Changes of Setting and the History of Mathematics: A New Study of Frege

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The following thesis was inspired by a series of conversations that I had with Dr Philip Catton and Dr Clemency Montelle over the summer of 2008/09. As such, this work would not have happened without them (nevertheless, I claim responsibility for any and all errors that occur within).

0.1 Abstract

This thesis addresses an issue in the philosophy of mathematics which is little discussed, and indeed little recognised. This issue is the phenomenon of a 'change of setting'. Changes of setting are events which involve a change in a scientific framework which is fruitful for answering questions which were, under an old framework, intractable. The formulation of the new setting usually involves a conceptual re-orientation to the subject matter. In the natural sciences, such re-orientations are arguably unremarkable, inasmuch as it is possible that within the former setting for one's thinking one was merely in error, and under the new orientation one is merely getting closer to the truth of the matter. However, when the subject matter is pure mathematics, a problem arises in that mathematical truth is (in appearance) timelessly immutable. The conceptions that had been settled upon previously seem not the sort of thing that could be vitiated. Yet within a change of setting that is just what seems to happen. Changes of setting, in particular in their effects on the truth of individual propositions, pose a problem for how to understand mathematical truth.

Thus this thesis aims to give a philosophical analysis of the phenomenon of changes of setting, in the spirit of the investigations performed in Wilson (1992) and Manders (1987) and (1989). It does so in three stages, each of which occupies a chapter of the thesis:

1. An analysis of the relationship between mathematical truth and setting-changes, in terms of how the former must be viewed to allow for the latter. This results in a conception of truth in the mathematical sciences which gives a large role to the notion that a mathematical setting must 'explain

itself' in terms of the problems it is intended to address.

2. In light of (1), I begin an analysis of the change of setting engendered in mathematical logic by Gottlob Frege. In particular, this chapter will address the question of whether Frege's innovation constitutes a change of setting, by asking the question of whether he is seeking to answer questions which were present in the frameworks which preceded his innovations. I argue that the answer is yes, in that he is addressing the Kantian question of whether alternative systems of arithmetic are possible. This question arises because it had been shown earlier in the 19^{th} century that Kant's conclusion, that Euclid's is the only possible description of space, was incorrect.

3. I conclude with an in-depth look at a specific aspect of the logical system constructed in Frege's *Grundgesetze der Arithmetik*. The purpose of this chapter is to find evidence for the conclusions of chapter two in Frege's technical work (as opposed to the philosophical). This is necessitated by chapter one's conclusions regarding the epistemic interdependence of formal systems and informal views of those frameworks.

The overall goal is to give a contemporary account of the possibility of setting-changes; it will turn out that an epistemic grasp of a mathematical system requires that one understand it within a broader, somewhat historical context.

1. SETTINGS IN MATHEMATICS: A HISTORICO-PHILOSOPHICAL OVERVIEW

This chapter sets forth a philosophical framework for the investigations which follow in chapters 2 and 3. The purpose of this framework is twofold:

- It seeks to explain why certain mathematical 'settings' - conceptual views on formal frameworks - seem more accommodating to certain ideas, both in terms of approaching subject matter and identifying that subject matter. This results in an approach to the philosophy of mathematics which centers on the language of mathematics, thus encompassing both the formal nature of actual mathematical investigation and the conceptual nature of historical investigations.

- It seeks to take the reflections carried out in the first investigation and using them to set up a historical methodology for an in-depth analysis of the work of a specific thinker, Gottlob Frege, and the change of setting which his work engendered.

Throughout I will emphasise the distinction between the formal language of mathematical symbolism and its informal counterpart, conceived as a metalanguage. These reflections are used in an investigation into both the conceptual and formal relationships between old and new settings.[1] It turns out that a semantic conception of mathematical truth best explains the foregoing conclusions while at the same time avoiding the pitfalls of traditional Platonism.

That a semantic conception of mathematical truth serves to explain how we can have *a priori* knowledge of mathematical truth whilst at the same time accommodating the phenomenon of setting changes is the main conclusion of the first part of chapter one. This then serves as the basis of a framework for historical investigation. This framework will emphasise the need to place a thinker in the intellectual climate of their time. With regard to the investigation of Frege's change of setting which will make up chapters 2 and 3, this shift in emphasis involves locating him in the context of the 19th-century tripartite division of the philosophy of mathematics into transcendental idealism (descended from Immanuel Kant), pre-Hilbert Formalism, and the Empiricism of John Stuart Mill. Such a reading of Frege reveals aspects of his thought which have not been fully appreciated in (all of) the literature on the subject.

While this kind of recognition of historical context is not a new desideratum in the History and Philosophy of Science, my reasons for adopting it are not the same as those often put forth by other thinkers. More often than not, this methodological principle is motivated by concerns of the historian. My reasons revolve around the conclusions reached in the non-historical, preliminary part of this work; they are *philosophical*, and are underwritten by reflections on the epistemology of mathematics. It is essential to a change of setting that it be able to explain its presence in terms of what has gone before; thus the historical context of a change of setting (such as Hilbert's work in axiomatic geometry, or

[1] The terms 'old' and 'new' are slightly misleading—no value judgement is intended when one setting is described as 'old' and another as 'new'; temporal occurrence is merely being used to mark one from the other. It is quite possible that an old setting may be superior to a new one. In such cases, the change of setting may be regarded as 'unsuccessful'.

Frege's innovations in mathematical logic) is just as vital to our understanding of the resulting system as its formal aspects. In this, the history of the philosophy of mathematics goes hand in hand with contemporary research into the subject.

1.1 Introduction

In the history of mathematics, there are many instances of statements appearing resistant to (dis)proof for long periods of time, until what may be termed 'a change of setting' takes place. Problematic theorems, which may be the product of conceptual difficulties or technically impoverished approaches, can take on a new, more tractable form. An example is Riemann's claim that elliptic functions exhibit more comprehensible behaviour when defined over a 'doughnut' of complex values, as opposed to the complex plane.[2] The complex doughnut is a new setting for elliptic functions—while not originally conceived in terms of such contexts, these functions are *reconceived* in terms which make them easier to understand and predict. In terms of the progress of mathematics as a science, this is *prima facie* a good thing. The mathematician Philip Davis notes how

> ...in mathematics there is a long and vitally important record of impossibilities being broken by the introduction of structural changes. (1987; cited in Wilson [1992], p.150)

However, from a philosophical point of view, this 'vitally important' phenomenon raises a problem. Mathematical truth is usually taken to be eternal and, as such, independent of events which occur in the physical world. When a change of setting takes place, it is often the case that while some statements in the original context benefit from the change, others that were true (maybe trivially so) will become false. One example of just this is that of moving from an affine geometrical setting to a projective one; the statement 'parallel lines never meet' is falsified. Thus, the idea that mathematical impossibilities can be 'broken' chal-

[2] Wilson (1992), p.151

lenges the supposed permanence, the Platonic timelessness, of mathematical truth.

In this discussion, I will focus on one aspect of this 'breaking of impossibilities'. The change of setting will only *fruitfully* break impossibilities if such changes result in what can be thought of as a more 'natural' conception of the problems which lead to the need for the changes in the first place. Now, in order for this to be the case, it needs to be the *same* question which is being addressed in the new setting as in the old one. What we have here is a new appearance of the old question as to what constitutes the difference between change *within* a framework, and a change *of* framework. Not only this, but it must also be the case that the conception of the new setting is a non-trivial intellectual accomplishment.

With these points in mind, in §1.2 I will give an example of a change of setting rendering possible new mathematical results. §1.3 will begin a philosophical analysis of this phenomenon, beginning with a treatment of the relation between the roles played by deductive rigour and conceptual clarity in constituting mathematical knowledge. Section 1.4 will sketch a philosophical view of the nature of mathematical knowledge which can incorporate, and account for, the conclusions of §§1.2 and 1.3. This philosophical approach will then, in chapters two and three, be brought to bear on a historical investigation of the contributions made by Gottlob Frege to the change of setting within the field of mathematical logic which occurred in the period from the late 1800's to the early 1900's.

1.2 Example of a Setting-Change: Hilbert and the Metamathematical Analysis of the Parallel Postulate

In 1904 David Hilbert published the standard-setting book *The Foundations of Geometry*. Although before this publication, results were known which are seen as leading up to it,[3] this volume contains what is probably the first comprehensive axiomatic application of what has come to be known as 'model-theoretic' techniques. Hilbert sought to answer a question which had been concerning mathematicians since the time of the ancient Greeks: what exactly is the logical relationship between Euclid's parallel postulate and his other postulates and common notions?[4] This question had become a more pressing since the early 1800's, for two reasons: firstly, Kant's doctrine of geometry as synthetic *a priori*[5] offered an impressive answer to the question of the applicability of geometry to experience. However, secondly, it was regarded as implying that Euclidean geometry (with the parallel postulate) is the *only* system of geometry—an implication that was regarded as intolerable by most accomplished mathematicians in Hilbert's time.

In order to conduct his investigation, Hilbert made two important innovations. The first is the notion of an implicit definition; the second is his technique of proving the independence of one statement from a collection of others. Both of these are fundamentally linked to the notion of an axiom. Traditionally, an axiom is taken to be a kind of *truth*, usually one which is 'self-evident'—either not requiring, or incapable of, proof. As such, axioms are the starting-point

[3] In axiomatics, mainly by the Italian school associated with Guiseppe Peano (see Toretti (1978), pp.223-227); in modeling non-Euclidean geometry, by Beltrami, Klein and Poincaré (see ibid p.125–137; p.169).

[4] From here on referred to by the modern term 'axioms'.

[5] While the actual arguments for Kant's conclusion will not be related at this stage, they will be related in the discussion of Frege which is to follow.

for any rigorous mathematical investigation; Euclid himself set the standard in his *Elements*. However, there had always been an element of suspicion regarding one of Euclid's axioms, the parallel postulate (also known as Euclid's 'fifth axiom'). The parallel postulate states that:

> If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (Heath (1956) Vol.I, p.155[6])

This is equivalent to the statement that, given a line and a point not on that line, there is one and only one line that runs through that point which does not intersect with that line. Geometrical systems developed in the 1820's, utilising the method of Cartesian co-ordinates, were developed that denied the parallel postulate but affirmed the rest of Euclid's axioms.[7] Before this, many mathematicians throughout history (from ancient Greece, through medieval Islamic programme of mathematics, to Europe in the eighteenth and nineteenth centuries) had tried to prove that the parallel postulate either is a consequence of Euclid's other axioms and definitions or that it contradicts them. Suffice to say, this was never accomplished (though some came near, as we shall see). However, there was not, until Hilbert's *Foundations of Geometry*, any rigorous way to prove that the statement about parallels is independent of the other statements—that it neither contradicts nor is a consequence of the other axioms. This is why Hilbert's work in geometry presents us with an instance of a change of setting.

[6] Cited in Toretti (1978), p.41
[7] Ibid., p.40

1.2.1 Implicit Definition

Hilbert's innovation was to look at the system of Euclidean geometry *itself* as a mathematical object. Now, this involved a certain level of abstraction from the then-typical views of geometry, in that statements made within the system are to be regarded as 'uninterpreted'. For Euclid, geometry was the science of space; and as such, geometrical objects (such as points, lines and planes) are entities which may possess properties which are incidental to geometry itself. In contrast, uninterpreted mathematical objects have no properties beyond those ascribed by the relevant axioms. For instance, the statement 'a straight line can always be drawn between two points' is no longer seen as actually being about points or lines, in the usual senses of the terms. It is to be seen as *uninterpreted*. The terms 'point', 'line', 'plane', 'angle', and so forth, are nothing more than what the axioms describe them to be. That is to say, a 'point' is anything that 'lies upon' a 'plane', where a 'plane' is anything which satisfies the conditions set forth in the axioms. This progressively more abstract conception of geometry is described by Historian of Geometry Roberto Torretti in the following passage:

> [O]ne can approximately see what [abstract axiomatics] means by recalling the well-known thesis of Hilbert, that the planes, lines and points of his *Grundlagen* may be taken to be *any* threefold collection of things—Hilbert once proposed chairs, tables and beer-mugs—which, given a suitable interpretation of the undefined properties of incidence, betweenness and congruence, happen to stand in the relations characterised by his axioms. (1978, p.190)

This is the notion of an implicit definition. The objects which are called 'points', 'lines', and 'planes' are not defined *explicitly*, as in 'a point is an object which

...'. Rather, the axioms are statements which 'define' or 'establish' the terms 'point', 'line', and 'plane' as whatever can be thought of as being referred to by those terms without the axioms becoming false. Thus, the axioms of geometry become stipulated conditions on the logical behaviour of these terms relative to one another. The point is that we are looking at the logical relationships between the axioms and theorems, rather than any intuitive content they may possess. The symbol 'point', a certain series of marks on a piece of paper, now behaves exactly as the axioms state that it does. Nothing more, nothing less.

1.2.2 Independence and Model Theory

The methods employed by Hilbert in this investigation show the beginnings of what is known today as model theory.[8] In model theory, a collection of objects is combined with a group of axioms (stated in a given language $\mathcal{L}$) defining those objects. Call the set of axioms Λ and the set of objects $\mathfrak{A}$. We then define a *satisfaction function* v as $v : \mathcal{L} \mapsto \mathfrak{A}$ from the variable symbols in $\mathcal{L}$ to the objects in $\mathfrak{A}$. The function v, together with the set of objects $\mathfrak{A}$, is called a *model* of Λ when a given formula S derived from the axioms can be 'satisfied' by the model (written $\Lambda \models_{\mathfrak{A}} S$). This happens just in case v is able to map all the free variables in S to objects in $\mathfrak{A}$ in such a way as to make S true. For example, say we have a language $\mathcal{M}$ which contains the usual logical symbols $(\forall, \rightarrow, \neg$ etc), a countable number of variable symbols $(v_1, v_2, \ldots v_n, \ldots)$, and the arithmetical symbols $+$, $\cdot$, and $=$. Assume that these arithmetical symbols are defined in the usual way. Then, the set $\mathbb{N}$ combined with the satisfaction function $s : \mathcal{M} \mapsto \mathbb{N}$ which maps v_n to $n - 1 \in \mathbb{N}$ (so that $s(v_1) = 0$, $s(v_6) = 5$,

[8] In what follows, I utilise notation and conventions which actually developed after the work of Hilbert. Thus, my exposition is in a certain sense anachronistic; however, nothing in the exposition contradicts the *spirit* of Hilbert's techniques. The notation is derived from that in Enderton's *A Mathematical Introduction to Logic* (2001).

etc) is a model of arithmetic on the natural numbers.

Now, a group of axioms Λ in a language $\mathcal{L}$ can have more than one model, and it is possible that not all models of Λ will be equivalent.[9] Without a model, a group of axioms in a language is uninterpreted; it could be 'about' anything (although for most axiom groups there are 'intended' or 'standard' models, such as $\mathbb{N}$ for the first-order Peano axioms). An expression K is *independent* relative to an axiom group Λ if and only if there is a model of $\Lambda \cup \{K\}$ and also a model of $\Lambda \cup \{\neg K\}$. It is important to note that it is desirable that all axioms in a group be independent of one another; if an axiom A can be derived from $\Lambda - \{A\}$ then we may as well de-list A as an axiom, and let it be a theorem. Also, knowing that each of a collection of axioms is independent from the others re-enforces the notion that they are logically basic, and cannot be reduced any further.

As mentioned above, mathematicians had been struggling for centuries with the status of the parallel postulate relative to the rest of geometry—especially in terms of whether it is a theorem of Euclid's axioms or contradicts them. It is, in fact, neither—it is *independent* from those other axioms. Let $\mathbf{E}$ denote Euclid's (or rather, Hilbert's) axioms, and let $\mathbf{p}$ be the parallel postulate. Hilbert showed that there is a model of $\mathbf{E} \cup \{\mathbf{p}\}$—the intended model, Euclidean space.[10] However, there is also a model of $\mathbf{E} \cup \{\neg\mathbf{p}\}$. This model is sometimes known as the 'Beltrami-Klein sphere'; it is characterised by the interior surface

[9] The main notion of equivalence between models is that of 'isomorphism'; two models are 'isomorphic' if and only if there is a one-one correspondence between the elements of the two models which preserves the functions and relations. Thus one model can be 'translated' into another without any significant alteration; it is more like an exercise in re-lettering.

[10] All of Hilbert's models are described analytically, that is, via functions on $\mathbb{R}^3$ (See Katz (1993) p.721 and Toretti (1978) pp.236–8 for details). Thus any supposed circularity in invoking Euclidean geometry as a model of the axioms is avoided; however, it also turns out that Hilbert's proofs are of *relative* consistency—the models are consistent if and only if arithmetic is.

of a sphere (points in non-Euclidean space are points on the interior surface of the sphere; lines in this space are chords along the surface).[11] Thus, the parallel postulate neither follows from nor contradicts the actual content of the other axioms, although it could well interfere with our intuitive interpretations of that content. Hilbert also gave models which showed the independence of some of the other axioms from each other—for instance, a model which was consistent with the negation of the Archimedean axiom.[12]

To summarise, Hilbert's work on the foundations of geometry constitute a change of setting because he made explicit a result which, though known at the time, had not been placed in the larger context of abstract axiomatic systems. Beltrami, Klein and Poincaré had all found ways to 'embed' non-Euclidean geometries into Euclidean ones, thereby showing that they were consistent if Euclid's was. However, Hilbert shifted the focus from the relationship of non-Euclidean geometries to Euclid's to the more abstract level of the relationships uninterpreted systems of axioms bear to real analysis. Furthermore, the new perspective leads to the new results in a somewhat natural manner. For example, while Hilbert himself did not address the notion of completeness—that every true statement of Euclidean geometry can be derived from his axioms—such results were soon found by other mathematicians.[13] Indeed, Hilbert was himself soon stating that an absolute consistency proof for arithmetic should be an important priority for mathematicians[14]—the very idea of the need for such a proof arises directly from the shift to abstract axiomatics of which Hilbert's

[11] Toretti (1978), p.237; cf. p.133

[12] Ibid., p.237/8

[13] "It is virtually certain... that Hilbert believed that his system was complete. In fact, several mathematicians soon showed that all of the theorems of Euclidean geometry could be proved using Hilbert's axioms." (Katz 1993, p.721)

[14] This was Hilbert's 'second problem' and was proved unsatisfiable by Gödel's Incompleteness Theorem (see Mac Lane 1983, p.379; pp.428–30 contains a discussion of some other problem's on Hilbert's list). It is important to note that Gödel would probably not have found his result if not for Hilbert's statement.

work on geometry was a direct demonstration. The change is not an ad-hoc, stopgap measure, but a fruitful restructuring of the whole inquiry. It is this phenomenon, of a change of setting resulting in the conceptual clarification which is displayed by the increased ease of solving old conjectures and the appearance of new insights, which I will seek to philosophically address.

1.3 Changes of Setting: A Philosophical View

This section commences a philosophically contemporary analysis of the phenomenon of changes of setting. As mentioned above, there are several main aspects of this investigation which will be of particular concern. First of all, we have two seemingly contradictory, but deeply connected, features of a change of setting which we must reconcile.

- That when a setting is changed and apparently results in 'new' solutions to 'old' problems, these new solutions are in fact solutions to the *same* problems which were present in the old setting. That is to say, change of setting $\neq$ change of topic.
- That the formulation of the new setting is a 'non-trivial intellectual achievement'. This essentially means that the new setting genuinely extends the old; it has not resulted from *ad hoc* stipulations, nor is it completely discontinuous with the old setting (i.e. is not simply a new field of study).

In terms of the discussion of Hilbert's work on the foundations of geometry, the first feature is the idea that Hilbert's proof of the independence of the parallel postulate does in fact apply to the parallel postulate which is present in Euclid's *Elements*, as well as its negation in non-Euclidean geometries. That is to say, the abstract axiomatic system analysed by Hilbert is 'the same as' that developed by Euclid (in some as-yet undefined notion of 'sameness'). The second is the notion that Hilbert's new setting—that in which the axiomatic geometry is uninterpreted—is a conceptually 'natural' extension of the work done by Klein, Beltrami and Poincaré, and sheds light on those approaches. In order to find a philosophical perspective which is able to accommodate and explain this phenomenon of change of setting (especially in view of the fact that setting-changes seem to violate the Platonic timelessness of mathematical truth), we must begin by looking at some of the overall features of mathematical knowledge which are largely beyond dispute; that it is in some sense 'global', and that it is *a priori*

The philosopher and mathematician René Descartes considered mathematical knowledge to be possessed of two defining qualities: clarity and distinctness. Mathematical objects are *distinct* because they are defined by all and only those statements which are true of them. For instance, the number '7' is defined by all the sums, differences, products, etc. which it produces when combined with other numbers.[15] This is in direct contrast to scientific objects (such as organisms, molecules, and planets) which are considered to be essentially constant even when statements formerly thought true of them are falsified (light is not considered to be a physically different phenomenon to what it was in the 17th century, even though we now know that it does not always travel in 'straight' lines).

[15] It is true that this seems to not be the case when it comes to transfinite numbers: $7 \cdot \aleph_0 = \aleph_0$, as does $n \cdot \aleph_0$ for any $n \leq \aleph_0$; however this can be taken as a defining characteristic of what it means to be 'a number less than or equal to $\aleph_0$'.

Mathematical knowledge is *clear* because it provides conceptual reification of whatever is being studied; for instance, the proof of the Pythagorean theorem adds to our concept of 'triangle' by making explicit general facts about certain types of triangles which implicitly follow from what it is for those triangles to be those types of triangles in the first place. Ideally, we should always know *why* a theorem holds. These two qualities, clarity and distinctness, can become opposed, as we shall see.

Philosopher of Mathematics Kenneth Manders[16] argues that the modern approach to the analysis of mathematical knowledge (via mathematical logic) has been concerned almost exclusively with the quality of distinctness; thus it is inadequate to the task of explaining setting-changes. This is because recent foundational work in mathematics singularly addresses the epistemological issue of *reliability*, in terms of analysing exclusively the methods of inference employed in mathematical reasoning. Under this conception, the hallmark of mathematical knowledge is the iron-clad reliability of its inferences; and as such, "...the aspect of distinctness [makes] sure that the objects of study are precisely determined and reliably reasoned about."[17] It is the critical evaluation of these inferences which provides the research programme for mathematical logic; and hence, distinctness and reliability readily combine into an approach to the foundations of mathematics (one which, while fruitful in certain aspects, cannot address the phenomenon of setting-changes; and hence cannot be the whole story).

One issue Manders raises with this conception is that it lends itself to taking individual propositions[18] as the units of investigation. There is little scope for

[16] In his (1987)

[17] Ibid., p.202

[18] and inferences: an inference from A to B can be rendered as the proposition 'if A, then B', or 'from A, infer B', although one would not want to iterate this process *ad infinitum*.

addressing mathematical theories[19] as wholes, and thus likewise for investigating the conceptual relationships *between* theories, especially in terms of why it is that some theories seem to be more accommodating to certain concepts than others. From the reliability-centred point of view, "...mathematical progress is nothing but piling up one theorem on another."[20] As long as it is the formal qualities of inferences which are the focus of foundational research (and not the content), little effort is made in the direction of *why* theories extend in one direction rather than another, or why insights sometimes rise from some investigations and not others.[21]

In opposition to this kind of approach, Manders advocates a return to the Cartesian emphasis on the *interdependence* of clarity and distinctness in characterising mathematical knowledge. As the conceptual clarity of a theorem is a product of the relationship between the theorem and its background theory, it is a *global* feature of the background theory. This means that the distinctness-come-reliability approach is by itself inadequate; this is due to its emphasis on individual propositions, over entire theories, as autonomous objects of knowledge.[22] This is doubly so when seeking to understand why some theories render

[19] Throughout this discussion, I will be talking of mathematical 'systems' and mathematical 'settings'. While these terms are not intended to be synonymous, they overlap considerably; in terms of the discussion in §§1.3 and 1.4, a setting is 'rigorised' into a system, through the introduction of deductively closed codifications of the setting into a formal language. The term 'theory' is intended to cover both cases.

[20] Ibid., p.195. Compare this with a remark by F.F. Bonsall on the validity of computer-assisted proofs in mathematics: "If such a problem, the four colour problem, for example, is solved by some clever new idea, that is magnificent; but a solution by the cumulative application of existing methods may do nothing more than demonstrate the cleverness of the solver." ('A Down-to-earth View of Mathematics', *American Mathematical Monthly* Vol. 89 (1982) p.13; reprinted in Fauvel & Gray (eds) 1987, *The History of Mathematics: A Reader*, p.599

[21] A prime example: "...I have recently heard it said that, whereas a complete proof of Fermat's theorem might prove mathematically insignificant, the older partial results on this problem (such as E.E. Kummer's) are "deep" in that they reveal hidden hidden aspects of the integers which become manifest only when they are imbedded within richer realms of "ideal numbers"." (Wilson 1992, p.112)

[22] Manders (1987), p.202

some theorems more comprehensible, as this involves a higher-level relationship between the setting where the theorem or conjecture was originally formulated, and the alternative setting which renders it further comprehensible. Hence, I will seek an explication of this notion of the clarity component of 'naturalness', by presenting it in contrast with distinctness-come-reliability. I will argue that seeing distinctness and clarity, or modern variants thereof, as working *in tandem* yields a way of viewing the connections between mathematical theories which can account for the seeming phenomenon of a result becoming more 'natural' once a change of setting has taken place.

1.3.1 Distinctness

In modern mathematics, the ultimate standard of reliability is the (possibility, in principle of) embedding the system into set theory (usually ZF or ZF + choice). Under this conception, all mathematical objects are sets, and all operations are operations on sets; the languages of individual theories are just convenient shorthands for the language of sets. The only non-logical operation is set membership—all other resources belong to the logical theory. However, mathematical objects are entirely constituted by what is *true* of them. There is nothing more to any number (or a set-theoretic representation of a number) than its set-theoretic interactions with other numbers—and thus, nothing more to any set than its interactions with other sets. From a logical point of view, the result of this conception of mathematics is that there is *no way to conceive* of there being any conceptual continuity between two different mathematical theories. For instance, in the system of natural numbers ($\mathbb{N}$), some differences cannot be calculated due to the absence of negative numbers. Thus

$$\forall x \forall y \exists z (x - y = z)$$

is false when x,y,z range over $\mathbb{N}$, but true when they range over $\mathbb{Z}$ (the integers). Hence these x,y,z denote different objects when those objects are taken from $\mathbb{Z}$ as opposed to $\mathbb{N}$;[23] and thus the above sentence, by virtue of its ability to have different truth-values in different theories, is actually a 'typographically ambiguous' representation of *two different sentences*. Hence '5' and '+5' denote different objects. To paraphrase Quine, 'deny the doctrine and you change the subject'. These considerations can be extrapolated to apply not just to numbers, but also to functions, sets, and all other mathematical objects. Thus, while objects *internal* to a mathematical theory are interdependent in terms of their properties and relations (if one property or relation is altered, the whole theory undergoes a systematic change), the total opposite is the case when it comes to an inter-theory perspective. All theories, conceived as wholes, are completely autonomous.

The outcome of this, as mentioned previously, is the complete loss of the ability to even conceive of any conceptual continuity between different settings; there cannot be reasons, internal to a theory, as to why that theory should be a preferred object of study over others. In the words of philosopher of science Mark Wilson, "...any self-consistent domain is equally worthy of mathematical investigation; preference for a given domain is justified only by aesthetic considerations, personal whim or its potential physical applications."[24]

It may be countered that this is where set theory is able to do some explain-

[23] This follows from the converse of Leibniz's identity of indisceriblesIf everything true of an object A is true of an object B, then $A = B$. The converse would be if at least one thing true of A is not true of B, then $A \neq B$.

[24] (1992), p.112

ing. If two theories are both able to be embedded in set theory, and thus all objects of both theories are treated as sets and all operations on those objects as operations on sets, then we have a way to conceive of continuity between theories. '5' and '+5' both denote the same set.[25] However, while this does supply continuity, it has the effect of completely doing away with the other aspect of of our investigation—that the formulation of new settings which leads to increased comprehensibility of theorems and conjectures is a non-trivial intellectual accomplishment. It seems that the bump in the rug has merely been shifted, not smoothed out. In Manders' words,

> [S]et theoretic definability [does not] *by itself* set apart those relationships between (set theoretically definable) conceptual settings which constitute successful reconceptualisations...from the infinitely many completely uninteresting ones. (1987, p.200; emphasis added)

Thus, we have a dichotomy. On the one hand, one setting cannot be a more natural home for a theorem than another because the expression in the new setting expresses something different. Elliptic functions do not achieve additional clarity when defined over a complex doughnut rather than the complex plane, because they are by definition different functions. On the other hand, one setting cannot be more natural than another because *all* settings are interpretations of set theory, and thus possess the same epistemic status. We have a dichotomous separation, with seemingly no middle path; and either way, the occurrence of fruitful changes of setting cannot be explained.

[25] Up to isomorphism: e.g. Zermelo vs. von Neumann ordinals.

1.3.2 Clarity

Over-emphasis on the reliability-theoretic approach (allied as it is with almost exclusive concern for distinctness) also results in the loss of our other major concern, *clarity*. Manders notes that "...fully formalised proofs are often unintelligible...increased precision is often achieved at the expense of clarity."[26] Likewise, mathematician Saunders Mac Lane observes that

> All mathematics can indeed be built up within set theory, but the description of many mathematical objects as structures [i.e., as sensitive to setting] is much more illuminating than some explicit set-theoretic description. (1996, p.182)

To state the obvious, most mathematicians do not 'think' in set theory. They think in what may be termed an informal metalangauge, where intuitive notions are used and explored in finding inspiration for taking research in certain directions, or using a certain proof strategy in some particular instance. The reliability-centered approach, and its attendant loss of conceptual continuity and/or respect for innovation, is an expression of the often implicit assumption that mathematics is purely a matter of rendering its objects distinct, in a formal and hence reliable manner.

We would do well to take note of the fact that when a new theory is being formulated and investigated, it is yet to be possessed of the rigorous formalisation into (usually) a formal (and hence *reliable*) language. Thus, maximal distinctness is not yet available as a fully cohesive view of the new theory. It is only once the theory has reached a certain point of maturation that it can be

[26] Manders (1987), p.202 [order reversed]

thus codified, and the formal claims can be stated in absence of the conceptual. However, there are historical cases where one or more of the developments which constitute formalisation have preceded the rest, and thus themselves served as the stimulus for research into a new area. Nevertheless, it seems that the majority of cases are those which arise from the search for conceptual clarity. In the words of Manders, "...genuine mathematical accomplishment consists primarily in *making* **clear** *by using new concepts*."[27]

Be that as it may, Manders goes on to observe that "...results also have to be correct to count as making something clear..."[28] But, if the new setting is yet to be mathematically rigourised to the point of constituting a new calculating system, how can these results be *known* to be correct? We are observing the return of the reliability condition. To get clear about the kind of conception we need, I will first outline the philosophical position known as 'naïve realism' regarding mathematical objects—also known as 'Platonism', often regarded as the default position of the working mathematician in contemporary philosophy of mathematics.

1.3.3 Naïve Realism, Set Theory, and Epistemology

The standard contemporary formulation of naïve realism about mathematics is a conjunction of two claims:[29]

[27] Manders (1987), p.193 (emphasis original)
[28] Ibid.
[29] See, for instance, Mac Lane (1986) p.3: "It is remarkable that (almost) all Mathematical objects can be constructed out of sets... Hence arises the view that Mathematics deals with just the properties of sets and that these properties can be deduced from a suitable list of axioms for sets... This set-theoretic description of mathematics is often coupled with a strong belief that these sets objectively exist in some ideal realm."

- that all mathematical theories can be reduced to set theory, and
- that sets exist in an objective, timeless manner which is ontologically independent of the material world (in particular, the beings and doings of mathematicians).

This can include existence claims for not unmysterious objects such as infinite sets, choice functions, and so on. Sometimes the claims about set theory are left to the side; existence claims are made directly about the subjects of specific mathematical theories.[30] The main point is that these objects are non-physical and non-temporal.

Such properties, usually associated with abstract objects, lead to many problems for naïve realism about mathematical objects. Among these is the well-known issue of epistemic access: if mathematical objects are timeless and do not occupy any spacial location, then it seems that we are incapable of the kind of interaction with them that is required for our gaining knowledge of their properties.[31] Also problematic is the (sometimes unexpected) applicability of mathematics to many real-world phenomena (again, a consequence of mathematical objects' non-physical existence). Due to the already vast extant literature on the subject, I will go no further than to state an alternative to naïve realism, the 'semantic conception'.[32]

[30] "There are other versions of platonism for mathematics, for example one in which the ideal realms are comprised of numbers and spatial forms (the "ideal triangle")." (Ibid.)

[31] An influential paper (among many) in this vein is Benacerraf (1973) 'Mathematical Truth', *The Journal of Philosophy*, Vol.70, No.19, pp.661–669

[32] This view, or a similar one, often goes by the moniker 'conventionalism'; however, this term has considerable historical baggage, which I wish to avoid.

Under the semantic conception, we know mathematical truths because they are truths of meaning; often described as 'true by definition,'[33] they are able to be applied to our physical surroundings because they are the result of the same phenomena as ordinary language usage. That is to say, mathematics is a system of interrelated terms and predicates, which are only different in degree (as opposed to being different in kind) to everyday terms and predicates. Mathematical truth is thus a variety of linguistic truth, and mathematical 'objects' are on par ontologically with linguistic terms. This means that our epistemic access to, and the empirical applicability of, mathematics becomes just as problematic as the existence and applicability of the term 'dog'. And while it is the case that ordinary natural language has its own philosophical issues, it should soon become clear that not only does the semantic conception of mathematics avoid the pitfalls of naïve realism, it also supplies us with an interesting and novel way of conceiving of the relationship between inter-theory conceptual continuity and its non-trivial attainability, thus supplying us with a coherent view of changes of setting.

Now, the semantic conception of mathematical truth can be described by opponents in terms so uncharitable as to make it seem unacceptable to working mathematicians. A particular instance is the claim that it makes the discovery/invention of new mathematical ideas 'arbitrary'. That is to say, if mathematical truth is of the same metaphysical category as linguistic truth, then that truth depends on us; specifically, on what *we accept* as the correct meaning/usage for any given mathematical term. As stated by philosopher S. G. Shanker:

[33] And thus analytic. However, as it is not especially pertinent to my case, I will not address the question as to whether mathematical truths are analytic or synthetic (requiring more than knowledge of the meaning of the sub-sentential terms in order for their truth to be determined); nor will I touch upon the other philosophical bugbear of whether the analytic/synthetic distinction is ultimately viable.

> The most common criticism levelled against [the semantic conception] is that mathematical truth must be sempiternal and universal: properties that outstrip the reach of conventions, which are rooted to the decisions of a speaker or community. (1987, p.303)

Moreover:

> The feeling behind [this] criticism is that the [semanticist] contends that a mathematical proposition only expresses either a speaker's or a community's decision to use symbols in a certain way. (Ibid., p.304)

Hence, if this truth is dependant on us, then it is inferred that we are 'free' to use the symbols in any way we want. We can make any mathematical statement true or false, with the justification for such decisions not originating from within the theory itself. In terms of the discussion of Hilbert and abstract axiomatics (§1.2), the objection would be that Hilbert's axiomatic analysis does not 'tell' us anything about the relationships between Euclidean and non-Euclidean geometries because the *reasons* we accept Hilbert's analysis are not 'geometrical' reasons. The parallel postulate is independent not because of any relationship between it and the other axioms, but because we act as if it were independent. Because this is a decision (to act in a certain way), and not a discovery (of a certain fact), it cannot *tell* us anything. For another example, all that is required for the Goldbach conjecture to be considered true (and thus, actually *be* true) is that all or most mathematicians in the world start considering it to be true. That is, that it becomes a property of all the symbols which denote natural numbers greater than two and divisible by two that they cannot appear in the formula

$$\forall n_{\in\mathbb{N}} \exists p_{\in\mathbb{P}} \exists q_{\in\mathbb{P}} n \neq p + q$$

where $\mathbb{P}$ is the set of primes. Because, under the semantic conception of mathematical truth, there is no gap between *accepted* truth and *actual* truth, opponents of this approach take it to be a position which trivialises or disregards the accomplishment inherent in a mathematician's solution of a difficult problem (and thus take it as a view which is totally incompatible with our intuitions regarding mathematical knowledge). Concurrent is the notion that when formulating a new mathematical system, we are free to make any axiom, rule or object behave in any way we want to, without the constraints normally seen as necessary to have mathematics 'make sense' (another supposedly counterintuitive conclusion). I argue that these conclusions do not follow; an important corollary of doing so supplies an explanation of the phenomenon of successful setting-changes.

That is the uncharitable description of the semantic conception. My description, considerably more conducive to the practice of the working mathematician, issues from the following idea: that while it is indeed the case that we are 'free' when constructing new mathematical theories, this does *not* mean that any old way of doing things will suffice. However, I cleave to the semantic conception's notion that mathematical truth does not exist prior to recognition; the truths of (pure) mathematics are not 'out there' to be discovered. We 'make' some mathematical statements true, others false, by regarding them as such. Nevertheless, this 'making' is not done by any force of will or raw desire, but by a process more akin to being 'convinced'. Not only do we have to regard a statement as a result of another for it to be so, we also have to regard it as a *mathematical* result for the result to be part of mathematics.

Now, it seems plausible to regard most mathematicians as believing that it is both possible and desirable for mathematics to one day be organised into a vast interlinking network, where every mathematical truth is logically connected to the sum total of all other mathematical truths. Thus it seems we have at least one constraint on the choices to be made when a new system is being constructed: the new system cannot be so strange and bizarre as to be unrecognisable as mathematics to the average mathematician.[34] This, in turn, leads to further constraints. For one, the rules of derivation (be they explicitly stated, or only used implicitly) must be possessed of some kind of constancy. For instance, 'take propositions A and B; flip a coin; if the result is heads, then B is a mathematical consequence of A' does not seem acceptable and, therefore, it is not acceptable. A system which says this about mathematics simply is *not* mathematical.

Furthermore, and more importantly, the new theory must be able to 'explain' its presence, usually in terms of theories already existent (which may or may not have served as the initial motivation for the formulation of the new theory). Thus, analytic geometry 'explains its presence' by being able to offer one singular proof of a situation which may require several synthetic proofs.[35] Hilbert's pioneering use of model theory explains its presence by giving a mathematical account of the independence of the parallel postulate that is in accord with our intuitive notions of what axioms are and how they should behave. Perhaps one of the most eloquent statements, by way of analogy with musical composition, of this situation is the following:

[34] It may be argued that there are historical exceptions to this rule; the introduction of 'imaginary' numbers, or maybe Kronecker's opposition to Cantor's explorations of completed infinities. My belief is that it can be counter-argued that (1) these cases will always be in the minority, and (2) the subsequent acceptance of these techniques bears them out. What is important is the attitude of the entire mathematical *community*, not the attitudes of one or two exceptionally vocal and well-placed critics.

[35] See Wilson (1992), pp.117–125 for an in-depth example in this particular change of setting.

> A succession of two musical notes is an act of choice; the first causes the second, not in the scientific sense of making it occur necessarily, but in the historical sense of provoking it, of providing it with a motive for occurring. A successful melody is a self-determining history; it is freely what it intends to be, yet is a meaningful whole, not an arbitrary succession of notes. (Auden, 1975, p.465-6)[36]

It is in this 'historical sense' that we should regard one mathematical theory as being 'caused' by another. Note that this implies two things—one, mathematical 'objects' are not required beyond being linguistic entities; the naïve realist's epistemic access and applicability worries vanish—and two, the 'explaining itself' that a mathematical theory needs to be seen to do is an ongoing process, one which accompanies the progression toward rigorisation hinted at above.

For example, up until modern times, the axioms of Euclidean geometry were held to be self-evident; this self-evidence was taken to be known through some kind of 'intuition'. The idea of intuition underpinning the (base) truths of mathematics found its culmination in the philosophy of Immanuel Kant, in his holding that mathematical truths are synthetic *a priori* (for Kant, all synthetic knowledge is reached via intuition). As is well known, the advent of non-Euclidean geometry had most philosophers of mathematics (and most mathematicians) concluding that it is in fact self-consistency which is the driving force of mathematical truth, and not some kind of psychological faculty or occurrence.[37] These events occurred mostly in the 19th century; the early 20th century saw the rise of model and proof theory, the mathematically rigorised counterparts to our pre-theoretic notion of axiomatics.

[36] Cited in Shanker, 1987, p.338

[37] Brouwer and his followers can be regarded as possible exceptions; however, their conclusions entailed a kind of revisionism which was seen as unacceptable by most of their contemporaries.

In his *Grundlagen der Geometrie*, Hilbert gave mathematical reasons for why certain mathematical statements are not even pre-theoretic logical consequences of one another; the existence of these models proved that it was metaphysically possible for one axiom to be true while the others are all false, and the mathematical rigour of his method gave compelling reasons to regard it rational to believe that this is the case. Hilbert quotes Kant in that work: "All human knowledge begins with intuitions, thence passes to concepts and ends with ideas".[38] However, philosopher of mathematics Stewart Shapiro notes in his analysis of the origins of model theory[39] that "...the plan executed in that work is far from Kantian. In Hilbert's hands, the slogan "passes to concepts and ends with ideas" comes to something like "is *replaced* by logical relations between ideas."" Furthermore, "In Hilbert's writing...the role of intuition is carefully and rigourously limited to *motivation* and *heuristic*."[40] Thus, we can see that the notion of 'intuitive' consequence operant in Kant's conception of the foundations of geometry gets replaced by 'logical consequence' in Hilbert's axiomatisation, where the 'logical' is understood as 'model-theoretic'.

It is possible to see Hilbert's *Grundlagen* as a historical, explicit instantiation of what occurs implicitly in most extensions and subsequent rigorisations of mathematical fields of inquiry (i.e. changes of setting). At the outset, intuitive notions of mathematical consequence and what objects/operations are present in the setting are primarily operant—witness how the propositions of Euclid's *Elements* are things that one is proposed to *do*. At this early stage, that a theorem follows from the axioms and definitions is ensured by the (possible) construction of a figure, not 'facts of the matter' as to what logically implies what. However, as rigorisation progresses, the intuitive, conceptual notions are

[38] *Critique of Pure Reason*, A702/B730. Quoted in Toretti (1978), p.227
[39] (1998), p.157
[40] Both ibid; emphasis in the second quotation is mine.

supplanted by their formal counterparts—with (in the present day) the limit case being the embedding of the new system into a set-theoretic metatheory. Such embedding can be seen as the ultimate way for a mathematical theory to 'explain itself' in terms of its antecedent theories. Nevertheless, the conceptual remains, to serve as a 'motivating' and 'heuristic' dimension of the theory—primarily useful in teaching and in exploring new directions for research. As emphasised above, intuition was not *totally* absent from Hilbert's *Grundlagen*.

As an interesting corollary of these considerations, we can view the gradual transition from the conceptual to the formal as being the source of the feeling of mathematical 'discovery' which makes naïve realism about mathematical objects such an intuitively attractive conception. Under my alternative, it is not the properties of timeless non-physical mathematical objects that are 'discovered', but rather the facts of the matter as to what formal notions best supplant the prior, intuitive ones. Such investigation will be subject to pragmatic concerns; for instance there is usually an implicit 'principle of minimal disturbance' dictating that the structural integrity of the theory is paramount. It was long held that in the postulation of new number systems, the arithmetical properties (uniqueness of sums, distributivity of multiplication over addition, etc) must always remain constant. However, in the inclusion in projective geometry of co-ordinate numbers which square to -1 (which results in the inability to form the resulting complex number system into an ordered field) the dictum that *all* properties of numbers be held invariant in *all* number systems is relaxed because of the proven possibility of elegant new theorems. Instances of the fact that such theorems can result from the inclusion of such 'problematic' concepts is what can be rightly said to be 'discovered' in mathematical investigation.

1.4 Conclusion

In §1.2 the idea that a new setting can result in conceptual simplification of a mathematical theory was illustrated by the example of Hilbert's abstract axiomatisation of geometry. Generally, changes of setting can result in certain theorems appearing more 'natural' when carried out successfully. §1.3 began the philosophical analysis of this phenomenon, and brought us to an effort to accommodate both conceptual continuity of objects and operations between theories *and* a regard for mathematical advances as genuine intellectual achievements. To this end, we recognised a role for an informal, conceptual metalanguage of mathematics in addition to the formal language(s) used to rigourously express mathematical theorems. Having reached these conclusions, we found that a semantic conception of mathematics, where both the conceptual and formal aspects of mathematical expression are fragments of natural language, enables us to both skirt the traditional problems attendant to naïve realism *and* account for our conclusions regarding the success that certain settings display over others. I will now offer an intuitive picture of the conception of mathematical advancement which our conclusions have led us to.

We begin with a fairly mature mathematical theory, either fully formalised or well on its way. Nevertheless, certain conjectures seem resistant to proof, or there are theorems which can be proved but are tediously difficult, sometimes requiring methods which are counter-intuitive and possess little explanatory power. A new setting is proposed, for instance one which contains new objects, or a reorientation toward the subject matter. At this early stage, research is mainly conducted along the lines provided by the conceptual content of the intensional view of the theorems and operations of the old system. However, as this research progresses, the proofs become increasingly formal, as the logical investigation of the new setting progresses. When alternative, non-equivalent formal definitions are proposed, their implications are explored and debate ensues over which best fits the intuitive concept being investigated. Eventually, purely formal rules dictating the relationships between the theorems of the new setting are proposed, codifying the results of the more informal proofs. As this progresses, the intuitive aspect plays less and less of a role: the limit case is that of Hilbert's *Grundlagen*, where intuition is relegated to heuristic and motivational purposes only. However, as still being motivational, the intuitive is still present to provide inspiration and hints at possible directions for new research in the future.

This perspective has consequences for the history of mathematics. As mathematical innovations, especially when they constitute changes of setting, are required to explain their presence in terms of what has gone before, historical investigations must be able to give this explanation. Thus the philosophical position outlined above has broad implications for the project of historical interpretation. In order to fully appreciate both the impact of and motivations behind a change of setting, we must view it as a historical occurrence, against a larger backdrop of events. We have seen that the two main aspects of a mathematical setting are on the one hand conceptual, on the other practical. It is the business of philosophy to investigate conceptual changes; it is that of mathematics to investigate formal changes. Investigations of changes should reflect this by seeking to explain changes in formal mathematical systems by the light of conceptual shifts; however, the current runs both ways—new formal approaches often influence shifts in conceptual viewpoints.

With these conclusions established, we can conduct just such an investigation—of Gottlob Frege, a pioneer in the field of mathematical logic in the late 19$^{\text{th}}$ century. Frege is known for both his technical and his philosophical innovations, which makes him an ideal case study for the kind of historical investigation that the semantic conception requires. We cannot view Frege's technical innovations as appearing *ex nihilo*, for then we have no ability to explain their presence in terms of providing new solutions to old problems. In order to show that Frege's contributions to mathematical logic constitute a change of setting, we will have to identify the motivations arising from concerns of his day. This requires not only investigating his 'philosophical' motivations, and their origins, but being able to discern evidence of these motivations in technical aspects of his work.

A minor caveat—care must be taken to not confuse the two levels at which we are working: that of the framework in which we are analysing Frege, and the analysis itself. I am *not* going to claim that Frege would have endorsed the conclusions of this chapter. What I will do is argue that if we view Frege's work in the context of its time, both in terms of the history of philosophy and the history of mathematics, then we can see that he is a part of a larger background; if we take note of this whilst reading Frege, then we can see that he is responding to his intellectual climate—a reading which will shed new light on hitherto unexplored aspects of both his philosophy and his mathematical logic. To successfully perform this kind of historical investigation affirms my philosophical position with regard to contemporary philosophy of mathematics: we can only understand our current settings if we know why they were developed.

Hence, we will first look at the reasons behind Frege's motivations; this will be done through an analysis of the relationship between Frege and Kant, the philosopher he most often credits positively in his work. Once these motivations have been placed in a context, we will continue investigating how they can be seen manifesting themselves in his technical innovations—specifically, his formal modeling of certain natural-language phenomena. We shall see that the many interlinking themes which we explore, encompassing philosophy, logic, and history, all come together in exemplifying Frege as a changer of setting.

2. A SETTING FOR FREGE

2.1 Introduction

Gottlob Frege (1848-1925) originally trained as a mathematician, and spent most of his career teaching mathematics at the University of Jena in his native Germany. Widely regarded as *the* pioneer in mathematical logic, his work is roughly contemporaneous with and responsive to that of Ernst Schröder, Guiseppe Peano, and George Boole; he wrote several articles detailing the differences between their respective systems and his own.[1] He also published reviews of works by Hilbert, Cantor, and Husserl. He was (through his work in logic) a considerable influence on the logical positivist movement of the early-to-mid 20th century, specifically on Rudolf Carnap (a former student) and (the early) Ludwig Wittgenstein (an erstwhile correspondent). Our main focus will be the more mature years which produced the *Grundlagen der Arithmetik* (1884) and the two volumes of the *Grundgesetze der Arithmetik* (1893/1903), plus several related articles written in the intervening period. Nevertheless, there are pertinent aspects of his first publication, the *Begriffsschrift* of 1879. The very title of this work suggests that this is a work in which he sets out his formal notation

[1] For instance, 'A Critical Elucidation of Some Points in E. Schroeder's *Algebra der Logic*', in Geach & Black (eds) (1970), pp.86–106; 'Boole's Logical Formula-Language and my Concept-Script', in Long & White (eds) (1991), 'On Herr Peano's Begriffsschrift and my Own', English translation in *Australian Journal of Philosophy*, 47, p.1-14. He was also temporally contemporaneous with the American logician C.S. Peirce—however, it appears that they carried out their research independently (cf. Legg [2008], p.213).

(also known as *Begriffsschrift*, as it translates into English as 'ideography' or 'concept-script').[2]

In order to verify whether Frege's contributions constitute a change of setting, we need to conduct an investigation of Frege's place with respect to an overall historical background utilising the reflections carried out in chapter one. To this end, we will interlink several aspects of Frege's thought, particularly in regard to his conception of the problems he was setting out to solve and how this conception manifests itself in his formal system. Because our aim is also to locate Frege's innovations within a larger framework of interpreting 'changes of setting' we shall also investigate the interdependencies between the logical system which is elucidated in the *Grundgesetze* and his philosophical views and motivations which are present in all three of his main works (and almost entirely make up the *Grundlagen*).

Thus the structure of our investigation will be as follows: §1 one of the present chapter will be an overview of how I intend to apply the general methodology for philosophically interpreting changes of setting which was set forth in chapter one. This involves consideration of the link between Frege and Kant, specifically with respect to their respective conceptions of mathematics, as this is the origin of the problems which Frege was seeking to address. Seeing Frege as concerned more with *mathematics* than the analytic, and hence not as exercised by Kant's synthetic *a priori* as many modern readings make out, leads us to an interesting new way to view Frege's contributions to both mathematics and philosophy.

[2] In what follows, I will cleave to the transliteration (e.g. 'In Frege's *Begriffsschrift*, negation is symbolised by '$\top A$"), with the book referred to as '*The Begriffsschrift*' ('In the *Begriffsschrift*, Frege claims that ...').

We will then progress to an analysis of some technical elements of Frege's notation, specifically with respect to Frege's doctrine of the possibility of 'multiple analyses' of the content of propositions. It will turn out that multiple analysability is a manifestation of Frege's concern with mathematics over and above the synthetic *a priori*, and this is reflected in various technical aspects of Frege's notation which have no counterpart in the logic that was available to Kant (and indeed, in other respects, in the notation of modern mathematical logic). The aim is to show that Frege can be viewed as part of a continuum of progress, both in the spheres of mathematics and philosophy; and this belonging to a larger tradition is what enables us to see Frege as changing the setting for both the philosophical and mathematical conceptions of arithmetic.

My reasoning is based on considerations of the state of the philosophy of mathematics at the time. We have, broadly speaking, the following three schools:

- Transcendental Idealism, descended from Kant
- 'Pre-Hilbert' Formalism,[3] epitomised in the idea that mathematics is a game, and has no ontology beyond the physical existence of symbols on paper, and
- Empiricism, of the variety due to John Stuart Mill. This is the view that the laws of mathematics are abstracted from empirical experience (i.e. are *a posteriori*), and do not obtain prior to it.

In his *Grundlagen der Arithmetik*, Frege argues forcefully and convincingly

[3] As opposed to Hilbert's Formalism—Hilbert's conception of mathematics was in fact a deeper view than that of the Formalists Frege saw himself as in opposition to. Unless otherwise noted, 'Formalism' always means 'pre-Hilbert Formalism'.

against the last school; his main arguments against the second are given in the later *Grundgesetze der Arithmetik*. He makes almost no mention of the first. I contend that this is because it was not whether arithmetic is synthetic or analytic which was his motivation in composing that work and his others—it was the very Kantian worry about the possibility of alternative systems of arithmetic.

2.2 *Frege, Kant, and the Rational Essentiality of Arithmetic*

In the *Critique of Pure Reason* (1781/1789), Immanuel Kant (1724-1804) argues that mathematics is a description of the formal qualities which we imbue things when we perceive them, rather than any description of properties of those things-in-themselves. This is an elegant explanation of the seemingly unreasonable applicability of mathematics—the laws of the mathematical sciences are descriptions of ourselves (specifically, our perceiving faculties) rather than the objects we perceive. Thus they apply uniformly and *a priori* to all possible experience. However, viewing mathematics in this way implies that the systems of which he was aware (arithmetic, mechanics and Euclidean geometry) are the only systems which are valid; there are no alternatives.[4] The advent of non-Euclidean geometries, which began with papers by Nikolai Lobachevsky[5] and Janos Bolyai[6] and was subsequently endorsed by C.F. Gauss and his student

[4] This is due to their being descriptions of the structural nature of our cognitive faculties; any alternative description must be able to be transformed into one of the three given descriptions. If it cannot, then it is describing something else, and is not applicable to empirical reality as we see it. Cf. *Prolegomena* §2: "...we must observe that properly mathematical propositions are always judgments *a priori*...because they carry with them necessity...". See also Toretti (1978): "For this science [geometry]—as Kant had taught in 1770, and again in the Transcendental Aesthetic of the *Critique of Pure Reason* and again in the chapter on pure mathematics in the *Prolegomena*—was revealed through pure intuition in full agreement with the *Elements* of Euclid." (p.33)

[5] 'On the Principles of Geometry', 1829-30

[6] 'Appendix Presenting the Absolutely True Science of Space', 1832

Bernhard Riemann in the mid 1800's,[7] greatly undermined Kant's conception. It became apparent that Euclidean geometry is *not* the only valid system of geometry. Now, Frege endorsed Kant's conclusions about geometry—that it is through some act of intuition that we see Euclidean geometry as valid for our immediate spatial perceptions[8]—but he is seen by some authors as denying that this is the case for arithmetic.[9] It is true that he saw arithmetic as analytic, unlike Kant who stated that it is synthetic. However, I will argue that this is not a reason for seeing a discontinuity between these two thinkers.

The goal of Frege's logicism is to rule out once and for all the possibility of alternative systems of arithmetic by proving that arithmetic is a direct consequence of what it is to be rational. Formalism and Empiricism both allow for the possibility of alternative arithmetics; and Kant's transcendental idealism cannot rule it out (as illustrated by alternative geometries to that of Euclid).

Frege's argument against Formalism is this: If mathematics is a game of symbol-manipulation where arithmetical equations are divorced from any contentful notion of meaning, as Formalism asserts, then there is absolutely no reason intrinsic to arithmetic itself as to why it is one way rather than another.

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[7] It is true that Gauss had, before the publication of the papes of Bolyai and Lobachevsky, been working on a non-Euclidean system. However, he refrained from publishing, as he was wary of how they might be received; upon reading Bolyai's paper he "…thereupon realized that he could spare himself the trouble of writing out his discoveries, for [Bolyai] had anticipated him." (Toretti [1978], p.53; the letter in which Gauss makes this statement is in Gauss, WW, vol.8, p.221)

[8] See *Grundlagen*, §14 and §89; also Currie (1982), p.29/30 and pp.75–8

[9] For instance, Coffa (1991) claims that "Frege's first book, his *Begriffsschrift* of 1879, put forth a program that *directly opposed* Kant. (p.64; emphasis mine). Likewise, "The distance between Frege and Kant is underscored by the paucity of remarks on that most disturbing of Kantian problems, the character of the objects of knowledge and their constitution through the categories [i.e., our epistemic relationship with external objects]. Objects are no problem for Frege—they are the tables and chairs of everyday experience, the numbers and classes of mathematical knowledge, the truth tables of his logic, and so on." (p.67). I would argue that such 'paucity', rather than underscoring a 'distance' between Frege and Kant, is actually evidence for a lack of such distance.

The ultimate consequence of this is Formalism's complete inability to *explain* the applicability of arithmetic to physical phenomena:

> Why can arithmetical equations be applied? Only because they express thoughts. How can we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? Now, it is the applicability alone which elevates arithmetic from a game to the rank of a science. So applicability necessarily belongs to it. Is it good, then, to exclude from arithmetic what it needs in order to be a science? ('Frege Against the Formalists' (*Grundgesetze* Vol. II §§86-137), §91, in Geach & Black (eds) *Translations from the Philosophical Writings of Gottlob Frege* (1970))

Frege is drawing from the applicability of mathematics as an objection to the kind of Formalism propounded by E. Heine and J. Thomae (the targets of the passage quoted). The fact that mathematics is possessed of such applicability is evidence of the first degree against taking it to be a game of symbol-manipulation; it is the *content* of mathematics which makes it *mathematics*. In this, Frege is in total agreement with Kant.[10]

The Empiricist view of arithmetic is descended from the denial that mathematics is *a priori*.[11] A well-known proponent is John Stuart Mill (1806–1873); his writings were the main target of Frege's polemic against the school. Arithmetic is *a posteriori* because its laws are observed empirically, and then recognised after the fact. For instance, one abstracts the rule for addition from

[10] Kant's reasons for arguing that mathematics describes formal properties of our experience are a direct result of appreciation for its applicability; this is one of the greatest points of difference between Kant and earlier philosophers such as David Hume.

[11] And thus not necessary.

repeated experiences of combining separate collections of objects (much as one abstracts the concept 'dog' from repeated interactions with dogs). Frege characterises such a view of arithmetic as about "...piles of pebbles and gingersnaps..."[12] rather than about number.

If mathematics is abstracted from experience (and is hence neither *a priori* nor necessary), then there is no reason why we can rationally assume that we will never come across an experience which falsifies a law of arithmetic. Frege thinks that this is manifestly impossible; a statement of arithmetic cannot be falsified:

> [H]e [Mill] holds that the identity $1 = 1$ could be false, on the ground that one pound weight does not always weigh precisely the same as another. But the proposition $1 = 1$ is not intended in the least to state that it does. (*Grundlagen*, §9)

The main confusion that Empiricists fall into is confusing the priority of statements of arithmetic with that of their applications. "Mill always confuses the applications that can be made of an arithmetical proposition, which are often physical *and do presuppose observed facts*, with the pure mathematical proposition itself."[13] Thus it is merely the applications of an arithmetical statement which depend on experience—not the truth of the proposition itself; this truth is above falsification. Once again, Frege is guided in his philosophy by an appropriate appreciation of the applicability of mathematics and its implications; and the tone of this appreciation is reminiscent of Kant.

The case with Transcendental Idealism is somewhat different. Because Kant

[12] *Grundlagen*, §27
[13] Ibid., §9; emphasis mine

saw the reason for the way arithmetic is in the structure of our perceptual apparatus, there is no *guarantee* that alternative systems cannot exist. It is, however, supposedly certain that we are incapable of conceiving of such systems—nevertheless this was also assumed to be the case with geometry until it was shown otherwise by the work of Bolyai and Lobachevsky. Thus, Frege sought assurance against the same happening in arithmetic in a source deeper than the anthro-specific nature of our perceptual apparatus—the very nature of rationality itself. If successful, he would have shown that so-called 'alternative' arithmetics could only have been carried out by fundamentally irrational beings; and as such, would have no claim to the title of 'arithmetic' at all. I will argue in what follows that this was Frege's goal, and that his statement that arithmetic is analytic was simply the best way to do this. Indeed, Frege ends his *Grundlagen* with the statement that "From all the preceding it thus emerged as a very probable conclusion that the truths of arithmetic are analytic and a priori; we achieved an *improvement* [as opposed to a *refutation*] on the view of Kant."[14]

2.2.1 Kant's Conception of Mathematical Knowledge

In §87 of the *Grundlagen*, Frege makes the curious statement that

> The laws of number...are not really applicable to external things; they are not the laws of nature. What they do apply to are judgements about things in the external world: they are the laws of the laws of nature. They do not assert connexions between phenomena, but connexions between judgements; and among judgements

[14] Ibid., §109; emphasis mine.

> are included the laws of nature.

There is a lot of content in this passage. Frege is saying that the 'laws of number' are not properties of external objects—rather, they govern what we are able to assert about such objects. Such a view is very close to that articulated by Kant almost a century earlier.[15]

Kant claimed that the mathematical sciences, construed by him to be exhaustively described as geometry, mechanics and arithmetic find their epistemic and metaphysical basis (i.e. why we believe them to be true, and what they actually say, respectively) in the 'pure forms of intuition'. Forms of intuition are the framework(s) that all of empirical experience is necessarily located within and restricted to. Geometry, for its part, is the description of space (the outer form of intuition)—which is to say, all human experience of outer objects takes place in space, and thus obeys the laws of (Euclidean) geometry. Arithmetic is the description of our pure form of inner intuition—time:

> [T]he intuitions which pure mathematics lays at the foundations of all its cognitions and judgments which appear at once apodeictic and necessary are space and time... Geometry is based on the pure intuition of space. Arithmetic attains its concepts of numbers by the successive addition of units of time, and pure mechanics especially can attain its concept of motion only by employing the representation of time. (*Prolegomena* §10)

In parallel with the case of space, all our experiences (regardless of whether they are of objects outside or within us [such as thoughts]) take place within

[15] See *Prolegomena* §§11 & 12

time, which is to say, they are ordered with respect to temporal progression.

Because we human beings can only experience outer objects as being spaced, and must experience every object as being temporal, we are entitled to infer that propositions regarding the forms themselves of inner and outer intuition (space and time, thus geometric and arithmetical propositions) are both necessary and *a priori*. Kant supplies many arguments for the idea that space is an original condition of our ability to perceive outer objects, and cannot be abstracted from those experiences after the fact. For instance:

> We can never represent to ourselves the absence of space, though we can quite well think it as empty of objects. It must therefore be regarded as a condition of the possibility of appearances, and not as a determination dependant upon them. It is an *a priori* representation, which necessarily underlies outer appearances. (*Critique of Pure Reason*, A24/B39)

It is important to notice that Kant takes space as 'imposed' upon reality by our act of perceiving—it is this mediating nature which allows us to regard space as applying in the same way to *all* human experiences, be they past, future, or even possible experiences which are never actualised:

> If our intuition [i.e. perception] had to be of such a nature that it represented things as they are in themselves, no intuition *a priori* would ever take place and intuition would be empirical every time. (*Prolegomena* §9)

Thus, it is the mediating nature of space—the fact that *all* experiences of outer phenomena are spatial experiences—which gives geometry, the science of describing the structure of space, its *a priority* and necessity. Similar arguments are given for the case of time and its acting as the basis for arithmetic; however, as the argument for space being a pure form of intuition is given first, and Kant claims that the two arguments are very similar, he gives only brief consideration to that which as to become Frege's main concern. Nevertheless, there are striking similarities between the doctrine of space and time as a pure forms of intuition and the idea that the laws of arithmetic are the 'laws of the laws of nature'.

2.2.2 Frege's Transcendental Idealism

The laws of nature are, for Frege, judgements; and the laws of arithmetic apply to all judgements. This is why they cannot be negated—for doing so would rob one of the ability to even think:

> [W]e have only to try denying one of [the laws of arithmetic], and complete confusion ensues. Even to think at all seems no longer possible. The basis of arithmetic lies deeper, it seems, than any of the empirical sciences...The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the existent...but everything thinkable. Should the laws of number, then, be connected very intimately with the laws of thought? (*Grundlagen* §14)

A more forceful statement of Frege's logicism would be hard to find. Frege is describing (among other things) his attitude toward the question of the *applicability* of the laws of arithmetic. 'The truths of arithmetic govern all that is numerable'—and everything that can be experienced is numerable. This is a point of intersection with Kant's attitude, because for Kant everything that can be experienced is also subject to the laws of arithmetic. However, the respective reasons which motivate the claims of Kant and Frege differ in an important respect, and we shall see that this difference is the source of the reason why Frege thinks he is entitled to make a claim about the applicability of the laws of arithmetic which is a lot stronger than that made by Kant. This stronger claim, in turn, implies the impossibility of any system of arithmetic which is not equivalent to our one.

Kant believes that the universal (empirical) applicability of arithmetic lies in its basis on time as a pure form of intuition, plus the fact that everything empirical is experienced temporally. Thus the applicability of arithmetic is a consequence of of the structure of the way the (human) mind interprets sense experience. For instance:

> [T]he difference between similar and equal things which are not congruent (for instance, helices winding in opposite ways) cannot be made intelligible by any concept, but only by relation to the left and right hands, which immediately refers to intuition. (*Prolegomena* §13)

Now, because all our knowledge is derived from intuitions (be it via their content as sense impressions, or their formal qualities as being spatiotemporal) our cognitive apparatus is very intimately linked with that of the senses: "it is only the form of sensuous intuition by which we can intuit things *a priori*".[16] For Kant, both the epistemic and metaphysical aspects of items of knowledge are interdependently consequent of the structure of our senses and minds. That is to say, the contingent fact-of-the-matter of how we come to know something—the context of discovery, or epistemic aspect—cannot be fully separated from the necessary facts—the context of justification, or metaphysical aspect—of *why* we think that something is true.

Frege's attitude is somewhat different. For him, the epistemic and the metaphysical are to be kept as separate as possible:

> In general...we should separate the problem of how we arrive at the content of a judgement from the problem of how its assertion is to be justified...When a proposition is called analytic or a posteriori in my sense, this is not a judgement about the conditions, psychological, physiological and physical, which have made it possible to form the content of the proposition in our consciousness; nor is it a judgement about the way in which some other man has come, perhaps erroneously, to believe it true; rather, it is a judgement about the ultimate ground upon which rests the justification for holding it to be true. (*Grundlagen* §3)

Thus the universal applicability of arithmetic lies not in how we arrive at its content—i.e. its relation to intuition—but in why we think that the content is accurate. In fact, Frege even *identifies* how content is arrived at with the psychological:

[16] *Prolegomena* §10

> [E]ven mathematical textbooks do at times lapse into psychology. When the author feels himself obliged to give a definition, yet cannot, then he tends to at least give a description of the way in which we arrive at the object or concept concerned. These cases can be easily recognised by the fact that such explanations are never referred to again in the course of the exposition. (*Grundlagen*, Introduction, p.VIII[e])

Thus it can be seen that Frege considered the contingent fact of the matter as to how content is arrived at to be irrelevant to the justifiability of the assertion of the proposition in question. He is taking the distinction made above, between the context of discovery and the context of justification, and forcing the two apart. Frege's project, the aim of his logicism, is to find all that is needed for the justification of the laws of arithmetic in that which is constitutive of what he termed the 'laws of thought'. And by 'laws of thought', Frege means something more wide-ranging than what the phrase might mean from Kant's pen. Kant admits that while his justifications for the possibility of synthetic *a priori* knowledge are based on pure forms of intuition, these pure forms only apply to creatures the structure of whose minds bear the same relationship between the cognitive and the sense-experiential as ours do. A creature that does not have left and right hands could not have the same justification for the judgment that 'two figures are equal though not congruent' that we do.[17] He concedes that a being with mind capable of a different relationship to the objects of experience could know things by other methods than we do; the standard example being God, for whom knowledge of the noumenal[18] is possible and thus whose power of intuition far surpasses ours.[19]

[17] *Prolegomena* §12

[18] Thing-in-itself; that which causes sensory perceptions

[19] By proceeding directly from the understanding, rather than standing as a prior condition for it. Cf. *Critique of Pure Reason*, A250

In holding that the laws of arithmetic are implied by what it is to think, *simpliciter*, Frege is entitled to claim that any being which is to be regarded as rational is to be bound by the laws of arithmetic.[20] Thus even a being whose powers of intuition far surpassed our own and whose knowledge far outstripped what we are even capable of, would still justify the laws of arithmetic in the same way that we do, by appeal to the same concepts. This is not to deny that such a being may come to believe arithmetical propositions in a different manner to us—it may arrive at '$2 + 3 = 5$' or '$a + (b + c) = (a + b) + c$' originally via the laws of second-order predicate logic (i.e. analytically), whereas we originally arrive at the proposition by counting objects and putting them into heaps (synthetically)—the nature of the ultimate *justification* for asserting such that such a proposition holds for all objects is to be found in the notion that it is implied by the nature of the laws of thought. That is to say, it can be justifiably asserted *because* it is analytic; this analyticity is what saves arithmetic from the Kantian worry that other systems of arithmetic may be possible, *á la* non-Euclidean geometries.

From here, it is a short step to the notion that *all* justifiably assertible propositions must not contradict any analytic proposition. No empirical observation is to be counted as contradicting a statement of logic over and above its simply being a bad observation.[21] Hence the idea that the laws of arithmetic are the 'laws of the laws of nature'. In performing the scientific investigations which (if successful) ultimately yield what Frege termed 'laws of nature', we must always respect the laws of arithmetic, which are (for Frege) covertly disguised laws of logic—which are in turn simply the laws of rational thought. To not do this is to risk losing the ability to be considered coherently thinking at all. Frege's

[20] Notice that it is agents (i.e. rational beings) which are bound by the laws of arithmetic, and not external objects. This is one instance of contact between the philosophies of Frege and Kant.

[21] A Quinean point.

logicism has come full circle, from the starting-point of a desire to take the laws of arithmetic as transcendentally ideal, and thus prior to experience (contra Formalism and Empiricism) but guard them against the possibility of plurality, as befell Kant's philosophy of geometry, to a framework in which to place all rational discourse—the language of the laws of thought themselves.

Thus, we can see that Frege saw himself as extending the Kantian framework, rather than seeking to overthrow or simply disregard it. His 'enlarging' of the purview of the analytic was an attempt to extend the Kantian conclusion that the laws of arithmetic are justified by the structural nature of our minds to apply to any rational mind whatsoever, and thus safeguard against the possibility of non-equivalent alternatives. In doing so, he has had to separate the sense-experiential and cognitive aspects of thought to a greater extent than that of Kant. This in turn leads Frege to place greater emphasis on the ultimate grounds for the justification of our knowledge, over and above the contingent facts surrounding how it is that we come to know/believe the content of that knowledge. However, such an interpretation would seem to leave, through its de-emphasis on the epistemic aspects of mathematical knowledge, little room for considerations of the role of practice in the founding of such knowledge. We shall see that in actual fact, Frege left plenty of room for such consideration, but he shifted the constitutive aspects from the conceptual to the practical domain—through the innovations presented in his formulation of a *Begriffsschrift*, or 'concept-script'. It was these innovations which led him to believe that he had found the resources to found arithmetical judgements in logic, the laws of thought, and enabled him to draw the stronger conclusion just discussed. It is to these practical aspects of Frege's thought to which we now turn.

2.3 Logic and Judgement

So, what is the source of Frege's confidence that he had in fact found an adequate re-orientation of the analytic, one which enabled him to revise Kant's conclusion that arithmetic is synthetic? A key point is that, according to our previous conclusion, Frege saw himself as *extending* the Kantian framework rather than refuting it. Following this train of thought, I contend that Frege relocated the 'definition' of the analytic/synthetic distinction to the realm of the practical, and this is what enabled him to perform his logicist extensions. This centers on the idea that for Kant, the distinction between analytic and synthetic knowledge is discursive, i.e. verbally stated—and thus firmly within the bounds of philosophy. Frege, however, was able to *show* rather than 'say' the distinction, via his powerful technical innovation, his *Begriffsschrift*. Hence Frege does not merely *define* the conditions that a statement must fulfil in order to be considered analytic, nor a list of checks that will verify the presence of such conditions; he shows how an analytic *construction* of such statements should be able to be performed. Consider the following statement from the *Grundlagen*:

> Kant obviously—as a result, no doubt, of defining them too narrowly—underestimated the value of analytic judgements, though it seems that he did have some inkling of the wider sense in which I have used the term... He seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, this is one of the least fruitful. ... What we find in [the definitions in mathematics] is not a simple list of characteristics; every element of the definition is intimately, I might almost say organically, connected with all the rest. (§88)

Frege is seeking to give a characterisation of mathematical concepts which is able to capture this ‘organic’ relatedness, rather than merely ‘defining’ them by stipulation. I conjecture that it is in Frege’s *Begriffsschrift* itself, his formal language and the use it is put to, that we find evidence for the idea that Frege is able to find a Kantian argument that arithmetic will brook no alternative.

Of considerable importance is the well-known fact that Frege’s *Begriffsschrift* is a far more powerful logical system than that Kant had at his disposal. Kant had the Aristotelian syllogism, supplemented with a theory of hypothetical and disjunctive propositions.[22] Frege, on the other hand, had full second-order polyadic logic. One example of the mathematical poverty of the syllogism is that it contains only one-place predicates. Predicates with two or more places (‘relations’) are required to formalise many mathematical phenomena, such as the definition of a dense ordering:

$$\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y))$$

The relation ‘$<$’ is two-place, in that it relates two variables (as in $x < y$). The above statement says that given any x which stands in the relation $<$ to any y, then there exists a z which also stands in that relation (’is between’) both x and y. This process can, in principle, be iterated an infinite number of times. Such an iteration would result in the proven existence of an infinite number of points. Friedman (1992) notes that in a logic of only one-place predicates, the possibility (in principle) of proving the existence of an infinite number of objects is conspicuously absent:

> A central difference between monadic logic and full polyadic logic is that the latter can generate an infinity of objects while the former cannot. . . Proof-theoretically, therefore, if we carry out deductions

[22] MacFarlane (2002), p.26

> from a given theory using only monadic logic, we will be able to prove the existence of at most 2^k distinct objects: after a given finite point we will run out of "provably new" individual constants. (*Kant and the Exact Sciences*, p.59)

We *need* two-place predicates in order to be able to construct new objects in infinite quantities. It is exactly this kind of access that we need if we wish to have dense orderings; if we can iterate the process of logically constructing them only a finite number of times, then our ordering will have 'gaps' (and thus no longer be dense). In syllogistic form, an attempted construction takes this form:

P.1. Every pair of rational numbers is a pair of rational numbers with a rational number between them.

P.2. $\langle A, B \rangle$ is a pair of rational numbers.

$\therefore$ C. $\langle A, B \rangle$ is a pair of rational numbers with a rational number between them.

Here we have a pair of premises and a conclusion which together contain only two (one-place predicates): 'is a pair of rational numbers' and 'has a rational number between them'. It is obvious from the language used to express these predicates that they apply to pairs of objects, and hence are in a way 'attempting' to be polyadic predicates; nevertheless the machinery of the syllogism compels us to view the required pairs as individual objects. Also, as there are only two predicates, according to Friedman the largest number of new constants (i.e. objects) which can be constructed is $2^2 = 4$! Surely, we need more than a total of 5 pairs-of-rational-numbers-with-a-rational-between-them if we are to have a dense ordering. In syllogistic logic there is no way to proceed from the assertion that '$\langle A, B \rangle$ is a pair of rational numbers with some rational number between them' to claim that there exists 'some rational number between A and B', even though we can infer '$\langle A, B \rangle$ is a pair of rational numbers with a rational number between them'.[23] This is because the statement 'there is a rational number *between* A and B' necessarily splits the single object $\langle A, B \rangle$ into two objects (A and B) and asserts that a third object is in the polyadic relation of 'betweenness' with them.

Syllogistic reasoning proceeds via the classification of parts of the premises into the major, middle and minor terms. The general form is this:

P.1 All A's are B

P.2 X is an A

$\therefore C$. X is a B

'A' is the middle term: it is what 'connects' P.1. and P.2. in such a way as to produce C. ('B' is the major term, the predicate of which the minor term ['X'] is predicated). However, in the (attempted) derivation of the property of belonging to a dense ordering, the assertion that the rational number between A and B *exists* cannot be inferred, because there is no middle term to mediate the deduction.[24] In that example, 'A'='pair of rational numbers', 'B'='a pair of rational numbers with a rational number between them', and 'X'='$\langle A, B \rangle$' (a specific pair of rational numbers, conceived as a single object). What is actually needed in the middle term is 'if $\langle A, B \rangle$ is a pair of rational numbers with a rational number between them, then there is a rational number between A and

[23] MacFarlane (2002), p.26
[24] Ibid.

B'. However the incorporation of this statement would beg the question—that there exists a rational between A and B is what we are trying to infer; also, the splitting of $\langle A, B\rangle$ into 'A and B' introduces the need for a dyadic (two-place) predicate, which as we have already seen is not possible in syllogistic logic. Thus, in order for Kant to think himself justified in asserting the existence of dense orderings, he would have seen no course but to 'model' the concept on that of an infinitely divisible line in space (the kind postulated by Euclid), thus rendering the judgement synthetic.[25] Generally, any inference which involves polyadic predicates is seen by Kant to rely on intuition, as his conception of what it is for something to be logical or analytic is restricted to the syllogism.[26]

In contrast to this situation, Frege was able to utilise his more powerful logical resources to derive the explicitly existential statement from the stated premises—without the need for any concepts modelled on experience or recourse to intuition. In light of the reflections of section one, we are to interpret Frege as merely 'shifting the burden' of the foundations of arithmetic to the practical, to found arithmetic *mathematically* rather than *philosophically*. However, this is not to mean that we should see Frege as endorsing as a foundation for arithmetic just *any* formal system capable of supplying the requisite deductions. I contend that Frege was in fact seeking to characterise the analytic, not merely to model it in the manner that Kant modelled his concepts on intuited phenomena. This interpretation will involve two strands. The first is a notion of an 'inferential correspondence' between a subject matter and the language used to talk about it; the second is the principle of multiple analysability, embodied in the two-dimensional, graphical nature of Frege's *Begriffsschrift* notation.

[25] Ibid; see also Friedman (1992), p.70/1 for an in-depth discussion.

[26] Friedman (1992), p.63: "For Kant logic is of course syllogistic logic or (a fragment of) what we call monadic logic. Hence for Kant, one cannot represent the infinity of points on a line by a formal theory such as [Hilbert's continuity axioms]. If logic is monadic, one can only represent such infinity intuitively—by an iterative process of spatial construction…".

2.3.1 Inferential Correspondence

On my re-interpretation, the idea of an 'inferential correspondence' between a subject-matter (i.e., collection of objects) and the language/notation used to talk about it, is essential to the link between Frege's practical innovation in the field of logic and his theoretical contributions to philosophy. This notion of inferential correspondence links to Frege's idea of what it is for something to be justified, or justifiably assertible; and the notion of justification is vital to Frege's project, as the preceding reflections have borne out.

Consider some field of natural science. The language used to talk about the objects and relationships which constitute that field is comprised, in a simplified manner, of two things: terms, which are the main ingredient for the forming of propositions (i.e. assertions about the objects), and rules for inferring new propositions from those already present. The idea of an isomorphism between the actual objects being investigated and the language used to talk about them is an expression of the intuitive notion that the two structures should mirror each other. When a science is at a mature level of development, one would hope that an inferential move sanctioned by the language would always (necessarily) be possessed of a structural counterpart in relations between the objects themselves. Conversely, every structural relationship that exists between objects should be able to be modelled in the language. Frege hints at such an idea when, in the *Begriffsschrift*, he notes when discussing Leibniz's notion of a '*calculus rationator*' that "... he contemplated the immense increase in the intellectual power of mankind that a system of notation *directly appropriate to objects themselves* would bring about..."[27] The phrase 'directly appropriate to objects themselves' is telling, for Frege is prioritising the relationships between

[27] Preface; emphasis mine.

objects to our subject-predicate apprehension of them.

In the *Grundlagen* Frege cautions us against 'letting our notation think for us' without being certain that the notation has a complete inferential correspondence with the subject under study (in the case of *Begriffsschrift*, the logical form of propositions and mathematical proofs).[28] In the *Grundgesetze*, he observes how

> It is remarkable how an inaccurate manner of speaking or writing, perhaps originally employed for convenience or brevity but with full consciousness of its inaccuracy, can end in a confusion of thought once that consciousness has disappeared... Such experiences teach us how necessary it is to place the highest demands on the accuracy of a manner of speaking and writing. (§0)

However, as Frege considers himself to have been adequately rigorous in his formulation of the *Begriffsschrift*, he is entitled to make the claim that any arithmetical judgement made with the aid of his notation can only be considered synthetic as an epistemic matter-of-fact; doing so does not affect the analytic *foundation* of the judgement[29] (one result of this is Frege's methodological stipulation that a function give a well-defined output for any well-defined input). Our reason for believing the judgement to be true (the metaphysical

[28] "It is possible, of course, to operate with figures mechanically, just as it is possible to speak like a parrot; but that hardly deserves the name of thought. It only becomes possible at all after the mathematical notation has, *as a result of genuine thought*, been so developed to that it does the thinking for us, so to speak." *Grundlagen*, Introduction, p.IV; emphasis mine.

[29] "...we divide all truths that require justification into two kinds, those for which the proof can be carried out purely by means of logic and those for which it must be supported by facts of experience. But that a proposition is of the first kind is surely compatible with the fact that it could nevertheless not have come to consciousness in a human mind without any activity of the senses." *Begriffsschrift*, Preface.

foundation which is a consequence of the structure of the rational mind) and the way in which we come to believe it to be true (the spatial intuitions we have of *Begriffsschrift* symbols written on a physical page) can be separated in the way desired by Frege and unavailable to Formalists, Empiricists, and indeed Kant himself.

Kant cannot separate the metaphysical and epistemically intuitive aspects of an arithmetical judgement because his logic is inadequate to the task of modeling the kinds of relations which occur in arithmetic. For example, the physical world as characterised by Euclidean geometry, is possessed of features which cannot be appropriately constructed in any manner which Kant would consider 'analytic'. Friedman states that

> The idea of infinite divisibility or denseness is not capturable by a formula or sentence [of monadic logic], but only by an intuitive procedure that is itself dense in the appropriate respect... Thus, the proposition that space is infinitely divisible is a priori because its truth—the existence of an appropriate "model"—is a condition for its very possibility. (*Kant and the Exact Sciences*, p.66)

Hence Kant cannot even envision the kind of correspondence between a purely logical language and (e.g.) Euclidean space which can be given in Frege's enhanced logic.[30] Likewise, certain arithmetical objects, such as what Frege terms

[30] However, Frege agreed with Kant about the synthetic truth of geometry. Nevertheless he worked in non-Euclidean geometry, regarding it as a logical system rather than a description of any actual space. Hence he was aware of a distinction between 'pure' and 'applied' geometry which was unavailable to Kant. Frege considered Euclidean geometry synthetically true because appeal to intuition is required to accept it as true of actual space over alternative systems (See Currie (1982), p.33/4). The very presence of alternative systems means that geometry cannot be analytically true; the opposite is the case for arithmetic.

'hereditary sequences',[31] cannot be captured in a logic which has monadic predicates only. This manifests itself in the way Kant draws the distinction between mathematics and philosophy. Philosophy is discursive, as shown by its analytic method of dissection of concepts, whereas mathematics is intuitive, or constructive, in that it relies on a constitutive role for intuition in progressing from one mathematical statement to another:

> [W]e find that all mathematical cognition has this peculiarity: it must first exhibit its concept in intuition, and do so *a priori*, in an intuition that it not empirical but pure. Without this mathematics cannot take a single step; hence its judgments are always *intuitive*; whereas philosophy must be satisfied with *discursive* judgments from mere concepts, and though it may illustrate its apodeictic [absolutely certain] doctrines through intuition, can never derive them from it. (*Prolegomena*, §7)

Frege's innovation was the realisation that the epistemic aspect of a judgment can be catered for by the practical tool of *Begriffsschrift*, whereas the theoretical aspect and its link to the structure of a language or notation are firmly metaphysical affairs. In other words, we know that any inference successfully made in *Begriffsschrift* is indeed a syntactic counterpart of a logical connection that does obtain. By using a symbolism which makes aspects of logical form and inference such as quantification and polyadic predication syntactically perspicuous, rather than at the level of (discursive) concepts, a *practical* aspect of the nature of judging—the move from a judgement to a consequence of that judgement—is being relegated to the fully epistemic nature of the judgement

[31] See *Begriffsschrift* §23, *Grundgesetze* §45; an illuminating discussion of whether 'following in a series' is a synthetic property occurs in Boolos (1985) 'Reading the *Begriffsschrift*', reprinted in *Frege's Philosophy of Mathematics* Demopoulos (ed), (1995); pp.163–179.

(the symbolism), rather than being mixed in with the metaphysical and its semantically constitutive role (the content). For example, the use of a rigorously formulated logical language (one which can 'do our thinking for us') frees us from the restriction that we be able to comprehend a proof in a single step. We can move from one step to another, safe in the knowledge that we are making only deductively sound transitions: "Its first purpose, therefore, is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed…".[32] The need for an intuitive manner to apprehend information can be divorced from playing a constitutive role in the content of that information; in a well-thought out symbolism, this is done in such a way as to pre-ensure the relation of inferential correspondence. We may let our notation think for us, if its structure is 'the result of genuine thought'.

Thus, we can interpret Frege's logicism as an attempt at a 'rational reconstruction' of the laws of arithmetic, in such a way as to show that they are derivable from the resources of his *Begriffsschrift* without the constitutive aid of any synthetic judgement. The basic laws and rules of inference of Frege's system were selected for their self-evidence; they were intended be such that even someone who thought that *arithmetic* was synthetic would still accept the basic laws as analytic. Anyone who rejected the derivation of arithmetic from those basic laws would be, in Frege's eyes, irrational.

[32] *Begriffsschrift*, Preface

2.3.2 The Principle of Multiple Analysability

I will now argue that Frege's notion of multiple analysability and its relation to one of Frege's better-known contributions to the philosophy of language, the context principle[33], are the result of his attempt to 'improve on the view of Kant'. Because the doctrine of multiple analysability is the lesser-known, I will begin with its explication, reserving the context principle for afterwards.

There is some amount of textual evidence for the fact that Frege held view that propositions are possessed of many different possibilities regarding what exactly they say; however, the main source of justification for this claim is to be found in the *Begriffsschrift* notation itself, and will be the subject of the next section. The main focus of the current discussion is textual evidence and its implications for Frege's philosophy as a whole. For instance, in §46 of the *Grundlagen*, Frege observes that

> While looking at one and the same phenomenon, I can say with equal truth both "It is a copse" and "It is five trees", or both "Here are four companies" and "Here are 500 men". Now what changes here from one judgement to to another is neither any individual object, nor the whole, the agglomeration of them, but only my terminology. But that is itself only a sign that one concept has been substituted for another.

This is a fairly straightforward statement of what I shall term Frege's doctrine of 'multiple analysability'. It is, broadly speaking, the idea that analysis of a state

[33] The other well known contribution being the distinction between Sense and Reference; see chapter three, §3.2.4

of affairs is prior to any assertion of that state of affairs's being true. According to the above quote, this doctrine holds for what may be termed 'unanalysed' states of affairs; situations where one is yet to formulate a propositional description. However, Frege also holds that the doctrine applies to *descriptions* of situations:

> The judgement "line a is parallel to line b", or, using symbols,
>
> $$a//b$$
>
> can be taken as an identity. If we do this, we obtain the concept of direction, and say: "the direction of line a is identical with the direction of line b". Thus we replace the symbol // by the more generic symbol =, through removing what is specific in the content of the former and dividing it between a and b. *We carve up the content in a way different from the original way, and this yields us a new concept.* (*Grundlagen*, §64 [my emphasis])

The idea here is that given a proposition, one can analyse it into different concepts and/or objects. This has affinities with Frege's statements in the *Begriffsschrift*, where he emphasises that in (what could broadly be termed) his 'embryonic philosophy of language', he is taking the notions of function and argument as *prior* to those of subject and predicate.

> A distinction between *subject* and *predicate* does *not occur* in my way of representing a judgement...[T]he contents of two judgements may differ in two ways: either the consequences derivable from the first, when it is combined with certain other judgements, always

> follow from the second, when it is combined with these same judgements, [and conversely,][34] or this is not the case. (*Begriffsschrift*, §3)

Frege is talking about propositions where the subject and predicate can be interchanged, as in the case of 'The A has property P' versus 'The property P is instantiated by A'. Frege gives the following example:

> The two propositions "The Greeks defeated the Persians at Plataea" and "The Persians were defeated by the Greeks at Plataea" differ in the first way [that the contents of either can be derived from the other]...Now I call that part of the content that is the *same* in both the *conceptual content*. Since *it alone* is of significance to our ideography, we need not introduce any distinction between propositions having the same conceptual content. (Ibid.)

Thus, the symbolising of relations in the *Begriffsschrift* takes place at a higher level than that of the subject-predicate form, as found in ordinary language (and the syllogism). The two objects a and b are related by R; hance we can equivalently say either that a R's b or that b is being R'd by a — in the former case, b is predicated with the property of being R'd by a, whereas in the latter a has the feature of being R'd by b. The two analyses are equivalent, however they yield different objects and predicates in each case. The statement has *multiple analyses*.

Indeed, Frege believed that natural language merely *symbolises* the various states of affairs regarding objects being related to one another, and incompletely

[34] Insertion original to translation.

at that. The statement 'All whales are mammals',[35] in its natural language form, looks like a statement about the subject, all whales. But, for Frege, it is not in fact about whales but about a *concept*:

> As a general principle, it is impossible to speak of an object without in some way designating or naming it; but the word "whale" is not a name for any individual creature. If it be replied that what we are speaking of is not, indeed, an individual definite object, but nevertheless an indefinite object, I suspect that "indefinite object" is only another term for concept, and a poorer, more contradictory one at that. (*Grundlagen*, §47)[36]

He reads this as 'take anything; if it is a whale, it is a mammal' which equates with the first-order sentence $\forall x(Wx \rightarrow Mx)$. On this reading, we find that the sentence is not 'about' whales *per se*, but rather is about the concept 'all whales', and states that any object falling under the concept 'whale' also falls under the concept 'mammal'. Thus, the sentence is stating that the two concepts 'whale' and 'mammal' stand in a certain (second-order) relation, namely that which Frege calls 'subsumption'.[37] The concept 'whale' is 'contained' in the concept 'mammal'.

However, in *Begriffsschrift*, many sentences are capable of multiple analyses which are equivalent in terms of truth-conditions, though this may not be immediately obvious in natural language form or in modern notation. In order to appreciate this, we will have to familiarise ourselves with the *Begriffsschrift*

[35] see *Grundlagen* §47

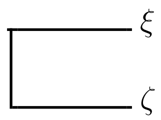
[36] Further discussion of Frege's attitude toward 'indefinite objects' will take place in chapter three, §3.2.5

[37] See Macbeth (2005), p.104 and *Grundgesteze* §22

notation. At this stage, we will concentrate on two main features: interchangeability of subcomponents, and contraposition.

2.3.3 Interchangeability of Subcomponents

In the 'Exposition of the Begriffsschrift', which comprises the first technical section of Frege's *Grundgeseze der Arithmetik*, Frege details how one can, in his *Begriffsschrift* or concept-script, freely alter the order of premises in an inference ('subcomponents'). For Frege, there is one main form of inference, of which any other (for instance, the traditional forms of argument associated with the method of Aristotelian syllogism) can be built up from. This form of inference Frege writes as[38]



which is a function "...that [takes] its value [to] be the False if the True be taken as ζ-argument and any object other than the True be taken as ξ-argument, and that in all other cases the value of the function shall be the True."[39] Not surprisingly, Frege calls the vertical component of this symbol the 'condition-stroke'; indeed, Frege's explanation of it in the earlier *Begriffsschrift* of 1879 makes out its similarities to our modern '$A \to B$' even clearer:

> If A and B stand for contents that can become judgments (§2[40]), there are the following four possibilities:
>
> (1) A is affirmed and B is affirmed;

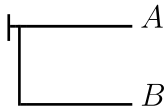
[38] See *Grundgesetze* §12
[39] Ibid.
[40] [Of the *Begriffsschrift*]

(2) A is affirmed and B is denied;

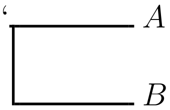
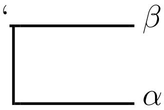
(3) A is denied and B is affirmed;

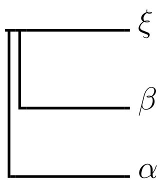
(4) A is denied and B is denied.

Now

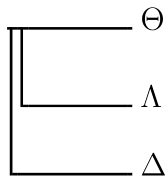


stands for the judgment that *the third of these possibilities does not take place, but one of the other three does.*(*Begriffsschrift* §5)

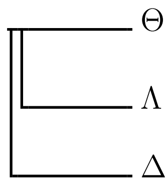
Clearly, Frege has described his 'condition-stroke' in a manner which makes the truth-conditions of '' identical to those of '$A \rightarrow B$'. They are both false only in the case where A is true and B is false. In the *Grundgesetze*, after introducing his condition-stroke, Frege goes on to describe a property of his system which he terms the 'interchangeability of subcomponents'. In a condition-stroke, the order of the premises is immaterial. While this makes no difference in the cases described above (there can only be one subcomponent), if we take as the ζ-argument the statement '', then the following formula is obtained:



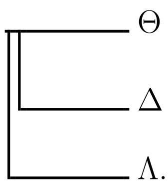
Now, as this statement is false only when α is true, β is true, and ξ is false, the *order* of α and β in the statement does not effect the resultant truth-value. Frege describes this situation thus:



is the False if Δ and Λ are the True while Θ is not the True; in all other cases it is the True. From this there follows the interchangeability of Λ and Δ;

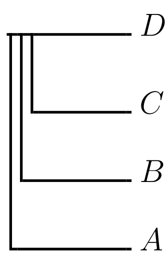


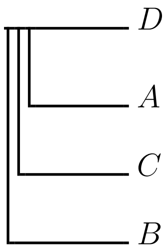
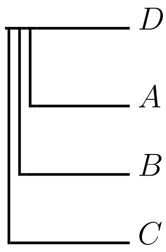
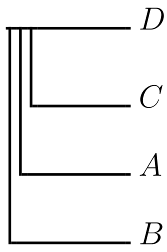
is the same truth-value as



.(*Grundgesetze* §12)

Thus we can see that a four-term statement has, via the interchangeability of subcomponents, many different ways of being expressed. For instance, the thought embodied in

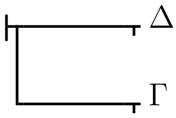


can be expressed as it is, or as  , or as  , or as 
without altering the truth-conditions of the statement. And, as Frege held that formal difference in the expression of a thought does not affect its content,[41] interchangeability of subcomponents is an instance of multiple analysability.

2.3.4 *Contraposition*

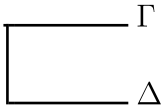
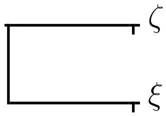
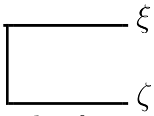
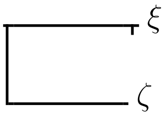
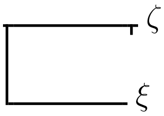
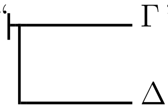
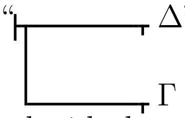
In Frege's concept-script it is possible to alter the order of premises in an inference without alteration of the content, i.e. the thought embodied in the proposition. However, it is also possible to change the position of the conclusion with a similar (non-)effect on the content of the proposition. This is done with the help of the negation sign ('┬'); it is the better-known technique of *contraposition.*

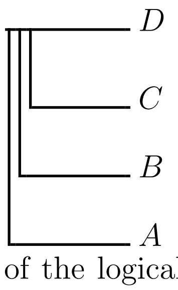
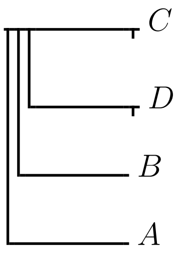
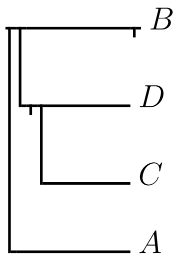
In modern logic, it can easily be shown that $P \to Q \vdash \neg Q \to \neg P$. This is merely a refined way of stating the argument form of 'Modus Tollendo Tollens', known since scholastic times. In the *Grundgesetze*, Frege states that

 is the False if ┬ Γ is the True and ┬ Δ is not the True;

i.e., if —— Γ is the False and Δ is the True. But the same holds

[41] Cf. MacBeth (2005), p.55: "[S]uch transitions are not acts of reason which that take one from one thought to another, but only transitions from a thought in one form to that thought in another form."

> for  ; thus the functions  and  always have the same value for the same arguments.... The functions  and  also have the same value for the same arguments. Thus we may pass from the proposition “” to the proposition “” and conversely.... *A subcomponent may be interchanged with the main component if the truth-value of each is simultaneously reversed.* This transition we call *contraposition.* (§15)

For example, consider the proposition  discussed above. There we gave four possible equivalent analyses of the logical relationships between the premsises$A$, B and C and the conclusion D which are embodied by this proposition. With our new tool of contraposition, we can add many more; too many to list here. There are many different ways to do this; one can merely interchange the order of the premises, as in . One can also ‘isolate’ a group of terms into a subcomponent and then contrapose, as in 

. It should be clear that through repeated iterations of this process, we can make any selection of the premises and/or conclusion into a subcomponent; we can then place the various components in any order we wish (provided we insert/remove the appropriate negation symbols). That the techniques of interchanging subcomponents and contraposition are for Frege rules of inference, rather than axioms—such techniques are processes which take us from a thought in one form to the same thought in another form—lends to a greater generality of application than analogous techniques in modern systems. In order to further explicate this difference, we shall now compare some proofs in Frege's notation with how the same proof would be carried out in a modern natural deduction systems.

2.3.5 Comparing Proofs in Begriffsschrift *and Natural Deduction*

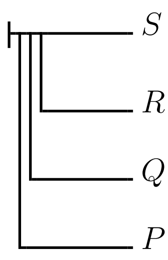
First of all, we we will look at a proof of a contraposition similar to those described in the previous section (using an ordinary natural deduction-style system).

Theorem: $P \to (Q \to (R \to S)) \vdash P \to (\neg(R \to S) \to \neg Q)$

Proof

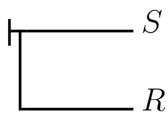
(1)	1.	$P \to (Q \to (R \to S))$	[Assumption]
(2)	2.	P	[A.]
$(1,2)$	3.	$Q \to (R \to S)$	[1, 2, Modus Ponens]
(4)	4.	Q	[A.]
$(1,2,4)$	5.	$R \to S$	[5, 4, M.P.]
(6)	6.	$\neg(R \to S)$	[A.]
$(1,2,4,6)$	7.	$(R \to S) \wedge \neg(R \to S)$	[5, 6, $\wedge$-introduction]
$(1,2,6)$	8.	$\neg Q$	[7, Reductio ad Absurdum]
$(1,2)$	9.	$\neg(R \to S) \to \neg Q$	[6, 8, Conditional Proof]
(1)	10.	$P \to (\neg(R \to S) \to \neg Q)$	[2, 9, C.P.]□

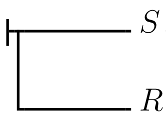
As we can see, this is a 10-line proof that involves making four assumptions (three not counting the premise), two uses of conditional proof, and one use of proof by contradiction. In Frege's symbolism, this proof can be carried out in two steps. The first involves analysing the proposition into components, and the second invokes the contraposition rule. Observe:

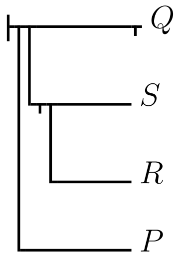


We analyse this into three components thus

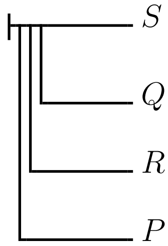
$$\vdash P$$

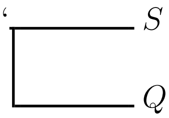


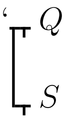
We now contrapose the Q and , taking care to insert a ‘’ in front of them when performing the contraposition, yielding



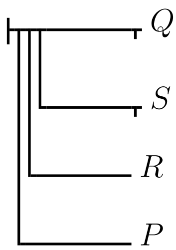
which is rendered into modern notation as $P \to \neg((R \to S) \to \neg Q)$, the desired formula. Also, by interchangeability of subcomponents, we can transform our proposition into



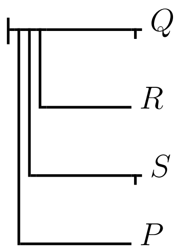
which is equivalent to $P \to (R \to (Q \to S)$. An application of contraposition to ‘

’ yields ‘

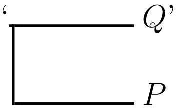
’; and our proposition becomes

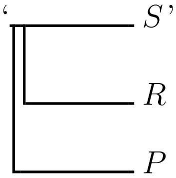


Via another change of order of subcomponents, we have



We can see that the *Begriffsschrift* equivalent of $P \rightarrow (Q \rightarrow (R \rightarrow S)) \vdash P \rightarrow (\neg S \rightarrow (R \rightarrow \neg Q))$ takes four steps. The Natural Deduction proof is slightly more complicated than that of the example given above; it is thirteen lines long, and requires four assumptions (not counting the initial premise), three uses of conditional proof, and one reductio ad absurdum. Furthermore, it would seem that there is, in general, no modern counterpart to Frege's interchangeability of subcomponents. Whilst it is intuitively obvious, and thus can be proven in each case, that the order of antecedents in a conditional statement is immaterial to its validity, it cannot be formulated as a general rule (at least not in the natural deduction system we are working with). This is because natural deduction systems work with what may be termed ‘individual’ components, such as $P \rightarrow Q$, rather than whole statements. When it comes to statements with only two terms, such as $P \rightarrow Q$, the order of premises is important. Frege acknowledges this, supplying the rule of contraposition to deal with interchanging antecedents and consequents. However, in modern systems, the '$P \rightarrow Q$' in '$P \rightarrow Q$', in '$P \rightarrow (Q \rightarrow (R \rightarrow S))$', and in '$\neg S \rightarrow (P \rightarrow (Q \rightarrow R))$' behave differently. The order of terms cannot be altered in the former; likewise, it cannot be altered in the latter in a manner characterised by a general rule—even though a proof specific to the proposition involved can be given to show that the order can be altered. The point is that the proofs of $P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)$ and $P \rightarrow (Q \rightarrow (R \rightarrow S)) \vdash Q \rightarrow (P \rightarrow (R \rightarrow S))$ will look different, and be specific to the propositions in question, even though there seems to be a general principle at work.

In Frege's system, the situation is exactly the opposite. While the order of terms in '

' is important, and can only be altered with the aid of negation symbols (i.e. via contraposition), when it comes to a more complex statements such as '

' we can freely interchange the antecedents, as noted above.[42] This holds no matter what terms are substituted for 'P', 'Q' or 'R', even if these are complex formulae—the same rule holds; all such proofs will be similar, both in terms of rules invoked and in appearance. This is because Frege's system seems to be more focussed on the 'form' of an inference, rather than its specific content. This is shown by Frege's employment of rules which are based purely on the form of an inference, such as the interchangeability of subcomponents rule; it is also shown by the fact that these rules can be applied similarly to statements with different numbers of terms—which is definitely not the case in modern natural deduction systems, as evidenced above. This form-over-content approach, in the most formal of endeavours, formal logic, can be

[42] See *Grundgesetze* §12

taken as evidence that Frege was bearing a form/content distinction reminiscent of Kant's when designing his notational system.

These two rules, interchangeability of subcomponents and contraposition, are both examples of Frege's principle of multiple analysability manifesting itself in technical aspects of his system. This is evidenced by the fact that the use they are put to has no easy counterpart in modern systems. Also serving as evidence is the way that they are applied to whole propositions rather than specific components, unlike the rules of inference found in modern natural deduction systems. However, in order to investigate the connection with Kant, we must now view Frege's context principle in terms of multiple analysability.

2.3.6 The Context Principle (In Context)

Frege's most forward statement of the context principle occurs at the beginning of the *Grundlagen*, when he is outlining his methodology:

> [I have resolved] never to ask for the meaning of a word in isolation, but only in the context of a proposition. . . (Introduction, p.X)

The principle also manifests itself in a form that iterates to the level of the relationship between propositions and whole contexts, as in the section of the *Grundlagen* quoted above (§2.3.2): "While looking at one and the same phenomenon, I can say with equal truth both "It is a copse" and "It is five trees", or both "Here are four companies" and "Here are 500 men"."[43] What Frege is observing is that questions to which number phrases are the answer, as in

[43] §46

'how many are there?' should not be taken as having an absolute answer, independent of any context. If one points to a company and asks, 'how many are there?', two 'equally correct' answers are possible (4 and 500). One object cannot both be four things and at the same time 500 things, as these are judgements ascribing different numbers to the same object. One would not say object x is one thing, and immediately assert that the same x is two things, without going on to fill out the variable phrases 'thing' and 'x':

> [W]hat changes here from one judgment to the other is neither any individual object, nor the whole, the agglomeration of them, but only my terminology. But that itself is only a sign that one concept has been substituted for another. (*Grundlagen* §46)

Providing an answer to the question 'how many are there?' requires specification of context; thus we find that in the context of filling out 'thing' with 'company' in the first instance and with 'men' in the second, the two assertions 'here are four companies' and 'here are 500 men' lose all appearances of contradicting each other. However, the object itself has not changed, only the way we are regarding it.

It seems that Frege is telling us that the question 'how many are there?' cannot be given a full interpretation[44] when supplemented only with an ostensive act of pointing—for instance, pointing at a pack of cards (the two answers '1' [pack of cards] and '52' [individual cards] are both 'correct'). Hence, the context principle and the notion of multiple analysability are two sides of the same coin. We are not to ask for the meaning of a word outside the context of a proposition, whilst analysis of a proposition is prior to the question of

[44] I.e. be regarded as 'saturated'; we will come to Frege's notion of 'saturation' in Chapter Three, §3.2.2

that proposition's truth (as evidenced by the possibility of altering the order of antecedents and contraposing the terms of a complex *Begriffsschrift* conditional). The meaning of a word *within* the context of a proposition is fixed by analysis of that proposition—for instance, whether it is to be taken as (part of the) function or (part of the) argument. In order for the meaning of a word to be given, it must be placed in the context of a proposition; and once the proposition is given one of its multiple analyses, the question of its truth can be answered. Thus meaning is the amorphous aspect within the context of a proposition prior to analysis, whilst truth is amorphous prior to an analysis of a proposition within the larger context of a whole body of knowledge. The idea behind truth being amorphous prior to analysis of a proposition within the context of a body of knowledge is that the different possible analyses of a proposition yield different objects (i.e. arguments for functions), and some analyses may be incompatible with others due to their asserting the existence of different collections of objects.

Multiple anaysability, as a general principle, is the result of Kant's dictum that in order for analysis to proceed something must be synthesised together prior to analysis being applied to a conception of language which goes beyond the subject-predicate form. As Hans Sluga writes, "Frege's principle that words only have meaning in the context of a sentence must be... interpreted as a linguistic version of Kant's principle of the transcendental unity of judgment."[45] We see that far from being an assertion of a *priority* of the analytic over the synthetic, Frege's statement that arithmetic is analytic is actually a more subtle claim. Kant claims that the synthetic holds priority over the analytic, as analysis separates wholes into parts, whereas this separation is possible only if the whole has already been synthesised out of parts:

[45] 'Frege's Alleged Realism', *Inquiry*, No. 20, 1977, p.238

> For where the understanding has not previously combined, it cannot dissolve, since only as having been combined *by the understanding* can anything that allows of analysis be given to the faculty of representation. (*Critique of Pure Reason*, B130)

On my reading, Frege does not dispute this; rather, his claim that arithmetic is analytic is a statement to the effect that given a context, *any* context, the nature of the rational mind ensures that there is one and only one way in which to reach correct arithmetical judgments about this context. Such judgments will be performed relative to the thinkers orientation to the context—i.e. whether one is looking for individual cards or whole packs. Thus the principle of multiple analysability is the link between Kant and the context principle which motivated Frege to (among other things) build interchangeability of subcomponents and contraposition into his logical system in the way he did.

2.4 Conclusion

This chapter has been a review of the 'philosophical' evidence for us to view Frege's innovations as a change of setting. We bear in mind the two major conditions, outlined in chapter one, which constitute the difference between a change of setting and a change *simpliciter*:

- that the questions addressed in the new settings show some continuity with the old setting, and
- that the formulation of the new setting is a non-trivial intellectual achievement.

The evidence for the satisfaction of the first condition is Frege's addressing his project in Kantian terms. This was argued for in §2, by identifying Frege's motivations for desiring to characterise arithmetic as 'analytic'; it was found that his reason was to safeguard arithmetic against the possibility of alternatives which had arisen for geometry—a possibility which is disallowed under Kant's philosophy of mathematics. This was then followed by a detailed exposition of some specific aspects of Frege's approach, especially the principle of multiple analysability. The second condition is addressed by Frege's technical innovation, the *Begriffsschrift*, or concept-script; it allowed him (he hoped) to *show*, rather than *say*, that arithmetic is possessed of defining attributes which ensure that a non-equivalent system of arithmetic is not rationally possible—a conclusion which was beyond the discursive musings of the Formalists and Empiricists.

However, this second conclusion is not, as yet, quite within our grasp. The final chapter will comprise a detailed study of a specific aspect of the system that Frege constructed, using his *Begriffsschrift*, in pursuit of his ultimate goal of deriving the laws of arithmetic from logic alone.

3. THE SYSTEM OF THE *GRUNDGESETZE*

3.1 Introduction

The system Frege expounded in his *Grundgesetze der Arithmetik*[1] is mammoth; as of 2009, only around one-fifth the two volumes have been published in English. This work was intended by Frege to substantiate the logicist claims made in his *Grundlagen* nine years previously. In concluding the *Grundlagen*, Frege had noted that "From the preceding it thus emerged as a very probable conclusion that the truths of arithmetic are analytic and a priori. . . We saw further what is still needed to raise this probability to a certainty, and indicated the path which must lead to that goal."[2] The *Grundgesetze* is the attempt to fulfil that promise—an absolutely thorough development of the natural number system from nothing more than the laws of thought.[3]

The logical system presented in the *Grundgesetze* has many technical features which are interesting from the point of view of the modern logician; for example, functions are taken as 'logically prior' to the sets which they deter-

[1] Two volumes, 1893/1903. Translated to English by Montgomery Furth as *The Basic Laws of Arithmetic: Exposition of the System* (1964).

[2] §109

[3] Frege also intended to develop the real number system in a like manner; however, this project was left only partially completed when Frege abandoned his logicist programme in the wake of Russell's paradox.

mine.[4] This is opposed to the modern conception where functions are defined as a special kind of set.[5] Unfortunately, this system is also inconsistent—a contradiction can be derived in it (Russell's paradox), using only logical resources. The nature of this inconsistency has been the subject of a great deal of scholarship over the last hundred years or so.[6] Therefore, I will not address the paradox beyond a brief note of its effects on the future development of foundational theories of mathematics; specifically, Bertrand Russell's Type Theory and the axiomatisation of Set Theory mainly due to Ernst Zermelo.

Russell's ramified theory of types was developed in his *Principles of Mathematics* (1903), 'Mathematical Logic as Based on the Theory of Types' (1908*a*),[7] and *Principia Mathematica* (1910/12/13), co-authored with A.N. Whitehead. Zermelo's axiomatisation of the set theory descended from Cantor and Dedekind occurs mainly in 'Investigations in the Foundations of Set Theory I' (1908*a*).[8] The former approach assigns all functions a 'type', or 'level'; functions can only take arguments which are of a lower type than they are.[9] This blocks the function 'set which is not a member of itself' from taking itself as argument (the origin of the inconsistency). While largely abandoned in modern times as a research programme in the foundations of mathematics, it has found many

[4] See 'A Critical Elucidation of Some Points in E. Schroeder's *Vorlesungen Ueber die Algebra der Logik*': "I do, in fact, maintain that the concept is logically prior to its extension; and I regard as futile the attempt to take the extension of a concept as a class, and make it rest, not on the concept, but on single things." (p.106 in Geach & Black [eds]) For a comparison between Frege's approach and that of Russell & Whitehead, see Furth's introduction to the *Grundgesetze*, p.*xli*/*ii*

[5] Specifically, a set of ordered pairs; for a similar discussion in a modern context, see Chapter One, §1.3.3.

[6] See for instance Boolos (1987) & (1986/87) and Parsons (1987) (all reprinted in Demopoulos (1995)). Aside from these, there are numerous articles addressing various aspects of Russell's paradox and its impact on Frege. Any introductory text will contain at least a chapter; for instance, Chapter 5 of Currie (1982), Chapter 5 of Macbeth (2005), and Section 2 of Zalta (2009). Link (2004) is an entire *book* dedicated to the paradox.

[7] Reprinted in van Heijenoort (1967), pp.150–182

[8] Reprinted in ibid., pp.199–215

[9] The type-level is indicated by an index located in the upper right corner of the function-name; e.g. $f^2(x^1)$ is a type-level 2 function taking a type-level 1 argument.

fruitful applications in computer science.

The latter approach is based upon a modification of Frege's Basic Law V (that which sanctions the derivation of the contradiction). The modified axiom, which is usually called 'comprehension' or 'abstraction', states that a set can be defined by a function only if that set is a subset of a larger set which has already been defined. One cannot create sets *ex nihilo*, or (equivalently) by unrestricted abstraction. While this is seemingly a more *ad hoc* answer to the problem, this kind of set theory remains the conventionally accepted 'foundation' of mathematics (see Chapter One for a contemporary discussion).

In the following I focus mainly on Frege's notion of 'logical objects' and the use he makes of them. My contention is that he uses them to connect the mind to the world, in that their existence is implied by the presence of a rational mind (a position with numerous affinities with Transcendental Idealism). Looking into Frege's ontology for particular evidence for this claim, we will find that his distinction between functions and objects easily accommodates such an approach. Furthermore, Frege appropriates the mathematical notion of a function, and applies it to philosophical aspects of the foundations of mathematics. Down the line we will come to the construction of the definition of one-to-one correspondence in the *Grundgesetze*; in following this construction we will see that Frege makes considerable use of the logical objects which are the 'extensions' of functions. These objects, extensions, lie at the very boundary of function and object; and this boundary, I will argue, is the most basic in Frege's ontology. Thus we begin with an introduction to the concepts necessary for this investigation.

3.2 Functions and Objects

3.2.1 Functions

By the late 19th century, the function had become one of the main focuses in philosophically-minded mathematical investigations in continental Europe. It had been discovered that functions can be defined which are nowhere continuous, such as ‘the function which takes the value zero for rational arguments and the value one for irrational arguments’ (known as the Dirichlet function). There are also functions which are everywhere continuous but nowhere differentiable, such as the Koch function. The existence of this sort of function showed that the intuitive connection between continuity and differentiability was not as close as it seemed;[10] differentiability implies continuity, but not vice versa. Mathematicians were becoming less confident in their understanding of what *exactly* a function *is*, despite the fact that what they could do with them was becoming more and more advanced.

In his works, both philosophical and technical, Frege appropriates the notion of a function for his own uses. For him, a function is the most basic unit, the link between his conception of mathematics and natural language; virtually everything which occurs in Frege’s system can be described in terms of functions. In the essay ‘Function and Concept’,[11] Frege states that

> . . . the field of mathematical operations that serve for constructing functions has been extended. Besides addition, multiplication, exponentiation, and their converses, the various means of transition to

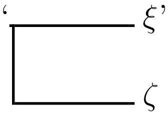
[10] MacBeth (2000), pp.173-4
[11] In Geach & Black, pp.21-41

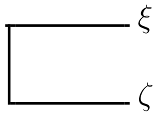
> the limit have been introduced—to be sure, without people's being always clearly aware that they were thus adopting something essentially new. People have gone further still, and have actually been obliged to resort to ordinary language, because the symbolic language of Analysis failed; e.g. when they were speaking of a function whose value is 1 for rational and 0 for irrational arguments.
>
> Secondly, the field of possible arguments and values for functions has been extended by the admission of complex numbers. In conjunction with this, the sense of the expressions 'sum,' 'product,' etc., had to be defined more widely.
>
> In both directions I go still further. (p.28)

By 'in both directions, I go still farther', Frege means that he has made the function the basic unit in his theory. The distinction between particular and general referring expressions is made out in terms of the properties of functions; this itself has a bearing on the distinction between an object and its name, as we shall see. Even the *Begriffsschrift* connectives in the *Grundgesetze*, such as $\begin{array}{l}\ulcorner B\\ \llcorner A\end{array}$ are conceived as functions. In modern logic, logical terms such as connectives and quantifiers are taken to be syncategorematic items, which means they are incapable of being meaningful by themselves. For example, '$\rightarrow$', or '$\neg$', are not meaningful terms (while both '$A \rightarrow B$' and '$\neg A$' are). An example of the modern approach to sentential connectives:

> The five symbols
>
> $$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$$

> are called *sentential connective* symbols. Their use is suggested by the English translation given above [example: $\wedge$ = connective symbol = English: and]. In translating to and from English, their role never changes. (Enderton, H.B. (2001) *A Mathematical Introduction to Logic*, 2nd edition)

We can see that this modern treatment defines the connectives in a rather intuitive manner, while the truth-value of a statement is defined in terms of a further function called a truth assignment. Thus, the connective symbols say nothing about truth in themselves. In contrast, Frege describes ‘’ as a “function of two arguments”[12] which is defined as



$$= \begin{cases} \text{the True, if } \zeta \text{ is the False or } \xi \text{ is the True, or} \\ \text{the False otherwise.} \end{cases}$$

The universal quantifier and negation are defined in similar terms. Thus we can see that Frege is conceiving of the conditional as a function which takes truth-values A and B as arguments, and gives the truth-value of ‘if A is the case, then B is the case’.

3.2.2 *Objects*

Besides functions, there is one other type of thing in Frege’s system, objects. The distinction between functions and objects is the most basic in Frege’s conception of logic, as evidenced by his early mention of it in the *Grundgesetze*:

[12] *Grundgesetze*, §12

> *Objects* stand opposed to functions. Accordingly I count as *objects* everything that is not a function. . . (§2)

The function/object distinction is a metaphysical one; it demarcates all entities into two categories. However, we also need a way of apprehending this distinction; a method of recognising, for any given thing, whether it is a function or an object. Thus, Frege supplies an epistemic counterpart to the function/object distinction, which is given by the notion of *saturation*.

Let us begin with an example of Frege's. In 'Function and Concept' Frege states that "I am concerned to show that the argument does not belong with the function, but goes together with the function to make a complete whole; for the function by itself must be called incomplete, in need of supplementation, or 'unsaturated.'"[13] What Frege has in mind is the following situation: there is a distinct difference between, say, '$2x^3 + x$' and '$2.1^3 + 1$'. The one which Frege focusses on is that the latter is an *instance* of the former. '1' has been substituted for 'x'. This is what Frege calls the act of 'saturating' the function. The former example is 'in need of completion' because it does not name any particular number, i.e. does not refer to a determinate *object*, while the latter does—it names the number 3.

Frege notes that his way of drawing this distinction would be better served by dispensing with the notation for variables altogether. Thus, '$2x^3 + x$' would be '$2(\ )^3 + (\ )$' (the reason Frege does not actually use this method of symbolising an unsaturated function is that it cannot deal with functions of two or more variables. '$2(\ )^3 + (\ )$' could be either '$2x^3 + x$' or '$2x^3 + y$'; we cannot tell if

[13] p.24

we are to be uniform in filling in the argument places or not).[14] Functions are 'incomplete names', which can be saturated with any complete name to yield a complete name.[15] In our example, we start with the unsaturated $2x^3 + x$; because this expression is unsaturated, it names a function. We then saturate it by putting the name '2' in the argument-places. This gives us the expression $2.2^3 + 2$. This expression is a name for an object, the number 18; it names an object because it is saturated. Indeed, '$2.2^3 + 2$' has the same reference as '18'; they both name the number eighteen.[16] Thus, the metaphysical distinction between functions and objects is signified to us by the logical distinction between a saturated expression and an unsaturated one.

3.2.3 Special Objects

There are two kinds of objects which play deep, structural roles in Frege's system.[17] These are

- the two truth-values, the True and the False, and
- 'courses-of-values', which are defined by their respective functions.

[14] It has been brought to my attention that Frege could have used *subscripts* to differentiate argument-places, as in '$2(\ )_1^3 + (\ )_2$'. However, this notation itself can be thought of as a shorthand for variables, or vice-versa. It *could* be more pedagogical for Frege's purposes; nevertheless in what follows we will use as much standard notation as possible.

[15] We shall see, in the exposition that follows, that indeed any object can fill an argument-place; for Frege, there is no such thing as a restricted domain of discourse. The laws of logic are the laws of thought, and they stand at the very centre of our ability to cognise. Thus, they have full universal applicability; any function which is claimed to be defined using only logical resources must have a well-defined value for any given input. We shall see that Frege expends some effort defining the values of functions which are 'naturally' intended to have numerical arguments for non-numerical arguments. For example, the addition function $\xi + \zeta$, is stipulated to be zero if either ξ or ζ is any object other than a number (such as Julius Caesar). 140+ Julius Caesar= 0. Cf. Furth's introduction to the *Grundgesetze*, p.x.

[16] However, they have different *senses*; we shall cover this in due course

[17] Which is not to say that ordinary objects like tables and chairs do not play *any* role; just that the roles they play are not as important to the *structure* of Frege's logic.

Both are intimately related to extensions. We begin with an explication of the role of the truth-values.

A saturated function (read 'name for an object') refers to some object; this is obvious for those expressions traditionally conceived as functions, such as numerical functions. However, for Frege, all expressions are functions. Consider what happens when we take an arithmetical function, and append a suspected result with an '=' sign; for instance, $2.2^3 + 2 = 18$. This expression refers not to a number, for instead it refers to the True; it is a different expression from '$2.2^3 + 2$':

> ...I call the number four the *denotation* of "4" and of "2^2", and I call the True the denotation of "$3 > 2$". (*Grundgesetze* §2)

Similarly,

> '$2^2 = 4$' stands for the True as, say, '2^2' stands for 4. And '$2^2 = 1$' stands for the False. ('Function and Concept', p.28/29)

All expressions which describe a situation in which something is the case refer to one or the other of the two truth-values. They are functions which take various things as arguments, but give either the True or the False as value. If one asserts something that is not the case, such as $2.2^3+2 = 19$, then one is referring to the False. These expressions are merely a particular type of saturated function, the object referred to being a truth-value. Frege calls the unsaturated form of such an expression a 'concept': "[I] call directly a *concept* a function whose value is always a truth-value."[18] Two functions which 'stand for the same thing' can

[18] *Grundgesetze* §3

flag both sides of an '=' sign to create a true statement.

The other type of object which plays a central role in the system of the *Grundgesetze* is what Frege calls 'the course-of-values of a function'.[19] It will suffice for our current purposes to note that the course-of-values of a function is the class of objects which can be used to saturate that function to give a name for the True; in this it is similar to the modern set-theoretic method of defining n-place functions as particular sets of ordered $n+1$-tuples.[20] For example, we can represent the course-of-values of the function $\xi^2 = \zeta$ over $\mathbb{N}$ (a 1-place function) by the class of ordered pairs $\{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,4\rangle, \langle 3,9\rangle, \ldots \langle 16,256\rangle \ldots\}$, or in other words, $\{\langle x,y\rangle : x^2 = y\}$. However, the set-theoretic notion of a set or class is not entirely equivalent to Frege's courses-of-values. One difference is that in modern set theory, a function is *defined* by the requisite set of ordered pairs. For Frege, a course-of-values is defined by the function. The order of priority between functions and classes is reversed.

This may, on the face of it, seem counter-intuitive; for is it not the case that a mathematician will be confronted with a set of data-points, and then take this as determining the function? In 'Function and Concept', Frege devotes a few paragraphs to this question, and so shall we. Also, this discussion further clarifies the relationship between functions and arguments. Frege says:

> People... recognise the same function again in
>
> '$2.1^3 + 1$,' '$2.4^3 + 4$,' '$2.5^3 + 5$,'

[19] 'Course-of-values' is, of course, a translated term; it has also been rendered as 'value-range'. I cleave to Furth's choice of 'course-of-values' as it his translation of the *Grundgesetze* which I make the most use of.

[20] See Chapter One, §1.3.3; n is the number of arguments the function takes, the representative tuple for a given function will need to also include the value of the function for the given inputs. Hence, 1-place functions will be represented by classes of ordered doublets, 2-place functions by triplets, and so on.

> only with different arguments, viz. 1, 4, and 5. From this we may discern that it is the common element of these expressions that contains the essential peculiarity of a function; i.e. what is present in
>
> $$`2.x^3 + x'$$
>
> over and above the letter 'x.' (p.24)

It seems that Frege is endorsing some kind of 'abstractionist' view in this passage, stating that we 'abstract' the general form from a collection of particular statements. However, such views about physical properties had been around since the time of Locke, and had been decisively refuted, not least by Frege himself. General terms simply cannot have their genesis in aggregates of particulars—the issue of underdetermination cannot be resolved. Nevertheless, an appeal to the recurring theme of this discussion, the distinction between the metaphysical and epistemic aspects of a judgement. We can hold it true that we do *epistemically* abstract the function. However, we can avoid the difficulties which plague pure abstractionist accounts such as Locke's account of general terms. We hold that the constitutive aspects of the judgement, the reasons why we hold it to be true, involve holding the function to be prior to the class it determines. But, all the counter-intuitive consequences of such a view are to do with aspects of how we come to acknowledge the actuality of the judgement. Thus, we accept the abstractionist account for describing the context of discovery, but reject for the context of justification. This brings us to anther aspect of Frege's system. To fully appreciate how functions are related to their courses-of-values, we must avail ourselves of Frege's distinction between *sense* and *reference*.

3.2.4 Sense and Reference

The distinction between sense and reference is a distinction between the metaphysical and epistemic aspects of a function (and thus, by extension, to all expressions). We shall first give the distinction for saturated functions, and then move on to unsaturated functions, which refer to courses-of-values. The reference of a saturated function (i.e., a name[21]) is the object that it claims to refer to, provided that it has this reference. The sense of a name is the manner in which we come to know what the reference is.[22] Consider the equivalent functions $y = (3+1)^2$ and $y = 3^2 + 2.3 + 1$; they have the same referent, the True for $y = 16$. However, the arithmetical operations which we must perform in order to ascertain the referent are different. For the first function, we add 3 and 1, and then multiply the result by itself. In the second case, we multiply 3 by itself, add 2 times 3 to the result, and then add 1 to this result. The practical aspects of determining the referent of these two functions are different; that is to say, the contingent facts-of-the-matter about how we come to recognise the referent are different. However, this does not interfere with the metaphysical, meaning-dependent fact that they do, in fact, have the same referent. Frege states that "Accordingly '$2^2 = 4$,' '$2 > 1$,' '$2^4 = 4^2$,' stand for the same thing, viz. the True, so that in $(2^2 = 4) = (2 > 1)$ we have a correct equation."[23] Now it is obvious that while '$2^2 = 4$' and '$2 > 1$' are both true, it does not follow that they actually make the same assertion. Frege accepts this, arguing that they have the same *reference*—the True—but different *senses*. The sense of the

[21] Cf. §2 of the *Grundgesetze*
[22] Cf. Currie (1982) p.85: "Roughly speaking, the sense of an expression is the information which it conveys. Every true sentence refers to the truth-value true. But two true sentences can say different things, and have different contents. The content of a sentence is the Thought [*Gedanke*] expressed by it—its sense."
[23] 'Function and Concept', p.29

former is the 'thought'[24] that two multiplied by itself is four, while that of the second that two is a greater quantity than one. "We see from this that from identity of reference there does not follow identity of the thought [expressed]."[25]

As mentioned above, functions are logically prior to their courses-of-values. What this means is that in order to be able to perform the epistemic act of *recognising* a function's course-of-values, we must be able to apprehend what the function says. In other words, in order to determine the function's referent, we must be able to comprehend its sense.[26] At this juncture it is quite important to realise that the sense of an expression is *not* the idea that it invokes. This is essential; for it serves to separate the subjective ideas or impressions which may be evoked by an intellect's discernment of a sense from the essentially objective content which actually makes up the sense.[27] For example, the fact that having to multiply 2 by itself may evoke, in a given individual, certain memories of being taught how to multiply, this in no way impinges upon or contributes to the difference in sense between '$y = (3+1)^2$' and '$y = 3^2+2.3+1$'. Such memories belong firmly to the context of discovery, and should not be associated with the context of justification. This is vitally important to Frege, as it is the subjectivism which is a consequence of an Empiricist orientation to mathematical truth which was one of his main polemic targets.

As courses-of-values are objects, they must be the value of some kind of function. And, indeed, they are; Frege defines the function $\acute{\epsilon}\Phi(\epsilon)$ to be the course-of-values of the function $\Phi(\xi)$. For instance, take an example given above, the squaring function. '$\acute{\epsilon}(\epsilon^2)$' names the class $\{\langle x, y\rangle : x^2 = y\}$. The

[24] "'$2^4 = 4^2$' and '$4.4 = 4^2$' have the same reference, but do not the same sense (which means, in this case: they do not contain the same thought)." (Function and Concept, p.29)

[25] Ibid., insertion by translator

[26] The exact nature of how we comprehend senses is one of the vague points in Frege's philosophy of language; cf. 'The Thought: A Logical Enquiry'.

[27] Cf. 'On Sense and Reference', in Geach & Black, p.60-61

function $\acute{\epsilon}\Phi(\epsilon)$ takes the function Φ as argument and gives Φ's course-of-values as its value. Thus, $\acute{\epsilon}\Phi(\epsilon)$ is a saturated function (when Φ is a determinate name of a function)—that is to say, it is an object. Here we have an instance of the sometimes subtle distinction Frege makes between an object and the name for it: because Frege never gives an *explicit* definition of what a course-of-values is, but only a contextual definition, there is in general no symbol for Φ's course-of-values other than $\acute{\epsilon}\Phi(\epsilon)$ itself. More will be said about this, and other aspects of courses-of-values, in the coming sections.

3.2.5 Generality

The distinction between saturated and unsaturated functions perfectly mirrors the distinction between particular and general expressions. A saturated function is a name of a particular object; if it names more than one object, then what is being named is not an object, but a predicate. For instance, let 'φ' be a property that is possessed by one and only one object (being the 22nd President of the United States of America, say). Then we can regard ourselves as able to talk about '*the* φ' (*the* 22nd President). On the other hand, unsaturated functions refer to their courses-of-values; they are general in a manner which saturated functions are not. An instance of this is the way in which Frege explains the nature of the variable.

In the essay 'What is a Function?', Frege's main polemic target is conceptions which regard variables as numbers that are capable of changing. He says "...there are not variable numbers...these letters [x, y, etc] are not proper names of variable numbers in the way that '2' and '3' are proper names of constant numbers...".[28] Consider the analogous example of the phrase 'the king of this realm' (Frege's example). Compare this with a phrase which describes a supposedly 'variable' number: 'the number that gives the length of this rod in millimeters'. The rod mentioned in the second sentence can be heated and cooled; thus, the number which the phrase describes can vary. The rod could be 110 millimeters long; we then subject it to heat; we re-measure and find that the rod now has a length of 110.5 millimeters.[29] Now, if this were the case, the phrase 'the king of this realm' could be taken as naming a variable *man*. Consider the statement "Ten years ago the king of this realm was an old man; at present the King of this realm is a young man". It is directly analogous to the rod-length example: "...by using this expression I have designated a man who was an old man and is now a young man."[30] This is implied by the king example, whilst 'I have designated a number which was 110 and is now 110.5' is implied by the rod example.

Frege notes that the idea that one can designate a man who is getting progressively younger is a sign that "[t]here must be something wrong here."[31] It is obvious that the phrase designating the king is in fact designating *two different* men, one old, one young; what determines the referent is the *time* at which the phrase 'the king of this realm' is uttered. Uttered ten years ago, it names the old man—uttered now, it names the young man. Likewise with the rod example. What we need to do is to incorporate a reference to a time-frame in order to determine exactly *which* number is being referred to. 'The length of this rod in millimeters at t_n' is capable of naming a different number to 'the length of this rod in millimeters at t_{n+1}'. It is the absence of a reference to a

[28] p.109

[29] 'What is a Function?', p.108: "...when I say "the number that gives the length of this rod in millimeters" I am naming a number; and this is variable, because the rod does not always keep the same length; so by using this expression I have designated a variable number."

[30] Ibid.

[31] Ibid.

time-frame which gives the illusion that the number being named is variable, that is to say, is a number which can alter its value.

Further arguments against the notion of 'variable' numbers are given in 'What is a Function?'. One such argument invokes the way we are able to differentiate numbers on recognition. The names of different natural numbers designate objects, the differences between which we are able to describe. We know that what makes '2' and '3' different numbers is that when added, subtracted, multiplied by or divided into some third number they will give different sums, differences, products and quotients. Also, we can determine what these different results will be. The situation is somewhat different when it comes to variables. Frege asks "... what is the difference between the variables that are said to be designated by 'x' and 'y'?" The answer is that "We cannot say."[32] We are unable to specifically name any differences between x and y.[33] Furthermore, it is not *a priori* impossible that $x = y$ (they could be the *same* number). What Frege is getting at here is that the symbols 'x' and 'y' do not actually designate anything beyond the 'empty spaces' (i.e. argument places) in unsaturated functions:

> [H]ere we come upon what distinguishes functions from numbers. 'Sin' requires completion with a numeral,[34] which, however, does not form part of the designation of the function. This holds good in general; the sign for a function is 'unsaturated'; it needs to be completed with a numeral, which we then call the argument-sign.

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[32] Ibid., p.109

[33] Except the trivial distinction between the symbols 'x' and 'y'; however, this distinction is irrelevant, as we are talking about what the symbols *designate*, not the symbols themselves. Frege was definitely aware of the necessity of such a distinction: cf. Currie (1982), p.55: "... the attention to semantic and syntactic distinctions [in the *Grundgesetze*] was something that logicians had not previously attained."

[34] [As in 'sin 2', sin 3', etc.]

('What is a function?', p.113/4)

It is because functions such as $y = mx+c$ are unsaturated that they have general content, and which gives their open argument-places the appearance of being 'variable'. It is not that an unsaturated numerical function contains numbers designated by the signs 'x', 'y', and so forth, which can vary and which gives a different value depending on what the variable numbers are at the moment of computation; rather, the function denotes a collection, rather than a single object:

> People have got used to reading the equation '$y = f(x)$' as 'y is a function of x.' There are two mistakes here: first, rendering the *equals*-sign as a copula; secondly, *confusing the function with its value for an argument.* From these mistakes has arisen the opinion that the function *is* a number, although a variable or indefinite one. (Ibid., p.115; all emphasis except on the first 'equals' mine.)

Here we see Frege re-stating his opinion that functions are essentially different from numbers. Numbers are objects, by virtue of their being designated by *saturated* functions. Unsaturated functions do not designate any individual numbers; this is why Frege says that 'the opinion that a function is a variable number' is a 'mistake'.

An unsaturated function is general in that it can, upon completion, designate a particular entity. However, until this happens, it does not designate any *particular* object; the closest thing which comes to being referred to is its course-of-values.[35] What it definitely does *not* designate is a particular entity which

[35] The exact nature of this relationship will have to wait until we analyse Frege's function

is somehow capable of change.

3.2.6 Conclusion

The metaphysical distinction between functions and objects gives Frege a way of explaining generality (in particular, the apparent variable nature of functions) which does not require entities which are capable of changing. However, Frege needs a way to explain how two seemingly different functions can have the same course-of-values without immediately becoming themselves identical. The sense/reference distinction serves this purpose by stating that two functions, such as $y = (x + 1)^2$ and $y = x^2 + 2x + 1$ are actually presentations of the same logical relationship—they both designate the same metaphysical entity, their identical courses-of-values—which have a different sense. That is to say, they are epistemically different presentations of the same metaphysical entity. However, this epistemic difference does not imply any metaphysical difference; Frege is not assenting to the view that a table viewed from different angles, resulting in different shaped sense-impressions, amounts to there being different tables.[36] Thus a common theme of our discussion of Frege remains constant: the context of discovery and the context of justification must be kept separate.

3.3 *Extensions as Objects in the system of the* Grundgesetze

In this part we are going to examine aspects of the formal system Frege presents in his *Grundgesetze der Arithmetik*. Logic, for Frege, comprises the laws of

for the definite article.

[36] As occurs in the subjective idealism of George Berkeley; see his *Treatise Concerning the Principles of Human Knowledge.*

thought; and as such, encompasses arithmetic. However, Frege's derivation of arithmetic from logic involves ineliminable use of objects, necessitated by his distinction between saturated and unsaturated functions. Thus, we will take particular interest in Frege's methods of formalising this relationship, and his ways of passing from concepts to objects and vice versa. We begin with his function for the definite article, and trace his use of courses-of-values in the stipulation of the conditions a function must fulfil in order to be a one-to-one correspondence.

However, before beginning, we shall quickly review Frege's conventions regarding the use of Gothic letters, Roman (Latin) letters, and Greek letters.

- Upper-case Greek letters: Γ, Δ, Φ etc. These are used as 'placeholder' names, e.g. '$\Gamma(a)$', or '$\vdash \Delta$'. Specifically, they are metalinguistic variables, and are absent from all asserted propositions of the formal language.[37]
- Lower-case Greek letters: ζ, ξ, φ etc. These are used as placeholder-names for arguments (although this is a different case than that for bound variables), much like free variables in mathematical formulae, e.g. '$F(\zeta)$'. Like capital Greek letters, these do not occur in any formal proofs.[38]
- Lower-case Greek vowels: ϵ, α etc. Used in the notation for courses-of-values, as in $\acute{\epsilon}\Phi(\epsilon) = \acute{\alpha}\Psi(\alpha)$. This will be discussed in due course.
- Lower-case Roman letters (from the beginning of the alphabet): a, b, c etc. Used as object-names, as in $\vdash \Phi(a)$.
- Upper-case Roman letters (from the middle of the alphabet): F, G, H etc. Used as function-names, as in $\vdash F(\varphi)$.

[37] See p.*xxviii* of Furth's introduction to the *Grundgesetze*, and footnote 15 of the main text.
[38] see ibid., §1.

- Lower-case Gothic letters (from the middle of the alphabet): $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ etc. Used in conjunction with the 'concavity' $\smile^{\mathfrak{f}}$ to serve as a bound functional variable, e.g. $\vdash\!\smile^{\mathfrak{f}}\, \mathfrak{f}(a)$—in modern notation, $\forall F Fa$.

- Lower-case Gothic letters (from the beginning of the alphabet): $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ etc. Analogous to the case with Upper-case Gothic letters, these signs serve as bound object-variables, as in $\vdash\!\smile^{\mathfrak{a}}\, \Phi(\mathfrak{a})$—in modern notation, $\forall x \Phi(x)$.

3.3.1 Courses-of-Values

Up until now, we have been talking 'around' courses-of-values, by viewing them informally as sets, without really addressing how Frege defines them. However, we are about to begin an in-depth formal examination of Frege's system; thus we need to avail ourselves of the notion that Frege actually uses, as opposed to an anachronistic stand-in. In the *Grundgesetze*, Frege first introduces the notion of a course of values early in the text, in §3. He initially gives the following intuitive definition:

> I use the words
>
> > "the function $\Phi(\xi)$ has the same *course-of-values* as the function $\Psi(\xi)$"
>
> generally to denote the same as the words
>
> > "the functions $\Phi(\xi)$ and $\Psi(\xi)$ have always the same value for the same argument".

Frege is giving a contextual definition to explicate his notion of a course-of-values.[39] That is to say, rather than explicitly defining what a course-of-values is, he merely gives the condition under which two courses-of-values can be recognised as the same. For instance, consider the two functions '$y = (x + 3)^2$' and '$y = x^2 + 6x + 9$'. It is easily verified that these two functions have the same y-value for any given x-value; thus, in Frege's terminology, they have the same 'course-of-values'.

When it comes to functions whose values are always either the True or the False, we can treat the phrases 'course-of-values of the function' and 'extension of the concept' as interchangeable. Likewise, these sorts of functions can be referred to as 'concepts'. So, for the two concepts 'x is a round square' and 'x is a piece of wooden iron', we can say both that the two have numerically identical[40] courses-of-values and that they share the same extension (as they assume the value of the False for any x).

§9 is where Frege introduces his notation for courses-of-values. He employs a symbolism where a small Greek vowel with an acute accent is followed by an expression in parentheses; this expression is that whose course-of-values is being denoted by the whole. Accordingly, the variables which are being evaluated are replaced by the same small Greek vowel (without the accent). For example, the course-of-values of the function '$x^2 + (x - 2)$' is denoted by $\acute{\epsilon}(\epsilon^2 + (\epsilon - 2))$. Likewise, the concept $\acute{\epsilon}[(\epsilon + 3)(\epsilon - 3)] = \acute{\alpha}(\alpha^2 - 9)$[41] denotes the True, as $\forall x([(x + 3)(x - 3)] = (x^2 - 9))$. Similarly, according to Frege:

[39] Cf. the formalisation of this definition as Basic Law V, given below.

[40] As opposed to qualitatively identical. Two objects are numerically identical if they are in fact the same object; two objects are qualitatively identical if they are of the same kind, belong to the same sortal, etc.

[41] Frege uses different Greek vowels on each side of the '=' as a notational device to remind us that we are equating the courses-of-values of two different functions, rather than amalgamating them into one. This is similar to the difference between '≡' and '='.

$$\acute{\epsilon}(\epsilon^2 - \epsilon) = \acute{\alpha}(\alpha \cdot (\alpha - 1))$$

is the value of the function

$$\xi = \acute{\alpha}(\alpha \cdot (\alpha - 1))$$

for the argument $\acute{\alpha}(\alpha^2 - \alpha)$.(§9)

Thus we reach the general symbolised definition of equality of course-of-values:

$$(\forall x Fx = Gx) \equiv (\acute{\epsilon}F(\epsilon) = \acute{\alpha}G(\alpha))$$

which is stated as Frege's Basic Law V:

$$\vdash (\acute{\epsilon}f(\epsilon) = \acute{\alpha}g(\alpha)) = (\smile^{\mathfrak{a}} f(\mathfrak{a}) = g(\mathfrak{a})).$$

We are now in a position to follow Frege's definition of one-one correspondence.

3.3.2 The Function-Name for the Definite Article: $\backslash\xi$

We begin with §11, where Frege explains the function-symbol '$\backslash\Delta$' for Δ as argument. This function designates the definite article. A definite article is an expression of the form '*the* such-and-such'. In such cases, the intended course-of-values for such a function contains one and only one object.[42] We want to be able to equate a function which is realised by only one object with that object itself, in order to, among other things, be able to use numbers (which are extensions of concepts) as arguments for first-level functions. We can paraphrase Frege's definition thus:[43]

$$\backslash\xi = \begin{cases} \Delta\text{, if there exists an object } \Delta \text{ such that the argument } \xi = \acute{\epsilon}(\Delta = \epsilon)\text{, or} \\ \xi\text{, if no such object exists.} \end{cases}$$

[42] Similar in spirit to Russell's definite descriptions. An *indefinite* article is a phrase of the form 'a such-and-such'.

[43] Cf. Ibid., §11

This function serves to equate concepts under which fall only one object with that object itself. If the concept has more than one object in its course-of-values, then the value of the function is the concept itself. The reason for this second clause is Frege's methodological stipulation that a function be well-defined for any argument; that is to say, that a function always have a determinate value for a determinate argument. Nevertheless, it is the first clause which does the work in the development of Frege's analytic definition of 'number'. As noted by Currie, "The second stipulation is simply a convenient way of making sure that the function is defined for *all* possible arguments. It is for the first class of cases that the function serves its intended purpose."[44]

Frege himself specifies the function thus:

> [T]wo cases [of the occurrence of $\backslash\xi$] are to be distinguished:
>
> 1. If to the argument there corresponds an object Δ such that the argument is $\acute{\epsilon}(\Delta = \epsilon)$, then let the value of the function $\backslash\xi$ be Δ itself;
>
> 2. if to the argument there does not correspond an object Δ such that the argument is $\acute{\epsilon}(\Delta = \epsilon)$, then let the value of the function be the argument itself. (*Grundgesetze* §11)

He then supplies some examples. We are told that '$2 = \backslash\acute{\epsilon}(\epsilon + 3 = 5)$' names the True. This is because the value of '$\backslash\acute{\epsilon}(\epsilon + 3 = 5)$' is, in fact, the unique object '2' which falls under the concept which is the argument for $\backslash\xi$. That is to say, '$\acute{\epsilon}(\epsilon + 3 = 5)$' names the course-of-values of the function '$\epsilon + 3 = 5$', which is the collection of objects which can each assume the position of 'ϵ' while keeping it

[44] (1982), p.127

the case that the whole, $\epsilon + 3 = 5$, names the True.[45] Now, there is only one object which can take the place of ϵ. It is the number 2. Thus, $\acute{\epsilon}(\epsilon + 3 = 5)$ only names one object: 2. Hence, by the definition of $\backslash\xi$, the first of the two cases is taking place. There *is* an object Δ, 2, such that the argument—$\acute{\epsilon}(\epsilon + 3 = 5)$—is of the form $\acute{\epsilon}(\Delta = \epsilon)$.[46] Thus, when $\xi = \acute{\epsilon}(\epsilon + 3 = 5)$, the value of $\backslash\xi$ is the unique object which comprises the course-of-values named as the argument; i.e., $\backslash\acute{\epsilon}(\epsilon + 3 = 5) = 2$.

So much for the first case; we shall now look at an example of the second. Frege gives the example of the concept 'being a square root of one'. We know that more than one object falls under this concept, as both $1^2 = 1$ and $(-1)^2 = 1$. Our situation is this: because both '1 is a square root of one' *and* 'negative one is a square root of one' are phrases which name the True, we know we are in a number two-type case with respect to the function $\backslash\xi$. Thus, no unique object is named; rather, the sign for this concept's course-of-values— '$\acute{\epsilon}(\epsilon^2 = 1)$', when taken as argument for the function $\backslash\xi$, gives us the value of the entire course-of-values—which is named by the symbol for the argument. Hence, $\backslash\acute{\epsilon}(\epsilon^2 = 1) = \acute{\epsilon}(\epsilon^2 = 1)$. So, the two cases run like this: if the course-of-values of a concept is comprised of only one object, then the value of the 'definite article' function for that course-of-values is that object; in all other cases (including empty courses-of-values and ill-defined arguments) the result is the argument—'$\backslash\xi$' works like the identity function. '\ schmoggle = schmoggle' is the True, given that there are no schmoggles.

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[45] This is not, strictly speaking, the case; there is some uncertainty regarding exactly what kind of collection Frege understood a course-of-values to be (if indeed he had any such understanding). Furth, in his introduction to the *Grundgesetze*, takes extensions of concepts to be collections of ordered n-tuples where the first $n - 1$ elements are objects which fill the argument-places of the concept and the nth element is a truth-value (p.*xxxviii*). I believe that my elucidation gives a maximally intuitive grasp with a minimum loss of detail, and thus will progress with it as given.

[46] If you need further convincing, consider it this way: we can *replace* the ϵ in $\acute{\epsilon}(\epsilon + 3 = 5)$ by Δ, and then 'balance' the equation, yielding $\acute{\epsilon}(\Delta = 5 - 3)$

3.4 Names, Objects, and the Definite Article

It is becoming apparent that there are multiple connections between Frege's approaches to functions and objects, generality, and the definite article. These themes will now be drawn closer together, making plain the effects of Frege's transcendental idealism on the foundations of his formal system. At the very root of this is his conception of the relationship between an object and its singleton set (or unit class), which is a particular case of that between functions and objects. This conception, it turns out, is opposed to that of modern set theory.

Frege's 'function-name' for the definite article[47] is constructed by specifying two cases. The second case amounts to an identity function. Hence, even if the argument is not of the intended sort, the function will provide a well-defined output (or, at least, the output will always be at least as well-defined as the input). The function itself cannot lead us astray.[48] The *Grundgesetze* contains an axiom[49] governing the behaviour of this function (Basic Law VI):

$$\vdash a = \backslash\acute{\epsilon}(a = \epsilon)$$

Basically, this axiom states that the name of an object and the name of the class which contains that and only that object can be interchanged when the object-name is prefixed by 'the'.[50] In §11 of the *Grundgesetze*, where Frege first

[47] Frege makes a distinction between 'functions' and 'function-names'. A *function* is an entity with a determinate ontological status; a *function-name* is the sign by which we denote a function (cf. *Grundgesetze* §26). As will become apparent, the syntactic category is prior to the ontological.

[48] Cf. the discussion of logic as the 'laws of thought' in §2.2.2

[49] Frege actually states in §18 that the axiom 'follows from' the denotation of $\backslash\xi$; hence, it probably should not be regarded as an axiom in the modern sense. Nevertheless, I shall follow Frege's convention of referring to it as such.

[50] Compare Currie's reading: "It says that the value of the function for an argument which is the extension of a concept under which exactly one thing falls is that thing itself." (1982), p.128

introduces this function (Basic Law VI does not appear until §18), he states that "$\backslash\acute{\epsilon}(\Delta = \epsilon) = \Delta$ is the True"; hence it appears that Frege intended the interchangeability of object-name and class-name to run both ways. Now, if we regard Frege's courses-of-values in a modern set-theoretic manner,[51] as in $\acute{\epsilon}\Phi(\epsilon) = \{x : x \text{ is a } \Phi\}$, then basic law VI essentially states that

$$\backslash x = \{x\}.$$

The definite article, for Frege, is the syntactic criterion for (claiming to be) referring to an individual or unique object;[52] moreover, courses-of-values are collections of objects, and are closely (though not absolutely) approximated by the modern notion of a set. The definition of the empty set is the same (the collection of all objects not identical to themselves; $\acute{\epsilon}(\top\, \epsilon = \epsilon)$ [*Grundgesetze*] and $\{x : x \neq x\}$ [ZF]); extensionality[53] (two courses of values with all the same members are the same course-of-values [though the same need not hold for functions]; likewise for sets); and so forth.

One point of difference is that modern Zermelo-Fraenkel set theory makes an important distinction between an object and its singleton;[54] in contrast, Frege's system contains a function which can (if applied) equivocate them. However, this occurs only if the relevant courses-of-values contains a single object (i.e. nonempty sets with only one member). Thus, an important distinction for Frege

[51] Fudging our notations in the process, for pedagogical purposes only. The notation for set theory is that of Enderton's (2001).

[52] Frege says as much in §38 of the *Grundlagen*: "When we speak of "the number one", we indicate by means of the definite article a definite and unique object of scientific study." See also 'Function and Concept', p.23: "We say '*the* number,' not 'a number'; by using the definite article, we indicate that there is only a single number."

[53] Cf. §0 of the *Grundgesetze*, where he states that Dedekind's assertion that a set S is 'completely determined' if for any given object it is determined whether it is a member of S or not (implying that $S = T$ if every member of S is a member of T) "...comes near the mark...".

[54] One reason for this distinction is the desire to not have any sets be self-membered; this is related to the representation of natural numbers by sets and the possibility of non-standard models of these representations.

is between 'singletons' and courses-of-values which contain *multiple* members. For instance:

$$\vdash \backslash\acute{\epsilon}(\Delta = \epsilon) = \Delta$$

is the True, whereas

$$\vdash \backslash\acute{\epsilon}(\Delta = \epsilon \text{ or } \Gamma = \epsilon) \neq \Delta \text{ or } \Gamma.$$

What we actually get when applying Frege's function for the definite article to collections not of one object is

$$\vdash \backslash\acute{\epsilon}(\Delta = \epsilon \text{ or } \Gamma = \epsilon) = \acute{\epsilon}(\Delta = \epsilon \text{ or } \Gamma = \epsilon).$$

If we fudge our notations a little by combining Frege's function for the definite article with some modern set-theoretic notation, and put aside concerns about the result of applying it to non-well-formed arguments, the situation becomes clearer still. Let Δ be some unique object and Φ some non-unique object. Then, the first of the two cases above is

$$\backslash\{x : x = \Delta\} = \Delta$$

whereas the second is

$$\backslash\{x : x = \Phi\} = \{x : x = \Phi\}.$$

The definite article function only equates a set with its member if that set *only has one member*. It equates what may be termed 'plural' sets with the set as a whole.[55] Collections of more than one member are *general*; hence, a function which names a plurality of objects must be unsaturated in some kind of way. On the other hand, a function which names a single object behaves just like a name for that object (and thus is saturated). The appearance of such generality-related considerations explains Frege's use of the definite article; and these

[55] Including the empty set.

considerations are an outcome of Frege's Transcendental Idealism. It turns out that these considerations bear on a debate in the current Frege literature—that of 'the concept *horse*'.

In Chapter Two I argued that Frege's approach to the ontology of arithmetic was in fact motivated by an approach reminiscent of Kant's Transcendental (read: *objective*) Idealism, rather than a straight-forward realist conception. This interpretation rests upon a claim that when Frege asserted that arithmetic is analytic, he was more concerned with establishing that there cannot be any alternative systems of arithmetic (as there are for geometry). He sought to ground arithmetic in logic, and thus in the very nature of rationality, much as Kant had sought such grounds in intuition. This leads naturally a claim made by Schirn in his (1990):

> It is generally agreed that the distinction between concept and object lies at the very heart of Frege's philosophy of logic. One of its key elements is the primacy of the syntactic categories *concept-word* (more generally: *function-name*) and *proper name* (*singular term*) over the ontological ones *concept* (*function*) and *object*. (p.46; emphasis original)

That Frege takes syntactic categories as prior to ontological ones is striking evidence for an interpretation which favours transcendental idealism. Coming from this is the idea that names and objects share a special kind of relationship; names are primary, with the object being somehow 'derived' from the name. But what kind of 'derivation' could this be?

A function-name which is saturated is a linguistic description of a determinate object. Hence, it is epistemically prior to that which it purports to name; we must know what the name refers to before we can recognise an object as *that* particular object. Likewise with functions (general names). The resulting difference between function-names and proper names is that the former are general, whilst the latter are not. This difference results from their relative saturation. Now, would a function which names a unit class[56] be saturated or not? Consider the phrase 'positive number which, when multiplied by itself, yields 4' Intuitively, we are speaking of *the* number 2. However, there is no definite article in the phrase; we may be naming a collection of numbers. For Frege, there is no *logical* difference between (i) 'positive number which when multiplied by itself yields the number 4' and (ii) 'positive number which, when multiplied by itself, yields a number less than 10'; this phrase definitely names a collection (1,2, and 3). Inserting a definite article into (i) gives 'the positive number which, when multiplied by itself, yields 4'; the presence of the definite article makes it clear that we are naming a single object. It is just this behaviour of the definite article which is captured by Frege's function:

$$\{x : x > 0 \text{ and } x^2 = 4\} = \{2\}$$

and hence

$$\backslash\{x : x > 0 \text{ and } x^2 = 4\} = 2;$$

whereas

$$\{x : x > 0 \text{ and } x^2 < 10\} = \{1, 2, 3\}$$

thus

$$\backslash\{x : x > 0 \text{ and } x^2 < 10\} = \{1, 2, 3\}.$$

Thus we have both $\{x : x > 0 \text{ and } x^2 = 4\} = \{2\}$ and $\backslash\{x : x > 0 \text{ and } x^2 = 4\} = 2$. Since equality is transitive, we can substitute '$\{2\}$' for '$\{x : x >$

[56] I.e. a course-of-values which contains a single object.

0 and $x^2 = 4\}$', obtaining $\backslash\{2\} = 2$; likewise for the second case, where we get $\backslash\{1, 2, 3\} = \{1, 2, 3\}$.

This shows why the definite article function will, when applied to a singleton, give the object which is the one and only member. *A singleton is a proper name*—'the collection of objects which have these properties', when it so happens that the conjunction of such properties serves to describe one and only one object.[57] In that case, we may pass (via an application of the function-name for the definite article) from the name to the object itself. A description of a collection of only one object can serve as a name for that object. On the other hand, a function which names a plurality of objects—any number more than one—is *general*, and refers to what may be properly called a 'collection'. One *cannot* pass from the collection (an epistemic entity) to an object, because it is not determinate *which* object to 'pass to' over and above the whole collection itself. Thus, in general, $\backslash\{x\} = x$, whereas $\backslash\{x, y\} = \{x, y\}$. Bearing this in mind, I will now argue that this perspective on the definite article function, motivated by a trascendental idealist interpretation of Frege, gives a new perspective on the so-called 'paradox of the concept *horse*'.

3.4.1 *The Paradox of 'the Concept* Horse*'*

This debate concerns the question of what exactly the reference of the expression 'the concept *horse*' is. This question has received considerable attention in the last few decades of Frege scholarship. One particular vein of this debate has taken place between Matthais Schirn and Marco Ruffino.

[57] A paradigm example of such a property for macroscopic objects would be a millimeter/millisecond specific description of spatiotemporal location.

Intuitively, the answer seems to be that it refers to the concept 'horse'—the position which would make it true that 'the concept *horse* is a concept'. However, in his 'On Concept and Object' Frege claims that 'the concept *horse*' must refer to an object. If it did not, then the expression could not be used as a grammatical subject, as it is in (apparently true) statements such as 'the concept *horse* is realised'. The reason that only *objects* can be grammatical subjects is that only objects are saturated, and only saturated entities can play the role of grammatical subjects. If an unsaturated entity is placed in the position of grammatical subject, then the statement as a whole remains unsaturated, and cannot name a truth-value. Thus, in order for 'the concept *horse* is realised' to have a truth-value at all 'the concept *horse*' must name an object. The discussion between Schirn and Ruffino concerns whether Frege's position, as stated in 'On Concept and Object', has implications which contradict broader aspects of Frege's philosophy (with Schirn arguing 'no', Ruffino arguing 'yes').

We begin with Schirn. In his paper 'Frege's Objects of Quite a Special Kind'[58], he argues that Frege's supposed identification of 'the concept *horse*' and 'the extension of the concept *horse*' leads to intolerable conclusions. The main argument for this 'identification thesis' is that if 'the concept *horse*' is in fact an object, then this object is concept's extension; hence we can substitute the expression 'the extension of the concept *horse*' for 'the concept *horse*' without a change of reference. This is then implies that the truth-value of any statement containing the latter expression will be the same as that of any expression which is the result of said substitution. This leads to a kind of regress, where the substitution can be iterated any given number of times, with the expression 'the extension of the extension of . . . the extension of the concept *horse*' having the same reference as 'the concept *horse*':

[58] *Erkenntnis (1975-)*, Vol. 32, No. 1 (January 1990), pp.27-60

> If "the concept F" and "the extension of the concept F" really did have the same reference, then in the expression "the extension of the concept F" the part "the concept F" could be replaced by "the extension of the concept F" without altering the reference. And in the resulting expression, "the extension of the extension of the concept F", the part "the concept F" could again be replaced by "the extension of the concept F", and so on *ad infinitum*. Obviously, no one would seriously assume that in Frege's view "the extension of the extension of the extension of the concept F" designates the same object as does "the concept F". (p.28)

I shall refer to this argument from now on as the 'regress argument'. Ruffino, in his 'Extensions as Representative Objects in Frege's Logic'[59] attempts to counter Schirn's conclusion that this particular consequence of the identification thesis—the regress argument—is unacceptable. He first addresses a slightly different argument given by Schirn, to the conclusion that 'the extension of A' is 'semantically vacuous'. Ruffino then gives a response, and states that "This reasoning provides an indication of how one could deal with the difficulty pointed out in Schirn's argument 1.2 as well."[60] In order to fully explicate Ruffino's response to the regress argument, I will first have to explain his response to the semantic vacuousness argument in order to reconstruct the 'indication' that he speaks of.

Schirn's explanation of the semantic vacuousness argument can be paraphrased as follows:[61]

[59] *Erkenntnis (1975-)*, Vol. 52, No. 2(2000), pp.239-252

[60] Ibid., p.148. Ruffino distinguishes between the 'semantic vacuousness' argument and the regress argument (stated above) in Schirn's paper; the former is termed argument 1.1, the latter 1.2.

[61] Schirn (1990), pp.29-30

1. The two sentences (1) ‘there is at least one square root of 4’ and (2) ‘the extension of the concept *square root of 4* is nonempty’ are synonymous.
2. The two sentences (3) ‘the concept *square root of 4* is realised’ and (4) ‘the concept *square root of 4* is not empty’ are also synonymous.
3. Since Frege says that (1) and (3) are synonymous, and synonymy is a transitive relation, so are (2) and (4).
4. Synonymy is a relation between both reference *and* sense; hence (2) and (4) express the same sense.
5. The sense of an expression is a function of the senses of its constituent terms.
6. Therefore, the function-name ‘the extension of A’ is ‘semantically vacuous’ (i.e. makes no difference to either sense or reference), since (2) and (4) are synonymous.

Schirn goes on to state that “The result is completely untenable, since for Frege the function-name “the extension of [A]”, which forms apart of the term “the extension of the concept *square root of 4*”, clearly has a sense and contributes to determining the sense of the more complex term.”[62] Thus ‘the concept A’ and ‘the extension of the concept A’ can only be synonymous if the term ‘the extension of X’ is semantically vacuous. I will outline Ruffino’s response to Schirn’s argument, before attempting to follow his ‘indications’ as to his response to the identification argument. From there I will be able to move onto my own response.

Ruffino’s response to Schirn’s semantic vacuousness argument is to allege

[62] Ibid., p.30

that "No evidence is provided for this claim"[63] (that quoted in the previous paragraph). He claims that in this case, Schirn is begging the question by presupposing the interpretation which he is seeking to endorse. Ruffino goes on to state that we can accept that 'the concept A' and 'the extension of the concept A' express the same sense without necessarily endorsing the idea that 'the extension of x' has no bearing on meaning:

> ... if 'the concept F' is a singular term in 'the extension of the concept F', then 'the extension of $[A]$' must be the name of a first order function.... Thus 'the extension of $[A]$' is a name of the identity-function, i.e., if we abbreviate by $E(x)$ the expression 'the extension of $[A]$', then '$E(a) = a$' for every singular term 'a'. Obviously a consequence of this is that '$E(E(E \ldots E(a)) \ldots) = E(a)$' and '$E(a) = a$' are true statements. This does not imply, however, that '$E(x)$' is semantically vacuous; it is not so because it is a name for a function (identity), and hence has both a reference and a sense. (p.247/8)

The moral that Ruffino draws from this is that the terms 'the concept A' and 'the extension of the concept A' are together one of the cases where two terms have the same reference without necessarily expressing the same sense.[64] Now, one of Frege's main motivations in introducing the sense/reference distinction was to deal with non-trivial instances of identity;[65] hence the identity function is a paradigm example of how two terms a and b can both name the same object without $a = b$ being an *a priori* truth in the way that $a = a$ is. Presumably, this is the route from Ruffino's 'indication' as to his response to

[63] Ruffino (2000), p.247

[64] Although Ruffino is slightly guarded about this conclusion, stating "[the extension of A] admittedly does not seem, at first sight, to add to the sense of the latter [the concept *square root of 4*]." (Ibid.)

[65] See 'On Sense and Reference' in Geach & Black (eds), pp.56-78

the identification argument. It is not *a priori* that (in Ruffino's symbolism) $E(E(E \dots E(a)) \dots) = E(a)$, because in order for this to be the case 'a' must be a singular term; and this could very well not be an *a priori* truth. There is only one sphinx; there could have been many. I now turn to my own take on the situation.

The application of the function for the definite article sanctions the base case of the regress argument. That is to say, $\backslash E(a) = a$ for a singular term a. Likewise for the step case: $\backslash E(E(a)) = E(a)$, again for a singular a[66] However, the fact that a must be a *singular* term casts doubt on whether the function for the definite article sanctions the regress argument for class terms such as 'man'.

In the *Grundgesetze*, Frege states that

> We have here [in the function for the definite article] a substitute for the definite article of ordinary language, *which serves to form proper names out of concept-words.* For example, we form from the words
>
> "positive square root of 2",
>
> which denote a concept, the proper name
>
> "the positive square root of 2".
>
> (§11, emphasis mine)

In 'On Concept and Object', Frege makes a related assertion:

[66] see the previous section; I am assuming that Ruffino's $E(a)$ is equivalent to my set-theoretic rendering of Frege, where 'the extension of Φ'$= \{x : x \text{ is a } \Phi\}$.

> We designate [an] object by prefixing the words ‘the concept’; eg.:
>
> ‘The concept *man* is not empty.’
>
> Here the first three words are to be regarded as a proper name. (p.46/7)

In a footnote to the quote above, Frege notes that he “. . . call[s] anything a proper name if it is a sign for an object.”[67] From these statements follows the conclusion that prefixing the definite article to a concept-word yields the name of an object. What object would this be? It seems that it should be the *extension* of the concept. If we take ‘the concept ‘horse’’, and apply Frege’s function for the definite article, we get

$$\backslash\text{the concept `horse'} = \begin{cases} \text{the single object named by `the concept `horse''}, \text{ if} \\ \text{there is one, or} \\ \text{the concept `horse' if there is not.} \end{cases}$$

We already know that ‘the concept ‘horse’’ purports to name an object due to the presence of the definite article; we should be in the first clause. However, there is no textual evidence stating *what* this object should be. The obvious candidate object is the *extension* of the concept ‘horse’, similar to the case where the a in $\backslash E(a) = a$ is a singular term. But in this case a is *not* a singular term, and thus the effect of the definite article is less straightforward. The best argument for ‘the concept ‘horse’’ naming its own extension is that extensions are objects, and singular objects at that (just as the set $\{1, 3, 5\}$ is a single *set*). Nevertheless, to invoke this in favour of equating definite articles of non-singular concepts with their extensions is to beg the question. Furthermore, Frege is often quite explicit in his denial of any kind of equating concepts and objects,[68] so perhaps (we can only speculate) he saw that in this case staying vague was his best course.

[67] ‘On Concept and Object’, first f/n on p.47

[68] Cf. ‘On Concept and Object’ p.x, *Grundgesetze* §2

On the other hand, 'the extension of the concept 'horse'' *does* name a single object—the extension of the concept 'horse'—which we can symbolise Ruffino-style as $E(h)$, faux-Frege-style as $\acute{\epsilon}H(\epsilon)$, and set-style as $\{x : x \text{ is a horse}\}$. This phrase names an *extension*, which is a single object, and hence we obtain the identity $\backslash E(h) = E(h)$. For non-singular terms, 'the extension of the concept a' = 'the extension of the concept a', but 'the extension of the concept $a \neq$ 'the concept a'.

The situation differs with $E(E(a))$ ($\{\{x : x \text{ is an } a\}\}$) cases, however. Consider 'the extension of the extension of the concept 'horse''. If this names a single object, then we are in a first-clause situation. Clearly, it refers to the extension of 'the extension of the concept 'horse'', and 'the extension of the concept 'horse'' names one and only one object—namely $\{x : x \text{ is a horse}\}$. Hence 'the extension of the extension of the concept 'horse'' has the same referent as 'the extension of the concept 'horse'':

$$\backslash E(E(a)) = E(a).$$

There is no reason to suppose that this particular equivalence cannot be iterated repeatedly.[69] Thus, Schirn's regress argument holds for extensions of non-singular concepts. It also holds for singular concepts and extensions of singular concepts. However, it does *not* hold for non-singular concepts—at least, the evidence for it is not as compelling as in the other three cases—and this means that Ruffino's claim that the expression 'the extension of A' is not semantically vacuous holds. For, if it were, then we would not be able to tell the cases when the regress argument holds from those when it does not.

Furthermore, while he is committed to saying that expressions which occur

[69] Though it may be the case that there are other expressions apart from 'the extension of' which, when combined with the function for the definite article, yield a complex expression which is not able to be iterated in the same manner.

as grammatical subjects must refer to objects, he can defer the question as to exactly what object they refer to. Frege only ever gives a contextual definition of the course-of-values function. Given that it is in the context of a proposition that expressions garner meaning, we need only be able to identify the reference of an entire proposition in which the sub-sentential term 'the extension of x' occurs. We cannot ask what 'the extension of x' refers to, as this is an incomplete expression, an unsaturated function Nevertheless, we *can* ask what 'the extension of 'the extension of *horse*" refers to. The iteration can be freely performed just so long as there are no empty argument places; we know that there are no empty argument-places exactly when the whole expression refers to an object.

Thus we can see that the definite article occupies an important, though perhaps peculiar, place in Frege's logic. It is a symptom of his latent Transcendental Idealism that Frege takes synactic categories as prior to ontological ones. This manifests formally as the function for the definite article, which imposes a strong relationship between objects and their proper names (equivalently, their their singleton sets). This relationship is of a finer nature that either Schirn or Ruffin give full credit too.

Nevertheless, this is but one corner of a much larger programme. Having taken note of the finer aspects of how the function for the definite article functions for Frege in his later period, we shall now survey some further aspects of the *Grundgesetze* in order to ascertain whether Frege may have actually been aware of anything along the lines of the reflections we have just carried out.

3.5 The Definition of One-One Correspondence: the Role of the Definite Article

Sections 34–40 of *Grundgesetze I*, roughly half of a section entitled 'Particular Definitions', form the beginning of Frege's absolutely rigorous attempt at the final proof that arithmetic brooks no equal—that for any rational being, the science of number is as we see it. In what follows, we will inspect each step in Frege argument for signs that he was possessed of the particular conception of the definite article which (if the argument above holds) is implied by the combination of his remarks in 'On Concept and Object' with the formal definition given in §11 of the *Grundgesetze*.

3.5.1 The Function $\xi \frown \zeta$: Representing a Concept by its Extension

This function is used to represent a first-level function by an object, its course-of-values. This technique of representability enables Frege to have objects stand in the roles of (first-level) concepts—thus validating his claim that numbers are both objects and the extensions of concepts. In §34, Frege defines[70] $\xi \frown \zeta$ as

$$\Vdash \setminus\acute{\alpha}\left(\begin{array}{l} \smile^{\mathfrak{g}}\!\!\top\!\!\!-\!\!\!\top\ \mathfrak{g}(a) = \alpha \\ \quad\ \llcorner\ u = \acute{\epsilon}\mathfrak{g}(\epsilon) \end{array}\right) = a \frown u.$$

The statement within the brackets contained in the definiens is of the existential form, which in modern notation is an instance of the unsaturated form

$$\exists G(u = \{x : x \text{ is a } G\} \wedge G(\ ) = (\ )).$$

However, this is prefixed by the functional operator '\', which takes the function described in the brackets as its argument. $\xi \frown \zeta$ is the definite article of what

[70] Frege uses the symbol '$\Vdash$' in place of '$\vdash$' when giving a definition.

is contained in the large brackets; hence $\xi \frown \zeta$ refers to a singular object.

Frege begins his elucidation with an example, and then proceeds to the definition given above. Just as in his definition of $\backslash\xi$, this function has two kinds of cases. He states:

> In the first instance it is only a matter of designating the value of the function $\Phi(\xi)$ for the arguments Δ, i.e., $\Phi(\xi)$, by means of "Δ" and "$\acute{\epsilon}\Phi(\epsilon)$". I do so in this way:
>
> $$\text{“}\Delta \frown \acute{\epsilon}\Phi(\epsilon)\text{”}$$
>
> which is to mean the same as [*Gleichbedeutend sein*] "$\Phi(\Delta)$". The object $\Phi(\Delta)$ thus appears as the value of the function of two arguments $\xi \frown \zeta$ for Δ as ξ-argument and $\acute{\epsilon}\Phi(\epsilon)$ as ζ-argument.

Thus, if Φ is a second-level function and Δ is Φ's course-of-values, then

$$\Delta \frown \acute{\epsilon}\Phi(\epsilon) = \Phi(\Delta)$$

names the True. In §35, Frege goes on to state the definition of $\xi \frown \zeta$ must always denote something (recall the universal nature of logic); this is the aim of the informal argument he gives in §34, which results in the following:[71]

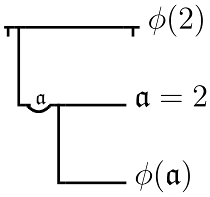
$$a \frown u = \begin{cases} \Phi(a) \text{ if } u = \acute{\epsilon}\Phi(\epsilon), \text{ or} \\ \acute{\epsilon}(\top\, \epsilon = \epsilon) \text{ otherwise.} \end{cases}$$

In cases where u is not a course-of-values we still obtain a singular, defined object: the empty set. This safeguards Frege's methodological stipulation that a function never give an output 'less well-defined' than the corresponding input.[72]

[71] Frege's explanation runs thus: "If the ζ-argument is a course-of-values, then the value of the function $\xi \frown \zeta$ is the value, for the ξ-argument as argument, of the function whose course-of-values is the ζ-argument. If on the other hand the ζ-argument is not a course-of-values, then the value of the function $\xi \frown \zeta$ for any ξ-argument is $\acute{\epsilon}(\top\, \epsilon = \epsilon)$." (§34)

[72] This is the result of inferential correspondence; cf. Chapter Two, §2.3.1

Another example of representing a function by its course-of-values is given in §35. It revolves around the second-level function symbolised as '$\phi(2)$'. This function is first introduced in §22, and its primary purpose is to serve as an elucidatory example. There it is defined as 'property of the number 2': we are given this function



as 'property of the number 2 that belongs to it exclusively'.[73] Back in §35 Frege says that 'property of the number 2' can be written as "$2 \frown \acute{\epsilon}\phi(\epsilon)$". We are told that "This is still the name of a second-level function; but if we write "ξ" for "$\acute{\epsilon}\phi(\epsilon)$", then we have in "$2 \frown \xi$" the name of a first-level function."[74] As a first-level function, only an object can serve as the ξ-argument in $2 \frown \xi$; thus we have two cases, one where the object ξ is a course-of-values, and another where it is not.

In the first case, the value of $2 \frown \xi$ is the same as that of $\Phi(2)$. This is because the argument for $2 \frown \xi$ is $\acute{\epsilon}\Phi(\epsilon)$, whereas the argument for $\phi(2)$ is $\Phi(\xi)$. Substituting Φ, a metalinguistic variable for a definite object, for ϕ, which denotes an empty argument place, yields $\Phi(2)$. Following the substitution procedure described in §34 (above) which is essentially $\Psi(\Delta) \Rightarrow \Delta \frown \acute{\epsilon}\Psi(\epsilon)$, we are given $2 \frown \acute{\epsilon}\Psi(\epsilon)$.

If, however, the ξ-argument of $2 \frown \xi$ is not a course-of-values, then there is no counterpart argument for the second-level function $\phi(2)$. This results in

[73] In modern notation: $\neg[\forall x(\phi x \rightarrow x = 2) \rightarrow \neg\phi(2)]$. This is equivalent to $\forall x(\phi x \rightarrow x = 2) \wedge \phi(2)$; we can reverse the order of the conjuncts, yielding $\phi(2) \wedge \forall x(\phi x \rightarrow x = 2)$, which is the intuitive rendering of 'property of the number 2 that belongs to it exclusively'. For example, ϕ could be 'is an even prime'.

[74] §35

the relationship of mutual representation between the two functions ($\phi(2)$ and $2 \frown \xi$) breaking down. To see why this is so, we shall now examine Frege's formal definition in greater detail.

Frege defines the function $a \frown u$ as the definite article of the function $\begin{array}{l} \overset{\mathfrak{g}}{\smile}\!\!\top\, \mathfrak{g}(a) = \alpha \\ \llcorner\, u = \acute{\epsilon}\mathfrak{g}(\epsilon) \end{array}$. As stated at the beginning of this section, this statement is of the '$\exists G(u = \{x : x \text{ is a } G\} \wedge G(\ ) = (\ ))$ form. Accordingly, we shall analyse the two statements contained within the parentheses, before evaluating what the appended existence claim amounts to.

The first statement, $u = \acute{\epsilon}\mathfrak{g}(\epsilon)$, states that an object u is identical with the course-of-values of the function $\mathfrak{g}$. As the function being defined is $a \frown u$, then we know that according to the definition, $a \frown \acute{\epsilon}\mathfrak{g}(\epsilon)$ (subject to the caveat described two paragraphs above; we shall come to this when discussing the existence claim). The second statement, $\mathfrak{g}(a) = \alpha$, is slightly more complex due to the presence of both Latin and Greek letters (a and α). Frege instructs us that "$\begin{array}{l} \overset{\mathfrak{g}}{\smile}\!\!\top\, \mathfrak{g}(\Theta) = \Delta \\ \llcorner\, \Gamma = \acute{\epsilon}\mathfrak{g}(\epsilon) \end{array}$ is the True if there exists a first-level function of one argument whose value for the argument Θ is Δ and whose course-of-values is Γ."[75] The 'function of one argument' is $\mathfrak{g}$, and we have that $\mathfrak{g}(\Theta) = \Delta$ is the True, where Θ is a member of $\acute{\epsilon}\mathfrak{g}(\epsilon)$. Now, because $\Gamma = \acute{\epsilon}\mathfrak{g}(\epsilon)$, and our statement is of a conjunctive form, we have that $\mathfrak{g}$ of any member of Γ is the True.

However, this whole statement is prefixed by (the *Begriffsschrift* equivalent of) a 'there exists', applying to $\mathfrak{g}$; thus, it is being entertained that there is a function which satisfies the two conditions discussed above. If such a function

[75] §34

does exist, then $\overset{\mathfrak{g}}{\smile}\!\!\top\, \mathfrak{g}(\Theta) = \Delta$ is the True. The next part of the definition

$$\llcorner\, \Gamma = \acute{\epsilon}\mathfrak{g}(\epsilon)$$

is the fact that the function is prefixed by a '$\acute{\alpha}$', and the Δ is replaced by an α. This means that we are now discussing this function's course-of-values. It takes two arguments, Γ and Θ; if Γ is a course-of-values and Θ is a member of this course-of-values, then the function is a name for the True. Thus, its course-of-values is $\mathfrak{g}$'s course-of-values.

However, the function for the course-of-values, $\acute{\alpha}\Phi(\alpha)$ is itself preceded by the function-name for the definite article, $\backslash\Delta$. Thus, we are talking about *the* course-of-values of the function '$\overset{\mathfrak{g}}{\smile}\!\!\top\, \mathfrak{g}(\Theta) = \Delta$'. This will be *the* collection

$$\llcorner\, \Gamma = \acute{\epsilon}\mathfrak{g}(\epsilon)$$

of objects which are members of the course-of-values of $\mathfrak{g}$, as long as the two conjunctive conditions of the definition are satisfied; otherwise, it will be the function itself. That is, the function for the course-of-values, taking the function contained within the parentheses as argument. If that which is contained within the parentheses is a name for the False, then by virtue of the fact that the course-of-values for the false is $\top\, \epsilon = \epsilon$, this becomes the argument for $\backslash\Delta$. Seeing as $\Delta = (\top\, \epsilon = \epsilon)$, then the value of the whole is $\top\, \epsilon = \epsilon$, as Frege argues.

Observe the vital role that the function for the definite article is playing here. Both cases of the function preceded by the definite article are names for singular objects; either the course-of-values of $\mathfrak{g}$, or the empty set. In both cases, there is an individual object taken as argument for $\backslash\Delta$.

3.5.2 Double Courses-of-Values: Extension of a Relation

The course-of-values of a function whose value is always a truth-value is termed the extension of a concept. We now move to concepts of two (or more) arguments, which Frege terms 'relations'. Thus the courses-of-values of of relations are likewise extensions.

Because relations require two arguments to become saturated, we have courses-of-values which are in some respects more complex. For instance, Frege gives the example of addition as a function of two arguments $(\xi + \zeta)$. We can supply only one argument, which we can tentatively term a 'partial saturation', as in '$\xi + 3$', where '3' is given as the ζ-argument. The course-of-values of this function is $\acute{\epsilon}(\epsilon + 3)$. Now, reversing our partial saturation, we have a course-of-values which looks like $\acute{\alpha}\acute{\epsilon}(\epsilon + \alpha)$.

Following the reflections of the previous section, we can apply the function $\xi \frown \zeta$ to $\acute{\alpha}\acute{\epsilon}(\epsilon + \alpha)$. Considering our operation of partial saturation, we can set Δ to be an argument (as opposed to the whole course-of-values) which gives us

$$\Delta \frown \acute{\alpha}\acute{\epsilon}(\epsilon + \alpha) = \acute{\epsilon}(\epsilon + \Delta).$$

This sentence describes the event in which $\Delta \frown \acute{\epsilon}\Phi(\epsilon) = \Phi(\Delta)$, where Φ is itself an instance of the course-of-values function, namely $\acute{\epsilon}(\epsilon + \xi)$. Now, as $\Delta \frown \acute{\alpha}\acute{\epsilon}(\epsilon + \alpha) = \acute{\epsilon}(\epsilon + \Delta)$, we can iterate the operation further by adding another argument, Γ, though this argument is for the above result and *not* for the addition function itself. This can be symbolised thus:

$$\Gamma \frown (\Delta \frown \acute{\alpha}\acute{\epsilon}(\epsilon + \alpha)) = \Gamma \frown \acute{\epsilon}(\epsilon + \Delta)$$

Now, '$\Gamma \frown \acute{\epsilon}(\epsilon + \Delta)$' is of the same form as $\Theta \frown \Phi$. Seeing as $\Theta \frown \Phi = \Phi(\Theta)$, we can obtain $\Gamma \frown \acute{\epsilon}(\epsilon + \Delta) = \Gamma + \Delta$, the addition function.[76] We now have

$$\Gamma \frown (\Delta \frown \acute{\alpha}\acute{\epsilon}(\epsilon + \alpha)) = \Gamma \frown \acute{\epsilon}(\epsilon + \Delta) = \Gamma + \Delta,$$

which gives us

$$\Gamma \frown (\Delta \frown \acute{\alpha}\acute{\epsilon}(\epsilon + \alpha)) = \Gamma + \Delta.$$

We have on the left a double course-of-values, and on the right, a function of two arguments. Frege tells us that "...in the double course-of-values [whose name occurs] on the left-hand side is captured what is peculiar to the function [whose name occurs] on the right-hand side, what distinguishes it from other first-level functions of two arguments."[77] Now, what is it that is peculiar to any given first-level function of two arguments? The collection of objects which when used to saturate the function will render a name of the True; that is to say, that which identifies a first-level function with another and separates it from all non-identical others—its course-of-values. It is this sort of 'identity of indiscernibles' which Frege codified in his basic law V.[78]

As in the case for single courses-of-values, we may ask what value $\Gamma \frown (\Delta \frown \Theta)$ assumes if Θ is not a double course-of-values. Frege distinguishes two such cases: the first, in which Θ is not a double, but a single course-of-values,

[76] It may be more pedagogical to explain the transition like this: we have $\Gamma \frown \acute{\epsilon}(\epsilon + \Delta)$, and wish to obtain $\Gamma + \Delta$, or alternatively, $+(\Gamma, \Delta)$. We have the general case, $\Theta \frown \Phi = \Phi(\Theta)$. The 'variable' in $\acute{\epsilon}(\epsilon + \Delta)$ is the ϵ; thus we substitute Γ for ϵ and drop the $\acute{\epsilon}$, as we are no longer talking about a course-of-values. This yields $\Gamma + \Delta$

[77] §36; additions by translator

[78] Basic Law V, with the identity sign conceived of as a biconditional, has both a 'safe' direction and an 'unsafe' direction. The safe direction is $\vdash \begin{array}{l} \acute{\epsilon}f(\epsilon) = \acute{\alpha}g(\alpha) \\ \smile^{\mathfrak{a}}\; f(\mathfrak{a}) = g(\mathfrak{a}) \end{array}$, which Frege calls (Va). The converse, (Vb), is the unsafe direction which leads to Russell's paradox. Thus, it seems that so long as we are *given* the equivalence of two functions for all arguments, we can infer the equivalence of their courses-of-values. It is the converse inference, from equivalent courses-of-values to equivalent functions, which proves inconsistent. This situation is handled in modern set theories such as ZF via a restricted comprehension axiom; the 'restriction' is to sets which are already given, or shown to exist. If we follow the discussion of courses-of-values above, and think of $\{x\}$ as the name for x, then equivalent objects will have the same name. However, two objects' bearing of the same name does not guarantee that they are the same object.

results in $\Gamma \frown (\Delta \frown \Theta) = \acute{\epsilon}(_{\top}\, \epsilon = \epsilon)$. The second, in which Θ is not a course-of-values at all, results in $\Delta \frown \Theta$'s coincidence with $\acute{\epsilon}(_{\top}\, \epsilon = \epsilon)$. In this case $\Gamma \frown (\Delta \frown \Theta) = \Gamma \frown \acute{\epsilon}(_{\top}\, \epsilon = \epsilon)$. Thus, $\Gamma \frown (\Delta \frown \Theta) = (_{\top}\, \Gamma = \Gamma)$; hence $\Gamma \frown (\Delta \frown \Theta)$ is the False. Either way, if Θ is not a double course-of-values, then $\Gamma \frown (\Delta \frown \Theta)$ is the False. To summarise:

$$\Gamma \frown (\Delta \frown \Theta) = \begin{cases} \text{the True, if } \Theta = \acute{\alpha}\acute{\epsilon}(\Phi(\alpha, \epsilon)) \text{ for some } \Phi \text{ where } \Phi(\Gamma, \Delta) \text{ is} \\ \text{the True;} \\ \text{the False otherwise} \end{cases}$$

Notice that the function for the definite article, $\backslash\Delta$ does not itself occur in the above definition. However, it is 'built in' to the function $\xi \frown \zeta$, which serves as the basis for $\Gamma \frown (\Delta \frown \Theta)$; we talk of *the* extension of a relation. We shall now see the use that Frege puts his extensions of relations to.

3.5.3 The Function $I\xi$: Many-One Relations

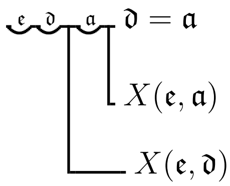
In §23 of the *Grundgesetze*, Frege stipulates the condition under which a relation is considered many-one. In modern terminology, a many-one relation is called *functional*. For a relation R to be many-one, it must be the case that if an object is R-related to any two (or more) objects, then those objects are identical (i.e. are the same object). The notion that all inputs have singular outputs is vital to the modern concept of a function—however, this does not bar the case in which a given output can be the result of distinct inputs.[79] An example is the squaring function, $x^2 = y$; for we have both $1^2 = 1$ and $(-1)^2 = 1$. However, it is impossible that squaring the same number multiple times could give different results. Richard Heck, in his (1993), gives a modern equivalent of

[79] When this case also is barred, the relation is called one-to-one; we consider this in due course

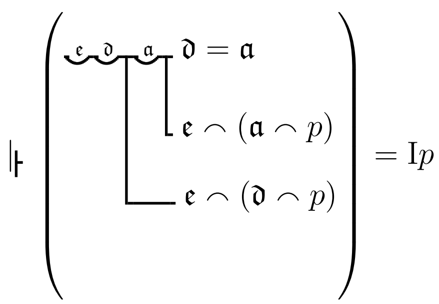
Frege's definition:[80]

$$\text{Func}_{\alpha\epsilon}(R\alpha\epsilon) \equiv_{df} \forall x \forall y(Rxy \rightarrow \forall z(Rxz \rightarrow y = z))$$

Heck notes that his definition makes no mention of courses-of-values, but deals directly with relations. This means that in Heck's modern formulation, no definite articles are required, becasue we are speaking directly of functions (in Frege's terminology), and not objects. Frege begins in §23 with a similar stipulation, which runs thus:



If the relation X is many-one, then the above statement is always the True. However, Frege wishes to use a first-level equivalent of this second-level function (it takes the relation X as argument), so makes use of his technique of representing first-level functions by their courses-of-values. Because we can represent relations by their extensions (that is, functions of two arguments by their double courses-of-values), we can replace the '$X(\mathfrak{e}, \mathfrak{d})$' in the above statement with '$\mathfrak{e} \frown (\mathfrak{d} \frown X)$'.[81] This yields his definition of a relation's being functional, given by the symbol '$\text{I}p$', where p is a relation:



This definition is object-oriented, unlike that given above; through the presence of courses-of-values we know that the definite article is present. Frege describes

[80] Heck (1993) 'The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*' in Demopoulos (ed, 1995) *Frege's Philosophy of Mathematics*

[81] Similar considerations hold for the other occurrences of X

the function-symbol Ip as being the truth-value of $\xi \frown (\zeta \frown p)$'s—that is, *the* $p(\xi, \zeta)$'s—being a many-one (functional) relation. This means that "for every ξ-argument there [is] either no ζ-argument, or only one, for which the value of the function is the true, or, as we can also say, for every object there being at most one object to which it stands in the relation $\vdash \xi \frown (\zeta \frown \Delta)$."[82].

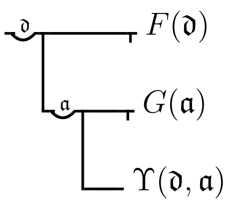
3.5.4 The Function $\rangle\xi$: The Relation of 'Being Many-One'

We are examining this part of the *Grundgesetze* with a view toward the role of the definite article in Frege's definition of equinumerosity. Frege states that his definition has not changed substantially since the *Grundlagen*. In that work, he says

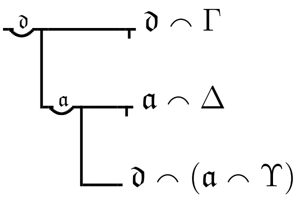
> . . . the proposition
>
> "to every object which falls under G there stands in the relation ϕ an object falling under F"
>
> means that the two propositions
>
> "a falls under F"
>
> and
>
> "a does not stand in the relation ϕ to any object falling under G"
>
> cannot, whatever a may be, both be true together. (*Grundlagen*, §72; cited in *Grundgesetze* §38)

[82] *Grundgesetze* §37

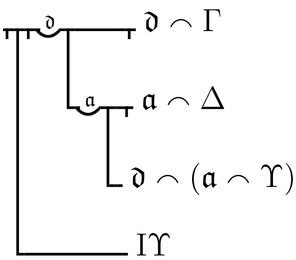
The relation ϕ that Frege speaks of is one which must obtain if the phrase '*the* concept F is equinumerate with *the* concept G' is to name the True (notice the definite articles).[83] In *Begriffsschrift*, this can be symbolised as



Or, in modern notation, $\forall x[\forall y(\Upsilon xy \rightarrow \neg Gy) \rightarrow \neg Fx]$. If in Frege's definition we represent the concept F with a course-of-values $\xi \frown \Gamma$, concept G with a course-of-values $\xi \frown \Delta$, and the relation by the double course-of-values $\xi \frown (\zeta \frown \Upsilon)$, then Frege's definition of equinumerosity can be written



If we append the condition that Υ is a many-one relation (which we surely want it to be, if it is to underpin equinumerosity), then we get the following formula:

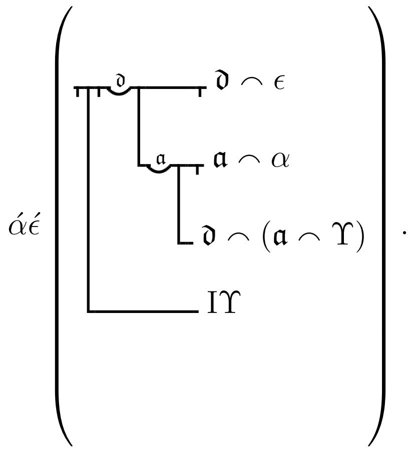


Remember that '$\mathrm{I}\Upsilon$' is an abbreviation for the formula which expresses (the truth-value of) the relation $\xi \frown (\zeta \frown \Upsilon)$'s being many-one. Hence, the above formula states that the relation Υ correlates all $\mathfrak{d}$'s which are Γ's to $\mathfrak{a}$'s which

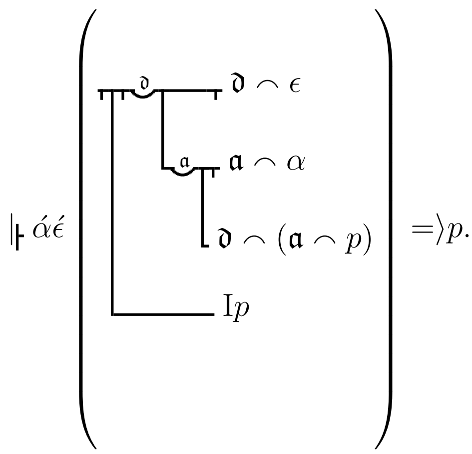
[83] Slightly more formally, $F = G$ iff $\forall x(Fx \wedge \neg\exists y(\phi(x, y) \wedge \neg Gy))$. We will come to a more rigorous formulation, including the conditions the relation ϕ must fulfill, in due course.

are Δ's, *and* that ϒ is many-one—for every $\mathfrak{d}$ which is a Γ, there is at most one $\mathfrak{a}$ which is a Δ that is ϒ-related to that $\mathfrak{d}$.

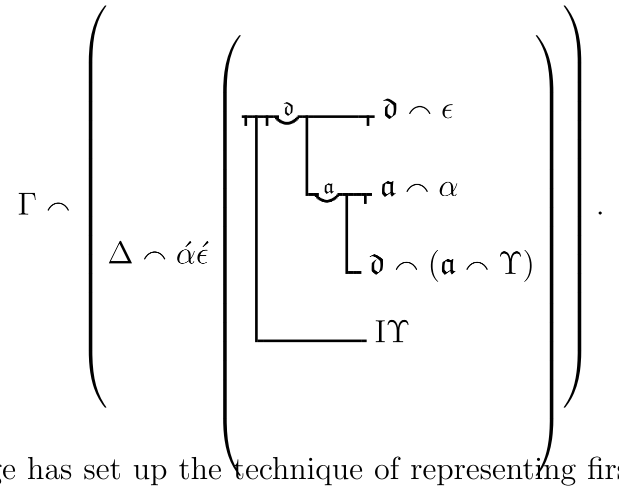
This is itself a relation, a two-place function for the arguments Γ and Δ. Thus, it has a (double) course-of-values, which is symbolised as



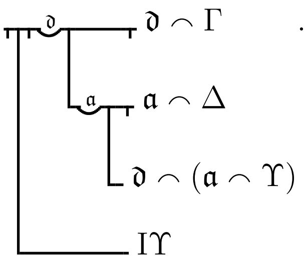
This double course-of-values is itself a value, for the above as a single-argument function on the argument ϒ (yet another example of Frege's utilisation of multiple analysability in his *Begriffsshrift*). Hence Frege introduces an abbreviation for this function, $\rangle p$, which is defined as



As this function takes the argument p and gives as its value a double course-of-values we can substitute it in $\Gamma \frown (\Delta \frown \xi)$ without predeterminately being given the False (see §3.5.2). So we will have $\Gamma \frown (\Delta \frown\rangle \Upsilon)$, which, by substitution of the definition of $\rangle p$, will be



By the way Frege has set up the technique of representing first-level functions by their courses-of-values, the above formula denotes the same truth-value as



Thus, the truth-value of the above formula, and hence that of $\rangle\Upsilon$, as that of the relation $\xi \frown (\zeta \frown \Upsilon)$ putting the objects which belong to $\xi \frown \Gamma$ in a many-one correspondence with those object belonging to $\xi \frown \Delta$. Frege calls this relation a 'mapping' relation. Heck puts it in modern notation thus:

$\mathrm{Map}_{\alpha\epsilon xy}(R\alpha\epsilon)(Fx, Gy) \equiv_{df} \mathrm{Func}_{\alpha\epsilon}(R\alpha\epsilon)$[84] $\wedge \forall x(Fx \rightarrow \exists y(Rxy \wedge Gy))$[85]

However, Heck notes that due to the nature of Frege's system—specifically, his extensive use of contraposition—this definition is more accurately rendered $\mathrm{Map}(R)(F, G) \equiv_{df} \mathrm{Func}(R) \wedge \forall x[\forall y(Rxy \rightarrow \neg Gy) \rightarrow \neg Fx]$.

---

[84] [Heck's '$\mathrm{Func}_{\alpha\epsilon}(R\alpha\epsilon)$' is equivalent to Frege's '$\mathrm{I}\Upsilon$'.]
[85] see p.266 of his (1993).

3.5.5 The Function $\mathcal{U}\xi$: Converse of a Relation

The function $\mathcal{U}\xi$ is that which names the *converse* of the relation ξ. If $R(\xi, \zeta)$ is the relation 'ξ is the parent of ζ', then $\mathcal{U}R(\xi, \zeta)$ will be 'ζ is the offspring of ξ'. Frege approaches the definition of a converse via the representation of a relation by its extension (its double course-of-values; an object, hence referred to by an expression containing a definite article).

Given a the extension of a relation ξ, $\mathcal{U}\xi$ will be the extension of the converse of ξ. The definition of $\mathcal{U}\xi$ is given in the following form:

$$\Vdash \grave{\alpha}\grave{\epsilon}(\alpha \frown (\epsilon \frown p) = \mathcal{U}p$$

It is then stated that if we have a relation $\xi \frown (\zeta \frown \mathcal{U}\Upsilon)$, which by the above definition is equivalent to $\xi \frown (\zeta \frown \grave{\alpha}\grave{\epsilon}(\alpha \frown (\epsilon \frown \Upsilon)$, then this relation is equivalent to $\zeta \frown (\xi \frown \Upsilon)$ (notice that the order of ξ and ζ has been reversed).[86] Thus, this function takes a relation as input and gives its extension as an output; this can then be used to obtain the extension of the extension of that relation, which yields the converse.

3.5.6 The Function $\mathcal{A}\xi$: Mappings Between Relations

By the previous definition, if we have $\xi \frown (\zeta \frown \Upsilon)$, we can 'say the same things' of the converse of this relation if we replace 'Υ' with '$\mathcal{U}\Upsilon$'. Hence, $\mathrel{\top\!\!\top} \Gamma \frown (\Delta \frown\rangle\Upsilon)$ $\llcorner\, \Delta \frown (\Gamma \frown\rangle\mathcal{U}\Upsilon)$ is "is the truth-value of the Υ-relation's mapping the Γ-concept into the Δ-concept, and its converse's mapping the Δ-concept into

[86] §39

the Γ-concept".[87] The upper conjunct states that the relation named by ϒ maps each Γ to to most one Δ. The lower conjunct states that the converse of ϒ maps the Δ's to the Γ's in the same manner. Thus, the conjunction of the two states that the Γ's and Δ's are in one-to-one correspondence. For each Γ, there is at most one Δ to which it bears the ϒ-relation, and vice versa.

Frege states that if Γ and Δ are to be equinumerate concepts, that is, concepts whose courses-of-values contain the same number of objects, then a relationship such as ϒ must obtain between Γ and Δ. That is, there must be a way of putting Γ and Δ into one-one correspondence. Now, all relations can be represented by their extensions in the double course-of-values manner: $\Phi(\xi, \zeta)$ can be represented by $\xi \frown (\zeta \frown \acute{\alpha}\acute{\epsilon}(\Phi(\alpha, \epsilon))$. This technique of representation makes ineliminable use of the definite article. Thus, a statement of the form 'there is a relation such that it puts Γ's and Δ's in a one-one correspondence' will be a statement which names the truth-value of the statement 'there is the same number of Γ's as Δ's'. And hence, we will know that whenever the first is true, the second is true also, and vice versa; thus we can deduce the concept of number purely out of that of a certain type of relation.

However, the statement 'there is a relation ξ such that it puts Γ's and Δ's in a one-one correspondence' is, for Frege, a statement about objects; this is because of his use of the function for the definite article in the definition of one-one correspondence. On the other hand, 'there is the same number of Γ's as Δ's' is about *concepts*. Nevertheless the truth-values of the two statements are in a biconditional relationship (one is only true when the other is; likewise for falsity). It is the definite article which underwrites this relationship.

[87] §40; italics removed

Now, in modern symbolism, the statement we want will be of the form

$$\exists R[\mathrm{Map}(R)(F,G) \wedge \mathrm{Map}(\mathrm{Conv}R)(G,F)].$$[88]

Seeing as we now that this statement is true under *all* and *only* the circumstances by which 'the number of F's = the number of G's' is true, we can proceed by combining the two statements into one of the form 'the no. of F's = the no. of G's *if, and only if*, there is a relation between the F's and the G's such that for every a which is an F, there is a b which is G, and vice versa'. Frege gives the following symbolisation of the second part of the phrase:

$$\overset{\mathfrak{q}}{\smile}\!\!\begin{array}{l} \Gamma \frown (\Delta \frown\rangle \mathfrak{q}) \\ \Delta \frown (\Gamma \frown\rangle \mathcal{U}\mathfrak{q}) \end{array} .$$

As in previous cases, this can be seen as a function of one argument (in this case, Γ) which gives either the True or the False as its value. It will give the True if there is a relation which puts the two collections in one-one correspondence, and the False otherwise. As this is a function of one argument which gives only the True or the False as values, it is a concept; and thus, it has an extension, which is

$$\acute{\epsilon}\left(\overset{\mathfrak{q}}{\smile}\!\!\begin{array}{l} \epsilon \frown (\Delta \frown\rangle \mathfrak{q}) \\ \Delta \frown (\epsilon \frown\rangle \mathcal{U}\mathfrak{q}) \end{array}\right).$$

Frege notes that this definition is in accord with that given in §68 of the *Grundlagen*, in that "this extension is the *Number* that belongs to the concept $\xi \frown \Delta$."[89] We can shorten the phrase 'Number that belongs to the Δ-concept' to 'Number of the Δ-concept'; likewise, we define the function $\mathcal{A}u$ as a function on the argument Δ, in the following manner:

$$\Vdash \acute{\epsilon}\left(\overset{\mathfrak{q}}{\smile}\!\!\begin{array}{l} \epsilon \frown (u \frown\rangle \mathfrak{q}) \\ u \frown (\epsilon \frown\rangle \mathcal{U}\mathfrak{q}) \end{array}\right) = \mathcal{A}u.$$

[88] This formulation is due to Heck (1993).
[89] §40, italics original

Thus, the function $\mathcal{A}$ for the argument u gives the extension of the concept which states that there is a relation that correlates the u's one-one with the objects in that concept's course-of-values. In other words, $\mathcal{A}u$ is the collection of objects—itself an object—which are in one-one correspondence with the ϵ's which form the course-of-values of the concept which states that the ϵ's and the u's are in one-one correspondence. To quote the passage of the *Grundlagen* which Frege refers to, "the Number which belongs to the concept [u] is the extension of the concept "equal to [i.e. in one-one correspondence with] the concept [u]." The number of u's—$\mathcal{A}u$—is the collection of courses-of-values of those concepts which fall under the concept 'in one-one correspondence with the members of u'.

So, to summarise: we have two concepts, Δ and Γ which (we hope) are equinumerous. In order to ascertain this, we must prove three things:

- that there is a relation φ between the extensions of Δ and Γ such that
- φ from the Γ's to the Δ's is many-one; for each Γ, there is at most one Δ such that $\varphi(\Gamma, \Delta)$, and
- that the converse holds: for each Δ, there is at most one Γ such that $\varphi(\Delta, \Gamma)$.

Once we know this, we know that the number of Γ's = the number of Δ's. However, we do not know *which* number this is: whether there is only one Γ (and hence one Δ), or hundreds, or an infinite number.[90] Frege has (so far) only given the truth-conditions for the recognition-judgement for the notion of cardinal number, not that for any *individual* cardinal numbers.

[90] This is Frege's 'Julius Caesar' problem.

This is why Frege does not stop here; he goes on to define in §41 the number zero as the extension of the concept ‘not equal to itself’. No object is non-self-identical; hence this extension is empty. Next is the number one, which is the extension of ‘the extension of the concept ‘not equal to itself’’. *This* extension contains one object—the empty set. Now all that Frege needs is a definition of the function ‘Λ follows directly in the number-series directly after Θ’.[91] With this done, we have the requirements that any two concepts must satisfy in order to be numbers that are immediately adjacent in the number-series; we already have zero and one, so we can progress to two, three, four...

3.5.7 Extensions and Frege's Transcendental Idealism

Frege needed logical objects in order to make logic ‘contentful’. That is to say, he needed it to be more than it was for Kant; something which is capable of generating its own content, rather than an empty structure which needs objects given by intuition to operate on. Via the function $\xi \frown \zeta$ Frege equivocates a function with its course-of-values. The definition of this function makes necessary use of the function for the definite article; in order to represent a function by its course-of-values, we must be able to speak of ‘*the* course-of-values of the function f’. From this come double-courses-values, and the specific case of extensions of relations.

Thus, Frege believed that he had found a way to derive the defining aspects of the science of number from the laws of thought alone. Such a conclusion, though it goes against Kant in that Frege claimed that arithmetic is analytic, also ensures that the laws of (cardinal) number are themselves unique. No ra-

[91] §43

tional alternative is possible; Kant's claim of the *uniqueness* of arithmetic is preserved. Far from being at a remove from Kant, this is in the very spirit of Kant's original arguments that all of mathematics is synthetic *a priori*—however, Frege has had to go beyond Kant, particularly in terms of epistemic bounds; Frege claims that arithmetic is a consequence of the structure of rationality itself, rather than merely that of the *human* mind. Nevertheless, such a strong conclusion could only be wrought from a philosophy such as Kant's Transcendental Idealism. The Empiricism and Formalism of the time were in no way deep enough to yield such strong conclusions, a fact that Frege himself ascertained in the negative portions of the *Grundlagen* and the *Grundgesetze*.

This is because Frege's definition is motivated by the very uses which arithmetic is put to: verifying whether there are the same number of x's as y's, the same number of x's as $y + z$'s, and so on. The first phrase Frege fixes the meaning of is 'has the same number as'; the contextual definition of number precedes the explicit definitions (of zero, one, and [Frege's version of] the successor function). And hence, this science of number is such that it needs no further explanation for its applicability. Frege's logicism meets the issue which confronts the Formalist. And as it is so intimately related to the very nature of rationality, it precedes all experience—the Empiricist is dispatched. Given a world where rational minds are present, there is one and only one way to arithmetically comprehend that world.

The existence of logical objects must also be a consequence of the structure of rationality of Frege's logicism is to work. However, the relationship between function and object is a sticking point in Frege's philosophical logic. At times, the distinction is stated categorically and forcefully; at other times, such as the exposition of the definite article, it seems as thought Frege wants to equivocate an object with the class containing only that object. In 'On Concept and Object'

Frege gives convincing reasons for seeing 'the concept *horse*' as referring to an object, rather than a concept. In the *Grundgesetze*, first-level concepts are represented y their courses-of-values. This technique of representation makes ineliminable use of the definite article, and is itself exploited extensively in the construction of the laws of arithmetic. And, as argued in §3.4.1, the behaviour of Frege's function for the definite article is not even that well understood in modern investigations of Frege's logic and philosophy.

4. CONCLUSION

We have come to the end of a long road. We began with the question, what are the constitutive factors in a change of mathematical setting? Two decisive aspects of such a conceptual shift were identified. These are (1) that the problems which motivate the development of the new setting spring from issues with the old, and (2) that addressing these problems makes use of resources unavailable to the old setting. Philosophical analysis revealed that standard contemporary attitudes toward the 'foundations' of mathematics, especially those which couple set theory with a Platonic realism about sets, are thoroughly inadequate to the task of accommodating the very real occurrences of setting-changes which have taken place in the history of mathematics.

We then moved onto an analysis of one of the greatest setting-changes in recent history: the almost spontaneous development of a mathematical logic very near to that in use today by Gottlob Frege in the last decades of the 19th century. However, our reflections cautioned us against seeing Frege's innovations as anything near 'spontaneous'. In order to fully understand *any* change of setting, we must be able to place it in a wider context. Doing so should explain *why* the change of setting had to occur in order to solve the problems which motivated its inception in the first place.

We found that Kant's influence on Frege was to motivate him to show that alternative sciences of number are not only inconceivable, but *impossible*. The development of non-Euclidean geometries had imploded Kant's philosophy of mathematics in the minds of most mathematicians; Frege's claim that arithmetic is analytic, rather than synthetic (as Kant had argued), was simply the best way to ensure that there could be no logically valid alternative—no 'non-Peano arithmetics', as it were. This serves to satisfy condition (1), for we know that Frege was addressing an issue which directly relates his work to a philosophical tradition that spans at least a hundred years. He was attempting to solve a problem which was already present in the old setting; 'can Kant's claims about the universal validity of arithmetic be dispatched in the same manner as his claims about geometry?'.

In order to verify condition (2), we had to look deeper, into Frege's formal system itself. There we found that his way of formalising the relationship between objects and our syntactic methods of identifying them is both in line with the view that he is changing settings (in terms of condition (1)) and is somewhat opposed to modern set theory's formalisation of the connection between an object and its unit class. We were then able to use these conclusions to comment on a contemporary debate in the history of philosophy—the so-called 'paradox of the concept *horse*'—and found that both sides of this debate had neglected to look for answers in the context of a wider reading of Frege, one which considers him both as, and from the perspectives of, a mathematician and a philosopher.

This last comment is important. The research carried out in this thesis does not belong entirely to mathematics, nor to philosophy, nor even to history. We have blended all three disciplines together in order to gain new perspective on a historical figure. We do this out of the desire for greater understanding, not only of why things happened the way they did, but why they *had* to happen that way. Furthermore, this historical research into mathematics has led to new conclusions in contemporary philosophy. Very few great figures of history can be fruitfully observed from only one angle. We must encourage more interdisciplinary investigations into the history of the sciences if we are to gain a more illuminating view of our past, and thus, ourselves.

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