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Ian R Wood and Kenneth Choo

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RESEARCH REPORT

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An extension to the theory of steady selective withdrawal for a two layer fluid

Ian R. Wood
Department of Civil Engineering
University of Canterbury, Christchurch, New Zealand

and

Kenneth Choo
Applied Computing Mathematics and Statistics Group
Lincoln University, Christchurch, New Zealand

Abstract

Most reservoirs contain stratified fluid and selective withdrawal is used to obtain water of the desired properties. Initially we review the case with an infinite upper layer with a sharp interface. When the total discharge is specified, then the ratio of the discharge from each layer is determined by the criteria of smoothness at the virtual control (i.e. the critical point). At this point, the long wave velocity on the interface is zero. For the case when the upper layer depth is large, we show that the control is in the valve and the virtual control (which determines the ratio of the discharge in each layer) moves further from the source as the total discharge increases. When there is a finite upper layer, a portion of the flow is in the duct and a portion of the flow is in the free surface. We derive the criteria for the virtual control in the free surface flow and show that the duct control occurs first. If we then assume that the flow is not over-specified, we determine the necessary conditions for a smooth transition between the duct and the free surface flow. This enables us to determine the minimum ratio of the upper layer depth to the lower layer depth for the steady duct solution to be valid. This contrasts with the conclusions of Bryant and Wood (1976).

1. Introduction

In most reservoirs at some time of the year, the water is stratified and it is well known that the stratification inhibits the vertical motion without restricting the horizontal motion. Given an appropriate intake tower with multiple ports, this allows selective withdrawal to manage the water quality from the outlet. This method has been used for sometime (e.g. Craya, 1949, Gariel, 1949, Harleman *et al.*, 1959), Nece, 1970). There is a very good review of the state of the art in Lawrence and Imberger (1979). This paper extends the work when there are sharp interfaces and the outlet is positioned on a vertical reservoir wall. The case considered is illustrated in Figure 1 when two layers are flowing and where the depth and densities of the layers at infinity are d_0 and ρ_0, d_1 and ρ_1 and d_2 and ρ_2 respectively. We then define

$$\rho_1 = \rho_0 + \Delta\rho_1 \; ; \rho_2 = \rho_0 + \Delta\rho_1 + \Delta\rho_2$$

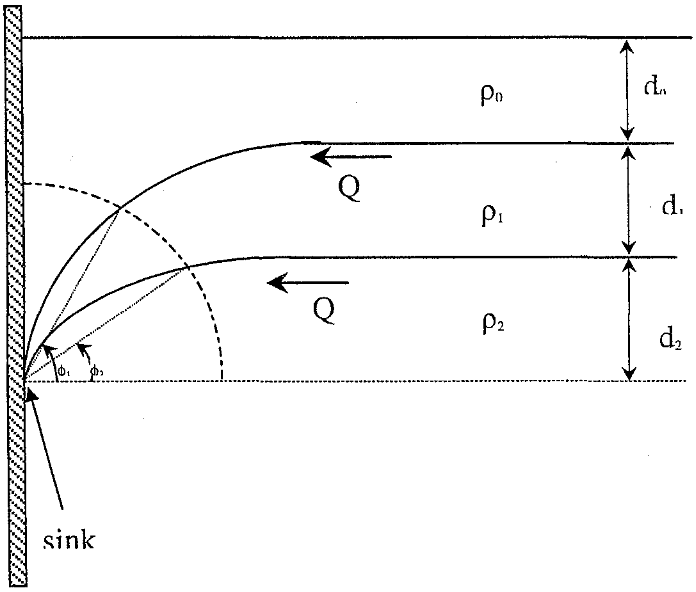

Figure 1: The reservoir with two layers flowing and where the depth of the layers at infinity are d_0 and ρ_0 , d_1 and ρ_1 and d_2 and ρ_2 respectively.

2. The critical discharge for a single layer

Assume that we have an outlet below the lowest sharp interface. If the discharge is small and the flow is assumed irrotational, then the interface is illustrated in Figure 2. In this figure, it is important to note the stagnation point is on the reservoir wall. As the discharge increases, the flow will suddenly change and the flows come from both layers. Craya (1949) assumed the flow was spherically symmetric and determined a critical flow without the stagnation point. In this case, the Bernoulli equation was used

$$\frac{1}{2}\left(\frac{\rho_2}{\Delta\rho_2 g}\right)\left[\frac{Q_2^2}{\pi^2 r^4 (1+\sin\phi_2)^2}\right]+r=d_2 \tag{1}$$

and differentiated with respect to r so as to determine the maximum discharge. Craya showed that the maximum discharge occurs when the depth above the sink is 0.8 times the upstream depth (d_2), and Q_c, the critical discharge[1] at which the upper layer is drawn into the flow was

$$Q_c^2 = 2\pi^2\left(\frac{4}{5}\right)^5 \frac{\Delta\rho_2 g d_2^5}{\rho_2} \tag{2}$$

[1] The critical discharge is defined such that for $Q_{Total} \leq Q_c$, flow is restricted to the lower layer, and for $Q_{Total} > Q_c$, flow is from both layers.

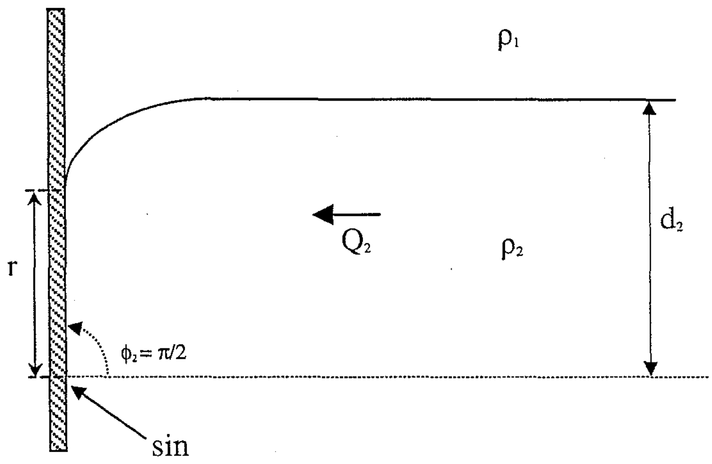

Figure 2: The flow when the discharge is less than the critical discharge for a single layer

Craya's results were verified by Gariel (1949). However, it must be noted that these are difficult experiments since there is always a boundary layer at the interface and this makes the determination of the critical discharge a matter of judgement. Indeed, if we had a truly irrotational fluid, the transition between a flow with a stagnation point through critical discharge to a two-layer discharge would be so fast that it is doubtful that this value could ever be determined.

3. The case when the upper layer is infinite but the discharges come from both layers

When the discharge at the valve exceeds this value then some of the fluid must come from both layers and in this case there is a virtual control that sets the discharge ratio. Wood (1978) modified the spherically symmetric assumption for the two layers in a sector of a sphere and determined the virtual control. Lawrence and Imberger (1979) experimentally showed that the spherically symmetric assumption worked reasonably well for half a sphere and Figure 1 illustrates this case. The Bernoulli equations for the two layers are subtracted and we get

$$\frac{1}{2}\left(\frac{\rho_2}{\Delta\rho_2 g}\right)\left[\frac{Q_2^2}{\pi^2 r^4 (1+\sin\phi_2)^2}\right]-\frac{1}{2}\left(\frac{\rho_2}{\Delta\rho_2 g}\right)\left[\frac{Q_1'^2}{\pi^2 r^4 (1-\sin\phi_2)^2}\right]+r\sin\phi_2 = d_2 \tag{3}$$

where

$$Q_1'^2 = \frac{\rho_1}{\rho_2} Q_1^2 \tag{4}$$

Differentiating (3) with respect to r and defining the Froude numbers

$$Fr_2^2 = \frac{\rho_2}{\Delta\rho_2 g}\left(\frac{Q_2^2}{\pi^2 r^5 (1+\sin\phi_2)^3}\right) \qquad Fr_1^2 = \frac{\rho_2}{\Delta\rho_1 g}\left(\frac{Q_1^2}{\pi^2 r^5 (1-\sin\phi_2)^3}\right)$$

and $\alpha = \dfrac{\Delta\rho_1}{\Delta\rho_2}$

we get

$$r\frac{d\sin\phi_2}{dr} = \frac{2Fr_2^2(1+\sin\phi_2) - \sin\phi_2 - 2\alpha Fr_1^2(1-\sin\phi_2)}{\left(1 - Fr_2^2 - \alpha Fr_1^2\right)} \tag{5}$$

We define the virtual control as the section where the denominator in (5) equals zero

$$1 - Fr_2^2 - \alpha Fr_1^2 = 0 \tag{6}$$

Close to the valve, the Froude numbers are large and the denominator tends to a large negative value; as r tends to infinity, the denominator tends to a value of one (Figure 3).

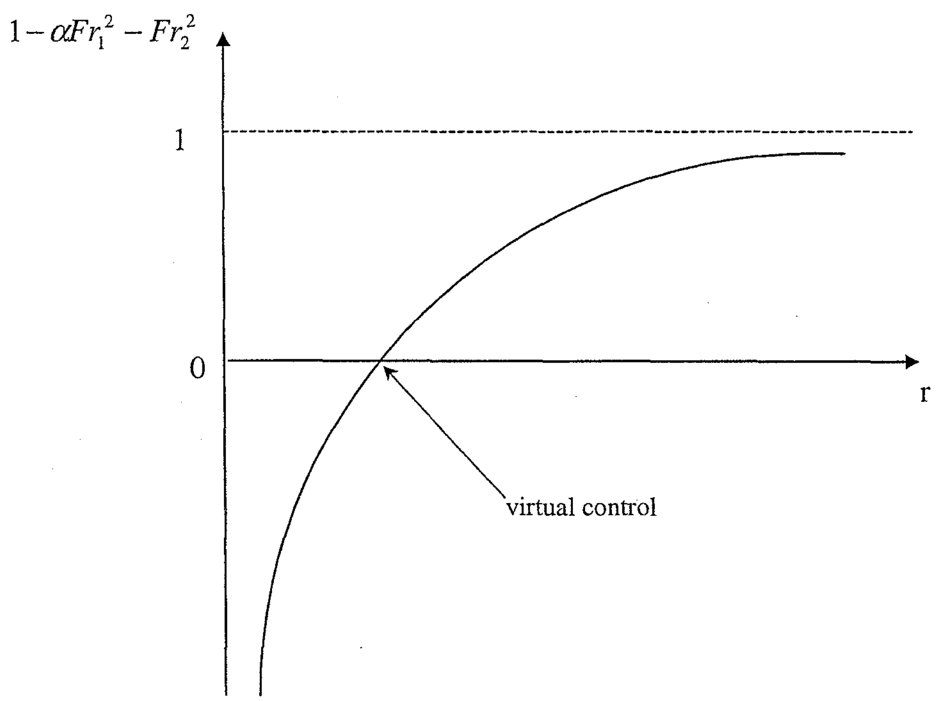

Figure 3: The values of DCC as a function of radius for the closed conduit

In order for the flow to remain determinate, the numerator in (5) must also equal zero at some point. If the subscript v is used to define values at the virtual control, then combining these conditions with the subtracted Bernoulli equations we get

$$\frac{r_v \sin\phi_{2v}}{d_2} = \frac{4}{5} \tag{7}$$

Then

$$\frac{Q_2^2}{Q_c^2} = \frac{1}{8}\left[\frac{(1+\sin\phi_{2v})^3(2-\sin\phi_{2v})}{\sin^5\phi_{2v}}\right] \tag{8}$$

and

$$\frac{Q_1'^2}{Q_c^2} = \frac{1}{8}\left[\frac{(1-\sin\phi_{2v})^3(2+\sin\phi_{2v})}{\sin^5\phi_{2v}}\right] \tag{9}$$

and hence

$$\frac{Q_T}{Q_c} = \frac{Q_2+Q_1'}{Q_c} = \frac{Q_2}{Q_c}\left[1+\frac{(2+\sin\phi_{2v})^{1/2}(1-\sin\phi_{2v})^{3/2}}{(2-\sin\phi_{2v})^{1/2}(1+\sin\phi_{2v})^{3/2}}\right] \tag{10}$$

In a very elegant experiment, Lawrence and Imberger (1979) verified this expression (see Figure 5).

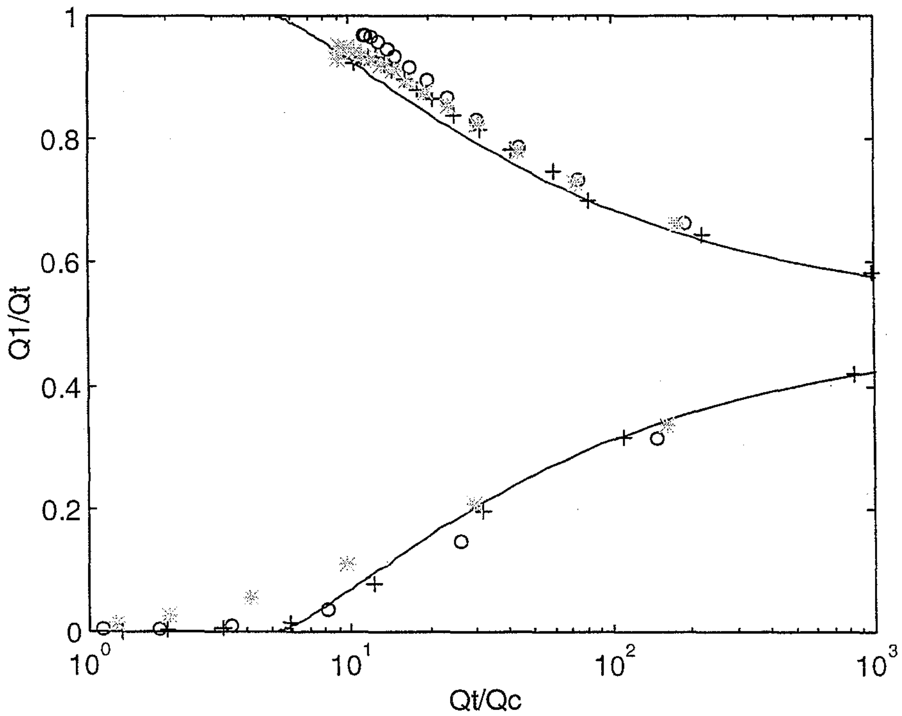

Figure 4: Comparison of experimental results of Lawrence and Imberger (1979) with the calculation of Q_1/Q_t as a function of Q_t/Q_c

4. The determination of the maximum discharge with a finite upper layer

So far we have assumed that the depth of the upper layer (d_1) is infinite. We now discuss the case when the upper layer is finite. We have shown that when the upper layer is infinite, the virtual control determines the density of the outlet discharges. When the upper layer is finite and the portion of the flow is a free surface flow, we must consider the virtual control for the free surface (Wood, 1968). When we take a spherical section in the open channel then the angle to the upper layer (ϕ_1) is a variable (see Figure 1). The Bernoulli equation for the upper layer can be written as

$$\frac{1}{2}\alpha\left(\frac{\rho_1}{\Delta\rho_1 g}\right)\left[\frac{Q_1^2}{\pi^2 r^4(\sin\phi_1-\sin\phi_2)^2}\right]+\alpha r\sin\phi_1=\alpha(d_1+d_2) \qquad (11)$$

The difference between the Bernoulli equations between the two flowing layers is

$$\frac{1}{2}\left(\frac{\rho_2}{\Delta\rho_2 g}\right)\left[\frac{Q_2^2}{\pi^2 r^4(1+\sin\phi_2)^2}\right]-\frac{1}{2}\left(\frac{\rho_1}{\Delta\rho_2 g}\right)\left[\frac{Q_1^2}{\pi^2 r^4(\sin\phi_1-\sin\phi_2)^2}\right]+r\sin\phi_2=d_2 \qquad (12)$$

Adding (11) and (12) we get

$$\frac{1}{2}\left(\frac{\rho_2}{\Delta\rho_2 g}\right)\left[\frac{Q_2^2}{\pi^2 r^4(1+\sin\phi_2)^2}\right]+r\sin\phi_2+\alpha r\sin\phi_1=\alpha(d_1+d_2)+d_2 \qquad (13)$$

Differentiating (11) and (13) with respect to r and defining the Froude numbers

$$Fr_1^2=\frac{\rho_1}{\Delta\rho_1 g}\left(\frac{Q_1^2}{\pi^2 r^5(\sin\phi_1-\sin\phi_2)^3}\right) \qquad Fr_2^2=\frac{\rho_2}{\Delta\rho_2 g}\left(\frac{Q_2^2}{\pi^2 r^5(1+\sin\phi_2)^3}\right)$$

we get

$$\left(1-Fr_1^2\right)r\frac{d\sin\phi_1}{dr}+Fr_1^2 r\frac{d\sin\phi_2}{dr}=2Fr_1^2(\sin\phi_1-\sin\phi_2)-\sin\phi_1 \qquad (14)$$

and

$$\alpha r\frac{d\sin\phi_1}{dr}+\left(1-Fr_2^2\right)r\frac{d\sin\phi_2}{dr}=2Fr_2^2(1+\sin\phi_2)-\alpha\sin\phi_1-\sin\phi_2 \qquad (15)$$

In the solution to the differentials (14) and (15), the denominator is given by the determinant, DOC=AD-BC, where

$$A = 1 - Fr_1^2;\ B = Fr_1^2;\ C = \alpha;\ D = 1 - Fr_2^2$$

and

$$DOC = 1 - \alpha Fr_1^2 - Fr_2^2 - Fr_1^2\left(1 - Fr_2^2\right) \tag{16}$$

When DOC equals zero, we then have a virtual control in the open channel. Now

$$DOC = DCC - Fr_1^2\left(1 - Fr_2^2\right) = -Fr_1^2\left(1 - Fr_2^2\right) \tag{17}$$

To ensure that the problem is not over-specified, there can only be one virtual control. In order to determine whether the virtual control lies in the closed conduit or the open channel, we need to investigate the nature of DOC at the point of closed conduit virtual control (i.e. DCC=0). Consider the case when the valve is turned on slowly. After the critical discharge, the virtual control is in the duct. When this is the case, we get

$$DCC = 1 - Fr_2^2 - \alpha Fr_1^2 = 0 \tag{18}$$

At this point,

$$DOC = -Fr_1^2\left(1 - Fr_2^2\right) = -Fr_1^2 \alpha Fr_1^2 \tag{19}$$

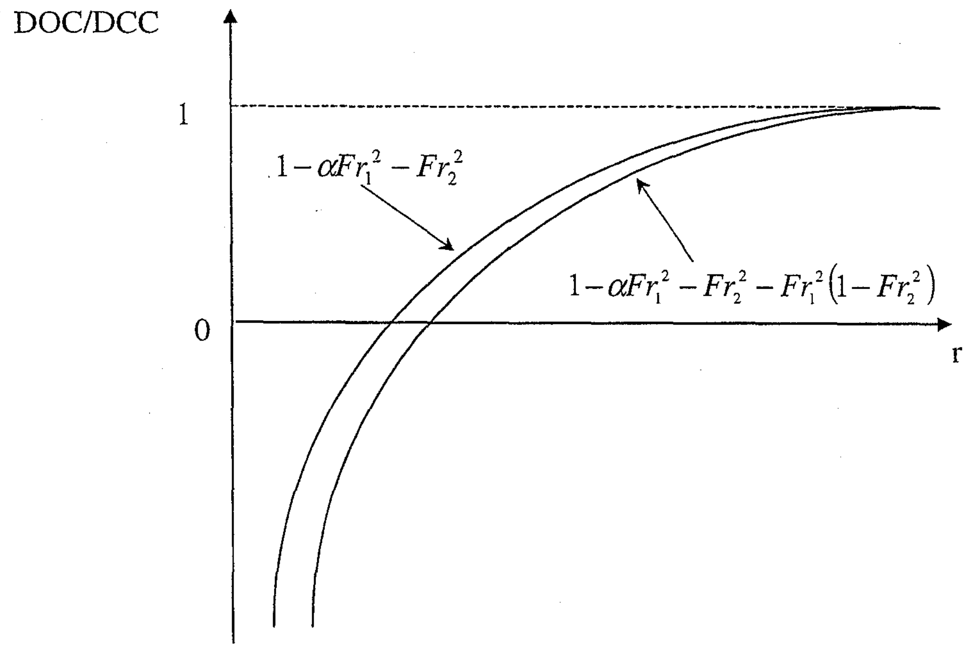

Figure 5: The comparison of values of virtual control in the closed conduit (DCC) and the virtual control in the open channel (DOC) with the radius

Thus the value of DOC at the virtual control in the closed conduit is negative. Hence for all cases, the closed conduit virtual control occurs first. Further, since there can only be one control (otherwise the problem is over-specified) the control must lie in the closed conduit. Figure 5 shows the variation for the value of DCC and DOC. In order to have a completely specified steady flow there must be only one control. For all flows we start at this closed conduit control and compute a backwater curve until we reach a value when DOC just turns positive. At this point, we use the difference Bernoulli equation for the open channel case to calculate the value of the ratio of the upstream depths of the two layers. In some cases, this value is negative and for this case, we proceed further in the closed conduit until the value becomes positive. Figure 6 shows the variation of the ratio d_1/d_2 as a function of Q_T/Q_c as a result of this calculation. This implies that when at a particular value of d_1/d_2, the maximum of Q_T/Q_c is calculated and if for that particular value of Q_T/Q_c is exceeded, then there will be a flow from layer 0. As with the case of a single layer flowing, there is a problem in observing the transition of the flow from two layers to three layers. Before the three-layer flow is initiated, there would be a stagnation point on the reservoir wall and this would suddenly change as the three-layer flow commences.

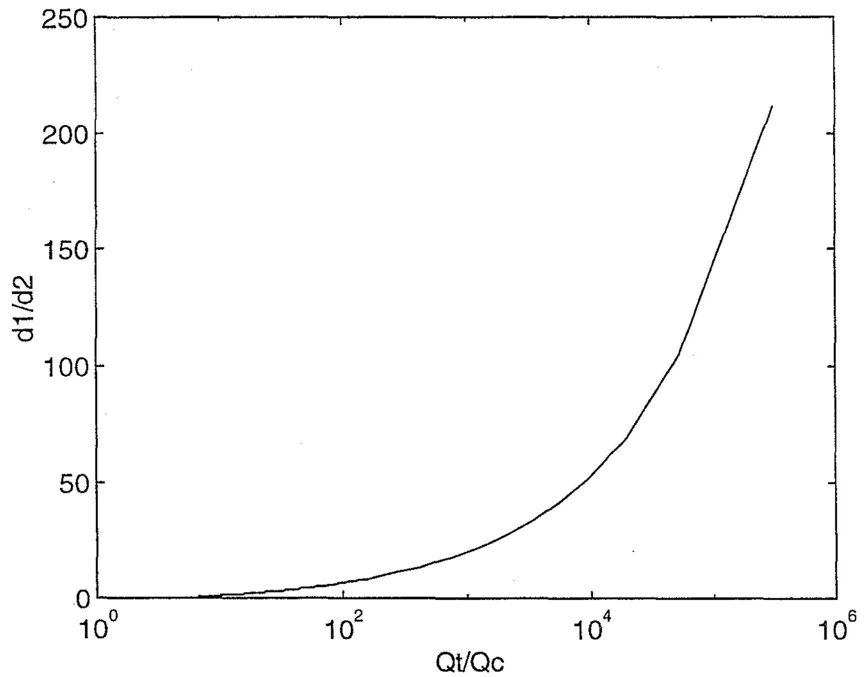

Figure 6: The calculation of d_1/d_2 as a function of Q_T/Q_c

5. Conclusions

When we have two layers of finite depth in a reservoir and the discharge exceeds the critical discharge, the ratio of the discharges from each layer is determined by the conditions at the virtual control. However, with a finite depth above the lower layer, there are two conditions for the virtual control: the first condition using the duct equations (DCC=0), and the second using the free surface equations (DOC=0). When the discharge just exceeds the critical discharge, we have shown that at the virtual control, the value of DOC is negative when DCC=0. As the discharge increases, the virtual control moves further away from the sink, but in all cases, the control is in the duct. For a set steady discharge, there can only be one virtual control and it is assumed that the transition between the closed conduit flow and the open channel cannot take place until DOC is positive. This contrasts with the finding of Bryant and Wood (1976). Further experiments are required.

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