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# Finite Element Analysis of Slab and a Comparative Study with Others Analytical Solution

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## ABSTRACT

*Slabs are one of the most widely used structural elements. The function of slabs is to resist loads normal to their plane. In many structures, in addition to support transverse load, the slab also forms an integral portion of the structural frame to resist lateral load. Inspite of their widespread use, there has never been a universally accepted method of designing all slabs systems. The paper presents finite element analysis of slabs. The finite element method is chosen as this is more powerful and versatile compared to other numerical methods. A slab element is developed on the basis of conventional slab theory expressed in terms of rectangular co-ordinates and displacement. The element incorporates 20 degree of freedom, namely, normal displacement with its first derivatives along longitudinal and transverse direction respectively and two tangential displacements. A computer program is developed for solution of finite element equations as well as to check rigid body modes and to obtain the results. The results are compared with analytical solution and previously developed finite element solution with the help of a table.*

**Keywords:** Slab, Finite Element Analysis, Degree of freedom, Analytical Solution.

## 1 INTRODUCTION

Slabs are most widely used structural elements of modern structural complexes and the reinforced concrete slab is the most useful discovery for supporting lateral loads in buildings. Slabs may be viewed as moderately thick plates that transmit load to the supporting walls and beams and sometimes directly to the columns by flexure, shear and torsion. It is because of this complex behavior that is difficult to decide whether the slab is a structural element or structural system in itself. Slabs are viewed in this paper as a structural element. The greatest volume of concrete that goes into a structure is in the form of slabs, floors and footings. Since slabs have a relatively large surface area compared with their volume, they are affected by temperature and shrinkage slabs may be visualized as intersecting, closely spaced, grid-beams and hence they are seen to be highly indeterminate. This high degree of indeterminacy is directly helpful to designer, since multiple load-flow paths are available and approximations in analysis and design are compensated by heavy cracking and large deflections, without significantly affecting the load carrying capacity. Slabs, being highly indeterminate, are difficult to analyze by elastic theories. Since slabs are sensitivity support restraints fixate, rigorous elastic solutions are not available for many practically important boundary conditions. More recently, finite difference and finite element methods have been introduced and this is extremely useful. Methods have also been innovated to find the collapse loads of various types of slabs through the yield line theory and strip methods. In addition to supporting lateral loads (perpendicular to the horizontal plane), slabs act as deep horizontal girders to resist wind and earthquake forces that act on a multi-storied frame. Their action as girder diaphragms of great stiffness is important in restricting the lateral deformations of a multi-storied frame. However, it must be remembered that the very large volume and hence the mass of these slabs are sources of enormous lateral forces due to earthquake induced accelerations.

## 2 OBJECTIVES OF THE STUDY

The objectives of this paper are:

1. To develop a generalized computer program in fortran77 for the analysis of slab.
2. To find element stiffness matrix and overall stiffness matrix for quarter slab.
3. To find the displacement components i.e. u, v, w, θx, θy.
4. To find stresses / moments at various points of the slab.
5. To perform finite element analysis of slab and a comparative study with others analytical solution.

## 3 HISTORICAL BACKGROUND OF FINITE ELEMENT METHOD

One of the most powerful numerical techniques that have been developed in the realm of engineering analysis is the finite element method. The advent of high-speed computers has enabled engineers to employ this method for approximate solutions of complex problems having complicated boundary conditions. It was the work of Turner et al (1956) that led the discovery of the finite element method. Clough (1960) appears to be the first to use the terminology ‘finite element’ and he gave the physical interpretation to the method. The growing need for a safe and rational structural design of a modern aircraft resulted in the development of the method in its present form. The method was recognized by Melosh (1963) as rigorously sound and it became a respectable area of study. Finite element method has got similarly with various classical mathematical procedures e.g. Rayleigh Ritz method, Collocation method, Galerkins method etc. and this may be viewed from all those perspectives [Bathe (1982), Desai and Abel (1972), Desai (1979), Zienkiewicz (1977].

## 4 FINITE ELEMENT METHOD

The concept of finite element is that a body or continuum is divided into smaller elements of finite dimensions called finite elements interconnected at a number of joints called ‘Nodes’ or ‘Nodal Points’. The original body or structure is then idealized as an assemblage of these elements connected at nodal points. The displacements of these nodal points will be the basic unknown parameters of the problem. In most popular approach, a simple displacement function is assumed in terms of the displacements at the prescribed nodal points of elements [Zienkiewicz (1977)]. Then the principle of virtual displacements is used to derive a set of linear simultaneous equations called stiffness equations. The method has been used in this paper. The first application of the finite element method to reinforced concrete structures dates back to the late 1960’s by Ngo and Scordelies (1967). In this paper, simple beams were analyzed in which the concrete and the steel reinforcement were represented by two-dimensional triangular finite elements. Special bond link elements were used to connect the steel to the concrete. Linear analyses were performed on beams with predefined crack patterns to determine principal stresses in the concrete, stresses in the steel reinforcement and bond stresses.

## 5 FORMULATION OF THE PROBLEM

### Finite Element Procedure

The finite element method can be considered as a generalized displacement method for two and three dimensional continuum problems. It is necessary to discrete the continuum into a system with a finite number of unknowns so that the problems can be solved numerically. The finite element procedure can be divided into the following steps:

1. Idealization of the continuous surface as an assemble of discrete elements.
2. Selection of displacement models.
3. Derivation of the element stiffness matrix.
4. Assembly of element stiffness matrix into an overall structure stiffness matrix.
5. Solution of the system of linear equations relating nodal points loads and unknown nodal displacements.
6. Computation of internal stress resultants by use of the nodal point displacements already found.

### Displacement Function

In order to assure convergence to a valid result by mesh reinforcement, the following three sacred rules have emerged for the assumed displacement functions:

1. The displacement must be continuous within the element and the displacements must be

compatible between adjacent elements. For plane stress and plane strain elements, continuity of the displacement functions along is sufficient, whereas for bending elements, continuity of both the displacement and slope is needed.

2. The displacement function must include the states of constant strain of the element. This seems to be the most sacred of all the rules, since eventually, by mesh reduction, one is evitable going to reach small region where the strains are constant.
3. The displacement function must allow the element to undergo rigid body motion without any internal strain. For plane stress and plate bending elements, it is easy to establish displacement functions satisfying all these three requirements.

The displacement functions used in deriving the 20 * 20 stiffness matrix are:

$$u\,(x,y) = a_1xy+a_2x+a_3y+a_4 \qquad \text{(i)}$$
$$v\,(x,y) = a_5xy+a_6x+a_7y+a_8 \qquad \text{(ii)}$$
$$w\,(x,y) = a_9x^3y+a_{10}x^3+a_{11}x^2y+a_{12}x^2+a_{13}xy^3+a_{14}xy^2 +a_{15}xy+a_{16}x+a_{17}y^3+a_{18}y^2+a_{19}y+a_{20} \qquad \text{(iii)}$$

Alternatively, in matrix form we can write this symbolically as follows:

$$\{\bar{u}\} = [P]\ \{a_i\} \qquad (1)$$

Where $\{\bar{u}\}$ is vector of slab displacement and $[P]$ is matrix of displacement functions. Here the rectangular co-ordinate system is considered. The degree of freedom considered at each node (corner) of the element is u, v, w, $w_x$ and $w_y$.

**Element Stiffness Matrix:**

To simplify the derivation of the element stiffness matrix, a more convenient form of nodal displacement parameters with five degrees of freedom per node is listed as follows:

$$[u_i]^T = u_1, v_1, w_1, w_{1x}, w_{1y}, u_2, v_2, w_2, w_{2x}, w_{2y}\ u_3, v_3, w_3, w_{3x}, w_{3y}, u_4, v_4, w_4, w_{4x}, w_{4x} \qquad \text{(iv)}$$

Where, $w_{ix} = (\delta_w/\delta_x)_i$, $w_{iy} = (\delta_w/\delta_y)_i$ ; $i$ = 1 to 4 stands for the node number of the node of an element.

Substituting the values of co-ordinates of four nodes in the three displacement finction and two derivatives of w stated above, we get the 20 nodal displacements of an element as follows:

$$\{u_i\}= [H]\ \{a_i\} \qquad (2)$$

Where, $\{u_i\}$is vector of nodal displacement co-ordinates and [H] is called transformation matrix.

The strain displacement relationships used in the analysis of this of slab element may be expressed as:

$$\{e\}= [\delta]\ \{\bar{u}\} \qquad (3)$$

Therefore substituting Eq.1 into Eq.3 we get the strain expressed in terms of displacement parameters as follows:

$$\begin{aligned}\{e\} &= [\delta]\ \{\bar{u}\} \\ &= [\delta]\ [P]\ \{a_i\} \\ &= [B]\ \{a_i\} \end{aligned} \qquad (4)$$

Where [B] is called strain matrix is a function of x and y co-ordinates.

The stress matrix can be expressed as follows:

$$\{AN\} = [D]\ \{e\} \qquad (5)$$

The strain energy developed in the element is expressed by

$$U_t = \tfrac{1}{2} * \iint [AN]^T\ \{e\}\, dx\, dy \qquad (6)$$

Substituting the expression for [*AN*] and {e} in the Eq.6 we get the strain energy

$$\begin{aligned} U_t &= \tfrac{1}{2} * \{a_i\}^T \left[ \iint [B]^T [D]\ [B]\ dx\, dy \right] \{a_i\} \\ &= \tfrac{1}{2}\ \{a_i\}^T\ [U]\{a_i\} \end{aligned} \qquad (7)$$

Where $[U] = \iint [B]^T [D]\ [B]\ dx\, dy$

Now substituting $\{a_i\}$ from Eq.2 into Eq.7 and finally making derivatives of strain energy $U_t$ with respect to the nodal displacement parameters, we get the required element stiffness matrix [*S*] and are given by

$$[S] = \left[H^{-1}\right]^T [U]\ [H]^T \qquad (8)$$

**Overall Stiffness Matrix:**

The element stiffness matrix relates quantities defined on the surface. Therefore, co-ordinate transformations are completely avoided and the overall stiffness matrix SFF of the slab structure is assembled by direct summation of the stiffness contributions from the individual elements. The degree of freedom for the overall stiffness matrix is obtained by substituting joint restraint form, the total number of displacement co-ordinates. The overall stiffness matrix is first partitioned so that the terms pertaining to the degrees of freedom are separated from those for the joint restraints. Then the matrix is rearranged by interchanging rows and columns in such a manner that stiffness corresponding to the degrees of freedom is listed first and those corresponding to joint restraints are listed second. Such a matrix is always symmetric. To computer time and storage, only the upper band of the stiffness matrix $S_{FF}$ (for free joint displacements) is constructed.

**Load Matrix**

The vertical gravity load (mainly self-weight) is the major load for roof slab. The load intensity 'QL' is uniform over the area of a slab of uniform thickness. This load intensity 'QL' can be resolved into three components at a point in the three directions x, y and z as follows in a matrix.

The above-distributed load is replaced by an equivalent nodal load matrix $\{AQ\}$ for each element. This load matrix $\{AQ\}$ is obtained by equating virtual work done by the uniform load $\{Q\}$ and the nodal loads $\{AQ\}$. According to the standard formulae from texts:

$$\{AQ\} = \int_{-A/2}^{A/2} \int_{-B/2}^{B/2} \left[H^{-1}\right]^{T} [P]^{T} [Q] \, dx \, dy \tag{9}$$

Such a consistent load matrix will truly represent the distributed gravity load 'QL'. But the laborious process of Eq.9 can be avoided by using approximate overall nodal matrix $\{AQ\}$. This can be worked out as follows:

The total vertical load on an element is assumed to be equally shared by its four nodes. The z components of this vertical load are the element nodal loads corresponding to displacements w. Contributions from all elements connected at a node together form the final values of nodal loads for that node. Hence in the overall load matrix, out of five load values for each node, only the third will be non-zero.

**Expression for Stresses / Moments**

From Equation 2 the expression for {ai} is found as follows:

$$\{a_i\} = [H]^{-1} \{u_i\} \tag{10}$$

Using these values of $\{a_i\}$ and combining Eq.4 with Eq.5, we get the matrix of resultant stresses / moments at any point (x, y) in terms of nodal displacements as follows:

$$\begin{aligned} \{AN\} &= [D]\{e\} \\ &= [D][B]\{a_i\} \\ &= [D][B][H]^{-1}\{u_i\} \end{aligned} \tag{11}$$

## 6 RESULTS AND DISCUSSIONS

In this section of the paper results of two problems (problem 1 is fix supported square plate and problem 2 is simple supported square plate) obtained by this analysis are compared with the result of other author obtained by analytical solution are shown in the table 1 and in the table 2. Normal displacements w along mid section at different points on the slab of problem 1 are shown in table 3.

Table 1. Comparison of results of problem 1 with other author

| No. of element The quarter slab is divided | Vertical deflection (cm) | | Moment about X-axis, Mx (kg-cm / cm) | |
|---|---|---|---|---|
| | Present finite element analysis | Analytical solution by S. Razasekaran | Present finite element analysis | Analytical solution by S. Razasekaran |
| 4 | -0.06058 | -0.03883 | 23.36 | 15.00 |
| 16 | -0.05870 | -0.03767 | 18.63 | 10.95 |
| 64 | -0.05532 | -0.03732 | 11.81 | 9.60 |

Table 2: Comparison of results of problem 2 with other author

| Points of deflection | Distance from center to the end (inch) | Deflection / Maximum Deflection | |
|---|---|---|---|
| | | Present finite element analysis | Analytical solution by Hinton & Owen |
| D | 0.000 | -1.00000 | -1.000 |
| E | 0.125 | -0.94160 | -0.933 |
| F | 0.250 | -0.76325 | -0.733 |
| G | 0.375 | -0.45517 | -0.433 |
| C | 0.500 | 0.00000 | 0.000 |

Table 3: Vertical Deflection along Mid Span of Problem 1

| Points of vertical deflection | Distance from center to the end (cm) | Vertical deflection (cm) |
|---|---|---|
| D | 00.0 | -0.05532 |
| E | 04.0 | -0.05276 |
| F | 08.0 | -0.04446 |
| G | 12.0 | -0.02834 |
| C | 16.0 | 0.00000 |

The results obtained for problem 1 are compared with the results of S. Razasekaran. From the table 1 it is seen that our obtained values vary about 20 percent from the author's obtained values. The results obtained for problem 2 are compared with E. Hinton & Dr. J.Owen values that are very close to each other. If we take a greater number of elements, result that is more accurate will be obtained.

## 7 CONCLUSIONS

This paper deals with the finite element analysis of slab. Numerical results obtained with a slab element are compared with analytical solution and previously developed finite element solutions. Observations of table 1 reveals that displacements and moments are deviating by an order of 20% than the analytical results. Accurate results for moments may be obtained if more refined mesh is taken. A computer program is developed for solving different slabs. The overall stiffness matrix is assembled in banded form and only the upper half-band is stored in the memory. These save computer time and memory requirement. A minor modification in the program can solve slab with boundary conditions. The program is thus general and is strongly recommended for analysis of slab. Experimental investigation using advanced techniques of experimental stress analysis, i.e., Photo-elasticity. Semiconductor gauges is likely to help getting a clear picture. The present investigation restricts to only the analysis for strength of slab for a given load. No rational basis is yet available for knowing the exact values of strength and load. A probabilistic approach defining the stress-strength relationship is recently emerging. An assessment for reliability of the slab is likely to give a better insight to the design engineer for enabling him to propose a safer and economical system for construction. The Reliability analysis is also likely to increase the confidence of site-incharge, since he will know in advance, the nature of redundancy available and its purpose so that he is in a position to execute the work in a planned and sequential manner. An investigation for Reliability estimate is recommended

## REFERENCES

[1]. Clough, R. W. “The finite element in plane stress analysis” Proc. A.S.C.E. Conf. on electronic computation, Pitusburg, Pa. Sept. 1960.
[2]. Zienkiewicz, O.C. and Cheung, Y.K. “The finite element method in structural and continuum mechanics”. McGraw-Hill (1967).
[3]. Moan, T. and Shinozuka, M. “Structural safety and Reliability”. Elsevier scientific Publishing Company, New York 1981.
[4]. Dr. Nizamud-doulah, S.M. “Nonlinear Analysis of Reinforced Concrete Skew Slab”, Ph.D. Thesis, Department of Civil Engineering. BUET, Dhaka, 2000.
[5]. Dr. Nizamud-doulah, S.M. “Finite Element Analysis of Cylindrical Shells”. M.Tech Thesis, Department of Civil Engineering, IIT Kharagpur, India, 1983.
[6]. Purushothaman, P. “Reinforced Concrete Structural Elements”.
[7]. Nilson, A.H. “Design of Concrete Structures”, McGraw-Hill International Editions (1997).
[8]. Razasekaran, S. “Finite Element Analysis in Engineering Design”.
[9]. Ram Kumar. “Programming with Fortran 77’.