GENERALIZED $\Delta - Y$ EXCHANGE AND k–REGULAR MATROIDS

JAMES OXLEY, CHARLES SEMPLE, AND DIRK VERTIGAN

Abstract. This paper introduces a generalization of the matroid operation of $\Delta - Y$ exchange. This new operation, segment-cosegment exchange, replaces a coindependent set of k collinear points in a matroid by an independent set of k points that are collinear in the dual of the resulting matroid. The main theorem of the first half of the paper is that, for every field, or indeed partial field, $\mathbf{F}$, the class of matroids representable over $\mathbf{F}$ is closed under segment-cosegment exchanges. It follows that, for all prime powers q, the set of excluded minors for $GF(q)$–representability has at least 2^{q-4} members. In the second half of the paper, the operation of segment-cosegment exchange is shown to play a fundamental role in an excluded-minor result for k–regular matroids, where such matroids generalize regular matroids and Whittle's near-regular matroids.

1. Introduction

The class of regular matroids is one of the best-known and most frequently studied classes of matroids. Moreover, this class plays a fundamental role within the class of binary matroids. The corresponding subclass of the class of ternary matroids is the class of near-regular matroids introduced by Whittle [28]. Consider $\mathbf{Q}(\alpha)$, the extension of the field $\mathbf{Q}$ by a transcendental α. A *near-regular matroid* is one that is representable over $\mathbf{Q}(\alpha)$ by a *near-unimodular matrix*, that is, a matrix all of whose non-zero subdeterminants are members of $\{\pm\alpha^i(\alpha-1)^j : i, j \in \mathbf{Z}\}$. An important result of Whittle [29] shows that, just as the class of regular matroids coincides with the class of matroids representable over all fields of size at least two, the class of near-regular matroids coincides with the class of matroids representable over all fields of size at least three. The importance of the classes of regular and near-regular matroids motivated Semple [16] to introduce the following class. For all non-negative integers k, a *k–regular matroid* is one that can be represented over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are algebraically independent transcendentals, by a *k–unimodular matrix*. The latter is a matrix for which every non-zero subdeterminant is in the set $\mathcal{A}_k$ that consists of all products of integer powers of differences of distinct members of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. Evidently, the classes of 0–regular and 1–regular matroids coincide with the classes of regular and near-regular matroids, respectively. Moreover, if $k' \leq k$, then the class of k'–regular matroids is a subset of the class of k–regular matroids. A matroid is called *ω–regular* if it is

Date: September 30, 1998.
1991 *Mathematics Subject Classification.* 05B35.

k–regular for some $k \geq 0$, and a matrix is ω–*unimodular* if it is k–unimodular for some $k \geq 0$.

It was noted above that, for k in $\{0, 1\}$, the set of k–regular matroids coincides with the class of matroids representable over all fields with at least $k+2$ elements. Regrettably this result is not true for any $k \geq 2$. Indeed, Semple [17] showed that the matroid that is obtained by freely adding a point on a 2-point line of $M(K_4)$ is representable over all fields of size at least four but is not k–regular for any k. However, it is easy to see that, for all k, the class of k–regular matroids is contained in the class of matroids representable over all fields of size at least $k+2$. Moreover, it is hoped that, for all prime powers q, the class of $(q-2)$–regular matroids will play the same role within the class of $GF(q)$–representable matroids that the classes of regular and near-regular matroids play within the classes of binary and ternary matroids. To explain some of the grounds for this hope, we shall need the following notion. A partial field $\mathbf{P}$ is a structure that behaves very much like a field except that addition may be a partial operation. More precisely, Vertigan [27] has shown that every partial field $\mathbf{P}$ can be obtained from a commutative ring R and a multiplicative group G of units of R for which $-1 \in G$. The *partial field* $\mathbf{P}$ associated with the pair (G, R) has $G \cup \{0\}$ as its set of elements, and has the binary operations of addition and multiplication restricted from R to $G \cup \{0\}$. Thus multiplication is a total binary operation, but addition is a partial binary operation; that is, if a and b are elements of $G \cup \{0\}$, then their product ab is always in $G \cup \{0\}$, but their sum $a + b$ need not be, in which case it is *undefined*. Partial fields were introduced in [19] where it was shown that one can develop a theory of matroid representation for them. Numerous properties of matroids representable over fields hold in the more general setting of partial fields and a number of natural classes of matroids can be characterized as classes of matroids representable over a fixed partial field. In particular, if $\mathcal{M}(\mathbf{P})$ is the class of matroids representable over the partial field $\mathbf{P}$, then $\mathcal{M}(\mathbf{P})$ is closed under the taking of duals, minors, direct sums, and 2–sums. Now the set $\mathcal{A}_k$ defined in the last paragraph is a subgroup of the multiplicative group of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$, and $-1 \in \mathcal{A}_k$, so there is a partial field $\mathbf{R}_k$ associated with the pair $(\mathcal{A}_k, \mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k))$. Furthermore, the class of k–regular matroids is precisely the class of matroids representable over $\mathbf{R}_k$ [16]. Semple and Whittle [19] showed that, for all partial fields $\mathbf{P}$, the matroid $U_{2,3} \in \mathcal{M}(\mathbf{P})$ if and only if $\mathcal{M}(\mathbf{P})$ contains the class of regular matroids, while $U_{2,4} \in \mathcal{M}(\mathbf{P})$ if and only if $\mathcal{M}(\mathbf{P})$ contains the class of near-regular matroids. One indication of the significance of k–regular matroids was provided by Semple [17] when he generalized the last two results by proving that $U_{2,k+3} \in \mathcal{M}(\mathbf{P})$ if and only if $\mathcal{M}(\mathbf{P})$ contains the class of k–regular matroids.

Since the class of k–regular matroids is minor-closed, a natural problem is to determine the set of excluded minors for this class. However, this problem seems very difficult. Tutte [24] showed that the set of excluded minors for the class of 0–regular, that is, regular, matroids is $\{U_{2,4}, F_7, F_7^*\}$. Recently, Geelen [6] has determined the set of excluded minors for the class of 1–regular matroids, that is, near-regular matroids, but his argument is hard and is based on the new techniques that were developed by Geelen, Gerards, and Kapoor [7] for finding the excluded minors for the class of quaternary matroids. Indeed, it appears unrealistic with currently available techniques to expect to find an explicit determination of all

the excluded minors for the class of k–regular matroids. The problem that we shall attack and, indeed, solve in this paper is to give an excluded-minor characterization of the class of k–regular matroids *within the class of ω–regular matroids*. The results of Tutte [24] and Geelen [6] noted above imply that an ω–regular matroid is 0–regular if and only if it has no minor isomorphic to $U_{2,4}$; and an ω–regular matroid is 1–regular if and only if it has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$. These two results may suggest that, for arbitrary k, the set of ω–regular excluded minors for the class of k–regular matroids consists only of uniform matroids. But this is not true even for $k = 2$. What is true, however, is that the non-uniform matroids in this set of excluded minors can be constructed from uniform matroids, indeed from lines, in a very predictable way, which we shall describe next. This construction seems both attractive and natural, and we think it will be of independent interest.

The operations of $\Delta - Y$ and $Y - \Delta$ exchange are of basic importance in graph theory. For matroids, these operations are defined in terms of the generalized parallel connection [4]. Let M_1 and M_2 be matroids such that $M_1|T = M_2|T$, where $T = E(M_1) \cap E(M_2)$. Let $N = M_1|T$ and suppose that T is a modular flat of M_1. The *generalized parallel connection* $P_N(M_1, M_2)$ of M_1 and M_2 across N is the matroid on $E(M_1) \cup E(M_2)$ whose flats are those subsets X of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_1)$ is a flat of M_1, and $X \cap E(M_2)$ is a flat of M_2. In the case when $M_1 \cong M(K_4)$ and N is a triangle T of this matroid, Akkari and Oxley [1] defined a $\Delta - Y$ *exchange* on M across T to be the matroid that is obtained from $P_N(M(K_4), M_2)$ by deleting T. Moreover, they proved that, for all fields $\mathbf{F}$ of size at least three, the set of excluded minors for $\mathbf{F}$–representability is closed under this operation.

To motivate our generalization of $\Delta - Y$ exchange, consider the following construction. In $PG(2, \mathbf{R})$, take a basis $\{b_1, b_2, b_3\}$ and a line L that is freely placed relative to this basis. By modularity, for each i in $\{1, 2, 3\}$, the hyperplane of $PG(2, \mathbf{R})$ that is spanned by $\{b_1, b_2, b_3\} - \{b_i\}$ meets L. Let a_i be the point of intersection. We shall denote by Θ_3 the restriction of $PG(2, \mathbf{R})$ to $\{b_1, b_2, b_3, a_1, a_2, a_3\}$. Clearly Θ_3 is isomorphic to $M(K_4)$ and has $\{a_1, a_2, a_3\}$ as a modular line. To generalize this construction, suppose that $k \geq 3$ and let $\{b_1, b_2, \ldots, b_k\}$ be a basis B of $PG(k-1, \mathbf{R})$ and let L be a line that is freely placed relative to B. As before, modularity implies that, for each i, the hyperplane of $PG(k-1, \mathbf{R})$ that is spanned by $\{b_1, b_2, \ldots, b_k\} - \{b_i\}$ meets L, and we let a_i be the point of intersection. Then Θ_k will denote the restriction of $PG(k-1, \mathbf{R})$ to $\{b_1, b_2, \ldots, b_k, a_1, a_2, \ldots, a_k\}$. Let $A = \{a_1, a_2, \ldots, a_k\}$. By construction, each hyperplane of Θ_k meets the line A, so this line is modular. It follows that if M is a matroid having a k–point line as a restriction and the points of this line are labelled by the elements of A, then the matroid $P_A(\Theta_k, M)$ is well-defined. Hence so too is $P_A(\Theta_k, M)\backslash A$. However, the restriction of the dual of the last matroid to B need not be isomorphic to a k–point line. The condition that one needs to ensure that $[P_A(\Theta_k, M)\backslash A]^*|B \cong U_{2,k}$ is precisely that A is coindependent in M. When this extra condition holds, we define $\Delta_A(M)$ to be $P_A(\Theta_k, M)\backslash A$. Moreover, this extra condition is needed in the following theorem. This theorem, the main result of the first half of the paper, generalizes the result of Akkari and Oxley noted above.

Theorem 1.1. *Let $\mathbf{P}$ be a partial field and M be an excluded minor for the class $\mathcal{M}(\mathbf{P})$ of matroids representable over $\mathbf{P}$. Let A be a subset of $E(M)$ such that $M|A$ is isomorphic to a rank–2 uniform matroid and A is coindependent in M. Then $\Delta_A(M)$ is an excluded minor for $\mathcal{M}(\mathbf{P})$.*

Since every partial field is a field, we obtain, as a straightforward consequence of this theorem, the following exponential lower bound on the number of excluded minors for representability over $GF(q)$.

Theorem 1.2. *For all prime powers q, the set of excluded minors for the class of $GF(q)$–representable matroids has at least 2^{q-4} distinct members.*

In the second half of the paper, we solve the problem of determining precisely which ω–regular matroids are k–regular. We shall denote by $\mathcal{U}_n$ the class of all n–element uniform matroids of rank and corank at least three. Note that $\mathcal{U}_n$ is empty when $n < 6$. The operation Δ_A is called *segment-cosegment exchange* and the dual operation ∇_A is called *cosegment-segment exchange*. The latter is defined as follows. Let M be a matroid having an independent set A such that $M^*|A$ is uniform of rank two. Then $\nabla_A(M) = [P_A(\Theta_k, M^*)\backslash A]^*$.

Theorem 1.3. *Let k be a non-negative integer and M be an ω–regular matroid. Then M is k–regular if and only if it has no minor isomorphic to a member of $\mathcal{U}_{k+4}$ or any matroid that can be obtained from $U_{2,k+4}$ by a sequence of segment-cosegment and cosegment-segment exchanges.*

Unique representability results are very powerful tools in matroid representation theory. Indeed, it is no coincidence that the only finite fields $GF(q)$ for which the sets of excluded minors have been completely determined are those over which every 3–connected $GF(q)$–representable matroid is uniquely representable [2, 5, 7, 10, 21, 24]. Theorem 1.3 is the main tool in the proof of the following unique representability result for k–regular matroids.

Theorem 1.4. *Let $k \geq 0$ and let M be a 3–connected k–regular matroid. Then all ω–unimodular representations of M are equivalent.*

The matroid terminology in this paper will follow Oxley [15]. The plan of the paper is as follows. In Section 2, the operation of segment-cosegment exchange is more formally defined and numerous properties of this and its dual operation are obtained. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we consider the class Λ_m of matroids that can be obtained from an m–point line by a sequence of segment-cosegment and cosegment-segment exchanges. We introduce a way of describing each such matroid via a vertex-labelled tree, and develop properties of the members of Λ_m by using these "del-con" trees. Theorems 1.3 and 1.4 are proved in Section 5. Finally, Section 6 answers a question of Oxley [14] by showing that the union of the classes Λ_m for all $m \geq 4$ is precisely the set of 3–connected matroids for which every 3-connected minor of rank and corank three is isomorphic to P_6, the matroid that is obtained from a 6-point line by a single $\Delta - Y$ exchange.

2. Generalized $\Delta - Y$ Exchange

In this section, we give a more formal definition of the operation of segment-cosegment exchange and we establish a number of properties of this operation and its dual. We begin by discussing the matroid Θ_k that plays the same role in the generalized operation to that played by $M(K_4)$ in the $\Delta - Y$ exchange.

For $k \geq 4$, let $\alpha_1, \alpha_2, \ldots, \alpha_{k-3}$ be algebraically independent transcendentals over $\mathbf{Q}$, and let Θ_k be the matroid that is represented over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$ by the matrix $[I_k|D_k]$, where D_k is the matrix

$$\begin{array}{c} \\ a_1 \\ a_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \\ b_k \end{array} \begin{array}{c} \begin{array}{ccccccc} b_1 & b_2 & a_3 & a_4 & a_5 & \cdots & a_k \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{k-3} \\ 1 & 1 & 0 & 0 & 0 & & 0 \\ 1 & \alpha_1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & \alpha_2 & 0 & 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \vdots \\ 1 & \alpha_{k-3} & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}.$$

Let Θ_2 and Θ_3 be the matroids represented over the rationals by the matrices $[I_2|D_2]$ and $[I_3|D_3]$, respectively, where D_2 and D_3 are the matrices

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \end{array} \text{ and } \begin{array}{c} \\ a_1 \\ a_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & a_3 \end{array} \\ \left[\begin{array}{ccc} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right] \end{array}.$$

Thus Θ_2 is isomorphic to the matroid obtained from $U_{2,2}$ by adding exactly one element in parallel with each member of the ground set of $U_{2,2}$, and Θ_3 is isomorphic to $M(K_4)$. Evidently, for all $k \geq 2$, the ground set of Θ_k equals $A \cup B$ where $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$.

The first lemma is easily deduced by looking at $[-D_k^T|I_k]$, a canonical representation of Θ_k^*, and scaling appropriate rows and columns.

Lemma 2.1. *For all $k \geq 2$, the matroid Θ_k is self-dual. In particular, $\Theta_k^* \cong \Theta_k$ under the map that interchanges a_i and b_i for all i.*

In order to describe the structural properties of Θ_k, it will be helpful to list its circuits.

Lemma 2.2. *For all $k \geq 2$, the collection of circuits of Θ_k consists of the following sets:*

(i) *all 3-element subsets of A;*
(ii) *all sets of the form $(B - b_i) \cup a_i$ for which $i \in \{1, 2, \ldots, k\}$; and*

(iii) *all sets of the form* $(B - b_u) \cup \{a_s, a_t\}$ *for which* s, t, *and* u *are distinct elements of* $\{1, 2, \ldots, k\}$.

Proof. The lemma is easily checked when $k = 2$. Now assume that $k \geq 3$. We show next that if σ is the permutation $(2, 3, \ldots, k, 1)$ of $\{1, 2, \ldots, k\}$, then the map that, for all i takes a_i and b_i to $a_{\sigma(i)}$ and $b_{\sigma(i)}$, respectively, is an automorphism of Θ_k. To see this, begin with the matrix $[I_k|D_k]$ as labelled above. Pivot on the $(1, 3)$-entry of D_k and then on the $(3, 1)$-entry of the resulting matrix, where each pivot includes the natural column interchange to return the matrix to standard form $[I_k|X]$. Next interchange the first two rows of the current matrix, and then interchange column 1 with column 2, and column $k + 1$ with column $k + 2$. After rescaling rows and columns, the resulting matrix is $[I_k|D'_k]$ where D'_k is

$$\begin{array}{c} \\ a_2 \\ a_3 \\ b_4 \\ b_5 \\ \vdots \\ b_k \\ b_1 \end{array} \begin{array}{c} \begin{array}{ccccccc} b_2 & b_3 & a_4 & a_5 & \cdots & a_k & a_1 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 1 & \frac{1-\alpha_1}{1-\alpha_2} & \cdots & \frac{1-\alpha_1}{1-\alpha_{k-3}} & 1-\alpha_1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \frac{1-\alpha_1}{1-\alpha_2} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \frac{1-\alpha_1}{1-\alpha_{k-3}} & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1-\alpha_1 & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \end{array}.$$

Now an immediate consequence of [16, Theorem 7] is that there is an automorphism φ of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$ such that, for all $i \in \{1, 2, \ldots, k-4\}$, $\varphi(\alpha_i) = \frac{1-\alpha_1}{1-\alpha_{i+1}}$ and $\varphi(\alpha_{k-3}) = 1 - \alpha_1$. Thus $[I_k|D'_k]$ can also be obtained from $[I_k|D_k]$ by applying an automorphism of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$ to each of its entries. It follows that Θ_k does indeed have the permutation $(b_2, b_3, \ldots, b_k, b_1)(a_2, a_3, \ldots, a_k, a_1)$ as an automorphism.

It is clear that every 3-element subset of A is a circuit of Θ_k. Hence, by Lemma 2.1, every 3-element subset of B is a cocircuit of Θ_k. It follows, by orthogonality, that every circuit of Θ_k that meets B contains at least $|B| - 1$ elements of B. From considering the matrix $[I_k|D_k]$, we deduce that $(B - b_1) \cup a_1$ is the unique k-element circuit of Θ_k containing $B - b_1$. Thus, by the symmetry noted above, $(B - b_i) \cup a_i$ is a circuit of Θ_k for all i.

All remaining circuits of Θ_k must have $k + 1$ elements and must contain exactly $k - 1$ elements of B. Thus it suffices to determine all such circuits containing $B - b_1$ and avoiding b_1. But, for every such circuit C, the set $C - \{b_3, b_4, \ldots, b_k\}$ is a circuit of $\Theta_k/\{b_3, b_4, \ldots, b_k\}\backslash b_1$ containing b_2. The last matroid is obtained from a k-point line on A by adding b_2 in parallel with a_1. To see this, observe what happens to $[I_k|D_k]$ when, for all $j \in \{3, 4, \ldots, k\}$, the j-th column and j-th row are deleted. The 3-element circuits of $\Theta_k/\{b_3, b_4, \ldots, b_k\}\backslash b_1$ containing b_2 consist of all sets of the form $\{b_2, a_s, a_t\}$ where s and t are distinct elements of $\{2, 3, \ldots, k\}$. Thus, for all such s and t, the set $\{b_2, a_s, a_t\} \cup \{b_3, b_4, \ldots, b_k\}$ contains a circuit of Θ_k. Since we have already identified all non-spanning circuits of Θ_k and none of

these is contained in the last set, we deduce that the last set itself is a circuit of Θ_k, and the lemma follows. □

The following is an immediate consequence of the last lemma.

Corollary 2.3. *For all $k \geq 2$ and all permutations σ of $\{1, 2, \ldots, k\}$, the map that, for all i, takes a_i and b_i to $a_{\sigma(i)}$ and $b_{\sigma(i)}$, respectively, is an automorphism of Θ_k.*

On combining Lemmas 2.1 and 2.2, we see that, geometrically, Θ_k can be obtained from a free matroid $U_{k,k}$ by adding a point to each hyperplane of the latter so that each of these hyperplanes becomes a circuit in the resulting matroid and so that the restriction of Θ_k to the set of added points is a k–point line. This is essentially the way that we described Θ_k in the introduction, and it is not difficult to see that these different descriptions of Θ_k are equivalent.

The operation of generalized parallel connection of two matroids relies on the presence of a modular flat in one of the matroids. Recall that a flat F of a matroid M is *modular* if $r(F) + r(F') = r(F \cup F') + r(F \cap F')$ for all flats F' of M.

Lemma 2.4. *For all $k \geq 2$, the set A is a rank–2 modular flat of Θ_k, and B is a basis of Θ_k.*

Proof. It is clear from Lemma 2.2 that A is a rank–2 flat and B is a basis of Θ_k. Now A is a modular flat of Θ_k if and only if $r(A) + r(F) = r(\Theta_k)$ for all flats F avoiding A such that $F \cup A$ spans Θ_k [4, Theorem 3.3] (see also [15, Proposition 6.9.2 (iii)]). For every such flat, $r(F) \geq r(\Theta_k) - 2$. If $r(F) = r(\Theta_k) - 2$, then, certainly, $r(A) + r(F) = r(\Theta_k)$. Moreover, by Lemmas 2.1 and 2.2, every hyperplane of Θ_k meets A. We deduce that A is indeed a modular flat of Θ_k. □

Now let M be a matroid such that M has a $U_{2,k}$–restriction. Label the elements of this restriction $a_1, a_2, \ldots, a_k$. As before, let $A = \{a_1, a_2, \ldots, a_k\}$. By Lemma 2.4, A is a modular line of Θ_k. Thus the generalized parallel connection $P_A(\Theta_k, M)$ of Θ_k and M across A exists. Hence the matroid $P_A(\Theta_k, M)\backslash A$ is certainly defined. If $|A| = 2$, then $P_A(\Theta_k, M)\backslash A$ is obtained from M by adding an element in parallel with each of the elements of A and then deleting the elements of A. Thus $P_A(\Theta_2, M)\backslash A \cong M$. If $|A| = 3$, then, since $\Theta_3 \cong M(K_4)$, the matroid $P_A(\Theta_3, M)\backslash A$ is exactly the matroid that is obtained by performing a $\Delta - Y$ exchange on M at A. While such a $\Delta - Y$ exchange is defined as long as A is a triangle of M, the set B will be a triad in $P_A(\Theta_3, M)\backslash A$ only if A is coindependent in M. Indeed, the following extension of this observation is straightforward to prove.

Lemma 2.5. *For all $k \geq 2$, the restriction of $(P_A(\Theta_k, M)\backslash A)^*$ to B is isomorphic to $U_{2,k}$ if and only if A is coindependent in M.*

Since we should like an operation whose inverse is the dual of the original operation, in defining this operation we shall impose the additional condition that A is coindependent in M. Thus let M be a matroid having a $U_{2,k}$ restriction on the set A and suppose that A is coindependent in M. We recall that $\Delta_A(M)$ is

defined to be $P_A(\Theta_k, M)\backslash A$. We call this operation a Δ_A*–exchange* or a *segment-cosegment exchange on* A. As $|A| = k$, such an operation will also be referred to as a Δ_k*–exchange* or a *segment-cosegment exchange of size* k. Thus, for example, the matroid $U_{4,6}$ can be obtained from $U_{2,6}$ by a segment-cosegment exchange of size 4.

In defining the dual operation of segment-cosegment exchange, we mimic the definition of $Y - \Delta$ exchange in terms of $\Delta - Y$ exchange or, indeed, the definition of contraction in terms of deletion. Let M be a matroid for which M^* has a $U_{2,k}$–restriction on the set A. If A is independent in M, then $\nabla_A(M)$ is defined to be $(\Delta_A(M^*))^*$, that is, $[P_A(\Theta_k, M^*)\backslash A]^*$. This operation is called a ∇_A*–exchange* or a *cosegment-segment exchange on* A. As $|A| = k$, the operation will also be referred to as a ∇_k*–exchange* or a *cosegment-segment exchange of size* k.

Lemma 2.6. *If* $|A| = k$*, then*

$$r(\Delta_A(M)) = r(M) + k - 2.$$

Proof. Now

$$r(P_A(\Theta_k, M)) = r(\Theta_k) + r(M) - r(A)$$

[4, Proposition 5.5] (see also [15, p. 418]). Since A is coindependent in M, it is coindependent in $P_A(\Theta_k, M)$. Thus $r(P_A(\Theta_k, M)) = r(\Delta_A(M)) = k + r(M) - 2$. □

The next lemma determines the bases of $\Delta_A(M)$ in terms of the bases for M. Recall that $E(\Theta_k) - A = B$, and B is a basis for Θ_k.

Lemma 2.7. *A subset* D *of* $E(\Delta_A(M))$ *is a basis of* $\Delta_A(M)$ *if and only if* D *satisfies one of the following:*

(i) D *contains* B*, and* $D - B$ *is a basis for* M/A*;*
(ii) $D \cap B = B - b_i$ *for some* i *in* $\{1, 2, \ldots, k\}$*, and* $D - (B - b_i)$ *is a basis of* $M/a_i\backslash(A - a_i)$*; or*
(iii) $D \cap B = B - \{b_i, b_j\}$ *for some distinct elements* i *and* j *of* $\{1, 2, \ldots, k\}$*, and* $D - (B - \{b_i, b_j\})$ *is a basis of* $M\backslash A$*.*

Proof. By Lemma 2.6, $r(\Delta_A(M)) = r(M) + k - 2$, where $k = |A|$, and therefore every basis of $\Delta_A(M)$ must contain at least $k - 2$ elements of B. First assume that D contains B. Then D is a basis of $\Delta_A(M)$ if and only if $D - B$ is a basis of $\Delta_A(M)/B$. Since B spans Θ_k in $P_A(\Theta_k, M)$, it is not difficult to show that $\Delta_A(M)/B = M/A$. Therefore D is a basis of $\Delta_A(M)$ containing B if and only if $D - B$ is a basis of M/A.

Now assume that D contains exactly $k - 1$ elements of B. Let $D \cap B = B - b_i$, where $i \in \{1, 2, \ldots, k\}$. Then D is a basis for $\Delta_A(M)$ if and only if $D - (B - b_i)$ is a basis for $\Delta_A(M)/(B - b_i)\backslash b_i$. By Lemma 2.2, $B - b_i$ spans a unique element a_i of A in $P_A(\Theta_k, M)$. Therefore $\Delta_A(M)/(B - b_i)\backslash b_i = M/a_i\backslash(A - a_i)$. Thus D is a basis of $\Delta_A(M)$ containing $B - b_i$ if and only if $D - (B - b_i)$ is a basis of $M/a_i\backslash(A - a_i)$.

Lastly, assume that D contains exactly $k - 2$ elements of B. Let $D \cap B = B - \{b_i, b_j\}$, where i and j are distinct elements of $\{1, 2, \ldots, k\}$. Then D is a basis of

$\Delta_A(M)$ if and only if $D-(B-\{b_i,b_j\})$ is a basis of $\Delta_A(M)/(B-\{b_i,b_j\})\backslash\{b_i,b_j\}$. From considering the representation $[I_k|D_k]$ of Θ_k and using Corollary 2.3, we deduce that $\Theta_k/(B-\{b_i,b_j\})$ is equal to the matroid that is obtained from $\Theta_k|A$ by placing b_i and b_j in parallel with a_j and a_i, respectively. Therefore, by [15, Proposition 12.4.14],

$$P_A(\Theta_k,M)/(B-\{b_i,b_j\}) = P_A(\Theta_k/(B-\{b_i,b_j\}),M).$$

Thus $\Delta_A(M)/(B-\{b_i,b_j\})\backslash\{b_i,b_j\} = M\backslash A$. Hence D is a basis of $\Delta_A(M)$ containing $B-\{b_i,b_j\}$ if and only if $D-(B-\{b_i,b_j\})$ is a basis of $M\backslash A$. □

A natural way of preserving the ground set of M in $\Delta_A(M)$ is by relabelling b_i with a_i, for all i in $\{1,2,\ldots,k\}$. For the rest of the paper, we adopt this convention to preserve the ground set of a matroid under both Δ_k– and ∇_k–exchanges.

Lemma 2.8. (i) *If $\Delta_A(M)$ is defined, then $\Delta_A(M)\backslash A = M\backslash A$ and $\Delta_A(M)/A = M/A$. Moreover, $\Delta_A(M)\backslash a_i/(A-a_i) = M/a_i\backslash(A-a_i)$ for all a_i in A.*

(ii) *If $\nabla_A(M)$ is defined, then $\nabla_A(M)\backslash A = M\backslash A$ and $\nabla_A(M)/A = M/A$. Moreover, $\nabla_A(M)/a_i\backslash(A-a_i) = M\backslash a_i/(A-a_i)$ for all a_i in A.*

Proof. It is clear that (ii) follows from (i) by duality. The first two assertions of (i) are straightforward to check. Moreover, the last follows from (ii) of the previous lemma. □

The next lemma simply restates Lemma 2.7 under the convention that M and $\Delta_A(M)$ have the same ground sets.

Lemma 2.9. *Let $\Delta_A(M)$ be the matroid with ground set $E(M)$ that is obtained from M by a Δ_A–exchange. Then a subset of $E(M)$ is a basis of $\Delta_A(M)$ if and only if it is a member of one of the following sets:*

(i) $\{A\cup B' : B' \text{ is a basis of } M/A\}$;
(ii) $\{(A-a_i)\cup B'' : 1\le i\le k \text{ and } B'' \text{ is a basis of } M/a_i\backslash(A-a_i)\}$; *or*
(iii) $\{(A-\{a_i,a_j\})\cup B''' : 1\le i<j\le k \text{ and } B''' \text{ is a basis of } M\backslash A\}$.

We shall classify each base of $\Delta_A(M)$ as being of type (i), (ii), or (iii) depending on which of the three sets in the last lemma contains the base. The remaining results in this section not only show some of the attractive properties of Δ_k– and ∇_k–exchanges but are also needed for the proofs of the main theorems of the paper. The proofs of these results make frequent use of Lemma 2.9. In particular, the first such result follows straightforwardly from that lemma, and its proof is omitted.

Lemma 2.10. *Let A be a coindependent set in a matroid M such that every 3-element subset of A is a triangle.*

(i) *If X is a subset of $E(M)$ avoiding A, then e is in the closure of X in M if and only if e is in the closure of X in $\Delta_A(M)$.*
(ii) *If $\{e,f\}$ is a cocircuit of M, then $\{e,f\}$ is a cocircuit of $\Delta_A(M)$. Conversely, if $\{e,f\}$ is a cocircuit of $\Delta_A(M)$ avoiding A, then $\{e,f\}$ is a cocircuit of M.*

Lemma 2.11. *Let A be a coindependent set in a matroid M such that every 3-element subset of A is a triangle. Then $\nabla_A(\Delta_A(M))$ is well-defined and*

$$\nabla_A(\Delta_A(M)) = M.$$

Proof. Lemma 2.9 implies that A is independent in $\Delta_A(M)$. Moreover, every 3–element subset of A is a minimal set meeting every basis of $\Delta_A(M)$ and hence is a triad of $\Delta_A(M)$. Therefore $\nabla_A(\Delta_A(M))$ is well-defined. Now, by definition,

$$\nabla_A(\Delta_A(M)) = [\Delta_A[(\Delta_A(M))^*]]^*.$$

To prove the rest of the lemma, we shall show that $[\Delta_A[(\Delta_A(M))^*]]^*$ and M have the same sets of bases. It follows from Lemma 2.9 that a subset of $E(M)$ is a basis of $[\Delta_A(M)]^*$ if and only if it is a member of one of the following sets:

(i)′ $\{E(M\backslash A) - B' : B' \text{ is a basis of } M/A\}$;
(ii)′ $\{(E(M\backslash A) - B'') \cup a_i : 1 \le i \le k \text{ and } B'' \text{ is a basis of } M/a_i\backslash(A - a_i)\}$; or
(iii)′ $\{(E(M\backslash A) - B''') \cup \{a_i, a_j\} : 1 \le i < j \le k \text{ and } B''' \text{ is a basis of } M\backslash A\}$.

Now consider the bases of $\Delta_A[(\Delta_A(M))^*]$. By Lemma 2.9, these bases are precisely the members of the following sets:

(i)″ $\{A \cup X' : X' \text{ is a basis of } (\Delta_A(M))^*/A\}$;
(ii)″ $\{(A - a_i) \cup X'' : 1 \le i \le k \text{ and } X'' \text{ is a basis of } (\Delta_A(M))^*/a_i\backslash(A - a_i)\}$; and
(iii)″ $\{(A - \{a_i, a_j\}) \cup X''' : 1 \le i < j \le k \text{ and } X''' \text{ is a basis of } (\Delta_A(M))^*\backslash A\}$.

Now X' is a basis of $(\Delta_A(M))^*/A$ if and only if X' is a basis of $[\Delta_A(M)\backslash A]^*$. The latter holds if and only if $E(M\backslash A) - X'$ is a basis of $\Delta_A(M)\backslash A$, and, by Lemma 2.8,this holds if and only if $E(M\backslash A) - X'$ is a basis of $M\backslash A$. Similarly, using Lemma 2.8 again, we obtain that X'' is a basis of $(\Delta_A(M))^*/a_i\backslash(A - a_i)$ if and only if $E(M\backslash A) - X''$ is a basis of $M/a_i\backslash(A - a_i)$. Finally, X''' is a basis of $(\Delta_A(M))^*\backslash A$ if and only if $E(M\backslash A) - X'''$ is a basis of M/A. Thus a subset of $E(M)$ is a basis of $[\Delta_A[(\Delta_A(M))^*]]^*$ if and only if it is a member of one of the following sets:

(i)‴ $\{E(M\backslash A) - X' : E(M\backslash A) - X' \text{ is a basis of } M\backslash A\}$;
(ii)‴ $\{(E(M\backslash A) - X'') \cup a_i : E(M\backslash A) - X'' \text{ is a basis of } M/a_i\backslash(A - a_i) \text{ and } 1 \le i \le k\}$;
(iii)‴ $\{(E(M\backslash A) - X''') \cup \{a_i, a_j\} : E(M\backslash A) - X''' \text{ is a basis of } M/A \text{ and } 1 \le i < j \le k\}$.

Since the union of the sets (i)‴-(iii)‴ is the collection of bases of M, the lemma is proved. □

The dual of the last result is the following.

Corollary 2.12. *Let A be an independent set in a matroid M such that every 3-element subset of A is a triad. Then $\Delta_A(\nabla_A(M)) = M$ is well-defined and*

$$\Delta_A(\nabla_A(M)) = M.$$

In the definition of a segment-cosegment exchange on a set A of M, we have insisted that A must be a coindependent set of M. As we have seen, this ensures that a cosegment-segment exchange can be performed on $\Delta_A(M)$ to recover M. From the perspective of the excluded-minor characterizations that will be discussed later in this paper, there is another good reason for imposing this condition. As we shall show, if we perform a segment-cosegment exchange on a matroid M that is an excluded minor for representability over a partial field $\mathbf{P}$, then we will obtain another excluded minor for the class of $\mathbf{P}$–representable matroids. However, if A is not coindependent in M, then there is no guarantee that $P_A(\Theta_k, M)\backslash A$ is such an excluded minor. For example, if $|A| = 3$, then $P_A(\Theta_3, U_{2,4})\backslash A \cong U_{3,4}$. However, although $U_{2,4}$ is an excluded minor for the class of binary matroids, $U_{3,4}$ is not.

Recall that, for a Δ_k–exchange to be defined, $k \geq 2$.

Lemma 2.13. *Suppose that $\Delta_A(M)$ is defined. If $x \in A$ and $|A| = k \geq 3$, then $\Delta_{A-x}(M\backslash x)$ is also defined and*

$$\Delta_A(M)/x = \Delta_{A-x}(M\backslash x).$$

Proof. By relabelling if necessary, we may assume that $x = a_1$. If D is a basis of $\Delta_A(M)/a_1$, then $D \cup a_1$ is a basis of $\Delta_A(M)$. Therefore, by Lemma 2.9, the collections of type (i)-(iii) bases of $\Delta_A(M)/a_1$ are

(i) $\{(A - a_1) \cup X' : X' \text{ is a basis of } M/A\}$;
(ii) $\{(A - \{a_1, a_i\}) \cup X'' : 2 \leq i \leq k \text{ and } X'' \text{ is a basis of } M/a_i\backslash(A - a_i)\}$; and
(iii) $\{(A - \{a_1, a_i, a_j\}) \cup X''' : 2 \leq i < j \leq k \text{ and } X''' \text{ is a basis of } M\backslash A\}$.

Now $\Delta_{A-a_1}(M\backslash a_1)$ is easily seen to be defined. By Lemma 2.9 again, the collections of type (i)-(iii) bases of $\Delta_{A-a_1}(M\backslash a_1)$ are

(i) $\{(A - a_1) \cup Y' : Y' \text{ is a basis of } M\backslash a_1/(A - a_1)\}$;
(ii) $\{(A-\{a_1, a_i\})\cup Y'' : 2 \leq i \leq k \text{ and } Y'' \text{ is a basis of } M\backslash a_1/a_i\backslash(A - \{a_1, a_i\})\}$; and
(iii) $\{(A-\{a_1, a_i, a_j\})\cup Y''' : 2 \leq i < j \leq k \text{ and } Y''' \text{ is a basis of } M\backslash a_1\backslash(A - a_1)\}$.

Since $|A| \geq 3$, the element a_1 is a loop of $M/(A-a_1)$. Hence $M\backslash a_1/(A-a_1) = M/A$. Furthermore, $M\backslash a_1/a_i\backslash(A-\{a_1, a_i\}) = M/a_i\backslash(A-a_i)$ and $M\backslash a_1\backslash(A-a_1) = M\backslash A$. Hence the collection of bases of $\Delta_A(M)/a_1$ is equal to the collection of bases of $\Delta_{A-a_1}(M\backslash a_1)$, and the lemma follows. □

Corollary 2.14. *Suppose that $\nabla_A(M)$ is defined. If $x \in A$ and $|A| \geq 3$, then $\nabla_{A-x}(M/x)$ is also defined and*

$$\nabla_A(M)\backslash x = \nabla_{A-x}(M/x).$$

Lemma 2.15. *Suppose $x \in \mathrm{cl}_M(A) - A$ and let a be an arbitrary element of the k–element set A. Then $\Delta_A(M)/x$ equals the 2-sum, with basepoint p, of a copy of $U_{k-1,k+1}$ with ground set $A \cup p$ and the matroid obtained from $M/x\backslash(A - a)$ by relabelling a as p.*

Proof. Clearly $\Delta_A(M)/x = P_A(\Theta_k, M)\backslash A/x$. Now let $\Theta'_k = P_A(\Theta_k, M)|(E(\Theta_k) \cup x)$. As A is a modular line of Θ_k, and x lies in the closure of this line in M, it follows that $A \cup x$ is a modular line of Θ'_k. Thus $P_A(\Theta_k, M) = P_{A\cup x}(\Theta'_k, M)$, so $P_A(\Theta_k, M)/x = P_{A\cup x}(\Theta'_k, M)/x$. Moreover, by [4, Proposition 5.11], the last matroid equals $P_{[M|(A\cup x)]/x}(\Theta'_k/x, M/x)$. But $M|(A\cup x) \cong U_{2,k+1}$, so $[M|(A\cup x)]/x \cong U_{1,k}$. It follows, since $a \in A$, that $P_A(\Theta_k, M)/x\backslash(A-a)$ is the parallel connection, with basepoint a, of $\Theta'_k/x\backslash(A-a)$ and $M/x\backslash(A-a)$. Thus $P_A(\Theta_k, M)/x\backslash A$ is the 2-sum of the last two matroids. When we recall the ground-set relabelling that is done in forming $\Delta_A(M)$, we obtain the lemma provided we can show that $\Theta'_k/x\backslash(A-a) \cong U_{k-1,k+1}$. To establish this isomorphism, it suffices to show that $\Theta'_k/x\backslash(A-a)$ has no non-spanning circuits.

Suppose that $\Theta'_k/x\backslash(A-a)$ has a non-spanning circuit C. Then either (i) $C\cup x$ is a non-spanning circuit of $\Theta'_k\backslash(A-a)$, or (ii) C is a circuit of $\Theta'_k\backslash(A-a)\backslash x$ of size at most $k-1$. But $\Theta'_k\backslash(A-a)\backslash x = \Theta_k\backslash(A-a)$ and the last matroid has no circuits of size less than k. Hence (ii) cannot occur. Suppose that (i) occurs. Then, since every hyperplane of Θ_k that is spanned by a proper subset of B meets A in exactly one element, C must contain a. It follows that C spans A in Θ_k, so $|C| = k$; a contradiction. □

Both parts of the next lemma can be proved by comparing collections of bases as above. We omit the straightforward details.

Lemma 2.16. *Suppose that $\Delta_A(M)$ is defined.*

(i) *If $x \in E(M)-A$ and A is coindependent in $M\backslash x$, then $\Delta_A(M\backslash x)$ is defined and*
$$\Delta_A(M)\backslash x = \Delta_A(M\backslash x).$$
(ii) *If $x \in E(M) - \mathrm{cl}(A)$, then $\Delta_A(M/x)$ is defined and*
$$\Delta_A(M)/x = \Delta_A(M/x).$$

The next result is a useful consequence of the last two lemmas.

Corollary 2.17. *Suppose that $x \in E(M) - A$, $|E(M) - A| \geq 3$, and $|A| \geq 3$.*

(i) *Suppose that $\Delta_A(M)$ is defined.*
  (a) *If $M\backslash x$ is 3–connected, then $\Delta_A(M\backslash x)$ is defined and*
  $$\Delta_A(M)\backslash x = \Delta_A(M\backslash x).$$
  (b) *If M/x is 3–connected, then $\Delta_A(M/x)$ is defined and*
  $$\Delta_A(M)/x = \Delta_A(M/x).$$
(ii) *Suppose that $\nabla_A(M)$ is defined.*
  (a) *If $M\backslash x$ is 3–connected, then $\nabla_A(M\backslash x)$ is defined and*
  $$\nabla_A(M\backslash x) = \nabla_A(M)\backslash x.$$
  (b) *If M/x is 3–connected, then $\nabla_A(M/x)$ is defined and*
  $$\nabla_A(M/x) = \nabla_A(M)/x.$$

Proof. By duality, it suffices to prove (i). Clearly (a) holds by Lemma 2.16(i) unless A is not coindependent in $M\backslash x$. But, in the exceptional case, since A is a coindependent rank–2 set in M, it follows that $\{A, E(M)-(A\cup x)\}$ is a 2–separation of $M\backslash x$; a contradiction. Part (b) is an immediate consequence of Lemma 2.16(ii) for, if $x \in \mathrm{cl}(A) - A$, then M/x is not 3–connected since it has A as a parallel class but has at least four elements. □

Lemma 2.18. *Let M be a matroid, and S and T be disjoint subsets of $E(M)$ such that $|S| \geq 2$ and $|T| \geq 2$. If $M|S \cong U_{2,|S|}$ and $M|T \cong U_{2,|T|}$, and both S and T are coindependent in M, then*

$$\Delta_S(\Delta_T(M)) = \Delta_T(\Delta_S(M)).$$

Proof. Since T is coindependent in M, there is a basis of M avoiding T. It follows, by Lemma 2.9, that $\Delta_S(M)$ has a basis avoiding T, so T is coindependent in $\Delta_S(M)$. Moreover, $\Delta_S(M)|T = M|T$. Hence $\Delta_T(\Delta_S(M))$ is well-defined and, similarly, so is $\Delta_S(\Delta_T(M))$. We now establish the equality of these two matroids. Using the fact that a set is a flat of a generalized parallel connection of two matroids if and only if its intersection with each of the matroids is a flat in that matroid [15, Proposition 12.4.13], it is routine to deduce that

$$P_S(\Theta_{|S|}, P_T(\Theta_{|T|}, M)) = P_T(\Theta_{|T|}, P_S(\Theta_{|S|}, M)).$$

As S and T are disjoint, this implies that

$$[P_S(\Theta_{|S|}, P_T(\Theta_{|T|}, M))\backslash T]\backslash S = [P_T(\Theta_{|T|}, P_S(\Theta_{|S|}, M))\backslash S]\backslash T.$$

Therefore, by a result of Brylawski [4, Proposition 5.11] (see also [15, Proposition 12.4.14]),

$$P_S(\Theta_{|S|}, P_T(\Theta_{|T|}, M)\backslash T)\backslash S = P_T(\Theta_{|T|}, P_S(\Theta_{|S|}, M)\backslash S)\backslash T,$$

which in turn implies that

$$P_S(\Theta_{|S|}, \Delta_T(M))\backslash S = P_T(\Theta_{|T|}, \Delta_S(M))\backslash T.$$

Hence

$$\Delta_S(\Delta_T(M)) = \Delta_T(\Delta_S(M))$$

as required. □

Corollary 2.19. *Let M be a matroid, and S and T be disjoint subsets of $E(M)$ such that $|S| \geq 2$ and $|T| \geq 2$.*

(i) *If $M^*|S \cong U_{2,|S|}$ and $M^*|T \cong U_{2,|T|}$, and both S and T are independent in M, then*

$$\nabla_S(\nabla_T(M)) = \nabla_T(\nabla_S(M)).$$

(ii) *If $M^*|S \cong U_{2,|S|}$ and S is independent in M, and $M|T \cong U_{2,|T|}$ and T is coindependent in M, then*

$$\nabla_S(\Delta_T(M)) = \Delta_T(\nabla_S(M)).$$

Proof. Part (i) follows without difficulty from the last lemma by using duality. Consider (ii). By Lemma 2.11,

$$\begin{aligned}\nabla_S(\Delta_T(M)) &= \nabla_S[\Delta_T[\Delta_S(\nabla_S(M))]],\\ &= \nabla_S[\Delta_S[\Delta_T(\nabla_S(M))]], && \text{by Lemma 2.18,}\\ &= \Delta_T(\nabla_S(M)), && \text{as required.}\end{aligned}$$

□

Two elements x and x' are *clones* in a matroid M if the map that fixes every element of $E(M) - \{x, x'\}$, but interchanges x and x', is an automorphism of M. Thus, up to labelling, two such elements are indistinguishable in M. The study of clones was initiated in [8, Section 4]. A straightforward consequence of the definition of clones is that if x and x' are clones of M, and N is a minor of M containing $\{x, x'\}$, then x and x' are clones in N. We use this property in the next result.

Lemma 2.20. *Let x and x' be clones in a matroid M. If $A \cap \{x, x'\}$ is empty or $A \supseteq \{x, x'\}$, then x and x' are clones in $\Delta_A(M)$. Moreover, if $\{x, x'\}$ is independent in M, it is independent in $\Delta_A(M)$, and if $\{x, x'\}$ is coindependent in M, it is coindependent in $\Delta_A(M)$.*

Proof. The lemma is straightforward if $A \supseteq \{x, x'\}$ and we omit the details. Now assume that $A \cap \{x, x'\}$ is empty. First suppose that $\{x, x'\}$ is independent in M. Since A is coindependent in M, there is a subset of $E(M) - A$ that contains $\{x, x'\}$ and is a basis of M. Therefore, by Lemma 2.9, there is a basis of type (iii) of $\Delta_A(M)$ containing $\{x, x'\}$, so $\{x, x'\}$ is independent in $\Delta_A(M)$. Now suppose $\{x, x'\}$ is coindependent in M. Then $E(M) - \{x, x'\}$ spans M and therefore spans $\Delta_A(M)$. Hence $\{x, x'\}$ is coindependent in $\Delta_A(M)$.

We show next that x and x' are clones in $\Delta_A(M)$. Let $\mathcal{B}(\Delta_A(M))$ denote the collection of bases of $\Delta_A(M)$ and let $\mathcal{B}'(\Delta_A(M))$ be the set obtained from $\mathcal{B}(\Delta_A(M))$ by interchanging the elements x and x', and fixing every other element of $E(M)$. By the definition of clones, it suffices to show that $\mathcal{B}(\Delta_A(M)) = \mathcal{B}'(\Delta_A(M))$. By Lemma 2.9, the collection of bases of $\Delta_A(M)$ consists of the union, over all subsets A' of A having size at least $|A|-2$, of the collection $\mathcal{B}_{A'}$ of bases that meet A in A'. But each such $\mathcal{B}_{A'}$ is obtained by adjoining A' to every basis of some fixed minor $M_{A'}$ of M, where $M_{A'}$ has ground set $E(M)-A$ and depends only on A'. Therefore, since x and x' are clones in each $M_{A'}$, it follows that $\mathcal{B}(\Delta_A(M)) = \mathcal{B}'(\Delta_A(M))$, as desired. □

The dual of the last lemma is as follows.

Corollary 2.21. *Let x and x' be clones in a matroid M. If $A \cap \{x, x'\}$ is empty or $A \supseteq \{x, x'\}$, then x and x' are clones in $\nabla_A(M)$. Moreover, if $\{x, x'\}$ is independent or coindependent in M, then it is independent or coindependent, respectively, in $\nabla_A(M)$.*

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. This proof will require some more preliminaries. These results are stated in the context of partial fields, which were defined in the introduction. The first two propositions contain elementary properties of determinants that generalize to partial fields.

Proposition 3.1. [19, Proposition 3.1] *Let X be a square matrix with entries in a partial field $\mathbf{P}$.*

(i) *If Y is obtained from X by interchanging a pair of rows or columns, then $\det(Y)$ is defined if and only if $\det(X)$ is defined. Moreover, when $\det(X)$ is defined, $\det(Y) = -\det(X)$.*
(ii) *If Y is obtained from X by multiplying each entry of a row or a column by a non-zero element q of $\mathbf{P}$, then $\det(Y)$ is defined if and only if $\det(X)$ is defined. Moreover, when $\det(X)$ is defined, $\det(Y) = q\det(X)$.*
(iii) *If $\det(X)$ is defined and Y is obtained from X by adding two rows or two columns whose sum is defined, then $\det(Y)$ is defined and $\det(Y) = \det(X)$.*

Proposition 3.2. [19, Proposition 3.2] *Let X be a square matrix (x_{ij}) with entries in a partial field $\mathbf{P}$. Let X_{ij} denote the submatrix obtained by deleting row i and column j from X.*

(i) *If X has a row or a column of zeros, then $\det(X) = 0$.*
(ii) *If x_{ij} is the only non-zero entry in its row or column, then $\det(X)$ is defined if and only if $\det(X_{ij})$ is defined. Moreover, when $\det(X)$ is defined, $\det(X) = (-1)^{i+j}x_{ij}\det(X_{ij})$.*

Recall that a matrix over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is k–unimodular if all its non-zero subdeterminants are products of positive and negative powers of differences of distinct elements of $\{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_k\}$, that is, they are members of the set

$$\mathcal{A}_k = \{\pm\prod_{i=1}^{k}\alpha_i^{l_i}\prod_{i=1}^{k}(\alpha_i - 1)^{m_i}\prod_{1\le i<j\le k}(\alpha_i - \alpha_j)^{n_{i,j}} : l_i, m_i, n_{i,j} \in \mathbf{Z}\}.$$

A k–regular matroid is one that can be represented by a k–unimodular matrix. In particular, a 0–regular matroid is just a regular matroid and a 1–regular matroid is exactly a near-regular matroid [28, 29].

The next lemma generalizes [16, Proposition 4] to partial fields. Moreover, the proof of [16, Proposition 4] will work for this generalization by replacing the field "$\mathbf{F}$" with the partial field"$\mathbf{P}$".

Lemma 3.3. *Let $\mathbf{P}$ be a partial field. If there are k distinct elements $a_1, a_2, \ldots, a_k$ in $\mathbf{P} - \{0, 1\}$ such that, for all distinct i and j in $\{1, 2, \ldots, k\}$, both $a_i - 1$ and $a_i - a_j$ are in $\mathbf{P}$, then the class of $\mathbf{P}$–representable matroids contains the class of k–regular matroids.*

Evidently both Θ_2 and Θ_3 are regular matroids.

Lemma 3.4. *Θ_k is $(k-3)$–regular for all $k \ge 4$.*

Proof. By our definition of Θ_k, it suffices to show that the matrix $[I_k|D_k]$ over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is $(k-3)$–unimodular. Thus we need to show that if X is an $m \times m$ submatrix of $[I_k|D_k]$, then $\det(X)$ is in $\mathcal{A}_{k-3} \cup \{0\}$. This is certainly true if $m \leq 2$. Now suppose that $m \geq 3$. If X avoids one of the first two rows or one of the first two columns of D_k, then, it follows by (3.2) and the fact that all non-zero 2×2 subdeterminants of D_k are in $\mathcal{A}_{k-3}$, that the determinant of X is either zero or in $\mathcal{A}_{k-3}$. Thus we may assume that X meets both the first two rows and the first two columns of D_k. Hence X is of the form

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & y_1 & y_2 & \cdots & y_n \\ 1 & x_1 & 0 & 0 & & 0 \\ 1 & x_2 & 0 & 0 & & 0 \\ \vdots & \vdots & & & \ddots & \\ 1 & x_n & 0 & 0 & & 0 \end{bmatrix},$$

where $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are elements of $\{1, \alpha_1, \alpha_2, \ldots, \alpha_{k-3}\}$.

Let X' be the matrix obtained from X by pivoting on the $(1,3)$–entry. Then X' is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ -1 & -y_1 & 0 & y_2 - y_1 & \cdots & y_n - y_1 \\ 1 & x_1 & 0 & 0 & & 0 \\ 1 & x_2 & 0 & 0 & & 0 \\ \vdots & \vdots & & & \ddots & \\ 1 & x_n & 0 & 0 & & 0 \end{bmatrix}.$$

By (3.1), the determinant of X is in $\mathcal{A}_{k-3} \cup \{0\}$ if and only if the determinant of X' is in $\mathcal{A}_{k-3} \cup \{0\}$. By expanding the determinant of X' down the last column, we see that $\det(X')$ is either zero or is in $\mathcal{A}_{k-3}$. We conclude that $[I_k|D_k]$ is $(k-3)$–unimodular and the lemma follows. □

Let X be the following matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{k-3} \end{bmatrix}$$

over $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{k-3})$. Then X is a $(k-3)$–unimodular representation for $U_{2,k}$ for all $k \geq 3$. Moreover, it is clear that we can extend this $(k-3)$–unimodular representation of $U_{2,k}$ to a $(k-3)$–unimodular representation of Θ_k. Up to permuting columns, this extended matrix is $[I_k|D_k]$, which we used to define the matroid Θ_k. Now let $\mathbf{P}$ be a partial field. Suppose there are $k-3$ distinct elements $x_1, x_2, \ldots, x_{k-3}$ in $\mathbf{P} - \{0, 1\}$ such that, for all distinct i and j in $\{1, 2, \ldots, k-3\}$,

both $x_i - 1$ and $x_i - x_j$ are in $\mathbf{P}$. Let X' be the matrix obtained from X by replacing α_i by x_i for all i. Then X' is a $\mathbf{P}$–representation for $U_{2,k}$. Consider the matrix $[I_k|D_k]'$ obtained from $[I_k|D_k]$ by replacing α_i by x_i for all i. Certainly $[I_k|D_k]'$ extends the matrix X'. Moreover, by Lemmas 3.3 and 3.4 and from the proof of [16, Proposition 4], $[I_k|D_k]'$ is a $\mathbf{P}$–representation for Θ_k. Thus, given a $\mathbf{P}$–representation of $U_{2,k}$ in the form displayed above, one can always extend it to a $\mathbf{P}$–representation for Θ_k. We make use of this property of $U_{2,k}$ and $\mathbf{P}$ in the next lemma.

A matrix X over a partial field $\mathbf{P}$ is a $\mathbf{P}$–*matrix* if $\det(X')$ is defined for every square submatrix X' of X.

Lemma 3.5. *Let $k \geq 2$ and let M be a matroid such that $M|A \cong U_{2,k}$. If M and Θ_k are both representable over $\mathbf{P}$, then the generalized parallel connection $P_A(\Theta_k, M)$ of Θ_k and M across A is representable over $\mathbf{P}$.*

Proof. The result is clear for $k = 2$. Therefore assume that $k \geq 3$. Since the independent sets of a $\mathbf{P}$–representation for M are preserved under the operations of interchanging a pair of rows or columns, multiplying a column or row by a non-zero scalar, and performing a pivot on a non-zero entry of the representation [19, Proposition 3.5], we may assume that M has as a $\mathbf{P}$–representation the matrix

$$Y = \left[\begin{array}{c|ccccccc} Y_1 & \multicolumn{7}{c}{0} \\ \hline Y_2 & 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ & 0 & 1 & 1 & y_1 & y_2 & \cdots & y_{k-3} \end{array}\right]$$

where $y_1, y_2, \ldots, y_{k-3}$ are distinct elements of $\mathbf{P} - \{0, 1\}$ such that, for all i and j in $\{1, 2, \ldots, k-3\}$, both $y_i - 1$ and $y_i - y_j$ are in $\mathbf{P}$. By Lemma 2.4, A is a modular line of Θ_k. Furthermore, by the remarks preceding the statement of this lemma, the $2 \times k$ submatrix in the bottom-right corner of Y can be extended to a $\mathbf{P}$–representation of Θ_k. Let Z be the matrix

$$\left[\begin{array}{c|ccccccc|c|cc} Y_1 & \multicolumn{7}{c|}{0} & 0 & \multicolumn{2}{c}{0} \\ \hline Y_2 & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 1 \\ & 0 & 1 & 1 & y_1 & y_2 & \cdots & y_{k-3} & & -1 & 0 \\ \hline 0 & \multicolumn{7}{c|}{0} & I_{k-2} & 1 & 1 \\ & & & & & & & & & 1 & y_1 \\ & & & & & & & & & 1 & y_2 \\ & & & & & & & & & \vdots & \vdots \\ & & & & & & & & & 1 & y_{k-3} \end{array}\right].$$

We shall show that Z is a $\mathbf{P}$–matrix. From this it will follow that Z is a $\mathbf{P}$–representation of $P_A(\Theta_k, M)$. To see this, let N be the matroid that is represented by Z. Then N/A is isomorphic to $(M/A) \oplus (\Theta_k/A)$. Thus, by the extension of [4, Proposition 5.9] to matroids [15, Proposition 12.4.15], $N = P_A(\Theta_k, M)$, as required.

To complete the proof, we now show that all subdeterminants of Z are defined. Label the last two columns of Z by b_1 and b_2, respectively. Also label the last $k-2$ rows of Z by $b_3, b_4, \ldots, b_k$. Let Z' be a square submatrix of Z. By (3.2), to verify that Z is a $\mathbf{P}$–matrix, we may assume that Z' avoids the third column of blocks of Z. If Z' avoids both of the columns b_1 and b_2, or all of the rows $b_3, b_4, \ldots, b_k$, then $\det(Z')$ is defined since Y is a $\mathbf{P}$–representation for M. Thus, by (3.2), we may assume that Z' meets the block B in the bottom-right corner of Z. Let Z'' be the matrix obtained from Z' by pivoting on a non-zero entry z'_{ij} of Z' that is also in B. Then, by (3.1), $\det(Z')$ is defined if and only if $\det(Z'')$ is defined. Now the only entries of Z' that are affected by this pivot are those that correspond to the last two columns of Z. Let Z''_{ij} denote the matrix obtained from Z'' by deleting the i-th row and j-th column. If Z' meets B in one column, then, by (3.2) and the fact that Y is a $\mathbf{P}$–representation for M, it follows that $\det(Z''_{ij})$ is defined and, therefore, so is $\det(Z')$. Therefore we may assume that Z' meets B in two columns. If Z' meets B in at least two rows, then, by pivoting twice in Z', once on z'_{ij} and once on another entry of B that is in a different row and column from z'_{ij}, we deduce that $\det(Z')$ is defined. Thus we may also assume that Z' meets B in exactly one row and two columns. Hence Z' is a submatrix of the matrix

$$\left[\begin{array}{c|ccccccc|cc} Y_1 & \multicolumn{7}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline Y_2 & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ & 0 & 1 & 1 & y_1 & y_2 & \cdots & y_{k-3} & -1 & 0 \\ \hline 0 & \multicolumn{7}{c|}{0} & 1 & z' \end{array}\right],$$

where z' is an element of $\{1, y_1, y_2, \ldots, y_{k-3}\}$. If Z' avoids either the second- or third-last rows of this matrix, then, by (3.2), it is easily seen that $\det(Z')$ is defined. Therefore Z' meets the last three rows and last two columns of the above matrix. Now let Z'' be the matrix obtained from Z' by adding the last row to the second-last row of Z' and then deleting the last row and second-last column of the resulting matrix. Then, by (3.1) and (3.2), $\det(Z')$ is defined if and only if $\det(Z'')$ is defined. Since Z'' is either a submatrix of Y or a submatrix of Y with one column repeated, the latter holds. Thus Z is a $\mathbf{P}$–matrix and so Z is a $\mathbf{P}$–representation for $P_A(\Theta_k, M)$. □

The next result generalizes [30, Lemma 5.7] from a Δ_3–exchange to a segment-cosegment exchange of arbitrary size. Two matrix representations of a matroid over a partial field $\mathbf{P}$ are *equivalent* if one can be obtained from the other by a sequence of the following operations: permuting rows; permuting columns (along with their labels); multiplying a row or column by a non-zero element of $\mathbf{P}$; replacing a row by the sum of that row and another; and applying an automorphism of $\mathbf{P}$ to the entries of the matrix. The two matrix representations are *strongly equivalent* if one can be obtained from the other by a sequence of these operations that avoids applying an automorphism of the partial field $\mathbf{P}$.

Corollary 3.6. *Let $\mathbf{P}$ be a partial field and let M be a matroid. If M is $\mathbf{P}$–representable, then the strong-equivalence classes of $\mathbf{P}$–representations of M are in one-to-one correspondence with the strong-equivalence classes of $\mathbf{P}$–representations of $\Delta_A(M)$.*

Proof. By Lemma 3.5, $\Delta_A(M)$ is $\mathbf{P}$–representable. Let Y and Z, respectively, denote the first two matrices in the proof of Lemma 3.5. Now consider the $\mathbf{P}$–representations of M and $\Delta_A(M)$ given, respectively, by the matrix Y and the matrix Z' obtained from Z by deleting the second column of blocks. Just as we may assume that a $\mathbf{P}$–representation of M has the same form as Y, we may also assume that a $\mathbf{P}$–representation of $\Delta_A(M)$ has the same form as Z'. The corollary now follows by observing the canonical bijection between these two $\mathbf{P}$–representations. □

Corollary 3.7. *Let $\mathcal{M}(\mathbf{P})$ be the class of matroids representable over the partial field $\mathbf{P}$. Let M be a matroid. Then M is in $\mathcal{M}(\mathbf{P})$ if and only if $\Delta_A(M)$ is in $\mathcal{M}(\mathbf{P})$.*

Proof. If M is in $\mathcal{M}(\mathbf{P})$, then, by Lemma 3.5, $\Delta_A(M)$ is in $\mathcal{M}(\mathbf{P})$. Now suppose that $\Delta_A(M)$ is in $\mathcal{M}(\mathbf{P})$. By Lemma 2.11, $\nabla_A(\Delta_A(M))$ is well-defined and equal to M. Therefore it suffices to show that $\nabla_A(\Delta_A(M))$ is in $\mathcal{M}(\mathbf{P})$. Now $\mathcal{M}(\mathbf{P})$ is closed under duality [19, Proposition 4.2]. Therefore, as $\nabla_A(\Delta_A(M)) = [\Delta_A[(\Delta_A(M))^*]]^*$ and $\Delta_A(M)$ is in $\mathcal{M}(\mathbf{P})$, it follows by Lemma 3.5 that $\nabla_A(\Delta_A(M))$ is in $\mathcal{M}(\mathbf{P})$. □

At last we prove Theorem 1.1, which we restate for convenience.

Theorem 1.1. *Let M be an excluded minor for $\mathcal{M}(\mathbf{P})$. Let A be a subset of $E(M)$ such that $M|A$ is isomorphic to a rank–2 uniform matroid and A is coindependent in M. Then $\Delta_A(M)$ is an excluded minor for $\mathcal{M}(\mathbf{P})$.*

Proof. Let $M' = \Delta_A(M)$ and let $|A| = k$. If $k = 2$, then $M' \cong M$ and so M' is an excluded minor for $\mathcal{M}(\mathbf{P})$. Therefore assume that $k \geq 3$. Suppose that M' is not an excluded minor for $\mathcal{M}(\mathbf{P})$. Then, by Corollary 3.7, there is an element x of $E(M')$ such that either $M'\backslash x$ or M'/x is not in $\mathcal{M}(\mathbf{P})$. The proof is partitioned into four cases:

(i) $x \in A$ and $M'/x \notin \mathcal{M}(\mathbf{P})$;
(ii) $x \in A$ and $M'\backslash x \notin \mathcal{M}(\mathbf{P})$;
(iii) $x \notin A$ and $M'/x \notin \mathcal{M}(\mathbf{P})$; and
(iv) $x \notin A$ and $M'\backslash x \notin \mathcal{M}(\mathbf{P})$.

In the proof of these cases, we freely use the fact that both the parallel connection and the 2-sum of two matroids in $\mathcal{M}(\mathbf{P})$ is also in $\mathcal{M}(\mathbf{P})$ [19, Proposition 4.2].

Case (i). $x \in A$ and $M'/x \notin \mathcal{M}(\mathbf{P})$.

By Lemma 2.13, $M'/x = \Delta_A(M)/x = \Delta_{A-x}(M\backslash x)$. Thus, as $M\backslash x \in \mathcal{M}(\mathbf{P})$, it follows that $\Delta_{A-x}(M\backslash x)$, and hence M'/x, is also in $\mathcal{M}(\mathbf{P})$; a contradiction.

Case (ii). $x \in A$ and $M'\backslash x \notin \mathcal{M}(\mathbf{P})$.

Since every 3-element subset of A is a triad of M', it follows that the elements of $A - x$ are in series in $M' - x$. Thus $M'\backslash x$ is isomorphic to the 2-sum of $M\backslash(A - x)$ and a circuit and so $M'\backslash x$ is certainly in $\mathcal{M}(\mathbf{P})$; a contradiction.

Case (iii). $x \notin A$ and $M'/x \notin \mathcal{M}(\mathbf{P})$.

First suppose that $r_{M/x}(A) = 2$. Then, by Lemma 2.16, $M'/x = \Delta_A(M)/x = \Delta_A(M/x)$. Now $M/x \in \mathcal{M}(\mathbf{P})$. Therefore, by Corollary 3.7, $\Delta_A(M/x)$, and hence M'/x, is in $\mathcal{M}(\mathbf{P})$. This contradiction implies that $r_{M/x}(A) \neq 2$. Hence we may assume that $r_{M/x}(A) = 1$, that is, $x \in \mathrm{cl}_M(A)$. Then $M|(A \cup x) \cong U_{2,k+1}$ and, since A is coindependent in M, the ground set of M properly contains $A \cup x$. Thus $U_{2,k+1} \in \mathcal{M}(\mathbf{P})$ and hence $U_{k-1,k+1} \in \mathcal{M}(\mathbf{P})$. Now, by Lemma 2.15, M'/x is isomorphic to the 2-sum of $M/x\backslash(A - a)$ and a copy of $U_{k-1,k+1}$, where a is some element of A. Since the last two matroids are both in $\mathcal{M}(\mathbf{P})$, we obtain the contradiction that $M'/x \in \mathcal{M}(\mathbf{P})$.

Case (iv). $x \notin A$ and $M'\backslash x \notin \mathcal{M}(\mathbf{P})$.

Since $M' = P_A(\Theta_k, M)\backslash A$, it follows that $M'\backslash x = P_A(\Theta_k, M\backslash x)\backslash A$. But $M\backslash x \in \mathcal{M}(\mathbf{P})$, so by Lemma 3.5, $P_A(\Theta_k, M\backslash x) \in \mathcal{M}(\mathbf{P})$. Hence $M'\backslash x \in \mathcal{M}(\mathbf{P})$; a contradiction. □

The following is the dual of the last result.

Corollary 3.8. *Let M be an excluded minor for $\mathcal{M}(\mathbf{P})$. Let A be a subset of $E(M)$ such that A is independent in M and $M^*|A$ is isomorphic to a rank–2 uniform matroid. Then $\nabla_A(M)$ is an excluded minor for $\mathcal{M}(\mathbf{P})$.*

4. Del-con Trees

In this section, we study a class of matroids that will be fundamental in solving the problem of which ω–regular matroids are k–regular.

Let M and N be matroids. Then M is $\Delta - \nabla$–*equivalent* to N if there is a sequence $M_0, M_1, \ldots, M_n$ of matroids such that, for all i in $\{1, 2, \ldots, n\}$, the matroid M_i is obtained from M_{i-1} by either a Δ–exchange or a ∇–exchange, $M_0 = N$ and $M_n \cong M$. Evidently, if M is $\Delta - \nabla$–equivalent to N, then N is $\Delta - \nabla$–equivalent to M.

For $m \geq 4$, let Λ_m denote the class of matroids that are $\Delta - \nabla$–equivalent to $U_{2,m}$. In other words, if M is a member of Λ_m, then M can be obtained from $U_{2,m}$ by a sequence of operations each of which consists of a segment-cosegment or a cosegment-segment exchange. Lemma 4.2 shows that Λ_m is closed under duality. As a step towards that result, we first show that a rank–2 uniform matroid is $\Delta - \nabla$–equivalent to its dual.

Lemma 4.1. *Let E be the disjoint union of sets X and Y, and let N be a rank–2 uniform matroid on E. If $|X| \geq 2$ and $|Y| \geq 2$, then*

$$\Delta_Y(\Delta_X(N)) = N^*.$$

Proof. By Lemma 2.6, $r(\Delta_Y(\Delta_X(N))) = |E| - 2$. Now every 3–element subset of Y is a triad of $\Delta_Y(\Delta_X(N))$ and, since $\Delta_Y(\Delta_X(N)) = \Delta_X(\Delta_Y(N))$, every 3–element

subset of X is a triad of $\Delta_Y(\Delta_X(N))$. Thus $[\Delta_Y(\Delta_X(N))]^*$ is a rank–2 uniform matroid on E unless it has a 2–circuit $\{x, y\}$ for some x in X and some y in Y. Hence we may assume that the exceptional case holds. Then, for x' in $X - x$, Lemma 2.20 implies that x and x' are clones in $\Delta_Y(\Delta_X(N))$. Hence $\{x', y\}$ is a circuit of $[\Delta_Y(\Delta_X(N))]^*$ and, therefore, so too is $\{x, x'\}$; a contradiction. □

Lemma 4.2. *Let $m \geq 4$. If $M \in \Lambda_m$, then $M^* \in \Lambda_m$.*

Proof. This is a straightforward consequence of the last lemma and the fact that $[\Delta_A(N)]^* = \nabla_A(N^*)$. The details are omitted. □

In general, 3–connectivity is not preserved under a Δ–exchange or, dually, under a ∇–exchange. To see this, consider the following example. Let Q_6 be the matroid obtained by placing a point on the intersection of two lines of $U_{3,5}$. Then the matroid obtained from Q_6 by performing a Δ_3–exchange on one of its triangles is not 3–connected. However, as we show next, every matroid in $\bigcup_{m\geq 4} \Lambda_m$ is 3–connected.

Lemma 4.3. *Let M be a matroid in $\bigcup_{m\geq 4} \Lambda_m$. Then M is 3–connected.*

Proof. For all $k \geq 0$, it follows from [18, Corollary 4.2] that $U_{2,k+4}$ is an excluded minor for the class of k–regular matroids. By Theorem 1.1, so too is every matroid that is $\Delta - \nabla$–equivalent to $U_{2,k+4}$. Thus every matroid in Λ_{k+4} is an excluded minor for the class of k–regular matroids. But, for all $k \geq 0$, the class of k–regular matroids is closed under the taking of direct sums and 2–sums. Hence every excluded minor for this class must be 3–connected. In particular, every member of Λ_{k+4} is 3–connected, and so every member of $\bigcup_{m\geq 4} \Lambda_m$ is 3–connected. □

Next we shall associate a particular type of labelled tree with every member of $\bigcup_{m\geq 4} \Lambda_m$. Before specifying this association, we begin by describing the class of trees being considered. A *del-con tree* is a tree T for which every vertex v is labelled by one of the ordered pairs (E_v, del) or (E_v, con) such that the following conditions hold:

(i) each E_v is a finite, possibly empty, set;
(ii) if u and v are distinct vertices, then E_v and E_u are disjoint;
(iii) if v is a degree-one vertex of T, then $|E_v| \geq 2$; and
(iv) if two vertices of T are adjacent, then the second coordinates of their labels are different.

A vertex v of a del-con tree T will be referred to as a *del* or *con vertex* in the obvious way, and the corresponding set E_v will be called a *del* or *con class* of T. Now suppose v is a degree-one vertex of T. Let T' be the tree obtained from T by deleting v and keeping all vertex labels inherited from T except on the unique neighbour u of v in T. In the exceptional case, we retain the second coordinate of the label, but change the first coordinate to $E_u \cup E_v$. This operation on T is called *shrinking*, and T' is said to be obtained from T by *shrinking v into u*.

Let T be a del-con tree and let $|V(T)| = n$. Let $E = \cup_{v \in V(T)} E_v$ and assume that $|E| \geq 4$. We now describe how to obtain, from T, a matroid $M(T)$ that is in Λ_m where $m = |E|$. Let $T_1, T_2, \ldots, T_n$ be a sequence of del-con trees such that $T_n = T$ and, for all i in $\{1, 2, \ldots, n-1\}$, the tree T_i has i vertices and is obtained from T_{i+1} by shrinking a degree-one vertex into its unique neighbour. We call such a sequence a *chain* of del-con trees. Since $E = \cup_{v \in V(T_n)} E_v$, it follows that $E = \cup_{u \in V(T_i)} E_u$ for all i in $\{1, 2, \ldots, n\}$. In particular, the unique vertex of T_1 is labelled (E, del) or (E, con). We define $M(T_1)$ to have ground set E and to be isomorphic to $U_{2,|E|}$ or $U_{|E|-2,|E|}$ depending on whether the vertex of T_1 is a del or a con vertex. In general, for all $i \geq 1$, if T_{i+1} is obtained from T_i by shrinking the vertex v into the vertex u, we define $M(T_{i+1})$ to be $\Delta_{E_v}(M(T_i))$ or $\nabla_{E_v}(M(T_i))$ according to whether v is labelled (E_v, con) or (E_v, del). Define $M(T) = M(T_n)$. We need to show that $M(T)$ is well-defined. The proof of this will use the following lemma, the straightforward proof of which follows from Lemma 4.1 and the definition of a ∇–exchange.

Lemma 4.4. *Let the ground set E of $U_{2,|E|}$ be the disjoint union of sets X and Y. If $|X| \geq 2$ and $|Y| \geq 2$, then*

$$\Delta_X(U_{2,|E|}) = \nabla_Y(U_{|E|-2,|E|}).$$

Lemma 4.5. *Let T be a del-con tree, let $E = \cup_{v \in V(T)} E_v$, and assume that $|E| \geq 4$. The matroid $M(T)$ is a well-defined member of $\Lambda_{|E|}$. Moreover, if v is a vertex of T and $|E_v| \geq 2$, then either v is a del vertex and $M(T)|E_v$ is uniform of rank two, or v is a con vertex and $M(T).E_v$ is uniform of corank two.*

Proof. We prove both parts of the lemma simultaneously, arguing by induction on $|V(T)|$. We note first that the result is certainly true if $|V(T)| = 1$. If $|V(T)| = 2$, let $V(T) = \{v_1, v_2\}$. Without loss of generality, we may assume that v_1 is a del vertex and v_2 is a con vertex. Then $M(T)$ can be constructed in exactly two ways: from the del-con tree obtained by shrinking v_2 into v_1, and from the del-con tree obtained by shrinking v_1 into v_2. The first of these constructions yields $\Delta_{E_{v_2}}(U_{2,|E|})$ and the second $\nabla_{E_{v_1}}(U_{|E|-2,|E|})$. But, by Lemma 4.4, these are equal and each is in $\Lambda_{|E|}$. Moreover, $M(T)|E_{v_1}$ is uniform of rank two and $M(T).E_{v_2}$ is uniform of corank two.

Now let $|V(T)| = n \geq 3$, and assume that every matroid obtained from a del-con tree T' with fewer vertices is well-defined and is in Λ_m where m is the cardinality of the union of the first coordinates of the vertex labels of T'. Assume also that, for every such T', the restriction to every del class of $M(T')$ of size at least two is uniform of rank two and the contraction to every con class of $M(T')$ of size at least two is uniform of corank two. We need to show that $M(T)$ is independent of the chain of del-con trees used in its construction. For each j in $\{1, 2\}$, let $T_{1j}, T_{2j}, \ldots, T_{nj}$ be a chain of del-con trees such that $T_{nj} = T$. We shall show next that $M(T_{n1}) = M(T_{n2})$ and that this matroid is in $\Lambda_{|E|}$.

Suppose first that $T_{(n-1)1} = T_{(n-1)2}$. Then, by the induction assumption, $M(T_{(n-1)1}) = M(T_{(n-1)2})$ and this matroid is $\Delta - \nabla$–equivalent to $U_{2,|E|}$. By Lemma 4.3, $M(T_{(n-1)1})$ is 3–connected. Let the vertex v be shrunk into the vertex u in T_{n1} to produce $T_{(n-1)1}$. Assume first that u is a del vertex of $T_{(n-1)1}$.

Then, by the induction assumption, $M(T_{(n-1)1})|(E_u \cup E_v)$ is uniform of rank two. Therefore, as $M(T_{(n-1)1})$ is 3–connected, E_v is a coindependent set of this matroid. Thus, when u is a del vertex of $T_{(n-1)1}$, the matroid $M(T_{n1})$, which equals $\Delta_{E_v}(M(T_{(n-1)1})$, is a well-defined member of $\Lambda_{|E|}$. A similar argument shows that $M(T_{n1})$ is a well-defined member of $\Lambda_{|E|}$ when u is a con vertex of $T_{(n-1)1}$.

We may now assume that $T_{(n-1)1} \neq T_{(n-1)2}$ and that $T_{(n-1)i}$ is obtained by shrinking v_i into u_i for each i where $v_1 \neq v_2$. Since $|V(T)| \geq 3$, the vertices v_1 and u_2 are distinct, as are v_2 and u_1. Let T'' be the del-con tree obtained from $T_{(n-1)1}$ by shrinking v_2 into u_2. Then T'' can also be obtained from $T_{(n-1)2}$ by shrinking v_1 into u_1. Now, by the induction assumption, each of $M(T'')$, $M(T_{(n-1)1})$, and $M(T_{(n-1)2})$ is a well-defined member of $\Lambda_{|E|}$ and hence is independent of the chain of del-con trees used to construct it. First suppose that v_1 and v_2 are both con vertices of T. Then

$$\begin{aligned} M(T_{n1}) &= \Delta_{E_{v_1}}(M(T_{(n-1)1})) \\ &= \Delta_{E_{v_1}}[\Delta_{E_{v_2}}(M(T''))] \\ &= \Delta_{E_{v_2}}[\Delta_{E_{v_1}}(M(T''))] \qquad \text{by Lemma 2.18,} \\ &= \Delta_{E_{v_2}}(M(T_{(n-1)2})) \\ &= M(T_{n2}). \end{aligned}$$

Moreover, $M(T_{n1})$ is certainly in $\Lambda_{|E|}$. Similar arguments establish that $M(T_{n1}) = M(T_{n2})$ and that this matroid is in $\Lambda_{|E|}$ when v_1 and v_2 are both del vertices, and when one is a del vertex and one a con vertex.

It remains to establish that the restriction of $M(T)$ to a del class of size at least two is uniform of rank two and the contraction of $M(T)$ to a con class of size at least two is uniform of corank two.

Recall that $T_{(n-1)1}$ is obtained from T_{n1} by shrinking v_1 into u_1. We shall only treat the case when v_1 is a con vertex, as a similar argument covers the other case. Clearly $M(T_{n1}).E_{v_1}$ is uniform of corank two and, if $|E_{u_1}| \geq 2$, then $M(T_{n1})|E_{u_1}$ is uniform of rank two. Now let w be a vertex of T other than u_1 or v_1. If w is a del vertex of T_{n1}, then it is a del vertex of $T_{(n-1)1}$ and so every 3–element subset X of E_w is a triangle of $M(T_{(n-1)1})$. Since $M(T_{(n-1)1})|X = M(T_{n1})|X$ for every such set X, it follows that $M(T_{n1})|E_w$ is uniform of rank two. If w is a con vertex of T_{n1}, then it is a con vertex of $T_{(n-1)1}$ and so every 3–element subset Y of E_w is a triad of $M(T_{(n-1)1})$ that is disjoint from $E_{v_1} \cup E_{u_1}$ and hence is disjoint from the closure in $M(T_{(n-1)1})$ of the last set. Thus Y is a triad of the generalized parallel connection across E_{v_1} of $M(T_{(n-1)1})$ and $\Theta_{|E_{v_1}|}$. Now $M(T_{n1})$ is a spanning restriction of this generalized parallel connection. Since $M(T_{n1})$ is 3–connected, it follows that Y, which must contain a cocircuit of this matroid, is actually equal to a cocircuit of $M(T_{n1})$. Thus $M(T_{n1}).E_w$ is uniform of corank two. □

A del-con tree T is *reduced* if there is no vertex v of $V(T)$ such that either $d(v) = 1$ and $|E_v| = 2$, or $d(v) = 2$ and E_v is empty. Given a del-con tree T that is not reduced, one can obtained a reduced del-con tree T' from T by a sequence of the following two operations:

(i) Suppose there is an element v of $V(T)$ such that $d(v) = 1$ and $|E_v| = 2$. Let u be the unique neighbour of v in T. Then T is replaced by the tree that is obtained from it by shrinking v into u.
(ii) Suppose there is an element v of $V(T)$ such that $d(v) = 2$ and E_v is empty. Let u and w be the neighbours of v in T. Then u and w have the same second coordinate. Let $T/\{uv, vw\}$ denote the tree obtained from T by contracting the edges $\{u, v\}$ and $\{v, w\}$. Then T is replaced by $T/\{uv, vw\}$ with all vertices of $T/\{uv, vw\}$ retaining their labels from T except the vertex corresponding to u, v, and w. That vertex has $E_u \cup E_w$ as its first coordinate, and its second coordinate is the second coordinate of u and w.

Lemma 4.6. *Let T be a del-con tree and let T' be obtained from T by applying either of the reduction operations above. Then $M(T) = M(T')$.*

Proof. Suppose there is a vertex v of T such that $d(v) = 1$ and $|E_v| = 2$. Let u be the unique neighbour of v in T and let T' be the del-con tree obtained from T by shrinking v into u. By definition, either $M(T) = \nabla_{E_v}(M(T'))$ or $M(T) = \Delta_{E_v}(M(T'))$ depending on whether v is a del or con vertex of T, respectively. Since $|E_v| = 2$, it follows that, in both cases, $M(T) = M(T')$.

Now suppose that v is a vertex of T such that $d(v) = 2$ and $|E_v| = 0$. Let u and w be the neighbours of v in T. The graph $T - v$ has exactly two components, T_u and T_w containing u and w, respectively. From T, we construct a sequence of del-con trees as follows. Pick a vertex of T_u that is the maximum distance from u, and hence has degree one, and, in T, shrink this vertex into its neighbour. Repeat this process until the only remaining vertex of T_u is u itself. Let T'_u be the del-con tree that is obtained at the conclusion of this process. Now consider T_w. Pick a vertex of it that is the maximum distance from w and, in T'_u, shrink this vertex into its neighbour. Repeat this process until the only remaining vertex of T_w is w itself. We now have a del-con tree T_3 with vertices u, v, and w whose second coordinates match their second coordinates in T and whose first coordinates are, respectively, E'_u, $\emptyset$, and E'_w where $E'_y = \cup_{x \in V(T_y)} E_x$. Finally, let T_2 and T_1 be obtained from T_3 and T_2, respectively, by shrinking u into v and shrinking w into v. We have now constructed a chain of del-con trees whose last term is T and whose first three terms are T_1, T_2, and T_3.

Let $E = E'_u \cup E'_w$. Now v is either a del or a con vertex of T. In the first case, $M(T_1)$ has ground set E and is isomorphic to $U_{2,|E|}$. Moreover, since $M(T_3) = \Delta_{E'_u}(\Delta_{E'_w}(M(T_1)))$, it follows by Lemma 4.1 that $M(T_3)$ has ground set E and is isomorphic to $U_{|E|-2,|E|}$. A similar argument shows that, if v is a con vertex of T, then $M(T_3)$ has ground set E and is isomorphic to $U_{2,|E|}$. In both cases, $M(T_3)$ is the dual of $M(T_1)$.

The sequence of shrinkings that produced T_3 from T induces a corresponding sequence when applied to T' and produces a tree T'_3 with a single vertex whose first coordinate is E and whose second coordinate matches that of u in T. Thus $M(T'_3) = M(T_3)$ and hence $M(T') = M(T)$. □

Our interest in del-con trees is that they give us a convenient way to deal with members of $\bigcup_{m\geq 4}\Lambda_m$. Indeed, every matroid in $\bigcup_{m\geq 4}\Lambda_m$ can be described by a del-con tree. To see this, note that if M is in $\bigcup_{m\geq 4}\Lambda_m$, then M can be obtained from $U_{2,m}$ by a sequence of operations each consisting of a Δ–exchange or a ∇–exchange. This sequence of matroids beginning with $U_{2,m}$ induces a chain of del-con trees beginning with a single-vertex tree whose vertex is labelled $(E(M), \text{del})$. The final tree in this chain is a del-con tree corresponding to M.

Now we consider some examples of del-con trees and their associated matroids. Let R_7 be the matroid whose geometric representation is shown in Figure 4. Let $E(R_7) = \{1, 2, \ldots, 7\}$ and let $\{1, 2, 3\}$ and $\{4, 5, 6\}$ be the triangles of R_7. If T_{R_7} is the del-con tree that is a path consisting of three vertices labelled, in order, $(\{1, 2, 3\}, \text{del})$, $(\{7\}, \text{con})$, and $(\{4, 5, 6\}, \text{del})$, then $R_7 = M(T_{R_7})$. Moreover, T_{R_7} is a reduced del-con tree. Note that we can also describe R_7 with the del-con tree that is a path consisting of four vertices labelled, in order, $(\{1, 2\}, \text{con})$, $(\{3\}, \text{del})$, $(\{7\}, \text{con})$, and $(\{4, 5, 6\}, \text{del})$, but this last del-con tree is not reduced.

We show next that the del-con tree corresponding to the dual $M^*(T)$ of $M(T)$ can be readily obtained from T. Let T^* denote the tree obtained from T by changing the second coordinate of the vertex labels so that all del vertices in T become con vertices in T^* and all con vertices in T become del vertices in T^*.

Lemma 4.7. *Let T be a del-con tree. Then $M^*(T) \cong M(T^*)$.*

Proof. We argue by induction on the cardinality of $V(T)$. Suppose that T consists of exactly one vertex v. If v is a del vertex, then $M(T)$ is $U_{2,|E_v|}$ and so $M^*(T)$ is $U_{|E_v|-2,|E_v|}$. Now v is a con vertex in T^* so $M(T^*)$ is $U_{|E_v|-2,|E_v|}$. Hence the lemma holds for $|V(T)| = 1$. Suppose that T consists of exactly two vertices u and v. Without loss of generality, we may assume that u is a del vertex and v is a con vertex. Let $E = E_u \cup E_v$. Then $M(T)$ is the matroid $\Delta_{E_v}(U_{2,|E|})$. By Lemma 4.4,

$$\begin{aligned}[\Delta_{E_v}(U_{2,|E|})]^* &= [\nabla_{E_u}(U_{|E|-2,|E|})]^* \\ &= [(\Delta_{E_u}(U_{2,|E|}))^*]^* \\ &= \Delta_{E_u}(U_{2,|E|}).\end{aligned}$$

The last matroid is $M(T^*)$. Hence the lemma also holds for $|V(T)| = 2$. Let T be a del-con tree such that $|V(T)| = n$, where $n \geq 3$. Suppose that the lemma holds for $|V(T)| = n-1$. Let v be a degree-one vertex of T and let u be the unique neighbour of v in T. Let T_v be the tree obtained from T by shrinking v into u. Since $|V(T_v)| = n - 1$, it follows by the induction assumption that $M^*(T_v) = M(T_v^*)$. Assume first that v is a con vertex of T. Then v is a del vertex of T^* and therefore, as u is a con vertex of T^*,

$$\begin{aligned}M(T^*) &= \nabla_{E_v}(M(T_v^*)) \\ &= [\Delta_{E_v}(M^*(T_v^*))]^* \\ &= [\Delta_{E_v}(M(T_v))]^* \qquad \text{by the induction assumption.}\end{aligned}$$

But $\Delta_{E_v}(M(T_v)) = M(T)$ and so $M^*(T) = M(T^*)$. Since $(T^*)^* = T$, it follows that the lemma also holds when v is a del vertex of T. This completes the proof of Lemma 4.7. □

We show next that the removal of an element e from a del-con tree T corresponds to the deletion or contraction of e from $M(T)$ depending on whether e is in a del or a con class of T.

Lemma 4.8. *Let v be a vertex of a del-con tree T and let $E = \bigcup_{u \in V(T)} E_u$. Suppose that $|E| \geq 5$ and that if v has degree one, then $|E_v| \geq 3$. Let e be an element of E_v and let $T\backslash e$ denote the tree obtained from T by removing e from E_v.*

(i) *If e is in a del class of T, then $M(T\backslash e) = M(T)\backslash e$.*
(ii) *If e is in a con class of T, then $M(T\backslash e) = M(T)/e$.*

Proof. We first prove (i). Let $|V(T)| = n$ and construct a chain of del-con trees as follows. Let $T_n = T$. For each i in $\{2, 3, \ldots, n\}$, find a vertex in T_i that is a maximum distance from v and shrink that vertex into its unique neighbour to produce T_{i-1}. Then T_1 has v as its unique vertex and this vertex is labelled (E, del). Moreover, if $T_i\backslash e$ is obtained from T_i by removing e from the del class corresponding to v, then it is clear that $T_1\backslash e, T_2\backslash e, \ldots, T_n\backslash e$ is a chain of del-con trees and $T_n\backslash e = T\backslash e$. Also, for all i, exactly the same Δ– or ∇–exchange that produced $M(T_i)$ from $M(T_{i-1})$ produces $M(T_i\backslash e)$ from $M(T_{i-1}\backslash e)$. We shall show, by induction, that $M(T_j\backslash e) = M(T_j)\backslash e$ for all j in $\{1, 2, \ldots, n\}$. Certainly $M(T_1\backslash e) = M(T_1)\backslash e$ since $M(T_1\backslash e)$ and $M(T_1)$ are rank-2 uniform matroids on $E - e$ and E, respectively. Assume that $M(T_{j-1}\backslash e) = M(T_{j-1})\backslash e$. Now either (a) $M(T_j) = \nabla_A(M(T_{j-1}))$, or (b) $M(T_j) = \Delta_A(M(T_{j-1}))$. Consider the first case. Clearly $M(T_j\backslash e) = \nabla_A(M(T_{j-1}\backslash e))$. Since this ∇_A–exchange is defined, it follows that A has rank two and is coindependent in $M^*(T_{j-1}\backslash e)$. Thus, by the induction assumption, A has rank two and is coindependent in $M^*(T_{j-1})/e$. But, since $\nabla_A(M(T_{j-1}))$ is also defined, A has rank two and is coindependent in $M^*(T_{j-1})$. Thus e is not in the closure of A in $M^*(T_{j-1})$. Hence

$$\begin{aligned} M(T_j\backslash e) &= \nabla_A[M(T_{j-1}\backslash e)] \\ &= \nabla_A[M(T_{j-1})\backslash e] && \text{by the induction assumption,} \\ &= \nabla_A[M(T_{j-1})]\backslash e && \text{by the dual of Lemma 2.16(ii),} \\ &= M(T_j)\backslash e \end{aligned}$$

A similar argument establishes that $M(T_j)\backslash e = M(T_j\backslash e)$ in case (b). We conclude, by induction, that $M(T_n)\backslash e = M(T_n\backslash e)$.

The proof of (ii) follows by a straightforward combination of (i) and the preceding lemma. □

The following is an immediate consequence of the last lemma.

Corollary 4.9. *Let T' be a del-con tree that is obtained from a del-con tree T by a sequence of operations each consisting of removing an element from a vertex class, or reducing the tree. Then $M(T')$ is a minor of $M(T)$.*

Recall from the introduction that P_6 is the matroid that is obtained from a 6–point line by a single $\Delta - Y$ exchange.

Lemma 4.10. *Let e be an edge of a reduced del-con tree T and let V_1 and V_2 be the vertex sets of the components of the graph obtained from T by deleting e. If $\{x_1, y_1, z_1\} \subseteq \bigcup_{v \in V_1} E_v$ and $\{x_2, y_2, z_2\} \subseteq \bigcup_{v \in V_2} E_v$, then either*

(i) *$M(T)$ has a P_6–minor on $\{x_1, y_1, z_1, x_2, y_2, z_2\}$ in which $\{x_1, y_1, z_1\}$ is a triangle or a triad; or*
(ii) *$M(T)$ or its dual has an R_7–minor in which $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ are both triangles.*

Proof. Suppose, to the contrary, that $M(T)$ has no such minor. Moreover, assume that $|E(M(T))|$ is minimal. We break the proof into two cases. In the first case, suppose that T has at least three degree-one vertices. Then, without loss of generality, $T[V_1]$, the subgraph of T induced by V_1, contains at least two degree-one vertices of T. Choose one of these vertices of $T[V_1]$, say v, so that E_v contains an element w where $w \notin \{x_1, y_1, z_1\}$. By condition (iii) in the definition of a del-con tree, such an element exists. Let T' be the tree obtained from T by first removing w and then, if possible, reducing the resulting tree. In T', the edge e still separates $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$. Therefore, by the last corollary, $M(T')$ has a minor of the required type. Since $|E(M(T'))| < |E(M(T))|$, the choice of $M(T)$ is contradicted. Hence T does not have at least three degree-one vertices.

For the second case, suppose that T has exactly two degree-one vertices. Then T is a path. If one of the degree-one vertices of T, say v, has the property that E_v contains an element w such that $w \notin \{x_1, y_1, z_1, x_2, y_2, z_2\}$, then w can be removed from T and, as in the first case, the choice of $M(T)$ is contradicted. Thus the subsets of $E(M(T))$ associated with the degree-one vertices of T are $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$. Suppose first that T has an even number of vertices. Then one degree-one vertex of T is a del vertex and the other is a con vertex. If T has no degree-two vertices, then $M(T)$ is isomorphic to P_6; a contradiction. If T has a degree-two vertex, then by removing an element from the corresponding vertex class and reducing the resulting tree, we again obtain a contradiction to the choice of $M(T)$. We conclude that T has an odd number of vertices. But a similar argument to that just given now shows that $M(T)$ has an R_7– or R_7^*–minor depending on whether the degree-one vertices of T are del or con vertices, respectively. This contradiction completes the proof of the lemma. □

As noted in [8], it is immediate from the definition of clones that elements x and x' are clones in M if and only if they are clones in M^*. Also, recall from the last section that if x and x' are clones of a matroid M, and N is a minor of M containing $\{x, x'\}$, then x and x' are clones in N. We shall use both these facts in the next result, the first of two corollaries of the last lemma.

Corollary 4.11. *Let T be a reduced del-con tree. Then elements x and x' of $M(T)$ are in the same vertex class of T if and only if x and x' are clones in $M(T)$.*

Proof. Suppose first that x and x' are in different vertex classes of T. Clearly T has at least two vertices and so T has at least two degree-one vertices. Let e be an edge of T such that x and x' are in different components of the graph obtained

from T by deleting e. Now T is a reduced del-con tree. Therefore, by Lemma 4.10, either $M(T)$ has a P_6–minor in which y_1 is in a triad, y_2 is in a triangle and $\{y_1, y_2\} = \{x, x'\}$, or $M(T)$ or its dual has an R_7–minor in which x and x' are in different triangles. In each case, x and x' are not clones in the distinguished minor. Hence x and x' are not clones in $M(T)$.

To prove the converse, suppose that x and x' are in the same vertex class of T. We argue by induction on the cardinality of $V(T)$ that x and x' are clones in $M(T)$. This is clearly true if T has exactly one vertex. Assume it true for $|V(T)| < n$ and let $|V(T)| = n \geq 2$. Let u be a degree-one vertex of T such that $\{x, x'\} \cap E_u$ is empty. By duality, we may assume that u is a del vertex of T. Let w be the unique neighbour of u in T and let T' be the reduced del-con tree obtained from T by shrinking u into w. By the induction assumption, x and x' are clones in $M(T')$. Therefore, as $\{x, x'\} \cap E_u$ is empty, it follows by Lemma 2.20 that x and x' are clones in $\Delta_{E_u} M(T')$. But this last matroid is $M(T)$ and so x and x' are clones in $M(T)$. This completes the proof of Corollary 4.11. □

Without the requirement that T is reduced, Corollary 4.11 may fail. For example, let T be a del-con tree consisting of three vertices u, v, and w, where $|E_v| = 0$ and u and w are degree-one con vertices such that $|E_u| = |E_w| = 3$. Then $M(T)$ is isomorphic to $U_{4,6}$. But, if $x \in E_u$ and $x' \in E_w$, then x and x' are clones in $M(T)$ belonging to different vertex classes of T.

Corollary 4.12. *Let T be a reduced del-con tree. If x, y, and z are three elements of $E(M(T))$ such that no vertex class of T contains all three, then $\{x, y, z\}$ is neither a triangle nor a triad of $M(T)$.*

Proof. Clearly, we may assume, without loss of generality, that there is an edge e of T such that x and y are in a different component from z in the graph obtained from T by deleting e. Then, by Lemma 4.10, $\{x, y, z\}$ is contained in a minor of $M(T)$ that is isomorphic to one of P_6, R_7, or R_7^* but has $\{x, y, z\}$ as neither a triangle nor triad. Since none of these three minors has a circuit or cocircuit of size less than three, it follows that $\{x, y, z\}$ is neither a triangle nor a triad of $M(T)$. □

Next we describe the 3–separations of the members of $\bigcup_{m \geq 4} \Lambda_m$. Since every matroid in this set is 3–connected, all such 3–separations are exact. But, as $\Lambda_4 = \{U_{2,4}\}$ and $\Lambda_5 = \{U_{2,5}, U_{3,5}\}$, every matroid in $\Lambda_4 \cup \Lambda_5$ has infinite connectivity and so has no 3–separations. Thus we shall confine attention to the members of $\bigcup_{m \geq 6} \Lambda_m$.

Lemma 4.13. *Let M be a member of Λ_m where $m \geq 6$, and let T_M be a reduced del-con tree for which $M = M(T_M)$. Let v be a vertex of T_M and let $\{X, Y\}$ be a partition of $E(M)$ into subsets each of size at least three such that, for every component T' of $T_M - v$, the set $\bigcup_{z \in V(T')} E_z$ is contained in either X or Y. Then $\{X, Y\}$ is a 3–separation of M.*

Proof. By Lemmas 4.7 and 4.8, it suffices to show that the result holds when v is a del vertex of T_M. We argue by induction on $|V(T_M)|$ noting first that if

$|V(T_M)| = 1$, then the result is clear. Now let $|V(T_M)| = n$ where $n \geq 2$, and assume that the lemma holds for all matroids that correspond to reduced del-con trees having fewer vertices. If v is a degree-one vertex of T_M, then the result certainly holds. Therefore we may assume that v is not a degree-one vertex. Let u be a degree-one vertex of T_M and let w be its unique neighbour in T_M. Let T_u be the tree obtained from T_M by shrinking u into w. Then M is either $\Delta_{E_u} M(T_u)$ or $\nabla_{E_u} M(T_u)$ depending on whether u is a con or a del vertex of T_M. Now, by the induction assumption, if $\{X, Y\}$ is a partition of $E(M)$ into subsets each of size at least three such that, for every component T'' of $T_u - v$, the set $\bigcup_{z \in V(T'')} E_z$ is contained in either X or Y, then $\{X, Y\}$ is a 3–separation of $M(T_u)$. Therefore, as u and w are in the same component of $T_M - v$, the lemma is proved provided we can show that $\{X, Y\}$ is also a 3–separation of M. But, by the definitions of segment-cosegment and cosegment-segment exchange, it is easy to deduce that this is indeed the case. □

The next lemma shows that the only 3–separations of a member of Λ_m are those described in the last lemma.

Lemma 4.14. *Let M be a member of Λ_m where $m \geq 6$, and let T_M be a reduced del-con tree for which $M = M(T_M)$. If $\{X, Y\}$ is a 3–separation of M, then there is a vertex v of T_M such that, for every component T' of $T_M - v$, the set $\bigcup_{z \in V(T')} E_z$ is contained in either X or Y.*

Proof. Assume that M has a 3–separation $\{X, Y\}$ that is not of the type described. Colour the elements of X red and the elements of Y green. Let v be a vertex of T_M. If E_v is empty, we call v *colourless*. If E_v is non-empty and all of its elements are the same colour, we assign that colour to v itself. A subgraph of T_M is *monochromatic* if it does not contain both red and green vertices.

We begin by showing the following.

4.14.1. *T_M has no edge e such that neither component of $T_M - e$ is monochromatic.*

Proof. Assume, to the contrary, that T_M has such an edge e. Let V_1 and V_2 be the vertex sets of the components of $T_M - e$. For each i in $\{1, 2\}$, let r_i and g_i, respectively, be a red and a green element of $\bigcup_{u \in V_i} E_u$. The last set has at least three elements as do both X and Y. Thus, by relabelling if necessary, we may assume that $\bigcup_{u \in V_1} E_u$ contains a red element r_1' such that $r_1' \neq r_1$ and $\bigcup_{u \in V_2} E_u$ contains a green element g_2' such that $g_2' \neq g_2$. Therefore, by Lemma 4.10, either

(i) M has a P_6–minor on $\{r_1, g_1, r_1', r_2, g_2, g_2'\}$ in which $\{r_1, g_1, r_1'\}$ is a triangle or a triad; or
(ii) M or M^* has an R_7–minor in which $\{r_1, g_1, r_1'\}$ and $\{r_2, g_2, g_2'\}$ are both triangles.

Furthermore, since this minor has at least three red and at least three green elements, the minor has a 3–separation induced by its sets of red and green elements. But the only 3–separation of P_6 has the triangle on one side and the triad on the

other. Moreover, the only 3–separations of R_7 contain a triangle on each side. By (i) and (ii), neither $\{r_1, r_1', r_2\}$ nor $\{g_1, g_2, g_2'\}$ is a triangle or a triad in the relevant minor. This contradiction completes the proof of (4.14.1). □

By (4.14.1), for each edge e in T_M, at least one component of $T_M - e$ is monochromatic. This implies that T_M has at most one vertex v for which E_v contains both red and green elements. If there is such a vertex v, then every component of $T_M - v$ must be monochromatic and so $\{X, Y\}$ is a 3–separation of the type described in the lemma. This contradiction implies that no such vertex exists in T_M. Next we show the following.

4.14.2. *If v is a vertex of T_M, then exactly one of the components of $T_M - v$ is not monochromatic. Moreover, the monochromatic components of $T_M - v$ all have the same colour as each other and, unless v is colourless, this colour matches that of v.*

Proof. Suppose first that $T_M - v$ has two components, T_1 and T_2, that are not monochromatic. Let e be the edge connecting T_1 to v in T_M. Then neither component of $T_M - e$ is monochromatic and (4.14.1) is contradicted. Thus there is at most one component of $T_M - v$ that is not monochromatic. If there is no such component, then $\{X, Y\}$ is a 3–separation of the type described in the lemma. This contradiction completes the proof of the first part.

To establish the second part, consider the component of $T_M - v$ that is not monochromatic, and let w be the neighbour of v in this component. Since there are both red and green elements in one component of $T_M - vw$, the other component must be monochromatic, and the second part of (4.14.2) follows. □

We now use (4.14.2) to complete the proof of the lemma. The choice of $\{X, Y\}$ ensures that T_M must have at least one red and at least one green vertex. Let $v_0 v_1 \dots v_n$ be a minimum-length path in T_M that begins at a red vertex and ends at a green vertex. Then all of $v_1, v_2, \dots, v_{n-1}$ are colourless. A straightforward induction argument shows that, for all i in $\{0, 1, \dots, n-1\}$, all the components of $T_M - v_i$ are red except for the one containing v_n, and the latter is non-monochromatic. By symmetry, for all i in $\{n, n-1, \dots, 1\}$, all the components of $T_M - v_i$ are green except for the one containing v_0, and the latter is non-monochromatic. In particular, if $n > 1$, then $T_M - v_1$ has two non-monochromatic components, one containing v_0 and the other containing v_n. This contradiction to (4.14.2) implies that $n = 1$. Now consider $T_M - v_0 v_1$. By (4.14.1), it certainly has a monochromatic component, and we may assume that it is the one containing v_0. But deleting the green vertex v_1 from T_M produces a red component, namely the one containing v_0. This contradiction to (4.14.2) completes the proof of Lemma 4.14. □

We shall say that the 3–separation $\{X, Y\}$ in the last lemma is *based on* a del or con class depending on whether the distinguished vertex v is a del or con vertex of T_M. The next lemma determines when a certain 3-separation of a member M of $\bigcup_{m \geq 6} \Lambda_m$ induces a 3-separation of a 3–connected single-element extension of M.

Lemma 4.15. *Let M' be a 3–connected matroid such that $M'\backslash e$ is a member M of $\bigcup_{m\geq 6}\Lambda_m$. Let $\{X, Y\}$ be a 3–separation of M based on a del class E_v of a reduced del-con tree T_M for which $M(T_M) = M$. Then either*

(i) *$\{X \cup e, Y\}$ or $\{X, Y \cup e\}$ is a 3–separation of M'; or*
(ii) *M' has a minor isomorphic to a single-element extension of R_7^* in which neither triad of R_7^* is preserved.*

Proof. Let M' be a counterexample to the lemma for which $|E(M')|$ is a minimum. As (i) fails, $r_{M'}(X\cup e) = r_M(X)+1$ and $r_{M'}(Y\cup e) = r_M(Y)+1$. Thus $r(M') > 2$, so T_M has more than one vertex.

Suppose that v has degree one. By Lemma 4.14, we may assume that X contains E_u for all u in $V(T_M) - v$. Then $r_M(X) = r(M)$, so

$$r(M') \geq r_{M'}(X \cup e) = r_M(X) + 1 > r(M);$$

a contradiction. Therefore the degree of v exceeds one, and hence T_M has at least three vertices. Assume that T_M has a non-empty del class E_u other than E_v. Let x be an element of E_u and assume, without loss of generality, that E_u is contained in X. By Lemma 4.8, $M\backslash x = M(T_M\backslash x)$, so $M\backslash x$ is a member of $\bigcup_{m\geq 5}\Lambda_m$. Hence, by Lemma 4.3, $M\backslash x$ is 3–connected. In particular, X is not a triad. As $r(X) + r(Y) - r(M) = 2$ and Y is non-spanning, it follows that $|X| \geq 4$. Thus $\{X - x, Y\}$ is a 3–separation of $M\backslash x$. Moreover, this 3–separation is based on the del class E_v of the reduced del-con tree obtained from $T_M\backslash x$. As $M'\backslash x$ is 3–connected, the choice of M' implies that $M'\backslash x$ obeys the lemma. But (ii) does not hold for M' so $M'\backslash x$ cannot have a minor of the specified type. Moreover, $r_{M'}((X - x) \cup e) = r_M(X - x) + 1$ and $r_{M'}(Y \cup e) = r_M(Y) + 1$, so neither $\{(X - x) \cup e, Y\}$ nor $\{X - x, Y \cup e\}$ is a 3–separation of $M'\backslash x$. This contradiction implies that T_M has no non-empty del classes other than, possibly, E_v. Therefore every degree-one vertex of T_M is a con vertex for which, since T_M is reduced, the associated con class has size at least three.

Now suppose that X contains two distinct triads X_1 and X_2 of M each of which is contained in a con class of T_M corresponding to a degree-one vertex. Then $r_M(Y) \leq r(M\backslash(X_1 \cup X_2)) \leq r(M) - 2$. Thus $X_1 \cup X_2$ contains an element c that is not in $\mathrm{cl}_{M'}(Y \cup e)$. Now, in M/c, we have

$$\begin{aligned} r_{M/c}(X - c) + r_{M/c}(Y) - r(M/c) &= r_M(X) - 1 + r(Y \cup c) - 1 - (r(M) - 1) \\ &= r_M(X) + r_M(Y) - r(M) \\ &= 2. \end{aligned}$$

Thus $\{X-c, Y\}$ is a 3–separation of M/c. Moreover, this 3–separation is based on a del class of the reduced del-con tree obtained from $T_M\backslash c$. Since $M(T_M\backslash c) = M/c$, Lemma 4.8 implies that M/c is 3–connected. We shall show next that M'/c is 3–connected and hence that M'/c obeys the lemma. If M'/c is not 3–connected, then, as M' and $M'\backslash e/c$ are both 3–connected, $\{e, c\}$ is contained in a triangle of M'. As $r_{M'}(X \cup e) = r_M(X) + 1$, the third element of this triangle is not in X; nor is it in Y since $c \notin \mathrm{cl}_{M'}(Y \cup e)$. Thus M'/c is indeed 3–connected. But, as is easily checked, neither $\{(X - c) \cup e, Y\}$ nor $\{X - c, Y \cup e\}$ is a 3–separation of

M'/c. Since M'/c certainly cannot have a minor of the type specified in (ii), we have a contradiction to the choice of M'. We conclude that X does not contain two distinct triads with the specified properties. By symmetry, nor does Y. Thus each con class corresponding to a degree-one vertex of T_M has size three. Moreover, T_M has exactly two such con classes, one in X and the other in Y. Also, since T_M is reduced and has more than one vertex but has at most one non-empty del class, it follows that T_M has exactly three vertices and $|E_v| \geq 1$.

Let x and y be the neighbours of v in T_M where $E_x \subseteq X$ and $E_y \subseteq Y$. Then $|E_x| = |E_y| = 3$. Since $|E(M)| \geq 7$, one side of the 3–separation of M, say X, has at least four elements. Thus there is an element f in $X \cap E_v$. Clearly $\{X - f, Y\}$ is a 3–separation of $M\backslash f$. Moreover, $r_{M'\backslash f}((X - f) \cup e) = r_{M\backslash f}(X) + 1$ and $r_{M'\backslash f}(Y \cup e) = r_{M\backslash f}(Y) + 1$, so neither $\{(X - f) \cup e, Y\}$ nor $\{X - f, Y \cup e\}$ is a 3–separation of $M'\backslash f$. If $|E_v| > 1$, then $E_v - f$ is non-empty and therefore $M'\backslash f$ contradicts the choice of M'. Thus we may assume that $|E_v| = 1$.

We now know that M is R_7^* and M has a 3–separation $\{X, Y\}$ such that neither $\{X \cup e, Y\}$ nor $\{X, Y \cup e\}$ is a 3–separation of M'. Let T_1^* and T_2^* denote the two triads of R_7^*, and let z denote the unique element of $E(R_7^*) - (T_1^* \cup T_2^*)$. By symmetry, we may assume that $(X, Y) = (T_1^*, T_2^* \cup z)$. Then

$$r_{M'}(T_1^* \cup e) = r_M(T_1^*) + 1 = 4$$

and

$$r_{M'}((T_2^* \cup z) \cup e) = r_M(T_2^* \cup z) + 1 = 4.$$

Hence neither T_1^* nor T_2^* is a triad of M'. We conclude that M' is a 3–connected single-element extension of R_7^* with no triads. This last contradiction completes the proof of the lemma. □

The next result shows that, for every member of $\bigcup_{m\geq 4} \Lambda_m$ except $U_{2,4}$, there is a unique associated reduced del-con tree.

Lemma 4.16. *Let T and T' be reduced del-con trees. If $M(T) = M(T')$, then either $M(T) \cong U_{2,4}$ and $|V(T)| = |V(T')| = 1$, or there is a bijection $\phi : V(T) \to V(T')$ such that, for all u and v in $V(T)$,*

(i) *u and v are neighbours in T if and only if $\phi(u)$ and $\phi(v)$ are neighbours in T'; and*
(ii) *the vertex labels of v and $\phi(v)$ are equal.*

Proof. Let $E = \bigcup_{v\in V(T)} E_v$. We prove the lemma by induction on $|V(T)|$. Suppose that $|V(T)| = 1$. Then $M(T)$ is isomorphic to a uniform matroid of rank 2 or corank 2. Since all reduced del-con trees associated with such matroids consist of a single vertex, it follows that if T' is a reduced del-con tree such that $M(T) = M(T')$, then either $M(T) \cong U_{2,4}$ and $|V(T')| = 1$, or there is a bijection from $V(T)$ into $V(T')$ with properties (i) and (ii). Thus the lemma holds for $|V(T)| = 1$. Now let $|V(T)| = n \geq 2$ and assume the lemma holds for all reduced del-con trees with fewer vertices. In particular, it follows that $|E| \geq 6$.

Let v be a degree-one vertex of T. By duality, we may assume that v is a del vertex of T. We first show that T' has a degree-one vertex with the same labelling as v in T. Since $M(T) = M(T')$, it follows by Corollary 4.11 that the non-empty vertex classes of T and T' coincide. Therefore, by Lemma 4.5, there is a vertex v' in T' with the same labelling as v in T. It remains to show that v' has degree one. Assume not and let T'_1 be a component of $T' - v'$ and X' be a proper non-empty subset of E_v. Let $X'' = X' \cup (\bigcup_{u \in V(T'_1)} E_u)$. Then, by applying Lemma 4.13, we deduce that $\{X'', E - X''\}$ is a 3–separation of $M(T')$ and hence of $M(T)$. Since E_v meets both X'' and $E - X''$, Lemma 4.14 implies that $\{X'', E - X''\}$ must be a 3–separation of $M(T)$ based on v. But v has degree one in T so every 3–separation of $M(T)$ based on v must have one part that is a subset of E_v. Since neither X'' nor $E - X''$ is a subset of E_v, we have a contradiction. We conclude that v' does indeed have degree one in T'.

Let T_v denote the tree that is obtained from T by shrinking v into its unique neighbour u. Then $M(T_v) = \Delta_{E_v} M(T)$. Let $T'_{v'}$ denote the tree that is obtained from T' by shrinking v' into its unique neighbour u'. Then $M(T'_{v'}) = \Delta_{E_v} M(T')$ and so $M(T'_{v'}) = M(T_v)$. Now $|V(T_v)| = n - 1$. Therefore, by the induction assumption and the fact that both u and u' are con vertices, it follows that there is a bijection $\phi_1 : V(T_v) \to V(T'_{v'})$ with properties (i) and (ii). Consider the function $\phi : V(T) \to V(T')$ defined by $\phi(u) = u'$, $\phi(v) = v'$, and $\phi(w) = \phi_1(w)$ for all $w \in V(T) - \{u, v\}$. As this function is clearly a bijection from $V(T)$ into $V(T')$ with properties (i) and (ii), Lemma 4.16 now follows. □

Evidently, the converse of Lemma 4.16 also holds. We end this section by proving Theorem 1.2, an exponential lower bound on the cardinality of the set of excluded minors for $GF(q)$–representability.

Proof of Theorem 1.2. Since $U_{2,q+2}$ is an excluded minor for $GF(q)$–representability, it follows by Theorem 1.1 that every member of Λ_{q+2} is an excluded minor for $GF(q)$–representability. We shall prove the theorem by bounding below the number of members of Λ_{q+2} for which the associated del-con tree is a path. To construct these paths, we first arrange the elements $1, 2, \ldots, q + 2$ consecutively in a line. There are $q - 3$ gaps between consecutive elements i and $i + 1$ such that $i \in \{3, 4, \ldots, q - 1\}$. In each of these gaps, we choose whether or not to insert a bar. Thus there are 2^{q-3} such sequences consisting of elements and inserted bars. With each of these sequences, we associate a reduced del-con tree, which is a path, defined as follows: for some $k \geq 1$, the bars partition $\{1, 2, \ldots, q + 2\}$ into k non-empty subsets $E_{v_1}, E_{v_2}, \ldots, E_{v_k}$ ordered in the natural way with $1 \in E_{v_1}$. Let $E_{v_1}, E_{v_2}, \ldots, E_{v_k}$ be the first coordinates of the vertex labels of consecutive vertices in a k–vertex path, where the second coordinates alternate between "del" and "con" beginning with "del". Clearly the number of such paths is 2^{q-3} and each is a reduced del-con tree. Dividing by 2 to account for a potential symmetry that arises by beginning the path at the right-hand instead of the left-hand end, we deduce, by Lemma 4.16, that there are at least 2^{q-4} non-isomorphic members of Λ_{q+2} for which the associated reduced del-con tree is a path. The theorem follows immediately. □

It is clear that the bound in Theorem 1.2 can be improved. The point of the theorem is not to provide a sharp bound but rather to show that the number of excluded minors for $GF(q)$–representability is at least exponential in q.

5. Two Theorems on k–regular Matroids

In this section, we prove Theorems 1.3 and 1.4. Each of these proofs relies on the theory of "stabilizers" and "universal stabilizers" initiated in [30] and [8], respectively. We now outline the definitions and results from these papers that will be used in proving Theorems 1.3 and 1.4.

Stabilizers. A *well-closed* class of matroids is one that is minor-closed, closed under isomorphism, and closed under duality. For example, the class of matroids representable over a certain partial field is a well-closed class. Recall that two matrix representations of a matroid over a partial field $\mathbf{P}$ are strongly equivalent if one can be obtained from the other by a sequence of the matrix operations that define equivalent representations, but without needing to apply an automorphism of $\mathbf{P}$.

Let $\mathbf{P}$ be a partial field and let M and N be matroids representable over $\mathbf{P}$ such that N is a minor of M. Then N *stabilizes* M over $\mathbf{P}$ if a $\mathbf{P}$–representation of M is determined up to strong equivalence by a $\mathbf{P}$–representation of any one of its N–minors. In other words, if a $\mathbf{P}$–representation of N can be extended to a $\mathbf{P}$–representation of M, then all such representations of M are strongly equivalent.

Let $\mathcal{N}$ be a well-closed class of $\mathbf{P}$–representable matroids and let N be a matroid in $\mathcal{N}$. Then N is a $\mathbf{P}$*–stabilizer* for $\mathcal{N}$ (or N *stabilizes* $\mathcal{N}$ *over* $\mathbf{P}$) if N stabilizes every 3–connected matroid in $\mathcal{N}$ with an N–minor. Surprisingly, determining whether a matroid is a $\mathbf{P}$–stabilizer is a finite task.

Theorem 5.1. ([30, Theorem 5.8]) *Let $\mathcal{N}$ be a well-closed class of matroids representable over a partial field $\mathbf{P}$ and let N be a 3–connected matroid in $\mathcal{N}$. Then N stabilizes $\mathcal{N}$ over $\mathbf{P}$ if and only if N stabilizes every 3–connected matroid M in $\mathcal{N}$ that has one of the following properties.*

(i) *M has an element x such that $M\backslash x = N$.*
(ii) *M has an element y such that $M/y = N$.*
(iii) *M has a pair of elements x and y such that $M\backslash x/y = N$, and both $M\backslash x$ and M/y are 3–connected.*

We can use stabilizers to bound the number of inequivalent representations of a matroid over a partial field. The next result combines Proposition 5.4 and Corollary 5.5 of [30]. As for fields, a matroid M is *uniquely representable* over a partial field $\mathbf{P}$ if all $\mathbf{P}$–representations of M are equivalent. The class of all $\mathbf{P}$–representable matroids will be denoted by $\mathcal{M}(\mathbf{P})$.

Proposition 5.2. *Let N be a $\mathbf{P}$–stabilizer for $\mathcal{M}(\mathbf{P})$.*

(i) *If N has n inequivalent $\mathbf{P}$–representations, then every 3–connected matroid in $\mathcal{M}(\mathbf{P})$ with an N–minor has at most n inequivalent $\mathbf{P}$–representations.*
(ii) *If N is uniquely representable over $\mathbf{P}$, then every 3–connected matroid in $\mathcal{M}(\mathbf{P})$ with an N–minor is uniquely representable over $\mathbf{P}$.*

Universal stabilizers. Let x be an element of the matroid M. The matroid M' is obtained from M by *cloning x with x'* if M' is a single-element extension of M by x', and x and x' are clones in M'. If it is not possible for x to be cloned with x' so that $\{x, x'\}$ is independent, then x is *fixed* in M. Dually, x is *cofixed* in M if no single-element coextension of M by x' has the property that $\{x, x'\}$ is a coindependent pair of clones in this coextension. The next result [8, Proposition 4.7] enables us to determine that an element is fixed in a matroid from the fact that it is fixed in certain minors of the matroid.

Proposition 5.3. *Let x be an element of a matroid M.*

(i) *If M has an element e such that x is fixed in $M\backslash e$, then x is fixed in M.*
(ii) *If M has distinct elements e and f such that $\{e, f, x\}$ is independent in M, and x is fixed in both M/e and M/f, then x is fixed in M.*

Let $\mathcal{N}$ be a well-closed class of matroids. Let N be a 3–connected member of $\mathcal{N}$. Then N is a *universal stabilizer* for $\mathcal{N}$ if the following holds: whenever M and $M\backslash x$ are 3–connected matroids in $\mathcal{N}$ for which $M\backslash x$ has an N–minor, the element x is fixed in M; and, whenever M and M/x are 3–connected matroids in $\mathcal{N}$ for which M/x has an N–minor, the element x is cofixed in M. Just as for stabilizers, the task of determining if a matroid is a universal stabilizer for a well-closed class of matroids can be decided by a finite case check.

Theorem 5.4. ([8, Theorem 6.1]) *Let N be a 3–connected matroid in a well-closed class of matroids $\mathcal{N}$ and suppose that $|E(N)| \geq 2$. Then N is a universal stabilizer for $\mathcal{N}$ if and only if the following three conditions hold.*

(i) *If M is a 3–connected member of $\mathcal{N}$ with an element x such that $M\backslash x = N$, then x is fixed in M.*
(ii) *If M is a 3–connected member of $\mathcal{N}$ with an element y such that $M/y = N$, then y is cofixed in M.*
(iii) *If M is a 3–connected member of $\mathcal{N}$ with a pair of elements x and y such that $M\backslash x/y = N$, and $M\backslash x$ is 3–connected, then x is fixed in M.*

Let N be a member of a well-closed class of matroids $\mathcal{N}$. The notion of a universal stabilizer was introduced in [8] to identify the underlying matroid structure that ensures that, whenever $\mathbf{P}$ is a partial field over which N is representable, N is a $\mathbf{P}$–stabilizer for all members of $\mathcal{N}$ which are $\mathbf{P}$–representable. Indeed, we have the following result [8, Theorem 5.1].

Theorem 5.5. *Let N be a 3–connected matroid that is a universal stabilizer for a well-closed class $\mathcal{N}$ of matroids and let $\mathbf{P}$ be a partial field over which N is representable. Then N is a $\mathbf{P}$–stabilizer for the class $\mathcal{N} \cap \mathcal{M}(\mathbf{P})$.*

One last set of preliminaries is required. A flat of a matroid is *cyclic* if it is the union of a set of circuits. Let x and y be elements of a matroid M. Then x is *freer than y in M* if every cyclic flat of M that contains x also contains y. Furthermore, if x is freer than y, but y is not freer than x, then x is *strictly freer* than y. The next, and last, result of these preliminaries is a combination of Proposition 4.4(i) and Proposition 4.5(iv) of [9].

Proposition 5.6. *Let x and y be distinct elements of a matroid M.*

(i) *If x is fixed in M/y, but not in M, then x is freer than y.*
(ii) *If x is strictly freer than y in M and x is not a coloop of M, then y is not cofixed in M.*

We noted in the introduction that the pair $(\mathcal{A}_k, \mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k))$ is a partial field denoted by $\mathbf{R}_k$. Moreover, a matroid is representable over $\mathbf{R}_k$ if and only if it is k–regular, that is, if and only if it can be represented by a k–unimodular matrix. Extending these ideas, we let $\mathcal{A}_\omega$ be the subset of $\mathbf{Q}(\alpha_1, \alpha_2, \ldots)$ consisting of all products of integral powers of differences of distinct elements in $\{0, 1, \alpha_1, \alpha_2, \ldots\}$. Then $(\mathcal{A}_\omega, \mathbf{Q}(\alpha_1, \alpha_2, \ldots))$ is a partial field, which we denote by $\mathbf{R}_\omega$. Clearly a matroid is $\mathbf{R}_\omega$–representable if and only if it is ω–regular.

Most of the work in proving Theorems 1.3 and 1.4 goes into the following two things: for all $k \geq 1$, (i) establishing that every member of Λ_{k+3} is a universal stabilizer for the class of k–regular matroids; and (ii) determining the minor-minimal 3–connected ω–regular matroids that are not stabilized over $\mathbf{R}_\omega$ by some member of Λ_{k+3}. These two tasks are completed in Lemmas 5.25 and 5.29, respectively. The ground work for these lemmas was laid in the last section. However, we still need to establish some results particular to ω–regular matroids before we are in a position to prove these lemmas. In particular, as we will use Theorems 5.1 and 5.4 in the proof, we need to determine all 3–connected ω–regular matroids that are single-element extensions of members of Λ_{k+3}.

The first two results were proved in [18]. Geometric representations for the matroids T_3^k, where $k \geq 0$, and S_{10} appearing in the statement of Lemma 5.8 are shown in Figures 1 and 2. A matroid M is *strictly k–regular* if M is k–regular but not $(k-1)$–regular.

Lemma 5.7. *Let $k \geq 0$. Then $U_{2,k+3}$ is strictly k–regular. Moreover, all ω–unimodular representations of $U_{2,k+3}$ are equivalent.*

Lemma 5.8. *Let M be a simple rank–3 k–regular matroid.*

(i) *If $k < 2$, then M is a restriction of T_3^k.*
(ii) *If $k = 2$, then M is a restriction of T_3^2 or S_{10}.*
(iii) *If $k > 2$, then M is a restriction of $U_{3,k+3}$, T_3^k, or S_{10}.*

The next result is a straightforward consequence of the last lemma.

Corollary 5.9. *For all $k \geq 3$, the unique 3–connected ω-regular single-element extension of $U_{3,k+3}$ is $U_{3,k+4}$.*

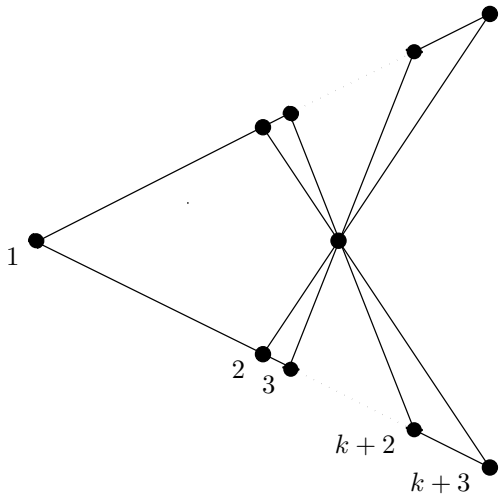

FIGURE 1. The matroid T_3^k.

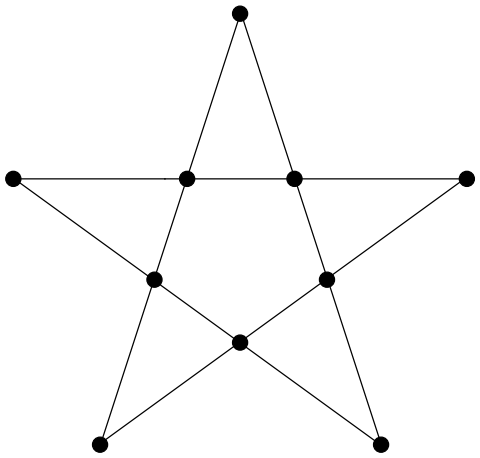

FIGURE 2. The matroid S_{10}.

The proof of Lemma 5.11 will make repeated use of the following result [16, Lemma 6].

Lemma 5.10. *Let z_1 and z_2 be distinct elements of $\mathbf{R}_k - \{0,1\}$ such that both $z_1 - 1$ and $z_2 - 1$ are in $\mathbf{R}_k$. Then $z_1 - z_2$ is in $\mathbf{R}_k$ if and only if*

(i) *there are distinct elements a, b, c, and d of $\{0,1,\alpha_1,\alpha_2,\ldots,\alpha_k\}$ such that $\{z_1,z_2\}$ is one of $\left\{\frac{a-b}{c-b},\frac{d-b}{c-b}\right\}$, $\left\{\frac{c-b}{a-b},\frac{c-b}{d-b}\right\}$, $\left\{\frac{a-b}{c-b},\frac{a-d}{c-d}\right\}$, or $\left\{\frac{a-b}{c-b},\frac{(a-b)(c-d)}{(c-b)(a-d)}\right\}$; or*

(ii) *there are distinct elements a, b, c, d_1, and d_2 of $\{0,1,\alpha_1,\alpha_2,\ldots,\alpha_k\}$ such that $\{z_1,z_2\} = \left\{\frac{(a-b)(c-d_1)}{(c-b)(a-d_1)},\frac{(a-b)(c-d_2)}{(c-b)(a-d_2)}\right\}$.*

Lemma 5.11. *For all $r, n-r \geq 2$, all ω–unimodular representations of $U_{r,n}$ are equivalent.*

Proof. The cases $r = 2$ and $n - r = 2$ follow by Lemma 5.7 and duality. Now consider the case where a representation of $U_{3,6}$ is obtained by coextending a representation of $U_{2,5}$. Using the fact that $U_{2,5}$ is uniquely representable over $\mathbf{R}_k$ and the fact that pivoting is an allowable operation on matrices over partial fields [19, Proposition 3.5], it follows that

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & 1 & x_1 & x_2 \end{bmatrix}$$

is an ω–unimodular representation for $U_{3,6}$, where x_1 and x_2 are non-zero elements of $\mathbf{R}_k$ such that both $x_1 - 1$ and $x_2 - 1$ are in $\mathbf{R}_k$. Therefore each of the subdeterminants $x_1 - \alpha_1$, $x_2 - \alpha_2$, and $x_2 - x_1$ must be a non-zero member of $\mathbf{R}_k$. Via a routine case analysis of the possibilities for x_1 and x_2 using Lemma 5.10, we deduce that, for some $j \geq 3$, we have $x_1 = \frac{\alpha_1(1-\alpha_j)}{\alpha_1-\alpha_j}$ and $x_2 = \frac{\alpha_2(1-\alpha_j)}{\alpha_2-\alpha_j}$. Thus all ω–unimodular representations of $U_{3,6}$ are equivalent.

The general case follows similarly by induction. □

Next we show that very little can happen in a 3–connected ω–regular matroid with a $U_{3,6}$–minor. The proof of this will rely on the following result that does not involve representability.

Lemma 5.12. *Let M be a 3–connected non-uniform matroid that is a single-element extension of a uniform matroid of rank and corank at least three. Then M has as a minor a 3–connected non-uniform single-element extension of $U_{3,6}$.*

Proof. We argue by induction on $|E(M)|$ noting that the result is immediate if $|E(M)| = 7$. Assume it true for $|E(M)| < n$ and let $|E(M)| = n \geq 8$. Let e be the element of M for which $M\backslash e$ is uniform, and let C be a minimum-sized circuit of M containing e. Then $|C| \leq r(M)$. If $r(M) = 3$ and $x \in E(M) - C$, then $M\backslash x$ is a 3–connected non-uniform single-element extension of a uniform matroid of rank and corank at least three, so the result follows by the induction assumption. Thus we may assume that $r(M) \geq 4$. In that case, choose an element y of M so that if $|C| = 3$, then $y \in E(M) - C$ and if $|C| \geq 4$, then $y \in C - e$. In each case, M/y is a 3–connected non-uniform single-element extension of a uniform matroid of rank and corank at least three, and the result follows by the induction assumption. □

Corollary 5.13. *If M is a 3–connected ω–regular matroid having a $U_{3,6}$–minor, then M is uniform.*

Proof. Let N be a uniform matroid that is a minor of M, has a $U_{3,6}$–minor, and has the maximum number of elements among such minors. We may assume that $N \neq M$ otherwise the corollary holds. Then, by the Splitter Theorem [22] (see also

[15, Corollary 11.2.1]), N has a 3–connected single-element extension or coextension N_1 that is a minor of M. Since N_1 is clearly non-uniform, Lemma 5.12 implies that N_1 or N_1^* has as a minor a non-uniform single-element extension of $U_{3,6}$. Thus, by Corollary 5.9, N_1 or N_1^* is not ω–regular; a contradiction. □

By combining the last result with the next one, we can deduce that the 3–connected ω–regular matroids with $U_{3,6}$ as a minor are precisely the members of $\bigcup_{n\geq 6} \mathcal{U}_n$, where we recall that $\mathcal{U}_n$ is the class of n–element uniform matroids whose rank and corank are both at least three.

Lemma 5.14. *All uniform matroids are ω–regular.*

Proof. By Corollary 5.9, all uniform matroids of rank or corank at most three are ω–regular. Now suppose that $M \cong U_{r,n}$ for some r and $n-r$ exceeding 3. Consider the $r \times n$ matrix $D_{r,n}$ that equals

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-3} \\ 0 & 0 & 1 & \alpha_1^2 & \alpha_2^2 & & \alpha_{n-3}^2 \\ 0 & 0 & 1 & \alpha_1^3 & \alpha_2^3 & & \alpha_{n-3}^3 \\ 0 & 0 & 1 & \alpha_1^4 & \alpha_2^4 & & \alpha_{n-3}^4 \\ \vdots & \vdots & & & & \ddots & \vdots \\ 0 & 1 & 1 & \alpha_1^{r-1} & \alpha_2^{r-1} & \cdots & \alpha_{n-3}^{r-1} \end{bmatrix}.$$

We show first that the determinant of every $r \times r$ submatrix X of $D_{r,n}$ is in $\mathcal{A}_{n-3} - \{0\}$. If such a matrix X avoids the second column of $D_{r,n}$, then it is a Vandermonde matrix with distinct columns and so its determinant is a non-zero member of $\mathcal{A}_{n-3}$. If X uses the second column of $D_{r,n}$, then, by expanding $\det(X)$ about that column, we have that $|\det(X)|$ is again the determinant of a Vandermonde matrix with distinct columns. Hence every $r \times r$ subdeterminant of $D_{r,n}$ is indeed in $\mathcal{A}_{n-3} - \{0\}$.

Now let A_r be the $r \times r$ matrix consisting of the first r columns of $D_{r,n}$. Consider the matrix $A_r^{-1} D_{r,n}$. Clearly this has the form $[I_r | D'_{r,n}]$. We shall complete the proof of the lemma by showing that this matrix is an $(n-3)$–unimodular representation of M. Let Y' be an $s \times s$ submatrix of $[I_r | D'_{r,n}]$ for some s in $\{1, 2, \ldots, r\}$. Let Y be the $r \times r$ submatrix of $[I_r | D'_{r,n}]$ whose columns are the s columns used by Y' together with the $r - s$ columns of I_r that are zero in all the rows of Y'. Then $|\det(Y)| = |\det(Y')|$. Moreover, $Y = A_r^{-1} X$ for some $r \times r$ submatrix X of $D_{r,n}$. Thus $\det(Y) = [\det(A_r)]^{-1} \det(X)$. Since both $\det(A_r)$ and $\det(X)$ are non-zero members of $\mathcal{A}_{n-3}$, so is $\det(Y)$. Thus so is $\det(Y')$ and the lemma follows. □

The next two results summarize some of the useful properties of the members of $\mathcal{U}_n$.

Lemma 5.15. *Let $k \geq 3$ and suppose $N \in \mathcal{U}_{k+3}$. Then*

(i) *N is strictly k–regular;*
(ii) *N is an excluded minor for the class of $(k-1)$–regular matroids; and*
(iii) *N is a splitter for the class of k–regular matroids.*

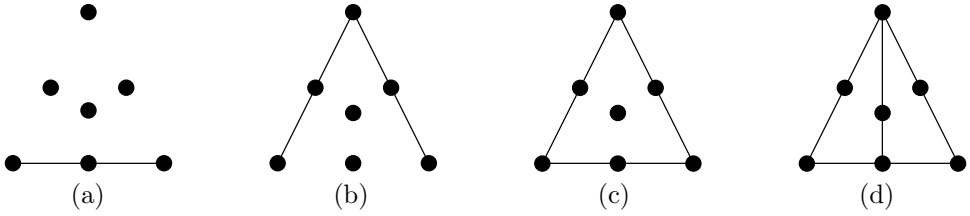

FIGURE 3. Four 7–element rank–3 matroids.

FIGURE 4. The matroids R_7 and U_7.

Proof. By Lemma 5.8 and the proof of Lemma 5.14, N is certainly k–regular. The first part follows because, by Lemma 5.11, a representation of N over $\mathbf{R}_\omega$ must use at least k of the transcendentals in $\{\alpha_1, \alpha_2, \ldots\}$. To prove (ii), we note first that, when $k = 3$, it follows from Lemma 5.8 that the unique member of $\mathcal{U}_{k+3}$, namely $U_{3,6}$, is an excluded minor for the class of 2–regular matroids. The proof of (ii) is completed by arguing by induction on k using (i). Finally, (iii) is a straightforward consequence of (i). □

Corollary 5.16. *Let $k \geq 3$ and suppose $N \in \mathcal{U}_{k+3}$.*

(i) *If N stabilizes a matroid M over $\mathbf{R}_\omega$, then $N = M$; and*
(ii) *N is not stabilized over $\mathbf{R}_\omega$ by any of its proper minors.*

For $k \geq 1$, let $\{X, Y\}$ be a 3–separation of a matroid N in Λ_{k+3}. If M is a 3–connected single-element extension of N, then, by Lemma 4.15, either (i) $\{X \cup e, Y\}$ or $\{X, Y \cup e\}$ is a 3–separation of M, or (ii) M has a minor isomorphic to a single-element extension of R_7^* in which neither triad of R_7^* is preserved. The next two results show that if M is ω–regular, then (i) must hold.

Lemma 5.17. *Let M be a single-element extension of R_7^* having no triads. Then M has a minor isomorphic to one of the matroids in Figure 3.*

Proof. Suppose, to the contrary, that M has no minor isomorphic to any of the matroids in Figure 3. Let $E(M) - E(R_7^*) = \{e\}$, and, for each i in $\{1, 2\}$, let $\{x_i, y_i, z_i\}$ be a triad T_i^* of R_7^*. Also let U_7 denote the second matroid shown in Figure 4. We first observe that, as M has no triads, e is not in the closure of either T_1^* or T_2^*. The proof is based on the following observation.

5.17.1. *If $u \in T_1^* \cup T_2^*$ and $\{e, u\}$ is in no triangles of M, then M/u is isomorphic to either R_7 or U_7.*

To see this, we first observe that R_7^*/u is isomorphic to P_6. Thus M/u is a 3–connected single-element extension of P_6. But M/u has no 4–point line restriction since e is in the closure of neither T_1^* or T_2^*. Moreover, M/u is not isomorphic to any of the matroids in Figure 3. Hence M/u is isomorphic to either R_7 or U_7.

If e is in neither a 3– nor a 4–circuit of M, then M/x_1 is isomorphic to the matroid in Figure 3(a). Thus there is either a 3– or 4–circuit of M containing e. Suppose that e is in a 3–circuit C of M. Without loss of generality, we may assume that $C = \{x_1, e, x_2\}$. Moreover, C is the only 3–circuit of M since circuit elimination using two 3–circuits containing e produces an immediate contradiction. Consider M/y_1. If y_1 is in no 4–circuit of M that contains e, then M/y_1 is isomorphic to the matroid in Figure 3(b); a contradiction. Therefore, by (5.17.1), M/y_1 must be isomorphic to U_7 and so $\{y_1, e, y_2, z_2\}$ is a circuit of M. But then it is not possible for M/z_2 to be isomorphic to either R_7 or U_7 contradicting (5.17.1). Thus M has no 3–circuits.

Now suppose that e is in a 4–circuit C' of M. Let w be the unique element of $E(R_7^*)$ that is not contained in a triad. There are two cases to consider: $w \in C'$ and $w \notin C'$. First assume that $w \in C'$. Then, without loss of generality, we may assume that $C' = \{w, x_1, x_2, e\}$. Consider M/x_1. If $\{x_1, e\}$ is contained in no 4–circuit of M other than C', then M/x_1 is isomorphic to the matroid in Figure 3(b); a contradiction. Therefore, by (5.17.1), M/x_1 is isomorphic to U_7 and $\{x_1, e, y_2, z_2\}$ is a 4–circuit C'' of M. By considering M/x_2 and applying the last argument to x_2 instead of x_1, we get that $\{x_2, e, y_1, z_1\}$ is a 4–circuit of M. Now, since M/y_1 must be isomorphic to U_7, it follows that $\{y_1, e, y_2, z_2\}$ is a 4–circuit C''' of M. Therefore, by the circuit elimination axiom, $(C'' \cup C''') - e$ contains a circuit of M; a contradiction. We conclude that $w \notin C'$. Then, we may assume, without loss of generality, that $C' = \{x_1, x_2, y_1, e\}$. Now arguing as above, we deduce, since M/x_1 and M/y_1 must both be isomorphic to U_7, that $\{e, x_1, y_2, z_2\}$ and $\{e, y_1, y_2, z_2\}$ are both circuits of M. Then circuit elimination again gives a contradiction. □

By Lemma 5.8, none of the matroids in Figure 3 is ω–regular. Using this, the next corollary follows immediately from the last lemma.

Corollary 5.18. *If M is a single-element extension of R_7^* having no triads, then M is not ω–regular.*

We remark here that we implicitly use Lemma 2.10 in the proof of the next lemma.

Lemma 5.19. *Let $m \geq 4$ and let M be a 3–connected single-element extension of a matroid N in Λ_m such that $M\backslash e = N$. Suppose that none of the matroids in Figure 3 is a minor of M. Then there is a sequence $M_0, M_1, \ldots, M_n$ of matroids with $M_0 = M$ and $M_n\backslash e \cong U_{m-2,m}$ such that, for all i in $\{0, 1, \ldots, n-1\}$,*

(i) *there is a set A_i that avoids e and has size at least three so that M_{i+1} is either $\Delta_{A_i}(M_i)$ or $\nabla_{A_i}(M_i)$;*
(ii) *M_{i+1} is 3–connected and $M_{i+1}\backslash e \in \Lambda_m$; and*
(iii) *the exchange that produced M_{i+1} from M_i can be applied to $M_i\backslash e$ and, when this is done, it produces $M_{i+1}\backslash e$.*

Proof. Let T_N be a reduced del-con tree for which $N = M(T_N)$. We prove all parts of the lemma simultaneously by induction on $|V(T_N)|$. Suppose that $|V(T_N)| = 1$. If T_N consists of a single con vertex, then the lemma certainly holds. Furthermore, if T_N consists of a single del vertex, then it is easily seen that the lemma also holds. Now let $|V(T_N)| = n \geq 2$ and assume that the lemma holds for every 3–connected single-element extension of a matroid in Λ_m for which there is an associated del-con tree with fewer vertices.

First suppose that T_N has a degree-one del vertex u. Since N is 3–connected, E_u is coindependent in N and hence in M. Therefore $\Delta_{E_u}(M)$ is well-defined since $N|E_u$, and hence $M|E_u$, is uniform of rank 2. If T_u is the tree that is obtained by shrinking u in T_N, then $N = M(T_N) = \nabla_{E_u}(M(T_u))$ and so $M(T_u) = \Delta_{E_u}(N)$. Now, by Lemma 2.16(i), $\Delta_{E_u}(M)\backslash e = \Delta_{E_u}(M\backslash e) = \Delta_{E_u}(N)$. The last matroid is certainly 3–connected. Suppose that $\Delta_{E_u}(M)$ is not 3–connected. Then $\Delta_{E_u}(M)$ has a 2–circuit. But this cannot occur since $\Delta_{E_u}(M)$ is a restriction of a generalized parallel connection of two simple matroids. We conclude that $\Delta_{E_u}(M)$ is a 3–connected single-element extension of $\Delta_{E_u}(N)$. Since the last matroid is equal to $M(T_u)$ and T_u has fewer vertices than T_N, the induction assumption implies that the lemma holds for $M(T_u)$ and hence for M.

We may now assume that all degree-one vertices of T_N are con vertices. Then, in particular, $|V(T_N)| \geq 3$, so T_N certainly has a del vertex v. Let $\{X, Y\}$ be a 3–separation of N that is based on v and chosen so that X and Y contain con classes E_x and E_y, respectively, each of which corresponds to a degree-one vertex of T_N. Since M has no minor isomorphic to one of the matroids in Figure 3, it follows by Lemmas 4.15 and 5.17 that either $\{X \cup e, Y\}$ or $\{X, Y \cup e\}$ is a 3–separation of M. Without loss of generality, we may assume the former. As $e \in \mathrm{cl}_M(E(M) - e - E_y)$, it follows that $e \notin \mathrm{cl}_{M^*}(E_y)$. Thus, as every 3–element subset of E_y is a triangle of N^*, and $N^* = M^*/e$, every 3-element subset of E_y is a triangle of M^*, that is, a triad of M. Since E_y is independent in N and hence in M, we deduce that $\nabla_{E_y}(M)$ is well-defined. Moreover, by the dual of Lemma 2.16, $\nabla_{E_y}(M)\backslash e = \nabla_{E_y}(M\backslash e) = \nabla_{E_y}(N)$. Thus $\nabla_{E_y}(M)$ is a single-element extension of $\nabla_{E_y}(N)$. But the last matroid equals $M(T_y)$ where T_y is the del-con tree obtained from T_N by shrinking y. Hence $\nabla_{E_y}(N)$ is 3–connected. If $\nabla_{E_y}(M)$ is also 3–connected, then, since it is a single-element extension of $\nabla_{E_y}(N)$, it follows by the induction assumption that the lemma holds for $\nabla_{E_y}(M)$ and hence for M.

It remains to consider when $\nabla_{E_y}(M)$ is not 3–connected. Then $\nabla_{E_y}(M)$ has a 2–circuit, $\{e, f\}$ say, containing e. But, since M, which equals $\Delta_{E_y}[\nabla_{E_y}(M)]$, has no 2–circuits, $\{e, f\}$ meets E_y. Hence $f \in E_y$. We show next that e must lie in the meet of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$ in M. Since M is obtained from $\nabla_{E_y}(M)$ by performing a Δ_{E_y}–exchange, the closure of E_y in M must contain e. Therefore, as $\{X \cup e, Y\}$

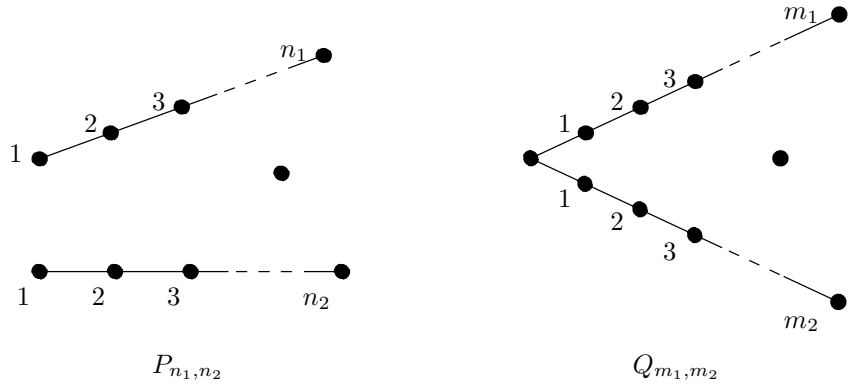

FIGURE 5. The matroids P_{n_1,n_2} and Q_{m_1,m_2}.

is a 3–separation of the 3–connected matroid M, and E_y is contained in Y, we get that $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Therefore $\{X, Y \cup e\}$ is a 3–separation of M. We may now apply the argument that began in the previous paragraph, interchanging X with Y and y with x, to deduce that the lemma holds for M unless $\nabla_{E_x}(M)$ has a 2–circuit $\{e, g\}$ containing e where $g \in E_x$. Assume the exceptional case occurs and consider $\nabla_{E_x}(\nabla_{E_y}(M))$ which is certainly defined and equals $\nabla_{E_y}(\nabla_{E_x}(M))$. Since e is parallel to f in $\nabla_{E_y}(M)$ and to g in $\nabla_{E_x}(M)$, it is not difficult to see that f is parallel to g in $\nabla_{E_y}(\nabla_{E_x}(M))\backslash e$, and that this matroid equals $\nabla_{E_y}(\nabla_{E_x}(N))$. This is a contradiction since the last matroid is in Λ_m. □

Let M be a 3–connected single-element ω–regular extension of a member of Λ_{k+3}, where $k \geq 1$. By the dual of Lemma 5.19, M^* is $\Delta - \nabla$–equivalent to a 3–connected single-element coextension of $U_{2,k+3}$ that is ω–regular. Figure 5 gives geometric representations for the matroids P_{n_1,n_2} and Q_{m_1,m_2}, which are defined for all integers n_1, n_2, m_1, and m_2 exceeding one.

Lemma 5.20. *Let $k \geq 1$. For a matroid M, the following two statements are equivalent:*

(i) *M is a 3–connected ω–regular matroid such that $M/x \cong U_{2,k+3}$.*
(ii) (a) *M is k–regular and, for some m_1 and m_2 with $m_1 + m_2 = k + 2$, there is an isomorphism between M and Q_{m_1,m_2} under which x maps to the element of Q_{m_1,m_2} that is on no non-trivial line; or*
(b) *M is strictly $(k+1)$–regular and M is isomorphic to $U_{3,k+4}$ or to a member of $\{P_{n_1,n_2} : n_1 + n_2 = k + 3\}$.*

Moreover, every matroid that is $\Delta - \nabla$–equivalent to a member of $\{P_{n_1,n_2} : n_1 + n_2 = k + 3\}$ is a member of Λ_{k+4}.

Proof. Using Lemma 5.8, it is routine to deduce that a matroid is a 3–connected single-element ω–regular coextension of $U_{2,k+3}$ if and only if it is isomorphic to a

member of

$$\{U_{3,k+4}\} \cup \{P_{n_1,n_2} : n_1, n_2 = k+3\} \cup \{Q_{m_1,m_2} : m_1, m_2 = k+2\}.$$

Furthermore, by the same lemma, every member of $\{Q_{m_1,m_2} : m_1 + m_2 = k+2\}$ is k–regular and every member of $\{P_{n_1,n_2} : n_1 + n_2 = k+3\}$ is strictly $(k+1)$–regular.

To prove the second part of the lemma, we need to show that every member of $\{P_{n_1,n_2} : n_1 + n_2 = k+3\}$ is in Λ_{k+4}. This is certainly true if either n_1 or n_2 is equal to two. Therefore assume that both n_1 and n_2 exceed two. Let X be the set of points of one of the non-trivial lines of P_{n_1,n_2}, and let x be the unique element of $E(P_{n_1,n_2})$ that is on no non-trivial lines. Using Lemma 2.9, it is straightforward to check that the bases of $\nabla_{X\cup x}[\Delta_X(P_{n_1,n_2})]$ coincide with the bases of $U_{2,k+4}$. Therefore P_{n_1,n_2} is indeed a member of Λ_{k+4}. □

In the proof of Lemma 5.21, we use the fact that X is a flat of a matroid M if and only if $E(M) - X$ is the union of a (possibly empty) set of cocircuits of M.

Lemma 5.21. *For $k \geq 1$, let M be a 3–connected matroid such that $M\backslash x \in \Lambda_{k+3}$. Suppose that x is not fixed in M. If $x \notin A$, then*

(i) *x is not fixed in $\Delta_A(M)$; and*
(ii) *x is not fixed in $\nabla_A(M)$.*

Proof. Let M' be a matroid obtained from M by independently cloning x with x'. Consider part (i). Since $\Delta_A(M)$ is well-defined, it follows that $\Delta_A(M')$ is also well-defined. By Lemma 2.20, the elements x and x' are independent clones in $\Delta_A(M')$. Therefore, by definition, x is not fixed in $\Delta_A(M)$ and part (i) is proved.

Now consider part (ii) of the lemma. As every 3–element subset of A is a triad of M, the set $E(M) - A$ is a flat F of M. First assume that x is in a circuit C of $M|F$. Then $(C - x) \cup x'$ is a circuit of $M'|(F \cup x')$ and so $F \cup x'$ is a flat of M' such that $r_M(F) = r_{M'}(F \cup x')$. Therefore every 3–element subset of A is a triad of M' and so, as A is independent in M', the operation $\nabla_A(M')$ is well-defined. By Corollary 2.21, it follows that x is not fixed in $\nabla_A(M)$.

Now assume that x is not in a circuit of $M|F$. Then x is a coloop of $M|F$ and so $F - x$ is a flat of M. Therefore $A \cup x$ is the union of a set of cocircuits of M. Let C^* be a cocircuit of M that contains x and is contained in $A \cup x$. Since every 3–element subset of A is a triad of M and M is 3–connected, it follows that there are exactly 2 elements of A in C^*. Thus every 3–element subset of $A \cup x$ is a triad of M. Therefore every 2–element subset of A is a cocircuit of $M\backslash x$, so $M\backslash x$ is not 3–connected, contradicting the fact that $M\backslash x$ is a member of Λ_{k+3}. This completes the proof of Lemma 5.21. □

We remark here that, in general, a ∇–exchange on a matroid M does not necessarily preserve the property of an element of $E(M)$ being not fixed. For example, suppose that M is isomorphic to $M(K_{2,3})$ and let A denote the set of elements of one triad of M. Now every element of M is not fixed. However, every element of $\nabla_A(M)$, which is isomorphic to $M(K_4)$, is fixed.

By Lemma 2.11 and its dual, the following corollary is an immediate consequence of Lemma 5.21.

Corollary 5.22. *For $k \geq 1$, let M be a 3–connected matroid such that $M\backslash x \in \Lambda_{k+3}$. Suppose that x is fixed in M. If $x \notin A$, then*

(i) *x is fixed in $\Delta_A(M)$; and*
(ii) *x is fixed in $\nabla_A(M)$.*

Lemma 5.23. *For $k \geq 1$, let M be a 3–connected k–regular matroid such that $M\backslash x = N$ and $N \in \Lambda_{k+3}$. Then*

(i) *x is fixed in M; and*
(ii) *N has an element x' such that either $M\backslash x'$ or M/x' is a member of Λ_{k+3} depending upon whether x' is a del or a con element, respectively, of a reduced del-con tree T_N for which $N = M(T_N)$.*

Proof. Since M is k–regular, it has none of the matroids in Figure 3 as a minor. Thus we may apply Lemma 5.19 to M. Let $M_0, M_1, \ldots, M_n$ be the sequence of matroids whose existence is established in that lemma. As $M_n\backslash x \cong U_{k+1,k+3}$ and M_n is k–regular, it follows, by Lemma 5.20, that there is an isomorphism between M_n and $Q^*_{m_1,m_2}$ under which x maps to the element of Q_{m_1,m_2} that is on no non-trivial lines. For convenience, we shall assume that this isomorphism is the identity. Let F_1 and F_2 be the complements of the two non-trivial lines of Q_{m_1,m_2}. Then it is not difficult to check that $\{F_1, F_2\}$ is a modular pair of flats in M_n meeting in $\{x\}$, so x is fixed in M_n. Hence, by Corollary 5.22, x is fixed in M.

Next we show, by induction on n, that the element x' of M_n that lies on both non-trivial lines of M_n^* has the property asserted in (ii) of the lemma. If $n = 0$, then $M \cong Q^*_{m_1,m_2}$ and $N \cong U_{k+1,k+3}$. Moreover, it is straightforward to deduce that M/x' is a member of Λ_{k+3}. The reduced del-con tree T_N associated with N has a single vertex, which is labelled "con", so (ii) holds for $n = 0$.

Now let $n \geq 1$ and suppose that (ii) holds for all smaller values of n. Let $N_1 = M_1\backslash x$. Then M_1 is 3–connected and k–regular, and $N_1 \in \Lambda_{k+3}$. Let T_{N_1} be the reduced del-con tree corresponding to N_1. By the induction assumption, either $M_1\backslash x'$ or M_1/x' is a member of Λ_{k+3} depending upon whether x' is a del or a con element, respectively, of T_{N_1}. There are four cases to consider depending on whether M is $\Delta_A(M_1)$ or $\nabla_A(M_1)$ and whether x' is or is not in A.

Case (1). $M = \Delta_A(M_1)$ and $x' \in A$.

Since $|A| \geq 3$, it follows that $M_1/x' \notin \Lambda_{k+3}$. Hence $M_1\backslash x' \in \Lambda_{k+3}$ and x' is a del element of T_{N_1}. By Corollary 2.17, $N = \Delta_A(N_1)$. Thus x' is a con element of T_N. Now $M/x' = \Delta_A(M_1)/x' = \Delta_{A-x'}(M_1\backslash x')$ by Lemma 2.13. As $M_1\backslash x' \in \Lambda_{k+3}$, we conclude that $M/x' \in \Lambda_{k+3}$.

Case (2). $M = \Delta_A(M_1)$ and $x' \notin A$.

In this case there are two possibilities. Suppose first that x' is a del element of T_{N_1}. Then x' is a del element of T_N. Moreover, by the induction assumption, $M_1\backslash x'$ is in Λ_{k+3} and so is 3–connected. Thus, by Corollary 2.17,

$$M\backslash x' = \Delta_A(M_1)\backslash x' = \Delta_A(M_1\backslash x').$$

We conclude that $M\backslash x'$ is a member of Λ_{k+3}.

Now suppose that x' is a con element of T_{N_1}. Then x' is a con element of T_N. Moreover, by the induction assumption, M_1/x' is in Λ_{k+3} and so is 3–connected. Thus, by Corollary 2.17,

$$M/x' = \Delta_A(M_1)/x' = \Delta_A(M_1/x').$$

We conclude that M/x' is a member of Λ_{k+3}, thereby completing case (2).

In the two cases that remain, $M = \nabla_A(M_1)$. In these cases, by applying the arguments just given with M^* replacing M, we obtain the desired conclusion. It follows, by induction, that (ii) holds. □

The next lemma is somewhat technical. It plays a crucial role in the proofs of Lemmas 5.25 and 5.29, the two main tools used to prove Theorems 1.3 and 1.4.

Lemma 5.24. *Suppose $k \geq 1$ and let M be a 3–connected ω–regular matroid such that $M\backslash x/y \in \Lambda_{k+3}$ for some elements x and y. Assume that every proper minor of M having a minor in Λ_{k+3} is k–regular. Then*

(i) *x is fixed in M/y; and*
(ii) *if $M\backslash x$ is 3–connected, then x is fixed in M.*

Proof. Part (i) is certainly true if $\{x, y\}$ is contained in a triangle of M. But if not, then M/y is a 3–connected extension by x of a member of Λ_{k+3} and it follows by Lemma 5.23(i) that (i) holds.

We prove (ii) by contradiction. Suppose that $M\backslash x$ is 3–connected, but x is not fixed in M. Since $(M^*/x)\backslash y \in \Lambda_{k+3}$, the matroid M^*/x is a 3–connected k–regular single-element extension of a member of Λ_{k+3}. Therefore, by Lemma 5.23(ii), either $(M^*/x)\backslash y'$ or $(M^*/x)/y'$ is a member of Λ_{k+3} for some $y' \neq y$. This implies that either $M\backslash x/y'$ or $M\backslash x\backslash y'$ is a member of Λ_{k+3}.

Suppose that $M\backslash x\backslash y' \in \Lambda_{k+3}$. Since $M\backslash y'$ is certainly 3–connected and k–regular, x is fixed in $M\backslash y'$ by Lemma 5.23(i). Thus, by Proposition 5.3(i), x is fixed in M; a contradiction.

Now suppose that $M\backslash x/y' \in \Lambda_{k+3}$. Then, by (i), x is fixed in M/y'. Since x is also fixed in M/y but x is not fixed in M, it follows by Proposition 5.3(ii) that $\{x, y, y'\}$ is a triangle of M.

Next we show that y is cofixed in M. Clearly, $M/y\backslash x \cong M/y\backslash y'$ so $M/y\backslash y' \in \Lambda_{k+3}$. Hence $M^*/y'\backslash y \in \Lambda_{k+3}$. Therefore, by (i), y is fixed in M^*/y', that is, y is cofixed in $M\backslash y'$. Similarly, y is also cofixed in $M\backslash x$. But $\{x, y, y'\}$ is a triangle

of M and $M \not\cong U_{2,4}$, so $\{x, y, y'\}$ is not a triad of M. Therefore, by the dual of Proposition 5.3(ii), y is cofixed in M.

Since x is fixed in M/y, but not in M, it follows by Proposition 5.6(i) that x is freer than y in M. Thus either $\{x, y\}$ are clones in M, or x is strictly freer than y in M. If x and y are clones in M, then, as M is 3–connected, x and y are coindependent clones in M and so y is not cofixed in M; a contradiction. If x is strictly freer than y, then, by Proposition 5.6(ii), y is not cofixed in M; a contradiction. □

Lemma 5.25. *Let $k \geq 1$. Then every member of Λ_{k+3} is a universal stabilizer for the class of k–regular matroids.*

Proof. Let N be a member of Λ_{k+3} and M be a k–regular matroid. We shall use Theorem 5.4. If $M\backslash x = N$, then, by Lemma 5.23(i), x is fixed in M. Dually, if $M/y = N$, then y is cofixed in M. Finally, if $M\backslash x/y = N$ and $M\backslash x$ is 3–connected, then, by Lemma 5.24, x is fixed in M. We now conclude using Theorem 5.4 that the lemma holds. □

The next corollary follows immediately from combining Lemma 5.25 with Theorem 5.5.

Corollary 5.26. *Let $k \geq 1$. Then every member of Λ_{k+3} is an $\mathbf{R}_\omega$–stabilizer for the class of k–regular matroids.*

Lemma 5.29, one of the two primary tools in the proofs of the main theorems of this section, will use two more preliminary results. The first of these is easily seen to be implicit in the first paragraph of the proof of Theorem 5.1 of [8].

Lemma 5.27. *Let $\mathbf{P}$ be a partial field. If M and N are both 3–connected $\mathbf{P}$–representable matroids such that $M\backslash x = N$ and x is fixed in M, then N stabilizes M over $\mathbf{P}$.*

Lemma 5.28. *An ω–regular matroid M that is not k–regular cannot be stabilized over $\mathbf{R}_\omega$ by a k–regular matroid.*

Proof. This follows immediately from the fact that an ω–unimodular representation of a matroid that is not k–regular requires at least $k+1$ algebraically independent transcendentals over $\mathbf{Q}$. □

Lemma 5.29. *Let $k \geq 1$. Suppose that M is a 3–connected ω–regular matroid that has as a minor a member of Λ_{k+3} that does not stabilize M over $\mathbf{R}_\omega$. Then M has a minor isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$.*

Proof. It suffices to consider the case when M is a minor-minimal 3–connected ω–regular matroid having a minor in Λ_{k+3} that does not stabilize M over $\mathbf{R}_\omega$. By Corollary 5.26, M is not k–regular. Moreover, by Theorem 5.1, for some member N of Λ_{k+3} that does not stabilize M over $\mathbf{R}_\omega$, there are elements x and y of M such that (i) $M\backslash x = N$, or (ii) $M/y = N$, or (iii) $M\backslash x/y = N$ and both $M\backslash x$ and M/y are 3–connected.

First let $M\backslash x = N$. By Lemma 5.20 and the remarks preceding it, either M is isomorphic to $U_{k+1,k+4}$, or M is $\Delta-\nabla$–equivalent to a member of $\{P^*_{n_1,n_2} : n_1+n_2 = k+3\}$. In the second case, by Lemma 5.20, M is a member of Λ_{k+4}. Thus, in both cases, M is isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$. By duality, if $M/y = N$, then, again, M is isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$.

Now assume that $M\backslash x/y = N$ and both $M\backslash x$ and M/y are 3–connected. Then Lemma 5.28 and the minimality of M imply that both $M\backslash x$ and M/y are k–regular. Therefore, by Lemma 5.25, y is cofixed in $M\backslash x$ and x is fixed in M/y. Furthermore, as $M\backslash x$ is k–regular but M is not k–regular, Lemma 5.28 implies that $M\backslash x$ does not stabilize M over $\mathbf{R}_\omega$. Thus, by Lemma 5.27, x is not fixed in M. Therefore, by Lemma 5.24(ii), M has a proper minor M' that is not k–regular and has a minor in Λ_{k+3}. Since $|E(M)| = k+5$, it follows that M' has an element z such that $M'\backslash z$ or $M'/z \in \Lambda_{k+3}$. Since M' is not k–regular, we conclude that M' is 3–connected and that no member of Λ_{k+3} stabilizes M'. Thus M' contradicts the choice of M. □

At last we are in a position to prove Theorems 1.3 and 1.4. Indeed, most of the work in proving these theorems has already gone into proving Lemmas 5.25 and 5.29.

The proof of Theorem 1.3 is by induction on k and relies on Theorem 5.1. Due to certain properties of the class of ω–regular matroids, it turns out that, for $k \geq 1$, the ω–regular excluded minors for the class of k–regular matroids can be determined from the ω–regular excluded minors for the class of $(k-1)$–regular matroids by simply performing the stabilizer check of Theorem 5.1 on each of the latter matroids. Before proving Theorem 1.3, we restate it for convenience. Recall that $\mathcal{U}_n$ is the class of n–element uniform matroids whose rank and corank are both at least three.

Theorem 1.3. *Let M be an ω–regular matroid and let $k \geq 0$. Then M is k–regular if and only if it has no minor isomorphic to a member of $\mathcal{U}_{k+4} \cup \Lambda_{k+4}$.*

Proof. We argue by induction on k. If $k = 0$, the theorem is an immediate consequence of Tutte's excluded-minor result for the class of regular matroids [24]. Now assume that the theorem holds for $k < n$ and let $k = n \geq 1$. First we note that, by Lemmas 5.7 and 5.15, $U_{2,k+4}$ and all the members of $\mathcal{U}_{k+4}$ are ω–regular excluded minors for the class of k–regular matroids. Hence, by Theorem 1.1 and Corollary 3.8, every member of Λ_{k+4} is also an excluded minor.

Now suppose that M is an ω–regular matroid that is an excluded minor for the class of k–regular matroids but is not isomorphic to a member of $\mathcal{U}_{k+4} \cup \Lambda_{k+4}$. Since M is not $(k-1)$–regular, the induction assumption implies that M has a minor N isomorphic to a member of $\mathcal{U}_{k+3} \cup \Lambda_{k+3}$. Moreover, by Lemma 5.28, M is not stabilized over $\mathbf{R}_\omega$ by N. Suppose that $N \in \Lambda_{k+3}$. Then, by Lemma 5.29, M has a minor isomorphic to a member of $\{U_{3,k+4}, U_{k+1,k+4}\} \cup \Lambda_{k+4}$; a contradiction. Thus we may suppose that $N \notin \Lambda_{k+3}$. Then $N \in \mathcal{U}_{k+3}$, so $k \geq 3$. Thus M is an ω–regular matroid having a $U_{3,6}$–minor. Thus, by Corollary 5.13, M is uniform. Since M has more than $k+3$ elements, it follows that M has a member of $\mathcal{U}_{k+4}$ as a minor. This contradiction completes the proof. □

A consequence of Theorem 1.3 is that, given a partial field $\mathbf{P}$, we can bound the number of inequivalent $\mathbf{P}$–representations of certain k–regular matroids.

Corollary 5.30. *Let $k \geq 1$. Let M be a 3–connected strictly k–regular matroid that is not in $\mathcal{U}_{k+3}$. Suppose that M is representable over a partial field $\mathbf{P}$ and let n be the number of inequivalent $\mathbf{P}$–representations of $U_{2,k+3}$. Then M has at most n inequivalent $\mathbf{P}$–representations.*

Proof. By Lemma 5.15, all members of $\mathcal{U}_{k+3}$ are splitters for the class of k–regular matroids. Therefore, by Theorem 1.3, M has a minor N isomorphic to a member of Λ_{k+3}. By Lemma 5.25, N is a universal stabilizer for the class of k–regular matroids, and so, by Theorem 5.5, N stabilizes M over $\mathbf{P}$. Thus, by Proposition 5.2, the number of inequivalent $\mathbf{P}$–representations of M is no more than the number of inequivalent $\mathbf{P}$–representations of N. Moreover, it is straightforward to deduce from Corollary 3.6 that there are exactly n inequivalent $\mathbf{P}$–representations of N. The corollary follows immediately. □

Next we prove Theorem 1.4.

Theorem 1.4. *Let $k \geq 0$ and let M be a 3–connected k–regular matroid. Then all ω–unimodular representations of M are equivalent.*

Proof. Suppose that M is strictly k–regular. Then, by Theorem 1.3, M has a minor N isomorphic to a member of $\mathcal{U}_{k+3} \cup \Lambda_{k+3}$. Since, by Lemma 5.15, all members of $\mathcal{U}_{k+3}$ are splitters for the class of k–regular matroids, if $N \in \mathcal{U}_{k+3}$, then $N = M$ and so, by Lemma 5.11, all ω–unimodular representations of M are equivalent. Thus we may assume that $N \in \Lambda_{k+3}$. Then, by Corollary 5.26, M is stabilized by N over $\mathbf{R}_\omega$. But, by Lemma 5.7 and Corollary 3.6, N is uniquely representable over $\mathbf{R}_\omega$. Hence, by Proposition 5.2, all ω–unimodular representations of M are equivalent. The theorem now follows readily. □

Let k be a positive integer and suppose that M is a 3–connected strictly k–regular matroid that is not in $\mathcal{U}_{k+3}$. If M is representable over a partial field $\mathbf{P}$, then, by Corollary 5.30, the number of inequivalent $\mathbf{P}$–representations of M is no more than the number of inequivalent $\mathbf{P}$–representations of $U_{2,k+3}$. The next corollary shows that a member of each equivalence class of $\mathbf{P}$–representations of M can be obtained via a k–unimodular representation of M.

Corollary 5.31. *Let k be a positive integer and $\mathbf{P}$ be a partial field with the property that there are k distinct elements $a_1, a_2, \ldots a_k$ in $\mathbf{P} - \{0, 1\}$ such that, for all distinct i and j in $\{1, 2, \ldots, k\}$, both $a_i - 1$ and $a_i - a_j$ are in $\mathbf{P}$. Let M be a 3–connected matroid that is strictly k–regular and has a minor N isomorphic to a member of Λ_{k+3}. Then the matrix obtained from a k–unimodular representation of M by replacing α_i with a_i for all i is a $\mathbf{P}$–representation of M. Moreover, up to equivalence, all $\mathbf{P}$–representations of M can be obtained in this way.*

Proof. The fact that the matrix obtained from a k–unimodular representation of M by replacing α_i with a_i for all i is a $\mathbf{P}$–representation for M follows from [16,

Proposition 4] and [19, Corollary 5.2]. We now show that all **P**–representations of M, up to equivalence, can be obtained in this way.

Consider a **P**–representation of $U_{2,k+3}$. Since all k–unimodular representations of $U_{2,k+3}$ are equivalent, it is clear that all **P**–representations of $U_{2,k+3}$ can be obtained from the following k–unimodular representation of $U_{2,k+3}$ by replacing α_i with a_i for all i in $\{1, 2, \ldots, k\}$.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_k \end{bmatrix}$$

Since $N \in \Lambda_{k+3}$, it follows by Corollary 3.6 that, up to equivalence, every **P**–representation of N can be obtained from a k–unimodular representation of N by replacing α_i with a_i for all i.

Let X be a k–unimodular representation of N and Y be the **P**–representation of N obtained by replacing α_i with a_i for all i. By combining Lemma 5.25 and Theorem 5.5, we deduce that N stabilizes M over **P**. Therefore if Y can be extended to a **P**–representation of M, then all such representations of M are strongly equivalent. Moreover, by Theorem 1.4, X is guaranteed to extend to some k–unimodular representation X' of M, so one of these representations can be obtained from X' by substituting a_i for α_i for all i. Corollary 5.31 is now proved. □

An immediate consequence of Corollary 5.31 is that if M is a non-binary 3–connected near-regular matroid representable over a partial field **P**, then all **P**–representations of M can be obtained in the way described in its statement. This result is [29, (2.12)] and has an important role to play in the theorems of [28, 29].

6. A Characterization of an Excluded-Minor Class of Matroids

There are exactly five non-isomorphic 3–connected matroids of rank and corank three, namely the matroids $M(\mathcal{W}_3), \mathcal{W}^3, Q_6, P_6$, and $U_{3,6}$. The first of these matroids is the rank–3 wheel. Each of the other four matroids in this sequence can be obtained from its predecessor by relaxing a line or, more formally, relaxing a circuit-hyperplane. In [14], it was noted that, of the five classes of matroids that result from excluding four of these matroids as minors, all have been described except $EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$, the class of matroids having no minor isomorphic to any of the matroids $M(\mathcal{W}_3)$, $\mathcal{W}^3$, Q_6, and $U_{3,6}$. In this section, we complete the picture by describing $EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$.

Lemma 6.1. *If M is in $\bigcup_{m\geq 4} \Lambda_m$, then $M \in EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$.*

Proof. Evidently, if $|E(M)| \in \{4, 5, 6\}$, then the lemma holds. Therefore assume that $|E(M)| \geq 7$. Suppose, to the contrary, that M has a minor N isomorphic to one of the matroids $M(\mathcal{W}_3)$, $\mathcal{W}^3$, Q_6, and $U_{3,6}$. If N is $M(\mathcal{W}_3)$, then replace N with the largest wheel minor of M, while if N is $\mathcal{W}^3$, then replace N with the largest whirl minor of M. Clearly, for all $r \geq 3$, neither the rank–r wheel nor the rank–r whirl

has two elements that are clones, hence neither matroid is a member of $\bigcup_{m\geq 4}\Lambda_m$. Since M is 3–connected, it now follows by Seymour's Splitter Theorem [22] (see also [15, Corollary 11.2.1]) that there is a sequence $M_0, M_1, \ldots, M_n$ of 3–connected matroids with $M_0 \cong N$ and $M_n = M$ such that, for all i in $\{0, 1, \ldots, n-1\}$, the matroid M_i is a single-element deletion or a single-element contraction of M_{i+1}. Since M_n is in $\bigcup_{m\geq 4}\Lambda_m$ but M_0 is not, there is clearly an index j in $\{1, 2, \ldots, n\}$ such that $\bigcup_{m\geq 4}\Lambda_m$ contains M_j but not M_{j-1}. Let $E(M_j) - E(M_{j-1}) = \{e\}$. Let T_{M_j} be a reduced del-con tree for which $M_j = M(T_{M_j})$, and let v be the vertex of T_{M_j} for which $e \in E_v$. Then, as $M_{j-1} \notin \bigcup_{m\geq 4}\Lambda_m$, it follows by Lemma 4.8 that either

(i) E_v is a del class of T_{M_j} and $M_j/e = M_{j-1}$, or
(ii) E_v is a con class of T_{M_j} and $M_j\backslash e = M_{j-1}$.

By duality, we may assume that (i) holds. Now v is not a degree-one vertex of T_{M_j}, otherwise M_j/e is non-simple. Let $\{X, Y\}$ be a 3–separation of M_j based on v and suppose that $e \in X$. Then both X and Y span E_v. Thus $\{X - e, Y\}$ is a 2–separation of M_j/e. Therefore M_j/e is not 3–connected and so the connectivity of M_{j-1} is contradicted thereby completing the proof of the lemma. □

The next theorem is the main result of this section. Its proof will use the following result.

Lemma 6.2. *If M is a connected matroid having a minor isomorphic to $U_{3,6}$ or Q_6, and $e \in E(M)$, then M has a $U_{3,6}$– or Q_6–minor that uses e.*

Proof. By a result of Seymour [20] (see also [15, Corollary 11.3.9]), it suffices to check that the lemma holds when $|E(M)| = 7$. We omit the routine case check. □

Theorem 6.3. *A matroid is a 3–connected member of $EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$ if and only if it is a member of $\bigcup_{m\geq 4}\Lambda_m \cup \{U_{0,0}, U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}, U_{2,3}\}$.*

The following lemma contains the core of the proof of this theorem.

Lemma 6.4. *Let N be a 3–connected member of $EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$ having an element e such that $N\backslash e \in \Lambda_m$ for some $m \geq 6$. Suppose that there is a sequence $N_0, N_1, \ldots, N_k$ of matroids with $N = N_0$ and $N_k\backslash e \cong U_{m-2,m}$ such that, for all i in $\{0, 1, \ldots, k-1\}$,*

(i) *there is a set A_i that avoids e and has size at least three so that N_{i+1} is either $\Delta_{A_i}(N_i)$ or $\nabla_{A_i}(N_i)$;*
(ii) *$N_{i+1}\backslash e \in \Lambda_m$; and*
(iii) *the exchange that produced N_{i+1} from N_i can be applied to $N_i\backslash e$ and, when this is done, it produces $N_{i+1}\backslash e$.*

Then N_k has neither a Q_6– nor a $U_{3,6}$–minor.

Proof. Assume the contrary taking a counterexample for which m is a minimum. Then, by Lemma 6.2, N_k has a Q_6– or a $U_{3,6}$–minor using e. Thus, since N_k has

corank 3, there is a subset D of $E(N) - e$ such that N_k/D is isomorphic to Q_6 or $U_{3,6}$.

Consider the sequence $N_0\backslash e, N_1\backslash e, \ldots, N_k\backslash e$. By Corollary 2.17, each member of this sequence can be obtained from its predecessor by a Δ–exchange or a ∇–exchange. Moreover, $N_k\backslash e \cong U_{m-2,m}$ and $N_0\backslash e = N\backslash e$. Construct the sequence $T_k, T_{k-1}, \ldots, T_0$ of del-con trees as follows. Let T_k have a single vertex labelled $(E(N) - e, \mathrm{con})$. Then $M(T_k) = N_k\backslash e$. In general, assume that $M(T_j) = N_j\backslash e$ for all j in $\{i, i+1, \ldots, k\}$ and let the Δ– or ∇–exchange that produced $N_{i-1}\backslash e$ from $N_i\backslash e$ determine how T_{i-1} is constructed from T_i.

Now let x be an element of D, and consider the sequence $T_0\backslash x, T_1\backslash x, \ldots, T_k\backslash x$ of del-con trees. Then $M(T_k\backslash x) \cong U_{m-3,m-1}$ and, by Lemma 4.8, $M(T_i\backslash x)$ is $M(T_i)\backslash x$ or $M(T_i)/x$ depending on whether x is in a del or a con class of T_i. In particular, $M(T_i\backslash x)$ is 3–connected. By hypothesis, the same sequence of Δ– and ∇–exchanges that produced N_k from N_0 also produces $N_k\backslash e$ from $N_0\backslash e$, that is, produces $M(T_k)$ from $M(T_0)$. Now $M(T_{i+1})$ is either $\Delta_{A_i}(M(T_i))$ or $\nabla_{A_i}(M(T_i))$. Thus $M(T_{i+1}\backslash x)$ is, respectively, $\Delta_{A_i - x}(M(T_i\backslash x))$ or $\nabla_{A_i - x}(M(T_i\backslash x))$.

Let N' be $N\backslash x$ or N/x depending on whether x is a del or con element of $N\backslash e$. Then it is not difficult to check, using Lemmas 2.13 and 2.16 and Corollary 2.14, that, by beginning with N', one can apply the same sequence of operations that produced $M(T_k\backslash x)$ from $M(T_0\backslash x)$ to obtain N_k/x.

Suppose that N' is 3–connected. Since $x \in D$, the matroid N_k/x has a Q_6– or a $U_{3,6}$–minor and $N_k/x\backslash e \cong U_{m-3,m-1}$. Thus N' contradicts the choice of N since (i)–(iii) hold for N', where we note that both the operations $\Delta_{A_i - x}$ and $\nabla_{A_i - x}$ equal the identity when $|A_i - x| = 2$. We conclude that N' is not 3–connected. But $N'\backslash e \in \Lambda_{m-1}$ so $N'\backslash e$ is 3–connected. Thus $N' = N/x$, otherwise $N' = N\backslash x$ and so N' is 3–connected since both N and $N\backslash x\backslash e$ are. As N/x is not 3–connected, but both N and $N/x\backslash e$ are, we deduce that $\{x, e\}$ is in a triangle of N.

We may assume the following:

6.4.1. *Every element x of D is a con element of $N\backslash e$ and lies in a triangle of N with e.*

We show next that

6.4.2. $|D| \leq 2$.

Proof. Suppose first that, in N, there are at least three non-trivial lines through e that each contains an element of D. If there are three such coplanar lines, then it is easy to see that N has one of $M(\mathcal{W}_3), \mathcal{W}^3, Q_6$, and $U_{3,6}$ as a minor; a contradiction. We may now assume that we have three non-trivial lines L_1, L_2, and L_3 through e whose union has rank four such that each contains an element of D. Let $\{e, d_i, f_i\} \subseteq L_i$ for each i, where $d_i \in D$. Then no two of d_1, d_2, and d_3 are clones in $N\backslash e$. Hence d_1, d_2, and d_3 are in different vertex classes of T_0. Thus, by two applications of Lemma 4.8, we deduce that $N\backslash e/d_1, d_2 = M(T_0\backslash d_1, d_2)$. But $M(T_0\backslash d_1, d_2)$ is in Λ_{m-2} and so is simple whereas $N\backslash e/d_1, d_2$ has $\{f_1, f_2\}$ as a circuit; a contradiction.

We conclude that there are at most two non-trivial lines through e that contain elements of D. Thus $|D| \leq 2$ unless there is a line through e containing two elements of D.

Now suppose that N has a line L containing $\{e, d, d'\}$ where $\{d, d'\} \subseteq D$. Then $\{e, d, d'\}$ is not a circuit of N_k otherwise $N_k/d\backslash e$, which is isomorphic to $U_{m-3,m-1}$, has a minor isomorphic to Q_6 or $U_{3,6}$; a contradiction. Thus, for some i, the set A_i meets $\{d, d'\}$ where we recall that N_{i+1} is either $\Delta_{A_i}(N_i)$ or $\nabla_{A_i}(N_i)$. For the first such i, since d and d' are con elements of T_0, the set A_i is a con class of T_i, and so $N_{i+1} = \nabla_{A_i}(N_i)$. In N_i, every 3–element subset of A_i is a triad. Since $e \notin A_i$ but $\{e, d, d'\}$ is a triangle of N_i, we deduce that $A_i \supseteq \{d, d'\}$ and $|A_i| = 3$. Since each T_{j+1} is obtained from T_j by shrinking a vertex and, for all degree-one vertices v of T_0, the set E_v has at least three elements, we deduce that A_i is a vertex class of T_0 and so is a triad of N. By orthogonality, there is no other element of N on the line L through $\{e, d, d'\}$. Thus $|D| \leq 2$ unless there is another non-trivial line through e meeting D. Consider the exceptional case, letting $\{e, d'', f\}$ be a subset of a line L' through e where $d'' \in D$ and $L' \neq L$. Then, by orthogonality, d'' is in a vertex class of T_0 that is different from A_i. Thus $N\backslash e/d, d'' = M(T_0\backslash d, d'')$ but the former has parallel elements, while the latter cannot. We conclude that (6.4.2) holds. □

Since N_k/D is isomorphic to Q_6 or $U_{3,6}$, it follows from (6.4.2) that N_k has at most 8 elements. Hence N_0 has at most 8 elements. Moreover, $N_0\backslash e \in \Lambda_m$ and, since N_k has a Q_6– or $U_{3,6}$–minor, $N_k \notin \Lambda_{m+1}$. Suppose that $N_0\backslash e$ has rank two. Then so does N_0 and hence $N_0 \in \Lambda_{m+1}$; a contradiction. Now suppose that $N_0\backslash e$ has corank two. Then $N_0\backslash e = N_k\backslash e$, so no Δ– or ∇–exchanges are used to produce $N_0\backslash e$ from $N_k\backslash e$. Hence no exchanges are used to produce N_0 from N_k, so $N_0 = N_k$. This is a contradiction since $N_0 \in EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$. We may now assume that both the rank and corank of $N_0\backslash e$ exceed two. Then T_0 has at least two vertices containing at least three elements and so $M(T_0)$, which equals $N_0\backslash e$, has a P_6–minor.

Assume that $|E(N_0)| = 7$. Then, since $N_0 \in EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$, it is not difficult to check that $N_0 \cong P_{2,4}$ or $N_0 \cong P_{3,3}$, where P_{n_1,n_2} is as shown in Figure 5. By Lemma 5.20, N_0 is in Λ_7 and hence so is N_k; a contradiction. We may now assume that $|E(N_0)| = 8$. Then, for some element f of N_0, either $N_0\backslash e\backslash f \cong P_6$ or $N_0\backslash e/f \cong P_6$. Moreover, $|D| = 2$. Thus either

(a) there is a line through e containing two con elements of $N_0\backslash e$, or
(b) there are two non-trivial lines through e each containing a con element of $N_0\backslash e$.

In particular, $N_0\backslash e$ has at least two con elements.

Suppose that $N_0\backslash e\backslash f \cong P_6$. Then, from the case when $|E(N_0)| = 7$, we deduce that both $N_0\backslash e$ and $N_0\backslash f$ are isomorphic to members of $\{P_{2,4}, P_{3,3}\}$. As $N_0\backslash e$ has at least two con elements, it follows that $N_0\backslash e \cong P_{2,4}$. It is not difficult to check that (a) holds and so $N_0 \cong P_{3,4}$. Thus N_0, and hence N_k, is in Λ_8; a contradiction.

Next, suppose that $N_0\backslash e/f \cong P_6$ and (b) holds. Then N_0 has rank and corank 4. Let C^* be a cocircuit that is the complement of a flat spanned by two non-trivial lines through e. Then C^* is a triad and C^* has no element g for which N_0/g is 3–connected otherwise N_0 has a Q_6–, $M(\mathcal{W}_3)$–, or $\mathcal{W}^3$–minor. Thus, by a result of Lemos [11], there are two triangles of N_0 meeting C^* in distinct subsets. Hence every element of C^*, and therefore every element of N_0, is in a triangle. Thus N_0^* is a minimally 3–connected matroid of rank 4 having 8 elements. Therefore, by [12], N_0^* is a wheel or whirl of rank 4; a contradiction.

Finally, suppose that $N_0\backslash e/f \cong P_6$ and (a) holds. Then, since $N_0\backslash e$ is 3–connected, it follows by duality from the case when $|E(N_0)| = 7$ that $N_0\backslash e$ is isomorphic to $P_{2,4}^*$ or $P_{3,3}^*$. Using geometric representations of these two matroids along with the fact that (a) holds, we obtain by a straightforward case analysis that either $N_0 \notin EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$ or $N_0 \in \Lambda_8$. Both possibilities yield contradictions. □

Proof of Theorem 6.3. Since the members of $\{U_{0,0}, U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}, U_{2,3}\}$ are all of the 3–connected matroids with at most three elements, it follows by Lemma 6.1 that we need to show that if M is a 3–connected member of $EX(M(\mathcal{W}_3), \mathcal{W}^3, Q_6, U_{3,6})$ with at least four elements, then M is a member of $\bigcup_{m\geq 4} \Lambda_m$. If either $r(M) = 2$ or $r^*(M) = 2$, then M is certainly in $\bigcup_{m\geq 4} \Lambda_m$. Therefore we may assume that both the rank and corank of M are at least three. If M is binary, then it follows, by Tutte's Wheels-and-Whirls Theorem [26] (see, for example, [15, Corollary 11.2.14]), that M has a minor isomorphic to $M(\mathcal{W}_3)$. Hence M is non-binary and so M has a $U_{2,4}$–minor [24]. Since M and $U_{2,4}$ are both 3–connected and since, for $r \geq 3$, M has no minor isomorphic to the rank–r whirl, it follows by the Splitter Theorem [22] that M has a 3–connected minor isomorphic to either a single-element extension or a single-element coextension of $U_{2,4}$. Thus M has either a $U_{2,5}$– or $U_{3,5}$–minor. But M has rank and corank at least three, so, by [14, Theorem 1.6], M has a minor isomorphic to $U_{3,6}$, Q_6, or P_6. The first two possibilities are excluded. Hence M has a P_6–minor. Thus there is a sequence $M_0, M_1, \ldots, M_n$ of 3–connected matroids with $M_0 \cong P_6$ and $M_n = M$, such that, for all i in $\{1, 2, \ldots, n\}$, the matroid M_{i-1} is a single-element deletion or a single-element contraction of M_i. Let j be the least index i for which M_{i-1} is in $\bigcup_{m\geq 4} \Lambda_m$ but M_i is not. Such an index certainly exists since P_6 is in $\bigcup_{m\geq 4} \Lambda_m$ and M is not. By duality, we may assume that $M_j\backslash e = M_{j-1}$.

Each of the matroids in Figure 3 has either Q_6 or $U_{3,6}$ as a minor. Therefore M_j has none of the matroids in Figure 3 as a minor. Thus, by Lemma 5.19, there is a sequence $N_0, N_1, \ldots, N_k$ of matroids with $M_j = N_0$ and $N_k\backslash e \cong U_{m-2,m}$ such that (i)–(iii) of Lemma 6.4 hold. Thus, by that lemma, N_k has neither a Q_6– nor a $U_{3,6}$–minor.

To complete the proof of the theorem, we now show that N_k^* is isomorphic to P_{n_1,n_2} for some n_1 and n_2 that sum to m. First, we observe that e is in no non-trivial lines of N_k^* since $N_k^*/e \cong U_{2,m}$. Moreover, since N_k^* has rank three, N_k^* does not have two intersecting non-trivial lines otherwise it has a Q_6–minor. Hence N_k^* has at most two non-trivial lines otherwise it has a $U_{3,6}$–minor. If N_k^* has exactly

two non-trivial lines having n_1 and n_2 points, respectively, than $N_k^* \cong P_{n_1,n_2}$. If N_k^* has at most one non-trivial line, then, since N_k^* has no $U_{3,6}$–minor, N_k^* must have exactly one such line and this line must use all but three of the elements of the matroid. In this case, $N_k^* \cong P_{2,m-2}$. We conclude that N_k^* is indeed isomorphic to P_{n_1,n_2} for some n_1 and n_2 that sum to m. Therefore N_k^*, and hence N_k, is in $\bigcup_{m\geq 4} \Lambda_m$. Thus M_j, which equals N_0, is in $\bigcup_{m\geq 4} \Lambda_m$; a contradiction. □

Acknowledgements

Oxley and Vertigan were partially supported by grants from the National Security Agency; Vertigan was also partially supported by a grant from the Louisiana Education Quality Support Fund; Semple was partially supported by a grant to Geoff Whittle from the Marsden Fund. The authors thank Joseph E. Bonin for several helpful suggestions including how to dramatically shorten the proof of Lemma 2.18. This work forms part of Semple's Ph. D. thesis [17], which was supervised by Geoff Whittle at Victoria University of Wellington.

References

[1] Akkari, S. and Oxley, J. (1993). Some local extremal connectivity results for matroids. *Combinatorics, Probability and Computing* **2**, 367–384.
[2] Bixby, R. E. (1979). On Reid's characterization of the ternary matroids. *J. Combin. Theory Ser. B* **26**, 174–204.
[3] Bixby, R. E. (1982). A simple theorem on 3–connectivity. *Linear Algebra Appl.* **45**, 123–126.
[4] Brylawski, T. H. (1975). Modular constructions for combinatorial geometries. *Trans. Amer. Math. Soc.* **203**, 1–44.
[5] Brylawski, T. H. and Lucas, D. (1976). Uniquely representable combinatorial geometries. In *Teorie Combinatorie* (Proc. 1973 Internat. Colloq.), pp. 83–104. Accademia Nazionale dei Lincei, Rome.
[6] Geelen, J. The excluded minors for near-regular matroids, in preparation.
[7] Geelen, J., Gerards, A. M. H., and Kapoor, A. The excluded minors for $GF(4)$–representable matroids, submitted.
[8] Geelen, J., Oxley, J., Vertigan, D., and Whittle, G. (1998). Weak maps and stabilizers of classes of matroids, *Adv. in Appl. Math.* **21**, 305–341.
[9] Geelen, J., Oxley, J., Vertigan, D., and Whittle, G. On totally free expansions of matroids, in preparation.
[10] Kahn, J. (1988). On the uniqueness of matroid representations over $GF(4)$. *Bull. London Math. Soc.* **20**, 5–10.
[11] Lemos, M. (1989). On 3–connected matroids. *Discrete Math.* **73**, 273–283.
[12] Oxley, J. G. (1981). On matroid connectivity. *Quart. J. Math. Oxford Ser. (2)* **32**, 193–208.
[13] Oxley, J. G. (1984). On the intersections of circuits and cocircuits in matroids. *Combinatorica* **4**, 187–195.
[14] Oxley, J. G. (1989). A characterization of certain excluded-minor classes of matroids. *European J. Combin.* **10**, 275–279.
[15] Oxley, J. G. (1992). *Matroid Theory.* Oxford University Press, New York.
[16] Semple, C. A. (1996). k–regular matroids. In *Combinatorics, Complexity and Logic*, Proceedings of the First International Conference on Discrete Mathematics and Theoretical Computer Science, Auckland, New Zealand, 1996, (eds. D. S. Bridges, C. S. Calude, J. Gibbons, S. Reeves, and I. H. Witten), pp. 376–386, Springer, Singapore.
[17] Semple, C. A. (1998). *k–regular matroids.* Ph.D. thesis, Victoria University of Wellington.
[18] Semple, C. A. On maximum-sized k–regular matroids. *Graphs and Combinatorics* **15**(1999), 441–462.

[19] Semple, C. A. and Whittle, G. P. (1996). Partial fields and matroid representation. *Adv. in Appl. Math.* **17**, 184–208.
[20] Seymour, P. D. (1977). A note on the production of matroid minors. *J. Combin. Theory Ser. B* **22**, 289–295.
[21] Seymour, P. D. (1979). Matroid representation over $GF(3)$. *J. Combin. Theory Ser. B* **26**, 305–359.
[22] Seymour, P. D. (1980). Decomposition of regular matroids. *J. Combin. Theory Ser. B* **28**, 305–359.
[23] Seymour, P. D. (1981). On minors of non-binary matroids. *Combinatorica* **1**, 387–394.
[24] Tutte, W. T. (1958). A homotopy theorem for matroids, I, II. *Trans. Amer. Math. Soc.* **88**, 144–174.
[25] Tutte, W. T. (1965). Lectures on matroids. *J. Res. Nat. Bur. Standards Sect. B.* **69B**, 1–47.
[26] Tutte, W. T. (1966). Connectivity in matroids. *Canad. J. Math.* **18**, 1301–1324.
[27] Vertigan, D. L. Matroid representation over partial fields, in preparation.
[28] Whittle, G. (1995). A characterisation of the matroids representable over $GF(3)$ and the rationals. *J. Combin. Theory Ser. B* **65**, 222–261.
[29] Whittle, G. (1997) On matroids representable over $GF(3)$ and other fields. *Trans. Amer. Math. Soc.* **349**, 579–603.
[30] Whittle, G. Stabilizers of classes of representable matroids. *J. Combin. Theory Ser. B*, to appear.

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana

E-mail address: oxley@math.lsu.edu, vertigan@math.lsu.edu

School of Mathematical and Computing Sciences, Victoria University, Wellington, New Zealand

Current address: Department of Mathematics and Statistics, University of Canterbury, Private Bag 4800, Christchurch, New Zealand

E-mail address: C.Semple@math.canterbury.ac.nz