Inferring Channel Bed Topography from Known Free Surface Data

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Abstract: *The knowledge of bed topography is important to study the hydrodynamics of open channel flows, fluvial hydraulics, or flood propagation to name but a few. Known channel bed topography allows one to computationally predict the free surface profile in different flow scenarios and predict critical quantities such as flood coverage. The direct measurement of the bed topography is a time-consuming and costly activity. This motivates the present work which proposes two algorithms which can reconstruct the bed topography from a known free surface elevation profile. We show that this inverse problem is governed by a differential equation which is only a slight modification of the standard shallow water equation. Hence, the inverse problem can be solved in "one shot" by solving this differential equation numerically. We also show that a one dimensional riverbed can be reconstructed using a pseudo-analytical approach. These reconstruction strategies are successfully tested on a set of experimental data.*

Keywords: *shallow water flows, bed topography, free surface data, inverse analytical solution.*

1. INTRODUCTION

Flows in rivers, estuaries and flood plains are natural open channel flows. The study of these kinds of flows has been a research interest for many decades. The underlying reasons include the need to map flood inundation over flood plains and for control of floods; the need of optimal design of open channels for irrigation purposes; the need for better understanding of natural stream hydrodynamics. Such flows are described by the shallow water equations. The shallow water approximation depicts the evolution of an incompressible fluid in response to gravitational and rotational accelerations in addition to the effects of friction and the slope of the channel. It is traditionally used to describe the behaviour of water flow in rivers, estuaries, open channels, flood plains and waves in lakes or downstream of dam breaks, Arico et al (2007), Wu (2008), and Cunge et al (1980).

Free surface elevation measurements of rivers using ground based surveys along a stream are often difficult and time consuming. However, recent developments show that it is possible to measure the free surface elevation of rivers using airborne optical remote sensing technology (LiDAR) and ground based close range photogrammetry (CRP). In Smart et al (2009) and (Hilldale & Raff, 2007), the potential use of LiDAR to measure the free surface elevation of rivers is well addressed. Smart et al (2009) used the LiDAR returns of the free surface elevation to reconstruct the underlying bed topography of the Waiau River in New Zealand. (Hilldale & Raff, 2007) assess the quality of bathymetric airborne LiDAR from the perspective of creating accurate, precise and complete river bathymetry of the Yakima and Trinity river basins in the USA. On the other hand Chandler et al (2008) report the use of digital close range photogrammetry (CRP) in combination with particle image velocimetry (PIV) to measure dynamic free surface elevation of real and flooded rivers.

In the following, we call finding the free surface profile of the flow from a given river bed topography the forward problem. The corresponding inverse problem is that of reconstructing the river bed from a known free surface profile. The study of the inverse problem and its application has similar advantages to the forward problem. Because there is a growing need to efficiently and accurately represent the river channel topography with high resolution to study fluvial hydraulics, flood routing and monitor geomorphological changes, (Marks & Bates, 2000). Different techniques have been implemented to reconstruct bed topography from a known free surface; the direct approach to reconstruct substrate topography in thin film flows, Sellier (2008), and (Heining & Aksel, 2009); Automatic differentiation technique to reconstruct channel bed topography Castaings et al (2006); Variational data assimilation based optimization technique to retrieve channel topography. Honnorat et al (2007). (Roux & Dartus, 2008) have also proposed a methodology to reconstitute information about the geometry of the river from top sight based on an optimization technique.

In this study, we present numerical and pseudo-analytical algorithms to reconstruct the bed profile and use analytical, numerical, and experimental data to demonstrate their applicability. These approaches have also a potential applicability for other geophysical flows due to the fact that it is a one step and easy to implement methodology. The numerical methodology relies upon the one dimensional unsteady shallow water equations to solve steady shallow water flows. Section 2 presents the governing equations with the description of the forward and the inverse problems. Section 3 presents the numerical and analytical solution methodologies. The discretization technique and its implementation are addressed. Section 4 presents an experimental study and its comparison with the numerical and analytical solution with discussion. In Section 5, the concluding remarks are addressed.

2. GOVERNING EQUATIONS

The Saint Venant shallow water equations govern unsteady incompressible one dimensional open channel flow. These equations are derived from the Navier Stokes equations based on the following assumptions: (1) the pressure distribution in the flow is hydrostatic, (2) the effect of wind stress on the free surface is neglected, (3) the effect of shear stress on the channel bottom neglected, (4) Coriolis forces are neglected, and (5) the depth of the flow is smaller than the other length scales (shallow aspect ratio). The equations can be written in the form of continuity and momentum equations

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0 \tag{1}$$

$$\frac{\partial Q}{\partial t}+\frac{\partial (Q^2/A)}{\partial x}+gA\frac{\partial \psi}{\partial x}+gAS_f=0 \tag{2}$$

Where Q is the flow rate (m^3/sec), A is the cross-sectional area in (m^2), h is the depth of the flow, $\psi = h + z$ is the free surface elevation which is the sum of the bed topography elevation and the depth of the flow, S_f is the frictional slope (frictional resistance) and g is the acceleration due to gravity. The frictional slope S_f can be given by the Manning formula

$$S_f=\frac{n^2 Q|Q|}{R^{4/3}A^2} \tag{3}$$

Where n is the manning friction coefficient, R is the hydraulic radius ($R = A/P$) and P the wetted perimeter of the channel.

The flow parameters Q and A can also be written as functions of the flow velocity u, the flow depth h and the channel width B.

$$Q = uhB \text{ and } A = hB \tag{4}$$

The terms incorporated in the momentum equation are the flow acceleration with respect to time, the convective acceleration of the flow, the potential energy and energy loss due to channel friction.

3. NUMERICAL AND ANALYTICAL SOLUTION

The above governing equations can be solved numerically and analytically to infer the channel bed topography from a measured free surface profile in shallow water flows. We restrict ourselves to the one dimensional shallow water equations. The following sections provide a detailed analysis of the numerical and analytical solutions respectively.

3.1. Numerical Approach

The forward problem is solved by substituting the variable ψ with $h + z$ in the momentum equation and solving for the flow depth h and flow velocity u for a given bed topography z. This is because the term ψ incorporates the unknown variable h and the known variable z. However, to solve the inverse problem the given governing equations are solved for h and u simultaneously. After substituting equation (4) into (1) and (2), an upwind conservative explicit numerical technique is used to discretize the governing equations. This methodology has previously been implemented by Ying et al (2004) for the forward problem and by the present authors for the inverse problem, Gessese et al (2011). The above governing equations are discretized on the spatial and temporal domains to simulate unsteady flows; however, the steady state solution can be obtained after a few iterations given a steady flow rate and boundary conditions. Hence, steady flow rate, depth at the inlet boundary and free surface data are the parameters needed for the algorithm in the reconstruction process of channel bed topography in one dimensional flow. The discretized form of the governing equations for the inverse problem can be written as:

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}(h_i^n u_i^n - h_{i-1}^n u_{i-1}^n) \tag{5}$$

$$u_i^{n+1} = \frac{1}{h_i^{n+1}} \left\{ \begin{array}{l} h_i^n u_i^n - \dfrac{\Delta t}{\Delta x}\left(\dfrac{(h_i^n u_i^n)^2}{h_i^n} - \dfrac{(h_{i-1}^n u_{i-1}^n)^2}{h_{i-1}^n} \right) - \\ \Delta t\left(g h_i^{n+1}\left[w_1\left(\dfrac{(\psi_i - \psi_{i-1})}{\Delta x} \right) + w_2\left(\dfrac{(\psi_{i+1} - \psi_i)}{\Delta x} \right) \right] - g\,\dfrac{n^2 h_i^n u_i^n \,|\, h_i^n u_i^n \,|}{(R_i^n)^{4/3} h_i^n} \right) \end{array} \right\} \tag{6}$$

where w_1 and w_2 are weighting factors for the upwind and downwind fluxes and they are evaluated by the following expressions.

$w_1 = 1 - \sqrt{Cr_{down}}$ and $w_2 = \sqrt{Cr_{up}}$

where $Cr_{down} = \frac{\Delta t}{\Delta x}\frac{u_{i+1} + u_i}{2}$ $Cr_{up} = \frac{\Delta t}{\Delta x}\frac{u_i + u_{i-1}}{2}$; and u is the depth averaged velocity of the flow.

Equations (12) and (13) are solved iteratively and simultaneously for h and u using a Matlab script given the initial and boundary conditions depending on the problem under study. Once the depth of the flow h is determined from the above sets of equations, the bed elevation z is determined by simply subtracting h from the given free surface elevation ψ. The full details of the algorithm are presented in Gessese et al (2011).

A set of numerical experiments using the forward problem has been done, and the results show that the methodology is capable of simulating standard benchmark problems such as dam break problems with dry and wet front conditions, and steady flows over a bump. These are not reported here for the sake of conciseness.

3.2. Analytical Approach

In this study, we show that for the steady state problem, an analytical solution exists to reconstruct the bed topography in shallow water flows by simplifying equations (1) and (2). The steady state governing equations can be written as

$$\frac{dQ}{dx} = 0 \tag{7}$$

$$\frac{d(Q^2/A)}{dx} + gA\frac{d\psi}{dx} + gAS_f = 0 \tag{8}$$

Equation (7) depicts the conservation of mass in the flow showing that there exists a constant flow rate along the channel length. For a known continuous function ψ, the term $\frac{d\psi}{dx}$ is also known by differentiating with respect to the independent variable x. However, the free surface data measurement is discrete and noisy in practice and a curve fitting technique is required to obtain reliable gradients of the free surface. Simplifying equation (8) and substituting equation (7) results in

$$\frac{Q^2}{A^3}\frac{dA}{dx} = g\frac{d\psi}{dx} + gS_f \tag{9}$$

Equation (9) is a simple ordinary differential equation which can be solved by simple integration.

$$\int \frac{Q^2}{A^3}dA = \int g\left(\frac{d\psi}{dx} + S_f\right)dx \tag{10}$$

$$\frac{Q^2}{2A^2} = -g\psi - g\int S_f dx + C_0 \tag{11}$$

Where C_0 is a constant of integration to account for the indefinite integral that can be determined from the known boundary condition. For a known area of the flow inlet and steady flow rate, the term S_f can be determined.

Assume a frictionless channel, the term with integral in equation (11) vanishes, thus C_0 can be evaluated by the remaining terms.

$$C_0 = \frac{Q^2}{2A_0^2} + g\psi_0 \tag{12}$$

Substituting the value of C_0 and introducing the friction term in the equation results in

$$\frac{Q^2}{2A^2} + g\psi + \int S_f dx = \frac{Q^2}{2A_0^2} + g\psi_0 \tag{13}$$

Equation (13) is the energy equation which governs one dimensional open channel flows with kinetic energy, potential energy and loss of energy due to channel friction terms respectively. This is illustrated in the following figure. The sum of the energies at point 1 is equal to the sum of the kinetic and potential energies at point 2 and the energy lost due to friction when the fluid particles flow from 1 to 2.

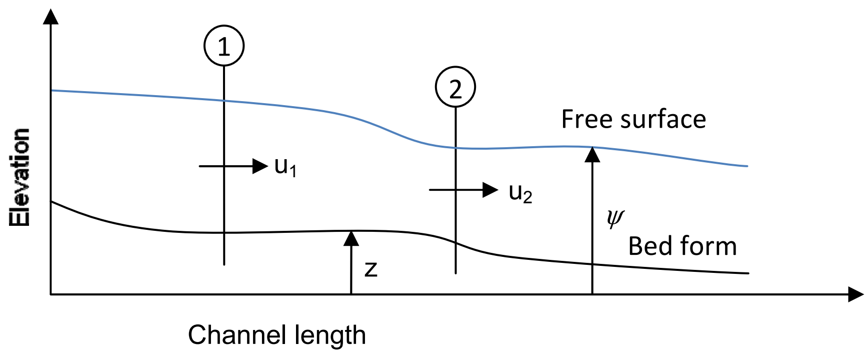

Figure 1 Description of open channel flow

The frictional energy loss between given points in a flow can be evaluated by considering an infinitesimal channel length over which we can conveniently assume a constant frictional loss. Hence, equation (13) can be rewritten between grid point (*i*) and (*i-1*) in the computational domain which relates the change in free surface elevation, kinetic energy and frictional energy loss between these grid points.

$$\left(\frac{Q^2}{2A^2}\right)_i - \left(\frac{Q^2}{2A^2}\right)_{i-1} + g\left(\psi_{(i)} - \psi_{(i-1)}\right) + g\Delta x\left(S_{f(i)} - S_{f(i-1)}\right) = 0 \tag{14}$$

Substituting equation (4) in to (14) produces the following simplified form.

$$u^2{}_i - u^2{}_{i-1} + 2g\left(\psi_{(i)} - \psi_{(i-1)}\right) + 2g\Delta x\left(S_{f(i)} - S_{f(i-1)}\right) = 0 \tag{15}$$

Thus by applying a marching approach from the upstream to the downstream boundary, we can solve for *u* and then for *h* from the continuity equation to generate an analytical solution of the channel bed topography in the one dimensional context. The results of analytical solution are compared with the experimental and numerical results in section 4.

4. EXPERIMENTAL RESULTS

A set of experiments was conducted to measure the free surface data for a given steady flow rate in a horizontal flume.The flume comprises a rectangular section of channel which is open at the top. It has clear acrylic sides which are bonded to the bed. A bed form with constant profile along its span is placed and mounted on the flat bed to give an uneven bed profile. Once a steady state flow rate is set, the free surface elevation is measured in the flow direction by point gauge. The bed topography under study has the form shown in figure 1 with a maximum elevation of 62 mm above the horizontal channel.

For given flow conditions, a steady flow rate is set at 25±0.5 litres/second, the depth of the flow at the unaffected upstream position is 144±3 mm; we measure the depth of the flow along the channel hence the free surface elevation can be determined. Figure 1 and table 1 show the stage of the flow and the measured channel bed topography from the experiment.

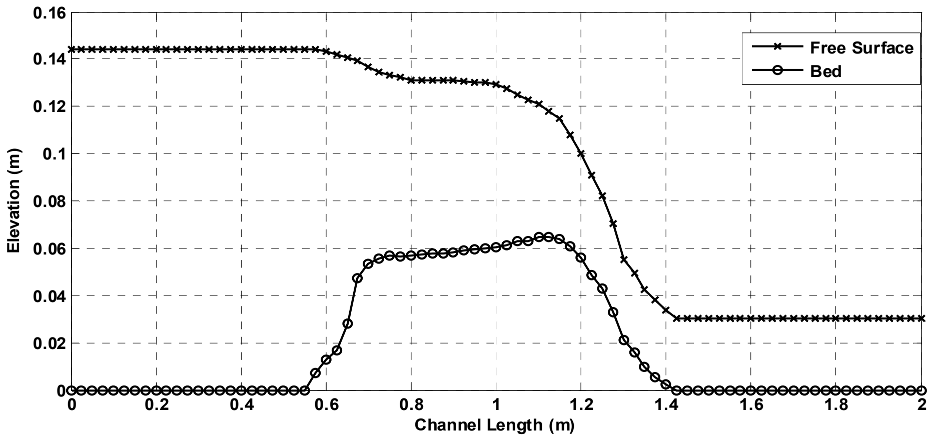

Figure 2 Free surface elevation and bed topography (flow is left to right)

Channel length (m)	Bed level (m)	Stage (m)	Channel length (m)	Bed level (m)	Stage (m)
0	0	0.144	1.1	0.0646	0.121
0.1	0	0.144	1.2	0.056	0.1
0.2	0	0.144	1.3	0.021	0.055
0.3	0	0.144	1.4	0.0025	0.0336
0.4	0	0.144	1.5	0	0.0304
0.5	0	0.144	1.6	0	0.0304
0.6	0.0128	0.143	1.7	0	0.0304
0.7	0.0532	0.1366	1.8	0	0.0304
0.8	0.057	0.131	1.9	0	0.0304
0.9	0.0584	0.131	2	0	0.0304
1	0.0604	0.1293			

Table 1 Bed and free surface elevation along the channel

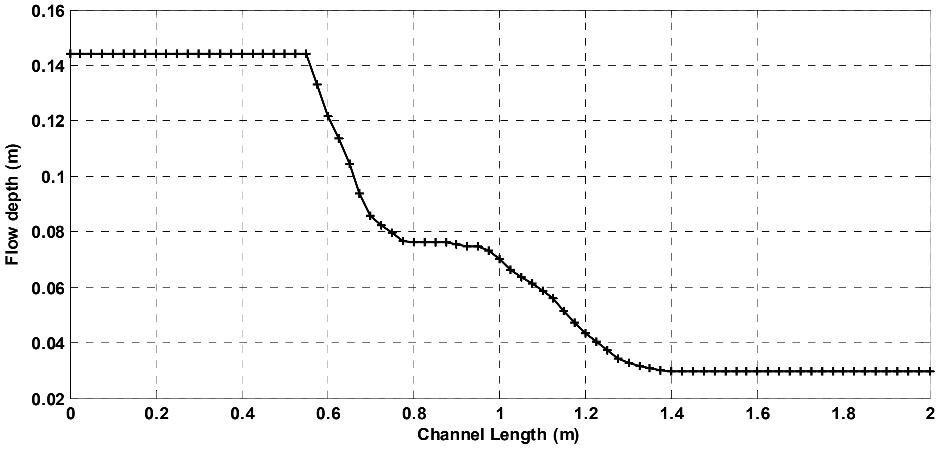

Figure 3 Flow depth variation along the channel length

The variation of depth of the flow is shown in figure 3. As can be seen from the above figures, the presence of the bed form affects the flow conditions. Downstream to the form, the flow has constant depth and it is supercritical in nature while at the upstream the flow is subcritical.

The known free surface along with the steady flow rate and boundary conditions can be used with the numerical reconstruction algorithm and analytical solution methodology to infer the channel bed topography.

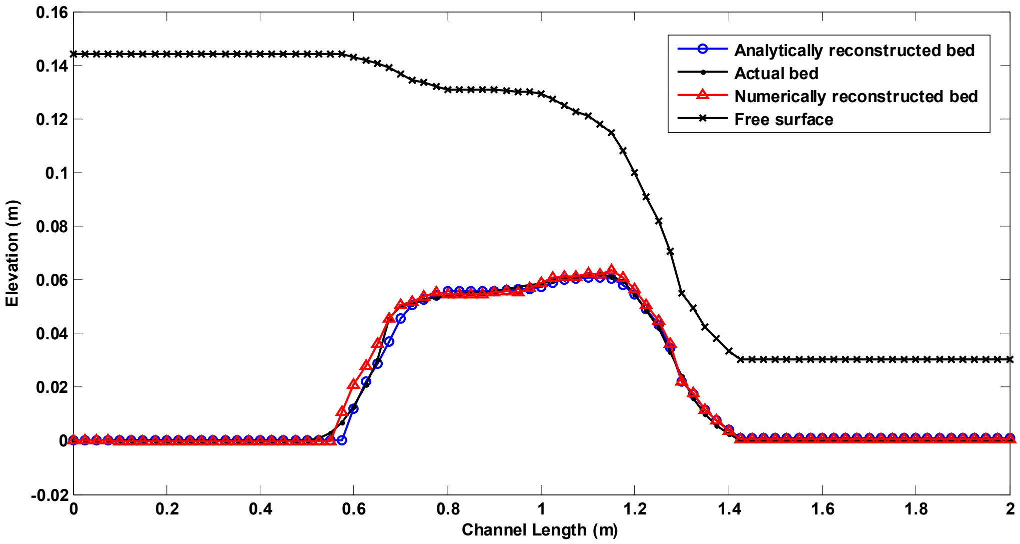

Figure 4 Comparison of numerically and analytically reconstructed bed with actual bed

As can be seen from figure 4, the numerically and analytically reconstructed channel bed topographies and actual bed topography are in a good agreement with each other. The slight difference on the left part of the topography arose from steady flow rate variations. The magnitude of the maximum difference in the reconstructed and actual bed topographies is approximately 5%. The uncertainty in the reconstructed bed topography is a consequence of the uncertainty in the steady flow rate measurement.

5. CONCLUSION

A direct numerical and analytical approach is proposed to reconstruct channel bed topography from a known free surface profile in open channel flows governed by the one dimensional shallow water equations. Inferring the bed topography requires knowledge of the flow rate and the depth of the flow at the inlet. The governing equations of the forward and inverse problems are very similar in form which allows the use of similar discretization procedures. The governing equations are conveniently integrated to give an analytical solution for steady one dimensional shallow water flow. Numerical and analytical solutions are tested against the experimental results and both results show good agreement with each other confirming that the numerical and analytical methodology is capable of reconstructing channel bed topography from measured free surface data for the one dimensional shallow flows.

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