UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

Department of Physics and Astronomy
University of Canterbury, Private Bag 4800
Christchurch, New Zealand

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Average cosmic evolution in a lumpy universe

by

James A.G. Duley

Supervisor: Assoc. Prof. David L. Wiltshire

Assistant Supervisor: Dr Teppo Mattsson

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To my parents, who without their help and encouragement I would not have made it this far.

Contents

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Abstract

The procedure of averaging and coarse-graining of the gravitational field equations with sources are investigated in both Newtonian gravity and in general relativity. In particular the schemes of Buchert and Korzyński are examined and compared in both situations. In Newtonian gravity it is shown how to calculate the tidal tensor given boundary conditions for it and how to average it given those boundary conditions. It is also shown that one can always choose boundary conditions to make the average tidal tensor vanish or take any value.

The problems of coarse-graining tensors in general relativity are critically examined, and a set of relevant conditions for such a procedure are enumerated. Korzyński's covariant coarse-graining procedure is reviewed and applied to a particular case. For the case of the Lemaître-Tolman-Bondi model it is shown that the backreaction was always zero for a centred spherical coarse-graining domain.

Wiltshire's timescape model, which applies a particular observational interpretation to Buchert's averaging scheme, is reviewed. The dust timescape model of Wiltshire is extended by the addition of a homogeneous radiation source. This model is solved numerically and it is shown not to vary significantly from the dust model since the redshift $z \approx 30$, which is when the backreaction and radiation density are equal. The model is integrated back in time from the surface of last scattering with results indicating a breakdown in aspects of the model at early times.

Conventions

Unless otherwise noted, the following conventions will be used. Units will be used such that $c = 1$. In reference to spacetime, Greek indices will be taken to be $0, 1, 2, 3$ and Latin indices will be taken to be $1, 2, 3$; the first half (a to g) will be used to denote Euclidean space and the middle letters (i to p) to denote the spatial indices of a 3+1 split of spacetime. In reference to coarse-graining on an arbitrary dimensional manifold, Greek indices will indicate a coordinate basis and Latin indices will indicate a non-coordinate basis. Einstein summation convention shall be assumed. Tensor signs will follow that of Misner, Thorne and Wheeler [1], i.e.,

$$\textit{Metric signature}: \; \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1) \tag{1}$$

$$\textit{Riemann tensor}: \; R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\nu\alpha} \tag{2}$$

$$\textit{Einstein tensor}: \; G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{3}$$

A comma shall denote a partial derivative, whereas a semicolon will denote a covariant derivative, i.e., $X^{\beta}{}_{,\alpha} \equiv \partial_\alpha X^\beta$ and $X^{\beta}{}_{;\alpha} \equiv \nabla_\alpha X^\beta$. Parentheses around indices shall denote symmetrization on those indices, i.e.,

$$A_{\cdots(\alpha_1 \cdots \alpha_p)\cdots} \equiv \frac{1}{p!} \sum_{\sigma \in S_p} A_{\cdots(\alpha_{\sigma(1)} \cdots \alpha_{\sigma(p)})\cdots}, \tag{4}$$

for permutations σ. Square brackets around indices shall denote antisymmetrization on those indices, i.e.,

$$A_{\cdots[\alpha_1\cdots\alpha_p]\cdots} \equiv \frac{1}{p!}\sum_{\sigma\in S_p}\mathrm{sgn}(\sigma)A_{\cdots[\alpha_{\sigma(1)}\cdots\alpha_{\sigma(p)}]\cdots}, \tag{5}$$

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ.

The Levi-Civita symbol is defined as

$$\epsilon_{\alpha_1\alpha_2\cdots\alpha_n} = \begin{cases} +1 & \text{if } \alpha_1\alpha_2\cdots\alpha_n \text{ is an even permutation of the index range} \\ -1 & \text{if } \alpha_1\alpha_2\cdots\alpha_n \text{ is an odd permutation of the index range} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and the Levi-Civita pseudotensor as

$$\eta_{\mu\nu\sigma\rho} = \sqrt{-\det(g_{\mu\nu})}\,\epsilon_{\mu\nu\sigma\rho}. \tag{7}$$

Also, we will define the following tensor derived from the metric,

$$g_{\mu\nu\sigma\rho} \equiv g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}. \tag{8}$$

CHAPTER 1

Introduction

1.1 Inhomogeneous cosmology

The assumption that the universe is homogeneous and isotropic was certainly very accurate at the time of last scattering, evident from the near perfectly smooth cosmic microwave background (CMB). During the intervening aeons formation of large scale structures has led to a universe that is no longer near to homogeneous, but rather dominated by voids with galaxy clusters in filaments and walls threading and surrounding these voids. This is seen in sky surveys such as the Sloan Digital Sky Survey (SDSS) [2], 2dF Galaxy Redshift Survey and others. These surveys show that voids with a characteristic mean effective radii of order $(15 \pm 3)h^{-1}$ Mpc[1] and a typical density contrast of $\delta\rho/\rho = -0.94 \pm 0.02$, where ρ is the average density of the observed volume, compose 40% of the volume of the nearby universe [3, 4]. A study [5] of the Sloan Digital Data Release 7 [6] found the median effective radius of voids in the survey volume of $17h^{-1}$ Mpc and 62% of the volume is occupied by voids with mean effective radii between $10h^{-1}$ Mpc and $30h^{-1}$ Mpc. Along with voids of this size there an abundant amount of smaller voids occupying the universe [7] meaning overall

[1]Due to the ellipticity some voids exhibit, the mean effective radius is defined as the radius of a sphere occupying the same volume.

the current universe is dominated by voids.

The non-linear nature of the Einstein equations makes trying to solve them for the full inhomogeneous geometry of the universe extremely difficult to do analytically with today's current mathematical knowledge, or even numerically with today's computing power. The difficulties in numerical relativity go beyond the simple limits imposed by hardware limitations. To solve Einstein's equations requires a splitting of spacetime into space and time in order to construct evolution equations. Such a splitting involves intrinsic ambiguities. Further problems arise when structures form and geodesics cross. Any numerical scheme has to deal with smoothing over singularities. In cosmology, in view of the complex hierarchy of observed structure, we have the additional problems of coarse-graining over these structures to define average symmetries of the global spacetime background.

The simplifying assumptions most cosmologists make is that, firstly, the universe is on average homogeneous and isotropic *and*, secondly, on average it evolves like an exactly homogeneous isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) model. The first assumption seems to be valid on scales of over $100h^{-1}$ Mpc, although there is some debate as what exactly this scale is [8, 9]. The second assumption concerning average evolution, however, has no direct physical justification. A further third assumption that is also often made is that our own measurements yield parameters which exactly coincide with those that describe the average cosmic evolution (which means those of a FLRW model if we also make the second assumption).

The general problem of relating our own measurements, which are related to invariants of our local metric, to some global average cosmological metric which describes the propagation of light on the largest scales, is known as the 'fitting problem' [10, 11]. This problem is a difficult fundamental problem which has not been solved, and which is simply ignored in the standard cosmology. Ideally we should match the local Schwarzschild type geometry of the solar system to the geometry of our Milk Way galaxy, then match the geometry of the Milky Way to that of the local group, and so on until we have matched geometries up to the scale which describes the average cosmic evolution and propagation of light. There are several possible relevant steps of coarse-graining in this hierarchical process, with qualitatively new physical questions entering when we make a transition from dealing with bound systems to regions of expanding cosmic fluid [12].

If we ignore the fitting problem and make the standard assumptions concerning homogeneity and isotropy then current cosmological observations indicate that the expansion rate of the universe appears to have begun accelerating in the relatively recent past, at redshifts $z < 1$. Given the intrinsically attractive nature of gravity, an acceleration of cosmic expansion is not possible if the universe contains only sources of mass–energy which obey the strong energy or "timelike convergence" condition, which for perfect fluids is characterized by an equation of state for which $p > -\frac{1}{3}\rho$. The strong energy condition must be satisfied in order for matter to focus light rays, and it is satisfied for all forms of matter which have been directly observed.

A form of matter which violates the strong energy condition is therefore required, and observationally the equation of state of such a fluid which best fits cosmological observations is found to be extremely close to the lowest possible bound, $p = -\rho$, allowed by the dominant energy condition[2]. The $p = -\rho$ bound is realised by a cosmological constant, which represents a pure vacuum energy. Generically any form of matter which violates the strong energy condition will not clump as a result of gravitational collapse, and is therefore called *dark energy*. Dark energy is, as of yet, not directly observed and its only manifestation is to change the expansion history of the universe to allow for cosmic acceleration.

Since the determination of the average expansion history of the universe is intimately related to the assumptions of homogeneity and isotropy in the

[2]Physically, the dominant energy condition, $|p| \leq \rho$, may be understood as saying that all physical energy fluxes are bounded by the speed of light, a condition which if violated would lead to fundamental physics drastically different from anything we know.

standard ΛCDM model, it is possible that the expansion history has been misinterpreted as a result of incorrect assumptions in a complex geometrical problem. This thesis will investigate what happens if we do not make such assumptions.

To perform such a task we need to first decide what is meant by an average. The concept of an average in general relativity is not a trivial one. We will present and analyse two methods that define the concept of an average, those of Korzyński [13] and of Buchert [14, 15]. These show that the Einstein equations of the average geometry and the average of the Einstein equations of the full geometry are not the same; the difference leads to a *backreaction* term. We will then generalise Wiltshire's timescape model [16] to include radiation. The timescape model drops the second and third assumption above and, after a particular physical interpretation of our own measurements relative to both cosmic averages and cosmic variance, fits observed data well without dark energy.

1.2 Thesis outline

Before beginning our discussions on averaging in general relativity, we will look at the simpler case of averaging in Newtonian cosmology first. This is presented in Chapter 2 where we start by reviewing the existing formulation of Newtonian gravity and proceed to averaging while deriving some new results along the way.

In Chapter 3 we review the kinematical description of spacetime and compare that with Newtonian gravity. We then review the 3+1 split of spacetime and some useful coordinate systems. This is followed by a review of hypersurfaces and then the Arnowitt-Deser-Misner (ADM) gauge [17, 18, 19].

In Chapter 4 we start by stating Korzyński's motivation for his procedure and setting out a list of properties that a coarse-graining procedure should satisfy. We then present Korzyński's coarse-graining procedure for the velocity gradient and apply it to the Bianchi I universe. Following Korzyński, we then develop the evolution equation for the coarse-grained velocity gradient and then apply the procedure to the Lemaître-Tolman-Bondi model.

In Chapter 5 we begin by reviewing the Buchert averaging formalism for dust. We then describe the timescape model and apply the Buchert averaging formalism and then describe how observables are related to variables of the timescape equations. Following this we proceed by adding homogeneous radiation to the model and analysing the results. Next we attempt to solve the model beyond the surface of last scattering and discuss the problems of doing so. This is followed by a discussion on the initial conditions and the merits of combining the timescape model without radiation with a homogeneous model with radiation at an earlier time.

In the summary, we analyse the work done and state what work is left to be done.

CHAPTER 2

Newtonian cosmology

2.1 Introduction

The approach we will adopt in this chapter predominantly follows the investigations of Buchert and Ehlers in 1997 [14] and by Korzyński in 2010 [13], supplemented by that of Zalaletdinov who wrote a rigorous series of papers on the subject in 2002 [20, 21, 22]. The work of Buchert and Ehlers was the first published work on the subject of averaging Newtonian cosmology and led to Buchert's extension of the scheme to general relativity [15, 23]. Whereas Buchert and Ehlers looked at differences relative to a homogeneous *and* isotropic cosmology, Korzyński generalised it to looking at the differences relative to just a homogeneous cosmology. This is preparation for Korzyński's method of coarse-graining in general relativity, which is the main subject of the same paper [13].

2.2 Governing equations

Consider a pressureless fluid, henceforth referred to by the term dust, in Euclidean space $\mathbf{E}^3$, interacting under the influence of Newtonian gravity. This system is described in Cartesian coordinates, x^a, by the local density function $\rho\left(x^a, t\right)$, velocity field $v^a\left(x^b, t\right)$ and the Newtonian potential

$\phi(x^a, t)$. The evolution of this system is governed by the following system of PDEs known as the *Euler-Poisson equations*,

$$\frac{\partial v^a}{\partial t} + v^b \frac{\partial v^a}{\partial x^b} = -\delta^{ab} \frac{\partial \phi}{\partial x^b} \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + v^b \frac{\partial \rho}{\partial x^b} = -\rho \frac{\partial v^a}{\partial x^a} \tag{2.2}$$

$$\delta^{ab} \frac{\partial^2 \phi}{\partial x^a \partial x^b} = 4\pi G \rho. \tag{2.3}$$

Following Buchert and Ehlers [14], we can define the gravitational acceleration by

$$g^a = -\delta^{ab} \frac{\partial \phi}{\partial x^b} \tag{2.4}$$

and rewrite equations (2.1)-(2.3) in terms of $\mathbf{g}$. Rewriting the LHS of (2.3) as the negative divergence of the gravitational acceleration, $-\nabla \cdot \mathbf{g}$, with the addition of a fourth equation requiring the gravitational acceleration be a conservative field, $\nabla \times \mathbf{g} = 0$, yields the desired result. We will, however, leave the equations in terms of the gravitational potential.

The position of a dust particle can be given in Eulerian coordinates by $x^a = f^a\left(X^b, t\right)$, where X^b denotes the Lagrangian coordinate of the dust particle which is constant with respect to any given dust particle. We define the total time derivative, $\frac{\mathrm{d}}{\mathrm{d}t}$, as the time derivative with respect to a dust particle, i.e., at fixed X^a, $\frac{\mathrm{d}}{\mathrm{d}t} \equiv (\dots)\dot{} = \frac{\partial}{\partial t}$ in Lagrangian coordinates. The velocity field is then $v^a = \frac{\mathrm{d}x^a}{\mathrm{d}t} = \frac{\partial}{\partial t} f^a$ and the total time derivative in Eulerian coordinates is $\frac{\mathrm{d}}{\mathrm{d}t} \equiv (\dots)\dot{} = \frac{\partial}{\partial t} + v^b \frac{\partial}{\partial x^b}$. The left-hand side of equations (2.1) and (2.2) can then be realised to be $\dot{v^a}$ and $\dot{\rho}$ respectively.

Equations (2.1)-(2.3) are invariant under the kinematical group of transformations,

$$x^a \to x^{a\prime} = A^a{}_b x^b + D^a(t) \quad t \to t' = t + b \tag{2.5}$$

where $A^a{}_b$ is a constant real-valued orthogonal matrix and $D^a(t)$ is an arbitrary function of time. Under this change of coordinates, the gravitational potential, $\phi(x^a, t)$, undergoes the following transformation,

$$\phi \to \phi' = \phi - \delta_{ab} \frac{\mathrm{d}^2 D^a(t)}{\mathrm{d}t^2} x^{b\prime}. \tag{2.6}$$

Zalaletinov [21] gives this group of transformations but erroneously scales t by a constant factor a and does not require ${A^a}_b$ to be orthogonal. The consequence of the invariance under this group of transformations is that inertial observers cannot be defined as the lack of invariance of the inertial and gravitational acceleration, $\frac{\mathrm{d}v^a}{\mathrm{d}t}$ and g^a respectively, means that it is impossible to distinguish between the two. We cannot say if the inertial acceleration, $\frac{\mathrm{d}v^a}{\mathrm{d}t}$, is zero which is what defines an inertial observer.

The reason for the undefined inertial frames is a result of the ill-posedness of equations (2.1)-(2.3) when only supplemented with initial conditions and *not* boundary conditions. Poisson's equation (2.3) does not have a unique solution unless we can place boundary conditions on ϕ, which in studying the infinite Newtonian cosmology we generally cannot. It is only when boundary conditions are placed on ϕ that it becomes uniquely defined and $\frac{\mathrm{d}^2 D^a}{\mathrm{d}t^2}$ must be zero. This results in the kinematical group of transformations reducing to the Galilean group of transformations,

$$x^a \to x^{a\prime} = {A^a}_b x^b + B^a t + C^a \quad t \to t' = t + b, \tag{2.7}$$

where ${A^a}_b$ is a constant real-valued orthogonal matrix and B^a and C^a are constants also. An example of such a case [21] is when we have an isolated fluid, and we demand the global boundary condition of vanishing potential at infinity,

$$\phi(x^a, t) \to 0 \quad \text{as} \quad (x^a x_a)^{\frac{1}{2}} \to \infty. \tag{2.8}$$

The ill-posedness is more evident with a more useful form of (2.1), obtained by taking a spatial partial derivative, so that (2.1)-(2.3) become

$$({v^a}_{,b})^{\cdot} = -{v^a}_{,c}{v^c}_{,b} - {\phi^{,a}}_{,b} \tag{2.9}$$

$$\dot{\rho} = -\rho {v^a}_{,a} \tag{2.10}$$

$${\phi^{,a}}_{,a} = 4\pi G\rho. \tag{2.11}$$

We can decompose the velocity gradient, as Buchert and Ehlers do, into its trace or expansion scalar, θ, traceless symmetric part or shear tensor,

σ_{ab}, and the antisymmetric part or vorticity tensor, ω_{ab},

$$v_{a,b} = \tfrac{1}{3}\theta\delta_{ab} + \sigma_{ab} + \omega_{ab}, \tag{2.12}$$

where

$$\theta = v^a{}_{,a} = \nabla \cdot \mathbf{v}, \tag{2.13}$$

$$\sigma_{ab} = v_{(a,b)} - \tfrac{1}{3}\theta\delta_{ab}, \tag{2.14}$$

and

$$\omega_{ab} = v_{[a,b]} = \delta_{[a}{}^c\delta_{b]}{}^d v_{c,d} = \tfrac{1}{2}\epsilon_{eab}\epsilon^{ecd}v_{c,d} = -\tfrac{1}{2}\epsilon_{eab}\zeta^e, \tag{2.15}$$

letting $\boldsymbol{\zeta} = \nabla \times \mathbf{v}$ be the curl of the velocity field. We can also perform a similar decomposition on $\phi_{,ab}$,

$$\Theta = \phi^{,a}{}_{,a} = \Delta\phi \tag{2.16}$$

$$E_{ab} = \phi_{,ab} - \tfrac{1}{3}\Theta\delta_{ab}, \tag{2.17}$$

where E_{ab} is referred to as the *tidal tensor*.

With the use of (2.13)-(2.17), equations (2.9)-(2.11) give, through some derivation, the transport equations,

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - \sigma^2 + \omega^2 - \Theta \tag{2.18}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\theta\sigma_{ab} - \sigma_{ac}\sigma^c{}_b - \omega_{ac}\omega^c{}_b + \tfrac{1}{3}\delta_{ab}\left(\sigma^2 - \omega^2\right) - E_{ab} \tag{2.19}$$

$$\dot{\omega}_{ab} = -\tfrac{2}{3}\theta\omega_{ab} - \sigma_{ac}\omega^c{}_b - \omega_{ac}\sigma^c{}_b \quad \text{or} \quad \dot{\boldsymbol{\zeta}} = -\tfrac{2}{3}\theta\boldsymbol{\zeta} + \bar{\sigma}\boldsymbol{\zeta} \tag{2.20}$$

$$\dot{\rho} = \rho\theta \tag{2.21}$$

$$\Theta = 4\pi G\rho, \tag{2.22}$$

where the scalar shear and vorticity are, $\sigma^2 = \sigma_{ab}\sigma^{ab}$ and $\omega^2 = \omega_{ab}\omega^{ab}$ respectively, and $\bar{\sigma}$ denotes the matrix composed of the elements σ_{ab}. We see that we have a system of ODEs governing the evolution of all of the above variables except the tidal tensor, E_{ab}. This can only be determined through boundary conditions placed on ϕ, as opposed to the trace, which is determined by the local matter density.

We note the following integrability conditions on account of θ, σ_{ab}, ω_{ab} and E_{ab} being derivatives,

$$\tfrac{1}{3}\delta_{a[b}\theta_{,c]} + \sigma_{a[b,c]} + \omega_{a[b,c]} = 0, \tag{2.23}$$

$$E_a{}^b{}_{,b} = \frac{8\pi G}{3}\rho_{,a} \tag{2.24}$$

and

$$E_{a[b,c]} = -\frac{4\pi G}{3}\delta_{a[b}\rho_{,c]}. \tag{2.25}$$

We can place boundary conditions on E_{ab} by specifying E_{ab} on the boundary, ∂G_t, of some domain G_t. These boundary conditions are not completely arbitrarily specifiable, however, one must ensure they satisfy the integrability conditions (2.24) and (2.25) on ∂G_t, as well as obviously being traceless and symmetric. Then one can solve for E_{ab} over G_t by using equations (2.24) and (2.25). This is performed using Helmholtz's theorem, treating E_{ab} as three separate vector fields labelled by a, $\mathbf{E}_a$. Equations (2.24) and (2.25) are then effectively the divergence and curl of $\mathbf{E}_a$ respectively. Helmholtz's theorem combined with

$$\int_{G_t} \frac{\partial A}{\partial x^a}\,\mathrm{d}^3x = \int_{\partial G_t} A\, n_a\,\mathrm{d}\sigma, \tag{2.26}$$

which is a form of Stokes' theorem, then leads to

$$E_{ab}(x^e,t) = B_{a,b}(x^e,t) + \epsilon_{bcd}A_a{}^{c,d}(x^e,t), \tag{2.27}$$

where

$$B_a(x^e,t) = \frac{1}{4\pi}\int_{\partial G_t} \frac{E_a{}^g(x'^f,t) - \frac{8\pi G}{3}\rho(x'^f,t)\,\delta_a{}^g}{\sqrt{(x^e - x'^e)(x_e - x'_e)}}\, n_g(x'^f,t)\,\mathrm{d}\sigma' \tag{2.28}$$

and

$$A_a{}^c(x^e,t) = \frac{1}{4\pi}\int_{\partial G_t} \epsilon^{hcg}\,\frac{E_{ah}(x'^f,t) + \frac{4\pi G}{3}\rho(x'^f,t)\,\delta_{ah}}{\sqrt{(x^e - x'^e)(x_e - x'_e)}}\, n_g(x'^f,t)\,\mathrm{d}\sigma'. \tag{2.29}$$

Here n_a is the outward pointing unit normal on ∂G_t. We can combine equations (2.27), (2.29) and (2.28) to become one surface integral by performing the derivatives with respect to the unprimed coordinates inside the integrals. However, the form does not become any more transparent, so we will not do so here.

The system will then have a unique solution up to the kinematical group of transformations (2.5) on the G_t. An example of such a case is the general Heckmann-Schücking boundary condition [24, 25],

$$E_{ab}(x^a, t) \to \mathring{E}_{ab}(t) \quad \text{as} \quad (x^a x_a)^{\frac{1}{2}} \to \infty, \tag{2.30}$$

where $\mathring{E}_{ab}(t)$ is some arbitrary function of t. By virtue of the integrability conditions, (2.24) and (2.25), this boundary condition is only valid if

$$\rho(x^a, t) \to \mathring{\rho}(t) \quad \text{as} \quad (x^a x_a)^{\frac{1}{2}} \to \infty, \tag{2.31}$$

where $\mathring{\rho}(t)$ is some function of t.

Alternatively, one could give an evolution equation for E_{ab} which must propagate the integrability conditions. This would then indirectly give boundary conditions for E_{ab}. An example of such an equation is given by Bertschinger and Hamilton [26], the local tidal approximation,

$$\dot{E}_{ab} = -\theta E_{ab} - \delta_{ab}\sigma^{cd}E_{cd} + 3\sigma^c{}_{(a}E_{b)c} + \omega^c{}_{(a}E_{b)c} - \Theta\sigma_{ab}. \tag{2.32}$$

At this stage, it is unclear to me whether (2.32) preserves the integrability conditions.

Solving the system

To solve the system we will assume that an evolution equation for E_{ab} has been specified, otherwise if boundary conditions are given explicitly equation (2.27) will couple all the equations making the following solution not just involve ODEs. Begin by solving the ODEs (2.18)-(2.22) and an evolution equation for E_{ab} from initial conditions

$$\begin{aligned}
&\theta(X^a, t_0) = \theta_0(X^a) \quad \omega_{ab}(X^c, t_0) = \omega_{ab0}(X^c) \quad \sigma_{ab}(X^c, t_0) = \sigma_{ab0}(X^c) \\
&\rho(X^a, t_0) = \rho_0(X^a) \quad E_{ab}(X^c, t_0) = E_{ab0}(X^c),
\end{aligned} \tag{2.33}$$

to give

$$v_{a,b}(X^c, t) \tag{2.34}$$

by equation (2.12). Compared to solving (2.1)-(2.3), where one can give a completely arbitrary initial velocity profile, $v^a(x^b, t_0)$, and density profile, $\rho(x^a, t_0)$, one must make sure (2.33) satisfy the integrability conditions (2.23)-(2.25). If one derives (2.33) from an arbitrary initial velocity profile and density profile, the integrability conditions are, of course, trivially satisfied. Because (2.33) are given in Lagrangian coordinates, one would need to use the inverse of the Eulerian-Lagrangian transformation (f_0^a below) to check this, alternatively one could give (2.33) in Eulerian coordinates and change to Lagrangian coordinates to do the solving. One can then show that

$$\frac{\partial}{\partial t}\left(\frac{\partial f^a}{\partial X^b}\right) = v^a{}_{,c}\frac{\partial f^c}{\partial X^b}, \tag{2.35}$$

which is a system of ODEs that one can solve for $\frac{\partial f^a}{\partial X^b}(X^c, t)$ given the initial condition $\frac{\partial f^a}{\partial X^b}(X^c, t_0) = \frac{\partial f^a}{\partial X^b}{}_0(X^c)$. That initial condition is a derivative of the initial Lagrangian coordinates, $f^a(X^b, t_0) = f_0^a(X^b)$. Usually one would let $f_0^a(X^b) = X^a$ so that the Lagrangian coordinates coincide with the Eulerian coordinates at t_0. Once $\frac{\partial f^a}{\partial X^b}(t, X^c)$ is obtained, one may solve the ODEs for $f^a(X^b, t)$ given an initial condition

$$f^a(\mathring{X}^b(t), t) = \mathring{f}^a(t), \tag{2.36}$$

where $\mathring{X}^b(t)$ and $\mathring{f}(t)$ are some arbitrary functions of t. Usually one will take $\mathring{X}^b(t) = \mathring{f}^a(t) = 0$ so that a particle at the origin stays at the origin in Eulerian coordinates. It is this freedom that gives rise to the kinematical group of transformations.

The homogeneous case

Korzyński [13] goes on to describe the homogeneous solutions to equations (2.1), (2.2) and (2.3). To create a homogeneous solution, we first set the density constant in space,

$$\rho(x^a, t) = d(t). \tag{2.37}$$

We also expect that the relative velocity of any two dust particles to be the same as any other two dust particles displaced by the same amount, which means v^a is linear in x^a. Setting $v^a(0,t) = 0$, to keep the particle at the origin at the origin, we have

$$v^a(x^b, t) = Q^a{}_b(t)x^b. \tag{2.38}$$

Using equation (2.37) in (2.3) and solving we obtain

$$\phi(x^a, t) = \tfrac{1}{2}\Phi_{ab}(t)x^a x^b + u_a(t)x^a + c(t). \tag{2.39}$$

By substituting (2.38) and (2.39) into equation (2.1) and evaluating at the origin we find $u_a(t) = 0$. So arbitrarily setting $c(t) = 0$ we have,

$$\phi(x^a, t) = \tfrac{1}{2}\Phi_{ab}(t)x^a x^b. \tag{2.40}$$

Note the antisymmetric part of Φ_{ab} does not play any part in the solution so is set to zero. The traceless part of Φ_{ab}, which equates to the tidal tensor, can be specified arbitrarily as a function of time.

Applying the equations of motion to these solutions, we obtain the following non-linear system of ODEs,

$$\dot{Q}^a{}_b = -Q^a{}_c Q^c{}_b - \Phi^a{}_b \tag{2.41}$$

$$\dot{d} = -dQ^a{}_a \tag{2.42}$$

$$\Phi^a{}_a = 4\pi G d. \tag{2.43}$$

We may create a homogeneous and isotropic solution by setting $Q_{ab} = H(t)\delta_{ab}$ or, alternatively, setting $\theta = 3H(t)$, $\sigma_{ab} = 0$ and $\omega_{ab} = 0$ as well as the tidal tensor vanishing, i.e., $\Phi_{ab} = \frac{4\pi G}{3} d\delta_{ab}$. This then gives the special case of the Friedmann-Lemaître-Robertson-Walker (FLRW) solutions, equations; (2.18)-(2.22) and (2.41)-(2.43) then yield,

$$3\dot{H} = -3H^2 - 4\pi G d \tag{2.44}$$

$$\dot{d} = -3Hd. \tag{2.45}$$

Defining the scale factor, a, by $H = \frac{\dot{a}}{a}$, equation (2.44) gives the second Friedmann equation (the acceleration equation) for a dust cosmology,

$$3\frac{\ddot{a}}{a} = -4\pi G d, \tag{2.46}$$

and equation (2.45) gives

$$\dot{d} = -3\frac{\dot{a}}{a}d. \tag{2.47}$$

Equation (2.47) can be solved and substituted into (2.46) to obtain

$$3\frac{\ddot{a}}{a} = -4\pi G\frac{M}{a^3}, \tag{2.48}$$

where $M = da^3$ is a constant, which represents the conserved mass inside the volume a^3.

2.3 Averaging of the Newtonian cosmology

We would like to construct an average or coarse-grained value of quantities on spatial surfaces. First, let us define a spatial domain, $G_t \in \mathbf{E}^3$, whose boundary is dragged by the dust particles. The volume average of a quantity, A, on that domain, is then defined by

$$\langle A\rangle_{G_t} = \frac{1}{V_{G_t}}\int_{G_t} A\,\mathrm{d}^3x, \tag{2.49}$$

where V_{G_t} is the volume of the domain. The quantity A may be a scalar or components of a Cartesian tensor. Now we would like to know how these averaged quantities evolve, but for that we need to know how the *evolution of an average quantity relates to the average evolution of the quantity.*

One approach is via the Lagrangian description of Buchert and Ehlers. The volume element in Lagrangian coordinates is related to the volume element in Eulerian coordinates by $\mathrm{d}^3x = J\,\mathrm{d}^3X$ where $J(X^a, t) = \left|\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\right|$ is the Jacobian determinant of $f^a\left(X^b, t\right)$. The derivative of the Jacobian determinant can be computed via Jacobi's formula,

$$\begin{aligned}\dot{J} &= \frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\right| \\ &= \left|\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\right| \operatorname{tr}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{X}}^{-1}\frac{\partial \dot{\mathbf{f}}}{\partial \mathbf{X}}\right) = J \operatorname{tr}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}^{-1}\frac{\partial \mathbf{v}}{\partial \mathbf{X}}\right) = J \operatorname{tr}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right) \\ &= J\nabla\cdot\mathbf{v} = J\theta. \end{aligned} \tag{2.50}$$

The derivative of volume of the comoving domain can then be shown to be

$$\dot{V}_{G_t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{G(t)} \mathrm{d}^3x = \int_G \dot{J}\,\mathrm{d}^3X = \int_{G(t)} \theta\,\mathrm{d}^3x, \tag{2.51}$$

or

$$\langle\theta\rangle_{G_t} = \frac{\dot{V}_{G_t}}{V_{G_t}}. \tag{2.52}$$

We may then derive the important commutation rule [14],

$$\langle A\rangle^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{V_{G_t}}\int_{G_t} A\,\mathrm{d}^3x\right) = -\frac{\dot{V}_{G_t}}{V_{G_t}}\langle A\rangle + \frac{1}{V_{G_t}}\int_{G_t}\left(\dot{A}J + A\dot{J}\right)\mathrm{d}^3X \tag{2.53}$$

or

$$\langle A\rangle^{\cdot} - \langle\dot{A}\rangle = \langle A\theta\rangle - \langle A\rangle\langle\theta\rangle. \tag{2.54}$$

Note the subscripts denoting the averaging region have been left off for simplicity and we will continue to do so.

Alternatively, one may start with Reynolds' transport theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{G_t} A\,\mathrm{d}^3x = \int_{G_t}\frac{\partial A}{\partial t}\mathrm{d}^3x + \int_{\partial G_t} A\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\sigma, \tag{2.55}$$

where ∂G_t denotes the boundary of the domain G_t and $\mathbf{n}$ is the outward pointing surface normal on ∂G_t. Although, Reynolds transport theorem usually requires a Lagrangian description to be proved.

It is then a fairly simple task to apply the commutation rule to equations (2.21), (2.18) and (2.20), giving

$$\langle\theta\rangle^{\cdot} = \tfrac{2}{3}\langle\theta^2\rangle - \langle\theta\rangle^2 - \langle\sigma^2\rangle + \langle\omega^2\rangle - 4\pi G\langle\rho\rangle \tag{2.56}$$

$$\langle\sigma_{ab}\rangle^{\cdot} = \tfrac{1}{3}\langle\theta\sigma_{ab}\rangle - \langle\theta\rangle\langle\sigma_{ab}\rangle - \langle\sigma_{ac}\sigma^c{}_b\rangle - \langle\omega_{ac}\omega^c{}_b\rangle + \tfrac{1}{3}\delta_{ab}\left(\langle\sigma^2\rangle - \langle\omega^2\rangle\right) - \langle E_{ab}\rangle \tag{2.57}$$

$$\langle\omega_{ab}\rangle^{\cdot} = \tfrac{1}{3}\langle\theta\omega_{ab}\rangle - \langle\theta\rangle\langle\omega_{ab}\rangle - \langle\sigma_{ac}\omega^{c}{}_{b}\rangle - \langle\omega_{ac}\sigma^{c}{}_{b}\rangle \text{ or } \langle\boldsymbol{\zeta}\rangle^{\cdot} = \tfrac{1}{3}\langle\boldsymbol{\zeta}\theta\rangle - \langle\boldsymbol{\zeta}\rangle\langle\theta\rangle + \langle\bar{\sigma}\boldsymbol{\zeta}\rangle \tag{2.58}$$

$$\langle\rho\rangle^{\cdot} = \langle\rho\rangle\langle\theta\rangle, \tag{2.59}$$

where we have substituted (2.22) into (2.18). Buchert and Ehlers derive equations (2.56) and (2.59) along with a slightly different version of the right side of (2.58). We may then average E_{ab} in terms of its boundary conditions by applying (2.49) to (2.27) and using (2.26) we find that it takes the form of a double surface integral over ∂G_t,

$$\begin{aligned}\langle E_{ab}\rangle(t) = \frac{1}{4\pi V}\int_{\partial G_t}\int_{\partial G_t}\Bigg[&\delta_b{}^d\left(E_a{}^g(x'^e,t) - \frac{8\pi G}{3}\rho(x'^e,t)\,\delta_a{}^g\right)\\ &+ 2\delta_{[b}{}^h\delta_{d]}{}^g\left(E_{ah}(x'^e,t) + \frac{4\pi G}{3}\rho(x'^e,t)\,\delta_{ah}\right)\Bigg]\\ &\times\frac{n^d(x^e,t)\,n_g(x'^e,t)\,\mathrm{d}\sigma'\,\mathrm{d}\sigma}{\sqrt{(x^f - x'^f)(x_f - x'_f)}}.\end{aligned} \tag{2.60}$$

One can show that if the boundary conditions are changed by $E_{ab}(x^e,t)|_{\partial G_t} \to E_{ab}(x^e,t)|_{\partial G_t} + A_{ab}(t)$, which will still satisfy the integrability conditions, then

$$\langle E_{ab}\rangle(t) \to \langle E_{ab}\rangle(t) + A_{ab}(t)\,\frac{1}{4\pi V}\int_{\partial G_t}\int_{\partial G_t}\frac{n^d(x^e,t)\,n_d(x'^e,t)\,\mathrm{d}\sigma'\,\mathrm{d}\sigma}{\sqrt{(x^f - x'^f)(x_f - x'_f)}}. \tag{2.61}$$

Thus, it is always possible to choose boundary conditions so that $\langle E_{ab}\rangle$ takes a specific value at any time.

Korzyński assigns coarse-grained quantities on G by the following averages,

$$\bar{Q}^a{}_b = \langle v^a{}_{,b}\rangle \tag{2.62}$$

$$\bar{\Phi}_{ab} = \langle\phi_{,ab}\rangle \tag{2.63}$$

$$\bar{d} = \langle\rho\rangle. \tag{2.64}$$

As a consequence of (2.26), the averages (2.62) and (2.63) are effectively surface integrals,

$$\bar{Q}^a{}_b = \frac{1}{V_{G_t}}\int_{\partial G_t} v^a n_b\,\mathrm{d}\sigma \tag{2.65}$$

$$\bar{\Phi}_{ab} = \frac{1}{V_{G_t}} \int_{\partial G_t} \phi_{,a} n_b \, \mathrm{d}\sigma, \tag{2.66}$$

which means they only depend on v^a and $\phi_{,a}$ respectively at the boundary of the coarse-graining domain. Now we can decompose the velocity, density and potential into their coarse-grained part and their deviations from such by

$$v^a = \bar{Q}^a{}_b x^b + \delta v^a \tag{2.67}$$

$$\phi = \tfrac{1}{2}\bar{\Phi}_{ab} x^a x^b + \delta\phi \tag{2.68}$$

$$\rho = \bar{d} + \delta\rho. \tag{2.69}$$

Substituting these definitions into the integrals (2.62)-(2.63), it follows that

$$\langle \delta v^a{}_{,b} \rangle = 0 \tag{2.70}$$

$$\langle \delta\phi_{,ab} \rangle = 0 \tag{2.71}$$

$$\langle \delta\rho \rangle = 0. \tag{2.72}$$

This leads us now to calculate the evolution equations for these coarse-grained quantities. Substituting the above into equations (2.1), (2.2) and (2.3), averaging and applying the commutation relation, (2.54), gives

$$\dot{\bar{Q}}^a{}_b = -\bar{Q}^a{}_c \bar{Q}^c{}_b - \bar{\Phi}^a{}_b + B^a{}_b \tag{2.73}$$

$$\dot{\bar{d}} = -\bar{d}\bar{Q}^a{}_a \tag{2.74}$$

$$\bar{\Phi}^a{}_a = 4\pi G \bar{d}, \tag{2.75}$$

where $B^a{}_b = \langle \delta v^a{}_{,b} \delta v^c{}_{,c} - \delta v^a{}_{,c} \delta v^c{}_{,b} \rangle$. We see that these are the same evolution equations as the homogeneous case, (2.41), (2.42) and (2.43), with the addition of a back-reaction term, $B^a{}_b$, that describes the influence of the inhomogeneities on the system. One can show that $B^a{}_b$ only depends on δv^a and its derivatives at the boundary by way of a surface integral,

$$\begin{aligned} B^a{}_b &= \langle \delta v^a{}_{,b} \delta v^c{}_{,c} - \delta v^a{}_{,c} \delta v^c{}_{,b} \rangle, \qquad (2.76) \\ &= \langle \delta v^a{}_{,b} \delta v^c{}_{,c} + \delta v^a{}_{,bc} \delta v^c - \delta v^a{}_{,bc} \delta v^c - \delta v^a{}_{,c} \delta v^c{}_{,b} \rangle, \\ &= \langle \left(\delta v^a{}_{,b} \delta v^c\right)_{,c} - \left(\delta v^a{}_{,c} \delta v^c\right)_{,b} \rangle \\ &= \frac{1}{V} \int_{\partial G_t} \delta v^a{}_{,b} \delta v^c n_c - \delta v^a{}_{,c} \delta v^c n_b \, \mathrm{d}\sigma. \qquad (2.77) \end{aligned}$$

Buchert and Ehlers derived a similar equation to (2.73) and (2.77) but they were looking at inhomogeneity relative to a homogeneous *and* isotropic cosmology, which is obtained by setting

$$3\bar{H}(t) = \langle v^a{}_{,a} \rangle = \frac{\dot{V}_{G_t}}{V_{G_t}} \tag{2.78}$$

and then

$$v^a = 3\bar{H}x^a + \delta v^a, \tag{2.79}$$

which gives

$$\langle \delta v^a{}_{,a} \rangle = 0. \tag{2.80}$$

Then in the same fashion to (2.73), we obtain [14]

$$3\dot{\bar{H}} = -3\bar{H}^2 - 4\pi G\bar{d} + \frac{1}{V_{G_t}} \int_{\partial G_t} (\delta\mathbf{v}\nabla \cdot \delta\mathbf{v} - (\delta\mathbf{v} \cdot \nabla)\,\delta\mathbf{v}) \cdot \mathbf{n}\,\mathrm{d}\sigma, \tag{2.81}$$

which is (2.56) in a different form. Also equation (2.74) now takes the form

$$\dot{\bar{d}} = -3\bar{H}\bar{d}. \tag{2.82}$$

These are the same equations as the homogeneous and isotropic case, (2.44) and (2.45), with the addition of a backreaction term. Similarly, we can define a scale factor by $\bar{a} = V_{G_t}{}^{1/3}$ and we find $\bar{H} = \frac{\dot{\bar{a}}}{\bar{a}}$ and [14]

$$3\frac{\ddot{\bar{a}}}{\bar{a}} = -4\pi G\frac{\bar{M}}{\bar{a}^3} + \frac{1}{V_{G_t}} \int_{\partial G_t} (\delta\mathbf{v}\nabla \cdot \delta\mathbf{v} - (\delta\mathbf{v} \cdot \nabla)\,\delta\mathbf{v}) \cdot \mathbf{n}\,\mathrm{d}\sigma, \tag{2.83}$$

where $\bar{M} = \bar{d}\bar{a}^3$ is the mass in G_t, which is conserved. This equation is the same as (2.48), with the addition of the backreaction term.

Equation (2.77) allows us to put bounds on the back-reaction if we can place bounds on the velocity inhomogeneities and their derivatives. Consider a spherical averaging domain with radius R, the volume divides the backreaction by $O(R^3)$ but the surface integral multiplies it by $O(R^2)$. Thus, we can place the bound on the back-reaction

$$|B^a{}_b| < \frac{C}{R}, \tag{2.84}$$

where C is some finite positive constant. This means the backreaction will tend to zero if a large enough volume is considered

Equation (2.77) may appear to involve derivatives in all directions, however, it only involves derivatives tangential to the surface because of the antisymmetrization in b and c. A spatial derivative can be split into a normal and tangential part with respect to a surface normal, n^a,

$$u_{,a} = n^b u_{,b} n_a + u_{||a}, \tag{2.85}$$

where the double vertical slash denotes a derivative tangential to the surface defined by the above equation. Therefore, (2.77) becomes

$$B^a{}_b = \frac{1}{V}\int_{\partial G(t)} n^d \delta v^a{}_{,d} n_b \delta v^c n_c - n^d \delta v^a{}_{,d} n_c \delta v^c n_b + \delta v^a{}_{||b} \delta v^c n_c - \delta v^a{}_{||c} \delta v^c n_b \, \mathrm{d}\sigma, \tag{2.86}$$

$$= \frac{1}{V}\int_{\partial G(t)} \delta v^a{}_{||b} \delta v^c n_c - \delta v^a{}_{||c} \delta v^c n_b \, \mathrm{d}\sigma \tag{2.87}$$

Equation (2.87) allows us to see that if the velocity inhomogeneities only vary perpendicularly to the bounding surface, then the backreaction is zero. An example of such a case is when averaging over a spherical region centred on the origin for a model that is isotropic around the origin. Such a model is the Newtonian equivalent of the Lemaître-Tolman-Bondi model in general relativity discussed in §4.8. Buchert proves this explicitly in reference [27].

CHAPTER 3

Congruences and the splitting of spacetime

3.1 A kinematical description of spacetime

Consider a timelike congruence on a Lorentzian manifold generated from some timelike unit vector field u^μ, which is usually (but not necessarily) the four-velocity of some fluid. We shall let an overdot denote the covariant derivative along u^μ, which is given by $\frac{D}{\mathrm{d}\tau} \equiv u^\rho \nabla_\rho \equiv \nabla_{\mathbf{u}}$, where τ is proper time if u^μ is a four-velocity. The acceleration of the congruence is denoted by $a^\mu = \dot{u}^\mu$ and is zero if it is geodesic. We define the transverse projection tensor,

$$h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu, \tag{3.1}$$

This has all the same properties as the hypersurface projection tensor in §3.3, e.g., orthogonality and idempotentness. It projects tensors on to a hyperplane orthogonal to u^μ at that point, the result being the transverse parts of the original tensors. We let

$$Z_{\mu\nu} = u_{\mu;\nu}, \tag{3.2}$$

which is known as the velocity gradient when u^μ is a fluid velocity, and calculate the transverse part which can be shown to be

$$Z^{\perp}_{\mu\nu} = h^{\sigma}{}_{\mu} h^{\rho}{}_{\nu} Z_{\sigma\rho} = Z_{\mu\nu} + a_\mu u_\nu \ = \theta_{\mu\nu} + \omega_{\mu\nu}, \tag{3.3}$$

where

$$\theta_{\mu\nu} = h^{\sigma}{}_{\mu} h^{\rho}{}_{\nu} Z_{(\sigma\rho)} = Z^{\perp}_{(\mu\nu)} \tag{3.4}$$

and

$$\omega_{\mu\nu} = h^{\sigma}{}_{\mu} h^{\rho}{}_{\nu} Z_{[\sigma\rho]} = Z^{\perp}_{[\mu\nu]} \tag{3.5}$$

are the expansion tensor and vorticity tensor respectively. The expansion tensor is decomposed into a trace, the expansion scalar,

$$\theta = \theta^{\mu}{}_{\mu} \tag{3.6}$$

and the shear tensor which is defined to be the traceless part of (3.4),

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \tfrac{1}{3}\theta h_{\mu\nu}, \tag{3.7}$$

so that

$$\theta_{\mu\nu} = \tfrac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu}. \tag{3.8}$$

The projected tensors are effectively 3-dimensional objects living locally in a hyperplane orthogonal to u^μ, that is,

$$Z^{\perp}_{\mu\nu} u^\mu = Z^{\perp}_{\nu\mu} u^\mu = 0. \tag{3.9}$$

We have

$$Z_{\mu\nu} = \tfrac{1}{3}\theta_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu u_\nu \tag{3.10}$$

and by use of the Ricci identity we can show that

$$\dot{Z}_{\mu\nu} = -Z_{\mu\sigma} Z^{\sigma}{}_{\nu} - R_{\mu\sigma\nu\rho} u^\sigma u^\rho + a_{\mu;\nu}. \tag{3.11}$$

Dust transport equations

We will now assume u^μ is the four-velocity of dust, so the energy-momentum tensor is

$$T_{\mu\nu} = \rho u_\mu u_\nu, \tag{3.12}$$

where ρ is the dust density. One may use the conservation law ${T^{\mu\nu}}_{;\nu} = 0$ to show

$$u_\mu {T^{\mu\nu}}_{;\nu} = 0 \iff \dot{\rho} = -\rho\theta \tag{3.13}$$

and

$$h_{\sigma\mu} {T^{\mu\nu}}_{;\nu} = 0 \iff a_\sigma = 0, \tag{3.14}$$

which gives us an evolution equation for the density and shows that the acceleration is zero for dust and hence $Z^\perp_{\mu\nu} = Z_{\mu\nu}$. This is the general relativistic analogy of the Newtonian cosmology in §2, where $Z_{\mu\nu}$ is effectively a 3-dimensional tensor analogous to $v_{a,b}$, with an analogous evolution equation to equation (2.9),

$$\dot{Z}_{\mu\nu} = -Z_{\mu\sigma}{Z^\sigma}_\nu - R_{\mu\sigma\nu\rho}u^\sigma u^\rho. \tag{3.15}$$

We can decompose the evolution equation for $Z_{\mu\nu} = \frac{1}{3}\theta_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$ in a similar way,

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - \sigma^2 + \omega^2 - \Theta \tag{3.16}$$

$$\dot{\sigma}_{\mu\nu} = -\tfrac{2}{3}\theta\sigma_{\mu\nu} - \sigma_{\mu\sigma}{\sigma^\sigma}_\nu - \omega_{\mu\sigma}{\omega^\sigma}_\nu + \tfrac{1}{3}h_{\mu\nu}\left(\sigma^2 - \omega^2\right) - E_{\mu\nu} \tag{3.17}$$

$$\dot{\omega}_{\mu\nu} = -\tfrac{2}{3}\theta\omega_{\mu\nu} - \sigma_{\mu\sigma}{\omega^\sigma}_\nu - \omega_{\mu\sigma}{\sigma^\sigma}_\nu, \tag{3.18}$$

where the scalar shear and vorticity are, $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}$ and $\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}$ respectively and we have used the following decomposition,

$$\Theta = {R^\mu}_{\sigma\mu\rho}u^\sigma u^\rho = R_{\sigma\rho}u^\sigma u^\rho \tag{3.19}$$

$$E_{\mu\nu} = R_{\mu\sigma\nu\rho}u^\sigma u^\rho - \tfrac{1}{3}\Theta h_{\mu\nu}. \tag{3.20}$$

One may then calculate the Ricci tensor via the Einstein equation (3) using (3.12),

$$R_{\mu\nu} = 8\pi G\rho\left(u_\mu u_\nu + \tfrac{1}{2}g_{\mu\nu}\right). \tag{3.21}$$

This gives

$$\Theta = 4\pi G\rho. \tag{3.22}$$

Now, using the definition of the Weyl tensor,

$$C_{\mu\sigma\nu\rho} \equiv R_{\mu\sigma\nu\rho} - \tfrac{1}{2}\left(g_{\mu\sigma\nu\lambda}R^{\lambda}{}_{\rho} + g_{\mu\sigma\lambda\rho}R^{\lambda}{}_{\nu}\right) + \tfrac{1}{6}g_{\mu\sigma\nu\rho}R, \tag{3.23}$$

and equations (3.21) and (3.20), one may show that

$$E_{\mu\nu} = C_{\mu\sigma\nu\rho}u^{\sigma}u^{\rho}, \tag{3.24}$$

which is also known as the electric part of the Weyl tensor. One may also define the magnetic part [28],

$$H_{\mu\nu} = \tfrac{1}{2}\eta_{\mu\sigma}{}^{\lambda\kappa}C_{\lambda\kappa\nu\rho}u^{\sigma}u^{\rho}. \tag{3.25}$$

The electric and magnetic parts are both traceless symmetric tensors containing 5 independent components each, which together define the 10 independent components of the Weyl tensor by [28]

$$C_{\mu\sigma\nu\rho} = (g_{\mu\sigma\alpha\gamma}g_{\nu\rho\beta\delta} - \eta_{\mu\sigma\alpha\gamma}\eta_{\nu\rho\beta\delta})u^{\alpha}u^{\beta}E^{\gamma\delta} + (\eta_{\mu\sigma\alpha\gamma}g_{\nu\rho\beta\delta} - g_{\mu\sigma\alpha\gamma}\eta_{\nu\rho\beta\delta})u^{\alpha}u^{\beta}H^{\gamma\delta}. \tag{3.26}$$

Together with the 10 additional independent components of the Ricci tensor, the 20 independent components of the Riemann tensor are given by (3.23). The Ricci tensor is given algebraically by the local matter content via the Einstein equation. The Weyl tensor, however, and hence its electric and magnetic parts, are only given differentially by the Bianchi identity. It has been shown [28, 29] that

$$\begin{aligned}\dot{E}^{\mu\nu} &= -h^{\sigma(\mu}\eta^{\nu)\kappa\lambda\rho}u_{\kappa}\nabla_{\lambda}H_{\sigma\rho} - \theta E^{\mu\nu} - h^{\mu\nu}\sigma^{\sigma\rho}E_{\sigma\rho}\\ &\quad + 3E^{\sigma(\mu}\sigma^{\nu)}{}_{\sigma} - E^{\sigma(\mu}\omega^{\nu)}{}_{\sigma} - 4\pi G\rho\,\sigma^{\mu\nu}\end{aligned} \tag{3.27}$$

$$\begin{aligned}\dot{H}^{\mu\nu} &= h^{\sigma(\mu}\eta^{\nu)\kappa\lambda\rho}u_{\kappa}\nabla_{\lambda}E_{\sigma\rho} - \theta H^{\mu\nu} - h^{\mu\nu}\sigma^{\sigma\rho}H_{\sigma\rho}\\ &\quad + 3H^{\sigma(\mu}\sigma^{\nu)}{}_{\sigma} - H^{\sigma(\mu}\omega^{\nu)}{}_{\sigma}\end{aligned} \tag{3.28}$$

$$h^\mu{}_\sigma h^\nu{}_\rho \nabla_\nu E^{\sigma\rho} + \eta^{\mu\nu\sigma\rho} u_\nu \sigma_{\sigma\kappa} H^\kappa{}_\rho + \tfrac{3}{2}\eta^{\nu\kappa\sigma\rho} u_\kappa \omega_{\sigma\rho} H^\mu{}_\nu = \frac{8\pi G}{3} h^{\mu\nu}\nabla_\mu \rho \quad (3.29)$$

$$h^\mu{}_\sigma h^\nu{}_\rho \nabla_\nu H^{\sigma\rho} - \eta^{\mu\nu\sigma\rho} u_\nu \sigma_{\sigma\kappa} E^\kappa{}_\rho - \tfrac{3}{2}\eta^{\nu\kappa\sigma\rho} u_\kappa \omega_{\sigma\rho} E^\mu{}_\nu = 8\pi G\rho\, \eta^{\mu\nu\sigma\rho} u_\nu \omega_{\sigma\rho}. \quad (3.30)$$

These are two evolution equations and two constraint equations for the Weyl tensor. Equation (3.29) is analogous to the Newtonian equation (2.24) and (3.28) is analogous to (2.25).

Equations (3.16), (3.17), (3.18), (3.13), (3.22) (3.27) and (3.28) constitute the transport equations for a dust filled space-time in general relativity that are analogous to the Newtonian cosmology equations (2.18)-(2.22). These transport equations, however, present a well-posed Cauchy problem due to the evolution equation for the tidal tensor. The initial conditions given on a Cauchy surface must satisfy the integrability conditions (3.29) and (3.30) and a few more derived from the Ricci identities on u^μ analogous to (2.23) (see reference [28] for such conditions). The are further differences from Newtonian gravity on account of the finite propagation velocity, c. This is due to the presence of $H_{\mu\nu}$, which enters the evolution equation of $E_{\mu\nu}$ as a spatial gradient and vice versa. This leads to other phenomena, for example gravitation waves which are not possible in the Newtonian case due to the infinite propagation velocity.

3.2 The 3+1 split of spacetime

Comoving coordinates

We will work in a *comoving coordinate system* $y^\mu = (t, y^i)$, one in which $u^\mu = (1, 0, 0, 0)$. Such a coordinate system can be constructed from a general coordinate system x^μ in the following way (see Figure 3.1). Take a spacelike hypersurface Σ_{t_0} that crosses all the trajectories of fluid under consideration, be it a small part of the manifold or the whole thing, and place coordinates y^i on the hypersurface. The fluid trajectories can then be given by $x^\mu = f^\mu(\tau, y^i)$, where y^i is the coordinate on Σ_{t_0} that it crossed and τ is the

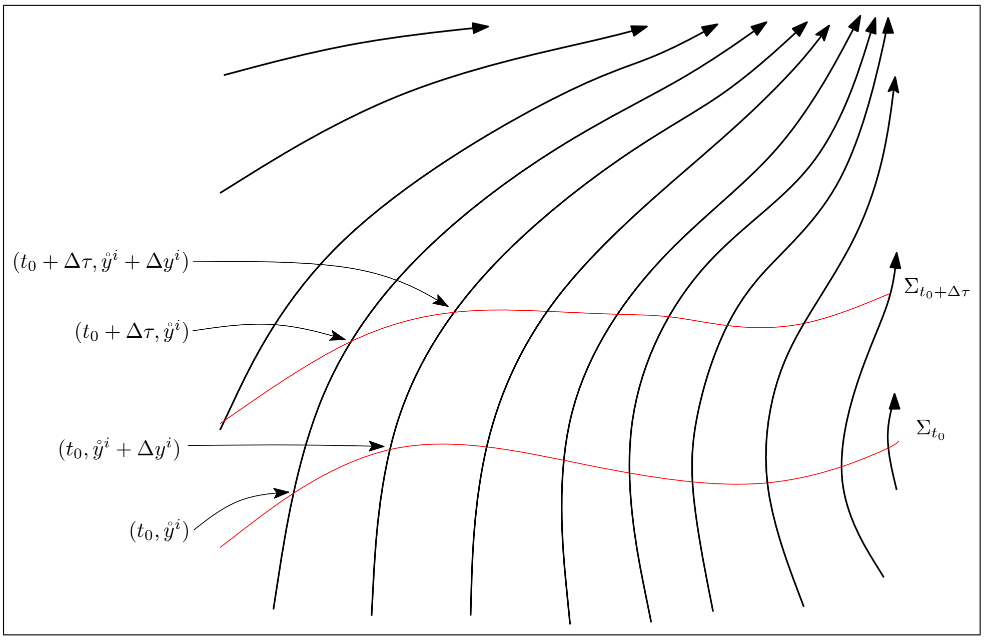

Figure 3.1: An illustration showing how one can construct a comoving coordinate system given the fluid trajectories parameterised by proper time, τ, and then choosing an arbitrary initial foliation Σ_{t_0} with coordinates y^i.

proper time along the trajectory relative to the crossing of Σ_{t_0} at $t = t_0$. Therefore, we can recognise y^i as Lagrangian coordinates. Provided there are no trajectory crossings, letting $t = \tau + t_0$, the inverse of f^μ gives the comoving coordinates y^μ in terms of the original coordinates x^μ.

Such a method gives constant time slices Σ_t that are dependent on the choice of the initial constant time slice, Σ_{t_0}. We can, however, in certain situations define a unique constant time slicing as one will see in the next sections.

Given that $g_{\mu\nu} u^\mu u^\nu = -1$, we have $g_{00} = -1$, so the metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + 2g_{0i}\mathrm{d}t\mathrm{d}y^i + g_{ij}\mathrm{d}y^i\mathrm{d}y^j. \qquad (3.31)$$

Orthogonal coordinates

When the flow is irrotational, we have the following three equivalent conditions [28]:

$$\omega_{\mu\nu} = 0, \tag{3.32}$$

which is the definition of irrotational flow,

$$u_{[\mu} \nabla_\gamma u_{\nu]} = 0, \tag{3.33}$$

which is the condition that says u^μ is hypersurface orthogonal, or equivalently

$$u_\mu = A\,\partial_\mu B \tag{3.34}$$

for some functions A and B of x^μ. The level surfaces $B = constant$ define the hypersurfaces orthogonal to u^μ and $A = \pm(-g^{\mu\nu}\partial_\mu B \partial_\nu B)^{-\frac{1}{2}}$ ensures $u^\mu u_\mu = -1$, where the sign is chosen to make $A\,\partial_\mu B$ future pointed.

If we take $B(y^\mu) = t$, so that the hypersurfaces are those of constant time (but not necessarily proper time), then

$$u_\mu = (A, 0, 0, 0), \tag{3.35}$$

where $A = -(-g^{00})^{-\frac{1}{2}}$. Hence,

$$\begin{aligned} u^\mu &= (g^{00} u_0, g^{0i} u_0) && (3.36) \\ &= (-\frac{1}{A}, C^i), && (3.37) \end{aligned}$$

where $C^i = -g^{0i}(-g^{00})^{-\frac{1}{2}}$ are functions that will depend on how spatial coordinates are placed on the hypersurfaces. Given that $C^i = \frac{\mathrm{d}y^i}{\mathrm{d}\tau}$, where τ is the fluid proper time, if we define the spatial coordinates such that the fluid stays at constant spatial coordinates, then $C^i = 0$. This is an *orthogonal coordinate system* (see Figure 3.2). We note y^i need not be orthogonal to each other but just with the time coordinate, i.e., $g^{0i} = 0$. In this coordinate system the metric is block diagonal, $g^{0i} = g_{0i} = 0$, $g^{00}g_{00} = 1$, $g^{ij}g_{ik} = \delta^i{}_k$ and

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + g_{ij}\mathrm{d}y^i\mathrm{d}y^j. \tag{3.38}$$

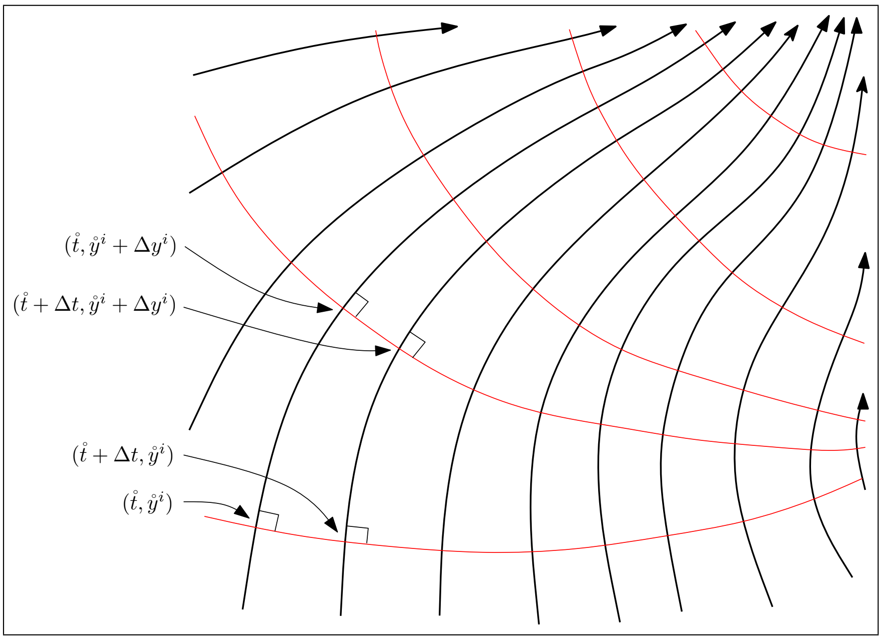

Figure 3.2: An illustration showing how one can construct an orthogonal coordinate system given the fluid trajectories that are irrotational. We note that in 1+1 dimensions, as depicted, fluid flow is always irrotational.

Comoving orthogonal coordinates

When working with orthogonal coordinates, if $\frac{\mathrm{d}t}{\mathrm{d}\tau} = -\frac{1}{A} = 1$ then the time coordinate t is equal to the fluid proper time and the coordinate system is also comoving. Such a choice is not possible in general, but can be made when $a^\mu = 0$ in addition to $\omega_{\mu\nu} = 0$, as we will now demonstrate. In an orthogonal coordinate system,

$$a^\mu = u^\nu \nabla_\nu u^\mu = -A^{-1} \nabla_0 u^\mu \tag{3.39}$$

$$= -A^{-1} \Big(\partial_0 \left(-A^{-1}\right) + {\Gamma^0}_{00} \left(-A^{-1}\right), {\Gamma^i}_{00} \left(-A^{-1}\right) \Big). \tag{3.40}$$

One can show

$${\Gamma^0}_{00} = \tfrac{1}{2} g^{00} \partial_0 g_{00} = \tfrac{1}{2}(-A^{-2}) \partial_0 (-A^2) = \frac{1}{A} \partial_0 A \tag{3.41}$$

and

$${\Gamma^i}_{00} = -\tfrac{1}{2} g^{ij} \partial_j g_{00} = -\tfrac{1}{2} g^{ij} \partial_j (-A^2) = A\, g^{ij} \partial_j A, \tag{3.42}$$

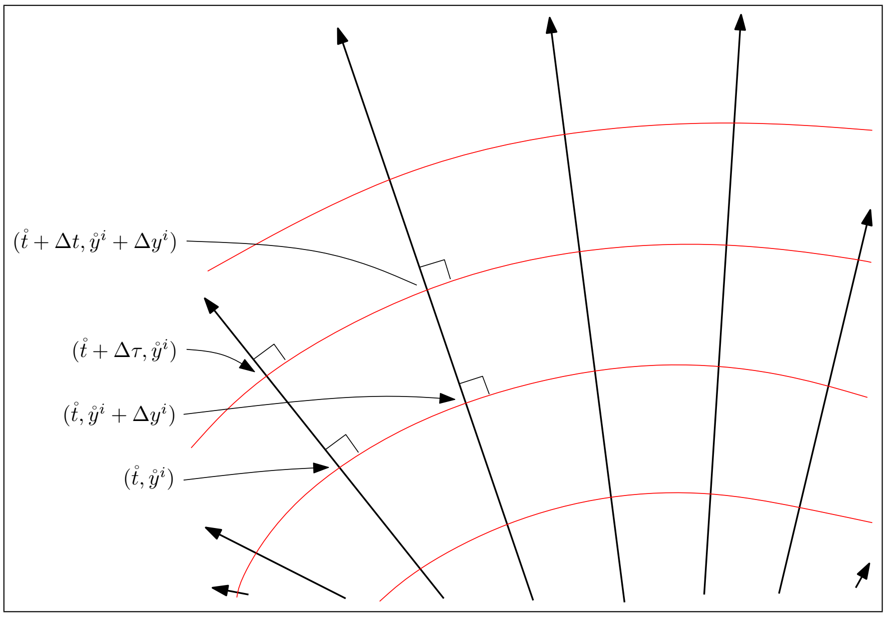

Figure 3.3: An illustration showing how one can construct a comoving orthogonal coordinate system given the fluid trajectories that are irrotational and geodesic. We note that in 1+1 dimensions, as depicted, fluid flow is always irrotational.

so that

$$a^\mu = (0, \frac{1}{A} g^{ij} \partial_j A). \tag{3.43}$$

So, if $a^\mu = 0$, then

$$\partial_i A = 0 \tag{3.44}$$

which implies A is constant with respect to the spatial coordinates, y^i. We are free to define $A(t)$ as we want as it is just a corresponds to a variable scaling of the time coordinate, t. Setting $A = -1$ we find that $\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1$ and we have a comoving *and* orthogonal coordinate system, or *Gaussian normal coordinates* (see Figure 3.3). The metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}\mathrm{d}y^i\mathrm{d}y^j. \tag{3.45}$$

3.3 Hypersurfaces

Consider an n-dimensional manifold, M, coordinates x^μ, with metric $g_{\mu\nu}$, and a $(n-1)$-dimensional submanifold, a hypersurface N with coordinates y^α. The hypersurface can be given in M, either implicitly by the points at which a function, $F(x^\mu)$, takes a constant value, or explicitly parametrized by the coordinates on N, $x^\mu = \psi^\mu(y^\alpha)$. The unit normal to the surface, η_μ, is then given by

$$\eta_\mu = \pm \frac{\pi_\mu}{|g^{\sigma\rho}\pi_\sigma\pi_\rho|^{\frac{1}{2}}}, \tag{3.46}$$

where

$$\pi_\mu = \partial_\mu F \quad \text{or} \quad \pi_\mu = \epsilon_{\mu\nu_1\nu_2\cdots\nu_{n-1}}\psi^{\nu_1}{}_{,1}\psi^{\nu_2}{}_{,2}\cdots\psi^{\nu_{n-1}}{}_{,n-1}. \tag{3.47}$$

One may then define the hypersurface projection tensor or *first fundamental form*,

$$P^\mu{}_\nu = \delta^\mu{}_\nu - \sigma\eta^\mu\eta_\nu, \tag{3.48}$$

where $\sigma = \eta^\mu\eta_\mu$ denotes whether the normal is timelike or spacelike. It is so called because its operation on tensors, denoted by a hat,

$$\widehat{A}^{\mu\nu\cdots}{}_{\rho\sigma\cdots} = P^\mu{}_\lambda P^\nu{}_\kappa \cdots P^\tau{}_\sigma P^\upsilon{}_\rho \cdots A^{\lambda\kappa\cdots}{}_{\tau\upsilon\cdots}\cdots, \tag{3.49}$$

can be shown to be orthogonal to the normal vector on every contraction,

$$\widehat{A}^{\mu\nu\cdots}{}_{\rho\sigma\cdots}\eta_\mu = \widehat{A}^{\mu\nu\cdots}{}_{\rho\sigma\cdots}\eta_\nu = \widehat{A}^{\mu\nu\cdots}{}_{\rho\sigma\cdots}\eta^\sigma = \widehat{A}^{\mu\nu\cdots}{}_{\rho\sigma\cdots}\eta^\rho = \cdots = 0. \tag{3.50}$$

It can be shown to act as the metric for vectors tangent to the hypersurface,

$$P_{\mu\nu}V^\mu W^\nu = g_{\mu\nu}V^\mu W^\nu \tag{3.51}$$

where V^μ and W^ν are tangent to the hypersurface. It also can be shown that it is idempotent,

$$P^\mu{}_\nu P^\nu{}_\lambda = P^\mu{}_\lambda, \tag{3.52}$$

as one would expect since projecting an already projected tensor should not change it.

If we have a family of hypersurfaces, as one would for the level sets of a function $F(x^\mu)$, we have a vector field of hypersurface normals. We can then define the extrinsic curvature or *second fundamental form*, $K_{\mu\nu}$. It describes how the projection tensor changes as we move along the integral curves of the normal vector field and is defined by

$$K_{\mu\nu} = -\tfrac{1}{2}\mathcal{L}_\eta P_{\mu\nu}. \tag{3.53}$$

One can then show this is equivalent to

$$K_{\mu\nu} = -P^\sigma{}_\mu P^\rho{}_\nu \nabla_{(\sigma}\eta_{\rho)}. \tag{3.54}$$

Given coordinates on the hypersurface N, y^α, such that $x^\mu = \psi^\mu(y^\alpha)$ defines the hypersurface, we can restrict or *pullback* any tensor in M to N by

$${}^{n-1}A_{\alpha\beta\cdots} = \psi^\mu{}_{,\alpha}\psi^\nu{}_{,\beta}\cdots {}^{n}A_{\mu\nu\cdots}, \tag{3.55}$$

where the ${}^{n-1}$ pre-superscript denotes the pulled-back tensor in N of the tensor in M denoted by the ^{n} pre-superscript.

3.4 The ADM gauge

We will henceforth work in the ADM gauge for a Lorentzian manifold, M, which is given in coordinates $x^\mu = (t, y^i)$. The hypersurfaces, Σ_t, are those of constant t, and the coordinates on the hypersurfaces are y^i. The unit normal, m^μ, which is timelike, is chosen to be future pointing. It is therefore given by

$$m_\mu = \frac{-\partial_\mu t}{(-g^{\sigma\rho}\partial_\sigma t\partial_\rho t)^{\frac{1}{2}}} = (-N, 0, 0, 0), \tag{3.56}$$

where $N \equiv \frac{1}{\sqrt{-g^{00}}}$ is known as the lapse function and

$$m^\mu = g^{\mu\nu}m_\nu = \frac{1}{N}(1, -N^i), \tag{3.57}$$

where $N^i \equiv g^{0i}N^2$ is the shift vector. One can then define the projection tensor,

$$p^\mu{}_\nu = \delta^\mu{}_\nu + m^\mu m_\nu, \tag{3.58}$$

$$= \begin{bmatrix} 0 & 0 \\ N^i & \delta^i{}_j \end{bmatrix}, \tag{3.59}$$

which projects tensors on to Σ_t and acts as the metric for tensors in the hypersurface. One can show by raising an index on equation (3.58) and rearranging with the use of equation (3.57) that

$$g^{\mu\nu} = \begin{bmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & p^{ij} - \frac{N^i N^j}{N^2} \end{bmatrix}. \tag{3.60}$$

It is then possible, using the fact that $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu{}_\lambda$, to show that

$$g_{\mu\nu} = \begin{bmatrix} -N^2 + N^k N_k & N_j \\ N_i & p_{ij} \end{bmatrix} \tag{3.61}$$

and

$$p^{ij}p_{jk} = \delta^i{}_k, \tag{3.62}$$

where $N_i = p_{ij}N^j$.

The pullback from M to Σ_t is given in terms of $\frac{\partial x^\mu}{\partial y^i} = \delta^\mu{}_i$. Thus, equation (3.55) gives

$${}^3A_{ij\cdots} = {}^4A_{ij\cdots}. \tag{3.63}$$

We can use ${}^3A_{ij\cdots}$ and ${}^4A_{ij\cdots}$ interchangeably but we cannot, however, interchange tensors when some or all of the indices are raised, e.g.,

$${}^3A^i{}_{j\cdots} = {}^3g^{ik}\,{}^3A_{kj\cdots} \neq {}^4A^i{}_{j\cdots} = g^{i\mu}A_{\mu j\cdots}, \tag{3.64}$$

in general. The equality only holds for tensors that are tangent to Σ_t, the hypersurface projection tensor for example. When a tensor is only given with Latin indices without a pre-superscript we will henceforth assume that it is the component of the 3-tensor. We have the 3-metric on Σ_t,

$${}^3g_{ij} = {}^4g_{ij} = p_{ij}. \tag{3.65}$$

Its inverse ${}^3g^{ij}$ satisfies ${}^3g^{ij}\,{}^3g_{jk} = \delta^i{}_k$, so using equations (3.65) and (3.62) we have

$$ {}^3g^{ij} = {}^4p^{ij} = {}^4g^{ij} + \frac{N^iN^j}{N^2}. \tag{3.66} $$

We will therefore use p_{ij} and p^{ij} to denote the 3-metric on Σ_t henceforth. We define the 3-covariant derivative, $D_i \equiv {}^3\nabla_i$, on Σ_t as the covariant derivative given by the 3-Christoffel symbols, ${}^3\Gamma^i{}_{jk}$, e.g., $D_iX^j = \partial_iX^j + {}^3\Gamma^k{}_{ij}X^j$ etc. The 3-Christoffel symbols are given in terms of the 3-metric, p_{ij},

$$ {}^3\Gamma^k{}_{ij} = \tfrac{1}{2}p^{kl}\left(\partial_ip_{jl} + \partial_jp_{il} - \partial_lp_{ij}\right). \tag{3.67} $$

We also have the extrinsic curvature of Σ_t given by

$$ K_{\mu\nu} = -p^\sigma{}_\mu p^\rho{}_\nu \nabla_{(\sigma}m_{\rho)}. \tag{3.68} $$

One may show with the use of equations (3.56) and (3.59) that

$$ K_{\mu\nu} = \begin{bmatrix} -NN^kN^l\Gamma^0{}_{kl} & -NN^k\Gamma^0{}_{ik} \\ -NN^k\Gamma^0{}_{kj} & -N\Gamma^0{}_{ij} \end{bmatrix} \tag{3.69} $$

Evaluating $\Gamma^0{}_{ij}$, we can show

$$ K_{ij} = {}^4K_{ij} = \frac{1}{2N}\left(D_iN_j + D_jN_i - \partial_tp_{ij}\right). \tag{3.70} $$

Evaluating the rest of the Christoffel symbols we obtain all the independent ones,

$$ {}^4\Gamma^0{}_{ij} = -\frac{1}{N}K_{ij} \tag{3.71} $$

$$ {}^4\Gamma^k{}_{ij} = {}^3\Gamma^k{}_{ij} + \frac{N^k}{N}K_{ij} \tag{3.72} $$

$$ {}^4\Gamma^0{}_{00} = \frac{1}{N}\left(\partial_tN + N^iD_iN - N^iN^jK_{ij}\right) \tag{3.73} $$

$$ \begin{aligned} {}^4\Gamma^k{}_{00} = {} & p^{ki}\partial_tN_i + ND^kN - N^iD^kN_i \\ & - \frac{1}{N}\left(N^k\partial_tN + N^kN^iD_iN\right) + \frac{N^iN^jN^k}{N^2}\left(2D_iN_j - NK_{ij}\right) \end{aligned} \tag{3.74} $$

$$ {}^4\Gamma^0{}_{i0} = \frac{1}{N}\left(D_iN - N^jK_{ij}\right) \tag{3.75} $$

$$ {}^4\Gamma^k{}_{i0} = -\frac{N^k}{N}D_iN + D_iN^k + \frac{N^kN^j}{N}K_{ij} - N\,K^k{}_i. \tag{3.76} $$

We now also note the scalar Gauss equation [30],

$$ {}^4R + 2\,{}^4R_{\mu\nu}m^\mu m^\nu = {}^3R + K^2 - K_{ij}K^{ij}, \tag{3.77}$$

where ${}^4R$ and ${}^3R$ are the Ricci scalars of $g_{\mu\nu}$ and p_{ij} respectively and $K = K^i{}_i$.

Comoving coordinates in the ADM gauge

Now if the coordinates are comoving with a fluid, we have $u^\mu = (1, 0, 0, 0)$ as the fluid 4-velocity and $g_{00} = -N^2 + N^k N_k = -1$. Using equation (3.61), we then have

$$u_\mu = g_{\mu\nu}u^\nu = (-1, N_i), \tag{3.78}$$

$$h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu = \begin{bmatrix} 0 & N_j \\ 0 & \delta^i{}_j \end{bmatrix}, \tag{3.79}$$

$$Z^\mu{}_\nu = \nabla_\nu u^\mu = \Gamma^\mu{}_{\nu 0} \tag{3.80}$$

and

$$a^\mu \equiv u^\nu \nabla_\nu u^\mu = Z^\mu{}_\nu u^\nu = \Gamma^\mu{}_{00}. \tag{3.81}$$

Also, one can show

$$Z_{ij} = {}^4Z_{ij} = \nabla_j u_i = D_j N_i - N K_{ij} = D_{[j}N_{i]} + \tfrac{1}{2}\partial_t p_{ij}. \tag{3.82}$$

So if $a^\mu = 0$, as is the case for a geodesic dust fluid, then $\Gamma^0{}_{00}$ and $\Gamma^k{}_{00}$ are zero. Also equation (3.3) gives

$$Z^\perp_{\mu\nu} = Z_{\mu\nu}. \tag{3.83}$$

and thus,

$$\theta_{ij} = {}^4\theta_{ij} = D_{(j}N_{i)} - N K_{ij} = \tfrac{1}{2}\partial_t p_{ij} \tag{3.84}$$

and

$$\omega_{ij} = {}^4\omega_{ij} = D_{[j}N_{i]}. \tag{3.85}$$

Comoving orthogonal coordinates in the ADM gauge

Now, if we have an irrotational geodesic fluid we can work in comoving orthogonal coordinates as is shown in §3.2. In this case the fluid 4-velocity *is* the hypersurface normal,

$$m^\mu = u^\mu = (1,0,0,0) \text{ and } m_\mu = u_\mu = (-1,0,0,0). \tag{3.86}$$

This implies $N = 1$, $N^i = 0$ and that the transverse projection tensor (3.1) and hypersurface projection tensor (3.58) coincide,

$$p^\mu{}_\nu = h^\mu{}_\nu = \begin{bmatrix} 0 & 0 \\ 0 & \delta^i{}_j \end{bmatrix}. \tag{3.87}$$

Equations (3.71) to (3.76) then give

$$\Gamma^0{}_{ij} = -K_{ij}, \quad {}^4\Gamma^k{}_{ij} = {}^3\Gamma^k{}_{ij}, \quad \Gamma^k{}_{0i} = -K^k{}_i, \quad \Gamma^0{}_{00} = \Gamma^0{}_{i0} = \Gamma^k{}_{00} = 0, \tag{3.88}$$

and equation (3.82) gives

$$Z_{ij} = -K_{ij} = \tfrac{1}{2}\partial_t p_{ij}. \tag{3.89}$$

We can then show the rest of the components of $Z_{\mu\nu}$ are zero, as well as

$$\theta \equiv \theta^\mu{}_\mu = \theta^i{}_i = \tfrac{1}{2}p^{ij}\partial_t p_{ij} \tag{3.90}$$

and

$$\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu} = \sigma_{ij}\sigma^{ij} = \tfrac{1}{4}p^{ij}p^{kl}\partial_t p_{ik}\partial_t p_{jl} - \tfrac{1}{12}\left(p^{ij}\partial_t p_{ij}\right)^2. \tag{3.91}$$

Using (3.88) and (3.89), the covariant derivative along u^μ can be shown to be

$$\dot{Z}_{ij} \equiv u^\mu \nabla_\mu Z_{ij} = \partial_t Z_{ij} - 2Z_{ik}Z^k{}_j. \tag{3.92}$$

From equation (3.15) we see that

$$\begin{aligned} \dot{Z}_{ij} &= -Z_{i\rho}Z^\rho{}_j - R_{i\rho j\sigma}u^\rho u^\sigma && (3.93) \\ &= -Z_{ik}Z^k{}_j - R_{i0j0}, && (3.94) \end{aligned}$$

and, therefore,

$$\partial_t Z_{ij} = Z_{ik} Z^k{}_j - R_{i0j0}. \tag{3.95}$$

Using the Ricci tensor for dust (3.21) we can show that the scalar Gauss equation (3.77), in these coordinates, becomes the *energy constraint* for dust,

$$16\pi G\rho = {}^3R + \tfrac{2}{3}\theta^2 - \sigma^2. \tag{3.96}$$

CHAPTER 4

Coarse-graining in general relativity

4.1 Introduction to coarse-graining tensors

Thus far there exists no natural method for coarse-graining tensors (above rank zero) in a covariant manner. Efforts have been made by Zalaletdinov [31, 32] and others [33], but these generally require the addition of much mathematical structure over and above that provided by general relativity. For a critical review of these approaches see, e.g., reviews of van den Hoogen [34], Ellis [35] and Wiltshire [12]. These methods also are not what we define as coarse-graining *per se*; they are methods for *smoothing* tensors. Smoothing defines a tensor field over the domain that is some continuous average of the original field, whereas coarse-graining defines a single averaged value of the original field. One could, however, choose the smoothed value at some single point as a coarse-grained value of the field.

4.2 Korzyński's method

Korzyński pushes the Newtonian cosmology analogy further by coarse-graining $Z_{\mu\nu}$ as was performed to $v_{a,b}$ Chapter 2. Consider a finite fluid

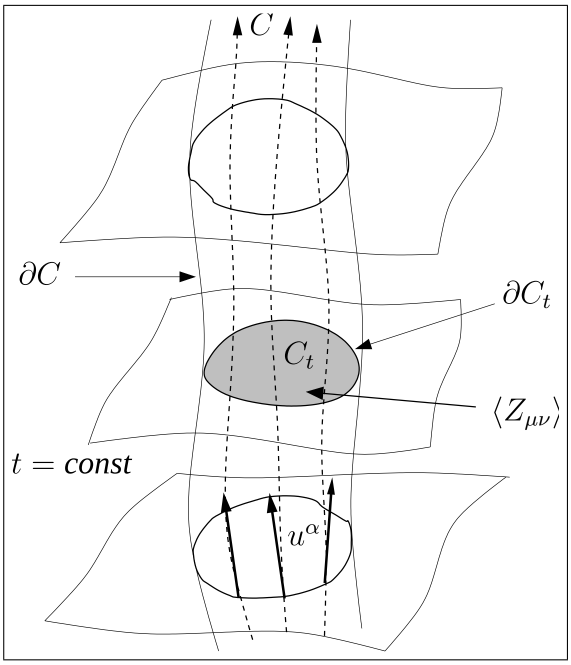

Figure 4.1: A four-dimensional cylinder C, generated by the collective trajectories of a finite volume of fluid, and its boundary ∂C_t are foliated by constant time slices. Image courtesy of Korzyński [13].

element travelling through spacetime (see Figure 4.1). The fluid element defines a four-dimensional cylinder C in spacetime and its boundary ∂C defines a three-dimensional tube. It can be foliated by suitably chosen constant time slices which will give three-dimensional spatial slices of the cylinder C_t bounded by a two-dimensional spatial boundary ∂C_t. One may then attempt to assign a coarse-grained value of $Z_{\mu\nu}$ for the domain C_t. A time evolution equation for the coarse-grained value of $Z_{\mu\nu}$ could then be derived and it should have a similar form to (3.15) with additional terms derived from inhomogeneities in the metric and 4-velocity field. The additional terms referred to as backreaction should reduce to zero for the FLRW solution as they did in the Newtonian case (2.73).

There are possibly unlimited ways in which one might assign a coarse-grained value to $Z_{\mu\nu}$ on a domain, but some conditions should be placed to give meaningful results. Korzyński states reasonable conditions that such a coarse-graining procedure should adhere to. The first condition states that if the volume of the fluid element is shrunk towards zero the coarse-grained velocity gradient should tend to the local one. The second condition states that such a procedure should be covariant in the sense that, apart from the choice of the fluid element itself and the 3+1 splitting of spacetime, *the result should not depend on any externally introduced structure, including the coordinate system.*

Korzyński does not explicitly state any further conditions. However, we will specify further natural conditions a coarse-graining procedure should satisfy. When we coarse-grain a n-tensor we coarse grain over some n-dimensional domain D on a n-dimensional (pseudo-)Riemannian manifold M. When applied to cosmology, this manifold will usually be some spatial hypersurface as illustrated in Figure 4.1 with $n = 3$. The conditions are as follows:

1. When one wants to describe a coarse-grained tensor they must give the components of the tensor with respect to some basis or alternatively in some abstract tangent space. This also defines a metric that one can use to raise and lower indices, $\bar{g}$ say. This basis or tangent space may or may not be related to some basis or tangent space on the manifold M.

   If the coarse-grained tensor basis is not related to any basis on the manifold it should naturally have an orthogonal basis. This would mean $\langle T\rangle^{ab\cdots}{}_{cd\cdots}$ should be unique up to orthogonal transformations,

   $$\langle T\rangle^{ab\cdots}{}_{cd\cdots} = \Lambda^{a}{}_{a'}\Lambda^{b}{}_{b'}\cdots\Lambda^{-1\,c'}{}_{c}\Lambda^{-1\,d'}{}_{d}\cdots\langle T\rangle^{a'b'\cdots}{}_{c'd'\cdots}, \tag{4.1}$$

   where

   $$\eta_{ab}\Lambda^{a}{}_{a'}\Lambda^{b}{}_{b'} = \eta_{a'b'}. \tag{4.2}$$

   When averaging over a spatial constant time slice the canonical metric is $\eta_{ab} = \mathrm{diag}(1, 1, 1)$ and $\Lambda^{a}{}_{a'} \in O(3)$.

   If the coarse-grained tensor basis is related to a basis on the manifold the coarse-grained tensor components could be completely unique given that basis on the manifold. However one generally has the freedom of choosing bases on a manifold so the coarse-grained tensor components generally will not be completely unique.

Either way, scalars which do not depend on a basis should be unique. Thus, scalars formed from coarse-grained tensors and the coarse-graining of scalars, e.g., $\langle T\rangle^{ab\cdots}{}_{ab\cdots}$, $\langle T\rangle^{ab\cdots}{}_{cd\cdots}\langle S\rangle^{cd\cdots}{}_{ab\cdots}$, $\langle U\rangle$, should be unique and not depend on the coarse-grained tensor basis.

The procedure should be covariant in the sense that the value of the coarse-grained tensor should not depend on coordinates placed on M. The coordinates might give the basis and the components of the coarse-grained tensor with respect to that basis. However, when one chooses different coordinates giving a different basis and components of the coarse-grained tensor, they should both be transformed the same way from the previous ones.

2. In the limit of the volume of the domain shrinking to zero around a point p the coarse-grained value should tend to the local value. In general, the coarse-grained tensor, $\langle T\rangle^{ab\cdots}{}_{cd\cdots}$, and the local tensor, $T^{\alpha\beta\cdots}{}_{\gamma\delta\cdots}$, on the manifold will be given in different bases, so more formally there exists some matrix $B^a{}_\alpha \in GL_n$ such that

$$\langle T\rangle^{ab\cdots}{}_{cd\cdots} \to B^a{}_\alpha B^b{}_\beta \cdots B^{-1\,\gamma}{}_c B^{-1\,\delta}{}_d \cdots T^{\alpha\beta\cdots}{}_{\gamma\delta\cdots}|_p \tag{4.3}$$

as the coarse-graining volume is shrunk to zero around p. Moreover, if $T^{\alpha\beta\cdots}{}_{\gamma\delta\cdots}$ is given in an orthonormal basis at p and $\langle T\rangle^{ab\cdots}{}_{cd\cdots}$ is given in an orthonormal basis, $B^a{}_\alpha$ should be orthogonal i.e., satisfy (4.2).

3. In the limit of the domain becoming flat the coarse-graining should just become volume averaging component-wise in a constant metric coordinate system and the coarse-grained tensor basis should just be the constant local basis. The 3-dimensional orthonormal case being the coarse-graining procedure defined for $\mathbf{E}^3$ §2.3. We note that the coarse-graining did not need to be performed in an orthonormal coordinate system, only in one with a constant metric.

4. When coarse-graining a scalar the procedure should just be a plain volume-average over the domain. The 3-dimensional case being the Buchert average defined in §5.2.

5. The procedure should be linear, that is,

$$\langle aT + bS\rangle^{ab\cdots}{}_{ab\cdots} = a\langle T\rangle^{ab\cdots}{}_{ab\cdots} + b\langle S\rangle^{ab\cdots}{}_{ab\cdots} \tag{4.4}$$

   should be true for any tensors T and S and constant scalars a and b.

The last three conditions we will not demand a coarse-graining procedure satisfy, but they are true in the flat case so one could demand them to possibly narrow down a number of coarse-graing methods that satisfy 1-4.

6. The coarse-grained metric, $\langle g\rangle_{ab}$, should be the metric given by the basis for the coarse-grained tensors, $\bar{g}_{ab}$.

7. The coarse-graining procedure should commute with contraction. That is, the following should hold,

$$\langle T\rangle^{ab\cdots}{}_{ad\cdots} = \langle T^{\alpha\beta\cdots}{}_{\alpha\delta\cdots}\hat{e}_\beta \cdots \hat{\omega}^\delta \cdots\rangle^{b\cdots}{}_{d\cdots}, \tag{4.5}$$

   where $\hat{e}_\alpha$ and $\hat{\omega}^\alpha$ are the basis vectors and dual vectors respectively for T on M.

8. When coarse-graining over a domain D, composed of sub-domains D_i, the coarse-grained tensor over D should be some volume weighted sum of the coarse-grained tensor over the sub-domains D_i. That is,

$${}_D\langle T\rangle^{ab\cdots}{}_{cd\cdots} = \frac{V_1}{V}\,{}_{D_1}\langle T\rangle^{ab\cdots}{}_{cd\cdots} \oplus \frac{V_2}{V}\,{}_{D_2}\langle T\rangle^{ab\cdots}{}_{cd\cdots} \oplus \cdots, \tag{4.6}$$

   where V is the volume of D and V_i is the volume of D_i such that $V = \sum_i V_i$. The summing operator $\oplus$ should reduce to the usual $+$ for the flat case when the bases on each sub-domain are the same and also for the curved scalar case.

Korzyński presents a method for coarse-graining Z_{ij} ($Z_{\mu\nu}$ pulled back to a constant time slice) over a comoving domain on constant time slices. One will see it satisfies 1 and 3. The reader is referred to reference [13], §4 for a proof that it satisfies 2. Korzyński does not define the procedure for scalars but one could make it the Buchert average by implicitly demanding 4. We shall see that Korzyński's method would not satisfy 7 in curved space if one were to do so. Korzyński's method does not define an average for the metric but one could do so implicitly by demanding 6 be true. That method is described in the following sections.

4.3 The coarse-graining boundary

The 2-dimensional surface in Σ_t that defines the boundary of the coarse-graining region, C_t, is denoted by ∂C_t. We can describe ∂C_t by $y^i = \xi^i(\theta^A)$ where θ^A are the two coordinates on the surface. We can make this hold for all t because y^i are comoving coordinates. The tube generated by ∂C_t is therefore parametrized by t and θ^A. The normal to ∂C_t in Σ_t is given by

$$\tilde{n}_i = \frac{\pm\tilde{b}_i}{\sqrt{p^{jk}\tilde{b}_j\tilde{b}_k}}, \tag{4.7}$$

where $\tilde{b}_i = \epsilon_{ijk}\xi^j{}_{,1}\xi^k{}_{,2}$ and the sign is chosen such that $\tilde{n}_i$ is outward pointing. In a similar manner to the 4-dimensional case, the projection tensor that projects from Σ_t to ∂C_t is then defined by

$$q^i{}_j = \delta^i{}_j - \tilde{n}^i\tilde{n}_j. \tag{4.8}$$

The pullback from Σ_t to ∂C_t is given in terms of $\frac{\partial y^i}{\partial\theta^A} = \xi^i{}_{,A}$ so that

$${}^2A_{AB\ldots} = \frac{\partial y^i}{\partial\theta^A}\frac{\partial y^j}{\partial\theta^B}\cdots{}^3A_{ij\ldots}. \tag{4.9}$$

This gives the 2-metric

$${}^2g_{AB} = \xi^i{}_{,A}\xi^j{}_{,B}\,{}^3g_{ij} = \xi^i{}_{,A}\xi^j{}_{,B}(q_{ij} - \tilde{n}_i\tilde{n}_j) = \xi^i{}_{,A}\xi^j{}_{,B}q_{ij} \equiv q_{AB}, \tag{4.10}$$

where the third equality comes from the fact $\xi^i{}_{,A}\tilde{n}_i = 0$, and where we have let q_{AB} denote the 2-metric to be consistent with Korzyński [13]. Taking a time derivative we obtain

$$\partial_t q_{AB} = \xi^i{}_{,A}\xi^j{}_{,B}\partial_t q_{ij}. \tag{4.11}$$

4.4 Isometric embedding in $\mathbf{E}^3$

The key idea Korzyński uses is the isometric embedding theorem for S^2 surfaces to embed ∂C_t into $\mathbf{E}^3$, which goes as follows [36]:

Theorem 1 (Isometric embedding theorem for S^2) *Given a compact, orientable surface S homeomorphic to S^2, with positive metric q whose scalar curvature $R > 0$. Then*

- *there exists an isometric embedding*

$$f : S \mapsto \boldsymbol{E}^3$$

  into the 3–dimensional Euclidean space;

- *the embedding is unique up to rigid transformations.*

We will call a surface that satisfies the conditions of the theorem *admissible*. Essentially, the theorem states that if ∂C_t is admissible there exists a map from ∂C_t to some surface in $\mathbf{E}^3$, ∂D_t say, such that metric induced on ∂C_t is equal to the metric induced on ∂D_t. Moreover, the map and surface are unique up to moving the surface around, rotating it as a whole or reflecting it.

Now consider a time-dependent embedding (general nonisometric at this stage),

$$f_t : \partial C_t \mapsto \partial D_t \subset \mathbf{E}^3, \tag{4.12}$$

where ∂D_t is the image of f_t. The surface is then given parametrically in Cartesian coordinates in $\mathbf{E}^3$ by

$$x^a = \chi^a(t, \theta^A). \tag{4.13}$$

Therefore, θ^A are coordinates on ∂D_t. The normal to ∂D_t in $\mathbf{E}^3$ is given by

$$n_a = \frac{\pm b_a}{\sqrt{\delta^{bc} b_b b_c}}, \tag{4.14}$$

where $b_a = \epsilon_{abc} \chi^b{}_{,1} \chi^c{}_{,2}$ and the sign is chosen such that n_a is outward pointing. In a similar manner to the Σ_t case, the projection tensor that projects from $\mathbf{E}^3$ to ∂D_t is then defined by

$$\bar{q}^a{}_b = \delta^a{}_b - n^a n_b. \tag{4.15}$$

Any tensor in $\mathbf{E}^3$ can be pulled back to ∂C_t by $\frac{\partial x^a}{\partial \theta^A} = \chi^a{}_{,A}$. We denote any tensor in ∂D_t by placing a $\bar{2}$ superscript so that,

$${}^{\bar{2}}A_{AB\ldots} = \frac{\partial x^a}{\partial \theta^A} \frac{\partial x^b}{\partial \theta^B} \cdots A_{ab\ldots}. \tag{4.16}$$

This gives the 2-metric

$${}^{\bar{2}}\delta_{AB} = \chi^a{}_{,A} \chi^b{}_{,B} \delta_{ab} = \chi^a{}_{,A} \chi^b{}_{,B} (\bar{q}_{ab} + n_a n_b) = \chi^a{}_{,A} \chi^b{}_{,B} \bar{q}_{ab} \equiv \bar{q}_{AB}, \tag{4.17}$$

where third equality follows on account of the fact $\chi^a{}_{,A} n_a = 0$. The ${}^{\bar{2}}\delta_{AB}$ is not the Kronecker delta on its indices. Rather, it is the pullback of the Kronecker delta, δ_{ab}. So we will use $\bar{q}_{AB}$ to denote it for clarity.

Provided ∂C_t is admissible, Theorem 1 states there exists an isometric embedding from ∂C_t to $\mathbf{E}^3$, i.e., an embedding such that $q_{AB} = \bar{q}_{AB}$. In terms of the embedding functions, that is,

$$q_{AB}(t, \theta^A) = \chi^a{}_{,A} \chi^b{}_{,B} \delta_{ab}, \tag{4.18}$$

which is a system of non-linear partial differential equations with, in general, non-analytical solutions. The solutions are unique up to

$$\chi^a(t, \theta^A) \to R^a{}_b(t) \chi^b(t, \theta^A) + W^a(t), \tag{4.19}$$

where $R^a{}_b$ is orthogonal.

The sub-manifolds ∂C_t and ∂D_t are now the exact same manifold only they are embedded in different spaces; they have the same intrinsic curvature which is defined by the manifold but different extrinsic curvature

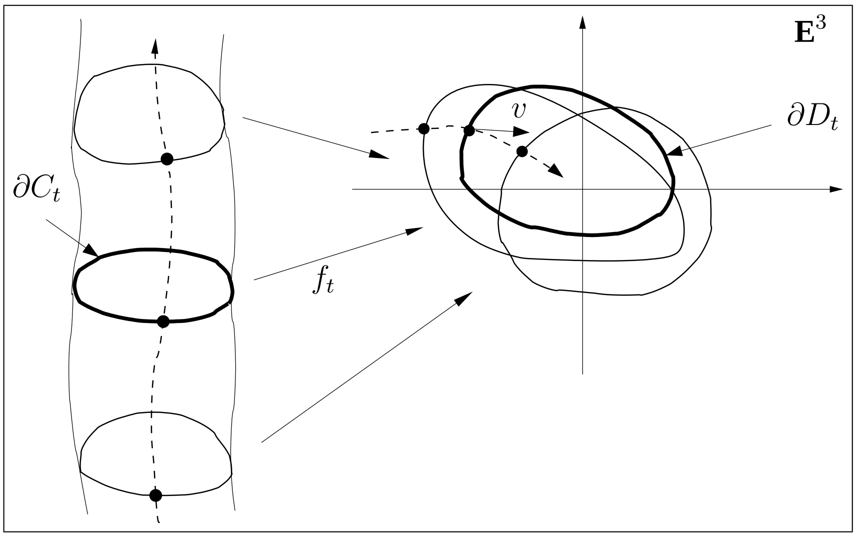

Figure 4.2: A sequence of embeddings in time and the trajectory of a particle in Euclidean space. Image courtesy of Korzyński [13].

which is defined by the embedding. We will drop all the bars using just the notation q_{AB} as we will do henceforth.

Once we have a sequence of embeddings in time, we can obtain a trajectory in $\mathbf{E}^3$ for each boundary particle, labelled by coordinates θ^A. Provided the trajectory is continuous, we can give each particle a fictitious velocity in $\mathbf{E}^3$, $v^a = \dot{\chi}^a$, at each time (see Figure 4.2). One may think that a sequence of embeddings is needed to generate v^a at any time t. However, if we just have the one embedding at t and know the time derivative of the metric on ∂C_t, $\partial_t q_{AB}$, then we can generate the velocity field in $\mathbf{E}^3$ at time t. This is guaranteed by the following theorem [13].

Theorem 2 *Given a compact, orientable surface S homeomorphic to S^2, embedded isometrically into $\boldsymbol{E}^3$ at time t, whose scalar curvature $R > 0$, and a symmetric tensor field r_{AB} on S. Then*

- *there exists a vector field v^a, $v^a(x^b) \in \boldsymbol{E}^3$, defined on $S \subset \boldsymbol{E}^3$, such that*

$$\partial_t \bar{q}_{AB} = r_{AB} \tag{4.20}$$

at time t when S dragged along v^a,

- *v^a is unique up to adding a vector field Y^a of the form*

$$Y^a = \Omega^a{}_b(t)\, x^b + W^a(t), \quad \textit{where} \quad \Omega_{ab} = -\Omega_{ba}. \tag{4.21}$$

It should be noted that the theorem only applies instantaneously. In general, when ∂D_t is dragged along v^a, it will no longer be an isometric embedding; only when $r_{AB} = \partial_t q_{AB}$ will it continue to be an isometric embedding. The vector field v^a is related to r_{AB} by the action of differential operator $\mathcal{P}$, such that

$$r_{AB} = \mathcal{P}[v^a]_{AB} \equiv 2v^a{}_{(,A}\chi^b{}_{,B)}\delta_{ab}, \tag{4.22}$$

which follows from the derivative of (4.17) along with (4.20).

4.5 Korzyński's coarse-graining

We are now ready to propose a coarse-graining method. Given an embedding at time t and a vector field v^a defined on ∂D_t, Korzyński proposes that the symmetric part of $\langle Z\rangle_{ab}$ be defined by

$$\langle Z\rangle_{(ab)} = \frac{1}{V_{D_t}} \int_{\partial D_t} v_{(a} n_{b)}\, \mathrm{d}\sigma, \tag{4.23}$$

where V_{D_t} is the Euclidean volume of the domain D_t enclosed by ∂D_t. This definition is motivated by the Newtonian cosmology equation (2.65) and, due to the symmetrization, it is unique. The addition of vector fields of the form (4.21) do not change it. This is reasonable as equation (3.84) shows that $\partial_t q_{ij}$ determines $Z_{(ij)}$ so it should also determine $\langle Z\rangle_{(ab)}$. One can see that it does by observing $\partial_t q_{ij}$ determines $\partial_t q_{AB}$ by (4.11) and $\partial_t q_{AB}$ determines the velocity, which is unique under the action of (4.23) by Theorem 2. Thus, we have a coarse-grained expansion tensor but still need to propose a way to coarse-grain the vorticity tensor.

We see from equation (3.85) that $Z_{[ij]}$ is determined by the shift vector N_i and is not influenced by $\partial_t q_{ij}$. To determine $\langle Z\rangle_{[ab]}$ we shall, therefore,

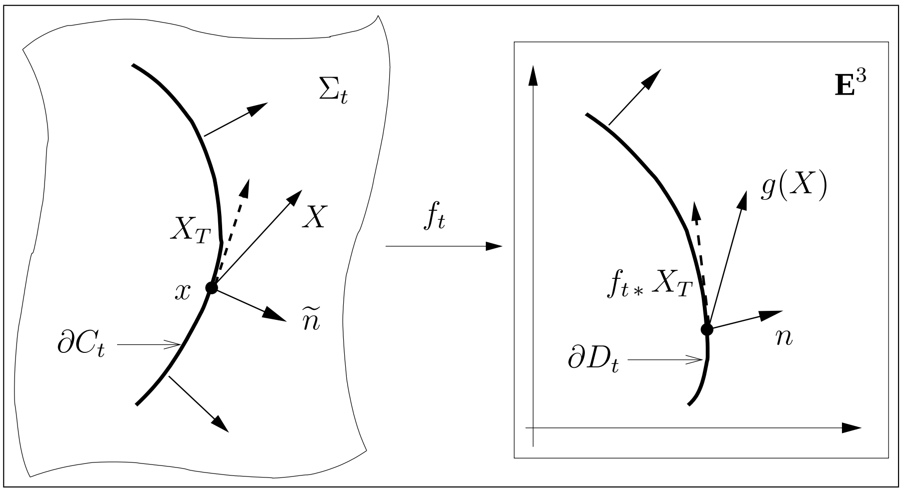

Figure 4.3: The "push" of a vector from Σ_t to $\mathbf{E}^3$. Image courtesy of Korzyński [13].

"push" N^i directly from Σ_t to $\mathbf{E}^3$. The "push" is performed via a canonical isometry between the tangent spaces $T_x\Sigma_t$, $x \in \partial C_t$ and $\mathbf{E}^3$, induced by the embedding f_t. It is defined as follows; for each index on a tensor, the tangent part in $\mathbf{E}^3$ is defined to be the pushforward from $\partial D_t = \partial C_t$ of the pullback from Σ_t and the normal part in $\mathbf{E}^3$ is defined to have the same magnitude as the normal part in Σ_t. It can be written

$$\begin{aligned} A^{ab\cdots}{}_{cd\ldots} = {} & A^{ij\cdots}{}_{kl\ldots} q_{mi} q_{nj} \cdots \delta_{gc}\delta_{hd} \cdots \\ & \times \left(\chi^a{}_{,A}\xi^m{}_{,M} q^{AM} + n^a \tilde{n}^m\right)\left(\chi^b{}_{,B}\xi^n{}_{,N} q^{BN} + n^b \tilde{n}^n\right)\cdots \\ & \times \left(\chi^g{}_{,G}\xi^k{}_{,K} q^{GK} + n^g \tilde{n}^k\right)\left(\chi^h{}_{,H}\xi^l{}_{,L} q^{HL} + n^h \tilde{n}^l\right)\cdots. \end{aligned} \tag{4.24}$$

Figure 4.3 gives a visual illustration for the vector case. It is invertible so it can also be used to "push" tensors from $\mathbf{E}^3$ to Σ_t, the inverse is given by just swapping $a, b, \ldots$ for $i, j, \ldots$, ξ for χ and $\tilde{n}$ for n. Equation (4.24) gives

$$N_a = N_i \left(\chi^b{}_{,B}\xi^i{}_{,I} q^{BI} + n^b \tilde{n}^i\right)\delta_{ab}. \tag{4.25}$$

Korzyński then proposes that

$$\langle Z\rangle_{[ab]} = \frac{1}{V}\int_{\partial D_t} N_{[a} n_{b]}\, \mathrm{d}\sigma. \tag{4.26}$$

We note the normal part of (4.25) drops out in (4.26) due to the antisymmetrization. We can use (4.26) to fix the rotation part of v^a by demanding

$$\int_{\partial D_t} v_{[a}n_{b]}\,\mathrm{d}\sigma = \int_{\partial D_t} N_{[a}n_{b]}\,\mathrm{d}\sigma. \tag{4.27}$$

In this case, Korzyński's proposed coarse-grained velocity gradient is then,

$$\langle Z\rangle_{ab} = \frac{1}{V_{D_t}}\int_{\partial D_t} v_a n_b\,\mathrm{d}\sigma. \tag{4.28}$$

We also note, that in a similar fashion to (2.52), one can show using Reynolds' transport theorem (2.55) that

$$\langle Z\rangle^a{}_a = \frac{\dot{V}_{D_t}}{V_{D_t}}. \tag{4.29}$$

We will now introduce the following notation as Korzyński does,

$$\mathcal{N}[A]_b = \frac{1}{V_{D_t}}\int_{\partial D_t} A n_b\,\mathrm{d}\sigma, \tag{4.30}$$

where A is any object on ∂D_t. We will also let $\mathcal{P}^{-1}$ be the unique inverse of $\mathcal{P}$ satisfying

$$\mathcal{N}\mathcal{P}^{-1}\left[r_{AB}\right]_{[cd]} = 0, \tag{4.31}$$

with some irrelevant term fixing the constant part W^a. Here we have introduced the notation

$$\mathcal{N}\mathcal{P}^{-1}\left[r_{AB}\right]_{ab} \equiv \mathcal{N}\big[\mathcal{P}^{-1}\left[r_{AB}\right]_a\big]_b. \tag{4.32}$$

Using this notation and using (4.11) and (3.84) we can write down the coarse-grained Z_{ij} in the following more compact form

$$\langle Z\rangle_{(ab)} = \mathcal{N}\mathcal{P}^{-1}\left[2Z_{(ij)}\xi^i{}_{,A}\xi^j{}_{,B}\right]_{ab} \tag{4.33}$$

$$\langle Z\rangle_{[ab]} = \mathcal{N}\big[N_{[a}\big]_{b]}. \tag{4.34}$$

4.6 Coarse-graining example: Bianchi I universe

We will now give an example of the coarse-graining procedure for a simple non-trivial case. Let us take the homogeneous anisotropic case, Bianchi I universe, which is given in comoving coordinates by the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R_x(t)^2\mathrm{d}x^2 + R_y(t)^2\mathrm{d}y^2 + R_z(t)^2\mathrm{d}z^2. \tag{4.35}$$

We can see this is also an orthogonal coordinate system so the constant time slice normal and the 4-velocity are given by

$$m^\mu = u^\mu = (1, 0, 0, 0). \tag{4.36}$$

One may then construct the hypersurface projection tensor and then write the 4-metric,

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & p_{ij} \end{bmatrix} \quad g^{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & p^{ij} \end{bmatrix}, \tag{4.37}$$

where the 3-metric is

$$p_{ij} = \begin{bmatrix} {R_x}^2 & 0 & 0 \\ 0 & {R_y}^2 & 0 \\ 0 & 0 & {R_z}^2 \end{bmatrix} \quad p^{ij} = \begin{bmatrix} {R_x}^{-2} & 0 & 0 \\ 0 & {R_y}^{-2} & 0 \\ 0 & 0 & {R_z}^{-2} \end{bmatrix}. \tag{4.38}$$

We will coarse-grain over a cube defined by $-a \leq x, y, z \leq a$ (see Figure 4.4). Now take the $x = a$ surface, we can calculate the normal,

$$\tilde{n}_i = \frac{\partial_i x}{(p^{jk}\partial_j x \partial_k x)^{\frac{1}{2}}} = (R_x, 0, 0). \tag{4.39}$$

We can then calculate the projection tensor on this side,

$$q_{ij} = p_{ij} - \tilde{n}_i\,\tilde{n}_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & {R_y}^2 & 0 \\ 0 & 0 & {R_z}^2 \end{bmatrix}. \tag{4.40}$$

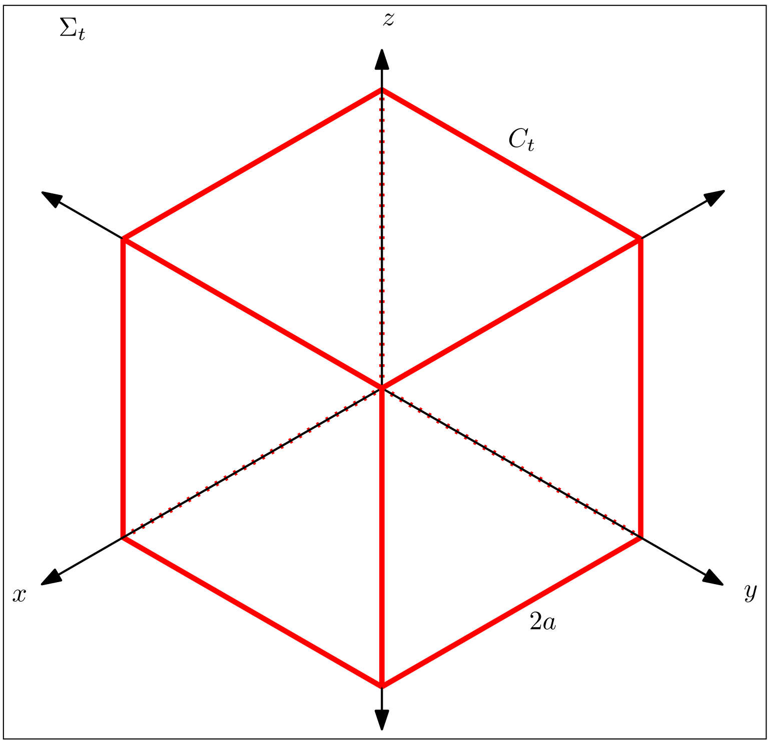

Figure 4.4: The chosen coarse-graining region, C_t, is a cube centred on the origin with side length $2a$.

Placing coordinates on this side, $\theta^A = \{Y, Z\}$, so that

$$y^i = \xi^i(\theta^A) = (a, Y, Z) \quad -a \leq Y, Z \leq a. \tag{4.41}$$

The coordinate transformation is given by

$$\xi^i{}_{,A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.42}$$

Then the 2-metric on this side is the

$$q_{AB} = q_{ij}\xi^i{}_{,A}\xi^j{}_{,B} = \begin{bmatrix} {R_y}^2 & 0 \\ 0 & {R_x}^2 \end{bmatrix}. \tag{4.43}$$

Now this surface is not admissible as the scalar curvature is zero on the sides so Theorem 1 does not guarantee the existence of an isometric embedding.

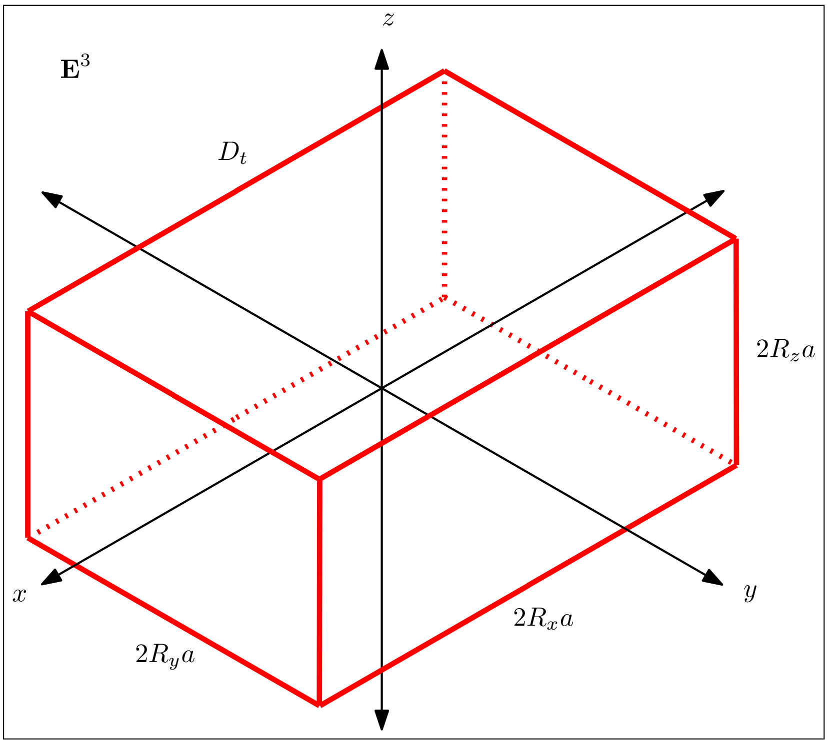

Figure 4.5: The isometric embedding of the cube is stretched by R_x, R_y and R_z in the x, y and z directions respectively.

However, we will show one exists. A reasonable guess for an isometric embedding is one in which the cube has been stretched by R_x, R_y and R_z in the x, y and z directions respectively as illustrated in Figure 4.5 . This is,

$$x^a = \chi^a(\theta^A, t) = \begin{bmatrix} R_x a \\ R_y Y \\ R_z Z \end{bmatrix} \tag{4.44}$$

for the side under consideration. The induced metric on this side ∂D_t from this embedding is then,

$$\bar{q}_{AB} = \delta_{ab}\, \chi^a{}_{,A}\, \chi^b{}_{,B} = \begin{bmatrix} {R_y}^2 & 0 \\ 0 & {R_x}^2 \end{bmatrix}. \tag{4.45}$$

Since $\bar{q}_{AB} = q_{AB}$, the proposed embedding is indeed isometric for this side. We can show, by using the same process for all the other sides, that it is an isometric embedding for all of them making the whole embedding isometric.

We now calculate the velocity,

$$v^a = \frac{\partial}{\partial t}\left(\chi^a(\theta^A, t)\right) = \begin{bmatrix} \dot{R}_x a \\ \dot{R}_y Y \\ \dot{R}_z Z \end{bmatrix} = \begin{bmatrix} \frac{\dot{R}_x}{R_x} x \\ \frac{\dot{R}_y}{R_y} y \\ \frac{\dot{R}_z}{R_z} z \end{bmatrix}, \tag{4.46}$$

where one can show the fourth term is the velocity for every side. The surface normal is obvious but one may calculate it via equation (4.14) if they wished,

$$n_a = (1, 0, 0). \tag{4.47}$$

We now have all the ingredients to calculate the coarse-grained velocity gradient, $\langle Z \rangle_{ab}$. We can split the surface integral of equation (4.23) up into the 6 sides of the cuboid D_t and then add them up to get the desired result. Thus, the contribution from the positive x side is

$$\langle Z \rangle_{(ab)} = \frac{1}{V} \int_{\partial D_t} \tfrac{1}{2}\left(v_a\, n_b + v_b\, n_a\right)\, \mathrm{d}\sigma \tag{4.48}$$

$$= \frac{1}{V} \int_{-R_z a}^{R_z a} \int_{-R_y a}^{R_y a} \tfrac{1}{2}\left(v_a\, n_b + v_b\, n_a\right)|_{x=a}\, \mathrm{d}y\mathrm{d}z \tag{4.49}$$

$$= \frac{1}{V} \begin{bmatrix} 4\dot{R}_x R_y R_z a^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.50}$$

Adding up the contribution from all sides, recognising $V = 8R_x R_y R_z a^3$ and noting the antisymmetric part is zero due to the vanishing shift vector we obtain

$$\langle Z \rangle_{ab} = \begin{bmatrix} \frac{\dot{R}_x}{R_x} & 0 & 0 \\ 0 & \frac{\dot{R}_y}{R_y} & 0 \\ 0 & 0 & \frac{\dot{R}_z}{R_z} \end{bmatrix} \tag{4.51}$$

As this does not depend on the size of the cube, we expect this to be some orthogonal transformation of $Z_{ij}(t)$ in orthonormal coordinates. That is because when the cube is shrunk towards zero we should recover the local velocity gradient, which is the same everywhere spatially because this solution is homogeneous. Using equation (3.82) we see that

$$Z_{ij} = \tfrac{1}{2}\dot{p}_{ij} \tag{4.52}$$

$$= \begin{bmatrix} \dot{R}_x R_x & 0 & 0 \\ 0 & \dot{R}_y R_y & 0 \\ 0 & 0 & \dot{R}_z R_z \end{bmatrix}. \tag{4.53}$$

This is not given in an orthonormal basis, however, so we need to transform to a basis that is. The transformation

$$B^i{}_{\hat{\imath}} = \begin{bmatrix} R_x^{-1} & 0 & 0 \\ 0 & R_y^{-1} & 0 \\ 0 & 0 & R_z^{-1} \end{bmatrix} \tag{4.54}$$

gives the metric

$$p_{ij} B^i{}_{\hat{\imath}} B^j{}_{\hat{\jmath}} = \delta_{\hat{\imath}\hat{\jmath}}, \tag{4.55}$$

so is a transformation to an orthonormal basis. The velocity gradient in this basis is then

$$Z_{\hat{\imath}\hat{\jmath}} = Z_{ij} B^i{}_{\hat{\imath}} B^j{}_{\hat{\jmath}} = \begin{bmatrix} \frac{\dot{R}_x}{R_x} & 0 & 0 \\ 0 & \frac{\dot{R}_y}{R_y} & 0 \\ 0 & 0 & \frac{\dot{R}_z}{R_z} \end{bmatrix}. \tag{4.56}$$

Since (4.56) and (4.51) are identical, we have automatically satisfied the requirement that they be related by an orthogonal transformation. Thus, Korzyński's coarse-graining worked as expected.

4.7 Evolution for the irrotational case

In this section we will now derive the evolution equation for $\langle Z \rangle_{ab}$ for irrotational dust following Korzyński [13]. Firstly, however, we note the following properties concerning the operators $\mathcal{N}$ and $\mathcal{P}$. The operators $\mathcal{N}$ and $\mathcal{P}$ are linear,

$$\mathcal{P}[Av_a + Bw_a]_{AB} = A\mathcal{P}[v_a]_{AB} + B\mathcal{P}[w_a]_{AB} \tag{4.57}$$

$$\mathcal{N}[Av_a + Bw_a]_b = A\mathcal{N}[v_a]_b + B\mathcal{N}[w_a]_b, \tag{4.58}$$

where A and B are constants, which is trivial to prove using their definitions. Also, it is trivial to prove that they commute with the metric, δ_{ab}, δ^{ab}, so that the index on v may be raised or lowered without worry and

$$\mathcal{N}[x_a]_b = \delta_{ab}. \tag{4.59}$$

One can also show

$$\mathcal{N}\mathcal{P}^{-1}\left[2A_{ab}\chi^a{}_{(,A}\chi^b{}_{,B)}\right]_{cd} = A_{(cd)} \tag{4.60}$$

and, more generally,

$$\mathcal{N}\mathcal{P}^{-1}\left[2A_{bc}v^c{}_{,a}\chi^a{}_{(,A}\chi^b{}_{,B)}\right]_{de} = \delta_{fg}A_{(d}{}^f\mathcal{N}[v^g]_{e)}, \tag{4.61}$$

where A_{ab} is constant, by considering the effects of $\mathcal{N}$ and $\mathcal{P}$ on $A_{ab}x^b$ and $A_{ab}v^b$ respectively. We then have two identities concerning the time derivatives of the operators. The first is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}[X_a]_b = -\mathcal{N}[v^c]_c\mathcal{N}[X_a]_b + \mathcal{N}\left[\dot{X}_a\right]_b + \mathcal{N}\left[v^c{}_{,c}X_a\right]_b - \mathcal{N}\left[v^c{}_{,b}X_a\right]_c, \tag{4.62}$$

for any vector field X^a defined on ∂D_t for some time interval. The overdot denotes the convective time derivative, i.e., at constant θ^A coordinates, which is $\frac{\partial}{\partial t} + v^a\frac{\partial}{\partial x^a}$ in $\mathbf{E}^3$. The second identity is

$$\begin{aligned}\dot{Y}_a = {} & \mathcal{P}^{-1}\left[\partial_t r_{AB}\right]_a - \mathcal{P}^{-1}\left[2v^c{}_{,A}Y_{c,B}\right]_a \\ & - \mathcal{N}\left[v^c{}_{,c}Y_{[a}\right]_{b]}x^b + \mathcal{N}\left[v^c{}_{[,b}Y_{a]}\right]_c x^b + W_a,\end{aligned} \tag{4.63}$$

where $Y_a = \mathcal{P}^{-1}\left[r_{AB}\right]_a$ and W_a denotes an irrelevant constant vector. The first identity follows by extending X_a arbitrarily over D_t, recognising $\mathcal{N}[X_a]_b = \langle X_{a,b}\rangle$, then using the commutation rule (2.54) and some rearrangement using the product rule. The second identity follows by calculating $\partial_t r_{AB}$, taking the inverse and demanding the uniqueness condition (4.31) with the use of (4.62).

We can then apply (4.62) to the velocity field v^a to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}[v_a]_b = -\mathcal{N}[v^c]_c\mathcal{N}[v_a]_b + \mathcal{N}[\dot{v}_a]_b + \mathcal{N}\left[v^c{}_{,c}v_a\right]_b - \mathcal{N}\left[v^c{}_{,b}v_a\right]_c. \tag{4.64}$$

Noting that $v_a = \mathcal{P}^{-1}\left[\partial_t q_{AB}\right]_a$, using it in equation (4.63) and then operating with $\mathcal{N}$, noting (4.59), gives

$$\begin{aligned}\mathcal{N}[\dot{v}_a]_b = &\mathcal{N}\mathcal{P}^{-1}\left[\partial_{tt} q_{AB}\right]_{ab} - \mathcal{N}\mathcal{P}^{-1}\left[2v^c{}_{,A}v_{c,B}\right]_{ab} \\ &- \mathcal{N}\left[v^c{}_{,c}v_{[a}\right]_{b]} + \mathcal{N}\left[v^c{}_{[,b}v_{a]}\right]_c.\end{aligned} \tag{4.65}$$

Combining equations (4.64) and (4.65), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}[v_a]_b = &-\mathcal{N}[v^c]_c\mathcal{N}[v_a]_b + \mathcal{N}\left[v^c{}_{,c}v_{(a}\right]_{b)} - \mathcal{N}\left[v^c{}_{(,b}v_{a)}\right]_c \\ &+ \mathcal{N}\mathcal{P}^{-1}\left[\partial_{tt}q_{AB}\right]_{ab} - \mathcal{N}\mathcal{P}^{-1}\left[2v^c{}_{,A}v_{c,B}\right]_{ab}.\end{aligned} \tag{4.66}$$

As we are considering irrotational dust we will work in comoving orthogonal coordinates so by using equations (3.84) and (4.11) one can show

$$\partial_{tt}q_{AB} = 2\partial_t Z_{ij}\,\xi^i{}_{,A}\xi^j{}_{,B}, \tag{4.67}$$

which with equation (3.95) yields

$$\partial_{tt}q_{AB} = 2Z_{ik}Z^k{}_j\xi^i{}_{,A}\xi^j{}_{,B} - 2R_{i0j0}\xi^i{}_{,A}\xi^j{}_{,B}. \tag{4.68}$$

Using equation (4.24), one can show that

$$Z_{ik}Z^k{}_j\xi^i{}_{,A}\xi^j{}_{,B} = Z_{ac}Z^c{}_b\chi^a{}_{,A}\chi^b{}_{,B}, \tag{4.69}$$

so that equation (4.66) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}[v_a]_b = &-\mathcal{N}[v^c]_c\mathcal{N}[v_a]_b + \mathcal{N}\left[v^c{}_{,c}v_{(a}\right]_{b)} - \mathcal{N}\left[v^c{}_{(,b}v_{a)}\right]_c \\ &+ \mathcal{N}\mathcal{P}^{-1}\left[2Z_{dc}Z^c{}_e\chi^d{}_{,A}\chi^e{}_{,B}\right]_{ab} - \mathcal{N}\mathcal{P}^{-1}\left[2R_{i0j0}\xi^i{}_{,A}\xi^j{}_{,B}\right]_{ab} \\ &- \mathcal{N}\mathcal{P}^{-1}\left[2v^c{}_{,A}v_{c,B}\right]_{ab}.\end{aligned} \tag{4.70}$$

We decompose the *local* Z_{ab}, pushed from Σ_t to $\mathbf{E}^3$ via (4.24), and the velocity field v^a into their coarse-grained part and local inhomogeneities on ∂D_t,

$$Z_{ab} = \langle Z\rangle_{ab} + \delta Z_{ab} \tag{4.71}$$

$$v^a = \langle Z\rangle^a{}_b x^b + \delta v^a, \tag{4.72}$$

so that one can show

$$\mathcal{N}\mathcal{P}^{-1}\left[\delta Z_{ab}\chi^{a}{}_{,A}\chi^{b}{}_{,B}\right]_{cd} = 0 \tag{4.73}$$

$$\mathcal{N}[\delta v_a]_b = 0. \tag{4.74}$$

Now we are at the point where we can substitute in the decompositions (4.71) and (4.72) into equation (4.70) and use equations (4.57)-(4.61), (4.73) and (4.74) to show

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Z\rangle_{ab} = -\langle Z\rangle_{ac}\langle Z\rangle^{c}{}_{b} - \langle R_{a0b0}\rangle + B_{ab} + \tilde{B}_{ab}, \tag{4.75}$$

where

$$\langle R_{a0b0}\rangle = \mathcal{N}\mathcal{P}^{-1}\left[2R_{i0j0}\xi^{i}{}_{,A}\xi^{j}{}_{,B}\right]_{ab} \tag{4.76}$$

and

$$B_{ab} = \mathcal{N}\left[\delta v^{c}{}_{,c}\delta v_{(a}\right]_{b)} - \mathcal{N}\left[\delta v^{c}{}_{(,a}\delta v_{b)}\right]_{c} \tag{4.77}$$

and

$$\begin{aligned}\tilde{B}_{ab} = &\,\mathcal{N}\mathcal{P}^{-1}\left[4\langle Z\rangle_{cd}\delta Z^{d}{}_{e}\chi^{e}{}_{(,A}\chi^{c}{}_{,B)}\right]_{ab} \\ &+\mathcal{N}\mathcal{P}^{-1}\left[2\delta_{cd}\left(\delta Z^{c}{}_{e}\delta Z^{d}{}_{f} - \delta v^{c}{}_{,e}\delta v^{d}{}_{,f}\right)\chi^{e}{}_{,A}\chi^{f}{}_{,B}\right]_{ab}.\end{aligned} \tag{4.78}$$

One can see that this is a coarse-grained version of (3.15) and is analogous to the Newtonian equation (2.73). We can see that $\langle R_{a0b0}\rangle$ is a coarse-graining of the Riemann tensor contracted with the 4-velocity, u^{μ}, which is a symmetric tensor on Σ_t. This can be compared to the coarse-graining of $Z_{(ij)}$, (4.33). There are two backreaction terms, the first, B_{ab}, takes the same form as (2.77) and we will call it the Newtonian backreaction. The second, $\tilde{B}_{ab}$, is more complicated and we will call it the relativistic backreaction. Korzyński has an extra term in the first part of (4.78) which one can show is zero, namely,

$$-\mathcal{N}\mathcal{P}^{-1}\left[4\langle Z\rangle_{cd}\delta v^{d}{}_{,e}\chi^{e}{}_{(,A}\chi^{c}{}_{,B)}\right]_{ab} = -2\delta_{cd}\langle Z\rangle_{(a}{}^{c}\mathcal{N}\left[\delta v^{d}\right]_{b)} = 0, \tag{4.79}$$

where the first and second equalities are on account of (4.61) and (4.74) respectively. Similar to the Newtonian case, both the backreaction terms take the form of a surface integral divided by a volume. However, $\tilde{B}_{ab}$, contains linear terms in the inhomogeneities. The term B_{ab} is exactly the same for an irrotational Newtonian case and hence also only the derivatives tangential to ∂D_t count. However, this must be so as δv^a is only defined on ∂D_t in $\mathbf{E}^3$. Similarly, the relativistic backreaction $\tilde{B}_{ab}$ also only involves derivatives along ∂D_t on account of the $\chi^a{}_{,A}$ terms. Therefore, whenever the inhomogeneities are perpendicular to coarse-graining domain boundary the backreaction will be zero. An example, as demonstrated in the next section, is that of the Lemaître-Tolman-Bondi model when coarse-graining over a ball with a centre on the point of isotropy.

4.8 Evolution example: The Lemaître-Tolman-Bondi model

We will now coarse-grain Z_{ij} and calculate the evolution equation for the Lemaître-Tolman-Bondi (LTB) solution. The metric in comoving coordinates is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{R'(r,t)^2}{1+2E(r)}\mathrm{d}r^2 + R(r,t)^2\mathrm{d}\Omega^2, \tag{4.80}$$

where the dash denotes a partial derivative with respect to r and $\mathrm{d}\Omega^2$ is the metric on a 2-sphere [37, 38, 39]. We can see this is also an orthogonal coordinate system so the constant time slice normal and the 4-velocity are given by

$$m^\mu = u^\mu = (1,0,0,0). \tag{4.81}$$

We have the following;

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & p_{ij} \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & p^{ij} \end{bmatrix}, \tag{4.82}$$

where

$$p_{ij} = \begin{bmatrix} \frac{R'^2}{1+2E} & 0 \\ 0 & R^2\Omega_{\Theta\Phi} \end{bmatrix}, \quad p^{ij} = \begin{bmatrix} \frac{1+2E}{R'^2} & 0 \\ 0 & R^{-2}\Omega^{\Theta\Phi} \end{bmatrix}, \tag{4.83}$$

where

$$\Omega_{\Theta\Phi} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}, \quad \Omega^{\Theta\Phi} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{bmatrix}. \tag{4.84}$$

We will now coarse-grain over a spherical volume, C_t, enclosed by $r = r_0$ and define $R_0(t) \equiv R(r_0, t)$, $R_0'(t) \equiv R'(r,t)|_{r=r_0}$, $E_0 \equiv E(r_0)$. The spacelike outward pointing normal to this surface is given by

$$\tilde{n}_i = \frac{\partial_i r}{(p^{jk}\partial_j r \partial_k r)^{\frac{1}{2}}} = \left(\frac{R_0'}{\sqrt{1+2E_0}}, 0, 0\right), \tag{4.85}$$

so

$$\tilde{n}^i = p^{ij}\tilde{n}_j = \left(\frac{\sqrt{1+2E_0}}{R_0'}, 0, 0\right). \tag{4.86}$$

The projection tensor is then

$$q_{ij} = p_{ij} - \tilde{n}_i\tilde{n}_j = \begin{bmatrix} 0 & 0 \\ 0 & R_0^2\Omega_{\Theta\Phi} \end{bmatrix}. \tag{4.87}$$

We will let $\theta^A = \{\theta, \phi\}$ be the coordinates on ∂C_t so that the tube is described by

$$y^i = \xi^i(\theta^A) = (r_0, \theta, \phi). \tag{4.88}$$

The metric induced on ∂C_t is then

$$q_{AB}(t, \theta^A) = p_{ij}\xi^i{}_{,A}\xi^j{}_{,B} = R_0^2(t)\Omega_{AB}, (\theta^A) \tag{4.89}$$

therefore the inverse is

$$q^{AB} = \frac{1}{R_0^2}\Omega^{AB} \tag{4.90}$$

We will now attempt to find an isometric embedding of ∂D_t in $\mathbf{E}^3$. We will try the embedding

$$x^a = \chi^a(t, \theta^A) = \begin{bmatrix} R_0(t)\sin\theta\cos\phi \\ R_0(t)\sin\theta\sin\phi \\ R_0(t)\cos\theta \end{bmatrix}, \tag{4.91}$$

which is a 2-sphere with radius $R_0(t)$. This gives

$$\chi^a{}_{,A} = \begin{bmatrix} R_0\cos\theta\cos\phi & -R_0\sin\theta\cos\phi \\ R_0\cos\theta\sin\phi & R_0\sin\theta\cos\phi \\ -R_0\sin\theta & 0 \end{bmatrix} \tag{4.92}$$

and one can show

$$\delta_{ab}\chi^{a}{}_{,A}\chi^{b}{}_{,B} = \begin{bmatrix} R_0^2 & 0 \\ 0 & R_0^2\sin^2\theta \end{bmatrix} = R_0^2\Omega_{AB}. \tag{4.93}$$

Therefore, the embedding (4.91) is isometric and is unique up to rigid transformations. One can easily show that the normal to ∂D_t in Σ_t is

$$n_a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{4.94}$$

and the boundary velocity is

$$v^a = \chi^{a}{}_{,t} = \begin{bmatrix} \dot{R}_0\sin\theta\cos\phi \\ \dot{R}_0\sin\theta\sin\phi \\ \dot{R}_0\cos\theta \end{bmatrix} = \frac{\dot{R}_0}{R_0}x^a. \tag{4.95}$$

We can then coarse-grain Z_{ij}, noting $N^i = 0$ because we are working in orthogonal coordinates. Thus, the antisymmetric part is zero and

$$\langle Z\rangle_{ab} = \frac{1}{V_0}\int_{\partial D_t} v_a n_b \,\mathrm{d}\sigma = \frac{\dot{R}_0}{R_0}\frac{1}{V_0}\int_{\partial D_t} x_a n_b \,\mathrm{d}\sigma = \frac{\dot{R}_0}{R_0}\delta_{ab}, \tag{4.96}$$

so $v^a = \langle Z\rangle^{a}{}_{b}x^b \Rightarrow \delta v^a = 0$.

The push of Z_{ij} to Σ_t is given by

$$Z_{ab} = Z_{ij}\left(\xi^{i}{}_{,I}\chi^{c}{}_{,C}q^{IC} + \tilde{n}^i n^c\right)\left(\xi^{j}{}_{,J}\chi^{d}{}_{,D}q^{JD} + \tilde{n}^j n^d\right)\delta_{ac}\delta_{bd}. \tag{4.97}$$

We have

$$Z_{ij} = \tfrac{1}{2}\partial_t p_{ij} = \begin{bmatrix} \frac{R'\dot{R}'}{1+2E} & 0 \\ 0 & R\dot{R}\Omega_{\Theta\Phi} \end{bmatrix} \tag{4.98}$$

and

$$\xi^{i}{}_{,A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.99}$$

So, by using equations (4.97), (4.98), (4.99), (4.90) and (4.86), one can show

$$Z_{ab} = \frac{\dot{R}_0}{R_0}\delta_{ab} + \left(\frac{\dot{R}_0{}'}{R_0'} - \frac{\dot{R}_0}{R_0}\right)n_a n_b = \langle Z\rangle_{ab} + \delta Z_{ab}, \tag{4.100}$$

where,

$$\delta Z_{ab} = \left(\frac{\dot{R}_0{}'}{R_0'} - \frac{\dot{R}_0}{R_0} \right) n_a n_b. \tag{4.101}$$

We can now evaluate the backreaction terms and we find

$$B_{ab} = 0 \tag{4.102}$$

because $\delta v^a = 0$ and

$$\tilde{B}_{ab} = 0 \tag{4.103}$$

on account of $n_a \chi^a{}_{,A} = 0$.

We will now calculate the coarse-grained Riemann tensor contracted with the 4-velocity (4.76). One can show that the relevant Riemann tensor components are [40]

$$R_{\theta t \theta t} = -R\ddot{R}, \tag{4.104}$$

$$R_{\theta t \phi t} = R_{\phi t \theta t} = 0, \tag{4.105}$$

$$R_{\phi t \phi t} = -R\ddot{R}\sin^2\theta. \tag{4.106}$$

Using (4.99) and then (4.10) we obtain

$$R_{i0j0}\xi^i{}_{,A}\xi^j{}_{,B} = \begin{bmatrix} -R_0\ddot{R}_0 & 0 \\ 0 & -R_0\ddot{R}_0\sin^2\theta \end{bmatrix} = -R_0\ddot{R}_0\Omega_{AB} = -\frac{\ddot{R}_0}{2\dot{R}_0}\partial_t q_{AB}, \tag{4.107}$$

so

$$\langle R_{0a0b} \rangle = -\frac{\ddot{R}_0}{\dot{R}_0}\mathcal{N}\mathcal{P}^{-1}\left[\partial_t q_{AB}\right]_{ab} = -\frac{\ddot{R}_0}{\dot{R}_0}\langle Z \rangle_{ab} = -\frac{\ddot{R}_0}{R_0}\delta_{ab}. \tag{4.108}$$

Therefore, equation (4.75) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{R}_0}{R_0}\delta_{ab}\right) = -\left(\frac{\dot{R}_0}{R_0}\delta_{ac}\right)\left(\frac{\dot{R}_0}{R_0}\delta^c{}_b\right) - \left(-\frac{\ddot{R}_0}{R_0}\delta_{ab}\right) + 0 + 0, \tag{4.109}$$

which one can easily confirm is true.

4.9 Discussion

The fact that the backreaction is always zero for the LTB model might come as a surprise to some. That is because when using the Buchert averaging formalism (see §5.2), the backreaction, in general, is not zero. To compare the two methods note the average expansion in both methods is the time derivative of the volume divided by the volume, that is,

$$_{\mathrm{K}}\langle Z\rangle^a{}_a = \frac{\dot{V}_{D_t}}{V_{D_t}} \quad \text{and} \quad _{\mathrm{B}}\langle Z^a{}_a\rangle = \frac{\dot{V}_{C_t}}{V_{C_t}}, \tag{4.110}$$

where the K and B denote the Korzyński and Buchert averages respectively. So the difference is that Buchert's method uses the actual volume of the averaging domain but Korzyński's method uses the volume of the embedded domain in $\mathbf{E}^3$. For the LTB model with a origin centred spherical domain discussed in the previous section these are

$$V_{C_t} = 4\pi \int_0^{r_0} \frac{R'R^2}{\sqrt{1+2E}}\,\mathrm{d}r \tag{4.111}$$

and

$$V_{D_t} = \frac{4\pi R_0^3}{3}. \tag{4.112}$$

We can see that only when $E = 0$ that these will coincide. The spatial scalar curvature is given by [41]

$$^3R = 4\frac{[RE]'}{R'R^2} \tag{4.113}$$

So they coincide when the spatial curvature is zero. One can see that whenever space is flat the two methods will coincide as the embedding is trivial. Whether the converse is true is unknown to me, but it will be whenever the actual volume and embedded volume coincide (and their time derivatives coincide also if their difference varies in time).

CHAPTER 5

Wiltshire's timescape model

5.1 Introduction

The timescape model proposed by Wiltshire [42, 16, 43, 44, 45] is a viable alternative to the currently accepted homogeneous cosmology with smooth dark energy, the ΛCDM model. The apparent acceleration of the universe is realised primarily by a difference in the calibration of the clocks of an observer in a bound system, such as ourselves, and the time parameter used to parameterise global average evolution.

In general, in the presence of significant inhomogeneities an apparent acceleration of the average expansion is in principle possible [46] regardless of the time parameter used. In particular, if the universe consists of an ensemble of voids and walls that decelerate at different rates, then even though all regions are locally decelerating, there is a transition in the average expansion as the structures which dominate the average change. At early epochs the wall regions dominate the ensemble and the fraction of voids is tiny. However, since the voids are very underdense they decelerate at a much smaller rate and their volume increases much more rapidly than the wall regions. Any volume average is thus eventually dominated by the faster expanding voids, and at the transition to void dominance the average expansion can appear to accelerate simply because the weighting of the faster expanding voids in the average increases rapidly.

Whether such an apparent acceleration of the average at the void dominance transition is actually observationally possible then depends on the both the initial conditions and the clock that is used to determine the relative deceleration of the walls and voids. In a scheme with backreaction such as that of Buchert [15], extra ingredients are required to relate observables to the statistical averages on spatial hypersurfaces which the averaging formalism deals with, as no observations are performed directly on spatial hypersurfaces. The issue of the interpretation of the Buchert formalism, and the reality of apparent acceleration has therefore been much debated, with some researchers arguing strongly that it is unlikely for reasonable initial conditions [47].

Wiltshire has responded to these challenges by revisiting the fitting problem [10, 11] from first principles [16, 48], and has developed a particular reinterpretation of physical observables in the Buchert formalism. Wiltshire argues that the relative volume deceleration of the walls and voids should play a physical role in the relative calibration of ideal clocks. Essentially, there are gravitational energy gradients related to spatial curvature gradients, and estimates of cosmological observables are affected not only by the average expansion of the universe but by the variance of local geometry from average geometry in the relative calibration of rulers and clocks.

In the timescape scenario Wiltshire finds that for realistic initial conditions a volume-average observer – namely one whose local spatial curvature matches the global average spatial curvature on a spatial hypersurface – will infer no cosmic acceleration, in accord with the arguments of Ishibashi and Wald [47]. However, on account of the cumulative integrated effects of a very tiny relative volume deceleration the time parameter used by observers in bound systems eventually differs appreciably from the time parameter used to define the Buchert average. What appears as deceleration according to one clock, can therefore appear as apparent acceleration when measured by a different clock.

The relative clock rate is directly related to the growth of spatial curvature gradients, as quantified by the fraction of the total volume in voids, f_{v}. The onset of apparent acceleration then depends on the void fraction reaching a critical value [16] $f_{\mathrm{v}} \simeq 0.59$, which typically occurs at a redshift $z \simeq 0.9$. Wiltshire's timescape scenario therefore provides a quantitative resolution of the *cosmic coincidence problem*. In the ΛCDM model it is a puzzle as to why the value of the cosmological constant is such that acceleration should have only begin in the recent past; it requires that the energy densities in ordinary matter and in vacuum energy should be roughly matched, whereas at more typical epochs the vacuum energy will eventually dominate. In the timescape scenario there is no vacuum energy, and the onset of apparent acceleration directly coincides with the epoch when the nonlinear structures associated with voids begin to dominate in determining the large-scale distribution of galaxies, as can be verified from galaxy surveys.

The timescape scenario therefore provides an interesting alternative to the standard ΛCDM model, and there are a number of observational tests which should make it possible to distinguish it from the standard homogeneous cosmology [45].

5.2 The Buchert averaging formalism

The Buchert averaging formalism [15, 23] is the most widely used averaging procedure used in general relativity. The averaging procedure itself is actually fairly trivial, that is, it is just a volume average on a spatial hypersurface. It is the choice of the spatial hypersurface and the evolution of the hypersurface and average quantities that Buchert develops that is the bulk of the formalism. We will now introduce the formalism for an irrotational geodesic dust fluid.

We will work in comoving orthogonal coordinates in the ADM gauge as outlined in §3.4 and preceding sections. The Buchert average of a scalar,

Ψ, over some domain C_t, on a spatial constant time slice Σ_t, at time t, is defined as the volume average of that scalar on that domain [15],

$$\langle\Psi\rangle_{C_t} = \frac{1}{V_{C_t}} \int_{C_t} \Psi \sqrt{|p|}\, \mathrm{d}^3 y, \tag{5.1}$$

where

$$V_{C_t} = \int_{C_t} \sqrt{|p|}\, \mathrm{d}^3 y \tag{5.2}$$

is the volume of C_t and p_{ij} is the 3-metric on Σ_t. One can then calculate the time derivative of the averaged quantity, remembering y^i are comoving coordinates and the averaging domain C_t is constant with respect to them.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi\rangle_{C_t} = &\frac{1}{V_{C_t}} \int_{C_t} \left(\tfrac{\partial}{\partial t}\Psi\sqrt{|p|} + \Psi\tfrac{\partial}{\partial t}\sqrt{|p|}\right) \mathrm{d}^3 y \\ &- \frac{1}{V_{C_t}} \int_{C_t} \tfrac{\partial}{\partial t}\sqrt{|p|}\, \mathrm{d}^3 y \times \frac{1}{V_{C_t}} \int_{C_t} \Psi\sqrt{|p|}\, \mathrm{d}^3 y.\end{aligned} \tag{5.3}$$

One can show via Jacobi's formula and then equation (3.90) that

$$\tfrac{\partial}{\partial t}\sqrt{|p|} = \tfrac{1}{2}\sqrt{|p|}p^{ij}\tfrac{\partial}{\partial t}p_{ij} = \sqrt{|p|}\theta \tag{5.4}$$

so that

$$\langle\theta\rangle_{C_t} = \frac{\dot{V}_{C_t}}{V_{C_t}} \tag{5.5}$$

and (5.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi\rangle_{C_t} = \langle\dot{\Psi}\rangle_{C_t} + \langle\Psi\theta\rangle_{C_t} - \langle\theta\rangle_{C_t}\langle\Psi\rangle_{C_t}, \tag{5.6}$$

where one should note the covariant derivative along u^μ, denoted by an overdot, of a scalar is equal to the partial derivative with respect to t, in these coordinates. Equation (5.6) is the Buchert commutation rule [23] which is analogous to the commutation rule in Newtonian cosmology (2.54). We can now apply the commutation rule to scalar quantities constructed from the Einstein equations for a universe filled with dust, assumed to be comoving with observers who measure proper time t. The application to equations (3.16) combined with (3.22) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\theta\rangle = \tfrac{2}{3}\langle\theta^2\rangle - \langle\sigma^2\rangle - \langle\theta\rangle^2 - 4\pi G\langle\rho\rangle, \tag{5.7}$$

the application to equation (3.13) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\rho\rangle = -\langle\theta\rangle\langle\rho\rangle, \tag{5.8}$$

and finally to the energy constraint (3.96) yields

$$\langle\mathcal{R}\rangle + \tfrac{2}{3}\langle\theta^2\rangle - \langle\sigma^2\rangle = 16\pi G\langle\rho\rangle, \tag{5.9}$$

where $\mathcal{R}$ is the scalar spatial curvature.

We can now define the volume-average scale factor, $\bar{a} = \left[\mathcal{V}(t)/\mathcal{V}(t_0)\right]^{1/3}$. One can then show using equations (5.2) and (5.4) that $\frac{\dot{\bar{a}}}{\bar{a}} = \frac{1}{3}\langle\theta\rangle$. We may then cast equations (5.7), (5.8) and (5.9) into the standard Buchert equations for a dust fluid [15],

$$\frac{3\dot{\bar{a}}^2}{\bar{a}^2} = 8\pi G\langle\rho\rangle - \tfrac{1}{2}\langle\mathcal{R}\rangle - \tfrac{1}{2}\mathcal{Q} \tag{5.10}$$

$$\frac{3\ddot{\bar{a}}}{\bar{a}} = -4\pi G\langle\rho\rangle + \mathcal{Q} \tag{5.11}$$

$$\dot{\langle\rho\rangle} + 3\frac{\dot{\bar{a}}}{\bar{a}}\langle\rho\rangle = 0 \tag{5.12}$$

where

$$\mathcal{Q} = \tfrac{2}{3}\left(\langle\theta^2\rangle - \langle\theta\rangle^2\right) - \langle\sigma^2\rangle \tag{5.13}$$

is the kinematic backreaction, a function of the expansion scalar, θ, and the shear scalar, $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}$. It denotes the departure from the standard Friedmann equations.

5.3 Buchert equations of the timescape model

In the timescape model [42, 45] Wiltshire considers a particular ensemble of structures - walls and voids - which matches well those in the observed universe, and after a careful consideration of the interpretation of observable quantities in the Buchert averaging formalism, a fit of observational data is performed. The large scale Buchert average is assumed to apply to the entire horizon volume, which at any epoch is given by $\mathcal{V} = \mathcal{V}_0\bar{a}^3$, represented by the disjoint union of walls and voids such that

$$a_{\mathrm{w}} = \left(\frac{\mathcal{V}_{\mathrm{w}}(t)}{\mathcal{V}_{\mathrm{w}}(t_0)}\right)^{\frac{1}{3}} \text{ and } a_{\mathrm{v}} = \left(\frac{\mathcal{V}_{\mathrm{v}}(t)}{\mathcal{V}_{\mathrm{v}}(t_0)}\right)^{\frac{1}{3}} \tag{5.14}$$

are the wall and void scale factors respectively where $\mathcal{V}_{\mathrm{w}}$ and $\mathcal{V}_{\mathrm{v}}$ are the volumes inside the horizon volume that are composed of wall and void regions respectively. We have $\mathcal{V} = \mathcal{V}_{\mathrm{w}} + \mathcal{V}_{\mathrm{v}}$ which gives,

$$\bar{a}^3 = f_{\mathrm{w0}} a_{\mathrm{w}}^3 + f_{\mathrm{v0}} a_{\mathrm{v}}^3 \tag{5.15}$$

where $f_{\mathrm{w0}} = \mathcal{V}_{\mathrm{w}}(t_0)/\mathcal{V}(t_0)$ and $f_{\mathrm{v0}} = \mathcal{V}_{\mathrm{v}}(t_0)/\mathcal{V}(t_0)$ are the present epoch wall and void fractions respectively. At any epoch the wall and void fractions are

$$f_{\mathrm{w}} = \frac{\mathcal{V}_{\mathrm{w}}}{\mathcal{V}} = f_{\mathrm{w0}}\frac{a_{\mathrm{w}}^3}{\bar{a}^3} \text{ and } f_{\mathrm{v}} = \frac{\mathcal{V}_{\mathrm{v}}}{\mathcal{V}} = f_{\mathrm{v0}}\frac{a_{\mathrm{v}}^3}{\bar{a}^3}. \tag{5.16}$$

We then have

$$\langle\Psi\rangle_{\mathcal{H}} = f_{\mathrm{w}}\langle\Psi\rangle_{\mathrm{w}} + f_{\mathrm{v}}\langle\Psi\rangle_{\mathrm{v}}, \tag{5.17}$$

for any scalar Ψ. The reader is directed to [49] if a more detailed explanation of volume partitioning is sought.

In the real universe the fluid approximation breaks down. In particular, whereas voids still contain ionic dust for which the fluid approximation is valid, within the wall regions dust geodesics cross as soon as structures such as stars form, and in any realistic scheme we would effectively have to coarse grain over galaxies, which are themselves growing and evolving. The Buchert scheme does not itself address the question of what is to be understood by the dust approximation. However, for consistency one should coarse-grain at least on scales on which the mass of a dust "particle" is not changing significantly over the life of the universe. Since we must include galaxies in any realistic approximation, this means that we should coarse-grain over the largest scales on which there are no average mass flows. In reality this implies coarse-graining on scales of order $100h^{-1}$ Mpc, over fluid "particles" which are themselves expanding. The essence of Wiltshire's approach is that the dust approximation breaks down within such cells, and the solutions of the Buchert formalism require careful interpretation when related to observable quantities. The fundamental issues relating to this are discussed by Wiltshire in references [48, 12].

The coordinate systems in §3.2 are not applied exactly on the local scales of galaxy clusters, but only for the largest macroscopic cosmological averages, where it is assumed that the average evolution of the universe can be approximated as being attributed to that of irrotational dust, even if the dust approximation itself requires careful interpretation. The non-exact nature of this system means that the time parameter t is not necessarily the proper time, τ, of any observer. Wiltshire interprets this as freedom to choose the *quasilocal uniformly expanding gauge* which is described below. For further discussion on this interpretation the reader is referred to reference [12, 48].

We define the wall and void Hubble parameters measured in the global average frame by

$$H_{\mathrm{w}} \equiv \frac{1}{a_{\mathrm{w}}}\frac{\mathrm{d}a_{\mathrm{w}}}{\mathrm{d}t} = \tfrac{1}{3}\theta_{\mathrm{w}} \quad \text{and} \quad H_{\mathrm{v}} \equiv \frac{1}{a_{\mathrm{v}}}\frac{\mathrm{d}a_{\mathrm{v}}}{\mathrm{d}t} = \tfrac{1}{3}\theta_{\mathrm{v}}, \tag{5.18}$$

then

$$\bar{H} = \tfrac{1}{3}\langle\theta\rangle_{\mathcal{H}} = f_{\mathrm{w}}H_{\mathrm{w}} + f_{\mathrm{v}}H_{\mathrm{v}}. \tag{5.19}$$

It should be noted that with Wiltshire's interpretation these are not the Hubble parameters that a wall or void observer would measure locally or globally. Locally wall and void observers measure Hubble parameters, $\frac{1}{a_{\mathrm{w}}}\frac{\mathrm{d}a_{\mathrm{w}}}{\mathrm{d}\tau_{\mathrm{w}}}$ and $\frac{1}{a_{\mathrm{v}}}\frac{\mathrm{d}a_{\mathrm{v}}}{\mathrm{d}\tau_{\mathrm{v}}}$, respectively. Phenomenological lapse functions[1], $\bar{\gamma}_{\mathrm{w}}$ (or just $\bar{\gamma}$) and $\bar{\gamma}_{\mathrm{v}}$, which relate local average proper times, τ_{w} and τ_{v}, to global average

[1] With the breakdown of the dust approximation, it assumed that there is not a single global ADM foliation of the whole universe with dust defined in an identical manner within walls and voids. Thus in Wiltshire's interpretation the lapse functions here are purely phenomenological, representing a degree of freedom relating to the cumulative effect of a relative regional volume deceleration. One should not confuse these phenomenological lapse functions with that of a single ADM lapse for the whole universe. Here it is assumed that the real universe is *not* globally hyperbolic.

proper time, t, are defined such that

$$\bar{\gamma}_{\mathrm{w}} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{w}}} \quad \text{and} \quad \bar{\gamma}_{\mathrm{v}} = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathrm{v}}}. \tag{5.20}$$

A quasilocal uniformly expanding gauge is used in which the local expansion ratios are equal over the entire averaging volume, $\frac{1}{a_{\mathrm{w}}}\frac{\mathrm{d}a_{\mathrm{w}}}{\mathrm{d}\tau_{\mathrm{w}}} = \frac{1}{a_{\mathrm{v}}}\frac{\mathrm{d}a_{\mathrm{v}}}{\mathrm{d}\tau_{\mathrm{v}}} = \frac{1}{\bar{a}}\frac{\mathrm{d}\bar{a}}{\mathrm{d}t}$. The underlying idea is that despite the observed inhomogeneity there is a notion of uniformity of the regional expansion of the universe, on scales of 1 to 10 Mpc, which preserves the isotropy of the CMB. Just as the strong equivalence principle allows the freedom of choosing the first derivatives of the metric at a point, the cosmological equivalence implies that, in the smoothing problem, there is a freedom in normalising the regional volume expansion of expanding regions which now involves first derivatives of the smoothed metric. In some sense the Hubble parameter is a "gauge choice". This guarantees a choice of uniform Hubble flow deep within the scale of statistical homogeneity, thereby implicitly solving the Sandage-de Vancoulers paradox that the statistical scatter in the Hubble flow is observed to be much smaller than naïve estimates might suggest. For further discussion on this interpretation the reader is referred to references [12, 48].

It can then be shown [16] that in the absence of shear the backreaction is given by

$$\mathcal{Q} = \frac{2\dot{f}_{\mathrm{v}}^{\,2}}{3f_{\mathrm{v}}(1-f_{\mathrm{v}})}, \tag{5.21}$$

where here the overdot denotes the derivative with respect to volume-average time, t. The walls are taken to be the regions containing bound systems and are assumed to be spatially flat. The voids are taken to be the remaining empty regions which have negative spatial curvature. Wiltshire makes the simplifying assumption that the voids are defined by a single curvature scale[2]. The average curvature contains then contribution from the voids diluted by the spatially flat walls,

$$\langle \mathcal{R} \rangle_{\mathcal{H}} = f_{\mathrm{v}} \frac{6k_{\mathrm{v}0}}{a_{\mathrm{v}}^2} = \frac{6k_{\mathrm{v}0} f_{\mathrm{v}0}^{2/3} f_{\mathrm{v}}^{1/3}}{\bar{a}^2}. \tag{5.22}$$

[2]One could generalise the model by introducing additional minivoids characterised by an additional curvature scale.

Noting that (5.12) is solved by $\langle\rho\rangle_{\mathcal{H}} = \bar{\rho}_{\mathrm{M0}}/\bar{a}^3$ the independent Buchert equations (5.10) and (5.11) are then found to reduce to

$$\frac{\dot{\bar{a}}^2}{\bar{a}^2} = \frac{8\pi G}{3}\frac{\bar{\rho}_{\mathrm{M0}}}{\bar{a}^3} - \frac{k_{\mathrm{v0}} f_{\mathrm{v0}}^{2/3} f_{\mathrm{v}}^{1/3}}{\bar{a}^2} - \frac{\dot{f_{\mathrm{v}}}^2}{9 f_{\mathrm{v}}(1-f_{\mathrm{v}})} \tag{5.23}$$

and

$$\frac{\ddot{\bar{a}}}{\bar{a}} = \frac{2\dot{f_{\mathrm{v}}}^2}{9 f_{\mathrm{v}}(1-f_{\mathrm{v}})} - \frac{4\pi G}{3}\frac{\bar{\rho}_{\mathrm{M0}}}{\bar{a}^3}. \tag{5.24}$$

It can be shown that there exists a first integral of these equations [16],

$$\epsilon_i = 1 - \frac{1-f_{\mathrm{v}}}{\bar{\gamma}^2 \bar{\Omega}_{\mathrm{M}}}, \tag{5.25}$$

where $\epsilon_i \ll 1$ is a small constant determined by the initial conditions and $\bar{\Omega}_{\mathrm{M}} = \frac{8\pi G\bar{\rho}_{\mathrm{M0}}}{3\bar{H}^2\bar{a}^3}$ is the bare matter density parameter where $\bar{H} = \frac{\dot{\bar{a}}}{\bar{a}}$ is the bare Hubble parameter. This has led Wiltshire to an exact solution of the system (5.23) and (5.24) [44].

5.4 Solution and observables

To analyse the solutions of (5.23) and (5.24) we must relate the given variables to observables. As wall observers we can be assumed to measure a time very close to wall time, τ_{w}, not the global average time, t, that is explicitly in equations (5.23) and (5.24). We also only measure observables along null geodesics coming from sources that are in other wall regions.

The local average geometry of the spatially flat wall regions is given by

$$\mathrm{d}s_{\mathrm{w}}^2 = -\mathrm{d}\tau_{\mathrm{w}}^2 + a_{\mathrm{w}}^2\left[\mathrm{d}\eta_{\mathrm{w}}^2 + \eta_{\mathrm{w}}^2\mathrm{d}\Omega^2\right] = -\mathrm{d}\tau_{\mathrm{w}}^2 + \frac{(1-f_{\mathrm{v}})^{2/3}\bar{a}^2}{f_{\mathrm{w0}}^{2/3}}\left[\mathrm{d}\eta_{\mathrm{w}}^2 + \eta_{\mathrm{w}}^2\mathrm{d}\Omega^2\right], \tag{5.26}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is the standard metric on a two-sphere, and the second equality is due to (5.16). Similarly the average local average geometry of the negatively curved void regions is given by

$$\mathrm{d}s_{\mathrm{v}}^2 = -\mathrm{d}\tau_{\mathrm{v}}^2 + a_{\mathrm{v}}^2\left[\mathrm{d}\eta_{\mathrm{v}}^2 + \sinh^2\eta_{\mathrm{v}}\mathrm{d}\Omega^2\right] = -\mathrm{d}\tau_{\mathrm{v}}^2 + \frac{f_{\mathrm{v}}^{2/3}\bar{a}^2}{f_{\mathrm{v0}}^{2/3}}\left[\mathrm{d}\eta_{\mathrm{v}}^2 + \sinh^2\eta_{\mathrm{v}}\mathrm{d}\Omega^2\right]. \tag{5.27}$$

Since no actual sources are present at densities very much less than the critical density, the metric (5.27) is not directly relevant. However, (5.26) is assumed to be the relevant geometry within a finite infinity region in the walls, where sources and observers in galaxies are located. To solve the fitting problem one must then relate measurements of cosmological observables made in this geometry to quantities integrated along the null geodesics that traverse the larger distances of the global average.

The Buchert equations define statistical quantities on spatial hypersurfaces, and therefore are not directly related to any geometry. In order to construct an average geometry Wiltshire therefore takes an operational interpretation of the Buchert equations. Since all average cosmological parameters are effectively determined by performing a spherically symmetric average on the past light, any fit of a geometry to the Buchert average quantities should in involve a spherically symmetric metric. The global volume-average geometry will therefore be written

$$\mathrm{d}\bar{s}^2 = -\mathrm{d}t^2 + \bar{a}^2(t)\left[\mathrm{d}\bar{\eta}^2 + A(\bar{\eta},t)\mathrm{d}\Omega^2\right] = -\bar{\gamma}^2(\tau)\mathrm{d}\tau^2 + \bar{a}^2\left[\mathrm{d}\bar{\eta}^2 + A\mathrm{d}\Omega^2\right], \tag{5.28}$$

where we have dropped the 'w' subscript on τ and $\bar{\gamma}$ as we will no longer need to refer to times in voids. The metric (5.28) has been written as comoving for an observer who locally measures the Buchert average time parameter. The varying area factor $A(\bar{\eta},t)$ ensures that the metric is inhomogeneous. It is chosen to normalize the Buchert average on the particle horizon volume but otherwise will not play a significant role. One might be tempted to think that (5.28) represents an LTB solution due to the spherical symmetry. However, this is not an exact metric that has been substituted into the Einstein field equations and solved. Rather it is a spherically symmetric *average* metric which is fit to observational data on the null cone to match the average expansion history described by the expansion scalar which solves the Buchert average of the full inhomogeneous Einstein equations.

A fit to observational data can only be performed by observers such as ourselves who use the local wall metric (5.26), and thus a matching to the geometry (5.28) is required in order to construct a dressed global metric similar to (5.28) in terms of the scale factor ($a(\tau)$ that wall observers would infer if they assumed that the global spatial curvature matched their local spatial curvature, rather than the global average spatial curvature dominated by the voids. To account for the relative volume deceleration in the normalization of clocks we conformally match (5.26) and (5.28) on radial null geodesics. The radial null sections are not isometric, but differ by a conformal factor of $\bar{\gamma}^2$. Taking a common centre for (5.26) and (5.28) in a wall region, null radial geodesics of the two geometries coincide provided that

$$\mathrm{d}\eta_{\mathrm{w}} = \frac{f_{\mathrm{w0}}^{1/3}}{\bar{\gamma}\,(1-f_{\mathrm{v}})^{1/3}}\mathrm{d}\bar{\eta}. \tag{5.29}$$

We can use (5.29) and its integral to extend the wall metric beyond the wall regions to obtain the dressed global metric

$$\begin{aligned}\mathrm{d}s^2 &= -\mathrm{d}\tau^2 + \frac{\bar{a}^2}{\bar{\gamma}^2}\left[\mathrm{d}\bar{\eta}^2 + \frac{\bar{\gamma}^2(1-f_{\mathrm{v}})^{2/3}}{f_{\mathrm{w0}}^{2/3}}\eta_{\mathrm{w}}^2(\bar{\eta},\tau)\mathrm{d}\Omega^2\right] \\ &= -\mathrm{d}\tau^2 + a^2(\tau)\left[\mathrm{d}\bar{\eta}^2 + r_{\mathrm{w}}^2(\bar{\eta},\tau)\mathrm{d}\Omega^2\right], \end{aligned} \tag{5.30}$$

where

$$a \equiv \bar{\gamma}^{-1}\bar{a} \tag{5.31}$$

and

$$r_{\mathrm{w}} \equiv \bar{\gamma}(1-f_{\mathrm{v}})^{1/3} f_{\mathrm{w0}}^{-1/3}\eta_{\mathrm{w}}(\bar{\eta},\tau). \tag{5.32}$$

For radial null geodesics we have

$$\bar{\eta} = \int_\tau^{\tau_0}\frac{\mathrm{d}\tau}{a} = \int_\tau^{\tau_0}\frac{\bar{\gamma}\,\mathrm{d}\tau}{\bar{a}} = \int_t^{t_0}\frac{\mathrm{d}t}{\bar{a}} \tag{5.33}$$

and

$$\eta_{\mathrm{w}} = \int_\tau^{\tau_0}\frac{f_{\mathrm{w0}}^{1/3}}{\bar{\gamma}\,(1-f_{\mathrm{v}})^{1/3}}\frac{\mathrm{d}\tau}{a} = \int_\tau^{\tau_0}\frac{f_{\mathrm{w0}}^{1/3}}{(1-f_{\mathrm{v}})^{1/3}}\frac{\mathrm{d}\tau}{\bar{a}} = \int_t^{t_0}\frac{f_{\mathrm{w0}}^{1/3}}{\bar{\gamma}\,(1-f_{\mathrm{v}})^{1/3}}\frac{\mathrm{d}t}{\bar{a}}, \tag{5.34}$$

so

$$r_{\mathrm{w}} = \bar{\gamma}(1-f_{\mathrm{v}})^{1/3}\int_t^{t_0}\frac{\mathrm{d}t}{\bar{a}\bar{\gamma}\,(1-f_{\mathrm{v}})^{1/3}}. \tag{5.35}$$

The luminosity distance we should measure from the corresponding redshift, $z = \frac{a_0}{a} - 1$, is given by the dressed global metric, (5.30),

$$d_{\mathrm{L}} = a_0(1+z)r_{\mathrm{w}} = \frac{\bar{\gamma}^2(1-f_{\mathrm{v}})^{1/3}}{\bar{a}} \int_t^{t_0} \frac{\mathrm{d}t}{\bar{a}\bar{\gamma}\,(1-f_{\mathrm{v}})^{1/3}}, \tag{5.36}$$

5.5 Data fitting

To minimize the number of free parameters we have to fit, conditions are placed at the redshift of last scattering consistent with the observed CMB. The relative Hubble rate is defined by,

$$h_{\mathrm{r}} \equiv \frac{H_{\mathrm{w}}}{H_{\mathrm{v}}} = \frac{f_{\mathrm{v}}}{\bar{\gamma} - 1 + f_{\mathrm{v}}} < 1 \tag{5.37}$$

Velocity perturbations can then be fixed by demanding $1 - h_{\mathrm{ri}} \simeq 10^{-5}$ and density perturbations by restricting f_{vi}. Here the 'i' subscript denotes the initial value at the the surface of last scattering. Since bound systems have not yet formed at this epoch the walls are taken to be all the density perturbations that can be averaged to critical density while the remaining underdense regions are taken to represent the void fraction. If this remaining fraction is viewed as a single density perturbation then

$$\delta_{\mathcal{H}} \equiv \left(\frac{\delta\rho}{\rho}\right)_{\mathcal{H}\mathrm{i}} = f_{\mathrm{vi}} \left(\frac{\delta\rho}{\rho}\right)_{\mathrm{vi}} \tag{5.38}$$

where $\mathcal{H}$ denotes the perturbation with respect to our present horizon volume. Demanding $-10^{-5} \lesssim \delta_{\mathcal{H}} \lesssim -10^{-6}$ means we might take $10^{-4} \lesssim f_{\mathrm{vi}} \lesssim 10^{-2}$, depending on what values of $(\delta\rho/\rho)_{\mathrm{vi}}$ are acceptable for the nonbaryonic dark matter power spectrum.

Combining the restrictions on h_{ri} and f_{vi} with the observed redshift of the CMB, $z \simeq 1100$, the 3 initial conditions required to solve equations (5.23) and (5.24) can then be constructed.

Solving the equations and comparing the calculated luminosity distances (5.36) to observations of type Ia supernovae the remaining unknown independent parameters, $\bar{H}_0$ and $\bar{\Omega}_{\mathrm{M0}}$ can be fitted.

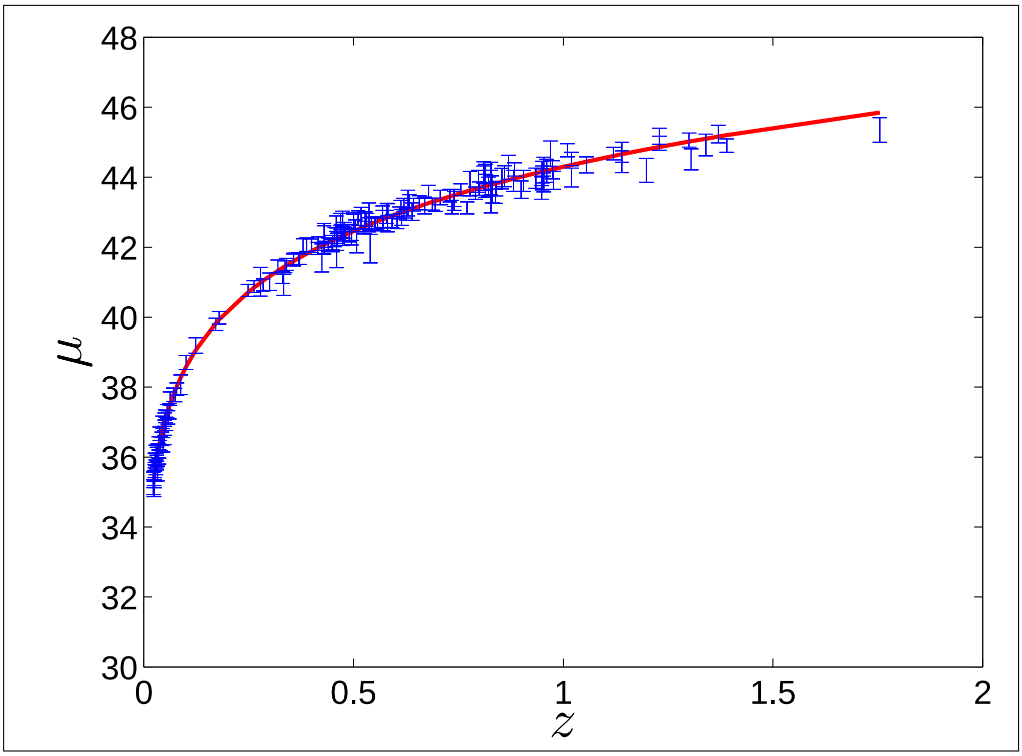

Figure 5.1: The distance modulus, $\mu \equiv 5\log_{10}(d_{\rm L}) + 25$, versus redshift, z. A reasonable fit is shown with dressed parameters $H_0 = 61.8\,{\rm km\,s^{-1}\,Mpc^{-1}}$ and $\Omega_{\rm M0} = 0.32$.

Figure 5.1 shows the distance modulus versus redshift for one example of parameters which yield a reasonable fit – with χ^2 per degree of freedom $\lesssim 1$ – to the 2007 "gold data set" of type Ia supernovae of Riess *et al.* [50]. This particular example has $\bar{\Omega}_{\rm M0} = 0.12$ and $\bar{H}_0 = 47.8\,{\rm km\,s^{-1}\,Mpc^{-1}}$. These values yield a present epoch lapse function and void fraction of $\bar{\gamma}_0 = 1.39$ and $f_{\rm v0} = 0.77$ respectively. Plots of the volume-average scale factor, the void fraction and phenomenological lapse function for this particular example are shown in 5.2 as a function of the Buchert time parameter. The bare cosmological parameters $\bar{\Omega}_{\rm M0}$ and $\bar{H}_0$ are defined with respect to the volume-average geometry (5.28), and their numerical value is somewhat different from similar quantities in the standard cosmology. It is possible to define a *conventional* density parameter and an *effective* global Hubble

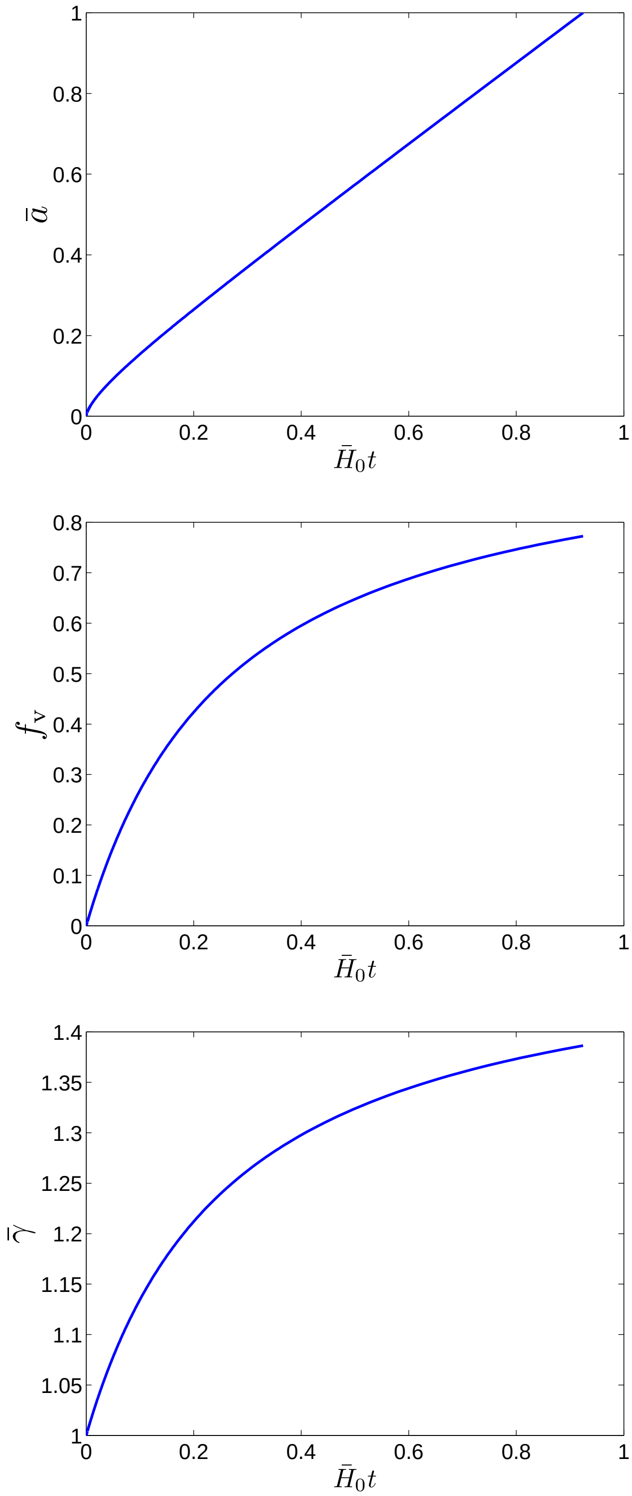

Figure 5.2: Plots of the three independent variables, volume-average scale factor, $\bar{a}$, void fraction, f_{v}, and relative clock rate of an observer in a galaxy to volume-average, $\bar{\gamma}$, against dimensionless volume-average time, $\bar{H}_0 t$, from the surface of last scattering to the the current epoch.

parameter which will be close to what one would measure with a ΛCDM model. These are the equivalent effective parameters for a wall observer who considers the metric (5.30) to be the global average metric. The two most relevant parameters are the dressed matter density parameter and the dressed Hubble parameter. The former is "dressed" by a factor $\bar{\gamma}^3$ relative to the bare parameter, and is given by

$$\Omega_\mathrm{M} \equiv \bar{\gamma}^3 \bar{\Omega}_\mathrm{M}. \tag{5.39}$$

The dressed Hubble parameter is derived by assuming the scale factor a to be the relevant global scale factor and taking time derivatives with respect to wall time so that,

$$H \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau} = \bar{\gamma}\bar{H} - \dot{\bar{\gamma}}. \tag{5.40}$$

For the particular example of Figure 5.1 the dressed parameters are $\Omega_\mathrm{M0} = 0.32$ and $H_0 = 61.8\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, which give numerical values comparable values to those expected in the ΛCDM model.

Best fit parameter values for the Riess 2007 gold data set [50] were determined by Leith, Ng and Wiltshire [51] and are very close to those given here. The void fraction and lapse function are found to be $f_\mathrm{v0} = 0.76^{+0.12}_{-0.09}$ and $\bar{\gamma}_0 = 1.381^{+0.061}_{-0.046}$ respectively. The bare Hubble and matter density parameters are $\bar{H}_0 = 48.2^{+2.0}_{-2.4}\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $\bar{\Omega}_\mathrm{M0} = 0.125^{+0.060}_{-0.069}$; while the corresponding dressed parameters[3] $H_0 = 61.7^{+1.2}_{-1.1}\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ $\Omega_\mathrm{M0} = 0.33^{+0.11}_{-0.16}$. Recently the analysis of supernova data has been greatly extended by Smale and Wiltshire [52] to consider more recent data sets. It turns out that the effects of unknown systematic uncertainties in the supernova data reduction

[3]The value of the Hubble constant fit to any supernova dataset depends on assumed normalization of the cosmic distance ladder implicit in the observed distance moduli that are published. The values of the Hubble constants given here therefore depend of a particular choice made in the Riess *et al.* [50], and should not be considered to be an absolute determination. An independent constraint on the Hubble constant can be made, however, if one demands cosmological parameters which simultaneously provide a good fit for: (i) the angular diameter distance of the sound horizon which affects the angular scale of the acoustic peaks in the cosmic microwave background anisotropy spectrum, as determined by WMAP; and (ii) the comoving distance of the baryon acoustic oscillation, as determined by measurements of galaxy clustering statistics. These measures lead to a broad constraint [51] $57 \lesssim H_0 \lesssim 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

are currently the greatest limiting factor. These include, in particular, a degeneracy between the effect of intrinsic colour variations in SneIa events and the effect of absorption by dust in the host galaxies, which is currently being investigated by astronomers. Smale and Wiltshire [52] find that one is led to different conclusions concerning the relative merits of the timescape and ΛCDM models depending on the method of light curve fitting by which the raw supernova data is reduced. For datasets reduced by the SALT and SALT-II fitters one finds that there is Bayesian statistical evidence in favour of the standard ΛCDM cosmology over the timescape cosmology, while alternatively there is Bayesian evidence in favour of the timescape cosmology over the ΛCDM cosmology for datasets which utilize the MLCS2k2 fitter. In other words, there are already enough supernova events to distinguish the two models, but the empirical treatment of supernova light curves to convert them to standard candles still needs to be understood before conclusions can be drawn.

5.6 Estimation of the effects of radiation

The assumption of a pure dust content is certainly very accurate in the late universe when matter dominates but to probe further back we must check if the effects of radiation are significant. The magnitude of the radiation density is typically calibrated against the current observed temperature, T_0, of the CMB. However an important consequence of the variance of clock rates between a wall observer and an idealised volume-average observer is that the volume-average observed temperature would be lower. This value is related to the wall temperature via the lapse function according to

$$\bar{T} = \bar{\gamma}^{-1} T, \tag{5.41}$$

at any epoch. Therefore as we measure a CMB temperature of $T_0 = 2.725\,\mathrm{K}$ with $\bar{\gamma}_0 = 1.39$ it is expected that measured from global average the CMB temperature would typically be $\bar{T}_0 = 1.975\,\mathrm{K}$. From this we can then work out a global average radiation density,

$$\bar{\rho}_{\mathrm{R0}} = \frac{\pi^2 g_*}{30} \frac{\left(k_{\mathrm{B}} \bar{T}_0\right)^4}{\hbar^3 c^5}, \tag{5.42}$$

where the degeneracy factor relevant for the standard model of particle physics, $g_* = 3.36$, is assumed. The bare radiation density parameter is then

$$\bar{\Omega}_{\mathrm{R}} = \frac{8\pi G \bar{\rho}_{\mathrm{R0}}}{3\bar{H}^2 \bar{a}^4}. \tag{5.43}$$

This allows us to add in an estimated radiation component to our solution to (5.23) and (5.24) which evolves assuming the radiation density to be zero. In addition to the bare density parameters $\bar{\Omega}_{\mathrm{M}}$ and $\bar{\Omega}_{\mathrm{R}}$ we can also define bare "density" parameters for the curvature and backreaction terms,

$$\bar{\Omega}_{\mathrm{k}} = -\frac{k_{\mathrm{v0}} f_{\mathrm{v0}}^{2/3} f_{\mathrm{v}}^{1/3}}{\bar{H}^2 \bar{a}^2} \tag{5.44}$$

$$\bar{\Omega}_{\mathrm{Q}} = -\frac{\dot{f}_{\mathrm{v}}^{\,2}}{9 f_{\mathrm{v}} (1 - f_{\mathrm{v}}) \bar{H}^2}, \tag{5.45}$$

so that equations (5.23) and (5.24) become

$$1 = \bar{\Omega}_{\mathrm{M}} + \bar{\Omega}_{\mathrm{k}} + \bar{\Omega}_{\mathrm{Q}}, \tag{5.46}$$

and

$$\bar{q} = \frac{1}{2} \bar{\Omega}_{\mathrm{M}} + 2\bar{\Omega}_{\mathrm{Q}}, \tag{5.47}$$

where $\bar{q} \equiv \frac{-\ddot{\bar{a}}}{\bar{H}^2 \bar{a}^2}$ is the volume-average deceleration parameter. Figure 5.3 shows the density parameters $\bar{\Omega}_{\mathrm{M}}$, $\bar{\Omega}_{\mathrm{k}}$ and $\bar{\Omega}_{\mathrm{Q}}$ along with an estimated $\bar{\Omega}_{\mathrm{R}}$ versus redshift. It is only a rough estimation at this stage as the effect of radiation is not included in the evolution of the model. It can be seen the magnitude of $\bar{\Omega}_{\mathrm{R}}$ surpasses that of $\bar{\Omega}_{\mathrm{Q}}$ at a redshift of $z \simeq 30$. This suggests that a Friedmann model with matter, radiation and curvature would be more accurate than the timescape model with a pure dust content from this epoch backwards. It is to be noted that $f_{\mathrm{v}} = 0.024$ and $\bar{\gamma} = 1.012$ at this epoch. It may be decided that we are satisfied by the near homogeneity at this epoch to make an approximation of joining the two models together

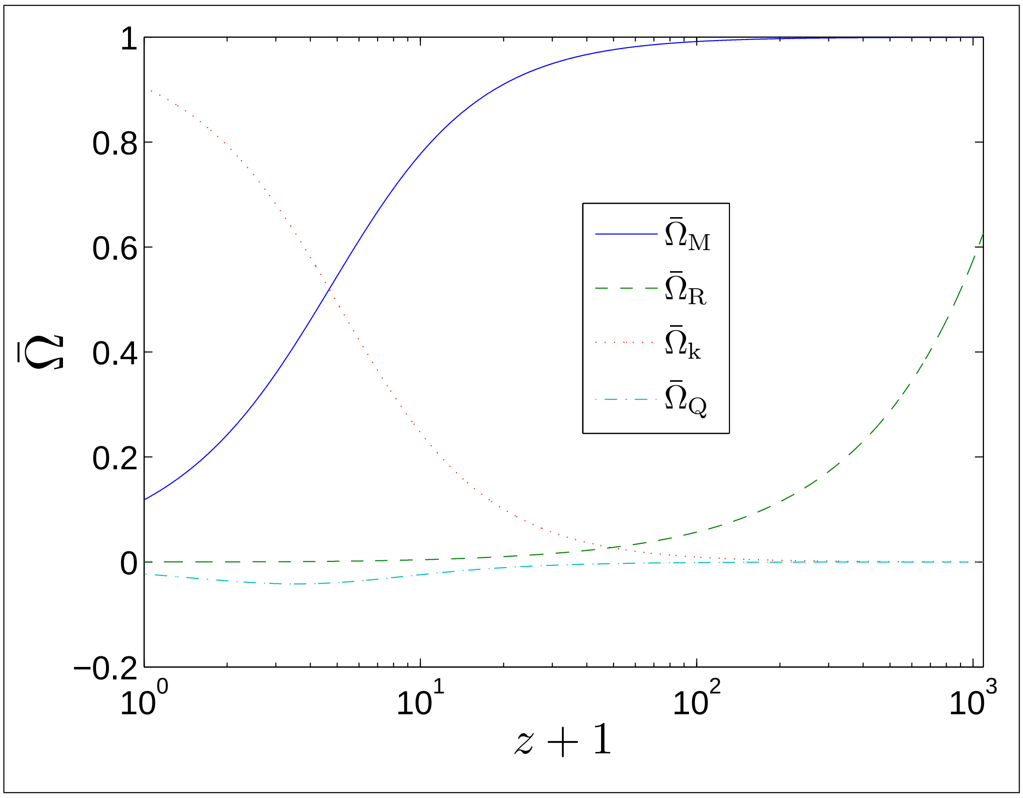

Figure 5.3: The density parameters, $\bar{\Omega}$, versus redshift, z, when system is evolved ignoring radiation.

here. However it is not clear what the initial conditions on the timescape model would be in such a case as previously the initial conditions were given at $z \simeq 1100$ which cannot be evolved forward in the Friedmann model. Thus we seek to add radiation inherently to the timescape model.

5.7 Adding radiation to the timescape model

Due to the non-clumping nature of radiation we make the assumption that it is homogeneous at the level of coarse-graining in the Buchert average. This means we do not need to consider Buchert's more generalised formalism that applies to inhomogeneous fluids with non-zero pressure gradient [23].

Buchert's equations for an inhomogeneous dust and homogeneous radiation content are therefore

$$\frac{3\dot{\bar{a}}^2}{\bar{a}^2} = 8\pi G\langle\rho_M\rangle + 8\pi G\langle\rho_R\rangle - \tfrac{1}{2}\langle\mathcal{R}\rangle - \tfrac{1}{2}\mathcal{Q} \tag{5.48}$$

$$\frac{3\ddot{\bar{a}}}{\bar{a}} = -4\pi G\langle\rho_M\rangle - 8\pi G\langle\rho_R\rangle + \mathcal{Q} \tag{5.49}$$

$$\langle\rho_M\rangle\dot{} + 3\frac{\dot{\bar{a}}}{\bar{a}}\langle\rho_M\rangle = 0 \tag{5.50}$$

$$\langle\rho_R\rangle\dot{} + 4\frac{\dot{\bar{a}}}{\bar{a}}\langle\rho_R\rangle = 0, \tag{5.51}$$

where $\rho_R = 3p_R$ has been used. In the timescape cosmology these can then be shown to take the form

$$\frac{\dot{\bar{a}}^2}{\bar{a}^2} = \frac{8\pi G}{3}\frac{\bar{\rho}_{\rm M0}}{\bar{a}^3} + \frac{8\pi G}{3}\frac{\bar{\rho}_{\rm R0}}{\bar{a}^4} - \frac{k_{\rm v0} f_{\rm v0}^{2/3} f_{\rm v}^{1/3}}{\bar{a}^2} - \frac{\dot{f}_{\rm v}^{\,2}}{9f_{\rm v}(1-f_{\rm v})} \tag{5.52}$$

$$\frac{\ddot{\bar{a}}}{\bar{a}} = \frac{2\dot{f}_{\rm v}^{\,2}}{9f_{\rm v}(1-f_{\rm v})} - \frac{4\pi G}{3}\frac{\bar{\rho}_{\rm M0}}{\bar{a}^3} - \frac{8\pi G}{3}\frac{\bar{\rho}_{\rm R0}}{\bar{a}^4}, \tag{5.53}$$

in similar fashion to (5.23) and (5.24). Equations (5.23) and (5.24) were however solvable analytically, this is now not possible. We proceed by solving (5.52) and (5.53) numerically by integrating from the surface of last scattering to the present epoch. Noting equation (29) of reference [16],

$$\dot{f}_{\rm v} = 3(1-f_{\rm v})(1-\bar{\gamma}^{-1})\bar{H}, \tag{5.54}$$

it can shown by differentiation, along with (5.52), its derivative and (5.53), that

$$\dot{\bar{\gamma}} = \bar{\gamma}\bar{H}\left[\tfrac{3}{2}\bar{\gamma}^{-1} - 1 - \tfrac{1}{2}\bar{\Omega}_{\rm M} - \left(1 - \tfrac{1}{2}\bar{\gamma}\right)\bar{\Omega}_{\rm R} - 2\bar{\Omega}_{\rm Q}\right]. \tag{5.55}$$

We can then recast equations (5.52) and (5.53) as a system of first order ODEs in dimensionless form,

$$\bar{a}' = \bar{a}\hat{\bar{H}} \tag{5.56}$$

$$f_{\rm v}' = 3(1-f_{\rm v})(1-\bar{\gamma}^{-1})\hat{\bar{H}} \tag{5.57}$$

$$\bar{\gamma}' = \bar{\gamma}\left(\frac{3}{2}\bar{\gamma}^{-1} - 1 - \frac{1}{2}\frac{\lambda}{\hat{\bar{H}}^2\bar{a}^3} - \left(1 - \frac{\bar{\gamma}}{2}\right)\frac{\phi}{\hat{\bar{H}}^2\bar{a}^4} + \frac{2(1-f_\mathrm{v})(1-\bar{\gamma}^{-1})^2}{f_\mathrm{v}}\right)\hat{\bar{H}}, \tag{5.58}$$

where

$$\hat{\bar{H}} \equiv \frac{\bar{a}'}{\bar{a}} = \sqrt{\frac{\frac{\lambda}{\bar{a}^3} + \frac{\phi}{\bar{a}^4} + \frac{f_\mathrm{v}^{1/3}}{\bar{a}^2}}{1 + \frac{1-f_\mathrm{v}}{f_\mathrm{v}}(1-\bar{\gamma}^{-1})^2}}. \tag{5.59}$$

The above derivatives are taken with respect to a dimensionless time $\hat{t} = \omega t$, where

$$\omega \equiv \sqrt{-k_{\mathrm{v}0}}f_{\mathrm{v}0}^{1/3} = \sqrt{\frac{\bar{\Omega}_{\mathrm{k}0}}{f_{\mathrm{v}0}^{1/3}}}\bar{H}_0 \simeq \bar{H}_0, \tag{5.60}$$

and the matter and radiation parameters are

$$\lambda \equiv \frac{8\pi G\bar{\rho}_{\mathrm{M}0}}{3\omega^2} = \frac{\bar{\Omega}_{\mathrm{M}0}f_{\mathrm{v}0}^{1/3}}{\bar{\Omega}_{\mathrm{k}0}} \tag{5.61}$$

and

$$\phi \equiv \frac{8\pi G\bar{\rho}_{\mathrm{R}0}}{3\omega^2} = \frac{\bar{\Omega}_{\mathrm{R}0}f_{\mathrm{v}0}^{1/3}}{\bar{\Omega}_{\mathrm{k}0}} \tag{5.62}$$

respectively.

Once conditions consistent with the observed CMB have been placed on the initial values of $\bar{a}$, f_v and $\bar{\gamma}$ and on the value of ϕ we are left we two free parameters, ω and λ, to fit to cosmological data. The equations are integrated forward in time from the initial conditions to the present epoch. Note, however, that $\bar{a}_i$ and ϕ depend on the present epoch value of the lapse, $\bar{\gamma}_0$, through equations (5.31) and (5.41) respectively. An estimate of $\bar{\gamma}_0$ must by made initially so that after a few iterations of integration it converges on an accurate value.

It is not surprising to find that the best fit parameters differ only slightly from the model without radiation as all of the supernovae data is at redshifts well within the matter dominated era. A comparison is now made between the two different models. Comparing Figure 5.4 to Figure 5.3, we see that at the surface of last scattering radiation density is quickly approaching the matter density and is expected to be equal not far beyond it. In Figure 5.5 we compare $\bar{a}$, f_v and γ for the timescape model with and without radiation

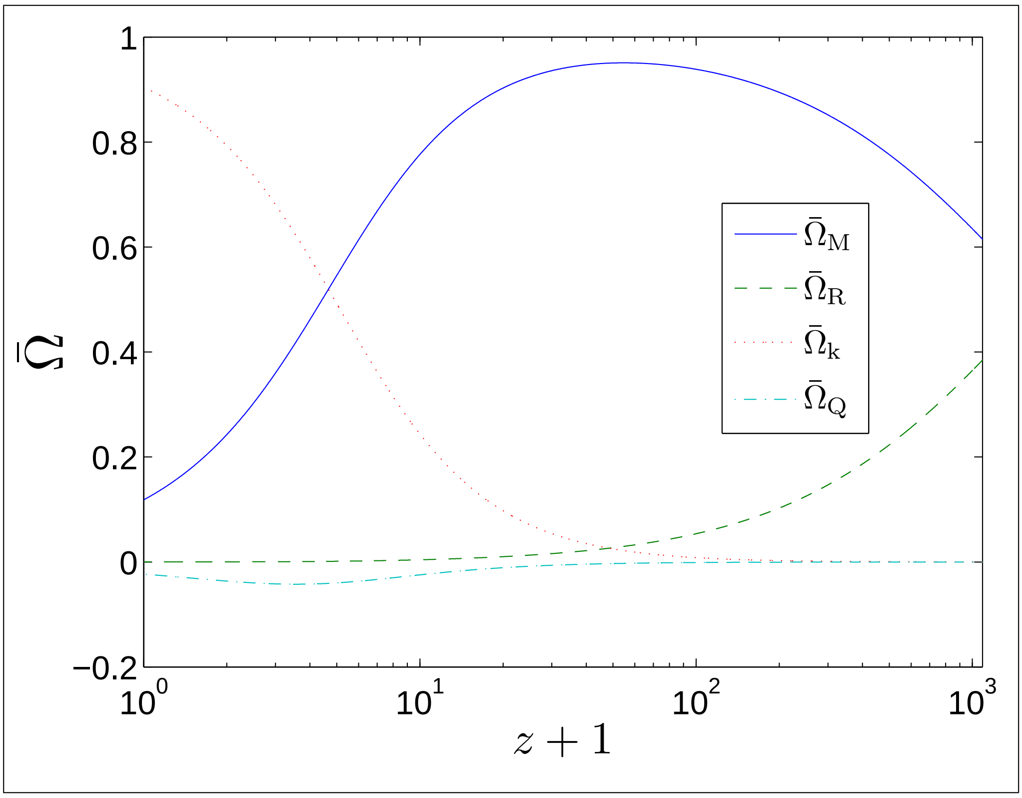

Figure 5.4: The density parameters, $\bar{\Omega}$, versus redshift, z when radiation is included in the evolution.

included. We see that only $\bar{a}$ and f_{v} are changed by any significant amount, $\bar{a}$ differing more at higher redshift as one would expect as the model moves closer to radiation dominance. The void fraction, f_{v}, differs only when it is very small, $\sim 10^3$, and not by any more than the accuracy that the initial condition is known to.

5.8 Beyond the surface of last scattering

Assuming the model is accurate to arbitrarily early time we would like to solve the equations starting with the initial conditions $f_{\mathrm{v}} \to 0$ and $\bar{\gamma} \to 1$ as $\bar{a} \to 0$ as we expect the model to approach a homogeneous one at early times, however, the equations governing the timescape model become

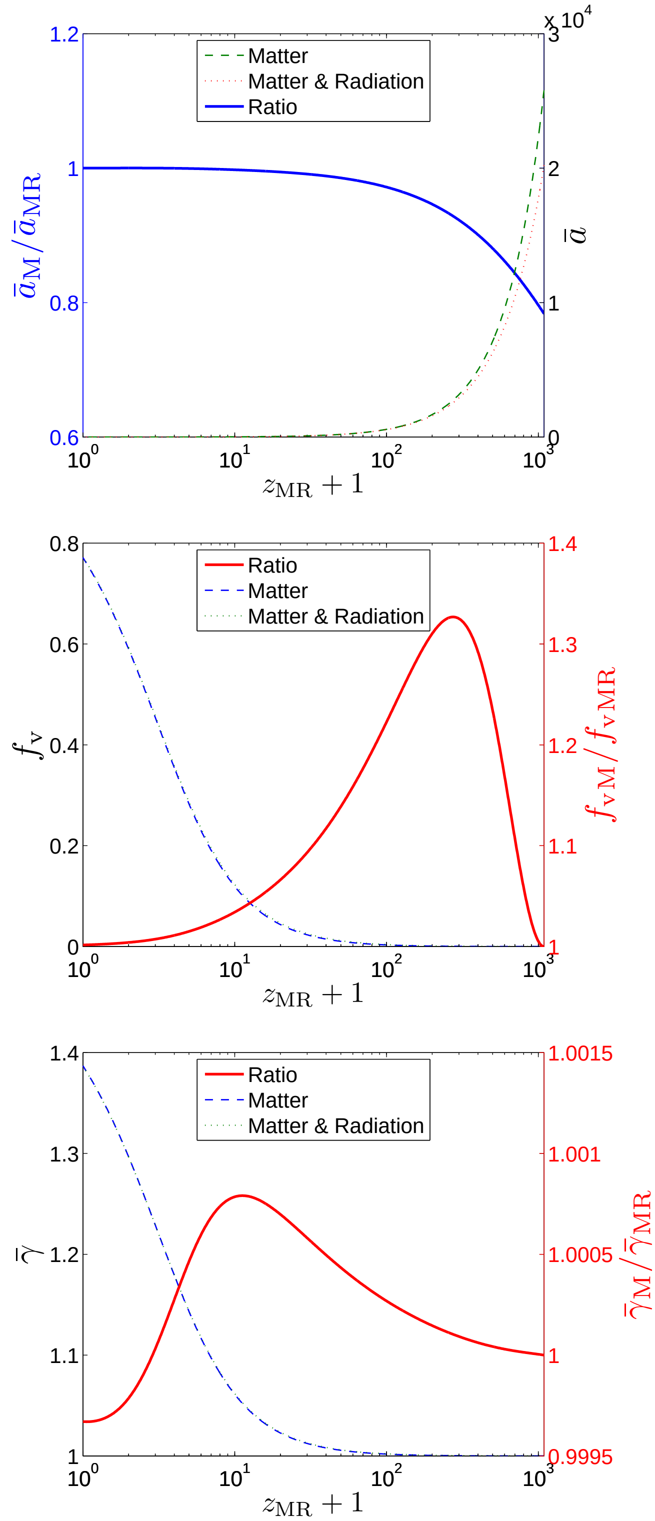

Figure 5.5: The three independent variables, volume-average scale factor, $\bar{a}$, void fraction, f_{v}, and relative clock rate of an observer in a galaxy to volume-average, $\bar{\gamma}$, against time, from the surface of last scattering to the the current epoch are plotted with and without radiation included in the model by the dashed and dotted lines respectively. The ratio of the two cases is plotted by the solid thick line. Although the two cases are plotted against time the redshift at that time for the model with matter and radiation is shown on the horizontal axis to make the graphs more viewable.

undefined at $\bar{a} = 0$ and there is no obvious transformation of variables that rectifies this problem. The timescape model without radiation does, however, have an analytic solution so we will analyse this before proceeding to numerically integrate the equations backward in time.

Equations (5.23) and (5.24) are solved by

$$\hat{t} + \hat{t}_\epsilon = \sqrt{u\left(u + C_\epsilon\right)} - C_\epsilon \ln\left(\sqrt{u} + \sqrt{u + C_\epsilon}\right) \tag{5.63}$$

$$v = D_\epsilon^{1/3}\left(\tfrac{3}{2}\hat{t}\right)^{2/3}, \tag{5.64}$$

where we have defined

$$u = f_{\rm v}^{1/3}\bar{a} \tag{5.65}$$

$$v = \left(1 - f_{\rm v}\right)^{1/3}\bar{a} \tag{5.66}$$

and $C_\epsilon = \epsilon_i \lambda$ and $D_\epsilon = \left(1 - \epsilon_i\right)\lambda$. We have set an arbitrary time origin to zero so that $v(0) = 0$ without loss of generality. It is then shown that the wall lapse function has the form

$$\bar{\gamma} = \frac{\sqrt{u^3 v^3 \frac{u + C_\epsilon}{D_\epsilon} + v^3}}{u^3 + v^3}. \tag{5.67}$$

The condition that $\bar{a}(0) = f_{\rm v}(0) = 0$ implies that $\hat{t}_\epsilon = -\frac{C_\epsilon}{2}\ln C_\epsilon$. We then see that setting $\epsilon_i = 0$ gives

$$C_\epsilon = 0 \quad \text{and} \quad D_\epsilon = \lambda, \tag{5.68}$$

which reduces equations (5.63) and (5.64) to

$$u = \hat{t} \tag{5.69}$$

$$v^3 = \tfrac{9}{4}\lambda \hat{t}^2 \tag{5.70}$$

and

$$\bar{a}^3 = \hat{t}^3 + \tfrac{9}{4}\lambda \hat{t}^2 \tag{5.71}$$

$$f_{\rm v} = \frac{\hat{t}}{\hat{t} + \frac{9}{4}\lambda} \tag{5.72}$$

$$\bar{\gamma} = \frac{\frac{3}{2}\hat{t} + \frac{9}{4}\lambda}{\hat{t} + \frac{9}{4}\lambda} = 1 + \tfrac{1}{2} f_{\rm v}. \tag{5.73}$$

This solution is then seen to be the solution that satisfies our third required initial condition $\bar{\gamma}(0) = 1$ and is known as the *tracker* solution [44]. It represents a solution in which the wall and void evolution in volume-average time is completely decoupled. The wall regions evolve in an Einstein-de Sitter manner governed by

$$\frac{\dot{a}_{\rm w}^2}{a_{\rm w}^2} = \frac{8\pi G}{3}\frac{\rho_{\rm Mw0}}{a_{\rm w}^3}, \tag{5.74}$$

where we have

$$\bar{\rho}_{\rm M} = (1 - f_{\rm v})\,\rho_{\rm Mw} + f_{\rm v}\rho_{\rm Mv}, \tag{5.75}$$

so

$$\rho_{\rm Mw0} = \frac{\bar{\rho}_{\rm M0}}{1 - f_{\rm v0}}, \tag{5.76}$$

given $\rho_{\rm Mv} = 0$. The void regions evolve in the manner of a Milne universe governed by

$$\frac{\dot{a}_{\rm v}^2}{a_{\rm v}^2} = \frac{-k_{\rm v0}}{a_{\rm v}^2}. \tag{5.77}$$

The relative expansion rate in this solution is constant $h_{\rm r} = 2/3$, the void regions expands one and a half times as fast as the wall regions in volume-average time (the same in local time).

One may think that a similar solution exists for the timescape model with radiation. However, the radiation exists in both the wall and void regions so it couples the two regions together. A reasonable ansatz for a solution would take the form

$$\frac{\dot{a}_{\rm w}^2}{a_{\rm w}^2} = \frac{8\pi G}{3}\rho_{\rm Mw} + \frac{8\pi G}{3}\rho_{\rm Rw} = \frac{8\pi G}{3}\frac{\rho_{\rm Mw0}}{a_{\rm w}^3} + \frac{8\pi G}{3}\frac{\bar{\rho}_{\rm R0}}{\bar{a}^4} \tag{5.78}$$

$$\frac{\dot{a}_{\rm v}^2}{a_{\rm v}^2} = \frac{-k_{\rm v0}}{a_{\rm v}^2} + \frac{8\pi G}{3}\rho_{\rm Rv} = \frac{-k_{\rm v0}}{a_{\rm v}^2} + \frac{8\pi G}{3}\frac{\bar{\rho}_{\rm R0}}{\bar{a}^4}, \tag{5.79}$$

the second equalities are on account of the homogeneity of radiation, $\rho_{\rm Rw} = \rho_{\rm Rv} = \bar{\rho}_{\rm R} = \bar{\rho}_{\rm R0}\bar{a}^{-4}$. These equations are coupled via (5.15) but can be shown to solve (5.52), they do not, however, solve (5.53).

Using l'Hôpital's rule as $\bar{a} \to 0$, $f_{\rm v} \to 0$ and $\bar{\gamma} \to 1$, equation (5.54) can be seen to take the form

$$\dot{f}_{\rm v} \to \frac{3\dot{\bar{\gamma}}}{1 + \bar{q}}. \tag{5.80}$$

where equation (5.47) becomes

$$\bar{q} = \tfrac{1}{2}\bar{\Omega}_{\rm M} + \bar{\Omega}_{\rm R} + 2\bar{\Omega}_{\rm Q} \tag{5.81}$$

in the presence of radiation. Combining the above with the use of l'Hôpital's rule on

$$h_{\rm r} = \frac{f_{\rm v}}{\bar{\gamma} - 1 + f_{\rm v}} \tag{5.82}$$

we obtain

$$h_{\rm r} \to \frac{3}{4 + \bar{q}} \tag{5.83}$$

under the above limit assuming $\dot{f}_{\rm v} \nrightarrow 0$. Thus for a pure matter content $h_{\rm r} \to 2/3$ as $\bar{q} \to 1/2$. For a model containing radiation, however, $\dot{f}_{\rm v} \to 0$, therefore, equation (5.83) is not valid in this case. Rather, one can use l'Hôpital's rule on $\frac{\dot{f}_{\rm v}}{\bar{a}}$ to show $\frac{\dot{f}_{\rm v}}{\bar{a}} \to \frac{\dot{\bar{\gamma}}}{\bar{a}}$, then using l'Hôpital's rule again on (5.82) to show $h_{\rm r} \to 1/2$.

This is in contradiction to assumption that $h_{\rm r} \to 1$ as velocity perturbations disappear at early times. Since the universe certainly is homogeneous at early epochs, we do expect that $h_{\rm r} \to 1$. Thus, it would appear that the defining relationship between $h_{\rm r}$, $f_{\rm v}$ and $\bar{\gamma}$ needs to be rectified at early epochs. Equations (5.54) and (5.82) were determined at late epochs with structure formation.

We now proceed by integrating backwards in time from the initial conditions placed at the surface of last scattering to see if they produced predicted results. Interestingly, integrating backwards in time the solution becomes non-physical, $f_{\rm v} \to 1$ and $\bar{\gamma}$ drops below 1 as shown in Figure 5.6. Again this suggests that relations (5.54) and (5.82) require modification at early times.

We could attempt to integrate the equations from a very early time as we know ${\bar{a}_{\rm i}}^{2\prime} \to 2\sqrt{\phi}$ by virtue of $\bar{\Omega}_{\rm R} \to 1$ and $\bar{\gamma}' \to f_{\rm v}'$. However, the problems with the $h_{\rm r} \to 1$ limit suggest that the first priority is to clarify the definition of the phenomenological lapse function, $\bar{\gamma}$, at early times. Before cosmic structure starts to form, it is assumed that the universe is globally hyperbolic, that means there will be not be a difference in clock

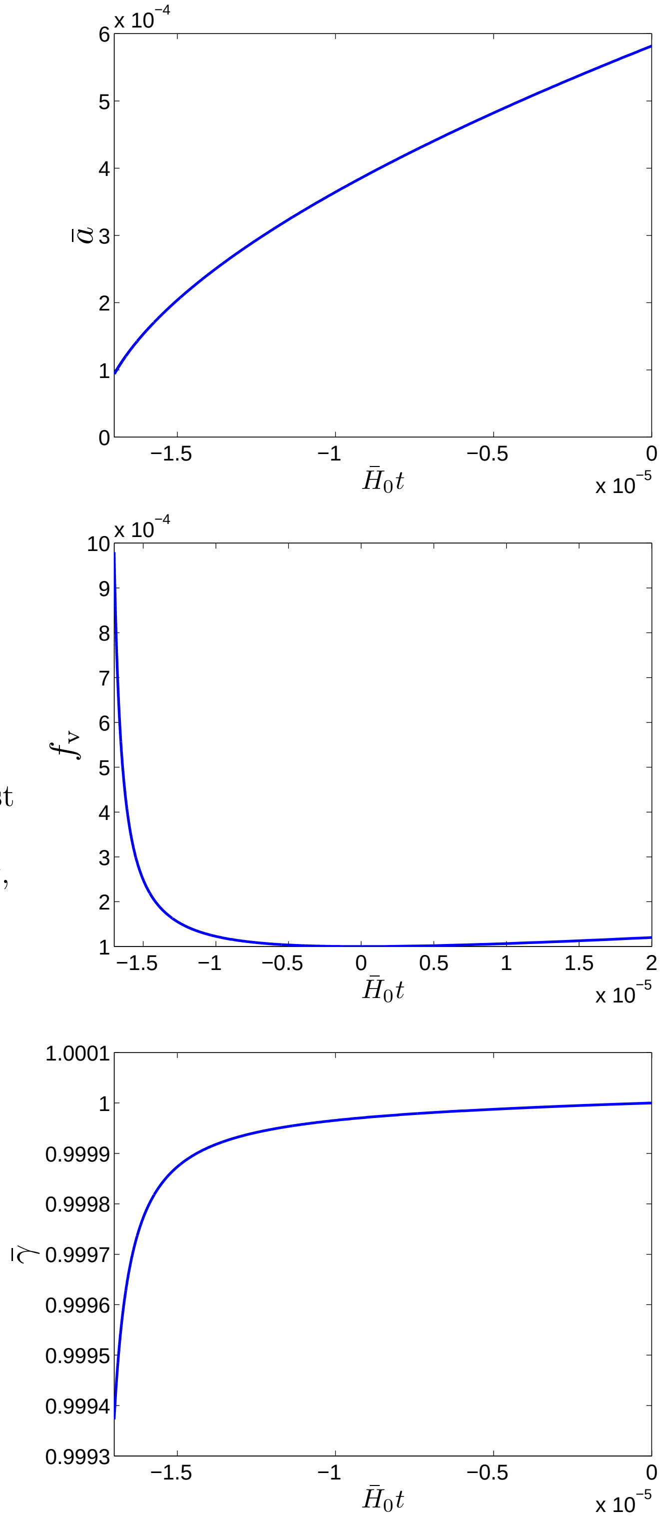

Figure 5.6: Plots of the three independent variables, volume-average scale factor, $\bar{a}$, void fraction, f_{v}, and relative clock rate of an observer in a galaxy to volume-average, $\bar{\gamma}$, against dimensionless volume-average time, $\bar{H}_0 t$, from the surface of last scattering backwards in time.

rates between the overdense and underdense regions that later form the wall and void regions respectively, unless there are spatial gradients in the radiation density. In a sense, a better set of initial conditions might be setting $\bar{\gamma} = 1$, and hence $h_{\rm r} = 1$, at some redshift when structure formation starts. The initial void fraction $f_{\rm v}$ could be possibly determined by using perturbation methods up to this stage as they should be valid up to this stage as non-linear structure has not formed. Using perturbation methods may also give slight spatial gradient in the radiation density and hence give a lapse slightly different from unity that one could use as an initial condition. We have not pursued this point, and leave it for further work.

5.9 Discussion

It is apparent the effects of radiation are only really noticeable at redshifts above 30, at this point it is to an extent questionable whether the two scale model is accurate as this is before galaxy clusters which define the walls have formed. At this point $\bar{\Omega}_{\rm Q} \approx 0.5\%$ so a homogeneous model with radiation from this epoch backwards in time would be very accurate anyway. Forwards of this epoch the late time *tracker* solution of the timescape model without radiation becomes very accurate. Note should be taken when using a homogeneous model at early times assuming the timescape model at late times of the fact that the present epoch lapse function, $\bar{\gamma}_0$, will increase observed redshifts and temperatures by that factor. This fact alone means that any calculations relating to the early universe that use the temperature of the CMB would need to be redone.

Wiltshire [16] has already performed such a recalibration in determining the angular diameter distance of the sound horizon relevant to the fitting of the CMB anisotropy spectrum, and in calibrating the baryon-to-photon ratio which is used to determine light element abundances that arise from primordial nucleosynthesis and, when combined with other cosmological parameters, the overall ratio of baryonic to nonbaryonic dark matter. (Bestfits values of such parameters were determined by Leith, Ng and Wiltshire [51].) The explicit inclusion of radiation is unlikely to change either of these estimates. However, it may be significant in determining other features of the full spectrum of acoustic peaks in the CMB anisotropy spectrum. Therefore, it is important that the outstanding issues we have uncovered should be resolved.

CHAPTER 6

Summary

6.1 Chapter 2: *Newtonian cosmology*

In Chapter 2, we started by reviewing the existing formulation of Newtonian gravity. This consisted of reviewing the governing equations and how they are ill-posed without boundary conditions and was followed by a discussion of the gauge freedoms associated with this lack of boundary conditions. We then gave the transport equations for the kinematical quantities, θ, σ_{ab} and ω_{ab}. Original work was then performed showing how to calculate the tidal tensor given boundary conditions for it. We then discussed how to solve the system in terms of the kinematical quantities, given an evolution equation for the tidal tensor. This followed by construction of the homogeneous case and then the homogeneous and isotropic case.

We then reviewed the averaging procedure set out by Buchert [14] and the evolution by deriving the commutation rule (2.54). This was then applied to the transport equations to construct their averaged forms. Following this, original calculations were performed to average the tidal tensor in terms of boundary conditions for it. It was shown to take the form of a double integral over the boundary and that one can always choose the average tidal tensor to take a specific value (e.g., zero) with a certain choice of boundary conditions.

We then examined Korzyński's work [13] and showed how the evolution of the average quantities differed from the the homogeneous case, which was shown to involve a backreaction term that took the form of a surface integral of the velocity inhomogeneities over the boundary. We then studied Buchert's scheme [14] and showed how the evolution of the average quantities differed from the the homogeneous and isotropic case, which was shown to involve a similar backreaction term. Finally, we showed that if the velocity inhomogeneities only varied perpendicularly to the boundary, the backreaction would be zero.

6.2 Chapter 3: *Congruences and the splitting of spacetime*

In Chapter 3, we started by reviewing the kinematical description of spacetime, including the transport equations for dust. Analogies were drawn with Newtonian gravity, notably that the general relativistic case presents a well-posed Cauchy problem, whereas the Newtonian case does not, and the finite propagation speed that gives $H_{\mu\nu}$ which is coupled with the tidal tensor.

We then reviewed the 3+1 split of spacetime and some useful coordinate systems. This was followed by a review of hypersurfaces and then ADM gauge later specialising to the previously described coordinate systems.

6.3 Chapter 4: *Coarse-graining in general relativity*

In Chapter 4, we started by reviewing Korzyński's motivation for his procedure. This followed by setting out a list of properties that, in my opinion, a coarse-graining procedure should satisfy. We then presented Korzyński's coarse-graining procedure for the velocity gradient with more detail than given in reference [13]. The procedure was then applied to the Bianchi I universe. (Korzyński does not explicitly do this example but states the result.)

Following Korzyński, we then developed the evolution equation for the coarse-grained velocity gradient. We showed that part of the relativistic backreaction term Korzyński gives is zero. That was followed by original calculations of applying the procedure to the Lemaître-Tolman-Bondi model. A comparison was made with Buchert's averaging formalism and it was shown that the two coincide when the spatial curvature is zero.

A lot of work remains to be done in applying Korzyński's method. An obvious generalisation of our calculation in §4.8 is to perform the coarse-graining over a non-centred spherical region of the Lemaître-Tolman-Bondi model to see what the backreaction is, which would, in general, be non-zero. A more ambitious program would be to construct an evolution equation for rotational dust, and generalise the procedure to perfect fluids with non-zero pressure. However, in my opinion a better coarse-graining procedure could be found that satisfies all of the properties described in §4.2. Potential aspects of such a procedure are outlined in §6.5.

6.4 Chapter 5: *Wiltshire's timescape model*

We began Chapter 5 by reviewing the Buchert averaging formalism for dust. We then reviewed the timescape model and the manner in which the Buchert averaging formalism is applied to derive the equations governing the average evolution of its expansion history. Following that, we described how observables are related to variables of the timescape equations. We reviewed the fitting of supernovae data, and the manner in which the radiation density has been estimated in work by Wiltshire [16].

Original calculations were then performed to systematically add radiation to the cosmic evolution. It was observed that the radiation began to dominate over the backreaction at $z \approx 30$. We extended the dust timescape model by adding a homogeneous radiation fluid into the evolution equations. These do not have a known analytical solution, so an explicit system of first-order ODEs was derived and solved numerically. This then allowed a more accurate calculation of the radiation density, along with the rest of the density parameters at redshifts above 30.

We then sought to solve the model beyond the surface of last scattering. Firstly, we reviewed the analytical solution for the timescape model without radiation, which exists back to the singularity, because, ideally, we want a solution with $f_{\mathrm{v}} \to 0$ and $\bar{\gamma} \to 1$ as $\bar{a} \to 0$. An attempt was made at an analogous analytical solution for the timescape model with radiation, however, it was shown to be incorrect. It was then shown that the relative Hubble rate had the limit $h_{\mathrm{r}} \to 1/2$, as opposed to $h_{\mathrm{r}} \to 2/3$ for the model without radiation. This is a contradiction to the assumption that $h_{\mathrm{r}} \to 1$ as velocity perturbations disappear at early times. This potentially means that aspects of the timescape model break down at early times. In particular the definition of the phenomenological lapse function to the void fraction may require revision. We then proceeded to integrate backwards in time from our initial condition at the surface of last scattering. The solution became non-physical which also suggests this.

We saw that the effects of radiation are only really noticeable at redshifts above 30, at this point it is, to an extent, questionable whether the two scale model is accurate as this is before galaxy clusters, which define the walls, have formed. At this point $\bar{\Omega}_{\mathrm{Q}} \approx 0.5\%$, so a homogeneous model with radiation from this epoch backwards in time would be very accurate anyway. We also saw that at the surface of last scattering, where initial conditions for the timescape model are placed, is well within the region that radiation has an effect. Thus, if one is using the timescape model without radiation, the initial conditions become questionable so one should just use the solution that all initial conditions tend to, that is, the tracker solution. They should also just use a homogeneous model beyond $z \approx 30$.

Until the problem of finding an analytic solution with radiation has been solved, the aforementioned method is the best compromise, as the initial conditions at the surface of last scattering are not completely consistent.

6.5 Proposed coarse-graining procedure

I will now briefly outline my views as to how some improvements might be made to the coarse-graining procedure in light of the investigations I have conducted in this thesis.

Let us start with what we know how to do properly, that is, firstly, the coarse-graining of scalars. The coarse-grained value of a scalar, A, on domain D in a manifold M is defined as

$$\langle A \rangle = \frac{1}{V_D} \int_D A \, \mathrm{d}V, \tag{6.1}$$

where $V_D = \int_D A \, \mathrm{d}V$ is the volume of D. So, in the limit of a scalar field, our procedure should give this.

Secondly, what we know how to do properly is coarse-graining any rank tensor in Euclidean space in Cartesian coordinates; we just volume average the components of the tensor. This gives the coarse-grained tensor in the orthonormal basis that is used everywhere on the domain. But what are we really doing here? Can we volume average the components in any coordinate system in Euclidean space to give a valid result? The answer is no: the coordinates we coarse-grain in must be a linear deformation of Cartesian coordinates. Another way of looking at it is that we can only volume average a tensor's components in a basis that has been parallel transported everywhere. There is a constant transformation matrix that will transform all these parallel transported bases to orthonormal bases. This same transformation will put the coarse-grained tensor components in the same orthonormal basis.

Now, how do we generalise this to curved space? We want to parallel transport a basis around M, but how do we do it? To do this we need a connection. Moreover, it must be a metric connection so it does not distort our basis as we parallel transport it. This leaves us with a few options, the first obvious one is the Levi-Civita connection. With respect to this connection, parallel transport, in general, is path dependent, so how do we choose a unique way of getting a basis at each point? Well, one way is to choose a special point on M, O say. Now we can choose a basis at O and then parallel transport it to every point on D along the geodesic connecting the point and O. We can then volume average component wise with respect to our parallelly transported coarse-graining basis,

$$\langle T\rangle^{ab\cdots}{}_{cd\cdots} = \frac{1}{V_D}\int_D T^{ab\cdots}{}_{cd\cdots}\,\mathrm{d}V, \tag{6.2}$$

where $T^{ab\cdots}{}_{cd\cdots}$ are the component of T in our parallelly transported coarse-graining basis. Given coordinates on M, x^μ, the unique parallel propagator $P^\mu{}_\sigma(x^\alpha)$, and the chosen coarse-graining basis vectors at O, $\mathring{\hat{e}}_{(a)} = \mathring{e}^\mu{}_a \mathring{\hat{e}}_{(\mu)}$ the coarse-graining basis at any point on M is then $\hat{e}_{(a)} = e^\mu{}_a \hat{e}_{(\mu)}$ where $e^\mu{}_a = P^\mu{}_\sigma \mathring{e}^\sigma{}_a$. Note, if we chose $\mathring{\hat{e}}_{(a)}$ to be an orthonormal basis, then our coarse-graining basis is a tetrad field. Equation (6.2), in terms of the coordinates, x^μ is then

$$\langle T\rangle^{ab\cdots}{}_{cd\cdots} = \frac{1}{V_D}\int_D e^{-1\,a}{}_\mu e^{-1\,b}{}_\nu \cdots e^\sigma{}_c e^\rho{}_d \cdots T^{\mu\nu\cdots}{}_{\sigma\rho\cdots}\sqrt{|g|}\,\mathrm{d}^n x. \tag{6.3}$$

Now, does this definition satisfy our condition given in §4.2? One can show that it satisfies 1, 2, 3, 5, 4, 6 and 7. Property 8 will also be satisfied if O is the same for every subdomain with $\oplus$ being the ordinary $+$.

The choice of the Levi-Civita connection was not a requirement, there is another one with relevant properties: the Weitzenböck connection [53] from teleparallel gravity. This metric connection is flat but unlike the Levi-Civita connection has non-zero torsion. The flatness of this connection means that parallel transport is path independent. This means we do not need to pick a special point to propagate geodesics out to all the other points. We can define our basis anywhere on M and propagate relative to that point and our answer will be the same as if we had chosen any other point, provided the starting bases at those points were parallel. This method will satisfy 1, 2, 3, 5, 4, 6 and 7. It will also satisfy 8, provided the bases on the subdomains are chosen so that they are parallel and $\oplus$ being the usual addition $+$. However, the Weitzenböck connection depends on the choice of some tetrad field $\epsilon^{\mu}{}_{\hat{a}}$ by [53]

$$\Gamma^{\sigma}{}_{\mu\nu} = g^{\sigma\rho}\eta_{\hat{a}\hat{b}}\epsilon^{\hat{a}}{}_{\rho}\partial_{\nu}\epsilon^{\hat{b}}{}_{\mu}. \tag{6.4}$$

Therefore it is not actually unique so does not strictly satisfy 1 of §4.2 unless there is a unique tetrad field on M. One could construct a tetrad field by parallel propagating with the Levi-Civita connection as described above and the coarse-graining procedure would be the same. It is only when there is some unique tetrad field on M is given that that procedure with the Weitzenböck connection becomes useful.

The method using the Levi-Civita connection turns out to be very similar to the smoothing procedure of Isaacson [54]. Isaacson, who was primarily interested in gravitational wave perturbations, was looking at smoothing tensors and integrated over the whole manifold, but had a weighting function in the integral that went to zero at some distance radially from the point O. This was calculated with O at every point on the manifold. This gives a smoothed tensor field once one assigns the value of the weighted average with O at the point under consideration. This method works well smoothing tensors on manifold[1]. It does not, however, smooth the manifold, that is, the smoothed metric is the same as the unsmoothed metric. This may or may not be a problem depending on the task one is trying to perform. Other people have attempted to generalise this method to smooth the manifold, the canonical example being that of Zalaletdinov [31, 32], and another example is that of Brannlund, Hoogen, and Coley who use the Weitzenböck connection for parallel propagation also [33].

These methods outlined above sketch potential ways for further developing our understanding of coarse-graining in general relativity. They should

[1]A procedure similar to Isaacson's, but more closely adapted to cosmological averages, has recently been proposed by Green and Wald [55]

then be applied to general relativity, in particular a spatial slice in some 3+1 split of spacetime and their time evolutions analysed in a similar manner to Buchert's formalism [15, 23].

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