^4\ 21^4\ 1^2)$ and $(2^5\ 21^4\ 2)$ and the sets of 3jm factors for these triads can all be calculated via the procedure (13.12) using the SU_6 6j symbols chosen in (14.7) and (14.9). (These 3jm factors were used in the calculation of reduced matrix elements in Part I.) It should be clear that these factors have no phase freedom, since we can explicitly calculate them, but it may be worthwhile for the reader to check that with all the restrictions made to date, the freedom in a factor such as

$$\begin{pmatrix} 1^4 & 21^4 & 1^2 \\ {}^1 2^2 & {}^3 21 & {}^3 1^2 \end{pmatrix}^{+}$$

completely cancels. An interesting case of multiplicity is afforded by the triad $(21^4\ 21^4\ 21^4)$. The 3jm factors for this case can again be obtained via the procedure (13.12) by simply solving a pair of simultaneous equations using the

6j symbols in (14.13) and (14.14) together with the pair obtained by conjugating their arguments. This provides an example of one of the cases mentioned in chapter 13 when proving that there are sufficient 6j symbols to give the required number of simultaneous equations. However, the triad $(1^2\ 1^2\ 1^2)$ poses something of a problem to a power-by-power calculation. The associated phase choice in the 6j calculation was not made until the power-three symbol

$$\begin{Bmatrix} 1^3 & 1^2 & 1 \\ 1^2 & 1 & 1^4 \end{Bmatrix}$$

and the direct procedure (13.12) must wait until we have calculated power-three primitive 3jms.

The power-three primitives provide some interesting examples. Consider the 3jm factor

$$\begin{pmatrix} 1^3 & 1^2 & 1 \\ {}^2 21 & {}^1 2 & {}^2 1 \end{pmatrix}^+ .$$

It has freedom expressed by

$$U\begin{pmatrix} 1^3 \\ {}^2 21 \end{pmatrix}\ U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}^3\ U_6(1^3\ 1^2\ 1)^*\ U_6(1^4\ 1\ 1)^*\ U_3(21\ 1^2\ 1)\ U_3(1\ 1\ 1)$$

but although $U\begin{pmatrix} 1^3 \\ {}^2 21 \end{pmatrix}$ is restricted to being a sign, most other transformation matrices are not and a complete phase choice is allowed. We choose

$$\begin{pmatrix} 1^3 & 1^2 & 1 \\ {}^2 21 & {}^1 2 & {}^2 1 \end{pmatrix}^+ = +(2/5)^{\frac{1}{2}} \qquad (14.45)$$

which results in the restriction

$$U\begin{pmatrix} 1^3 \\ {}^2 21 \end{pmatrix} = U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}^3 U_6(1^3\ 1^2\ 1)^* U_6(1^4\ 1\ 1)^* U_3(21\ 1^2\ 1)\ U_3(1\ 1\ 1)\ . \tag{14.46}$$

This in turn implies that

$$U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}^3 U_6(1^3\ 1^2\ 1)^* U_6(1^4\ 1\ 1)^* U_3(1\ 1\ 1) = \pm 1\ . \tag{14.47}$$

After this choice the freedom in

$$\begin{pmatrix} 1^3 & 1^2 & 1 \\ {}^2 21 & {}^3 1^2 & {}^2 1 \end{pmatrix}^+$$

vanishes but its value may be found using the procedure (13.13)

$$\begin{pmatrix} 1^3 & 1^2 & 1 \\ {}^2 21 & {}^3 1^2 & {}^2 1 \end{pmatrix}^+ \begin{Bmatrix} 1^3 & 1^2 & 1 \\ 1^5 & 1^4 & 1 \end{Bmatrix} = \ldots\ .$$

Now consider

$$\begin{pmatrix} 1^3 & 1^2 & 1 \\ {}^4 0 & {}^3 1^2 & {}^2 1 \end{pmatrix}^-\ .$$

After (14.47), the freedom in this 3jm factor is only a sign choice, but symmetry does not tell us whether the factor is real or imaginary. However, expanding

$$\begin{pmatrix} 1^3 & 1^2 & 1 \\ {}^4 0 & {}^3 1^2 & {}^2 1 \end{pmatrix} \begin{Bmatrix} 1^3 & 1^2 & 1 \\ 1^3 & 1^2 & 1^5 \end{Bmatrix} = \ldots$$

shows that

$$\begin{pmatrix} 1^3 & 1^2 & 1 \\ {}^4 0 & {}^3 1^2 & {}^2 1 \end{pmatrix}^2 = \frac{1}{2} \cdot \begin{pmatrix} 1^3 & 1^2 & 1 \\ {}^2 21 & {}^3 1^2 & {}^2 1 \end{pmatrix}^2$$

and hence we choose

$$\begin{pmatrix} 1^3 & 1^2 & 1 \\ {}^4 0 & {}^3 1^2 & {}^2 1 \end{pmatrix}^- = -(1/5)^{\frac{1}{2}} \qquad (14.48)$$

(Note the generic form of the 6j symbol used.) This now implies the restriction

$$U\begin{pmatrix} 1^3 \\ {}^4 0 \end{pmatrix} = U\begin{pmatrix} 1 \\ {}^2 1 \end{pmatrix}^3 U_6(1^3\ 1^2\ 1)^*\, U_6(1^4\ 1\ 1)^*\, U_2({}^3_2\ 1\ \tfrac{1}{2})\, U_2(1\ \tfrac{1}{2}\ \tfrac{1}{2})$$

$$\times\, U_3(1\ 1\ 1)\, U_3(0\ 1^2\ 1) \quad . \qquad (14.49)$$

(These 3jms were obtained by Butler et al. (1979) but the techniques used by those authors offer no justification for the above choices.) The remaining power-three phase choices are straightforward. For instance we can choose four phases in the set of 3jms associated with the antecedent triad $(2^4 1\ \ 1^2\ 1)$ and the remaining 3jm is obtained via the procedure (13.13) using the SU_6 6j symbol

$$\begin{Bmatrix} 2^4 1 & 1^2 & 1 \\ 1^5 & 1^4 & 1 \end{Bmatrix} \quad .$$

Then the entire class of 3jms for the non-antecedent triad $(2^4 1\ \ 2\ \ 1)$ is readily obtained via the procedure (13.11) using the 6j symbol chosen in (14.17)

$$\begin{pmatrix} 2^4 1 & 2 & 1 \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{Bmatrix} 2^4 1 & 2 & 1 \\ 1^5 & 1^4 & 1 \end{Bmatrix} = \cdots \quad .$$

These examples should suffice to acquaint the reader with the techniques presented in chapters 12 and 13. The 3jm factors for $U_{18} \supset SU_{12} \times U_6$ and $SU_{12} \supset SU_2 \times SU_3$ can be calculated similarly. For $U_{18} \supset SU_{12} \times U_6$ all 2jm factors can be chosen +1. The 2jm factors for $SU_{12} \supset SU_2 \times SU_6$ are chosen to be (for $n \leqslant 6$)

$$(1^n)_{I\lambda, I\lambda^*} = +1$$

$$(1^{12-n})_{I\lambda^*, I\lambda} = \begin{cases} +1 & n \text{ } even \\ -1 & n \text{ } even \end{cases} \qquad (14.50)$$

Note that negative 2jm factors slightly complicate the use of the Wigner relation (13.10) and also for some 3jm factors they lead to a sign change on conjugation of the arguments dictated by the Derome-Sharp lemma.

In tables II4,5 and 6 we list some of those 3jm factors necessary for the dissociation calculations of Part I. These tables suffice to do all dissociations of $q^2\bar{q}^2$ and $q^4\bar{q}$ multiquark hadrons and q^6 multiquark hadrons for strangeness $\leqslant -2$. For convenience the tables include the permutation and conjugation symmetries of these factors.

Table II1: Some SU_3 phase choices

choice	restriction
$\left\{\begin{matrix}21 & 2 & 1\\ 1^2 & 1 & 1\end{matrix}\right\}_{0000} = \frac{+1}{2\sqrt{2.3}}$	$U(21\ 2\ 1) = U(21\ 1^2\ 1)\ U(2\ 1^2\ 1^2)$ $\times\ U(1\ 1\ 1)$
$\left\{\begin{matrix}21 & 2 & 1\\ 2 & 1 & 2^2\end{matrix}\right\}_{0000} = \frac{+1}{4\sqrt{3}}$	$U(2\ 2\ 2) = U(2\ 1^2\ 1^2)^3\ U(1\ 1\ 1)^2$
$\left\{\begin{matrix}21 & 2 & 1\\ 21 & 1^2 & 2\end{matrix}\right\}_{0000} = \frac{+\sqrt{5}}{8.3}$	$U(21\ 2^2\ 2) = U(21\ 1^2\ 1)$
$\left\{\begin{matrix}21 & 21 & 21\\ 1 & 1 & 1\end{matrix}\right\}_{0000} = \frac{+\sqrt{5}}{8\sqrt{2.3}}$	$U(21\ 21\ 21)^0_0 = U(21\ 1^2\ 1)$
$\left\{\begin{matrix}21 & 21 & 21\\ 1 & 1 & 1\end{matrix}\right\}_{0001} = \frac{+i\sqrt{3}}{8\sqrt{2}}$	$U(21\ 21\ 21)^1_1 = U(21\ 1^2\ 1)$
$\left\{\begin{matrix}3 & 21 & 21\\ 1^2 & 2 & 1^2\end{matrix}\right\}_{0000} = \frac{+1}{4\sqrt{3}}$	$U(3\ 21\ 21) = U(3\ 2^2\ 1^2)\ U(2\ 1^2\ 1^2)$ $\times\ U(1\ 1\ 1)$
$\left\{\begin{matrix}3^2 & 3 & 21\\ 1^2 & 1^2 & 2\end{matrix}\right\}_{0000} = \frac{+1}{2\sqrt{3.5}}$	$U(3^2\ 3\ 21) = U(21\ 1^2\ 1)$
$\left\{\begin{matrix}31 & 21 & 1^2\\ 1 & 1^2 & 2^2\end{matrix}\right\}_{0000} = \frac{+1}{2.3\sqrt{2}}$	$U(31\ 21\ 1^2) = U(31\ 2^2\ 1)\ U(21\ 1^2\ 1)$ $\times\ U(2\ 1^2\ 1^2)$
$\left\{\begin{matrix}31 & 31 & 1\\ 1^2 & 1 & 21\end{matrix}\right\}_{0000} = \frac{+1}{2\sqrt{3.5}}$	$U(31\ 31\ 1) = U(31\ 2^2\ 1)^2\ U(2\ 1^2\ 1^2)^2$ $\times\ U(1\ 1\ 1)$
$\left\{\begin{matrix}31 & 3^2 & 1^2\\ 1 & 1^2 & 2^2\end{matrix}\right\}_{0000} = \frac{+1}{3\sqrt{5}}$	$U(31\ 3^2\ 1^2) = U(31\ 2^2\ 1)\ U(3^2\ 2\ 1)$ $\times\ U(1^2\ 1^2\ 1)$
$\left\{\begin{matrix}4^2 & 31 & 21\\ 1^2 & 1^2 & 3\end{matrix}\right\}_{0000} = \frac{-1}{4\sqrt{5}}$	$U(4^2\ 31\ 21) = U(4^2\ 3\ 1)\ U(3^2 2\ 1)$ $\times\ U(31\ 2^2\ 1)\ U(21\ 1^2\ 1)\ U(1^2\ 1^2\ 1^2)$
$\left\{\begin{matrix}4^2 & 4 & 21\\ 1^2 & 1^2 & 3\end{matrix}\right\}_{0000} = \frac{-\sqrt{7}}{4.3\sqrt{5}}$	$U(4^2\ 4\ 21) = U(21\ 1^2\ 1)$

Table II2: Some SU_6 phase choices

choice		restriction
$\begin{Bmatrix} 21^4 & 1^4 & 1^2 \\ 1 & 1^5 & 1^5 \end{Bmatrix}_{0000} = \frac{+1}{3.5}$		$U(21^4\ 1^4\ 1^2) = U(21^4\ 1^5\ 1)$
$\begin{Bmatrix} 21^4 & 2 & 1^4 \\ 1^5 & 1 & 1 \end{Bmatrix}_{0000} = \frac{+1}{\sqrt{2.3.5.7}}$		$U(21^4\ 2\ 1^4) = U(21^4\ 1^5\ 1)\ U(2\ 1^5\ 1^5) \times U(1^4\ 1\ 1)$
$\begin{Bmatrix} 21^4 & 2^5 & 2 \\ 1 & 1^5 & 1^5 \end{Bmatrix}_{0000} = \frac{+\sqrt{2}}{3.7}$		$U(21^4\ 2^5\ 2) = U(21^4\ 1^5\ 1)$
$\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0000} = \frac{+\sqrt{3}}{5.7}$		$U(21^4\ 21^4\ 21^4)^0{}_0 = U(21^4\ 1^5\ 1)$
$\begin{Bmatrix} 21^4 & 21^4 & 21^4 \\ 1 & 1 & 1 \end{Bmatrix}_{0001} = \frac{2i\sqrt{2}}{5.7\sqrt{3}}$		$U(21^4\ 21^4\ 21^4)^1{}_1 = U(21^4\ 1^5\ 1)$
$\begin{Bmatrix} 1^3 & 1^2 & 1 \\ 1^2 & 1 & 1^4 \end{Bmatrix}_{0000} = \frac{+1}{2.5\sqrt{3}}$		$U(1^2\ 1^2\ 1^2) = U(1^3\ 1^2\ 1)^2\ U(1^2\ 1^5\ 1^5)$
$\begin{Bmatrix} 1^3 & 1^3 & 21^4 \\ 1 & 1 & 1^4 \end{Bmatrix}_{0000} = \frac{+1}{2.5\sqrt{2.3}}$		$U(1^3\ 1^3\ 21^4) = U(21^4\ 1^5\ 1)$
$\begin{Bmatrix} 21 & 2^5 & 1^5 \\ 1 & 1^2 & 1^5 \end{Bmatrix}_{0000} = \frac{+1}{2\sqrt{3.5.7}}$		$U(21\ 2^5\ 1^5) = U(21\ 1^4\ 1^5)\ U(2^5\ 1\ 1) \times U(1^2\ 1^5\ 1^5)$
$\begin{Bmatrix} 21 & 1^3 & 21^4 \\ 1 & 1 & 1^4 \end{Bmatrix}_{0000} = \frac{+1}{5\sqrt{3.7}}$		$U(21\ 1^3\ 21^4) = U(21\ 1^4\ 1^5)\ U(1^3\ 1^2\ 1) \times U(21^4\ 1^5\ 1)$
$\begin{Bmatrix} 2^41 & 21 & 21^4 \\ 1^5 & 1^5 & 1^2 \end{Bmatrix}_{0000} = \frac{+3}{2.5.7}$		$U(2^41\ 21\ 21^4)^0{}_0 = (21^4\ 1^5\ 1)$
$\begin{Bmatrix} 2^41 & 21 & 21^4 \\ 1^5 & 1^5 & 1^2 \end{Bmatrix}_{0001} = 0$		$U(2^41\ 21\ 21^4)^0{}_1 = 0 = U(2^41\ 21\ 21^4)^1{}_0$ i.e. block diagonal
$\begin{Bmatrix} 2^41 & 21 & 21^4 \\ 1^5 & 1^5 & 2 \end{Bmatrix}_{0000} = \frac{+1}{2.9.5.7}$		———
$\begin{Bmatrix} 2^41 & 21 & 21^4 \\ 1^5 & 1^5 & 2 \end{Bmatrix}_{0001} = \frac{+8i\sqrt{2}}{9.5.7}$		$U(2^41\ \ 21\ \ 21^4)^1{}_1 = (21^4\ 1^5\ 1)$

$$\left\{\begin{matrix} 21^3 & 21^4 & 1 \\ 1^2 & 1^4 & 1^2 \end{matrix}\right\}_{0000} = \frac{+1}{5\sqrt{3.7}}$$

$U(21^3\ 21^4\ 1) = U(21^3\ 1^2\ 1^5)\ U(21^4\ 1^5\ 1)$
$\times\ U(1^4\ 1\ 1)$

$$\left\{\begin{matrix} 2^3 1^2 & 21 & 1 \\ 1^4 & 1^3 & 1 \end{matrix}\right\}_{0000} = \frac{+1}{15\sqrt{7}}$$

$U(2^3 1^2\ 21\ 1) = U(2^3 1^2\ 1^3\ 1)\ U(21\ 1^4\ 1^5)$
$\times\ U(1^3\ 1^2\ 1)$

$$\left\{\begin{matrix} 2^4 & 2^2 & 21^4 \\ 1^5 & 1^5 & 21 \end{matrix}\right\}_{0000} = \frac{+2}{3.35}$$

$U(2^4\ 2^2\ 21^4) = U(21^4\ 1^5\ 1)$

$$\left\{\begin{matrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 2^4 1 & 1 \end{matrix}\right\}_{0000} = \frac{+2}{3.35}$$

$U(2^3 1^2\ 21^2\ 21^4)^0{}_0 = U(21^4\ 1^5\ 1)$

$$\left\{\begin{matrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 2^4 1 & 1 \end{matrix}\right\}_{0001} = 0$$

$U(2^3 1^2\ 21^2\ 21^4)^0{}_1 = 0 = U(2^3 1^2\ 21^2\ 21^4)^1{}_0$
i.e. block diagonal

$$\left\{\begin{matrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 1^3 & 1 \end{matrix}\right\}_{0000} = \frac{-1}{4.3.5.7}$$

———

$$\left\{\begin{matrix} 2^3 1^2 & 21^4 & 21^2 \\ 1 & 1^3 & 1 \end{matrix}\right\}_{0001} = \frac{-i3\sqrt{3}}{4.5.7}$$

$U(2^3 1^2\ 21^2\ 21^4)^1{}_1 = U(21^4\ 1^5\ 1)$

$$\left\{\begin{matrix} 3^4 2 & 3 & 1 \\ 2^5 & 2^4 1 & 1 \end{matrix}\right\}_{0000} = \frac{+1}{3.7\sqrt{2.5}}$$

$U(3^4 2\ 3\ 1) = U(3^4 2\ 21\ 1)\ U(2^4 1\ 1^2\ 1)$
$\times\ U(1^4\ 1\ 1)\ U(3\ 2^5\ 1^5)\ U(2\ 1^5\ 1^5)$

Table II3: Phases of non-trivial 6j symbols for U_{18}, SU_{12} and U_6.

6j symbol						phase	method of calculation		
λ_1	λ_2	λ_3	λ_4	λ_5	λ_6		SU_{18}	SU_{12}	SU_6
1^{3*}	1^2	1	1^*	1^{2*}	1	+	RBC	RBC	RBC
1^{4*}	1^3	1	1^*	1^{2*}	1^2	+	chosen	chosen	chosen
			1^{2*}	1^{3*}	1	-	RBC	RBC	RBC
1^{5*}	1^3	1^2	1^*	1^{3*}	1^2	-	RBC	RBC	RBC
1^{5*}	1^4	1	1^*	1^{2*}	1^3	+	chosen	chosen	symmetry
			1^{2*}	1^{3*}	1^2	+	BE	BE	BE
			1^{3*}	1^{4*}	1	+	RBC	RBC	RBC
1^{6*}	1^4	1^2	1^*	1^{3*}	1^3	+	BE	BE	trivial *or* BE
			1^{2*}	1^{4*}	1^2	+	RBC	RBC	trivial *or* RBC
1^{6*}	1^5	1	1^*	1^{2*}	1^4	-	chosen	chosen	trivial
			1^{2*}	1^{3*}	1^3	+	chosen	chosen	trivial
			1^{3*}	1^{4*}	1^2	+	BE	BE	trivial *or* BE
			1^{4*}	1^{5*}	1	-	RBC	RBC	trivial *or* RBC

Table II4: $U_{18} \supset SU_{12} \times U_6$ 3jm factors

1^{2*}	1	1	$1^{10}.0$	1.0	1.0	+†	$+(2.11/3.17)^{\frac{1}{2}}$
			$1^{11}.1*$	1.0	0.1	-	$+(4/17)^{\frac{1}{2}}$
			0.1^{2*}	0.1	0.1	+	$+(5/3.17)^{\frac{1}{2}}$
1^{3*}	1^2	1	$1^9.0$	$1^2.0$	1.0	+	$+(5.11/4.3.17)^{\frac{1}{2}}$
			$1^{10}.1*$	$1^2.0$	0.1	-	$+(11/4.17)^{\frac{1}{2}}$
			$1^{10}.1*$	1.1	1.0	+	$-(11/2.17)^{\frac{1}{2}}$
			$1^{11}.1^{2*}$	1.1	0.1	+	$+(5/2.17)^{\frac{1}{2}}$
			$1^{11}.1^{2*}$	0.1^2	1.0	-	$+(5/4.17)^{\frac{1}{2}}$
			$0.1^{3}*$	0.1^2	0.1	+	$+(5/4.3.17)^{\frac{1}{2}}$
1^{4*}	1^3	1	$1^8.0$	$1^3.0$	1.0	+	$+(11/4.17)^{\frac{1}{2}}$
			$1^9.1*$	$1^3.0$	0.1	+	$+(11/2.3.17)^{\frac{1}{2}}$
			$1^9.1*$	$1^2.1$	1.0	-	$-(11/2.17)^{\frac{1}{2}}$
			$1^{10}.1^{2*}$	$1^2.1$	0.1	-	$+(11/4.17)^{\frac{1}{2}}$
			$1^{10}.1^{2*}$	1.1^2	1.0	-	$+(11/4.17)^{\frac{1}{2}}$
			$1^{11}.1^{3*}$	1.1^2	0.1	-	$+(1/17)^{\frac{1}{2}}$
			$1^{11}.1^{3*}$	0.1^3	1.0	+	$-(1/3.17)^{\frac{1}{2}}$
			0.1^{4*}	0.1^3	0.1	+	$+(1/4.3.17)^{\frac{1}{2}}$
1^{6*}	1^3	1^3	$1^6.0$	$1^3.0$	$1^3.0$	+	$+(11/13.17)^{\frac{1}{2}}$
			$1^7.1*$	$1^3.0$	$1^2.1$	+	$+(2.9.11/7.13.17)^{\frac{1}{2}}$
			$1^8.1^{2}*$	$1^3.0$	1.1^2	-	$-(9.5.11/4.7.13.17)^{\frac{1}{2}}$
			$1^8.1^{2*}$	$1^2.1$	$1^2.1$	+	$+(27.5.11/4.7.13.17)^{\frac{1}{2}}$
			$1^9.1^{3*}$	$1^3.0$	0.1^3	-	$-(5.11/3.7.13.17)^{\frac{1}{2}}$
			$1^9.1^{3*}$	$1^2.1$	1.1^2	-	$-(3.5.11/7.13.17)^{\frac{1}{2}}$
			$1^{10}.1^{4*}$	$1^2.1$	0.1^3	-	$+(3.11/2.7.13.17)^{\frac{1}{2}}$
			$1^{10}.1^{4*}$	1.1^2	1.1^2	+	$+(9.11/2.7.13.17)^{\frac{1}{2}}$
			$1^{11}.1^{5*}$	1.1^2	0.1^3	+	$-(3/7.13.17)^{\frac{1}{2}}$
			0.1^{6*}	0.1^3	0.1^3	+	$+(1/4.3.7.13.17)^{\frac{1}{2}}$

† column interchange symmetry.

Table II5: $SU_{12} \supset SU_2 \times SU_6$ 3jm factors

1^{10}	1	1	0.2^5	$\frac{1}{2}.1$	$\frac{1}{2}.1$	+	$+(7/2.11)^{\frac{1}{2}}$
			1.1^4	$\frac{1}{2}.1$	$\frac{1}{2}.1$	+	$+(3.5/2.11)^{\frac{1}{2}}$
1^9	1^2	1	$\frac{1}{2}.2^4 1$	0.2	$\frac{1}{2}.1$	+*	$+(7/2.11)^{\frac{1}{2}}$
			$\frac{1}{2}.2^4 1$	1.1^2	$\frac{1}{2}.1$	+*	$-(7/2.11)^{\frac{1}{2}}$
			$\frac{3}{2}.1^3$	1.1^2	$\frac{1}{2}.1$	-*	$+(4/11)^{\frac{1}{2}}$
1^8	1^3	1	0.2^4	$\frac{1}{2}.21$	$\frac{1}{2}.1$	-	$+(7/3.11)^{\frac{1}{2}}$
			$1.2^3 1^2$	$\frac{1}{2}.21$	$\frac{1}{2}.1$	-	$+(2.7/3.11)^{\frac{1}{2}}$
			$1.2^3 1^2$	$\frac{3}{2}.1^3$	$\frac{1}{2}.1$	+	$+(7/3.11)^{\frac{1}{2}}$
			2.1^2	$\frac{3}{2}.1^3$	$\frac{1}{2}.1$	+	$+(5/3.11)^{\frac{1}{2}}$
1^8	1^2	1^2	0.2^4	0.2	0.2	+	$-(7/2.3.11)^{\frac{1}{2}}$
			0.2^4	1.1^2	1.1^2	+	$-(7/2.3.11)^{\frac{1}{2}}$
			$1.2^3 1^2$	0.2	1.1^2	+	$-(7/3.11)^{\frac{1}{2}}$
			$1.2^3 1^2$	1.1^2	1.1^2	+	$+(7/3.11)^{\frac{1}{2}}$
			2.1^2	1.1^2	1.1^2	+	$+(5/3.11)^{\frac{1}{2}}$
1^7	1^3	1^2	$\frac{1}{2}.2^3 1$	$\frac{1}{2}.21$	0.2	+*	$+(7/3.11)^{\frac{1}{2}}$
			$\frac{1}{2}.2^3 1$	$\frac{1}{2}.21$	1.1^2	+*	$-(7/3.11)^{\frac{1}{2}}$
			$\frac{1}{2}.2^3 1$	$\frac{3}{2}.1^3$	1.1^2	-*	$+(7/2.3.11)^{\frac{1}{2}}$
			$\frac{3}{2}.2^2 1^3$	$\frac{1}{2}.21$	1.1^2	+*	$+(7/3.11)^{\frac{1}{2}}$
			$\frac{3}{2}.2^2 1^3$	$\frac{3}{2}.1^3$	0.2	-*	$-(7/2.3.11)^{\frac{1}{2}}$
			$\frac{3}{2}.2^2 1^3$	$\frac{3}{2}.1^3$	1.1^2	-*	$-(7/2.3.11)^{\frac{1}{2}}$
			$\frac{5}{2}.1$	$\frac{3}{2}.1^3$	1.1^2	-*	$-(1/2.11)^{\frac{1}{2}}$

* changes sign under conjugation of the arguments

Table II6: $SU_6 \supset SU_2 \times SU_3$ 3jm factors

1^4	1	1	$^1 2^2$	$^2 1$	$^2 1$	+	$+(2/5)^{\frac{1}{2}}$
			$^3 1$	$^2 1$	$^2 1$	+	$+(3/5)^{\frac{1}{2}}$
2^5	1	1	$^1 1$	$^2 1$	$^2 1$	+	$+(1/7)^{\frac{1}{2}}$
			$^3 2^2$	$^2 1$	$^2 1$	+	$+(2.3/7)^{\frac{1}{2}}$
1^3	1^2	1	$^2 21$	$^1 2$	$^2 1$	+	$+(2/5)^{\frac{1}{2}}$
			$^2 21$	$^3 1^2$	$^2 1$	+	$+(2/5)^{\frac{1}{2}}$
			$^4 0$	$^3 1^2$	$^2 1$	-	$-(1/5)^{\frac{1}{2}}$
$2^4 1$	1^2	1	$^2 0$	$^3 1^2$	$^2 1$	+	$+(1/5.7)^{\frac{1}{2}}$
			$^2 21$	$^1 2$	$^2 1$	+	$-(4/5.7)^{\frac{1}{2}}$
			$^2 21$	$^3 1^2$	$^2 1$	+	$+(4/5.7)^{\frac{1}{2}}$
			$^2 3^2$	$^1 2$	$^2 1$	-	$-(2/7)^{\frac{1}{2}}$
			$^4 21$	$^3 1^2$	$^2 1$	-	$-(16/5.7)^{\frac{1}{2}}$
$2^4 1$	2	1	$^2 0$	$^1 1^2$	$^2 1$	+	$-(1/5.7)^{\frac{1}{2}}$
			$^2 21$	$^1 1^2$	$^2 1$	+	$+(4/5.7)^{\frac{1}{2}}$
			$^2 21$	$^3 2$	$^2 1$	+	$+(4/5.7)^{\frac{1}{2}}$
			$^2 3^2$	$^3 2$	$^2 1$	-	$-(2/7)^{\frac{1}{2}}$
			$^4 21$	$^3 2$	$^2 1$	-	$+(16/5.7)^{\frac{1}{2}}$
$2^3 1^2$	1^3	1	$^1 1^2$	$^2 21$	$^2 1$	+	$+(1/5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^2 21$	$^2 1$	-	$-(1/5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^4 0$	$^2 1$	+	$+(2/5.7)^{\frac{1}{2}}$
			$^3 2$	$^2 21$	$^2 1$	+	$+(2.3/5.7)^{\frac{1}{2}}$
			$^5 1^2$	$^4 0$	$^2 1$	-	$-(1/7)^{\frac{1}{2}}$
$2^3 1^2$	21	1	$^1 1^2$	$^2 0$	$^2 1$	+	$+(1/2.5.7)^{\frac{1}{2}}$
			$^1 1^2$	$^2 21$	$^2 1$	+	$-(1/2.5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^2 0$	$^2 1$	-	$-(1/2.5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^2 21$	$^2 1$	-	$-(2/5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^4 21$	$^2 1$	+	$-(1/2.5.7)^{\frac{1}{2}}$
			$^3 2$	$^2 21$	$^2 1$	+	$-(3/5.7)^{\frac{1}{2}}$
			$^3 2$	$^4 21$	$^2 1$	-	$+(3/5.7)^{\frac{1}{2}}$
			$^5 1^2$	$^4 21$	$^2 1$	-	$-(1/7)^{\frac{1}{2}}$
2^4	21	1	$^1 2$	$^2 21$	$^2 1$	+	$+(2/5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^2 0$	$^2 1$	+	$-(1/5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^2 21$	$^2 1$	+	$+(1/5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^4 21$	$^2 1$	-	$-(1/5.7)^{\frac{1}{2}}$
			$^5 2$	$^4 21$	$^2 1$	-	$-(2/7)^{\frac{1}{2}}$

1^2	1^2	1^2	$^1 2$	$^1 2$	$^1 2$	+	$-(1/5)^{\frac{1}{2}}$
			$^1 2$	$^3 1^2$	$^3 1^2$	+	$+(1/5)^{\frac{1}{2}}$
			$^3 1^2$	$^3 1^2$	$^3 1^2$	+	$-(1/5)^{\frac{1}{2}}$
$2^3 1^2$	1^2	1^2	$^1 1^2$	$^3 1^2$	$^3 1^2$	+	$-(1/5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^1 2$	$^3 1^2$	-	$-(3/2.5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^3 1^2$	$^3 1^2$	-	0
			$^3 2$	$^3 1^2$	$^3 1^2$	+	$+(2.3/5.7)^{\frac{1}{2}}$
			$^5 1^2$	$^3 1^2$	$^3 1^2$	+	$+(1/7)^{\frac{1}{2}}$
$2^3 1^2$	2	1^2	$^1 1^2$	$^1 1^2$	$^1 2$	+	$-(3/4.5.7)^{\frac{1}{2}}$
			$^1 1^2$	$^3 2$	$^3 1^2$	+	$+(1/4.5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^1 1^2$	$^3 1^2$	-	$-(3/2.5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^3 2$	$^3 1^2$	-	$+(3/2.5.7)^{\frac{1}{2}}$
			$^3 2$	$^1 1^2$	$^3 1^2$	+	$+(3/2.5.7)^{\frac{1}{2}}$
			$^3 2$	$^3 2$	$^1 2$	+	$+(9/2.5.7)^{\frac{1}{2}}$
			$^5 1^2$	$^3 2$	$^3 1^2$	+	$+(1/7)^{\frac{1}{2}}$
2^4	1^2	1^2	$^1 2$	$^3 1^2$	$^3 1^2$	+	$+(1/5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^1 2$	$^3 1^2$	+	$-(1/2.5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^3 1^2$	$^3 1^2$	+	$-(2/5.7)^{\frac{1}{2}}$
			$^5 2$	$^3 1^2$	$^3 1^2$	+	$+(2/7)^{\frac{1}{2}}$
2^4	2	2	$^1 2$	$^1 1^2$	$^1 1^2$	+	$-(1/5.7)^{\frac{1}{2}}$
			$^1 2$	$^3 2$	$^3 2$	+	$+(1/5.7)^{\frac{1}{2}}$
			$^3 1^2$	$^1 1^2$	$^3 2$	+	$+(3/2.5.7)^{\frac{1}{2}}$
			$^5 2$	$^3 2$	$^3 2$	+	$-(2/7)^{\frac{1}{2}}$
2^4	1^2	1^2	$^1 2$	$^1 2$	$^1 2$	+	$+(1/5.7)^{\frac{1}{2}}$

CHAPTER 15

CONCLUSIONS

In Part II we have studied the relationship between coupling (and recoupling) coefficients and transformation coefficients. We have found that general factorization properties of transformation coefficients make the important Racah factorization lemma look almost trivial. Similarly we have seen that antilinear transformations provide a natural framework within which to discuss the conjugation properties of general transformation coefficients and this leads to a profound view of the Derome-Sharp lemma. This approach shows that the Derome-Sharp A matrix is essentially a 2jm factor for the embedding of the inner-product group in the outer-product group.

We have discussed the origins of phase-freedom, which lie in the resolution of branching multiplicity, and have been able to use the relationship between coupling and transformation coefficients to precisely describe the freedom in 2jm factors, 3jm factors, 6j symbols and 3j permutation matrices. This leads to analytic techniques for making phase choices. The canonical choices of Derome (1966) and Butler and King (1974) have been reviewed with emphasis on consistency and determination of the freedom remaining after choices have been made. Included is a discussion of choices for trivial 3jm factors and trivial 6j symbols which have not been adequately treated before

because the definition of the 2jm factor generally used embodies phase choices. The same analytic techniques employed in making canonical choices can then be used to show how to determine the freedom remaining in non-trivial 6j symbols and 3jm factors and to make admissible choices. A key concept here is that of antecedent phase freedoms with respect to which all other phase choices are made.

Armed with these phase-fixing procedures we then discussed Butler's method of calculating 6j symbols and 3jm factors. Some procedures for calculating these coefficients were presented which are considerable improvements on those described by Butler and Wybourne (1976a). The symbols in which phase choices were made play a key role in these procedures. This is not altogether surprising but it is remarkable that in the calculation of 3jm factors, these appear to be the only group 6js involved. For the 3jms we have been able to give what must come very close to being a "first-strike" algorithm, however the calculation of 6j symbols remains a rather haphazard affair in that many equations one might use result in further unknown 6js. We have found that many 6j symbols and most 3jm factors can be calculated power by power and this is far more appealing than the technique used by Butler (1980b, Butler and Wybourne 1976a) which requires knowledge of primitives up to power 2p before one can calculate non-primitives of power p. We have encountered some problems with a power-by-power calculation though in that (apparently) all non-primitive

power-two SU_6 6j symbols and a few non-primitive power-two 3jm factors for $SU_6 \supset SU_2 \times SU_3$ seem to require knowledge of power-three primitive SU_6 6j symbols for their calculation. However, one can calculate all SU_6 6j symbols up to power three without requiring power-four primitives (Haase, private communication 1980) and since Haase's (computer) calculation made choices for all power-three non-antecedent triads it follows that all non-primitive 3jms for $SU_6 \supset SU_2 \times SU_3$ up to power three can also be calculated, since we have been able to calculate (by hand) all primitive 3jm factors up to this power. Thus it is not clear yet whether the problem with power two is a quirk, associated perhaps with complex-conjugation of the primitive, or whether it is a general problem which will recur again at higher powers. Apart from these cases though we have been able to calculate, power by power, most of those 3jm factors needed in Part I including quite a few for power four and even some up to power six.

There remains the outstanding problem of proving the completeness of Butler's method. However, the fact that it is known to work so well in so many cases belittles any qualms we may have about this. The main deficiency in the method is that it is based on a building-up principle but in the absence of analytic formulae one cannot do much better. Its considerable appeal lies in its applicability to all compact groups. Further, it has the virtue of readily allowing one to demand maximum symmetry when making phase choices.

APPENDIX IIA

TENSOR OPERATORS AND THE WIGNER-ECKART THEOREM

A key result in many applications of group theory to physical problems is the Wigner-Eckart theorem (Wigner 1931, Eckart 1930, see also Koster 1958) which allows one to make quantitative predictions using only the symmetry of the system. We present here a brief formal derivation of this theorem to illustrate some points made in the text.

A linear operator, $T^{x\lambda}_{i}$ mapping the Hilbert space $\mathcal{H} = \bigoplus_{x\lambda} V_{x\lambda}$ onto itself is said to be the ith component of a group tensor operator transforming as the representation λ if (e.g. Stone 1961) it has the same transformation properties under the group operations O^{λ}_{R} as the basis vectors of $V_{x\lambda}$ i.e.

$$O^{\lambda}_{R}\, T^{x\lambda}_{i}\, O^{\lambda^{-1}}_{R} = \sum_{i'} T^{x\lambda}_{i'}\, \lambda(R)^{i'}{}_{i} \,. \qquad \text{(IIA.1)}$$

(Again, the parentage label x merely distinguishes different tensors with the same transformation properties.) The action of the tensor operator $T^{x_2\lambda_2}_{i_2}$ on the vector $|x_3\lambda_3 i_3\rangle$ is such as to give a vector in a space transforming as $\lambda_2 \times \lambda_3$. A transformation of basis in this space then yields

$$T^{x_2\lambda_2}_{i_2}|x_3\lambda_3 i_3\rangle = \sum_{r\lambda_1 i_1} |(T^{x_2\lambda_2}, x_3\lambda_3) r\lambda_1 i_1\rangle \times \langle(\lambda_2\lambda_3) r\lambda_1 i_1 | \lambda_2 i_2; \lambda_3 i_3\rangle. \qquad \text{(IIA.2)}$$

The vectors $|(T^{x_2\lambda_2}, x_3\lambda_3)r\lambda_1 i_1\rangle$ span a representation vector space of λ_1 and therefore by Schur's lemmas (cf. chapter 10)

$$\langle x_1\lambda_1' i_1' | (T^{x_2\lambda_2}, x_3\lambda_3)r\lambda_1 i_1\rangle = \delta^{\lambda_1'}_{\lambda_1}\,\delta^{i_1'}_{i_1}\langle x_1\lambda_1 | (T^{x_2\lambda_2}, x_3\lambda_3)r\lambda_1\rangle \ . \tag{IIA.3}$$

(Readers worried about the parentage not matching are reminded of the definition and referred to Butler, 1975.) Hence the matrix elements of $T^{x_2\lambda_2}_{i_2}$ are given by

$$\langle x_1\lambda_1 i_1 | T^{x_2\lambda_2}_{i_2} | x_3\lambda_3 i_3\rangle = \sum_r \langle(\lambda_2\lambda_3)r\lambda_1 i_1 | \lambda_2 i_2; \lambda_3 i_3\rangle\langle x_1\lambda_1 | (T^{x_2\lambda_2}, x_3\lambda_3)r\lambda_1\rangle \ . \tag{IIA.4}$$

Defining a "reduced matrix element" by

$$\langle x_1\lambda_1 \| \underset{\sim}{T}^{x_2\lambda_2} \| x_3\lambda_3\rangle_r = |\lambda_1|^{\frac{1}{2}} \sum_{r'} M\{(123), \lambda_1^* \lambda_2 \lambda_3\}^{r'}{}_r \langle x_1\lambda_1 | (T^{x_2\lambda_2}, x_3\lambda_3)r'\lambda_1\rangle \tag{IIA.5}$$

and employing 3jm symbols leads to a symmetric form of the Wigner-Eckart theorem

$$\langle x_1\lambda_1 i_1 | T^{x_2\lambda_2}_{i_2} | x_3\lambda_3 i_3\rangle = \sum_{r\bar{i}_1'} \langle\lambda_1\rangle^{\bar{i}_1' i_1} \begin{pmatrix} \lambda_1^* & \lambda_2 & \lambda_3 \\ \bar{i}_1' & i_2 & i_3 \end{pmatrix}^r \langle x_1\lambda_1 \| \underset{\sim}{T}^{x_2\lambda_2} \| x_3\lambda_3\rangle_r \ . \tag{IIA.6}$$

For nonsimple-phase triads the cyclic permutation matrix in (IIA.5) can not be chosen diagonal and its presence can therefore

not be ignored. The necessity for this permutation matrix arises from our desire for a symmetric form of (IIA.6) and the ordering of columns in the definition (11.14) of the 3jm symbol. (We note that some authors do not include the factor $|\lambda_1|^{\frac{1}{2}}$ in their definition of reduced matrix elements.)

The resolution of the multiplicity, r in the reduced matrix elements is clearly the same as in the 3jm symbols and therefore reduced matrix elements have no phase-freedom unless the triad $(\lambda_1^* \ \lambda_2 \ \lambda_3)$ is an antecedent.

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