What is dust? – Physical foundations of the averaging problem in cosmology

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Abstract. The problems of coarse-graining and averaging of inhomogeneous cosmologies, and their backreaction on average cosmic evolution, are reviewed from a physical viewpoint. A particular focus is placed on comparing different notions of average spatial homogeneity, and on the interpretation of observational results. Among the physical questions we consider are: the nature of an average Copernican principle, the role of Mach's principle, the issue of quasilocal gravitational energy and the different roles of spacetime, spatial and null cone averages. The observational interpretation of the timescape scenario is compared to other approaches to cosmological averaging, and outstanding questions are discussed.

1. Introduction

Observations show that although the universe was remarkably homogeneous at the epoch of last scattering, when the cosmic microwave background (CMB) radiation was laid down, at the present epoch the matter distribution displays a very complex structure with significant inhomogeneities up to scales of at least $100h^{-1}$ Mpc, where h is the dimensionless parameter related to the Hubble constant by $H_0 = 100h\,\mathrm{km\ sec^{-1}\,Mpc^{-1}}$. The present universe is dominated in volume by voids [1, 2, 3], with galaxy clusters grouped in sheets and filaments that surround the voids, and thread them. At the largest of scales we see a few peculiar structures, such as the Sloan Great Wall.

At the same time, despite a number of nagging puzzles, most of the gross features of the universe are extremely well described by a spatially homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) model, with additional Newtonian perturbations evolved by N–body computer simulations to model the structure. The price that is paid for observational concordance is that most of the matter content in the universe must be in forms that have never been directly observed: 20–25% in the form of clumped nonbaryonic dark matter, and 70–75% in the form of a smooth dark energy, with an equation of state, $P = w\rho c^2$, extremely close to that of a cosmological constant, $w = -1$.

The dichotomy that the universe displays considerable inhomogeneity, while still being phenomenologically well fit by an average spatially homogeneous evolution, has led to considerable interest in the averaging problem in inhomogeneous cosmology. Is it possible that one or more of the components of the dark stuff introduced for the purposes of a phenomenological fit to observations are simply an artefact of us misunderstanding the workings of gravity on the largest scales? Whereas some researchers immediately leap to the extreme of modifying the whole theory of gravity, those more intimately acquainted with general relativity are aware that the implementation of the physical ingredients of Einstein's theory has not been precisely specified on all scales. There are many unsolved problems provided by the questions of coarse-graining, fitting, averaging and the statistical notions of gravitational energy and entropy, which must inevitably enter when dealing with the complex many–body problem that observational cosmology presents us. These are hard problems. However, we should try to understand the universe we observe rather than inventing toy models purely because they are simple to solve.

It is my view that future progress in the averaging problem demands advances in conceptual understanding. This paper will therefore review the present state of play in averaging with a strong conceptual bias, focusing on questions rather than answers. It is not a review of the details of mathematical techniques in averaging; there are already a number of recent reviews of that nature – including, for example, those of Buchert [4, 5] and van den Hoogen [6]. The possible mathematical choices one can make are many, but they each entail physical choices, either explicitly or implicitly. It is the nature of these choices that I wish to focus on.

2. The fitting problem: On what scale are Einstein's equations valid?

Einstein's field equations

$$G^{\mu}{}_{\nu} = \frac{8\pi G}{c^4} T^{\mu}{}_{\nu} \tag{1}$$

define the structure of general relativity as a relationship between geometry and matter. However, the scale over which matter fields are coarse–grained to produce the energy–momentum tensor on the r.h.s. of (1) is not prescribed, leaving an inherent ambiguity in the theory. Observation provides no direct guide in this matter, since general relativity is only well tested for isolated systems – such as the solar system or binary pulsars – for which $T^{\mu}{}_{\nu} = 0$. Indeed, Wheeler's aphorism that *"matter tells space how to curve"* is really only tested to the extent that matter is defined by boundary conditions and symmetry assumptions as long as the vacuum Einstein equations apply‡.

Einstein's equations are designed to reduce to Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{2}$$

in the Newtonian limit that the spacetime geometry is that of a weak field near a flat Minkowski background, and all characteristic velocities are much smaller than that of light, so that $g_{00} = 1 - 2\Phi/c^2$, with $\Phi \ll c^2$, and $\rho c^2 \equiv T^{00} \gg |T^{0i}| \gg |T^{ij}|$, where Latin indices denote spatial components.

There is no ambiguity in applying the full Einstein equations (1) to a fluid of particles with well-defined properties, such as ions, atoms and molecules in the early phases of the universe's expansion. However, as soon as gravitational collapse occurs then the geodesics of atoms and molecules cross. There are phase transitions, and the definition of the particles in the fluid approximation must change, giving rise to at least the following layers of coarse-graining in the epochs following last scattering:

(i) Atomic, molecular, ionic or nuclear particles: applicable with

- dust equation of state within any expanding regions which have not yet undergone gravitational collapse;
- fluid equation of state within relevant collapsed objects (stars, white dwarfs, neutron stars) for periods of time between phase transitions that alter the nongravitational particle interactions and the equation of state;

(ii) Collapsed objects such as stars and black holes coarse-grained as isolated objects;

(iii) Stellar systems coarse-grained as dust particles within galaxies;

(iv) Galaxies coarse-grained as dust particles within clusters;

(v) Clusters of galaxies coarse-grained as bound systems within expanding walls and filaments;

‡ In the Schwarzschild geometry, for example, a stellar interior solution is assumed to be matched with junction conditions at the spherical surface of the star. The nature of the stellar interior is irrelevant for the exterior vacuum geometry, however, on account of Birkhoff's theorem.

(vi) Voids, walls and filaments combined as as expanding regions of different densities in a single smoothed out cosmological fluid.

General relativity with the vacuum Einstein equations is well–tested at level (ii), and it generally accepted that the Einstein equations with a microscopic fluid $T^{\mu\nu}$ apply at level (i), with the small caveats that the equation of state of objects with the density of neutron stars is not completely understood, and also that general relativity must break down near singularities in the extreme strong field regime. However, once we proceed to higher levels in this fitting problem [7, 8] the physical issues become more and more murky. Provided that we can ignore any galactic magnetic fields etc, and only consider the effects of gravity, then at levels (iii)–(vi) we are generally only dealing with dust sources. However, as the definition of dust becomes less and less clear with successive coarse-grainings, the scale on which the Einstein equations should apply becomes open to question.

2.1. Coarse–graining

One outstanding problem is that the mathematical problem of coarse-graining in general relativity is very little studied. Any coarse-graining procedure amounts to replacing the the microphysics of a given spacetime region by some collective degrees of freedom of those regions which are sufficient to describe physics on scales larger than the coarse-graining scale. Einstein's equations were originally formulated with the intent that the energy-momentum tensor on the r.h.s. of (1) should either describe fundamental fields, such as the Maxwell field, or alternatively to the coarse-graining of the purely nongravitational interactions described by such fields in the fluid approximation.

Einstein originally imagined a universe with the density of the Milky Way; the complex hierarchy of galaxies, galaxy clusters, filaments, walls and voids was unknown when he wrote down his equations (1). The fundamental problem then, is that since the universe is composed of a hierarchy of long-lived structures much larger than those of stars, we must also coarse-grain over gravitational interactions within that hierarchy to arrive at a fluid description for cosmology. With such a coarse-graining, geometry no longer enters purely on the left hand side of Einstein's equations but in a coarse-grained sense can be hidden inside effective fluid elements on the right hand side.

The most fundamental quantities of interest as the sources of the right hand side of Einstein's equations are those of mass–energy, momentum and angular momentum. Effectively, if we demand that equations (1) should also apply in a coarse-grained version on cosmological scales, then it means that we are seeking collective parameters such as mass–energy which average over local spatial curvature, rotational kinetic energy etc. Furthermore we must approach the problem more than just once, on a succession of scales. This necessarily involves the issue of quasilocal gravitational energy, and more particularly statistical properties of the gravitational interactions of bound systems.

Since we are no longer dealing with a fixed spatial metric this problem is far more complicated than it is in Newtonian theory, and indeed it is largely unexplored territory.

Before surveying what has been done, let me outline the challenges presented to us by observation at each of the levels in the hierarchy presented above.

In going from level (i) to level (ii), the simple coarse-graining problem for nongravitational interactions can be ignored for stellar system astrophysics, since by symmetry assumptions in a vacuum spacetime we can solve the Einstein equations exactly, and leave the matter to be assumed to be an interior solution with a fluid equation of state matched at the timelike boundary which defines the surface of the star.

To the best of my knowledge the formal coarse-graining of vacuum geometries as dust to proceed from level (ii) to level (iii) is not a problem that has been directly studied. However, quasilocal local mass definitions are very well developed for stationary asymptotically flat systems, and provided we are sufficiently far from any isolated source then gravity is generally assumed to coincide with its Newtonian limit via (2). Thus one can easily envisage a smoothing procedure in which one excises a timelike worldtube with S^2 spatial topology around an isolated source, and replaces it by a density in terms of an ADM-like mass divided by the excised spatial volume. The coarse-graining procedure of Korzyński [9] provides a formalism in which this might be realised.

2.1.1. Galactic dynamics Already at the level (iii) of galactic dynamics general relativity offers the possibility that dynamics is more interesting than Newtonian dynamics in a global asymptotically flat background. In the standard model, Newtonian dynamics is naïvely assumed to apply at the scale of galaxies and galaxy clusters. However, this may not be the case. For example, Cooperstock and Tieu [10, 11] have shown that stationary axisymmetric rotating dust solutions§ may be solved to phenomenologically match the rotation curves of certain spiral galaxies, whose observed density distribution might be plausibly approximated by circular symmetry (neglecting the density contrasts of spiral arms or bars). Although various details of the Cooperstock–Tieu model are debated [11, 12], it does demonstrate that the nonlinearity of the Einstein equations is a potentially significant complicating feature for extended matter distributions, even in the weak field limit.

2.1.2. Galaxy cluster dynamics In proceeding to level (iv) – galaxy clusters – the fundamental issues become obviously nontrivial. Since many galaxy clusters are spherical in shape there is a temptation to model them using the spherically symmetric dust Lemaître–Tolman–Bondi (LTB) solutions [13, 14, 15]. While LTB models have certainly been applied to structure formation [16, 17], their applicability is constrained by the uniform spherical shell approximation remaining valid, without shell crossings or the growth of angular momentum perturbations. This is probably unrealistically constraining for the case of a generic collapse, and LTB models are most obviously applicable to expanding spherical voids [18] with ionic or molecular sized dust. If we

§ The solutions are circular: i.e., with zero expansion and shear, but nonvanishing vorticity.

consider *virialized* spherical clusters of galaxies then there is no obvious reason for the LTB model to be applicable. In many galaxy clusters the motion of individual galaxies may be close to radial – however, the phases of the galaxies relative to passage through the centre of the cluster are completely uncorrelated. Individual galaxies will pass close to the core of the cluster and emerge from the other side; at any instant the number of galaxies moving out from the centre might be comparable to the number falling in. Thus virialized galaxy clusters certainly do *not* have the symmetry of a spherically symmetric dust solution if the galaxies are to be identified as the dust.

The real question is: can we nonetheless model such systems as spherically symmetric solutions of Einstein's equations with an effective purely radial pressure, and possibly also effective heat flow terms? Or do we have to go beyond spherically symmetric solutions to consider possible effective anisotropic stresses? Since the interaction between individual galaxies in a virialized cluster is purely gravitational, we see that this question is really one of the statistical nature of gravity under coarse–graining. Namely, for virialized systems with a manifestly spherical shape, can the statistical properties of the individual gravitational interactions of the dust particles (galaxies) be described by Einstein's equation with a spherically symmetric effective fluid, or otherwise?

2.1.3. Cosmological dynamics   The final levels (v) and (vi) of coarse-graining in going to cosmological averages involve qualitatively new fundamental questions. If we require that a single model should describe the evolution of the universe from last scattering to the present day, then we must coarse grain on scales over which the notion of a dust "particle" has a meaning from last scattering up to the present day. The description of a galaxy composed of stellar particles, or of a virialized galaxy cluster composed of galaxy particles is only valid for those epochs after which the relevant "particles" have formed and are themselves relatively unchanging. Over cosmological timescales we do not have well-defined invariant dust particles. The nature of galaxies and galaxy clusters changes through growth by accretion of gas and by mergers.

The problem of cosmological dynamics is therefore essentially different to that of the dynamics of galaxies or virialized galaxy clusters, as we can no longer make a stationary approximation. In order to circumvent the problem of ill-defined particle-like building blocks, an alternative strategy is that we coarse-grain the dust on scales large enough that the *average* flow of mass from one particle to another is negligible up to the present epoch. Although galaxy clusters vary greatly in their size and complexity, there are no common virialized structures larger than clusters. Thus any level of coarse-graining on scales larger than clusters necessarily means dealing with dust "particles" that are themselves *expanding*, i.e., with objects more akin to fluid elements in hydrodynamics. This feature gives the first fundamental qualitative difference for the cosmological problem as compared to that of galaxies or galaxy clusters.

Although we can receive signals from anywhere within our particle horizon, if we make the reasonable assumption that the amount of matter absorbed from cosmic rays

from distant galaxies is negligible, then the region which has contributed matter particles to define the local geometry of our own galaxy is very small. This bounding sphere, which Ellis and Stoeger [19] call the *matter horizon*, is estimated by them to be of order 2 Mpc for the Milky Way using assumptions about the growth of perturbations from the standard cosmology. This scale coincides roughly with the scale at which the Hubble flow is believed to begin in the immediate neighbourhood of the local group of galaxies. It is one way of realising the concept of *finite infinity*, introduced qualitatively by Ellis in his first discussion of the fitting problem [7].

The second qualitative difference is that we have to deal with expanding fluid elements that have vastly different densities at the present epoch, and which evolve more or less independently. For galaxy clusters some sort of finite infinity notion, with a variable scale of order 2–10 Mpc depending on the size of cluster might be useful for defining independent fluid elements. By combining such regions we arrive at the walls and filaments that contain most of the mass of the universe. However, to this we must also add the voids which dominate the volume of the universe at the present epoch. These are the regions in which structures have never formed, and which still contain the same ionic, atomic and molecular dust content that has existed since very early epochs, only greatly diluted by expansion.

Any relevant averaging scale is therefore phenomenologically related to the observed statistical distribution of void sizes. A precise definition of a void fraction of course depends on the definition of a void. Surveys indicate that voids with characteristic mean effective radii‖ of order $(15 \pm 3)h^{-1}\,\mathrm{Mpc}$ (or diameters of order $30h^{-1}\,\mathrm{Mpc}$), and a typical density contrast of $\delta\rho/\rho = -0.94 \pm 0.02$, make up 40% of the volume of the nearby universe [1, 2]. A very recent study [3] of the Sloan Digital Sky Survey Data Release 7 finds a median effective void radius of $17h^{-1}\,\mathrm{Mpc}$, with voids of effective radii in the range $10h^{-1}\,\mathrm{Mpc}$ to $30h^{-1}\,\mathrm{Mpc}$ occupying 62% of the survey volume. In addition to these there are numerous smaller minivoids [98], which combined with the dominant voids ensure that overall voids dominate the present epoch universe by volume.

For the purposes of coarse-graining, what is important is not the overall volume fraction of voids, but the size of the typical largest structures. Any minimal scale for the cosmological coarse-graining of the final smoothed density distribution has to be substantially larger than the diameter of the largest structures. Void statistics [3] indicate an effective cutoff of $60h^{-1}\,\mathrm{Mpc}$ for the largest mean effective diameters of voids, i.e., twice the scale of the typical dominant void diameters. Thus observationally, the relevant scale for coarse-graining appears to be at least three times the dominant void diameter, i.e., of order $100h^{-1}\,\mathrm{Mpc}$, which coincides roughly with the Baryon Acoustic Oscillation (BAO) scale.

‖ Voids display a degree of ellipticity. The *mean effective radius* of a void is that of a sphere with the same volume as occupied by the void [1, 2, 3], which is typically larger than the maximal sphere enclosed by the same void.

2.2. Approaches to coarse-graining

The physical degrees of freedom which we must coarse grain are contained in the curvature tensor and the sources of the field equations (1). In principle coarse-graining the curvature tensor might involve steps other than simply coarse-graining of the metric. However, if a metric description of gravity is assumed at each level, then schematically the hierarchy of coarse-graining might be heuristically described as

$$
\left.\begin{array}{c}
g_{\mu\nu}^{\text{stellar}} \to g_{\mu\nu}^{\text{galaxy}} \to g_{\mu\nu}^{\text{cluster}} \to g_{\mu\nu}^{\text{wall}} \\
\vdots \\
g_{\mu\nu}^{\text{void}}
\end{array}\right\} \to g_{\mu\nu}^{\text{universe}}
\tag{3}
$$

where the ellipsis denotes the fact that the metric of more than one type of wall or void might possibly be relevant. In this scheme the lowest members are assumed to be well modeled by exact solutions of Einstein's field equations: $g_{\mu\nu}^{\text{stellar}}$ being a solution to the vacuum field equations in a stellar system with a star or black hole as source (typically the Schwarzschild solution or Kerr solution), and $g_{\mu\nu}^{\text{void}}$ being that of a region filled with low density ionic dust with whatever symmetries are relevant.

Dealing with this problem is obviously very complicated, and the models that have been studied to date typically treat just one level in the hierarchy.

2.2.1. Discretized universes One of the few approaches to tackling the dust approximation head-on is the Lindquist–Wheeler model [21], which has received new interest recently [22, 23, 24]. In this approach the coarse–graining hierarchy (3) is replaced by the simplified scheme

$$
g_{\mu\nu}^{\text{stellar}} \to g_{\mu\nu}^{\text{universe}}
\tag{4}
$$

with the understanding that here $g_{\mu\nu}^{\text{stellar}}$ denotes the Schwarzschild solution and could be taken as a substitute for either galaxies or clusters of galaxies, depending on the masses assigned to the objects in the lattice. Furthermore, here $g_{\mu\nu}^{\text{universe}}$ does not play a tangible geometrical role. No cosmological metric is assumed in Einstein equations at the outset. Rather, by matching the spherical boundaries of radially expanding geodesics in the Schwarzschild geometries of a regular lattice of equal point masses, the Friedmann equations are obtained [21, 22]. The matching is exact only at the points where the radial spheres intersect and is approximate in the regions in which spheres overlap or are excluded.

This model is analogous to the Swiss cheese models [25, 26] in the sense that the point group symmetry of the lattice is a discretized version of overall global spatial homogeneity. The principal difference from Swiss cheese models is that one is not cutting and pasting spheres into a pre-existing continuum spacetime. Rather the continuum geometry is only realised as an approximate description of the underlying discretized space. Since the approximate continuum geometry is a FLRW model, the Lindquist–Wheeler model has much in common with the models listed in section 3.1 below. The symmetry of the lattice is such an integral part of the construction that it is difficult to

envisage how such models could be easily generalized to more typically inhomogeneous cases. Nonetheless, it is an extremely interesting toy model in which questions such as light propagation can be investigated with detailed control.

Clifton and Ferreira have carefully studied the propagation of light in a Lindquist–Wheeler model which approximates a spatially flat Einstein–de Sitter model [22]. They find a significant deviation of the redshift, z, of the lattice universe as compared to that of the FLRW universe, z_{FLRW}. For typical null geodesics, numerical calculations show that $1 + z \simeq (1 + z_{\mathrm{FLRW}})^{7/10}$. Essentially, this might be considered as a difference from the focusing arising from Weyl curvature in the case of the point masses, as compared with Ricci curvature focusing for a continuous dust fluid. While the change in the luminosity distance–redshift relation is in the opposite direction as compared to what is required for a viable explanation of the expansion history of the universe without dark energy, the large difference between the discrete and continuum cases demonstrates that we cannot confidently claim to have reached an era of "precision cosmology" as long as such fundamental issues as that of coarse-graining and the dust approximation are not understood.

2.2.2. Korzyński's covariant coarse-graining

Korzyński [9] has recently proposed a covariant coarse-graining procedure to be applied to dust solutions. This procedure could conceivably be applied to any step in the hierarchy (3) for which the starting point is the metric of a known dust solution. Korzyński also discusses the special limit of replacing a dust world tube by a single worldline [9], which might be viewed as proceeding in the opposite direction to that taken in the Lindquist–Wheeler model.

Korzyński's idea is to isometrically embed the boundary of a comoving dust-filled domain – required to have S^2 topology with positive scalar curvature – into a three-dimensional Euclidean space, and to construct a "fictitious" three-dimensional fluid velocity which induces the same infinitesimal metric deformation on the embedded surface as the "true" dust flow does on the domain boundary in the original spacetime. This velocity field is used to uniquely assign coarse-grained expressions for the volume expansion and shear to the original domain. An additional construction using the pushforward of the ADM shift vector is used to similarly obtain a coarse-grained vorticity. The coarse-grained quantities are quasilocal functionals which depend only on the geometry of the boundary of the relevant domain.

Korzyński's approach represents an interesting new way of attacking the fitting problem, and may also provide a useful framework for formulating the problem of backreaction.

2.3. Averaging and backreaction

Averaging and coarse-graining are of course intimately related. The basic distinction is that with averaging one is interested in the overall average dynamics and evolution, most often without direct consideration of the details of the course-graining procedure.

Whereas coarse-graining is little studied, much more attention has been paid to averaging, and a number of different approaches have been pursued. These approaches are also discussed in the reviews of van den Hoogen [6], Ellis [27] and Clarkson *et al* [28].

Whereas coarse-graining is generally a bottom-up procedure, averaging is top-down¶ as it usually starts from the assumption that a well-defined average exists, with a number of assumed properties. Generically, if one assumes that the Einstein field equations (1) are valid for some general inhomogeneous geometry, $g_{\mu\nu}$, then given some as yet unspecified averaging procedure denoted by angle brackets, the average of (1) gives

$$\langle G^{\mu}{}_{\nu}\rangle = \langle g^{\mu\lambda}R_{\lambda\nu}\rangle - \tfrac{1}{2}\delta^{\mu}{}_{\nu}\langle g^{\lambda\rho}R_{\lambda\rho}\rangle = \frac{8\pi G}{c^4}\langle T^{\mu}{}_{\nu}\rangle. \tag{5}$$

At this point a number of choices are possible since there is no reason to necessarily assume that $\langle G^{\mu}{}_{\nu}\rangle$ is the Einstein tensor of an exact geometry. In other words, on cosmological scales there is no a priori necessity for (5) to correspond to an exact solution of Einstein's equations.

In his "macroscopic gravity" approach, Zalaletdinov [31, 32, 33] chooses to work with the average inverse metric $\langle g^{\mu\nu}\rangle$ and the average Ricci tensor $\langle R_{\mu\nu}\rangle$ and to write

$$\langle g^{\mu\lambda}\rangle\langle R_{\lambda\nu}\rangle - \tfrac{1}{2}\delta^{\mu}{}_{\nu}\langle g^{\lambda\rho}\rangle\langle R_{\lambda\rho}\rangle + C^{\mu}{}_{\nu} = \frac{8\pi G}{c^4}\langle T^{\mu}{}_{\nu}\rangle, \tag{6}$$

where the correlation functions $C^{\mu}{}_{\nu}$ are defined by the difference of the left hand sides of (6) and (5). Zalaletdinov provides additional mathematical structure to prescribe a covariant averaging scheme, thereby defining properties of the correlation functions.

Another way of formulating the problem is to work in terms of the difference between the general inhomogeneous metric and the averaged metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{7}$$

where $\bar{g}_{\mu\nu} \equiv \langle g_{\mu\nu}\rangle$, with inverse $\bar{g}^{\lambda\mu} \neq \langle g^{\lambda\mu}\rangle$. We can now determine a connection $\bar{\Gamma}^{\lambda}{}_{\mu\nu}$, curvature tensor $\bar{R}^{\mu}{}_{\nu\lambda\rho}$ and Einstein tensor $\bar{G}^{\mu}{}_{\nu}$ based on the averaged metric, $\bar{g}_{\mu\nu}$, alone. The differences $\delta\Gamma^{\lambda}{}_{\mu\nu} \equiv \langle\Gamma^{\lambda}{}_{\mu\nu}\rangle - \bar{\Gamma}^{\lambda}{}_{\mu\nu}$, $\delta R^{\mu}{}_{\nu\lambda\rho} \equiv \langle R^{\mu}{}_{\nu\lambda\rho}\rangle - \bar{R}^{\mu}{}_{\nu\lambda\rho}$, $\delta R_{\mu\nu} \equiv \langle R_{\mu\nu}\rangle - \bar{R}_{\mu\nu}$ etc, then represent the *backreaction* of the average inhomogeneities on the average geometry determined from $\bar{g}_{\mu\nu}$. Furthermore, the average Einstein field equations (5) may be written

$$\bar{G}^{\mu}{}_{\nu} + \delta G^{\mu}{}_{\nu} = \frac{8\pi G}{c^4}\langle T^{\mu}{}_{\nu}\rangle. \tag{8}$$

This expresses the fact that the Einstein tensor of the average metric is not in general the average of the Einstein tensor of the original metric. The processes of averaging and constructing the Einstein tensor do not commute.

¶ The terms "averaging" and "coarse-graining" are often used interchangeably in a loose sense. One might view averaging as a "top-down coarse-graining procedure"; just there are many ways of coarse-graining, and here I have reserved the term for the bottom-up approaches.

Equations (5) and (8) are of course very similar, but may differ in both the overall average represented by the angle brackets, and also in the split between the background averaged Einstein tensor and the correlation or backreaction terms.

There are three main types of averaging schemes that have been considered. They can be classed as:

- Perturbative schemes about a given background geometry;
- Spacetime averages;
- Spatial averages on hypersurfaces based on a $1 + 3$ foliation of spacetime.

I will briefly describe each case in turn.

2.3.1. Perturbations about exact cosmological spacetimes A vast literature exists on inhomogeneous models treated as perturbations of the exact FLRW models. In this approach one assumes that the average geometry $\bar{g}_{\mu\nu}$ of (7) is exactly a FLRW model, and the quantities $\delta g_{\mu\nu}$ are to be treated as perturbative corrections.

The issue of whether backreaction is significant or insignificant in the perturbative FLRW context is a matter of much debate, with different authors coming to different conclusions, which may be traced to various differences in assumptions made. Since these issues are discussed in many other reviews, such as those of Clarkson *et al* [28], Kolb [29] and the paper of Clarkson and Umeh [30], I will not discuss the perturbative approach in detail here.

The perturbative approach of course relies on the assumption that a FLRW model exactly describes the average evolution of the universe at the largest scales, and this may be incorrect. Related physical issues are further discussed in section 3.3.

2.3.2. Spacetime averages: Zalaletdinov's macroscopic gravity   General covariance is generally seen as a desirable property, since it is an essential feature of general relativity that physical quantities should not depend on arbitrary choices of coordinates. However, any process of taking an average will in general break general covariance. Furthermore, if an average geometry on cosmological scales no longer satisfies the Einstein equations, which is a distinct possibility given that solutions of (1) are only directly tested on the scale of stellar systems, then the role that general covariance plays in defining spacetime structure on the largest scales may need to be revisited from first principles. In particular, although we might still desire that physical quantities should not depend on choices of coordinates, the relationship of the coordinates of a "fine–grained manifold" relative to those of an average "coarse–grained manifold" need to be carefully considered.

Zalaletdinov views general covariance as paramount in determining macroscopic spacetime structure, and he introduces additional mathematical structure to perform averaging of tensors in a covariant manner on a given manifold, $\mathcal{M}$ [31, 32, 33]. His aim is to consistently average the Cartan equations from first principles, in analogy to the averaging of the microscopic Maxwell–Lorentz equations in electromagnetism. However,

whereas electrodynamics is linear in the fields on the fixed background of Minkowski spacetime, gravity demands an averaging of the nonlinear geometry of spacetime itself.

The additional structure introduced by Zalaletdinov [31, 32, 33] takes the form of bilocal averaging operators, $\mathcal{A}^{\mu}{}_{\alpha}(x, x')$, with support at two points $x \in \mathcal{M}$ and $x' \in \mathcal{M}$, which allow one to construct a bitensor extension, $\mathbf{T}^{\mu}{}_{\nu}(x, x')$, of a tensor $T^{\mu}{}_{\nu}(x)$ according to

$$\mathbf{T}^{\mu}{}_{\nu}(x, x') = \mathcal{A}^{\mu}{}_{\alpha'}(x, x')T^{\alpha'}{}_{\beta'}(x')\mathcal{A}^{\beta'}{}_{\nu}(x', x).\tag{9}$$

The bitensor extension is then integrated over a 4-dimensional spacetime region, $\Sigma \subset \mathcal{M}$, to obtain a regional average according to

$$\bar{T}^{\mu}{}_{\nu}(x) = \frac{1}{\mathcal{V}_{\Sigma}} \int_{\Sigma} \mathrm{d}^4 x' \sqrt{-g(x')}\, \mathbf{T}^{\mu}{}_{\nu}(x, x'),\tag{10}$$

where $\mathcal{V}_{\Sigma} \equiv \int_{\Sigma} \mathrm{d}^4 x \sqrt{-g(x)}$ is the spacetime volume of the region Σ. The bitensor transforms as a tensor at every point but is a scalar when integrated over a region for the purpose of averaging.

In the macroscopic gravity approach, much effort has been expended [31, 32, 33, 34] in developing a mathematical formalism which in the average bears a close resemblance to general relativity itself. Indeed, apart from the fact that the macroscopic scale is assumed to be larger than the microscopic scale, there is no scale in the final theory. As Clarkson *et al* [28] have already commented, this is also potentially a weakness of the macroscopic gravity approach. Observations suggest a complex hierarchy of averaging, given by the scheme (3), which may involve physical issues more complex than simply one step from a microscopic theory to a macroscopic theory of gravity. Indeed, a number of the steps associated with the observed scales of coarse-graining phenomenologically involve going from background solutions of Einstein's field equations *with particular symmetries* to other solutions of Einstein's field equations *with particular symmetries*. Therefore it is a highly nontrivial question as to whether the physically relevant mathematical framework is one which takes us from one version of a diffeomorphism invariant theory of gravity with no specific symmetries to another diffeomorphism invariant theory of gravity with no specific symmetries, which is precisely what Zalaletdinov has constructed. Specific cosmological questions involve the choice of specific macroscopic scales.

In practice, cosmological applications of Zalaletdinov's formalism have involved making additional assumptions, such as those which lead to a spatial averaging limit [35], or additionally in *assuming* that the average geometry is a FLRW geometry [36, 37, 38, 39]. In this case it is found that the macroscopic gravity correlation terms take the form of a spatial curvature, even though a spatially *flat* FLRW geometry is assumed for the average geometry [36].

Recently an alternative covariant spacetime averaging scheme has been put forward by Brannlund, van den Hoogen and Coley [40]. It treats the manifold as a frame bundle as a starting point for the averaging of geometric objects.

2.3.3. Spatial averages: Buchert's formalism Building on earlier work [41, 42, 43] in the late 1990s Buchert developed an approach [44, 45] for the spatial averaging of scalar quantities associated with the Einstein field equations (1) in the $1 + 3$ ADM formalism, with cosmological averages in a fully nonperturbative relativistic setting in mind at the outset. The $1 + 3$ setting is natural if the Einstein field equations (1) are to be viewed as evolution equations.

Rather than introducing additional structure to fully tackle the mathematically difficult problem of averaging tensors, Buchert approached the problem by just averaging scalar quantities associated with spacetimes with inhomogeneous perfect fluid energy–momentum sources. Such scalars include the density, ρ, expansion, θ, and scalar shear, $\sigma^2 = \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ etc. For an arbitrary manifold, ADM coordinates

$$\mathrm{d}s^2 = -\boldsymbol{\omega}^0 \otimes \boldsymbol{\omega}^0 + g_{ij}(t, \mathbf{x})\, \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j, \tag{11}$$

where

$$\begin{aligned} \boldsymbol{\omega}^0 &= \mathcal{N}(t, \mathbf{x})\, c\, \mathrm{d}t, \\ \boldsymbol{\omega}^i &= \mathrm{d}x^i + \mathcal{N}^i(t, \mathbf{x})\, c\, \mathrm{d}t. \end{aligned} \tag{12}$$

can always be chosen locally but not necessarily globally. Buchert restricted the evolution problem to that of irrotational flow in order that (11), (12) can be assumed to apply over global $t = \mathrm{const}$ spatial hypersurfaces. For a dust source[+] we can then choose synchronous coordinates with $\mathcal{N} = 1$ and $\mathcal{N}^i = 0$. With these choices, the Einstein equations may be averaged on a domain, $\mathcal{D}$, of the spatial hypersurfaces, Σ, to give

$$3\frac{\dot{\bar{a}}^2}{\bar{a}^2} = 8\pi G\langle\rho\rangle - \tfrac{1}{2}c^2\langle\mathcal{R}\rangle - \tfrac{1}{2}\mathcal{Q}, \tag{13}$$

$$3\frac{\ddot{\bar{a}}}{\bar{a}} = -4\pi G\langle\rho\rangle + \mathcal{Q}, \tag{14}$$

$$\partial_t\langle\rho\rangle + 3\frac{\dot{\bar{a}}}{\bar{a}}\langle\rho\rangle = 0, \tag{15}$$

where the overdot denotes a t–derivative,

$$\mathcal{Q} \equiv \frac{2}{3}\left\langle(\theta - \langle\theta\rangle)^2\right\rangle - 2\langle\sigma^2\rangle = \frac{2}{3}\left(\langle\theta^2\rangle - \langle\theta\rangle^2\right) - 2\langle\sigma^2\rangle, \tag{16}$$

is the *kinematic backreaction*, and angle brackets denote the spatial volume average of a quantity, so that $\langle\mathcal{R}\rangle \equiv \left(\int_{\mathcal{D}}\mathrm{d}^3x\sqrt{\det{}^3g}\,\mathcal{R}(t, \mathbf{x})\right)/\mathcal{V}(t)$ is the average spatial curvature, for example, with $\mathcal{V}(t) \equiv \int_{\mathcal{D}}\mathrm{d}^3x\sqrt{\det{}^3g}$ being the volume of the domain $\mathcal{D} \subset \Sigma$. It is important to note that $\bar{a}$ is *not* the scale factor of any given geometry, but rather is defined in terms of the average volume according to

$$\bar{a}(t) \equiv \left[\mathcal{V}(t)/\mathcal{V}(t_0)\right]^{1/3}. \tag{17}$$

[+] The extension to perfect fluids was introduced in [45], and to other matter sources in [46]. Further extensions that deal with general hypersurfaces tilted with respect to the fluid flow have been discussed by various authors [47]–[51].

It follows that the Hubble parameter appearing in (13)–(15) is, up to a factor, the volume average expansion scalar, θ,

$$\frac{\dot{\bar{a}}}{\bar{a}} = \tfrac{1}{3}\langle\theta\rangle. \tag{18}$$

The following condition

$$\partial_t\left(\bar{a}^6\mathcal{Q}\right) + \bar{a}^4c^2\partial_t\left(\bar{a}^2\langle\mathcal{R}\rangle\right) = 0, \tag{19}$$

is required to ensure that (13) is the integral of (14).

In Buchert's scheme the non-commutativity of averaging and time evolution is described by the exact relation [41, 42, 44, 52]

$$\partial_t\langle\Psi\rangle - \langle\partial_t\Psi\rangle = \langle\Psi\theta\rangle - \langle\theta\rangle\langle\Psi\rangle \tag{20}$$

for any scalar, Ψ.

The operational interpretation of Buchert's formalism poses many questions, which we will return to in section 4. Leaving these issues aside, (14) is already suggestive since it implies that if the backreaction term is large enough – for example, in the case of a large variance in expansion rate with small shear – then the volume average acceleration, $\bar{a}^{-1}\ddot{\bar{a}} = \tfrac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\langle\theta\rangle + \tfrac{1}{9}\langle\theta\rangle^2$, could be positive, even if the expansion of all regions is locally decelerating. Essentially, the fraction of the volume occupied by the faster expanding regions is initially tiny but may become significant at late epochs, skewing the average to give an illusion of acceleration during the period in which the voids start to dominate the volume average. Whether this is observationally viable, however, depends crucially on how large the variance in expansion rates can grow given the initial constraints on density perturbations, and on the operational interpretation of the Buchert average.

Another big question is the extent to which the truncation of the averaging problem from the full Einstein equations to scalar evolution equations can be derived from a more fundamental basis. If density perturbations are the most important phenomenologically, then the Buchert scheme may well be justified, but it then needs to be understood as an appropriate limit in a more general scheme. Coley [53] suggests that since 4-dimensional Lorentzian manifolds can be completely characterized by their scalar polynomial curvature invariants, this might provide a suitable mathematical basis for an averaging scheme based on scalars. In practice, however, we are still faced with the same observational interpretation problems when dealing with a single null cone average versus spatial averages of statistical ensembles.

Korzyński's covariant coarse-graining approach [9], discussed in section 2.2.2, when applied to irrotational dust might be viewed as a generalization of the Buchert approach, which gives rise to additional backreaction terms. The extent to which Buchert's scheme might be viewed as a limit of Korzyński's approach remains to be determined.

Within the context of $1 + 3$ formalisms, there have been a number of studies of averaging and backreaction which focus principally on associated mathematical issues. These include Ricci flow [43, 54, 55, 56], group averaging of the FLRW isometry group [57] and the characterization of constant mean (extrinsic) curvature (CMC) flows

[58, 59, 60]. Such approaches might provide further insights into the general problem of averaging tensors. For example, the Ricci flow is a well studied procedure in Riemannian geometry which may be used to realize a regional smoothing through a rescaling of the metrical structure [43, 54, 55, 56], in the spirit of renormalization group flows. Since the primary motivation of this review is physical, I will not further discuss these approaches here.

3. Average spatial homogeneity: How do we define it?

The very near isotropy of the CMB demonstrates that when photons travel to us from the surface of last scattering, then to a very good approximation the geometry of the universe must be isotropic in some average sense. If we assume some sort of statistical Copernican principle, then we can also expect some sort of average notion of spatial homogeneity. The hard question is: how do we convert the observed averaged isotropy of the geometry on our past light cone into an appropriate notion of average spatial homogeneity?

It is my own view that whereas a lot of effort has been expended in defining the mathematics of averaging, not enough attention has been given to the foundational physics underlying the appropriate notion of an average. I will outline my views as to the best way to proceed in section 3.3, but will first describe the two approaches that have received the most attention.

3.1. The Friedmann–Lemaître universe as the average

The remarkable success of the standard cosmology, albeit with sources of dark matter and dark energy for which there is no direct evidence on the scale of the solar system, understandably leads most researchers to assume that it must be correct, even if only in an average sense.

As a consequence, even when researchers study inhomogeneity then, putting aside exact inhomogeneous solutions, the FLRW model is simply *assumed* as the average in a majority of approaches. A partial list of models for which this is the case includes:

- All perturbative calculations about the FLRW universe (whether based on the standard ΛCDM cosmology or otherwise);
- any LTB models for which the universe is "asymptotically FLRW" with a core spherical inhomogeneity;
- the Dyer-Roeder approach [61];
- Swiss cheese [25] and meatball [26, 62] models;
- studies of spatial averaging in a (1 + 3) setting which derive more general mathematical results, but then assume the FLRW model as the average when drawing specific conclusions in a cosmological context [52, 63];

- studies based on Zalaletdinov's macroscopic gravity which derive more general mathematical results, but then assume the FLRW model as the average when drawing specific conclusions in a cosmological context [35, 36, 37, 38, 39];

- studies of CMC flows which derive more general mathematical results, but then assume the FLRW model as the average when drawing specific conclusions in a cosmological context [58, 59].

I will not deal further with the details of these approaches, since several other papers in this special issue deal with them. The main comment I wish to make is that at the point that the FLRW model is introduced these approaches effectively assume that on large enough scales the average geometry is described by Einstein's equations (1) with a spatially homogeneous perfect fluid source, or more specifically a spatially homogeneous dust source if we consider late epoch cosmic evolution. In other words, although many of the general results derived in some of these approaches may be quite broadly applicable, assumption of the FLRW average demands very specific properties of dust in the unsolved processes of coarse-graining discussed in section 2.1. Since the large scale averages involve coarse–graining on scales on which space is expanding, it generally means extrapolating the dust approximation to scales on which usual notions of dust particles as bound systems cannot apply.

Furthermore, the notion of an average that these approaches implicitly demand is also very restrictive, since it involves (at least) three conditions:

 (i) The notion of average spatial homogeneity is described by a class of ideal comoving observers with synchronized clocks.

 (ii) The notion of average spatial homogeneity is described by average surfaces of constant spatial curvature (orthogonal to the geodesics of the ideal comoving observers).

(iii) The expansion rate at which the ideal comoving observers separate within the hypersurfaces of average spatial homogeneity is uniform.

Already at the level of perturbation theory about FLRW models, one can specialize to spacetime foliations which preserve one of the notions (i)–(iii) of average spatial homogeneity more fundamentally than the other two, as was already discussed many years ago in the classic work of Bardeen [64]. Since spatial curvature is not specified by a single scalar in general, there are many ways of realizing spacetime foliations which preserve one notion of average spatial homogeneity more strongly than the others. Among the foliations discussed by Bardeen we can recognise those of each type above: the *comoving hypersurfaces* (and related synchronous gauge) take property (i) as more fundamental; the *minimal shear hypersurfaces*[*] (and related Newtonian gauge) are one

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[*] For scalar perturbations this becomes a zero–shear condition, i.e., $\mathcal{K}_{ij} - \frac{1}{3}g_{ij}\mathcal{K} = 0$, where $\mathcal{K}_{ij}$ is the extrinsic curvature, g_{ij} the intrinsic metric, and $\mathcal{K} \equiv \mathcal{K}^\ell{}_\ell$. For general perturbations the hypersurfaces are defined by $\left(\mathcal{K}_{ij} - \frac{1}{3}g_{ij}\mathcal{K}\right)_{|ij} = 0$, where the bar denotes a covariant derivative with respect to the intrinsic metric.

type of foliation for which property (ii) is more fundamental; and finally the *uniform Hubble flow hypersurfaces* take property (iii) as more fundamental.

The possible foliations of perturbed FLRW models were recently considered in considerable detail by Bičák, Katz and Lynden-Bell [65], with a view to making gauge choices that provide a realization of Mach's principle, in the sense that the rotations and accelerations of local inertial frames can be determined directly from local energy–momentum perturbations $\delta T^{\mu}{}_{\nu}$. The choices of hypersurfaces they consider are: uniform Hubble flow hypersurfaces; uniform intrinsic scalar curvature hypersurfaces; and minimal shear hypersurfaces. The *uniform intrinsic scalar curvature hypersurfaces* provide a foliation in addition to those considered by Bardeen, which also take property (ii) as more fundamental. In addition to a choice of hypersurface Bičák, Katz and Lynden-Bell further fix the gauge by an adopting a condition similar to the "minimal shift distortion condition" of Smarr and York [66]. With this condition, for each choice of hypersurface the coordinates of local inertial frames are more or less uniquely determined by the energy–momentum perturbations $\delta T^{\mu}{}_{\nu}$. These "Machian gauges" are therefore substantially more restrictive than the commonly used synchronous gauge and generalized Lorenz–de Donder gauge [65].

3.2. Constant time hypersurfaces as the average

If we abandon the assumption that the average notion of spatial homogeneity is given by an exact solution of Einstein's field equations with a perfect fluid source, then there is no reason to assume that all of the conditions (i)–(iii) described in the last section need to apply. In general, we need just one condition to characterize the average; the question is which one?

Perhaps for the same historical reasons that led to the popularity of the synchronous gauge, the most studied choice of spacetime split beyond the perturbative regime is that of constant time hypersurfaces orthogonal to ideal observers "comoving with the dust", even though the nature of the dust is not generally prescribed. The Buchert approach to spatial averaging [44, 45] grew as a generalization of averaging in Newtonian cosmology [41, 42]. Since the split of space and time is unique in Newtonian theory, from the Newtonian viewpoint this is the only natural choice one can make.

If the particles of dust were observationally identifiable and invariant from the time of last scattering until today, then there would be no physical ambiguity about the notion of "comoving with the dust". In such a case, the choice of constant time hypersurfaces with a synchronous gauge would be well motivated. However, as discussed in section 2.1, in order to consistently deal with both the particles of ionic dust in voids, and also with "particles" of dust larger than galaxies, we have to coarse-grain over fluid elements which are themselves expanding. This demands a detailed understanding of the general statistical nature of general relativity in the presence of complex sources, which is as yet unavailable. What is true of Einstein's theory is that slicings based on any fixed time or space cannot be expected to be the most natural choice of average background.

In my view, to understand the formidable interlocking problems of averaging and the statistical nature of general relativity, we must go back to first principles.

3.3. Mach's principle, the equivalence principle, and the average

In this section I will outline my views about an alternative physically motivated approach to defining average spatial homogeneity [67, 68], which underlies the timescape cosmology [69, 70, 71]. Whether or not the current version of the timescape cosmology is observationally viable as an alternative to the standard ΛCDM cosmology without dark energy, the particular questions I wish to raise are key ones which must be fully understood if we are to make progress with the averaging problem.

In formulating general relativity as a dynamical theory of spacetime, Einstein was guided philosophically by Mach's principle – namely, the broad notion that spacetime does not have a separate existence from the material objects that inhabit it, but is a relational structure between things. As Einstein put it [72]: *"In a consistent theory of relativity there can be no inertia relatively to 'space', but only an inertia of masses relatively to one another"*. This after all is the physical principle that underlies general covariance: there is no absolute space or time, and so the basic laws of physics should not depend on arbitrary choices of coordinates.

Einstein did not, however, fully succeed in implementing Mach's principle in general relativity, since he never solved the global problem of uniquely determining the structure of spacetime on large scales. As things stand, his equations admit many cosmological solutions such as general Bianchi models or the Gödel universe, which do not look in the least like the universe we actually inhabit. Indeed, such solutions might be viewed as running counter to the spirit of Mach's principle. From the cosmological viewpoint Mach's principle may be phrased [73]: *"Local inertial frames are determined through the distributions of energy and momentum in the universe by some weighted average of the apparent motions"*. Although it is clear from this statement that any attempt to tackle the averaging problem in cosmology must necessarily deal with the issue of Mach's principle, relatively few authors [65, 67, 68, 74, 75, 76] have approached the averaging problem in these terms.

In considering the averaging problem, we should take the underlying physical principles of relativity as a guide to constructing the correct mathematical formalism, rather than simply trying to mimic mathematical properties such as general covariance, which might be relevant for formulating the microscopic nongravitational laws of physics but are not necessarily relevant for very large scale averages of gravitational degrees of freedom.

To prescribe the rules for defining spacetime in relational terms, it is necessary to define the relationship between inertial frames and any relevant mathematical structure. The strong equivalence principle (SEP) stands as the concrete legacy of Einstein's attempts to come to grips with Mach's principle, and it is the physical cornerstone of general relativity. By the SEP the nongravitational laws of physics should reduce

to those of special relativity in local inertial frames (LIFs) in the neighbourhood of an arbitrary spacetime point. The principle of general covariance is a means of formulating the nongravitational laws to achieve this.

However, when it comes to averages on large scales over which nongravitational fields are negligible and only gravitational degrees of freedom prevail, we are still left with the problem of defining a suitable "weighted average of the apparent motions". To be consistent with the broad principles of relativity, such an average must be limited by initial conditions and causality. If Einstein's field equations (1) are viewed as evolution equations which determine the geometry dynamically, then causality should limit the geometry at any event to only depend on the geometry within the past light cone of all possible observers at that event. For realistic large scale cosmological applications, given that energy absorbed from null signals provides a negligible contribution as compared to local matter densities, it is the local matter horizon [19] which is actually the most relevant domain in determining local average geometry.

In the perturbative FLRW framework, Bičák, Katz and Lynden-Bell [65] have identified choices of hypersurfaces and coordinates within those hypersurfaces, which are most uniquely restricted in terms of being determined by local energy–momentum perturbations $\delta T^\mu{}_\nu$ and thereby represent "Machian gauges". If the average geometry is not an exact FLRW model, then in going beyond the perturbative regime the question is which of these notions best embodies Mach's principle?

3.3.1. The cosmological equivalence principle I have argued [67, 68] that a nonperturbative generalization of the "uniform Hubble flow" [64, 65] or CMC [77, 78] slicing is the best choice for a further refinement of the notion of inertia consistent with Mach's principle. The reader is referred to the essay version [68] for a first introduction to these ideas, which I shall only very briefly sketch here.

Since gravity sourced by matter obeying the strong energy condition is universally attractive, any solution of Einstein's field equations (1) on scales which only involve gravity is necessarily dynamical. In my view there is a further principle of relativity that is lacking in general relativity with the SEP alone, which is a consequence of taking this dynamical nature as fundamental to the relational structure. In particular, for the regionally homogeneous and isotropic volume–expanding part of geodesic deviation *it should be impossible to distinguish whether particles are at rest in an expanding space or alternatively are moving in a static Minkowski space.*

The relation to inertia can be understood in terms of the semi–tethered lattice thought experiments [67, 68] in which a uniformly expanding lattice of observers in Minkowski space, connected by freely spooling isotropic tethers, apply brakes to the tethers with an impulse which is the same function of the synchronized local proper time at each lattice site. Deceleration takes place and work is done in converting the kinetic energy of expansion to heat in the brakes. However, since the magnitude of the impulse on each tether is identical at each lattice site, isotropy guarantees that there is *no net force* on any observer of the lattice, and so the motion is inertial in that sense,

although a deceleration and conversion of energy has taken place. In the gravitational case the regionally homogeneous isotropic part of the density plays the same role as the brakes on the tethers.

From the point of view of the averaging problem, my proposal is to restrict the global geometry in the final step of the average (3) to one which can be decomposed into average domains which obey the *cosmological equivalence principle* (CEP) [67]: *At any event, always and everywhere, it is possible to choose a suitably defined average spacetime region, the cosmological inertial frame (CIF), in which average motions (timelike and null) can be described by geodesics in a geometry that is Minkowski up to some time-dependent conformal transformation,*

$$\mathrm{d}s^2_{\mathrm{CIF}} = a^2(\eta) \left[-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \right]. \tag{21}$$

While this statement of the CEP would reduce to the standard SEP if $a(\eta)$ were constant, or alternatively over very short time intervals during which the time variation of $a(\eta)$ can be neglected, it is important to realize that the averaging region represented by the CIF (21) is very much larger than the neighbourhood of a point as assumed in a LIF. Relative to bound systems, the spatially flat FLRW metric (21) is to be viewed as applicable only on scales larger than galaxy clusters which correspond to finite infinity [7, 69] or the matter horizon [19]. Alternatively, within void regions, which might be regionally modeled by a portion of an open FLRW universe, a CIF would be applicable only on spatial scales which are small with respect to the scalar curvature radius.

Although the spatial extent of a CIF would be much smaller within void regions than within walls, it is important to recognise that unlike a LIF (21) is intended to apply on arbitrarily long timescales which capture the volume–decelerating part of the average geodesic deviation. Rather than being a geometry in the neighbourhood of a point, is an average asymptotic geometry for spatial regions of order 1–10 Mpc. On these scales (21) provides a suitable geometry to replace the usual notion of an asymptotically flat geometry for isolated systems. Although it is a spatially flat FLRW metric, it is not a global geometry for the whole universe as is the case in the standard cosmology.

A more detailed discussion of the rationale behind the CEP, including the roles of Weyl curvature and Ricci curvature in the averaging process, is provided in [67]. The key idea is that the CIF isolates a notion of inertia that only exists as a result of a *collective* degree of freedom of the regional background. In a sense we are dealing with the conformal mode of the 3–geometry which has been identified before in discussions that attempt to isolate the true gravitational degrees of freedom [77, 78, 79] and related discussions of Mach's principle‡ [81].

It is well–known that for the exchange of photons between comoving observers in the background (21), to leading order the observed redshift of one comoving observer relative to another yields the same local Hubble law, whether the exact relation, $z + 1 = a_0/a$,

‡ The separation of the conformal degree of freedom of the 3–metric from shape degrees of freedom is at the heart of Barbour's Machian approach to gravity [80, 81], which he has approached from many angles including the N-body problem [82].

is used or alternatively the radial Doppler formula, $z + 1 = [(c+v)/(c-v)]^{1/2}$, of special relativity is used, before making a local approximation. Making this a feature of regional averages allows for forms of inhomogeneity that admit such an indistinguishability of whether "particles are moving" or "space is expanding", while disallowing global coherent anisotropic flows of the sort typified by Bianchi models. Bianchi models single out preferred directions in the global background universe, thereby imbuing spacetime with absolute qualities that go beyond an essentially relational structure. To make general relativity truly Machian such backgrounds need to be outlawed by principle, and the CEP is one means to achieving this while allowing inhomogeneity.

To combine such regional average CIFs requires something akin to the introduction of a CMC slicing to preserve a uniform Hubble flow condition. By the SEP the first derivatives of the metric can always be set to zero in the neighbourhood of a point; it is not the connection that corresponds to the physical observables but the curvature tensor which is derived from it. The possibility of always being able to choose a uniform Hubble flow slicing extends this to regional scales – the first derivatives of the regional metric which correspond to the volume expansion are a degree of freedom upon which physical observables do not depend. The Hubble parameter is thus recognized as a "gauge choice" that can be made within the limits set by evolution from initial conditions at early epochs. We are always allowed to make a choice of coordinates of the averaging regions which keeps the Hubble parameter uniform in expanding regions despite the presence of large variations in regional densities and curvature.

The mathematical procedures required to construct such a uniform flow slicing in terms of patching one CIF region to the next have not been developed yet. Since one is dealing with a nonperturbative regime without prescribed dust the problem is likely to require mathematical constructions which go beyond the treatment of CMC foliations in a globally well-defined background with a prescribed energy-momentum tensor. The framework should involve a statistical description of geometry closely related to the coarse-graining procedure.

4. Cosmic averages versus cosmic variance

In any description of cosmic averages we must ask the question of how local observables are to be related to average quantities. There are two aspects to this question:

(i) How do our own measurements relate to some canonically defined average quantity?

(ii) How do statistically average quantities defined on spatial slices that define average cosmic evolution relate to average distances and angles on the past light cone from which all cosmological measurements are inferred?

These are significant questions which must be answered to build a viable cosmological model. It was recognized early on in the Buchert approach that this is a nontrivial problem, so that in general observed cosmological parameters will be "dressed" when compared to the bare cosmological parameters of the averaging

scheme [55, 83]. However, additional assumptions are required to achieve such a dressing. Rather than tackling this problem, a common approach has been simply to identify the volume average expansion parameter (18) in the Buchert scheme with our observed Hubble parameter [84]–[92], and to identify the observed redshift according to $1 + z = \bar{a}_0/\bar{a}$. However, such an assumption remains to be rigorously justified.

In my view it is important that we think carefully about these questions since once there is inhomogeneity and a variance in geometry, then not every observer can be the same average observer. Understanding and quantifying our observed measurements in relation to cosmic variance is as fundamentally important as understanding cosmic averages.

Structure formation provides us with a natural separation of scales which enable us to attack this problem from first principles [69]. In particular, we and all the galaxies we observe are bound structures which necessarily formed from density perturbations that were greater than critical density. Yet the volume of the universe is dominated at the present epoch by voids. Thus an average position by volume – which is operationally what the Buchert average defines – will be located in a void unbound to any structure, in a region whose local density and spatial curvature differs markedly from those in a galaxy where the actual objects we observe are located. The mass average therefore does not coincide with the volume average, and there can be systematic differences between the geometry of galaxies and the average geometry, which must be taken into account.

We are therefore led to a *statistical Copernican principle*: we are observers in an average galaxy, and in this sense our position is not special. However, by virtue of being in a galaxy our local environment is not the same density as the local environment of an average position by volume and this fact has to be taken into account in our interpretation of cosmological measurements. By analogy, in the original heliocentric solution Copernicus realised that we are not in the centre of the universe, but the fact that we are on the surface of a planet that rotates means our view of the universe is different from that at a random point in the solar system, and must be taken into account when interpreting astronomical observations.

The averaging problem has been studied in the Buchert formalism in various approaches which partition the universe into regions of different density [69, 70, 71, 85, 89, 92]. I will discuss the timescape cosmology in particular, since among the various approaches it is unique in focusing on the importance of the position of the observer in relation to average cosmic evolution, rather than the average cosmic evolution alone. It provides a prescription for quantifying the apparent variance in the expansion rate, and for the interpretation of volume average cosmological quantities by observers in galaxies, whose local geometry does not coincide with the volume-average metric on account of the mass-biased selection effect.

4.1. The timescape model and the Buchert formalism

To date the timescape model [69, 70, 71] has been developed by adapting the Buchert formalism. Since the timescape model is based on the idea that average spatial homogeneity is related to a uniform Hubble flow condition, the use of the Buchert formalism – with its choice of comoving hypersurfaces and a synchronous gauge – may seem contradictory. However, dust is not prescribed in the Buchert scheme, and the timescape model does not apply Buchert averaging to exact solutions with prescribed dust.

The timescape model begins from the premise that dust is coarse-grained at the $100h^{-1}$ Mpc scale§ of statistical homogeneity‖. It is hypothesized that within such a cell there is a notion of average homogeneous expansion when CIFs in regions of different density, which have undergone different relative volume decelerations, are patched together appropriately. The proper volume of void regions increases more rapidly, but there is a compensating increase of the clock rate of isotropic observers¶ within voids, as compared to isotropic observers in the denser wall regions. In this way there is always a choice of rulers and clocks for which an average uniform expansion is maintained.

The dust "particles" are then regions of the cosmic fluid which contain great variations in density and spatial curvature and which can be described by different alternative choices of time and space coordinates in a smoothed out description. The different coordinate systems are those adapted to isotropic observers in different locations in the fluid element, who have undergone different relative amounts of regional volume deceleration, and who extend coordinates to the whole element with a time coordinate assumed synchronous to their own. By any one set of clocks it appears that the void regions expand faster than the wall regions, and thus observers working with a single clock will assign a variance to the expansion rate within each fluid cell, even though there is another gauge in which the expansion is uniform.

It is the equivalence of the different descriptions of the coarse-grained fluid cell

§ As we will discuss in section 4.2, the specification of length scales depends on the average metric description any observer chooses. Here I assume a normalization of spatial distances to a spatially flat metric, in accord with the conventions of the standard cosmology.

‖ Here statistical homogeneity is understood as a scale at which the variance in density from one dust cell to another is bounded, rather than a scale at which a FLRW model is approached. The fact that this coincides with the BAO scale is a natural consequence of initial conditions on the density perturbation spectrum at last scattering. On scales smaller than the BAO scale the initial perturbations were slightly amplified by acoustic oscillations in the primordial plasma, leading to a greater variance in density contrasts on scales below the BAO scale as compared with scales larger than the BAO scale. A crude estimate [71] of the variance in density on scales larger than the BAO averaging scale gives a variance of order 6% in density on scales $\gtrsim 100h^{-1}$ Mpc, which accords well with observations of order 7–8% from galaxy clustering statistics [93, 94].

¶ Isotropic observers are those who to leading order see an isotropic CMB. Unlike the standard cosmology, however, in the timescape scenario the mean CMB temperature will differ for ideal observers within surfaces of average spatial homogeneity who have undergone varying amounts of regional volume deceleration and consequently have differently normalized clocks.

by isotropic observers in regions of different local Ricci scalar curvature that replaces diffeomorphism invariance of the microscopic metric as a relevant "coordinate freedom" of the coarse-grained metric description. The Buchert time parameter is assumed to apply to those isotropic observers whose locally measured spatial curvature is the same as the volume average spatial curvature when averaged on horizon scales. The Buchert formalism is assumed to apply insofar as the Buchert time parameter is a collective coordinate of the coarse-grained fluid element, and the variance in expansion rate refers to that attributed to the internal degrees of freedom of the fluid element by an isotropic observer whose local spatial curvature scalar matches the volume average one.

It is certainly true that the assumptions about use of the Buchert formalism here represent an ansatz, which needs to be rigorously demonstrated in a mathematical scheme for coarse-graining. However, since dust is not prescribed in the Buchert formalism any attempt to use the Buchert formalism in application to any realistic cosmology must invariably make assumptions about how dust is to be defined. I take the view that one should begin by making physically well-motivated assumptions, to see whether any phenomenologically realistic model universe can be constructed. If that is the case, then an appropriate mathematical formalism needs to be developed.

4.2. Spatial averages versus null cone averages illustrated by the timescape cosmology

If the Einstein equations (1) are to be viewed as evolution equations then a statistical description of average evolution would appear to have to involve spatial averages, especially since the geometry at any event is more influenced by the domain within the matter horizon [19] than by events on or close to the null cone. Nonetheless, almost all information about cosmic evolution comes to us on our past null cone, and thus any attempt to test a model of average cosmic evolution must relate the average cosmological parameters to observations made on the past null cone[+].

Even in the case of simple LTB models with prescribed dust, it is possible to demonstrate that an average of cosmic expansion and acceleration on spatial hypersurfaces does not in general coincide with the expansion and apparent acceleration as measured on the past light cone [97]. The determination of averages on the past light cone demands taking the position of the observer into account. Consequently in any spatial averaging formalism, including the Buchert formalism, specific arguments need to be presented for the identification of cosmological parameters in terms of measurements on our past light cone as observers in a galaxy.

Here I will briefly outline how these steps are achieved in the concrete example of the timescape model [69, 70, 71]. The timescape model assumes that within dust cells coarse-grained at the $100h^{-1}$ Mpc scale of statistical homogeneity there are spatially flat wall regions and negatively curved void regions. It is assumed we can always enclose the bound structures which formed from over-critical density perturbations within regions which are spatially flat on average, and marginally expanding at the boundary. These

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[+] For some other recent discussions of null cone averages see [95, 96].

boundaries are called *finite infinity* regions [7, 69], with local average metric

$$\mathrm{d}s_{fi}^2 = -c^2\mathrm{d}\tau_\mathrm{w}^2 + a_\mathrm{w}^2(\tau_\mathrm{w})\left[\mathrm{d}\eta_\mathrm{w}^2 + \eta_\mathrm{w}^2\mathrm{d}\Omega^2\right] \ . \tag{22}$$

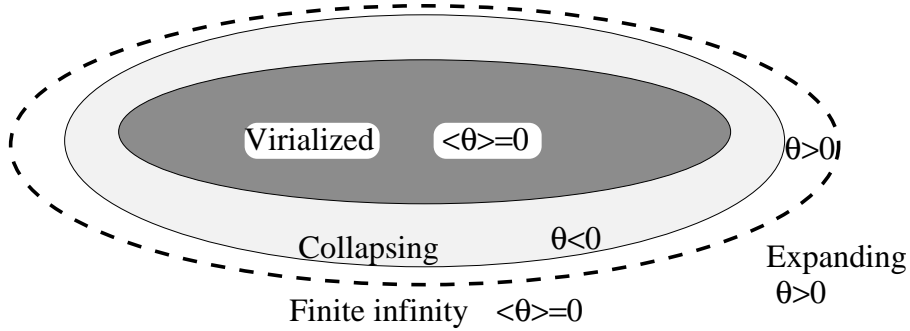The walls constitute the disjoint union of such finite infinity regions.

Figure 1. *A schematic illustration of the notion of finite infinity, fi [69]: the boundary (dashed line) to a region with average zero expansion inside, and positive expansion outside. It may or may not contain collapsing regions.*

Different finite infinity regions will have different spatial extents, being of order 1–2 Mpc for our Local Group of galaxies, while being one order of magnitude larger for rich clusters of galaxies. Although the overall universe is inhomogeneous, finite infinity provides a demarcation between bound systems and expanding regions (see figure 1). It also provides a notion of critical density regionally defined as the mass in a finite infinity region divided by its volume. The boundaries of separate finite infinity regions, although of different spatial extents, will have undergone the same amount of volume deceleration since the epoch of last scattering and the parameter τ_w is therefore assumed synchronous at all finite infinity boundaries.

In addition to the walls there are the voids which dominate the present epoch universe by volume. Voids will also have different spatial extents, being characterized by regional negatively curved metrics of the form

$$\mathrm{d}s_{\mathcal{D}_\mathrm{v}}^2 = -c^2\mathrm{d}\tau_\mathrm{v}^2 + a_\mathrm{v}^2(\tau_\mathrm{v})\left[\mathrm{d}\eta_\mathrm{v}^2 + \sinh^2(\eta_\mathrm{v})\mathrm{d}\Omega^2\right] \ . \tag{23}$$

Generally the voids will have different individual metrics (23). However, in the centres of the $30h^{-1}$ Mpc voids [1, 2] the regional metrics will rapidly approach that of an empty Milne universe for which the parameters τ_v can be assumed to be synchronous. One could potentially use different curvature scales for dominant voids and minivoids to characterize the average scalar curvature $\langle\mathcal{R}\rangle$. However, in the two–scale approximation of [69, 70] a single negative curvature scale is assumed as a simplification.

Although the construction is reminiscent of the Swiss cheese model, the important difference is that the metrics (22) and (23) both represent the regional geometries in disjoint regions of different spatial extents. Since there is no global FLRW metric there is no "cheese" in this construction. The global spacetime structure is determined instead by a Buchert average.

The Buchert average is constructed as a disjoint union of wall and void regions over the entire present horizon volume $\mathcal{V} = \mathcal{V}_i \bar{a}^3$, where

$$\bar{a}^3 = f_{vi}a_v^{\,3} + f_{wi}a_w^{\,3}, \tag{24}$$

f_{vi} and $f_{wi} = 1 - f_{vi}$ being the respective initial void and wall volume fractions at last scattering. It is convenient to rewrite (24) as $f_v(t) + f_w(t) = 1$, where $f_w(t) = f_{wi}\, a_w^3/\bar{a}^3$ is the *wall volume fraction* and $f_v(t) = f_{vi}\, a_v^3/\bar{a}^3$ is the *void volume fraction*. Within the dust particles the metrics (22) and (23) are assumed to be patched together with the condition of uniform quasilocal bare Hubble flow [67, 69] discussed in section 3.3,

$$\bar{H} = \frac{1}{a_w}\frac{\mathrm{d}a_w}{\mathrm{d}\tau_w} = \frac{1}{a_v}\frac{\mathrm{d}a_v}{\mathrm{d}\tau_v}. \tag{25}$$

Since this bare Hubble parameter is uniform within a dust particle, it will also be equal to the Buchert average parameter Hubble parameter (18). For the purpose of the Buchert average it is convenient to refer all quantities to the set of volume–average clocks that keep the time parameter t of (13)–(16) so that

$$\bar{H} = \frac{\dot{\bar{a}}}{\bar{a}} = \bar{\gamma}_w H_w = \bar{\gamma}_v H_v \tag{26}$$

where

$$H_w \equiv \frac{1}{a_w}\frac{\mathrm{d}a_w}{\mathrm{d}t}, \qquad H_v \equiv \frac{1}{a_v}\frac{\mathrm{d}a_v}{\mathrm{d}t}, \tag{27}$$

and

$$\bar{\gamma}_w \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau_w}, \qquad \bar{\gamma}_v = \frac{\mathrm{d}t}{\mathrm{d}\tau_v}, \tag{28}$$

are phenomenological lapse functions of volume–average time, t, relative to the time parameters of isotropic wall and void–centre observers respectively. The ratio of the relative Hubble rates $h_r = H_w/H_v < 1$ is related to the wall lapse function by

$$\bar{\gamma}_w = 1 + \frac{(1 - h_r)f_v}{h_r}, \tag{29}$$

and $\bar{\gamma}_v = h_r \bar{\gamma}_w$.

In this two-scale approximation the Buchert equations for pressureless dust with volume–average density $\bar{\rho}_M$ can be solved analytically [70] if we make the assumption that volume–average shear is negligible and that there is no backreaction within walls and voids separately*, but only in the combined average. With this assumption, the kinematic backreaction term becomes [69]

$$\mathcal{Q} = 6f_v(1 - f_v)\,(H_v - H_w)^2 = \frac{2\dot{f}_v^{\,2}}{3f_v(1 - f_v)}. \tag{30}$$

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* In the model of Wiegand and Buchert [92], two components of overdense and underdense regions are similarly identified, but with two additional parameters representing internal backreaction within these regions. Furthermore, as well as having a different observational interpretation of their solutions, Wiegand and Buchert's choice of the values of the parameters equivalent to the initial fractions f_{wi} and f_{vi} is different, as they do not formally identify walls and voids [92]. Since walls and voids do not exist at the surface of last scattering, I assume that the vast bulk of the present horizon volume that averages to critical density gives $f_{wi} \simeq 1$, while $f_{vi} = 1 - f_{wi}$ is the small positive fraction of the present horizon volume that consists of *uncompensated* underdense regions at the last scattering surface.

The independent Buchert equations (13), (19) then reduce to two coupled nonlinear ordinary differential equations [69] for $\bar{a}(t)$ and $f_{\mathrm{v}}(t)$, namely

$$\bar{\Omega}_M + \bar{\Omega}_k + \bar{\Omega}_{\mathcal{Q}} = 1, \tag{31}$$

$$\bar{a}^{-6}\partial_t\left(\bar{\Omega}_{\mathcal{Q}}\bar{H}^2\bar{a}^6\right) + \bar{a}^{-2}\partial_t\left(\bar{\Omega}_k\bar{H}^2\bar{a}^2\right) = 0\,, \tag{32}$$

where

$$\bar{\Omega}_M = \frac{8\pi G\bar{\rho}_{M0}\bar{a}_0^3}{3\bar{H}^2\bar{a}^3}\,, \tag{33}$$

$$\bar{\Omega}_k = \frac{-k_{\mathrm{v}}c^2 f_{\mathrm{vi}}^{2/3} f_{\mathrm{v}}^{1/3}}{\bar{a}^2\bar{H}^2}\,, \tag{34}$$

$$\bar{\Omega}_{\mathcal{Q}} = \frac{-\dot{f}_{\mathrm{v}}^2}{9 f_{\mathrm{v}}(1 - f_{\mathrm{v}})\bar{H}^2}\,, \tag{35}$$

are the volume–average or "bare" matter density, curvature density and kinematic backreaction density parameters respectively, $\bar{a}_0$ and $\bar{\rho}_{M0}$ being the present epoch values of $\bar{a}$ and $\bar{\rho}_M$. The average curvature is due to the voids only, which are assumed to have $k_{\mathrm{v}} < 0$. Equations (31), (32) are readily integrated to yield an exact solution [70], which also possesses a very simple tracking limit at late times with $h_r \to 2/3$.

The exact solution [70] of the Buchert equations is of course just a statistical description which is not directly related to any physical metric that has been specified thus far. Since all cosmological information is obtained by a radial spherically symmetric null cone average, we retrofit a spherically symmetric geometry relative to an isotropic observer who measures volume-average time, and with a spatial volume scaling as $\bar{a}^3(t)$,

$$\mathrm{d}\bar{s}^2 = -c^2\mathrm{d}t^2 + \bar{a}^2(t)\,\mathrm{d}\bar{\eta}^2 + \mathcal{A}(\bar{\eta}, t)\,\mathrm{d}\Omega^2. \tag{36}$$

Here the area quantity, $\mathcal{A}(\bar{\eta}, t)$, satisfies $\int_0^{\bar{\eta}_{\mathcal{H}}} \mathrm{d}\bar{\eta}\,\mathcal{A}(\bar{\eta}, t) = \bar{a}^2(t)\mathcal{V}_{\mathrm{i}}(\bar{\eta}_{\mathcal{H}})/(4\pi)$, $\bar{\eta}_{\mathcal{H}}$ being the conformal distance to the particle horizon relative to an observer at $\bar{\eta} = 0$. The metric (36) is spherically symmetric by construction, but is not a LTB solution since it is not an exact solution of Einstein's equations, but rather of the Buchert average of the Einstein equations.

In terms of the wall time, τ_{w}, of finite infinity observers in walls the metric (36) is

$$\mathrm{d}\bar{s}^2 = -\bar{\gamma}_{\mathrm{w}}^2(\tau_{\mathrm{w}})\,c^2\mathrm{d}\tau_{\mathrm{w}}^2 + \bar{a}^2(\tau_{\mathrm{w}})\,\mathrm{d}\bar{\eta}^2 + \mathcal{A}(\bar{\eta}, \tau_{\mathrm{w}})\,\mathrm{d}\Omega^2\,. \tag{37}$$

This geometry, which has negative spatial curvature is not the locally measured geometry at finite infinity, which is given instead by (22). Since (22) is not a global geometry, we match (22) to (37) to obtain a *dressed* wall geometry. The matching is achieved in two steps. First we conformally match radial null geodesics of (22) and (37), noting that null geodesics are unaffected by an overall conformal scaling. This leads to a relation

$$\mathrm{d}\eta_{\mathrm{w}} = \frac{f_{\mathrm{wi}}^{1/3}\mathrm{d}\bar{\eta}}{\bar{\gamma}_{\mathrm{w}}(1 - f_{\mathrm{v}})^{1/3}} \tag{38}$$

along the geodesics. Second, we account for volume and area factors by taking η_{w} in (22) to be given by the integral of (38).

The wall geometry (22), which may also be written

$$ds_{fi}^2 = -c^2 d\tau_w^2 + \frac{(1-f_v)^{2/3}\,\bar{a}^2}{f_{wi}^{\,2/3}} \left[d\eta_w^2 + \eta_w^2 d\Omega^2 \right] ,$$ (39)

on account of (24), is a local geometry only valid in spatially flat wall regions. We now use (38) and its integral to extend this metric beyond the wall regions to obtain the dressed global metric

$$\begin{aligned} ds^2 = & -c^2 d\tau_w^2 + \frac{\bar{a}^2}{\bar{\gamma}_w^2}\,d\bar{\eta}^2 + \frac{\bar{a}^2\,(1-f_v)^{2/3}}{f_{wi}^{\,2/3}}\,\eta_w^2(\bar{\eta},\tau_w)\,d\Omega^2 \\ = & -c^2 d\tau_w^2 + a^2(\tau_w) \left[d\bar{\eta}^2 + r_w^2(\bar{\eta},\tau_w)\,d\Omega^2 \right] \end{aligned}$$ (40)

where $a \equiv \bar{\gamma}_w^{-1}\bar{a}$, and

$$r_w \equiv \bar{\gamma}_w\,(1-f_v)^{1/3}\,f_{wi}^{\,-1/3}\eta_w(\bar{\eta},\tau_w).$$ (41)

While (22) represents a local geometry only valid in spatially flat wall regions, the dressed geometry (40) represents an average effective geometry extended to the cosmological scales, parametrized by the volume–average conformal time which satisfies $d\bar{\eta} = c\,dt/\bar{a} = c\,d\tau_w/a$. Since the geometry on cosmological scales does not have constant Gaussian curvature the average metric (40), like (36), is spherically symmetric but not homogeneous.

If as wall observers we try to fit a FLRW model synchronous with our clocks that measure wall time, τ_w, we are effectively fitting the dressed geometry (40), which is effectively the closest thing there is to a FLRW geometry adapted to the rulers and clocks of wall observers. The cosmological parameters we infer from taking averages on scales much large than the $100h^{-1}$ Mpc scale of statistical homogeneity will not be the bare parameters $\bar{H}$, $\bar{\Omega}_M$, $\bar{\Omega}_k$, and $\bar{\Omega}_Q$, but instead the *dressed Hubble parameter*

$$H \equiv \frac{1}{a}\frac{da}{d\tau_w} = \frac{1}{\bar{a}}\frac{d\bar{a}}{d\tau_w} - \frac{1}{\bar{\gamma}_w}\frac{d\bar{\gamma}_w}{d\tau_w} = \bar{\gamma}_w\bar{H} - \frac{d\bar{\gamma}_w}{dt} ,$$ (42)

and the *dressed matter density parameter*

$$\Omega_M = \bar{\gamma}_w^3\bar{\Omega}_M .$$ (43)

There is similarly a dressed luminosity distance relation

$$d_L = a_0(1+z)r_w,$$ (44)

where $a_0 = \bar{\gamma}_{w0}^{-1}\bar{a}_0$, and the *effective comoving distance* to a redshift z is $D = a_0 r_w$, where

$$r_w = \bar{\gamma}_{w0}\,(1-f_v)^{1/3} \int_t^{t_0} \frac{c\,dt'}{\bar{\gamma}_w(t')(1-f_v(t'))^{1/3}\bar{a}(t')} ,$$ (45)

and $1+z \equiv a_0/a = (\bar{a}_0\bar{\gamma}_w)/(\bar{a}\,\bar{\gamma}_{w0})$.

It was demonstrated in [69, 70] that for realistic initial conditions at last scattering the dressed deceleration parameter is negative at late epochs, even though the volume-average bare deceleration parameter is positive. In particular, the general solution [70] possesses a tracking limit which is reached to within 1% by a redshift $z \sim 37$. For the

tracker solution [70] the phenomenological lapse function, $\bar{\gamma}_{\rm w}(t)$, void fraction $f_{\rm v}(t)$, and bare Hubble parameter, $\bar{H}(t)$, are related by

$$\bar{\gamma}_{\rm w} = \tfrac{3}{2}t\bar{H}(t) \tag{46}$$

$$= 1 + \tfrac{1}{2}f_{\rm v}(t) \tag{47}$$

$$= \frac{9f_{\rm v0}\bar{H}_0 t + 2(1 - f_{\rm v0})(2 + f_{\rm v0})}{2\left[3f_{\rm v0}\bar{H}_0 t + (1 - f_{\rm v0})(2 + f_{\rm v0})\right]} , \tag{48}$$

where $\bar{H}_0$ and $f_{\rm v0}$ are the present epoch values of $\bar{H}(t)$ and $f_{\rm v}(t)$. From the bare Hubble parameter we can construct a bare deceleration parameter,

$$\bar{q} \equiv \frac{-\ddot{\bar{a}}}{\bar{H}^2\bar{a}} = \tfrac{1}{2}\bar{\Omega}_M + 2\bar{\Omega}_{\mathcal{Q}} = \frac{2\left(1 - f_{\rm v}\right)^2}{(2 + f_{\rm v})^2} . \tag{49}$$

For the tracker solution, the time parameter $\tau_{\rm w}$ of wall observers is related to the Buchert volume–average time parameter, t, by

$$\tau_{\rm w} = \tfrac{2}{3}t + \frac{4\Omega_{M0}}{27f_{\rm v0}\bar{H}_0}\ln\left(1 + \frac{9f_{\rm v0}\bar{H}_0 t}{4\Omega_{M0}}\right) , \tag{50}$$

where $\Omega_{M0} = \tfrac{1}{2}(1 - f_{\rm v0})(2 + f_{\rm v0})$ is the present epoch dressed matter density. At early epochs, as $t \to 0$, $\tau_{\rm w} \sim t$, but at later epochs the two parameters differ by an amount restricted to the range $\tfrac{2}{3}t < \tau_{\rm w} < t$. The dressed Hubble parameter (42) then satisfies

$$H = \frac{2}{3t} + \frac{f_{\rm v}(t)[4f_{\rm v}(t) + 1]}{6t} = \bar{H}(t) + \frac{f_{\rm v}(t)[4f_{\rm v}(t) - 1]}{6t} , \tag{51}$$

and can be used to construct a dressed deceleration parameter,

$$q \equiv \frac{-1}{H^2a^2}\frac{{\rm d}^2a}{{\rm d}\tau_{\rm w}^2} = \frac{-\left(1 - f_{\rm v}\right)\left(8f_{\rm v}^3 + 39f_{\rm v}^2 - 12f_{\rm v} - 8\right)}{\left(4 + f_{\rm v} + 4f_{\rm v}^2\right)^2} . \tag{52}$$

At early epochs when the void fraction is small both deceleration parameters take the value, $q \sim \bar{q} \sim \tfrac{1}{2}$, as would be expected for an Einstein–de Sitter universe. However, whereas the bare deceleration parameter (49) is always positive, the dressed deceleration parameter changes sign at a value of $f_{\rm v} \simeq 0.59$ corresponding to a zero of the cubic in the numerator of (52). At very late epochs both deceleration parameters approach the Milne value, but with opposite signs: $\bar{q} \to 0^+$, while $q \to 0^-$. For parameter values with a good fit to supernovae data [99, 100], $f_{\rm v0} \sim 0.76$, and apparent acceleration typically begins at a redshift $z \sim 0.9$. Apparent acceleration is therefore a transitional feature, which is largest in the transition period during which voids become dominant. During this epoch the variance of local geometry in galaxies from the volume average geometry grows large, leading to a variance in the relative calibration of clocks and rulers.

For parameter values which phenomenologically fit the observed expansion history of the universe [71, 99, 100], we find $\bar{\gamma}_{\rm w0} \sim 1.37$, meaning that the variance in the two time parameters $\tau_{\rm w}$ and t in (50) typically grows to 37% by the present epoch [69]. Such a large difference is counter-intuitive, but can be understood as the cumulative integrated

effect resulting from a tiny relative deceleration of whose magnitude is defined in terms of the phenomenological lapse function by [67]

$$\frac{\alpha}{c} = \frac{1}{\sqrt{\bar\gamma_{\rm w}^2 - 1}} \frac{{\rm d}\bar\gamma_{\rm w}}{{\rm d}\tau_{\rm w}} = \frac{\rm d}{{\rm d}t}\sqrt{\bar\gamma_{\rm w}^2 - 1}. \tag{53}$$

Substituting the tracker solution (48) in (53) gives a relative deceleration as plotted in figure 2. Its value at the present epoch is $\alpha_0 \sim 7 \times 10^{-11}{\rm m\,s}^{-2}$ and is typically of order $10^{-10}{\rm m\,s}^{-2}$ for most of the life of the universe. Although the absolute value of α was higher in the past, the relative expansion rate was even higher in the past. As a fraction of the Hubble parameter, $\alpha/(Hc)$, is highly suppressed at early epochs: this dimensionless ratio ranges from 0.1 at the present epoch to 6×10^{-6} at the epoch of last scattering. Thus even though we are dealing with an effect whose instantaneous magnitude is extremely small and well within the weak field regime, it can still lead to a significant *cumulative* difference when one has the age of the universe to integrate over.
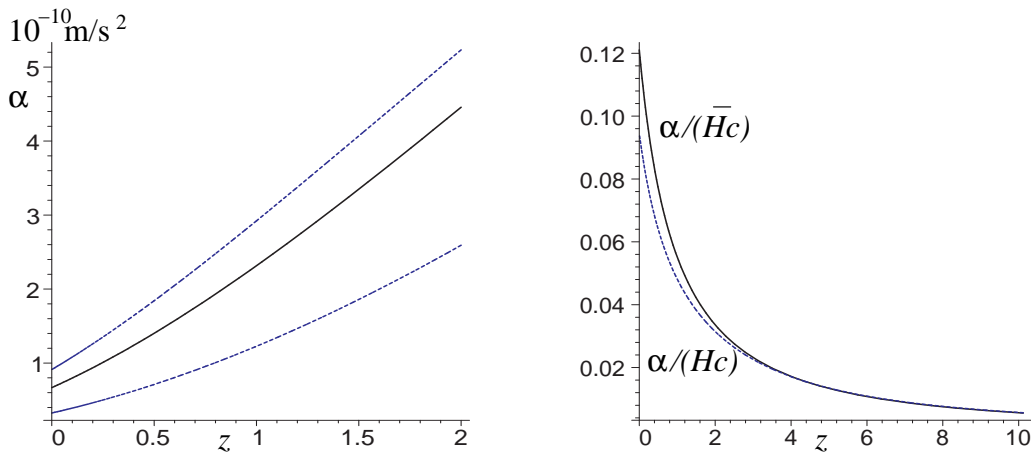

Figure 2. *The magnitude of the relative deceleration scale [67], α:* **(a)** *in terms of its absolute value for redshifts $z < 0.25$;* **(b)** *in terms of the dimensionless ratios $\alpha/(c\bar H)$ (solid curve) and $\alpha/(cH)$ (dashed curve) for redshifts $z < 10$. In panel (b) just the best fit value $f_{\rm v0} = 0.76$ is shown, whereas in panel (a) the solid and dashed represent the best fit value and 1σ uncertainties, $f_{\rm v0} = 0.76^{+0.12}_{-0.09}$ [99] respectively. A value $H_0 = 61.7\,km\ sec^{-1}\,Mpc^{-1}$ is assumed.*

4.3. Observational tests of the timescape cosmology

A variety of inhomogeneous model universes, including the LTB model and other exact solutions [101], can be fit to observational data which measure the expansion history of the universe in a variety of ways, with varying degrees of success. One must take care with such fitting, however, since the raw data has often been reduced assuming the standard homogeneous cosmology in ways which can sometimes be very subtle. Type Ia supernovae (SneIa), for example, are not standard candles but rather are standardizable candles, and one widely used light curve fitter [102, 103] marginalizes over parameters of

the standard cosmology as well as empirical light curves parameters when reducing the data. Naïvely using the reduced data in studies of inhomogeneous models and averaging, which a number of researchers unfortunately do, can therefore be problematic.

At the very least one needs a clear idea about how inhomogeneity limits the derivation of an average expansion. In the timescape scenario, the minimum scale on which an average isotropic Hubble law is expected is the $100h^{-1}$Mpc scale of statistical homogeneity. Since many SneIa data sets contain significant numbers of points below this scale, care must be taken to remove such points, or more generally to consider how the inclusion of such events might affect the overall calibration of light curves.

Observational tests of the timescape cosmology are relatively well developed [71, 99, 100], and to the extent that it has been tested the timescape model is competitive with the standard ΛCDM cosmology. Luminosity distances of SneIa [99, 100] are the most well tested. It was recently shown [100] that in terms of current data sets the differences between the luminosity distance predictions of the ΛCDM and timescape cosmologies are at the same level as systematic uncertainties in the light curve fitters, due to unknown properties of SneIa. These include, in particular, a degeneracy between the effect of intrinsic colour variations in SneIa events and the effect of absorption by dust in the host galaxies, which is currently being investigated by astronomers. Depending on which light curve fitting method one uses, one can find that there is Bayesian statistical evidence in favour of the standard ΛCDM cosmology over the timescape cosmology, or alternatively in favour of the timescape cosmology over the ΛCDM cosmology. In other words, there are already enough SneIa events to distinguish the two models, but the empirical treatment of SneIa light curves to convert them to standard candles still needs to be understood before conclusions can be drawn.

Our recent study of SneIa luminosity distances [100] finds that making cuts to the data below the $100h^{-1}$Mpc scale of statistical homogeneity can significantly affect the analysis. Furthermore, in terms of SneIa systematics, we find that the timescape scenario would appear to be more obviously favoured over the ΛCDM model if the reddening law for dust in other galaxies has a reddening parameter, R_V, close to the value $R_V \simeq 3.1$ observed for dust in the Milky Way [100], rather than half this value. Since the reddening law in nearby galaxies can be tested independently, future investigations of such issues will have the power to falsify or strongly constrain the timescape scenario. Thus far such studies find values $R_V = 2.82 \pm 0.38$ and $R_V = 2.71 \pm 0.43$ for two samples of eight and seven nearby galaxies respectively [104, 105], consistent with the Milky Way value.

In the timescape model parameter values have also been determined which fit the angular diameter distance of the sound horizon in the CMB anisotropy spectrum, and the BAO scale in galaxy clustering statistics [71, 99, 100]. These estimates are crude ones at this stage, as the raw data still needs to be reanalysed from first principles using the timescape methodology before statistical bounds can be obtained, and this is likely to be a very involved process. Several potential tests of the expansion history were proposed in [71], including, for example, variants of the Alcock-Paczyński test [106], the inhomogeneity test of Clarkson, Bassett and Lu [107], and the time drift of

cosmological redshifts [108, 109, 110]. Furthermore, three separate tests with indications of results in possible tension with the ΛCDM model on the basis of existing data are found to be consistent with the expectations of the timescape cosmology [71]. Since these observational tests have already been briefly reviewed elsewhere [111], I will not discuss them further here.

The greatest observational challenge for the timescape model is the value of dressed Hubble constant on scales larger than that of statistical homogeneity. If we compare the angular diameter distance of the sound horizon seen in the CMB anisotropy spectrum and the effective comoving scale of the BAO as seen in galaxy clustering statistics, then a range of values of the dressed Hubble constant, $57 \lesssim H_0 \lesssim 68 \,\mathrm{km\ sec^{-1}\,Mpc^{-1}}$, would be admissible in the timescape scenario [100]. This is at odds with the recent measurement of $H_0 = 73.8 \pm 2.4 \,\mathrm{km\ sec^{-1}\,Mpc^{-1}}$ by the SHOES survey [112]. However, it is a feature of the timescape model that a 17–22% variance in the apparent Hubble flow will exist on local scales below the scale of statistical homogeneity, and this may potentially complicate calibration of the cosmic distance ladder.

Further quantification of the variance in the apparent Hubble flow in relationship to local cosmic structures would provide an interesting possibility for tests of the timescape cosmology for which there are no counterparts in the standard cosmology. There is evidence from the study of large-scale bulk flows that apparent peculiar velocities determined in the FLRW framework have a magnitude arguably in excess of the statistical expectations of the standard ΛCDM model [113, 114, 115]. In the timescape model it conceptually better to think in terms of varying expansion rates, rather than peculiar velocities. Nonetheless, given that our location is right on an edge between a wall and a dominant void [116] the effective equivalent maximum peculiar velocity can be estimated as

$$v_{\mathrm{pec}} = (\tfrac{3}{2}\bar{H}_0 - H_0)\frac{30}{h}\,\mathrm{Mpc} = 510^{+210}_{-260}\,\mathrm{km/s}\,. \tag{54}$$

This estimate assumes the typical diameter of $30\,h^{-1}\,\mathrm{Mpc}$ for the local void, and uses the tracker solution relation $\bar{H}_0 = 2(2 + f_{\mathrm{v}0})H_0/\left(4f_{\mathrm{v}0}^2 + f_{\mathrm{v}0} + 4\right)$ [70] with the best fit values $f_{\mathrm{v}0} = 0.76^{+0.12}_{-0.09}$ [99]. This rough estimate is of a magnitude consistent with observation.

In any inhomogeneous cosmology the manner in which we estimate peculiar velocities from the data needs to be carefully considered. In the case of the kinematic Sunyaev-Zel'dovich effect [114, 115] the standard FLRW model is assumed in the data reduction in possibly subtle ways. One important question is whether the CMB dipole is purely due to our peculiar velocity with respect to the surface of average homogeneity, or whether it also contains some fraction, perhaps just at the percent level, which is due to foreground inhomogeneities within the scale of statistical homogeneity.

A more insidious problem with conceptual thinking is that many researchers, particularly observationalists, tend to think in terms of a uniform FLRW expansion of the universe in Euclidean space with an action–at–a–distance Newtonian gravitational force embedded on top, so that we experience "infall" towards the Virgo cluster or even towards the much more distant Shapeley concentration even though the physical

distances to these objects are always increasing. There is no fundamental reason to expect spacetime to arrange itself so that the intuition we have about gravity from the solar system repeats itself on the very largest of scales. If we use just one set of clocks, then in any inhomogeneous model it makes more sense conceptually to think about variations of the expansion rate in regions of different density (and spatial curvature) which decelerate by different amounts. We need to think a bit more deeply in analysing the data, particularly given the conundrum that observed peculiar velocities of galaxies with respect to a FLRW background do not match statistical expectations.

5. Discussion

In this review I have discussed what I believe are some of the most important physical questions in relation to the averaging problem. In my view, we stand at a very exciting juncture for the development of general relativity, as the mystery of dark energy indicates that deep fundamental questions remain concerning our understanding of spacetime on the largest of scales. Even if dark energy is "just" a cosmological constant, then that would be of profound significance, since quite apart from the problem of explaining its magnitude we would have to understand why there is a field which permeates spacetime without reference to other matter.

I believe that it is time to more seriously consider Einstein's dictum that *"In a consistent theory of relativity there can be no inertia relatively to 'space', but only an inertia of masses relatively to one another"* [72]. In particular, rather than modifying gravity to add exotic fields in the vacuum in ways which potentially violate the weak equivalence principle, we should consider modifications that do not violate any existing principles but which might add limiting principles to give a deeper realisation of spacetime as a relational structure, consistent with Mach's principle. The cosmological equivalence principle (CEP) [67, 68] is proposed with such an end in mind. Although there may ultimately be better ways of framing relevant principles, one cannot escape from the fact that if the averaging problem is to be thought of in physical terms, then it is intimately related to the cosmological statement of Mach's principle [73].

The observed universe has a very complex hierarchical structure [117], and is very clearly inhomogeneous on scales $\lesssim 100h^{-1}\,\mathrm{Mpc}$. This has led many researchers to consider both exact inhomogeneous solutions of general relativity [101], as well as the averaging problem for general inhomogeneous metrics. Yet the vast majority of this effort is mathematically driven, rather than physically driven.

Why is the expansion so close to that of a FLRW model despite the observed inhomogeneity? A potential answer is that there is a canonical choice of clocks and rulers that can always be made in the averaging problem to make the regional expansion uniform – analogously to the freedom of choosing Riemann normal coordinates to make the first derivatives of the metric zero near a point – and it is this choice which is made by nature to define average spatial homogeneity and preserve the near isotropy of the CMB.

Without additional limiting principles inhomogeneous geometries offer so many potential parameters that it is difficult to see how they could be constrained. Many researchers choose to limit their models by the demand that the average evolution is a FLRW one, as discussed in section 3.1. Yet the FLRW universe is not singled out by any physical principle, and it embodies three separate notions of average spatial homogeneity which may well be overly restrictive.

I suggest that the CEP, or something close to it, is a limiting principle which singles out a notion of uniform expansion as the condition of average spatial homogeneity without necessarily leading to FLRW evolution. A CMC-like slicing generalizes the notion of relating inertial frames by a uniform velocity, and has been independently recognised by a number of researchers [65, 67, 81] as embodying Mach's principle.

The timescape scenario is a framework which attempts to put such physical principles into a simple cosmological model. As a phenomenological model it is interesting to note that it is competitive with the standard ΛCDM model, in as far as it has been tested to date [71, 100, 111], with the only obvious major challenge at present being the value of the global average Hubble constant [112]. However, there are many outstanding issues in the timescape scenario, which prevent many researchers from giving it further consideration.

One clear problem is the issue of junction conditions and the patching together of CIFs to realise the uniform Hubble flow condition within a dust "particle". In the two scale model outlined in section 4.2 the wall and void regions are combined in a disjoint union without applying junction conditions♯. The reason this has not yet been done is that it would require the development of mathematical tools for the coarse-graining of geometries in a statistical sense, and this is far from trivial. It is not a simple case of cutting and pasting exact solutions for prescribed dust by well-known techniques. Rather the solution of the problem is intimately related to the question of what a dust fluid element is in general relativity when we have to coarse grain over gravitational degrees of freedom themselves.

One feature of the formalism that is required to tackle this problem is that it should deal with regional symmetries. In particular, whereas general relativity deals with diffeomorphism invariance on one hand, and the point symmetries of the Lorentz group on the other, establishing CIFs requires us to deal with the collective degree of freedom corresponding to a regional volume expansion in particular. Different CIFs which have undergone different amounts of relative volume deceleration will have differing

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♯ The effect of junction conditions can be seen in the case of LTB models with prescribed dust or networks of such LTB voids. In these cases the shear in hypersurfaces of constant comoving time counteracts the variance in volume expansion leading to a greatly suppressed backreaction [118, 119]. The timescape scenario deals with a rather different situation: in particular, it does not deal with prescribed dust constrained to avoid shell cross singularities, nor with highly symmetric exact solutions which have been cut and pasted together. Finally, the surfaces of average spatial homogeneity are not assumed to be surfaces of constant comoving time. In the LTB void model a "uniform Hubble flow" slicing of the prescribed dust for reasonable density contrasts can only be introduced at the price of taking hypersurfaces which are not necessarily purely spacelike [120].

phenomenological lapse functions.

The construction of a phenomenological lapse function†† from a more rigorous mathematical basis requires careful thought. In particular, there is no single global ADM metric covering the whole universe which adequately describes the metric degrees of freedom associated with galaxy clusters. Thus it is not a simple matter of taking a single ADM lapse function and integrating it out when coarse-graining. Furthermore, since we are dealing with a collective degree of freedom corresponding to a regional volume expansion, the phenomenological lapse function is also subtly different to the gamma factor in a Lorentz boost. By the semi-tethered lattice analogy [67, 68] it is not associated with a boost in any particular direction but has more the character of a boost which is orthogonal to every spatial direction. Since a rigorous treatment requires a new as yet undeveloped methodology, it is difficult to convince sceptics of its necessity. However, from a physical standpoint a relative deceleration implies a difference in the amount of kinetic energy of expansion that is converted to other forms of energy through gravitational collapse, which must have very tangible consequences for physical processes. To ignore this problem – simply because it involves the thorny issue of the nature of gravitational energy – is to ignore some of the most fundamental principles of physics.

Dealing with the regional average symmetries that emerge in coarse-graining inevitably means that we must consider quasilocal quantities, and in particular quasilocal mass–energy and angular momentum. On account of the strong equivalence principle we can always get rid of gravity near a point, and so the definition of quasilocal energy is a subtle problem, which has been studied for decades without any clear consensus emerging. (For a review see [121].) The quasilocal energy problem has principally been studied for isolated systems, where conventional notions of mass associated with asymptotically flat systems are well grounded. The issue of quasilocal energy is relatively little studied for cosmological solutions, and what work there is usually makes reference to specific exact solutions such as the FLRW models [122]– [126]. Sussman has considered the specific case of defining relevant quasilocal variables for generalized LTB models [127] and their relationship to averaging and backreaction for the case of spherically symmetric dust [128, 129, 130].

Rather than always working with the same set of exact cosmological solutions, more effort is needed to understand quasilocal variables that might be relevant for more general coarse-graining procedures. Korzyński's approach [9] represents an interesting idea, which still remains to be fully developed. The statistical nature of gravitational energy and entropy on the largest scales is an unsolved fundamental problem which might be better understood by thinking more carefully about these procedures.

††If the only symmetries that are allowed are diffeomorphisms of the global metric on one hand, and local Lorentz transformations corresponding to rotations and boosts on the other, then realistically there is no room for clock rate variations of the order of magnitude dealt with in the timescape scenario [51]. However, the suggestion here is that additional mathematical ingredients are required to define regional symmetries when coarse-graining.

In summary, it is my view that the apparently accelerated expansion of the universe demands that we take a fresh look at the foundations of cosmological general relativity from first principles. In particular, we face the very real possibility that "dark energy" is simply an illusion due to our misunderstanding of gravitational energy gradients in a complex hierarchical geometry. To attack a problem as fundamental as gravitational energy we must think fundamentally.

The argument about whether it is better to use spacetime averages (section 2.3.2) as opposed to spatial averages (section 2.3.3) cannot really be addressed without asking the more basic question of what is the structure of spacetime on the largest of scales, especially over scales larger than that of the matter horizon [19] beyond which the exchange of particles and energy between observers is minimal. Rather than simply taking a principle such as general covariance as being paramount, we have to ask why was general covariance introduced? The reason was that it is a way of characterizing physical laws which combine gravity with the nongravitational interactions of nature in such a way that spacetime geometry is a relational structure between the elementary particles which interact via nongravitational forces.

In seeking to coarse grain gravitational degrees of freedom themselves, we have to be prepared for the possibility that realising spacetime as a relational structure might involve new ingredients beyond those which apply to nongravitational microphysics or general relativity on the scale of isolated systems. For example, the Bianchi I universe picks out preferred directions in space, and is at odds with observation; but would not be admitted by the CEP. Whether the CEP or some other principle is the correct one, what is most important is that we take up the challenges offered by cosmological observations to think more deeply about the foundations of general relativity as a physical theory of the universe.

Acknowledgments

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