Groundwater Management Tools:
Analytical Procedure and Case Studies

October 2003

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Executive Summary

This report addresses an issue of groundwater management that was identified by regional council staff, as part of a project conducted by Ministry of Agriculture & Forestry and the Ministry for the Environment for encouraging and ensuring effective and efficient water allocation in New Zealand. The issue is how to manage groundwater allocation under conditions of increasing abstraction and imperfect, but developing, knowledge of the resource. The overall objective is to maintain sustainability of the groundwater resource in terms of acceptable environmental effects.

The first part of this report is a draft Best Practice Guideline, which sets the context of the nature of the groundwater resource, quality and availability of data, and an appropriate resource management approach. A recommendation from the water allocation project was that an adaptive approach to groundwater management was required, and that there was a need for appropriate analytical tools to support this approach. A companion report addresses the origin and philosophy of adaptive management in water resources.

The second part of the report is concerned with the development and demonstration of a suitable analytical method, and guidelines for its implementation, which supports the recommended adaptive management strategy.

The "eigenmodel" method is concerned primarily with the amount of water stored in an aquifer, and how this responds to recharge and abstraction. The resulting information about groundwater levels can be related to environmental effects such as low flow in streams, for example. It is a "whole aquifer" approach and does not purport to be suitable for detailed investigation of local effects caused by abstraction stresses. These problems require other well established modelling techniques, and their compatibility with the eigenmodel method is discussed.

The issue of sparse data is addressed by the simplicity of the analytical format, which enables identification of fundamental properties of aquifer storage, sometimes from only one observation well record. Implementation of the procedure is ideally suited to spreadsheet software. These simple models can also be expressed in a form that incorporates continual monitoring of groundwater levels for "real-time" forecasting as decision support for adaptive management. Several demonstrations with observed data from two aquifer systems are presented to illustrate the capabilities of the procedure.

Introduction

This report has been commissioned as a result of a two-year project conducted by Ministry of Agriculture & Forestry and the Ministry for the Environment, to encourage and ensure efficient and effective water allocation in New Zealand. One of the outcomes of that project was the identification of issues in groundwater management, as a result of a survey of regional councils and consultation with the Regional Groundwater Forum of council groundwater scientists. These issues include:

- Knowledge about an aquifer often follows demand on the groundwater resource;
- Allocation decisions, with potential long-term effects, must be made on the basis of information that is perceived to be insufficient;
- The amount of information about an aquifer may increase with time, but may never have included the state of zero demand
- An adaptive management approach was suggested for optimal decision-making when both demand and information are increasing with time.

This report presents an approach to groundwater assessment and allocation that enables the application of relatively simple analytical methods that are appropriate to the availability of data. This approach is derived from the same mathematical theory as groundwater modelling software, such as MODFLOW, so that compatibility with these methods is intended where there are sufficient data for more refined analyses.

The groundwater allocation issues that are addressed in this report are based on a whole-aquifer point of view that accounts for time-varying environmental effects such as changes to the low-flow regime of streams. There are of course other effects such as stream depletion and salt water intrusion, which may be caused primarily by short-term local groundwater demand. There are established analytical methods for assessing these latter effects, and the interface between these methods and the new approach will be discussed.

The report is presented in two parts. Section A describes a Best Practice Guideline for groundwater management, which is intended to be a general framework for analysing and resolving issues of groundwater allocation. Section B provides a detailed description of one method of analysing a groundwater resource for the purpose of supporting the adaptive management approach proposed in Section A.

Section A – Groundwater Management: Best Practice Guideline

1. The Management Objective

The amount of groundwater stored in an aquifer at any instant of time depends on the dynamic relationship between recharge inputs, through the overlying land surface and from rivers, and outflow to surface waters and pumped abstraction. Aquifer storage provides a buffer between highly variable, climatically-driven recharge processes and the less variable outflow that supports surface water ecology. Abstraction of groundwater for human use, and some kinds of land use changes, alters the dynamic balance between "natural" recharge and the state of surface waters. The resource management objective is to determine the regime of abstraction that results in acceptable environmental effects.

2. Concepts of Sustainable Yield

2.1. A MISCONCEPTION

Occasionally, there appears in the media a statement to the effect that "...we must conserve groundwater for our grandchildren...". This may arise from a concept of groundwater as a static, finite body of water, which is mined by any abstraction. What should be passed on to future generations are the beneficial results of a well-managed, dynamic water resource system.

2.2. GROUNDWATER BUDGETS

One resource management approach is to attempt to estimate all the input and output components of the groundwater budget, and then make a decision about what can be safely allocated for abstraction on a "sustainable" basis. In practice, it is very difficult to measure independently the quantity of recharge from rivers or the natural outflow from an aquifer to surface waters, for example. Selection of the recharge proportion to be used as sustainable yield can become an arbitrary choice that may bear no relation to the effects of abstraction.

2.3. DYNAMIC STORAGE

The amount of groundwater present in an aquifer at any instant of time depends on the recent (months or years) history of climatically-driven recharge processes, natural outflow to surface waters, and abstractions for human use. The recharge processes are highly variable through the land surface, and less so for river recharge. Natural outflows are less variable than the recharge history, because of the smoothing effect of groundwater storage, and abstraction is potentially measurable and manageable.

This is the picture of a dynamic water storage responding to inflows that have varying degrees of randomness. At most locations, this water storage is intimately connected to the surface water environment and the ecology has adapted to the local regime of natural variability of water supply. Abstraction of groundwater for human use alters the regime of variability, and therefore affects the natural environment. The conceptual basis for management should be understanding of system behaviour rather than a budgetary approach.

3. Scale of Abstraction Effects

The theory of groundwater wells shows that the piezometric effect of groundwater abstraction propagates away from the well in a radial direction, and that this effect decreases in magnitude with distance. Most applications of this theory are directed to estimating the effects of drawdown from a well, in terms of interference with other wells or influence on stream-aquifer interaction. The distance scale of these investigations is usually up to a few kilometres, and the time scale is up to a few months. These methods are well established and are not considered any further in this report. However, it is important to realise that these applications of well theory limit the estimation of effects to magnitudes that are significant or are practically measurable.

Any abstraction from an aquifer has an effect that eventually propagates throughout the whole aquifer. This effect may be a lowering of piezometric levels or induced additional recharge from a river. The effect from any one well may be infinitesimal in terms of practical measurement, but the cumulative long-term effects of many wells can be very significant. The result is that every user of groundwater from an aquifer is a contributor to environmental effects such as reduction of low flows in streams or salt water intrusion, which are determined by natural outflow to surface waters at the whole-aquifer scale.

4. Limitations of Data and Information

4.1. QUANTIFYING RECHARGE

Aquifers have two sources of recharge: through the land surface, and from rivers.

- Land surface recharge can be estimated from water balance models to an accuracy suitable for most resource management decisions.
- River recharge is very difficult to estimate, and it can be very expensive to conduct measurements of sufficient quality and frequency for useful resource assessment.

4.2. GEOHYDROLOGICAL INFORMATION

Groundwater models usually require information about the nature of aquifer boundaries and the aquifer properties of transmissivity and storativity. The latter are sometimes estimated from pumping tests, but these are subject to a high degree of variability and may not be appropriate for aquifer-scale application. For these reasons, aquifer properties tend to become model parameters for calibration by means of available piezometric data.

Identifying the nature of aquifer boundaries can sometimes be difficult, especially for aquifer-river boundaries in alluvial aquifers. Poor definition of boundaries has a significant effect on the reliability of a mathematical groundwater model.

4.3. PIEZOMETRIC DATA

Measurement of groundwater levels in observation wells provides high quality data about the dynamic behaviour of an aquifer, in terms of its value as a resource. For assessment of the whole-aquifer resource, length of record is generally more important than numbers of wells observed. The principal data limitation in many areas is length of record.

4.4. ABSTRACTION

Most use of groundwater is controlled by the resource consent process. However, these controls are usually specified as maximum rates and volumes. There are few data about actual abstraction from aquifers, but this situation is likely to improve as implementation of water metering is advanced.

4.5. CONCLUSIONS ABOUT GROUNDWATER RESOURCE DATA

The high quality data about an aquifer that are relatively easy to acquire, are estimates of land surface recharge and observations of groundwater level. It would be desirable to use analytical methods that can build on these strengths.

Inadequate data about current abstractions, and the general paucity of observations in many aquifers, suggests that management of these aquifers should proceed in a manner that allows for changes in strategy as more information becomes available.

5. Monitoring Methods

5.1.   GROUNDWATER LEVELS – THE KEY INDICATOR

Groundwater level, as monitored at observation wells, is the most important indicator of the state of the resource in terms of availability for use and the likely effects on the environment. Environmental effects, such as the low-flow regime of streams and salt water intrusion, are determined primarily by piezometric levels in the connecting aquifers.

The dynamic variability of groundwater levels at observation wells located throughout an aquifer has some components common to all the wells. This means that even one well, suitably located, can provide a significant amount of information about the overall state of the resource. The best locations are furthest from outflow boundaries to surface waters, because the amplitude of level variation is greatest in relation to measurement noise and other influences. It is also desirable to select sites that are less likely to be affected significantly by local abstraction, or local recharge from surface irrigation and flood events in streams perched above the aquifer.

The value of groundwater level observations increases more with length of record than with number of observation sites, because of the common dynamic components.

5.2.   LAND SURFACE RECHARGE

The defining characteristic of an aquifer as a groundwater resource is its dynamic behaviour as a leaky storage for natural recharge. This behaviour can be determined from the observed response of groundwater levels to land surface recharge, because the influence of river recharge is usually attenuated to a steady piezometric effect within a few kilometres of the recharge zone.

Land surface recharge is estimated as the soil-water drainage component of soil-plant-atmosphere processes at the land surface, in response to climate. There are a number of water balance models available, but these are not critiqued in the present report. The emphasis here is on necessary and sufficient aspects of these models:
- Water balance must be calculated on a daily basis, and then totalled to the selected time interval such as a month;
- Particular crop-soil combinations, can be expressed as a water-holding capacity;
- Only significant areas (as a percentage of total) of a particular soil-crop combination need be considered.

The strongest recharge signal for analysis of an aquifer comes from winter recharge when abstraction (for irrigation) is least. Therefore, estimation of land surface recharge need not take abstraction into account, for initial assessment, even if the aquifer is not in a "virgin" state.

5.3.   ABSTRACTION

Abstractions are limited to maximum rates and volumes by means of the resource consent process, but this information is not directly useful for management. There is an increasing trend to requiring significant water users to meter their abstractions. For the purpose of managing groundwater for sustainable use, abstraction data recorded as monthly volumes would usually be adequate.

6. Adaptive Management

6.1. WHAT IS ADAPTIVE MANAGEMENT?

The companion report to the present one (Lowry, 2002[1]) provides a comprehensive description of the nature of adaptive management in the context of managing groundwater resources in New Zealand. Material is quoted from that report, in this section, as a means of describing the role of analytical methods that support management.

> *... adaptive management develops management polices as experiments that test the responses of ecosystems to changes in people's behaviour.*

> *... shall be thought of as managing the people who interact with the ecosystem, not management of the ecosystem itself.*

> *Adaptive management is a process of 'learning while doing'.*

The emphasis is on cooperative management by stakeholders who need to understand the reasoning behind the possible range of outcomes. The role of models is seen as expressing the collective understanding of the participants about how the groundwater system operates, assessing the uncertainties, and predicting the effects of various management actions.

6.2. ANALYTICAL TOOLS FOR ADAPTIVE MANAGEMENT

The analytical tools that support adaptive management require at least the following characteristics:

- Conceptual plausibility to the stakeholders, which means that the physical basis and assumptions can be clearly presented
- Ability for implementation in a "real-time" mode consistent with the time scale of adaptive decision-making
- Suitable for use with the available data.

It is unlikely that any one analytical method would meet all these requirements in all situations, and therefore it is desirable that appropriate methods are, conceptually, upwardly compatible in terms of data availability and scales of time and space. In practical terms this means that method concepts can be presented to stakeholders as being consistent and appropriate views of physical reality.

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[1] Lowry, T, Bright, J (2002): Groundwater Management Tools: Draft Guideline for Groundwater Allocation Management. Lincoln Environmental Report No 4563/1

Section B – The Eigenmodel Method

1. Eigenmodel – a New Name

As a clear and convenient label for the approach presented in this report, we have coined the name "eigenmodel". The term "eigen-" is a German word that is long established in mathematical theory, for which it has the meaning of "characteristic". The eigenmodel approach is derived from the partial differential equations of groundwater flow by means of mathematical concepts called eigenvalues and eigenvectors (e.g. Sahuquillo, 1983)[2], but has not previously had a specific name. Sloan (2000)[3] applied the theory to groundwater discharge at catchment scale.

2. Assumptions

2.1.   THE MANAGEMENT OBJECTIVE

The amount of groundwater stored in an aquifer at any instant of time depends on the dynamic relationship between recharge inputs, through the overlying land surface and from rivers, and outflow to surface waters and pumped abstraction. Aquifer storage provides a buffer between highly variable, climatically driven recharge processes and the less variable outflow that supports surface water ecology. Abstraction of groundwater for human use, and some kinds of land use changes, alters the dynamic balance between "natural" recharge and the state of surface waters. The resource management objective is to determine the regime of abstraction that results in acceptable environmental effects.

2.2.   CAUSES OF ENVIRONMENTAL EFFECTS

The eigenmodel method for aquifer management depends on three assumptions:
- Recharge through the land surface overlying an aquifer can be estimated from water balance models based on climatic data and land-use parameters
- Most of the temporal variation in piezometric levels throughout an aquifer is caused by temporal variations in land surface recharge, together with the effects of pumped abstraction
- Environmental effects, such as the low-flow regime of streams are usually related to piezometric levels in the aquifer.

In summary, if the dynamic response of an aquifer to land surface recharge is quantified then environmental effects can be related to abstractive demand on the aquifer. Each of the above factors and the concepts of dynamic response will be explained in more detail.

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[2]   Sahuquillo, A (1983): An eigenvalue numerical method for solving unsteady linear groundwater models continuously in time. *Water Resources Research* **19:** 87-93.
[3]   Sloan, WT (2000): A physics-based function for modeling transient groundwater discharge at the watershed scale. *Water Resources Research* **36(1)**:225-241.

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3. Dynamic Behaviour of Aquifers

3.1.   LINEAR STORAGE – THE BUILDING BLOCK OF DYNAMIC RESPONSE

The concept of a linear water storage element is illustrated in Figure 1. We will examine the dynamic behaviour of this simple element in some detail, because complex linear systems are analysed as assemblies of this basic unit. Subsequently, we will show that the dynamic response of an aquifer to recharge can be analysed as a complex linear system.
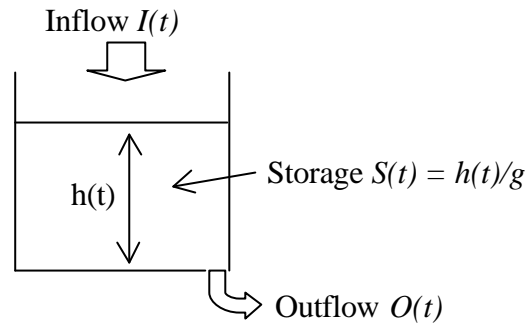
Figure 1: The linear water storage element

The water balance of the linear storage element (Figure 1) can be written as a differential equation:

$$\frac{dS(t)}{dt} = I(t) - O(t) \tag{1}$$

Since this is a linear water storage, the outflow $O(t)$ is proportional to the amount of stored water $S(t)$. This relationship can be expressed with a proportionality constant k as:

$$O(t) = kS(t) \tag{2}$$

By substituting equation (2) into equation (1):

$$\frac{dS(t)}{dt} + kS(t) = I(t) \tag{3}$$

The transition from equation (1) to equation (3) is quite significant. Equation (1) is a mathematical statement of the water budget without any other knowledge about processes, whereas equation (3) incorporates knowledge about the process expressed as the dynamic parameter k. Equation (3) no longer involves the outflow $O(t)$. A further step is to incorporate the relationship between water depth $h(t)$ and storage $S(t)$ in the form:

$$h(t) = g\,S(t) \tag{4}$$

The constant g, called the **gain**, could incorporate the cross-sectional area of the storage and other properties such as porosity of the storage medium, for example. By combining equations (3) and (4):

$$\frac{d h(t)}{dt} + k h(t) = g I(t) \tag{5}$$

Equation (5) provides the relationship between temporal variations in water depth $h(t)$ and temporal variations in water inflow $I(t)$. Equations of this kind are the reason that groundwater management can be conducted without knowing all the recharge and outflow quantities, because these are substituted for by knowledge of dynamic behaviour. Now we examine some of the practical implications of equation (5).

3.1.1. Computational Equations

In order to use equation (5) in a spreadsheet calculation with time-series data, a solution of this differential equation is required for time intervals of $?t$. For the case where the input $I(t)$ is averaged over the time interval from $t - ?t$ to t and the output $h(t)$ is the instantaneous value at the end of the interval, the solution is:

$$h_n = a h_{n-1} + b I_n \tag{6}$$

for which:

$$
\begin{aligned}
a &= \exp(-k\Delta t) \\
b &= g\left[1 - \exp\left(-k\Delta t\right)\right] \\
n &= \frac{t}{\Delta t}
\end{aligned}
\tag{7}
$$

Equation (6) is a difference equation suitable for use in a spreadsheet with data observed at discrete time intervals. The coefficients a and b are related to the process properties k and g by the relationships (7).

3.1.2. Steady-state Conditions

If the linear water storage receives a steady input of I_s, then $h_n = h_{n-1} = h_s$. If these values are substituted onto equation (6):

$$
\begin{aligned}
h_s &= a h_s + b I_s \\
h_s &= \frac{b}{(1-a)} I_s
\end{aligned}
\tag{8}
$$

The ratio $b/(1-a)$ is called the **steady-state gain** (ssg). This concept will be used in the eigenmodel assessment of groundwater resources.

3.1.3. Storage Residence Time and Eigenvalue

The residence time of a water storage is defined as the ratio of storage volume to mean flow. In the case of a single linear storage, the outflow is always related to storage by the coefficient k, as shown by equation (2). Therefore the **residence time** T_R of a single linear storage is $1/k$.

If there is no input to the storage, then the initial contents will be reduced to half in a time equal to $0.69T_R$.

For this single linear storage, considered as a dynamic system, the **eigenvalue** is equal to k and has the dimensions of 1/time.

3.1.4. Combinations of Linear Storages

Single linear storages, each with different values of k and g, can be interconnected into networks to form complex linear systems. The mathematical techniques of eigenvalue analysis can be used to convert any network into a parallel set of linear elements. The resulting eigenvalues provide the k values of the elements. This is the approach that we use to analyse the dynamic behaviour of aquifers by means of eigenmodels.

3.2. THE AQUIFER AS A COMPLEX LINEAR SYSTEM

Our approach to assessment and management of groundwater is based on a two-dimensional concept of aquifers, in which groundwater flow is essentially horizontal. Most applications of numerical models, such as MODFLOW, are for problems of this kind. In these numerical models, the horizontal extent of the aquifer is divided up into (usually) rectangular cells. Sets of equations are generated which specify the relationship between the piezometric heads on the cell corners, groundwater flow through the cell, abstraction and recharge in each cell, and aquifer properties of transmissivity and storativity.

These sets of equations that are the core of a numerical model are actually the mathematical description of a complex linear system, given the assumptions inherent in the 2-D concept of an aquifer. It is possible to convert these sets of equations into the mathematically equivalent eigenvalue-eigenvector form but there may be little computational advantage. The reason for this is mainly due to the ability of the numerical model to simulate localised variations in recharge and abstraction, which is the strength of these models. However, this modelling capacity demands appropriate levels of data. For many groundwater resource assessments these data are not available, and application of the numerical modelling packages may be inappropriate. If we are prepared to sacrifice some of this modelling flexibility then the eigenvalue-eigenvector analysis can yield models that are simple, have lower data requirements, and are suitable for implementation in spreadsheets. These are our eigenmodels.

4. Eigenmodels

The conference paper in Appendix I, about the eigenvalue approach to modelling aquifers, provides the theoretical basis for this section.

4.1. ASSUMPTIONS

Figure 2 illustrates the concepts and mathematical symbols used in the following discussion.
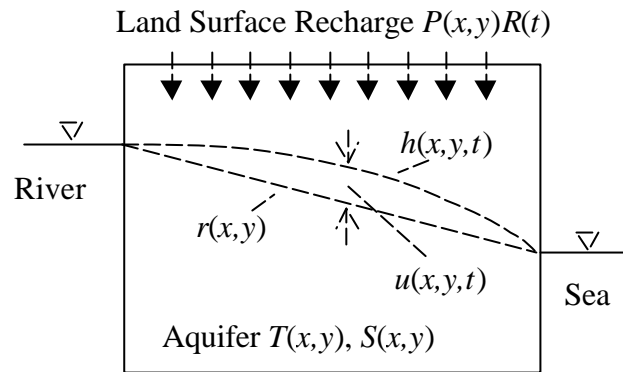
Figure 2:   **Variables and parameters for eigenmodel assumptions**

4.1.1. Dynamics of Recharge Processes

There is a fundamental difference in the responses of piezometric levels in an aquifer to changes in river recharge and land surface recharge.

- Time-varying piezometric effects near a river, that are associated with changes in river recharge, are rapidly attenuated with distance from the river into the aquifer. The time lag for piezometric effect also increases with distance from the river.
- Changes in land surface recharge cause relatively rapid piezometric effects everywhere in the aquifer, and the magnitude of these effects increases with distance from the fixed-head boundaries of the aquifer.

  Therefore we assume that at any observation well, the piezometric effect of river recharge $r(x,y)$ is a constant value for that location and that temporal variations in piezometric head $u(x,y,t)$ are caused by the effects of land surface recharge and time-varying abstraction. The validity of this assumption improves with distance from river recharge sources.

4.1.2. Land Surface Recharge Pattern

Land surface recharge is assumed to have a fixed spatial distribution, but the overall magnitude varies with time. This is expressed mathematically by stating that the recharge $R_{LS}(x,y,t)$ at time t for location x, y is given by:

$$R_{LS}(x, y, t) = P(x, y)R(t) \tag{9}$$

where $P(x,y)$ is a spatial distribution pattern and $R(t)$ is a time-varying magnitude. This means that the output from one water balance model may be quite satisfactory for a region with

varying amounts of rainfall because the spatial variations are incorporated into the eigenmodel calibration, as will be shown later.

4.1.3. Aquifer Properties

There are no assumptions required about the variation of transmissivity $T(x,y)$ and storativity $S(x,y)$ throughout an aquifer. The geological structure may be quite heterogeneous. However, the assumption of 2-D groundwater flow means that piezometric levels in areas with a significant vertical component of groundwater flow may be less well estimated.

4.1.4. Dynamic Effect of the Vadose Zone

Land surface recharge must travel through the unsaturated region (vadose zone) between the land surface and the groundwater before it has any piezometric effect. Recharge into the top of the vadose zone displaces water from the bottom of the zone into the groundwater surface, by means of hydraulic wave propagation. This process introduces a time delay and some attenuation of the estimated land surface recharge signal. We include an additional linear storage in the eigenmodel to account for this effect. This storage element can also account for the dynamic effect of groundwater perching above an aquitard as it leaks into a semi-confined aquifer.

4.2.   MODEL STRUCTURE

An eigenmodel of the dynamic behaviour of an aquifer can be represented as a set of linear storages connected in parallel, as shown in Figure 3. The dynamic effect of the vadose zone is represented by a single linear storage in series with the parallel set.
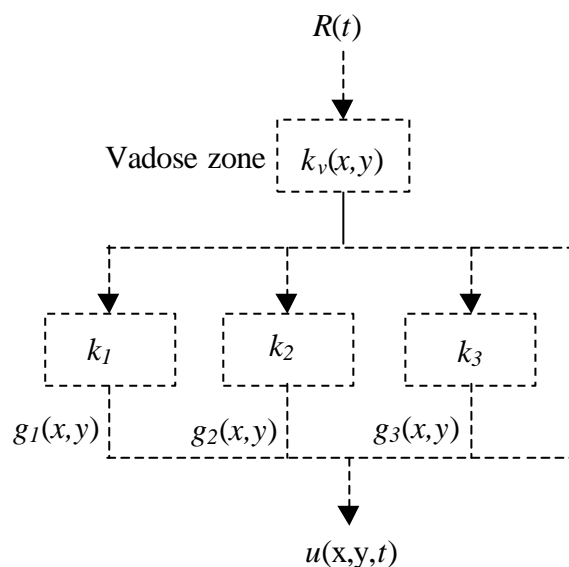
Figure 3:   Eigenmodel structure

4.2.1. Characteristics of an Eigenmodel

The eigenmodel structure shown in Figure 3 has some important characteristics that are relevant to practical applications:

1. The dynamic parameters k_1, k_2 ..., which are the eigenvalues of this linear system, are the same at all locations in an aquifer. This property enables transfer of these parameter values from locations with good data to other locations with poorer data.

2. Although the number of parallel elements is theoretically large, in practice only a few elements are required. Our experience is that for data at monthly time intervals only three need be considered.

3. The smallest eigenvalue k_1 has the most influence on long-term dynamic behaviour, and the other, larger, eigenvalues account for the more rapid response of piezometric head to changes in recharge.

4. The larger eigenvalues are more likely to be significant near fixed-head boundaries, because they can account for the rapid dynamics of the shorter drainage paths in these regions.

5. The vadose zone dynamic parameter k_v does vary with location, but its effect is usually less significant than the smallest eigenvalue, so it can be neglected (set to an arbitrary large value) during the initial stage of model calibration.

6. The gain coefficients $g_i(x,y)$ do depend on location.

4.3.   EQUATIONS FOR SPREADSHEETS

4.3.1. Prediction Equation

Differential equations, similar to equation (1), can be written for each of the components of the eigenmodel structure shown in Figure 3 but these would not be suitable for direct solution in a spreadsheet. We need a difference equation, similar to equation (6), which can be used with discrete time-series data such as monthly totals of recharge and monthly values of groundwater levels. Equation (6) is a first-order difference equation, because it has only one "a" coefficient. The four-component structure of Figure 3 will yield a fourth-order difference equation by use of a mathematical technique called the method of transforms (in this case, z-transforms). The development of the equation is shown in Appendix II, and the resulting form is:

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + a_3 u_{n-3} + a_4 u_{n-4} + b_1 R_n + b_2 R_{n-1} + b_3 R_{n-2}$$
$$\hat{h}_n = u_n + r(x,y) \tag{10}$$

where h_n is the prediction of the observed piezometric head h_n.

4.3.2. Parameter Relationships

The parameters a_i and b_i of equation (10) do not have obvious physical meanings, because they are derived from the more physically relevant parameters k_i and $g_i(x,y)$ by the mathematical transformation processes. The parameters in equation (10) are also interdependent, which can be a problem for model calibration, and it is difficult to set meaningful initial trial values during calibration. Therefore, calibration is conducted on the z-transform model, which has parameters a_i and β_i, and these values are converted to the a_i and b_i of the difference equation by the following relationships (developed in Appendix II) in the spreadsheet:

$$a_1 = (a_1 + a_2 + a_3 + a_4)$$
$$a_2 = -(a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4)$$
$$a_3 = (a_1 a_2 a_3 + a_1 a_2 a_4 + a_2 a_3 a_4 + a_1 a_3 a_4)$$
$$a_4 = -a_1 a_2 a_3 a_4$$

(11)

$$b_1 = (1 - a_4)(b_1 + b_2 + b_3)$$
$$b_2 = -(1 - a_4)(b_1 a_2 + b_1 a_3 + b_2 a_1 + b_2 a_3 + b_3 a_1 + b_3 a_2)$$
$$b_3 = (1 - a_4)(b_1 a_2 a_3 + b_2 a_1 a_3 + b_3 a_1 a_2)$$

(12)

The calibrated values of a_i and β_i are also converted to the equivalent eigenmodel parameters k_i and $g_i(x,y)$ by the relationships:

$$k_1 = -\frac{\ln(a_1)}{\Delta t} \qquad k_2 = -\frac{\ln(a_2)}{\Delta t} \qquad k_3 = -\frac{\ln(a_3)}{\Delta t} \qquad k_v = -\frac{\ln(a_4)}{\Delta t}$$

(13)

$$g_1(x, y) = \frac{b_1}{(1 - a_1)} \qquad g_2(x, y) = \frac{b_2}{(1 - a_2)} \qquad g_3(x, y) = \frac{b_3}{(1 - a_3)}$$

(14)

Equations (11) to (14) are shown above only for the purpose of illustrating the analytical process, and are already embedded in the example spreadsheets. Figure 4 shows the results header from an example spreadsheet.

Eigenmodel for Well M35/1080

							Storage residence times			
Aquifer properties	Datum (amsl) 61.04	LSR effect 3.17	GWL base 44.03	ssq 0.187	R2 0.823	T1 20.52	T2 1.59	T3 N/A	Tv 0.62	
Eigenmodel parameters	k1 0.049	k2 0.629	k3 N/A	kv 1.605	g1 0.151	g2 0.036	g3 0.000			
Transform model	**Alpha4** 0.201	**Base** -17.01	**Alpha1** 0.952	**Beta1** 0.007	**Alpha2** 0.533	**Beta2** 0.017	**Alpha3** 0.000	**Beta3** 0.000	**Obj. fn.** 115.32	
Difference equation	a1 1.686	a2 -0.806	a3 0.102	a4 0.000	b1 0.019	b2 -0.016	b3 0.000			

Figure 4: **Results header from an eigenmodel spreadsheet example**

The parameter values for the transform model, in the shaded line, have been obtained by calibration and the other parameter values have been calculated by means of equations (11) to (14).

4.4. MODEL CALIBRATION

Eigenmodels expressed in spreadsheet form are calibrated by use of the *Solver* routine in the *Tools* menu of Microsoft Excel. There are no problems with having missing data in the time-series of observed piezometric head, other than some loss of information, because the objective function in the spreadsheet ignores blanks.

4.4.1. Objective Function

The *Tools* user-form prompts for a *Target Cell*, and this should be set to the cell *Obj. fn.* shown by dark shading in Figure 4, and the option *Min* selected. The objective function is the sum of squares of the model error. This error is called *noise N_n* and is defined as the difference between the observed and predicted values of piezometric levels:

$$N_n = h_n - \hat{h}_n \tag{15}$$

4.4.2. Parameter Constraints

In the *Subject to constraints* box of the user-form, the Transform Model parameters (shaded line) should be set as follows:
- *Alpha*-parameters: >=0, <=1
- *Beta*-parameters: >=0
- *Base*:   no constraint

4.4.3. Initial Parameter Values

All the Transform Model parameters should be set to have an initial value of zero. This convenient setting is one of the advantages of conducting calibration on the z-transform version of the eigenmodel.

4.4.4. Calibration Procedure

The difference equation parameters are arranged in a particular order in the spreadsheet to facilitate a stepwise calibration procedure that minimises ambiguity and instability of parameter values. The *Solver* tool should be applied successively to the following selections of parameters shown in the shaded line of Figure 4:
- Base
- Base, Alpha1, Beta1
- Base, Alpha1, Beta1, Alpha2, Beta2
- Base, Alpha1, Beta1, Alpha2, Beta2, Alpha3, Beta3
- The best of the above selections plus Alpha4

The above groupings are based on selecting, in a stepwise manner, parameters in order of their likely effect on variations in piezometric levels. The inclusion of *Alpha4* is to apply the vadose zone element to the best groundwater model.

5. Aquifer Characteristics and Eigenmodel Parameters

In order to use the parameter values of the eigenmodel to assist with assessment of a groundwater resource, it is important to appreciate the meaning of these parameters in terms of the physical characteristics of an aquifer. We will be concerned primarily with the eigenvalues k_i and the gain coefficients $g_i(x,y)$, shown in Figure 3. The values of these parameters are provided by the eigenmodel calibration procedure described in Section 4.4.

5.1. EIGENVALUES OF A SIMPLE AQUIFER

Although the eigenmodel theory can be applied to any aquifer, a useful insight is obtained by considering the eigenvalues of a rectangular-shaped aquifer with homogeneous properties of transmissivity T and storativity S. The horizontal dimensions are L_x, L_y, and there is a fixed-head boundary on all edges (such as a surface water boundary all around the aquifer). This simple case can be solved analytically, and the general formula for the eigenvalues is:

$$k_{ij} = \frac{\mathbf{p}^2 T}{S}\left(\frac{i^2}{L_x^2} + \frac{j^2}{L_y^2}\right)$$

$$i,j = 1,2,\ldots$$

$$(16)$$

We now examine some of the implications of equation (16) for groundwater assessment.

5.1.1. Significance of Eigenvalues

The relative magnitudes of the eigenvalues given by equation (16) depends on the squares of the integers 1,2,... This means that the second eigenvalue is at least four to five times the magnitude of the first, depending on the relative magnitudes of L_x and L_y. The storage residence time (Section 3.1.3) of an aquifer, which is the inverse of the eigenvalue, will therefore be dominated by the first eigenvalue. The larger eigenvalues can be important for more accurate simulation of the dynamics of piezometric response, but they do not contribute very significantly to water storage.

5.1.2. Effect of Horizontal Scale

Equation (16) states that the eigenvalues are inversely proportional to the square of the horizontal dimensions. This means that, for example, an aquifer 16 km by 10 km has four times the storage residence time (inverse of eigenvalue) of an aquifer 8 km by 5 km that has similar geological properties and boundary conditions.

5.1.3. Effect of Aquifer Boundaries

Equation (16) has been derived for an ideal simple aquifer that drains to all four boundaries (fixed head). If this aquifer is square, $L_x = L_y = L$, and the first eigenvalue is:

$$k_1 = 2\frac{\mathbf{p}^2 T}{SL^2}$$

$$(17)$$

If the aquifer drains only to two opposite boundaries, this is equivalent to the 1-D case where $L_x = L$ and L_y ? 8, in which case equation (16) gives:

$$k_1 = \frac{\boldsymbol{p}^2 T}{SL^2} \tag{18}$$

If the aquifer drains to only one boundary, it can be shown that:

$$k_1 = \frac{1}{4} \frac{\boldsymbol{p}^2 T}{SL^2} \tag{19}$$

Comparison of equations (17) to (19) shows that changing only the boundary conditions of this simple aquifer results in an eight-fold variation of storage residence time.

5.1.4. Effect of Geological Properties

The ratio T/S appears in each of the above equations for the eigenvalues of the homogeneous aquifer. Analysis of heterogeneous aquifers by eigenvalue methods has the potential to provide estimates of the large-scale value of this ratio, if aquifer geometry and boundaries are well defined. However, it is unlikely that this value would be the same as properties derived from pumping tests, because the up-scaling process is quite complex. It is worth noting that the T/S value appears only to the first power in the above equations, and the likely effect of geological properties needs to be kept in perspective with the effects of horizontal scale and aquifer boundaries.

5.2. RELEVANCE OF THE GAIN COEFFICIENTS

The gain coefficients $g_i(x,y)$ at each well location not only indicate the relative significance of the aquifer eigenvalues at that location, but can also provide information about aquifer boundaries and storativity. The total steady-state gain $ssg(x,y)$ at a location is given by:

$$ssg(x, y) = \sum_i g_i(x, y) \tag{20}$$

and can also be derived from the difference equation (10) as:

$$ssg(x, y) = \frac{b_1 + b_2 + b_3}{1 - a_1 - a_2 - a_3 - a_4} \tag{21}$$

The average piezometric effect of land surface recharge (*LSR effect* in the eigenmodel results), $h_a(x,y)$, is given by:

$$h_a(x, y) = ssg(x, y) R_a \tag{22}$$

where R_a is the long-term average of the recharge magnitude series $R(t)$.

5.2.1. Effect of Location and Boundaries

The value of $ssg(x,y)$ is zero at fixed-head boundaries, and increases with distance from those boundaries. This property can be used, for example, to indicate whether a river is interacting directly with an aquifer or is perched above the groundwater surface. In the latter case, the value of $h_a(x,y)$ will be relatively large.

5.2.2. Estimating Aquifer Storativity

If the availability of data allows, then a plot of $h_a(x,y)$ would permit estimation of the average thickness, H_a, of the aquifer that is occupied by the average amount of water originating from land surface recharge. This average amount of water can be reasonably estimated by means of the average recharge R_a and the average storage residence time $1/k_1$. Then the large-scale aquifer storativity S is given by:

$$S = \frac{R_a}{k_1 H_a} \tag{23}$$

In cases where there are few observation well records, there is an approximation that may be useful. This relies on the assumption that the shape of the groundwater surface under steady-state conditions is a paraboloid, which degenerates to a parabola for 1-D groundwater flow. The average height of a paraboloid is 4/9 the maximum height. The corresponding ratio for a parabola is 2/3. If the value of $h_a(x,y)$ is known for a well that is furthest from a fixed-head boundary then, as a first approximation, the average thickness H_a is half this "maximum" value.

5.3. THE "BASE" PARAMETER AND RIVER RECHARGE

The *Base* parameter value in the eigenmodel results (Figure 4) is the piezometric effect of river recharge $r(x,y)$ at the particular well location, which is assumed to be steady. If there are sufficient locations with piezometric records, these base values can provide a map of the river recharge effect as an additional set of information about the aquifer. This piezometric surface could be used to estimate the quantity of river recharge if values of transmissivity are available. However, it would be simpler to express river recharge as a proportion of land surface recharge, which is more accurately calculated, by means of the relative piezometric gradients, because the transmissivity in both cases is the same.

6. Case Histories

In each of the following case studies, the only sources of data are:
- Daily values of land surface recharge estimated from a water balance model. These are summed to monthly totals for use with monthly groundwater level data.
- Records of groundwater (piezometric) level from a few observation wells, either at daily or monthly time intervals.

Reprints of portions of topographical maps are shown, to illustrate the location of observation wells and the scale of the aquifer.

Details of the water balance model and geological information about the aquifers are not presented here.

6.1. MOTUEKA AQUIFER

6.1.1. Preliminary Assessment with Limited Data

Figure 6 shows the results of an eigenmodel calibration with one year of daily data for Rossiters Well (Figure 5). This is a good set of data because there is a significant land surface recharge event that has caused a strong piezometric response in the aquifer.
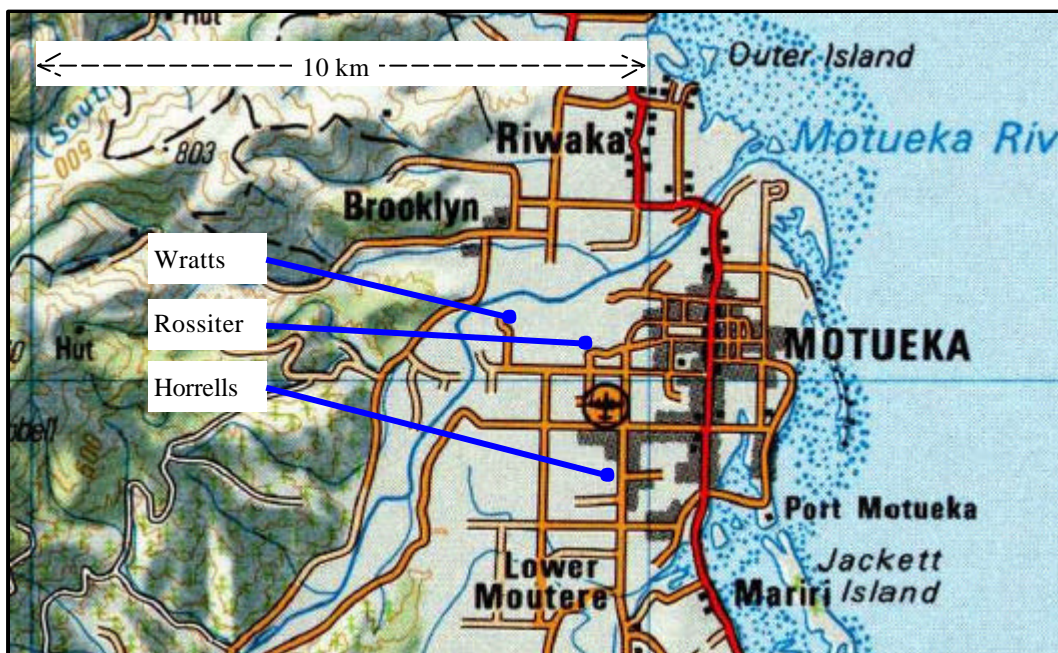
Figure 5:   **Location of observation wells in the Motueka Aquifer**

The results (Figure 6) show that only one eigenvalue was required to provide a good fit (R^2 = 0.91) to the data. The important result is that the storage residence time (T_1) is only 22 days. This means that the contribution of land surface recharge is unlikely to be adequate because half of this storage is lost by natural drainage in about 15 days (0.69×22). Therefore, this aquifer appears to be primarily a transport medium for river recharge. Now we examine the additional information provided by the use of five years of daily data for the three observation wells shown in Figure 5.

6.1.2. Assessment with Additional Data

The eigenmodel results and simulation plots for Wratts, Rossiters and Horrells observation wells (Figure 5) are shown in Figures 7 to 9. The relevant parameter values from these results are summarised in Table 1.

Table 1:   Summary of eigenmodel results for the Motueka Aquifer

Observation well	Wratts	Rossiters	Horrells
Eigenvalues (d^{-1})	0.260, 0.010	0.039	0.031
Residence time (d)	99	25	32
LSR effect (mm)	373	368	510
GWL base amsl (mm)	8697	3212	1576
Eigenmodel fit R^2	0.53	0.75	0.85

At Rossiters and Horrells, there is only one significant eigenvalue and it is similar for both locations, as are the storage residence times (inverse of the smallest eigenvalue). This similarity is expected from aquifer dynamics (Section 5.2.1). These values are also comparable to the eigenvalue of 0.045/d (residence time of 22 days) determined from the short data record (Section 6.1.1).

The well at Wratts exhibits an apparently anomalous value for the smallest eigenvalue, of 0.01/d. This could be caused by a piezometric response to the adjacent Motueka River, which reflects the storage characteristics of the river catchment rather than the aquifer. This signal is likely to be rapidly damped with distance from the river and therefore not detected at the other wells. The larger eigenvalue at this well (0.260/d) is an order of magnitude greater than the first eigenvalue of any of the wells, which is typical (Section 5.1.1).

The average piezometric effect of land surface recharge (LSR effect) is similar for Wratts and Rossiters (373 mm, 368 mm). This indicates that the Motueka River is not a fixed-head boundary near Wratts, otherwise the LSR effect would be relatively smaller.

Horrells Well has the largest value of LSR effect (510 mm), and this suggests that it is further from fixed-head outflow boundaries. This may suggest the presence of a no-flow boundary to the south of this well.

The GWL base and the LSR effect at Horrells can be compared with the bed level and stage of the Motueka River as part of any investigation of the nature of the connection between river and aquifer. This is not included in the present case study.

Comparison of the average piezometric heads (GWL base + LSR effect) at the three wells indicates groundwater flow to the southeast quarter.

The eigenmodel fit (R^2) is poorest at Wratts (0.53) because of the influence of the Motueka River, and because the eigenmodel treats river effects as a constant. The fit improves with distance from the river at Rossiters (0.75) and Horrells (0.85). The primary source of model error appears to be the low piezometric head during summer drought, when abstractions are high. These periods are indicated (Figures 7 to 9) by piezometric levels (GWL) below the GWL base values.

The conclusions from this eigenmodel assessment of the Motueka aquifer are as follows:

- There is insufficient aquifer storage to carry-over land surface recharge from winter months through the dry spring-summer period;
- The aquifer is a transport medium for river recharge, which is the primary water resource;
- The connection between the Motueka River and the aquifer is probably not directly interactive, but the nature of this connection is very important for managing abstraction from the aquifer, in terms of the magnitude of the resource.

Rossiters

Eigenmodel for Rossiters Well

						Storage residence times			
Aquifer properties	Datum (amsl)	LSR effect	GWL base	ssg	R2	T1	T2	T3	Tv
	0.00	314.78	3333.87	244.317	0.914	22.41	N/A	N/A	2.47
Eigenmodel parameters	k1	k2	k3	kv	q1	q2	q3		
	0.045	N/A	N/A	0.405	244.317	0.000	0.000		
Transform model	Alpha4	Base	Alpha1	Beta1	Alpha2	Beta2	Alpha3	Beta3	Obj. fn.
	0.667	3333.87	0.956	10.661	0.000	0.000	0.000	0.000	9798705.17
Difference equation	a1	a2	a3	a4	b1	b2	b3		
	1.623	-0.638	0.000	0.000	3.551	0.000	0.000		

Figure 6: **Eigenmodel results for short data record at Rossiters Well, Motueka**
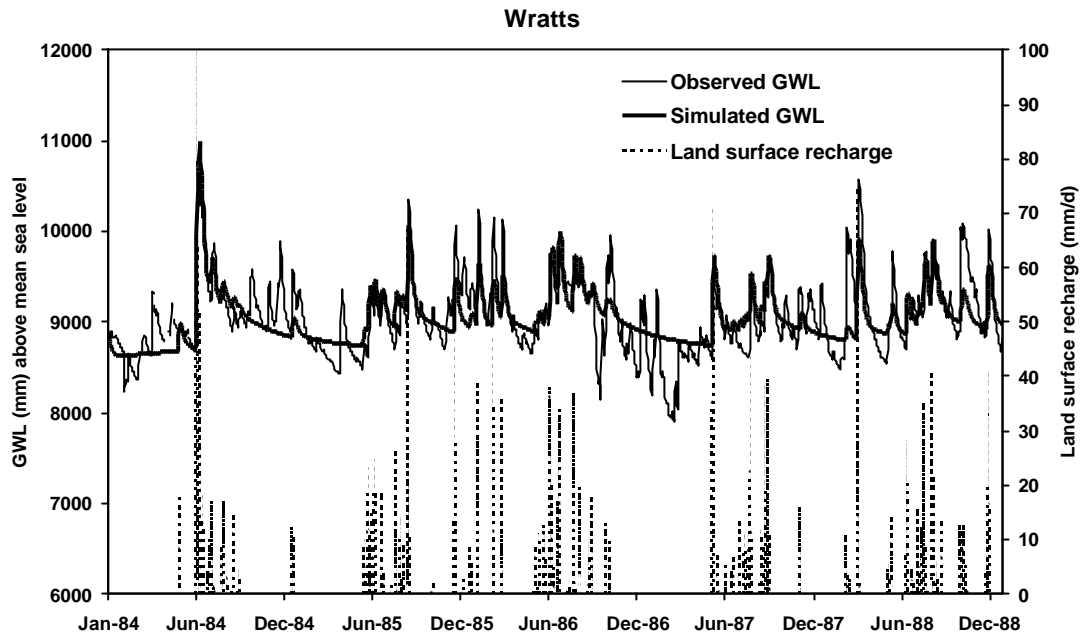
Wratts

Eigenmodel for Wratts Well

							Storage residence times		
Aquifer properties	Datum (amsl)	LSR effect	GWL base	ssq	R2	T1	T2	T3	Tv
	0.00	372.80	8697.38	289.356	0.530	3.83	98.80	N/A	3.83
Eigenmodel parameters	k1	k2	k3	kv	q1	q2	q3		
	0.261	0.010	N/A	0.261	100.658	188.697	0.000		
Transform model	**Alpha4**	**Base**	**Alpha1**	**Beta1**	**Alpha2**	**Beta2**	**Alpha3**	**Beta3**	**Obj. fn.**
	0.770	8697.38	0.770	23.125	0.990	1.900	0.000	0.000	144618188.94
Difference equation	a1	a2	a3	a4	b1	b2	b3		
	2.530	-2.118	0.587	0.000	5.751	-5.597	0.000		

Figure 7:   Eigenmodel results for Wratts Well, Motueka
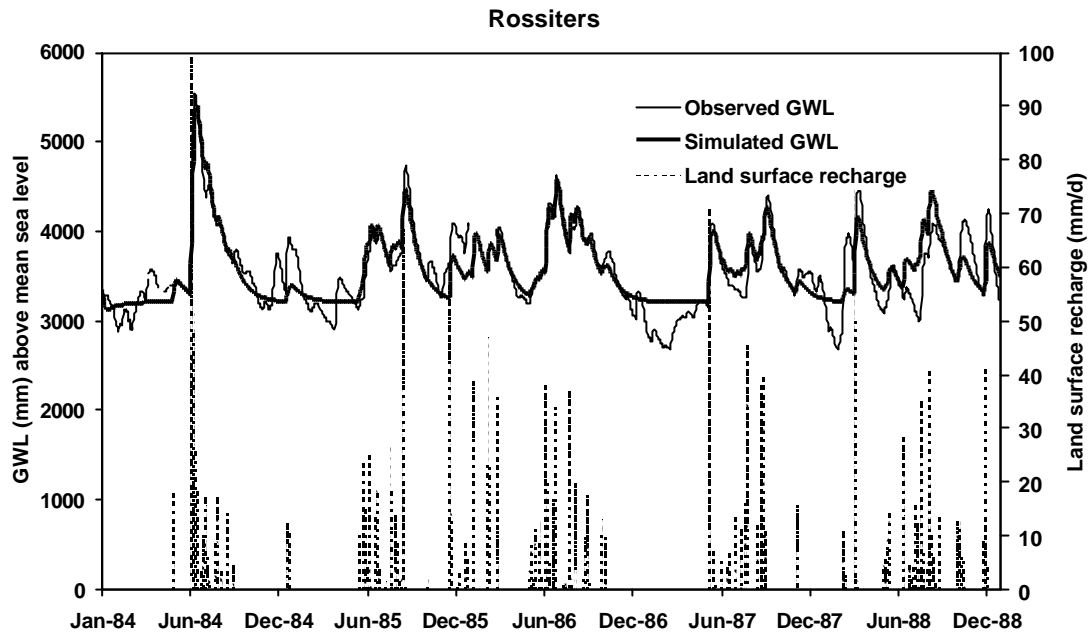
Rossiters

Eigenmodel for Rossiters Well

Aquifer properties	Datum (amsl)	LSR effect	GWL base	ssq	R2	T1	T2	T3	Tv
							Storage residence times		
	0.00	368.36	3211.95	285.909	0.752	25.45	0.05	N/A	2.88

Eigenmodel parameters	k1	k2	k3	kv	q1	q2	q3		
	0.039	19.460	N/A	0.347	285.909	0.000	0.000		

Transform model	Alpha4	Base	Alpha1	Beta1	Alpha2	Beta2	Alpha3	Beta3	Obj. fn.
	0.707	3211.95	0.961	11.017	0.000	0.000	0.000	0.000	84724126.30

Difference equation	a1	a2	a3	a4	b1	b2	b3		
	1.668	-0.679	0.000	0.000	3.232	0.000	0.000		

Figure 8: **Eigenmodel results for Rossiters Well, Motueka**
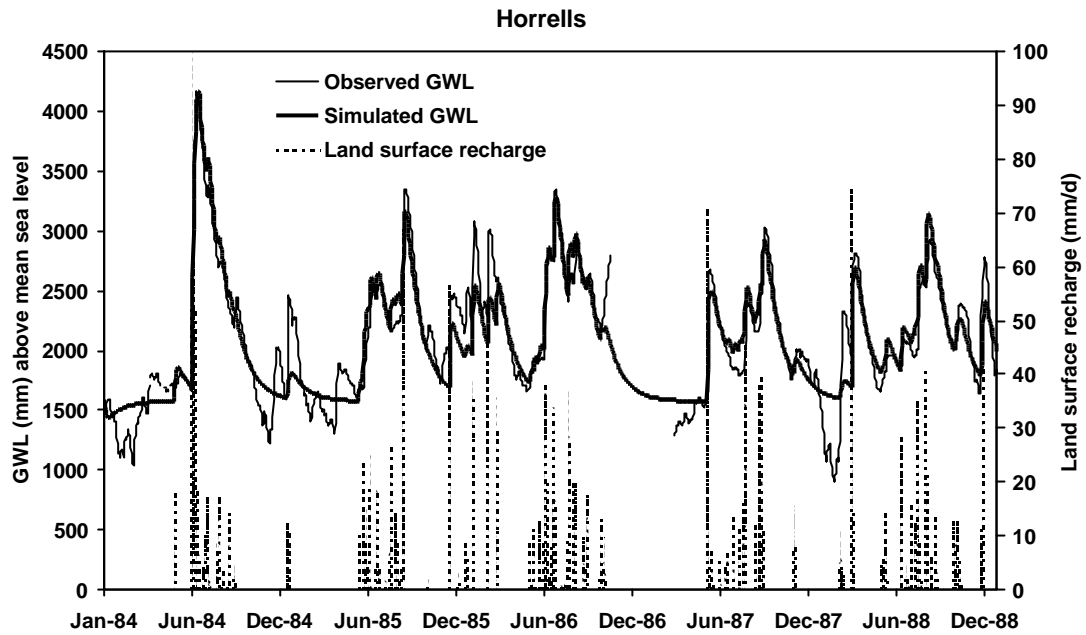
Horrells

Eigenmodel for Horrells Well

Aquifer properties	Datum (amsl)	LSR effect	GWL base	ssq	R2	T1	Storage residence times T2	T3	Tv
	0.00	509.74	1576.18	395.637	0.849	31.97	N/A	N/A	2.68

Eigenmodel parameters	k1	k2	k3	kv	g1	g2	g3		
	0.031	N/A	N/A	0.373	395.637	0.000	0.000		

Transform model	Alpha4	Base	Alpha1	Beta1	Alpha2	Beta2	Alpha3	Beta3	Obj. fn.
	0.689	1576.18	0.969	12.184	0.000	0.000	0.000	0.000	72213109.91

Difference equation	a1	a2	a3	a4	b1	b2	b3		
	1.658	-0.667	0.000	0.000	3.793	0.000	0.000		

Figure 9:   Eigenmodel results for Horrells Well, Motueka

6.2. CENTRAL PLAINS AQUIFER, CANTERBURY

Figure 10 shows the location of four observation wells in the Central Canterbury Plains, which will be the basis of the next case study. The Central Plains aquifer is much larger than the Motueka aquifer, and the dynamic behaviour can be adequately described by the use of monthly values of piezometric head. The land surface recharge is still estimated from a daily water balance model but the results are summed to provide monthly totals. The piezometric data are in units of metres (rather than millimetres as for the Motueka data).
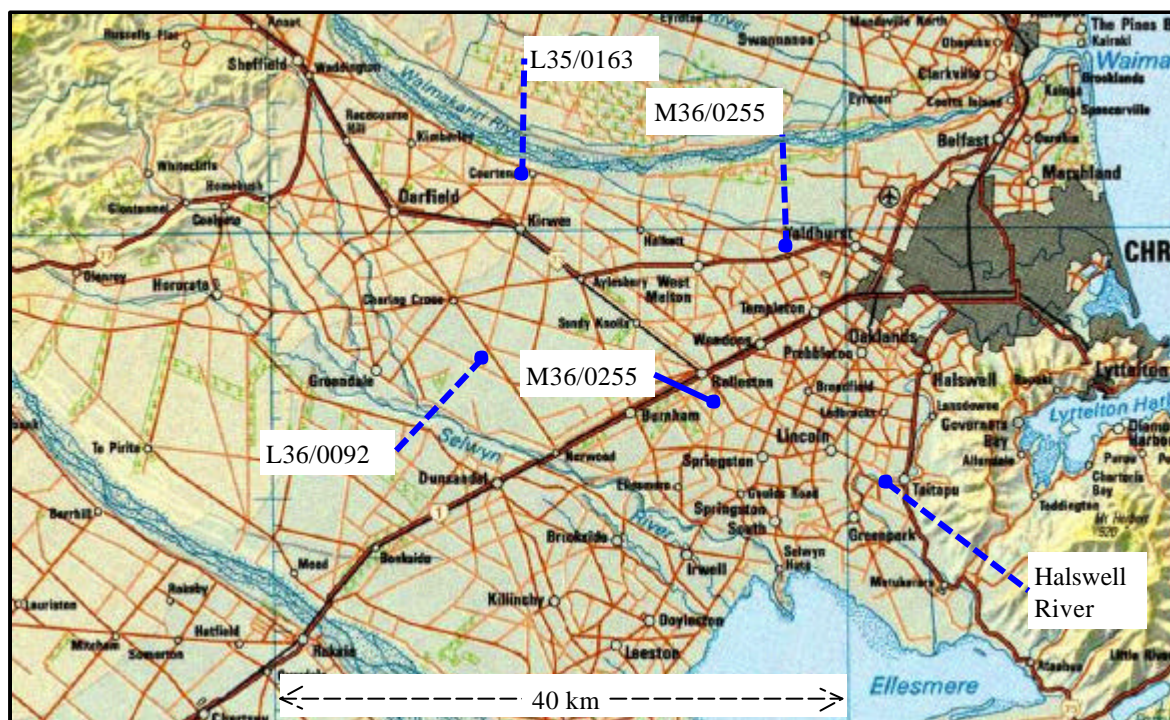
Figure 10: Location of observation wells and a river gauging station in the Central Canterbury Plains

Figure 10 also shows the location of a flow gauging station on the Halswell River. The relationship between streamflow and piezometric levels in the aquifer will be examined in Section 7.4.

6.2.1. Preliminary Assessment with Individual Short Records

Figures 11 to 14 show the simulation results and parameter values for the four wells (Figure 10), based on five years of monthly data for 1995-1999. This is quite a short data record for an aquifer of this size, given the climatic variability of the Canterbury Plains. The results are summarised in Table 2.

Table 2: Eigenmodel results for 5 years' data, Central Canterbury Plains, for individual calibration at each location

Observation well	L35/0163	L36/0092	M35/1080	M36/0255
Eigenvalues (mth^{-1})	N/A	0.011	0.189	0.043, 0.339
Residence time (mth)	N/A	95	5.3	23
LSR effect (m)	N/A	62.3	2.0	3.6
GWL base amsl (m)	N/A	0	44.9	30.0
Eigenmodel fit R^2	N/A	0.95	0.90	0.90

In general, these results are quite unsatisfactory. The following explanations include some additional physical knowledge of the aquifer:

- The eigenmodel could not be fitted to L35/0163 during this period of record because the variations in piezometric head were caused primarily by recharge events from the adjacent Waimakariri River. There is also a general decline in piezometric head which looks as though it is associated with decreasing land surface recharge but this signal has been swamped by the river recharge effect.
- The model fit (R^2 = 0.95) appears to be very good for L36/0092, but physical realism is poor. The GWL base of zero corresponds to sea level, and this was because this value is a constraint in the calibration procedure. The LSR effect of 62.3 m is questionably large. The physical explanation of this poor performance may be that this well is in a semi-confined layer and the response is damped by the storage characteristics of perched groundwater. This is usually simulated by the vadose zone storage element of the eigenmodel.
- The results for M35/1080 and M36/0255 are physically realistic, but the storage residence times are very different. According to eigenmodel theory (Section 4.2.1), the dominant residence times should be similar.

6.2.2. Reassessment with Combined Short Records

The eigenmodels for the four observation wells were then optimised simultaneously, with incorporation of the requirement that the eigenvalues are the same at all locations. The revised results are shown in Table 3.

Table 3:   Eigenmodel results for 5 years' data, Central Canterbury Plains, for simultaneous calibration at all locations

Observation well	L35/0163	L36/0092	M35/1080	M36/0255
Eigenvalues (mth-1)	0.015, 0.197	0.015	0.197	0.197
Residence time (mth)	67	67	5.1	5.1
LSR effect (m)	11.1	26.1	2.4	2.0
GWL base amsl (m)	68.1	33.7	44.9	30.4
Eigenmodel fit R2	0.89	0.95	0.90	0.89

The effect of combining the data is that Well L35/0163 now has a result, because the eigenvalues have been forced to "sensible" values by the influence of knowledge gained from the other observation wells. The model fit has hardly changed for these wells, but the realism of the parameter values remains questionable. In the next section, the influence of a longer data record will become apparent.

L35/0163

Eigenmodel for Well L35/0163

						Storage residence times			
Aquifer properties	Datum (amsl) 170.24	LSR effect 0.00	GWL base 100.61	ssq 0.000	R2 0.003	T1 N/A	T2 N/A	T3 N/A	Tv N/A
Eigenmodel parameters	k1 N/A	k2 N/A	k3 N/A	kv N/A	g1 0.000	g2 0.000	g3 0.000		
Transform model	Alpha4 0.000	Base -69.63	Alpha1 0.000	Beta1 0.000	Alpha2 0.000	Beta2 0.000	Alpha3 0.000	Beta3 0.000	Obj. fn. 1599.35
Difference equation	a1 0.000	a2 0.000	a3 0.000	a4 0.000	b1 0.000	b2 0.000	b3 0.000		

Figure 11:   Eigenmodel results for the 5-year record at well L35/0163
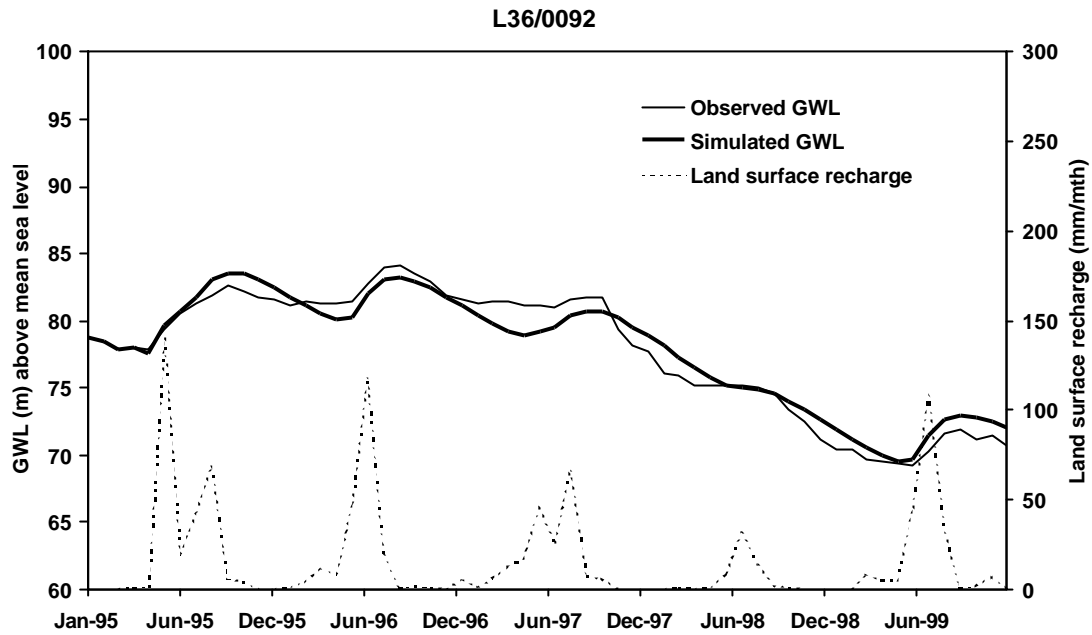
Eigenmodel for Well L36/0092

Aquifer properties	Datum (amsl)	LSR effect	GWL base	ssg	R2	Storage residence times T1	T2	T3	Tv
	119.09	62.29	0.00	3.882	0.950	95.06	0.03	N/A	1.68

Eigenmodel parameters	k1	k2	k3	kv	g1	g2	g3		
	0.011	29.720	N/A	0.595	3.882	0.000	0.000		

Transform model	Alpha4	Base	Alpha1	Beta1	Alpha2	Beta2	Alpha3	Beta3	Obj. fn.
	0.551	-119.09	0.990	0.041	0.000	0.000	0.000	0.000	65.73

Difference equation	a1	a2	a3	a4	b1	b2	b3		
	1.541	-0.546	0.000	0.000	0.018	0.000	0.000		

Figure 12: **Eigenmodel results for the 5-year record at well L36/0092**
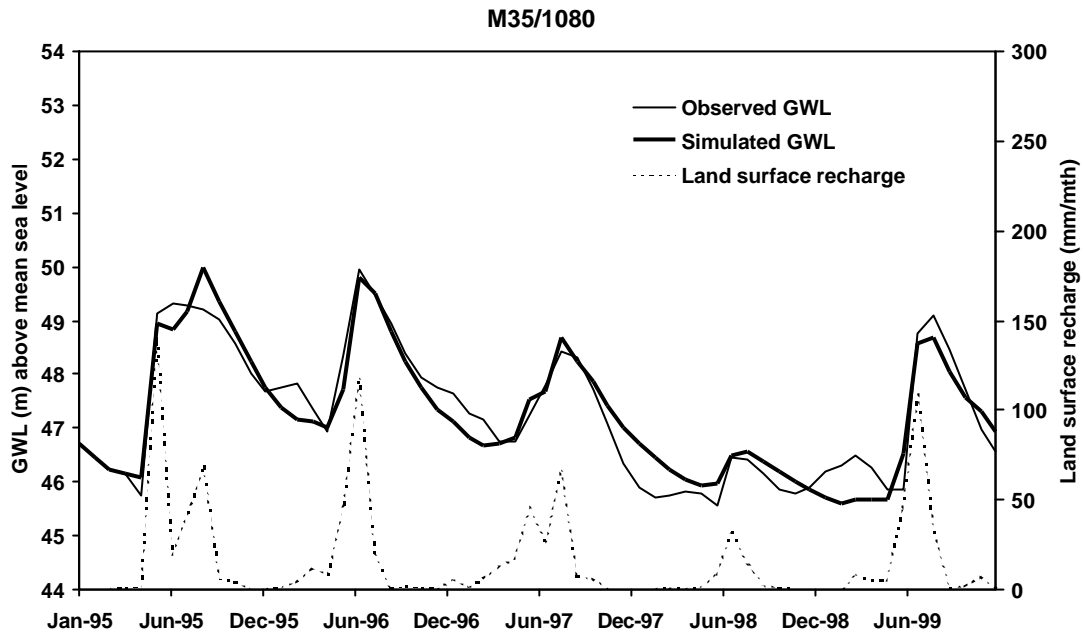
M35/1080

Eigenmodel for Well M35/1080

							Storage residence times		
Aquifer properties	Datum (amsl)	LSR effect	GWL base	ssq	R2	T1	T2	T3	Tv
	61.04	1.99	44.85	0.124	0.901	36.92	5.29	N/A	0.23
Eigenmodel parameters	k1	k2	k3	kv	g1	g2	g3		
	0.027	0.189	N/A	4.380	0.000	0.124	0.000		
Transform model	Alpha4	Base	Alpha1	Beta1	Alpha2	Beta2	Alpha3	Beta3	Obj. fn.
	0.013	-16.19	0.973	0.000	0.828	0.021	0.000	0.000	8.75
Difference equation	a1	a2	a3	a4	b1	b2	b3		
	1.813	-0.828	0.010	0.000	0.021	-0.021	0.000		

Figure 13:   Eigenmodel results for the 5-year record at well M35/1080

M36/0255

Eigenmodel for Well M36/0255
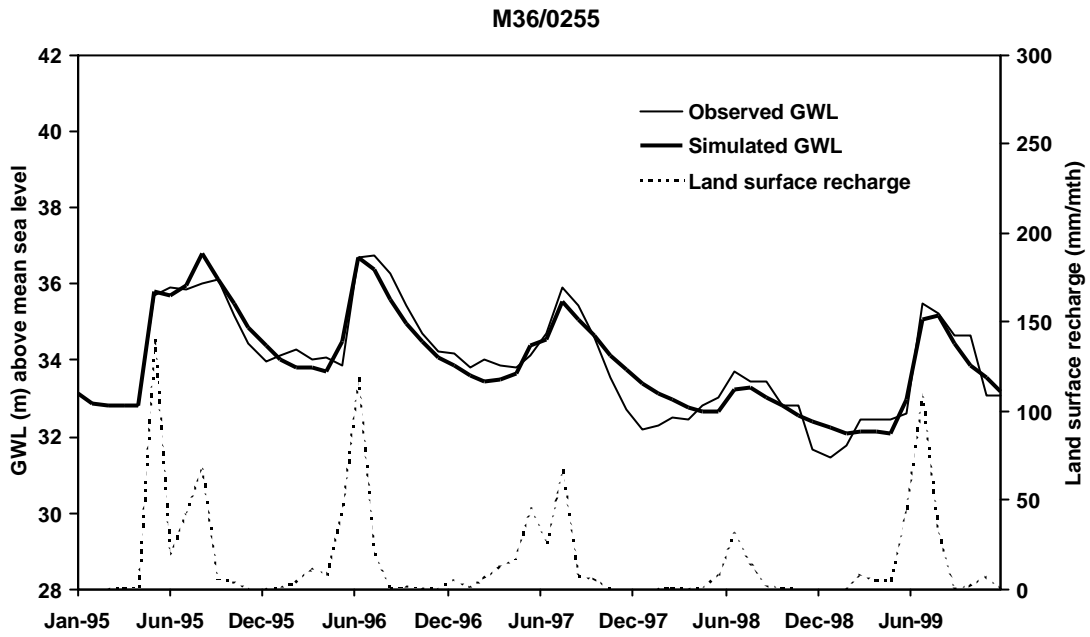
							Storage residence times		
Aquifer properties	Datum (am	LSR effect	GWL base	ssg	R2	T1	T2	T3	Tv
	43.68	3.56	29.97	0.222	0.896	23.38	2.95	N/A	0.38
Eigenmodel parameters	k1	k2	k3	kv	g1	g2	g3		
	0.043	0.339	N/A	2.604	0.163	0.059	0.000		
Transform model	**Alpha4**	**Base**	**Alpha1**	**Beta1**	**Alpha2**	**Beta2**	**Alpha3**	**Beta3**	**Obj. fn.**
	0.074	-13.71	0.958	0.007	0.713	0.017	0.000	0.000	11.03
Difference equation	a1	a2	a3	a4	b1	b2	b3		
	1.745	-0.806	0.051	0.000	0.022	-0.020	0.000		

Figure 14:   Eigenmodel results for the 5-year record at well M36/0255

6.2.3. Assessment with Longer Time Series

Table 4 shows the eigenmodel results for the four wells (Figure 10) from up to 28 years of monthly data (June 1972 to May 2000). The corresponding simulation plots and detailed results are shown in Figures 15-18.

Table 4: Eigenmodel results for 28 years' data, Central Canterbury Plains, for individual calibration at each location

Observation well	L35/0163	L36/0092	M35/1080	M36/0255
Eigenvalues (mth^{-1})	0.053	0.048	0.049, 0.629	0.052, 1.649
Residence time (mth)	19.0	20.8	20.5	19.1
LSR effect (m)	20.8	19.1	3.2	4.9
GWL base amsl (m)	80.1	59.5	44.0	28.9
Vadose storage (mth)	2.7	4.4	0.6	0.6
Eigenmodel fit R^2	0.87	0.91	0.82	0.85

These results are much more satisfactory than those reported in Section 6.2.1 for the 5-year record. Discussion of particular features follows:

- The first eigenvalues are all similar, as are the corresponding storage residence times. This residence time of about 20 months suggests that the aquifer has significant inter-seasonal storage, and therefore land surface recharge is an important water resource.
- The values of GWL base correspond to a gradient of about 0.002 up the Plains from the lower fixed-head boundary at Lake Ellesmere. This suggests that river recharge is also a significant contribution to groundwater.
- The values of LSR effect correspond to the distance of wells from surface water boundaries, except for L35/0163. This well has the highest LSR effect (20.8 m), which suggests that the aquifer does not directly interact with the Waimakariri River at this location. In other words, the river is perched above the aquifer.
- The vadose residence time is much larger for L35/0163 and L36/0092, which is probably due to the slow drainage of perched groundwater into the semi-confined aquifers tapped by theses two wells.
- A second eigenvalue is significant only for M35/1080 and M36/0255. This is because these two wells are closer to fixed head, surface-water boundaries, for which there is a component of more-rapid transient drainage.

The simulation plot for L35/0163 (

-

Figure 15) shows several peaks in the piezometric record, which are not simulated by the eigenmodel. Examination of the dates of these events suggests that the peaks are caused by recharge during floods in the Waimakariri River, over and above the more steady component of river recharge. A similar departure, caused by a major flood in the Selwyn River, can be seen in the 1986-1988 period of the record for L36/0092 (Figure 16).

In general, the improved model results for the longer records are due to including the earlier years when land surface recharge was more significant than during the recent years of relative drought.

6.2.4. Assessment with Combined Longer Series

The eigenmodels for the four observation wells were then optimised simultaneously, with incorporation of the requirement that the eigenvalues are the same at all locations. The revised results are shown in Table 5.

Table 5:   Eigenmodel results for 28 years' data, Central Canterbury Plains, for simultaneous calibration at all locations

Observation well	L35/0163	L36/0092	M35/1080	M36/0255
Eigenvalues (mth^{-1})	0.949	0.949, 0.285	0.949, 0.285	0.949, 0.285
Residence time (mth)	19.3	19.3	19.3	19.3
LSR effect (m)	20.9	18.7	3.1	5.0
GWL base amsl (m)	80.0	59.9	44.1	28.8
Vadose storage (mth)	2.7	5.3	1.3	0.1
Eigenmodel fit R^2	0.87	0.91	0.82	0.85

Comparison of Table 4 with Table 5 shows that simultaneous calibration of the four longer well records has not made a significant difference to the model parameters. This result demonstrates the value of a long period of record for assessing the overall storage capability of a large aquifer. Increasing the number of observation well records has only a marginal effect on defining this particular property.
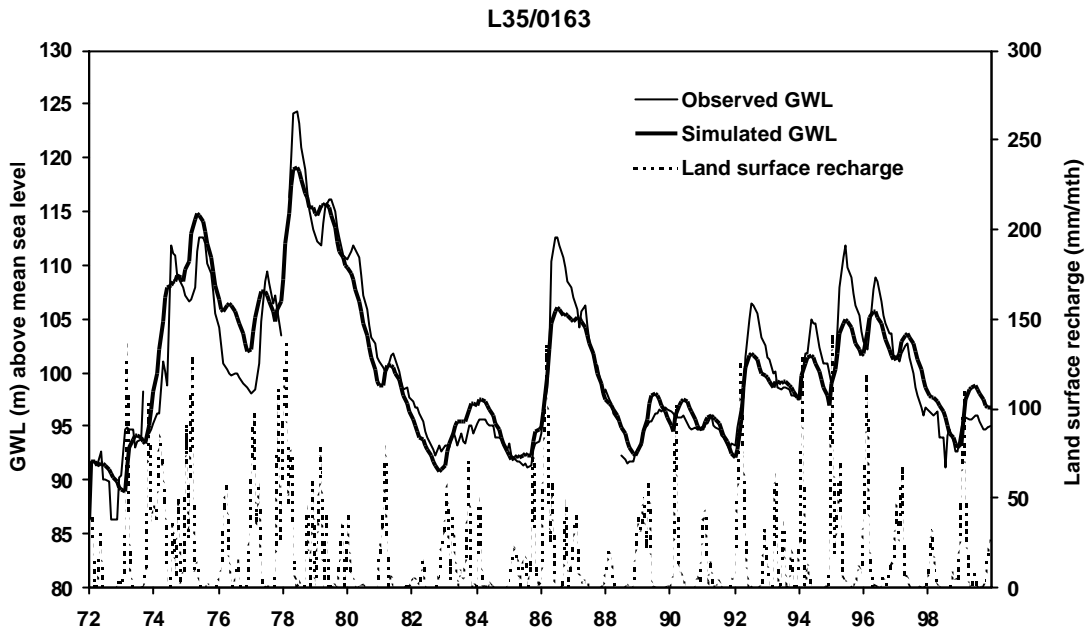
L35/0163

Eigenmodel for Well L35/0163

Aquifer properties	Datum (amsl)	LSR effect	GWL base	ssq	R2	T1	Storage residence times T2	T3	Tv
	170.24	20.78	80.09	1.225	0.873	19.01	N/A	N/A	2.68

Eigenmodel parameters	k1	k2	k3	kv	g1	g2	g3		
	0.053	N/A	N/A	0.373	1.225	0.000	0.000		

Transform model	Alpha4	Base	Alpha1	Beta1	Alpha2	Beta2	Alpha3	Beta3	Obj. fn.
	0.688	-90.15	0.949	0.063	0.000	0.000	0.000	0.000	2209.67

Difference equation	a1	a2	a3	a4	b1	b2	b3		
	1.637	-0.653	0.000	0.000	0.020	0.000	0.000		

Figure 15:   Eigenmodel results for the 28-year record at well L35/0163

L36/0092

Eigenmodel for Well L36/0092

Aquifer properties	Datum (amsl)	LSR effect	GWL base	ssg	R2	T1	Storage residence times T2	T3	Tv
	119.09	19.06	59.53	1.123	0.911	20.75	N/A	N/A	4.40

Eigenmodel parameters	k1	k2	k3	kv	g1	g2	g3		
	0.048	N/A	N/A	0.228	1.123	0.000	0.000		

Transform model	Alpha4	Base	Alpha1	Beta1	Alpha2	Beta2	Alpha3	Beta3	Obj. fn.
	0.797	-59.56	0.953	0.053	0.000	0.000	0.000	0.000	1121.01

Difference equation	a1	a2	a3	a4	b1	b2	b3		
	1.749	-0.759	0.000	0.000	0.011	0.000	0.000		

Figure 16: **Eigenmodel results for the 28-year record at well L36/0092**

M35/1080

Eigenmodel for Well M35/1080

					Storage residence times				
Aquifer properties	Datum (amsl) 61.04	LSR effect 3.17	GWL base 44.03	ssq 0.187	R2 0.823	T1 20.52	T2 1.59	T3 N/A	Tv 0.62
Eigenmodel parameters	k1 0.049	k2 0.629	k3 N/A	kv 1.605	g1 0.151	g2 0.036	g3 0.000		
Transform model	Alpha4 0.201	Base -17.01	Alpha1 0.952	Beta1 0.007	Alpha2 0.533	Beta2 0.017	Alpha3 0.000	Beta3 0.000	Obj. fn. 115.32
Difference equation	a1 1.686	a2 -0.806	a3 0.102	a4 0.000	b1 0.019	b2 -0.016	b3 0.000		

Figure 17:   Eigenmodel results for the 28-year record at well M35/1080
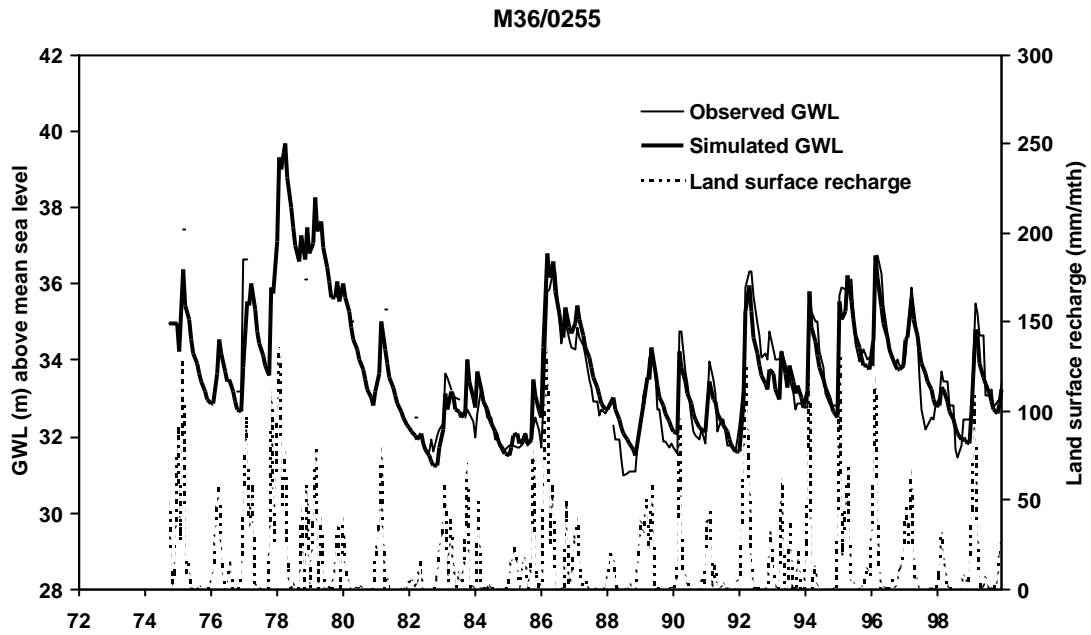
M36/0255

Eigenmodel for Well M36/0255

Aquifer properties	Datum (amsl)	LSR effect	GWL base	ssg	R2	T1	Storage residence times T2	T3	Tv
	43.68	4.93	28.89	0.302	0.847	19.14	0.61	N/A	0.61

Eigenmodel parameters	k1	k2	k3	kv	q1	q2	q3		
	0.052	1.649	N/A	1.649	0.285	0.018	0.000		

Transform model	Alpha4	Base	Alpha1	Beta1	Alpha2	Beta2	Alpha3	Beta3	Obj. fn.
	0.192	-14.79	0.949	0.014	0.192	0.014	0.000	0.000	85.20

Difference equation	a1	a2	a3	a4	b1	b2	b3		
	1.333	-0.402	0.035	0.000	0.023	-0.013	0.000		

Figure 18: **Eigenmodel results for the 28-year record at well M36/0255**

7. Application to Groundwater Management

7.1. SUSTAINABLE LIMITS TO GROUNDWATER ALLOCATION

There has been much debate about definitions of "sustainable yield" from aquifers. We take the view that sustainable yield is meaningful only in the context of potential effects on values to be protected. Some of the more important effects of groundwater abstraction, such as low flow in spring-fed streams and salt water intrusion, depend on piezometric levels in the aquifer. As discussed in Section 4.1.1, variations in piezometric levels are caused primarily by variation in land-surface recharge. If abstractions can be considered as negative land-surface recharge, then the eigenmodel method can be used to estimate the piezometric effects and hence the environmental effects. We examine this concept in more detail in Section 8.2.

This approach seems to imply that river recharge is not being considered as a groundwater resource, but this not the case. River recharge, in some aquifers, provides the steady piezometric surface upon which is superimposed the dynamic fluctuations caused by land surface recharge. The long-term effect of abstractions is to lower the steady piezometric surface to an amount that depends on the connection between river and aquifer. The eigenmodel approach assumes that the river does not increase recharge in response to abstraction, which is a conservative view. However, if this assumption proves to be grossly incorrect then the model is easily updated when new piezometric data become available.

7.2. SHORT-TERM AND LONG-TERM EFFECTS OF ABSTRACTION

The theory of groundwater wells shows that the piezometric effect of groundwater abstraction propagates away from the well in a radial direction, and that this effect decreases in magnitude with distance. Most applications of this theory are directed to estimating the effects of drawdown from a well, in terms of interference with other wells or influence on stream-aquifer interaction. The distance scale of these investigations is usually up to a few kilometres, and the time scale is up to a few months. These methods are well established and are not considered any further in this report. However, it is important to realise that these applications of well theory limit the estimation of effects to magnitudes that are significant or are practically measurable.

Any abstraction from an aquifer has an effect that eventually propagates throughout the whole aquifer. This effect may be a lowering of piezometric levels or induced additional recharge from a river. The effect from any one well may be infinitesimal in terms of practical measurement, but the cumulative long-term effects of many wells can be very significant. This is an issue that can be addressed by means of the eigenmodel approach, because the cumulative effects can be considered as a (negative) change to the land surface recharge over the whole aquifer.

One of the assumptions of the eigenmodel (Section 4.1.2) is that the spatial pattern of recharge is fixed, and that the time-variation is the same everywhere. It is obvious that this does not appear to be true for a developing pattern of abstraction wells that operate out of season to winter recharge from soil water drainage, for example. Our experience to date suggests that this assumption is not a serious limitation to application of the method. As aquifer development proceeds, the eigenmodel can be re-calibrated to account for any systematic changes. Therefore, our initial working assumptions for estimating the aquifer-wide, long-term effects of abstraction are:
- The cumulative effect of abstractions is spread throughout the whole aquifer

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- The time pattern of total abstraction can be applied as an input to the eigenmodel.

7.3. ANALYSIS OF AQUIFERS WITH UNKNOWN ABSTRACTION

Many of the aquifers that are likely to require improved management are already under stress from abstractions for which there are insufficient data on actual use in contrast to permitted allocation. This situation is demonstrated in the case histories considered in Section 6, for which the simulation plots show marked departures from the eigenmodel predictions, especially during drought periods. Our experience is that the eigenmodel calibrations are quite robust for aquifers with climatically driven, land surface recharge that is relatively larger than total abstraction.

7.4. CASE STUDY: EFFECTS OF GROUNDWATER ABSTRACTION ON THE HALSWELL RIVER

7.4.1. Relationship between Groundwater Level and Streamflow

Figure 19 shows the relationship between flow at a gauging station on the Halswell River and piezometric levels at Well M36/0255, which is about 14 km from the river (Figure 10).
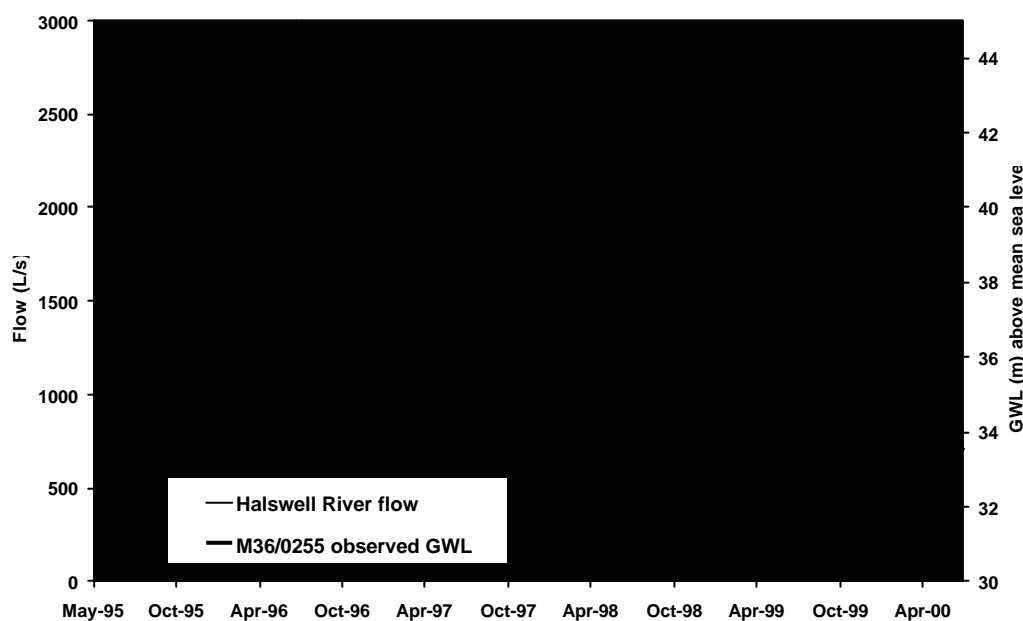

Figure 19: **Relationship between groundwater levels and low flow in the Halswell River**

The low-flow base of the Halswell River is clearly related to the piezometric level at the observation well, and this base is given by the linear scaling:

$$Halswell\ River\ low\text{-}flow\ (\text{L/s}) = 200 \times GWL@M36/0255\ (\text{m amsl}) - 6000 \quad (24)$$

We have an eigenmodel for predicting groundwater level at this indicator well M36/0255 (Table 3, Figure 18), and therefore it seems feasible to use it, in combination with equation (24), to predict the effects of Central Plains land surface recharge on the low-flow regime of the Halswell River.

7.4.2. Eigenmodel Predictions of Low Flow

Figure 20 shows the eigenmodel prediction of piezometric levels at M36/0255 for the same time period as in Figure 19. Predictions are quite satisfactory for the earlier part of the record but during the two consecutive "drought" seasons of 1997/98 and 1998/99 the observed groundwater levels were well below predictions. Therefore, low flow in the Halswell River would have been overestimated. Was this just a local effect caused by nearby groundwater abstraction, or was it part of an overall depletion of aquifer resource? Figure 21 shows the four well records, standardised by means of the *Base* and *LSR effect* parameters, for convenient comparison.
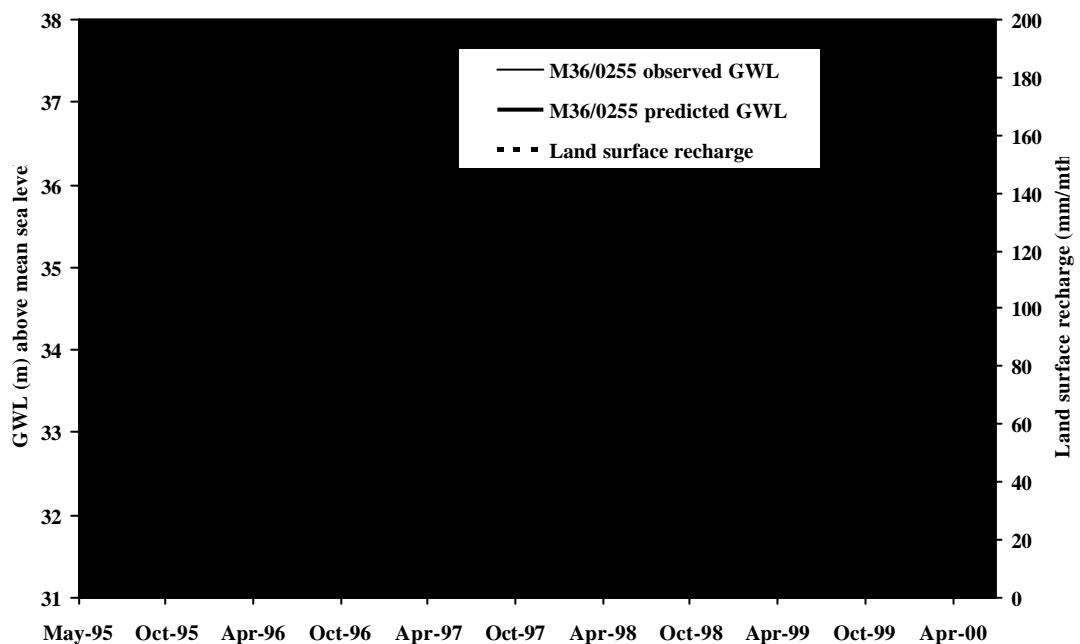

Figure 20: **Eigenmodel predictions at indicator well for low-flow effects**
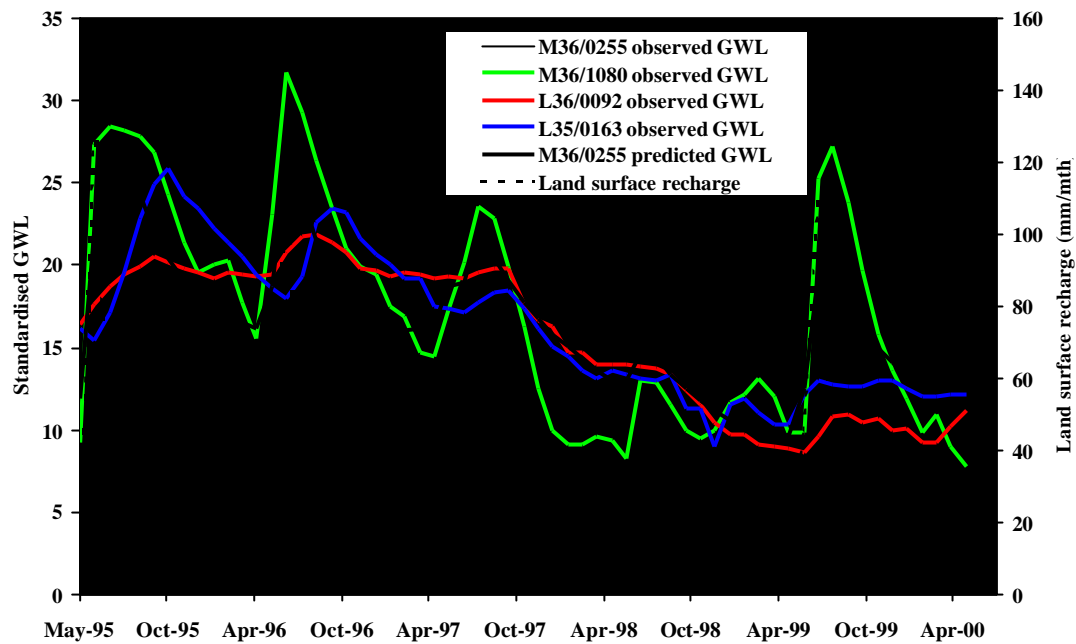
Figure 21: **Comparison of standardised well records for the droughts of 1997/98 and 1998/99**

For the first season (1997/98), the two well records for the upper part of the Central Plains (L35/0163, L36/0092) follow the eigenmodel prediction, whereas the two well records for the lower part of the plains (M35/1080, M36/0255) are significantly lower. This suggests that there is a component of "short-range, short-term" effect caused by abstraction in the lower plains, which has a direct effect on low-flow in the Halswell River. What is the role of the eigenmodel predictions in this scenario?

One solution would be to update the eigenmodel to account for observed departures from what has been predicted, and to use this information in operational management of the groundwater resource. In fact the mathematical structure of the eigenmodel is well suited to this kind of "real-time" updating.

7.4.3. A First Look at "sustainable yield"

Although the example shown in Figures 20 and 21 demonstrates that an environmental effect may be determined by local and temporal stresses, it is useful to be able to quantify the expected effect of allocating a proportion of the total groundwater resource to a particular use. Figure 22 shows the effect on the predicted indicator well levels of allocating 15% of average land surface recharge to be used for irrigation each year during November to February. The total land surface recharge can be estimated from the total land area overlying the Central Plains aquifer, multiplied by the average recharge calculated from the water balance model. Then, the allocation can be expressed as an annual volume of water, for example.
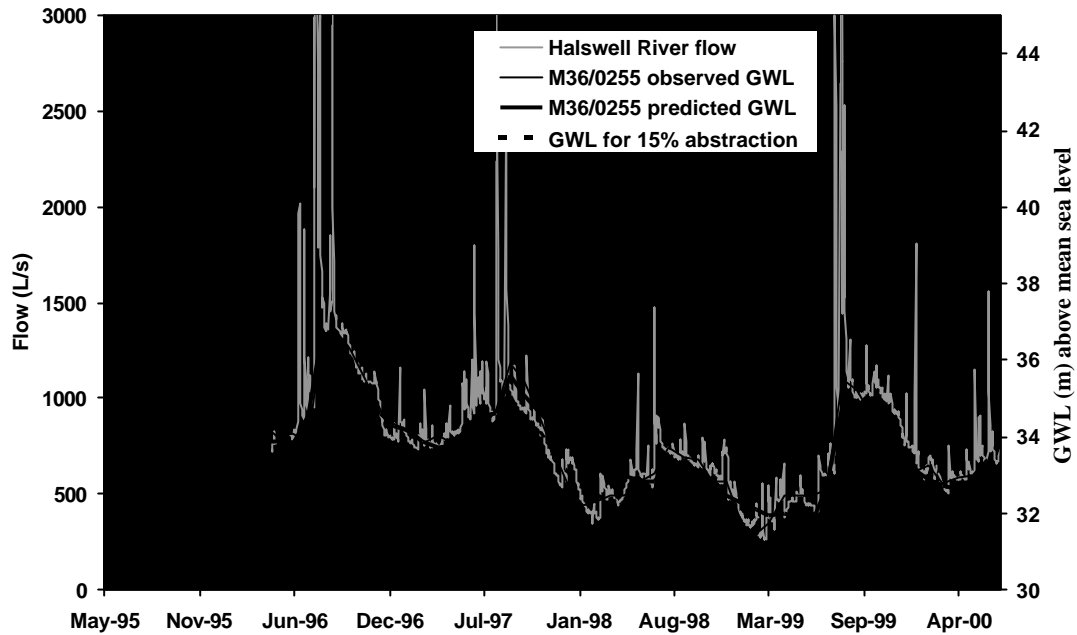
Figure 22: **Effect on groundwater level of a specified water allocation**

7.5. A TOOL FOR ADAPTIVE MANAGEMENT

The eigenmodel can be a useful tool for adaptive management of groundwater because it provides a means of expressing, as a simple concept, the overall dynamic behaviour of the groundwater resource in response to natural recharge and the abstraction stresses imposed by human use of the resource. This simple conceptual model is expressed in a mathematical form that is the key to a large body of theory that has been developed in the applied fields of control engineering and econometrics. Some of these theories are concerned with issues related to managing dynamic systems with uncertain knowledge of processes and poor quality data. The following section illustrates one example of these applications.

7.6. UPDATING THE EIGENMODEL FOR REAL-TIME FORECASTING

Appendix II describes how the difference equation (10) of the eigenmodel can be further augmented to a "forecast" form that is suitable for real-time operations in which the model is continually updated with the latest observations of piezometric levels. The resulting difference equation is:

$$
\begin{aligned}
U_n &= c_1 U_{n-1} + c_2 U_{n-2} + c_3 U_{n-3} + c_4 U_{n-4} + c_5 U_{n-5} \\
&+ d_1 R_n + d_2 R_{n-1} + d_3 R_{n-2} + d_4 R_{n-3} \\
&- a_1 e_{n-1} - a_2 e_{n-2} - a_3 e_{n-3} - a_4 e_{n-4} + e_n \\
h_n &= U_n + r(x, y)
\end{aligned}
\tag{25}
$$

in which the coefficients are related to the coefficients of equation (10), as given in Appendix II. The complete procedure for forecasting is not presented in this report and equation (25) is shown above only to illustrate the process. The input data for the forecasting procedure are the recent history of:

- Observed piezometric data h_n relative to the *Base* parameter (U_n)
- Estimated land surface recharge R_n
- Forecasting errors e_n

Future values of land surface recharge are not usually known because the climatic events are yet to occur. However, in a drought situation the future recharge can be assumed to be zero in order to forecast the worst-case. If restrictions on abstraction are being considered, then these can be estimated in terms of negative land surface recharge and used in the forecast equation.

The effect of incorporating the recent history of forecasting errors is to produce a feedback signal for the model to help it track unknown influences such as unrecorded abstractions and recharges. In order to operate the forecasting procedure, these data must be collected and processed in every time interval (e.g., monthly). This requirement can act as a focal point for the stakeholders involved in adaptive management.

The case study described in the next section illustrates the benefit of applying a forecast model to managing groundwater for environmental objectives. A forecast model of this type had previously been developed for Well L36/0092 by Bidwell *et al.* (1991)[4].

7.7.   FORECASTING THE EFFECTS OF ABSTRACTION

We now reconsider the case study of Section 7.4.2 in which prediction of low flow in the Halswell River was considered to be unsatisfactory during the 1997-99 droughts because unrecorded abstraction caused unexpected low piezometric levels in the indicator well (Figure 20). Figure 23 shows the result of applying the forecast version of the eigenmodel to the data from this well, in order to obtain forecasts of piezometric level one month ahead. These forecasts can than be transformed to low-flow estimates for the Halswell River by use of equation (24).
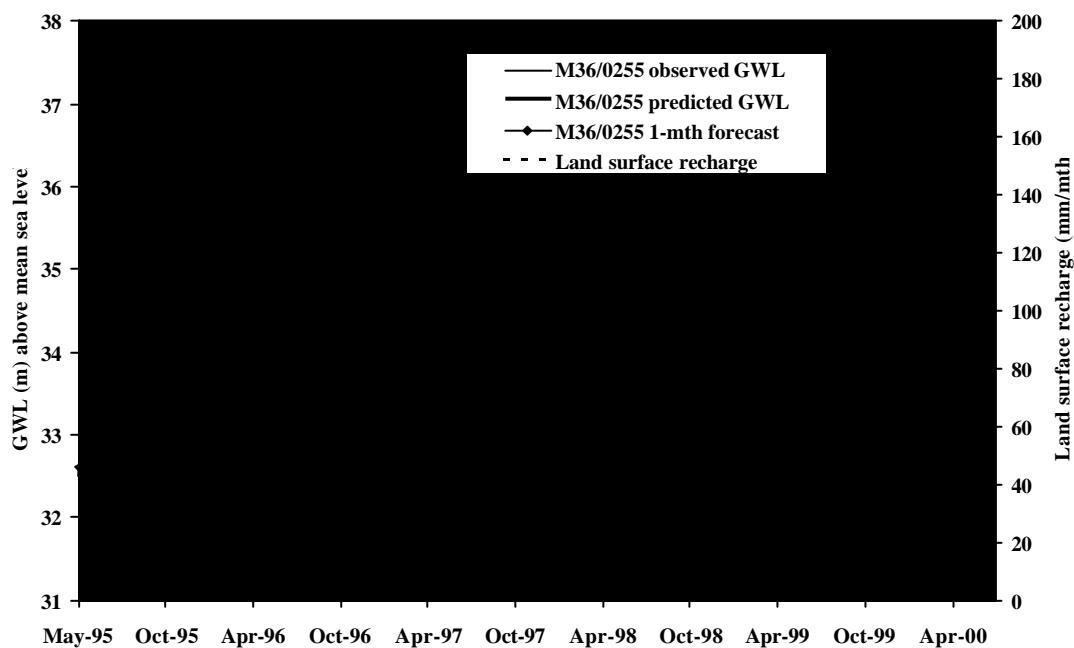

Figure 23:   Eigenmodel forecasts at indicator well for low-flow effects

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[4]   Bidwell, VJ; Callander, PF and Moore, CR (1991): An application of time-series analysis to groundwater investigation and management in Central Canterbury, New Zealand. *Journal of Hydrology (NZ)* **30(1)**:16-36.

Figure 23 shows that the one-month forecasts provide warning of significant departures from model prediction, quite similar to what was subsequently observed. This kind of information could be used to justify implementation of restrictions on abstraction if the low-flow forecasts were to violate agreed environmental values.

8. Summary of Good Practice

8.1.   ESTIMATION OF LAND SURFACE RECHARGE

The defining characteristic of an aquifer as a groundwater resource is its dynamic behaviour as a leaky storage for natural recharge. This behaviour can be determined from the observed response of groundwater levels to land surface recharge.

Land surface recharge is estimated as the soil-water drainage component of soil-plant-atmosphere processes at the land surface, in response to climate. There are a number of water balance models available, but these are not critiqued in the present report. The emphasis here is on necessary and sufficient aspects of these models:
- Water balance must be calculated on a daily basis, and then totalled to the selected time interval such as a month;
- Particular crop-soil combinations, can be expressed as a water-holding capacity;
- Only significant areas (as a percentage of total) of a particular soil-crop combination need be considered.

The strongest recharge signal for analysis of an aquifer comes from winter recharge when abstraction (for irrigation) is least. Therefore, estimation of land surface recharge need not take abstraction into account, for initial assessment, even if the aquifer is not in a "virgin" state.

8.2.   WHEN TO USE THE EIGENMODEL METHOD

The eigenmodel method performs best for land surface recharge that has a fixed spatial pattern and time-varying magnitude. This property is generally satisfied by natural recharge into an aquifer for which abstraction is a small fraction of the total resource. Under these conditions, the dynamic behaviour of the total groundwater resource can be assessed quite accurately.

As abstraction increases, generally with its own spatial pattern, the eigenmodel becomes less accurate but still provides useful information for management, especially in the forecasting format (Section 7.6). We have some experience of this effect of spatial distortion of recharge in Mid-Canterbury, where the piezometric response to land surface recharge is dominated by soil-water drainage from border-dyke irrigation. In other parts of this same region, pumped irrigation abstraction dominates the piezometric response, but forecasting is still feasible.

The purpose of the eigenmodel method is to provide:
- Whole-aquifer assessment of the useful availability of groundwater;
- Relationships between climate and environmental effects such as low flow in streams;
- Indicative relationships between abstraction and environmental effects;
- Estimates of the piezometric effect of river recharge.

The method is not suitable for assessment of short-range effects such as stream-depletion by groundwater pumping, or upconing of a salt water interface near the coast. However, some

salt water intrusion is determined by the rate of natural outflow to the coastal boundary, and the eigenmodel may be useful for estimating the time-variability of this flow.

8.3.   THE TRANSITION TO MORE COMPLEX MODELS

The effects of local abstraction stresses on an aquifer are best examined with models that simulate the continuous aquifer space ("continuum" models), such as MODFLOW. These models have the potential to use as much data and geological information as is available, but can require significant modelling resources in terms of expertise and time. Some of the time resource can arise from difficulties in obtaining unambiguous model calibration.

In situations where piezometric data are sparse, the benefits of the continuum model may be achieved only with considerable insight from the modeller. The eigenmodel approach offers the ability to appreciate the "big picture" of the dynamic response of an aquifer to recharge and abstraction. This insight can assist with appropriate reduction of the parameter options in the continuum model.

The small scale of a particular groundwater problem may mean that the areal extent of the continuum model is restricted by computational limits, and does not extend to the natural boundaries of the aquifer. Specification of boundary conditions for these continuum models can be difficult, and sometimes assumptions are made that may undermine the validity of the model. The eigenmodel method can be applied to several observation wells throughout the aquifer to assist in determining the nature of natural aquifer boundaries and how these influence the boundary conditions of the continuum model.

Appendix I: Conference Paper on Eigenmodel Theory

Conference paper presented at ModelCARE 2002, 4th International Conference on Calibration and Reliability in Groundwater Modelling, Prague, Czech Republic, 17-29 June 2002.

The Eigenvalue Approach to Groundwater Modelling for Resource Evaluation at Regional Scale

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Abstract
The dynamic response of piezometric head at any location in an aquifer to time variation of regional land surface recharge and abstraction can be expressed as a linear system comprising only a few conceptual water storages. Model structure and parameters are related to aquifer characteristics and spatial pattern of recharge by the eigenvalues and eigenfunctions of a general analytical solution to the linearised Boussinesq equation. The model is implemented as a stochastic ARMA difference equation, independently for each location. This modelling approach is demonstrated for an aquifer of 2000 km^2 area, yielding additional information about unobservable recharge and aquifer boundaries.

INTRODUCTION

Evaluation of an aquifer as a water resource is concerned primarily with the effect on stored groundwater of pumped abstraction in relation to climatically driven, highly variable recharge processes. The resulting dynamic variations in stored groundwater determine environmental effects such as the low-flow regime of streams. The available data for such evaluations often comprise estimates of the history of land surface recharge from water balance models, and piezometric records at a few locations in a poorly understood aquifer. In these circumstances, finite difference or finite element aquifer models are at a disadvantage because of their demands on knowledge of aquifer properties and specification of boundary conditions. These models also have high computational requirements because the piezometric head is calculated for every element and the time step is constrained by numerical accuracy and stability.

The purpose of our modelling approach was to identify the dynamic behaviour of an aquifer as a storage reservoir, and to quantify the relative recharge contributions from rivers and through the land surface. These requirements favour solution of the groundwater flow equation in terms of the eigenvalues, which determine the dynamic response in continuous time, and the corresponding eigenfunction components of spatial variation. When the model is to be applied to only a few locations and for large time increments, the complete set of eigenvalues and eigenfunctions is not required (Sahuquillo 1983). Sloan (2000) uses a similar approach to demonstrate that only a few eigenvalues are required for adequate simulation of the dynamic behaviour of an aquifer considered as a lumped system.

MODEL THEORY

The following theory can be applied to the general two-dimensional aquifer problem (Sahuquillo 1983), but we illustrate the argument by means of a one-dimensional aquifer (Fig.1) with fixed-head boundaries at different levels. The resulting piezometric surface, in the absence of land surface recharge, is defined by $\bar{h}(x)$ and may also include the effect of existing pumped abstractions that are relatively steady.
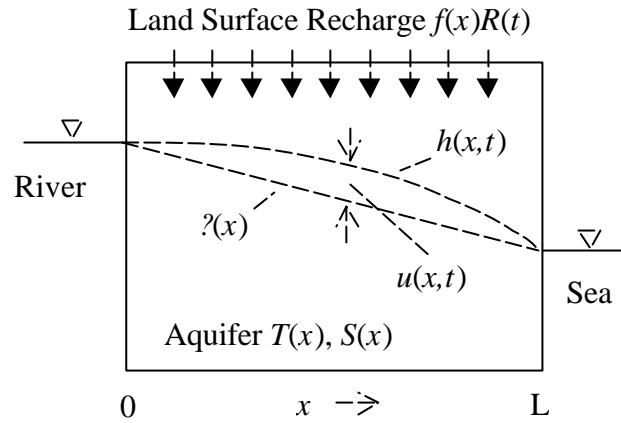

Fig. 1 One-dimensional heterogeneous aquifer with time-varying land surface recharge and fixed-head boundaries.

Land surface recharge is vertical drainage from the vadose zone overlying the aquifer, which can be defined by a spatial distribution $f(x)$ multiplied by a time-varying magnitude $R(t)$. Pumped abstractions that are unsteady and not correlated in time with $R(t)$ contribute to model error. The response $u(x,t)$ to land surface recharge is the observed piezometric level $h(x,t)$ relative to the unobservable level $\bar{h}(x)$:

$$u(x,t) = h(x,t) - \boldsymbol{h}(x)$$

(1)

Variations of $u(x,t)$ with time are assumed to be small in comparison to the aquifer depth, and therefore transmissivity $T(x)$ is only spatially variable, as is storativity $S(x)$. The governing equation for the aquifer shown in Fig.1 is:

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial u}{\partial x}\right] + f(x)R(t) = S(x)\frac{\partial u}{\partial t} \qquad x \in (0,\mathrm{L}), \quad t \in (0,\infty)$$

(2)

with boundary conditions:

$$u(0,t)=0, \quad u(\mathrm{L},t)=0 \qquad t \in (0,\infty)$$

(3)

Sloan (2000; Appendix A) shows that a general solution of (2) can be expressed in terms of the eigenvalues λ_i and eigenfunctions $p_i(x)$ derived from the physical characteristics of the aquifer as:

$$u(x,t) = \sum_{i=1}^{\infty} p_i(x) w_i(t)$$

$$\frac{dw_i(t)}{dt} + \lambda_i w_i(t) = c_i R(t)$$

$$w_i(0) = W_i$$

(4)

The general solution (4) can be illustrated in conceptual form (Fig. 2), for a location $x = X$, as the weighted output $u(X,t)$ from an infinite set of linear reservoirs in parallel, with input $c_i R(t)$ to the ith reservoir. The water content of each conceptual reservoir is represented by $w_i(t)$, with initial value W_i. The mean residence times are given by λ_i^{-1}, the reciprocals of the eigenvalues. Sloan (2000) reports that only a few of these reservoirs, corresponding to the smallest eigenvalues, are required to simulate adequately the dynamic behaviour of an aquifer. We use an additional reservoir (Fig. 2) in series with the parallel set to simulate storage in the vadose zone and the dynamic effect of leakage through aquitards.
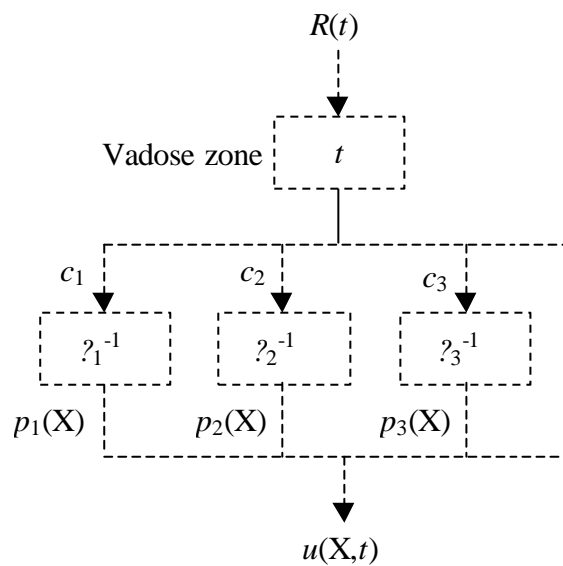
Fig. 2 Linear-system water storage model of aquifer response to land surface recharge.

We implemented the model structure of Fig. 2 within Microsoft Excel by converting (4) to a stochastic difference equation for each observation well, which relates monthly totals of recharge R_k, estimated from a water balance model, to monthly observations of groundwater level h_k. Only the first three eigenvalues were considered, so that with the inclusion of the vadose zone element the resulting difference equation was of fourth order. The model structure of Fig. 2 can be expressed in discrete time intervals by means of z-transforms as:

$$u_k = \left[\frac{b_1}{\left(1 - a_1 z^{-1}\right)} + \frac{b_2}{\left(1 - a_2 z^{-1}\right)} + \frac{b_3}{\left(1 - a_3 z^{-1}\right)} \right] \frac{\left(1 - a_4\right)}{\left(1 - a_4 z^{-1}\right)} R_k$$

$$h_k = u_k + h + e_k$$

(5)

in which $u_k + ?$ is the model estimate of the observation h_k, and e_k is an error term. The values of a_i are determined from the eigenvalues $?_i$ ($i=1,2,3$) and the mean hydraulic residence time t of the vadose zone, for the data observation interval $?t$, as:

$$a_i = \exp\left(-l_i ? t\right) \quad i = 1,2,3$$

$$a_4 = \exp\left(-\frac{\Delta t}{t}\right)$$

(6)

The parameter β_i is a function of c_i, $p_i(x)$ and $?_i$ in (4). For computation, (5) is converted to an autoregressive-moving-average (ARMA) difference equation:

$$u_k = a_1 u_{k-1} + a_2 u_{k-2} + a_3 u_{k-3} + a_4 u_{k-4} + b_1 R_k + b_2 R_{k-1} + b_3 R_{k-2}$$

$$e_k = h_k - u_k - h$$

(7)

for which the relationships between coefficients a_i, b_i in (7) and a_i, β_i in (5) are obtained by multiplying out (5) and equating powers of z^{-1}. The error term series e_k is the basis for model calibration and analysis of performance. The steady-state gain (ssg) of (7) is a useful measure of the mean piezometric response (m) at the particular location to mean land surface recharge (mm mth^{-1}), and is calculated from:

$$ssg = \frac{b_1 + b_2 + b_3}{1 - a_1 - a_2 - a_3 - a_4}$$

(8)

In general, the ssg increases with distance from fixed-head boundaries, and is a useful parameter for mapping and analysing aquifer characteristics.

Model calibration is conducted on the parameters $?_i$, β_i, t and $?$ in (5) and (6) because this form has minimal parameter interdependence and physically realistic initial values can be set. An important contribution to groundwater modelling is that the eigenvalues are the same at every location in the aquifer, and therefore calibrated values of $?_i$ from locations with good data records may be transferred to assist model calibration at locations with sparse data. We used the optimisation routine "solver", provided within Microsoft Excel, for model calibration at each location for which a groundwater level record was available.

MODEL DEMONSTRATION

We demonstrate the eigenvalue approach with some results obtained from a water resource study of the aquifer underlying part of the Canterbury Plains in New Zealand. The plains are 160 km long and 50 km wide between the mountains of the Southern Alps and the Pacific Ocean, formed by coalescing glacial outwash and alluvial deposits up to 600 m thick (Brown 2001). The study area is the 2000 km^2 Central Plains region bounded by the Waimakariri and

Rakaia Rivers that traverse the plains from mountains to sea. These large braided rivers are perched on alluvial fans and their interaction with the underlying aquifer is not completely understood, so that it is difficult to define boundary conditions for the aquifer.

The recharge series R_k for the region, as monthly totals, was estimated from a daily water balance model. This series, with mean annual total of 204 mm, was applied at all locations even though there is a rainfall gradient from about 950 mm y^{-1} near the mountains to about 650 mm y^{-1} near the ocean. The effect of this spatial variation in rainfall on land surface recharge is assumed to be time-invariant, as illustrated by $f(x)$ in (2), and therefore R_k is simply scaled by the model coefficients at each location.

The model (5)-(7) was calibrated independently at 14 observation wells in the region, with up to 28 years record of groundwater level at monthly intervals. The first four observations of groundwater level were used to compute the initial values of u_{k-i} in (7). The objective function in all cases was to minimise the sum of e_k^2.

RESULTS

We have selected four observation wells with the longer and more reliable records to present the model results in Table 1. The well locations form an approximate rectangle of 20 km x 15 km. Some of the remaining records were too short to capture the dynamic response, or were strongly influenced by local abstraction for spray irrigation and by recharge from surface irrigation. In the course of our resource study the eigenvalues were transferred from other locations, as supported by the theory, to assist with modelling these additional effects, but these results are not discussed in this paper.

Table 1 Parameters of the linear system model of aquifer response to land surface recharge.

Observation well reference	L35/0163	L36/0092	M36/0255	M35/1080
Piezometric response ssg (m mm^{-1} mth)	1.22	1.12	0.30	0.19
Dominant residence time $?_1^{-1}$ (mth)	18.9	20.4	19.2	20.4
Secondary residence time $?_2^{-1}$ (mth)	-	-	0.61	0.62
Dominant coefficient $ß_1$	0.063	0.053	0.014	0.007
Secondary coefficient $ß_2$	-	-	0.014	0.034
Vadose zone residence time t (mth)	2.7	4.8	0.6	1.6
Steady recharge datum $?$ (m amsl)	80.24	59.55	28.89	44.03
Model performance R^2	0.873	0.911	0.852	0.823

DISCUSSION

The results in Table 1 are presented in order of the piezometric response ssg of the model at each well, which is theoretically related to the distance of the well from a fixed-head boundary. Well L35/0163 is adjacent to the Waimakiriri River on the northern boundary of the study region, but the high value of ssg suggests that the river is not a fixed-head boundary at this location. In contrast, Well M35/1080 is near the lower reaches of this same river and the relatively low value of ssg suggests a more direct connection between the aquifer and the river at this location. The intermediate values of the other two wells are appropriate for their locations relative to the river and ocean boundaries.

The dominant residence time $?_1^{-1}$ at all wells is not significantly different from a value of 20 mth. This means that the aquifer can be considered, for management purposes, as a reservoir with a mean residence time of 20 mth, which receives a time-varying input of about 200 mm y^{-1} of recharge over the 2000 km^2 region. This concept is useful for quantifying the effect on lowland stream regimes of future abstraction from the aquifer. Only one other

eigenvalue was significant, and then only for the two wells nearest to the fixed-head boundaries as indicated by the smaller values of *ssg*. The values of residence time $?_2^{-1}$ for Wells M36/0255 and M35/1080 are not significantly different from 0.6 mth.

Comparison of the relative magnitudes of β_1 and β_2 for Wells M36/0255 and M35/1080 demonstrates the increasing importance of the larger eigenvalues (smaller residence times) in explaining aquifer dynamics near fixed-head boundaries, as indicated by the smaller values of *ssg*. The physical explanation is that initial response to perturbations of the piezometric surface is rapid near these boundaries because of the shorter drainage path, but subsequently is determined by water draining from the interior of the aquifer.

The residence time *t* of the vadose zone element varies with location from about 1 to 5 mth, and is loosely associated with depth to groundwater (data not shown). We know that an aquitard overlies the aquifer at Well L36/0092, and therefore slow drainage of perched groundwater may be a significant component of the 5-month residence time. It is worth noting that *t* is a measure of the hydraulic response of the vadose zone, which is much less than the transit time of a tracer particle.

The values of *?*, relative to mean sea level (amsl), contribute to defining the piezometric surface due to river recharge and steady abstractions. We propose using this information, in conjunction with lumped aquifer properties obtained from the dynamic analysis of the surface recharge response, to estimate the approximate magnitudes and locations of river recharge to the aquifer.

The explained variance R^2 provides a measure of the ability of the eigenvalue modelling approach to simulate the dynamic behaviour of an aquifer. However, there is still considerable information contained in the error series e_k at each of the observation sites, and further analysis may assist in quantifying the effects of abstractions and recharges not previously considered.

CONCLUSIONS
The benefits of the eigenvalue approach to groundwater modelling depend on aquifers behaving as distributed dynamic systems with a high degree of interdependence among the components. This means that the dynamic effects of heterogeneous aquifer properties and spatial variations in recharge can be condensed into a much smaller number of independent model parameters. The most significant of the dynamic system parameters are observable at any location in the aquifer, and useful information can be obtained from simple models calibrated with time-series of piezometric head at single locations. The reliability of this information is quantifiable in terms of the error component of a stochastic difference equation. Some of the model parameters are location-specific, and these can provide information about unobservable sources of recharge and the characteristics of aquifer boundaries.

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Appendix II:
Eigenmodel Mathematics for Spreadsheet Implementation

SYSTEM EQUATION

The following material supports the mathematical equations in Section 5.3 and Section 9.2. Figure 3 (from Section 5) is reproduced below, to illustrate the mathematical development. Equations developed in this appendix are numbered with the prefix "A", whereas references to equations in the main report are to their respective equation numbers.
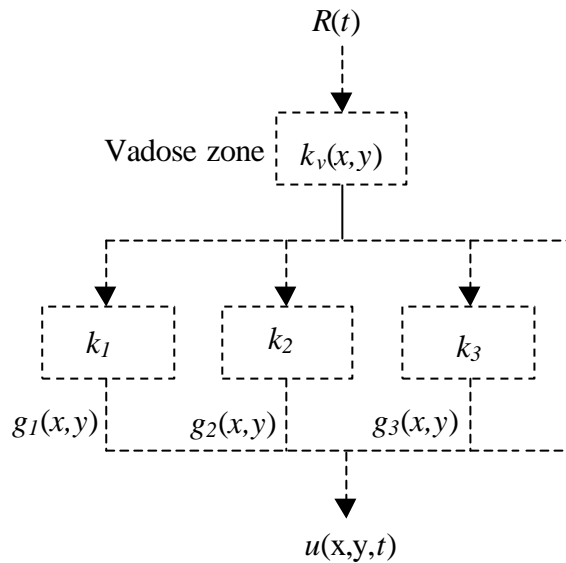

Figure 3. Eigenmodel structure

The dynamic behaviour of each component i of the linear system shown in Figure 3 can be described by a first-order differential equation that relates output $y_i(t)$ to input $x_i(t)$ by:

$$\frac{dy_i(t)}{dt} + k_i y_i(t) = g_i x_i(t)$$

(A1)

that is similar to equation (5). If the continuous time record of input and output is sampled at intervals of $?t$, such that the input values are averages or totals during the interval and the outputs are instantaneous values at the end of each interval, then equation (A1) may be expressed as a difference equation:

$$y_i(n) = \boldsymbol{a}_i y_i(n-1) + \boldsymbol{b}_i x_i(n)$$

(A2)

for which:

$$a_i = \exp(-k_i \Delta t)$$

$$b_i = g_i \left[1 - \exp\left(-k_i \Delta t\right)\right]$$

$$= g_i \left(1 - a_i\right)$$

$$n = \frac{t}{\Delta t}$$

(A3)

similar to equations (6) and (7).

We will now use z-transforms, which are the discrete equivalent of Laplace transforms, to manipulate and solve systems of first-order difference equations. The z-transformation of equation (A2) is:

$$y_i(n) = \frac{b_i}{\left(1 - a_i z^{-1}\right)} x_i(n)$$

(A4)

The vadose zone component in Figure 3, which is assigned the subscript $i = 4$, converts the land-surface recharge input $R(n)$ to a smoother output $Q(n)$ entering the groundwater:

$$Q(n) = \frac{\left(1 - a_4\right)}{\left(1 - a\ z^{-1}\right)} R(n)$$

(A5)

because for this component, $g_4 = 1$, and from equation (A3), $\beta_4 = (1 - a_4)$.

Each of the three groundwater components in Figure 3 receives the input $Q(n)$ and produces a piezometric output $h_i(n)$ given by:

$$h_i(n) = \frac{b_i}{\left(1 - a_i z^{-1}\right)} Q(n)$$

(A6)

The total piezometric effect in the aquifer $u(n)$ is:

$$u(n) = h_1(n) + h_2(n) + h_3(n)$$

(A7)

By combining equations (A5), (A6), and (A7), the z-transform equation of the system shown in Figure 3 can be written as:

$$u(n) = \left[\frac{b_1}{\left(1 - a_1 z^{-1}\right)} + \frac{b_2}{\left(1 - a_2 z^{-1}\right)} + \frac{b_3}{\left(1 - a_3 z^{-1}\right)}\right] \frac{\left(1 - a_4\right)}{\left(1 - a_4 z^{-1}\right)} R(n)$$

(A8)

Equation (A8) is a polynomial in z^{-1} that can be multiplied out to the form:

$$u(n) = \frac{\left(b_1 + b_2 z^{-1} + b_3 z^{-2}\right)}{\left(1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3} - a_4 z^{-4}\right)} R(n)$$

(A9)

The relationships between the coefficients of equation (A9) and equation (A8) are obtained by equating powers of z^{-1}, to give equations (11) and (12):

$$a_1 = \left(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 + \boldsymbol{a}_4\right)$$
$$a_2 = -\left(\boldsymbol{a}_1\boldsymbol{a}_2 + \boldsymbol{a}_1\boldsymbol{a}_3 + \boldsymbol{a}_1\boldsymbol{a}_4 + \boldsymbol{a}_2\boldsymbol{a}_3 + \boldsymbol{a}_2\boldsymbol{a}_4 + \boldsymbol{a}_3\boldsymbol{a}_4\right)$$
$$a_3 = \left(\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3 + \boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_4 + \boldsymbol{a}_2\boldsymbol{a}_3\boldsymbol{a}_4 + \boldsymbol{a}_1\boldsymbol{a}_3\boldsymbol{a}_4\right)$$
$$a_4 = -\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3\boldsymbol{a}_4$$

(11)

$$b_1 = \left(1 - \boldsymbol{a}_4\right)\left(\boldsymbol{b}_1 + \boldsymbol{b}_2 + \boldsymbol{b}_3\right)$$
$$b_2 = -\left(1 - \boldsymbol{a}_4\right)\left(\boldsymbol{b}_1\boldsymbol{a}_2 + \boldsymbol{b}_1\boldsymbol{a}_3 + \boldsymbol{b}_2\boldsymbol{a}_1 + \boldsymbol{b}_2\boldsymbol{a}_3 + \boldsymbol{b}_3\boldsymbol{a}_1 + \boldsymbol{b}_3\boldsymbol{a}_2\right)$$
$$b_3 = \left(1 - \boldsymbol{a}_4\right)\left(\boldsymbol{b}_1\boldsymbol{a}_2\boldsymbol{a}_3 + \boldsymbol{b}_2\boldsymbol{a}_1\boldsymbol{a}_3 + \boldsymbol{b}_3\boldsymbol{a}_1\boldsymbol{a}_2\right)$$

(12)

Equation (A9) can be written in difference equation form, by inverse z-transformation, as:

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + a_3 u_{n-3} + a_4 u_{n-4} + b_1 R_n + b_2 R_{n-1} + b_3 R_{n-2}$$

(10)

where for convenience we have written u_n and R_n for $u(n)$ and $R(n)$.

Equation (10) is the difference equation that is used as a spreadsheet formula to simulate the system shown in Figure 3.

SYSTEM NOISE AND THE FORECAST MODEL

Equation (10) provides the model prediction u_n of the piezometric response to land surface recharge, and therefore the predicted value of total piezometric level is:

$$\hat{h}_n = u_n + r(x, y)$$

(A10)

where $r(x,y)$ is the steady piezometric effect of river recharge, calibrated as the *Base* parameter (Section 5.4.4). This predicted value h_n differs from the observed piezometric level h_n by an error N_n that we call the noise term, such that:

$$N_n = h_n - \hat{h}_n$$

(15)

The noise contains the effects of model assumptions, imperfections in the data, and especially the influence of unknown recharge to and abstraction from the aquifer. These unknown effects are not completely random, and this introduces time dependence into the noise series. This time dependence can be estimated from the data and used as an additional source of information to improve model predictions over a short time into the future, or what is called forecasting.

The simplest model time of dependence is a first-order equation that transforms an uncorrelated series of values e_n into the correlated noise N_n, so that the z-transform model is:

$$N_n = \frac{e_n}{\left(1 - \boldsymbol{g} z^{-1}\right)}$$

(A11)

The complete model for the real, but unknown, dynamic response U_n to land surface recharge is obtained by combining equations (A9) and (A11):

$$U_n = \frac{\left(b_1 + b_2 z^{-1} + b_3 z^{-2}\right)}{\left(1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3} - a_4 z^{-4}\right)} R_n + \frac{e_n}{\left(1 - \boldsymbol{g} z^{-1}\right)}$$

(A12)

Equation (A12) can also be multipied out to a polynomial in z^{-1} and transformed to the difference equation:

$$\begin{aligned}
U_n &= c_1 U_{n-1} + c_2 U_{n-2} + c_3 U_{n-3} + c_4 U_{n-4} + c_5 U_{n-5} \\
&\quad + d_1 R_n + d_2 R_{n-1} + d_3 R_{n-2} + d_4 R_{n-3} \\
&\quad - a_1 e_{n-1} - a_2 e_{n-2} - a_3 e_{n-3} - a_4 e_{n-4} + e_n \\
h_n &= U_n + r(x, y)
\end{aligned}$$

(25)

in which $r(x,y)$ is the *base* parameter for the well and:

$$\begin{aligned}
c_1 &= a_1 + \boldsymbol{g} \\
c_2 &= a_2 - a_1 \boldsymbol{g} \\
c_3 &= a_3 - a_2 \boldsymbol{g} \\
c_4 &= a_4 - a_3 \boldsymbol{g} \\
c_5 &= -a_4 \boldsymbol{g} \\
d_1 &= b_1 \\
d_2 &= b_2 - b_1 \boldsymbol{g} \\
d_3 &= b_3 - b_2 \boldsymbol{g} \\
d_4 &= -b_3 \boldsymbol{g}
\end{aligned}$$

(A14)

This is the forecast version of the eigenmodel.