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Testing Linearity in Cointegrating Relations With an Application to Purchasing Power Parity

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This article shows that when applied to nonstationary time series, the conventional Regression Error Specification Test (RESET) leads to severe size distortion and its asymptotic distribution involves a mixture of noncentral χ^2 distributions. Nonstationarity introduces bias terms in the limit distribution, and appropriate corrections for the bias are presented leading to a modified RESET test that has a central χ^2 limit distribution. In simulations, this modified test is shown to have power not only against nonlinear cointegration but also against the absence of cointegration. In an empirical illustration, the linear purchasing power parity (PPP) specification is tested using five Organization for Economic Cooperation and Development (OECD) countries.

KEY WORDS: Noncentral χ^2 distribution; Nonlinear cointegration; RESET test; Specification test.

1. INTRODUCTION

Since the introduction of the cointegration concept, linear models have dominated practical work in cointegration analysis. This emphasis has arisen, not so much because the underlying economic theory suggests linearity, but rather because the cointegration concept and associated econometric methodology was developed largely for linear models of integrated processes. Correspondingly, the tools of econometric analysis are available in this case and there is great convenience in computation for applied work.

Empirical applications, however, often stimulate an interest in nonlinear specifications and, as a consequence, many nonlinear models (and almost as many specification tests) have been developed for stationary time series modeling. Many recent nonlinear model applications of nonstationary time series have focused on nonlinear short-run dynamics around linear long-run equilibria in error correction models (ECM), as in Berben and Dijk (1999), Lo and Zivot (2001), and Teräsvirta and Eliasson (2001) among others. However, few attempts have been made to study nonlinear cointegrating relations directly and the methods that have been tried in practical work often require restrictive conditions on the DGP (e.g., Haug and Basher 2003). Such extensions also await a corresponding development in tests of specification.

Neglecting the possible nonlinearity in a long-run relationship can be particularly detrimental in nonstationary cases. For stationary time series, linear models can often provide workable approximations at least locally to nonlinear models. Unlike mean-reverting stationary processes, nonstationary time series have a tendency to wander with no fixed mean or locality in the sample space and, like random walks, revisit points distant from the origin an infinite number of times. In such cases, local linear approximations can only poorly represent the global characteristics of the process, producing a high risk of faulty inference about a misspecified long-run equilibrium.

Consideration of the possibilities suggests three cases—linear cointegration, some form of nonlinear cointegration, or complete absence of cointegration. Existing cointegration tests essentially presume a form of linear cointegration and do not effectively distinguish between linear and nonlinear cointegration (e.g., Granger and Hallman 1989). So, traditional linear cointegration analysis requires an additional test of specification to address this particular issue of functional form. However, in the absence of more appropriate specification tests, applied economists have treated existing cointegration tests as tests for *linear cointegration* and all subsequent analysis rests on this assumption.

Furthermore, existing linearity tests also fail to provide any reliable guidance concerning the type of relationship that may be present between nonstationary time series (Granger 1995; Lee, Kim, and Newbold 2005). It is not surprising to find that existing linearity tests developed for stationary processes work poorly with nonstationary time series and this was well recognized earlier in the case of the Regression Error Specification Test (RESET).

The RESET test by Ramsey (1969) is a convenient device for testing general misspecification (e.g., Vitaliano 1987; Baghestani 1991; Peters 2000, among others), but is known not to be robust to autocorrelated disturbances, especially when the regressor is itself highly autocorrelated (Porter and Kashyap 1984) or contains a deterministic time trend (Leung and Yu 2001). Using simulation, Porter and Kashyap showed that the presence of serially correlated disturbances combined with an AR(1) regressor leads to size distortions, and the more autocorrelated the regressor is, the less robust the RESET test is to error autocorrelation. Naturally, we might expect this size distortion problem to become worse in the cointegrating case where the regressor has an autoregressive unit root and the errors are typically serially dependent. This article analyzes the source of this test failure and shows how the original test can be modified for empirical use with nonstationary time series using asymptotic tools from Park and Phillips (1999, 2001).

The rest of the article is organized as follows. The next section introduces the model and the maintained assumptions and shows how nonstationarity of the data changes the limit theory of existing tests. Section 3 discusses the modifications that are needed when the RESET test is applied to evaluate cointegrating relations. Section 4 discusses the behavior of the modified test under alternatives. Section 5 summarizes simulation results and Section 6 presents an empirical application of the modified test to purchasing power parity (PPP). Section 7 concludes and the proofs are collected in the Appendix.

2. MODELS AND BACKGROUND IDEA

Suppose that we want to test the linear conditional mean specification $H_0 : \mathbb{P}[\mathbb{E}(Y_t|X_t) = \theta_1 X_t] = 1$, $\forall t$ with a specific nonlinear alternative model in mind, such as $f(X_t)$. Then one can use some conventional tests such as a Wald or Lagrange multiplier (LM) test of $H_0 : \theta_2 = 0$ in

$$Y_t = \theta_1 X_t + \theta_2 f(X_t) + u_t. \tag{1}$$

In many practical cases, however, theory fails to provide a specific functional form, and the focus of attention is some convenient linear model (such as that implied by purchasing power parity considerations) with no specific nonlinear alternative. Numerous specification tests have been developed so far, and one of the most frequently used approaches is the so-called residual based procedure. The residuals from regressing Y on X, if the null is true, should not contain any systematic part of $f(X)$ and the many residual based tests arise from different methods of detecting such leftover signals in residuals. For example, the KPSS (Kwiatkowski et al. 1992) test and the CUSUM test (Ploberger and Kramer 1992 for structural change, Xiao and Phillips 2002 for cointegration) are based on the excess variations and the Park (1990) test and ANN (Artificial Neural Network; White 1992; Lee, White, and Granger 1993) test are based on the approximation to an unspecified $f(\cdot)$.

In the case of the approximation-based tests, first the unspecified nonlinear function $f(X_t)$ is replaced with its partial sum approximation $\hat{f}_k(X_t) = \sum_{j=1}^{k} \beta_j F_j(X_t)$ for some given basis functions $\{F_j(x)\}$ that form a complete set in L_2. Now we proceed to test the validity of the linear specification by testing whether a linear combination of $\{F_j(\cdot)\}_1^k$ can detect any nonlinearity in the regression residuals $\hat{u}_t$ in

$$Y_t = \theta X_t + u_t \qquad \text{and} \qquad \hat{u}_t = \sum_{j=1}^{k} \beta_j F_j(X_t) + e_t. \tag{2}$$

In general, there are two ways to test the linear specification in this setting. For an approximation based test with $\sum_{j=1}^{k} \beta_j F_j(X_t)$, we can either directly test its statistical significance with $H_0 : \beta_j = 0, \forall j$ (Ramsey 1969; White 1992; Kapetanios 2003; DeBenedictis and Giles 1998), or find whether the optimal $k = 0$, using a version of order selection criterion (Eubank and Hart 1992). In this article, we take the first approach with the polynomial basis functions $F_j(X_t) = X_t^{j+1}$ for the RESET test.

Note that estimation of (2) involves working with the sample moments of nonlinearly transformed integrated time series whose asymptotic behavior must be characterized. Before examining these quantities, we first specify the data generating processes and some assumptions that will facilitate the development of a limit theory.

Assumption A. Let $\Delta X_t = v_t$ and u_t be general linear processes satisfying the following conditions:

$$u_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j} = C(L)\varepsilon_t, \qquad v_t = \sum_{j=0}^{\infty} d_j \eta_{t-j} = D(L)\eta_t,$$

where $\tilde{\zeta}_t = (\eta_{t+1}, \varepsilon_t)'$ is a stationary and ergodic martingale difference sequence with natural filtration $\mathcal{F}_t = \sigma(\{\tilde{\zeta}_s\}_{-\infty}^t)$ satisfying

1. $\sup_{t\geq 1} \mathbb{E}(\|\tilde{\zeta}_t\|^r|\mathcal{F}_{t-1}) < \infty$ a.s. for some $r > 4$
2. $\mathbb{E}(\tilde{\zeta}_{i,t}^2 \tilde{\zeta}_{j,t-l}) = 0$ for all i, j and for all $l \geq 1$
3. ε_t is *iid* with $\mathbb{E}|\varepsilon_t|^r < \infty$ for some $r > 8$, its distribution is absolutely continuous with respect to Lebesgue measure and its characteristic function φ satisfies $\varphi(\lambda) = o(\|\lambda\|^{-\delta})$ as $\lambda \to \infty$ for some $\delta > 0$.

In addition, $\{c_j, d_j\}$ satisfy the summability conditions: $\sum_{j=0}^{\infty} j \times |d_j| < \infty$, $\sum_{j=0}^{\infty} j^{1/2}|c_j| < \infty$, and $D(1) \neq 0$.

These assumptions on the innovation processes are fairly standard and are satisfied by a wide class of processes; for example, an invertible Gaussian autoregressive moving average (ARMA) model. Similar conditions are employed in deriving the results of Park and Phillips (1999, 2000, 2001) and Chang, Park, and Phillips (2001). However, de Jong's (2002) more relaxed conditions are sufficient for the modification of the RESET test presented in this article.

Under Assumption A, the invariance principle holds for $\tilde{\zeta}_t$ so that $n^{-1/2} \sum_{t=1}^{[nr]} \hat{\zeta}_t \Rightarrow^d BM(\Sigma)$. Using the Beveridge–Nelson decomposition (Phillips and Solo 1992), we can show that a similar result holds for the time series $\zeta_t = [v_{t+1}, u_t]'$,

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} \zeta_t \Rightarrow^d B(r) \equiv \begin{pmatrix} B_x(r) \\ B_u(r) \end{pmatrix} \equiv BM(\Omega),$$

$$\text{with } \Omega = \begin{pmatrix} \Omega_{vv} & \Omega_{vu} \\ \Omega_{uv} & \Omega_{uu} \end{pmatrix}.$$

Here the covariance matrix $\Omega = \sum_{h=-\infty}^{\infty} \Gamma_\zeta(h)$, where $\Gamma_\zeta(h) = \mathbb{E}(\zeta_0 \zeta_h')$. Also, define the one-sided long-run covariance matrices with similar partitions as $\Lambda = \sum_{h=1}^{\infty} \Gamma_\zeta(h)$ and $\Delta = \Gamma_\zeta(0) + \Lambda$.

Among the wide variety of possible nonlinear functions, Park and Phillips (1999, 2001) provide tools for asymptotic analysis with some classes of functions (of integrated processes) satisfying certain regularity conditions. The simple basis functions $\{X_t^j\}$ of a Taylor series expansion fall within the so-called H-regular class (or Class H). Functions in this class have homogeneous, asymptotically dominating components, that is, $f(\lambda x) \cong \kappa(\lambda) H(x)$, that are locally integrable. $H(x)$ is referred as the asymptotic homogeneous function of $F(x)$ and $\kappa(\lambda)$ as the asymptotic order of $F(x)$. Park and Phillips (1999) provided various examples that belong to this class, such as finite order polynomials, and distribution-like functions, including their linear combinations and products. The polynomial basis functions $\{X^{m+1}\}$ from a Taylor series expansion belong to this class with $H(x) = x^{m+1}$ and $\kappa(\lambda) = \lambda^{m+1}$.

Another important class of nonlinear transformation is the I-regular (or Class I) transformation. Roughly speaking, functions in this class are bounded, integrable, and (piecewise) smooth. All pdf-like functions belong to this class. See Park and Phillips (1999) for further details on these classifications.

2.1 Nonlinear Sample Covariances & RESET

Testing a linear specification with $H_0: \beta_j = 0, \forall j$ in (2) involves the sample covariance between u_t and the polynomials of X_t. The following lemma is a special case of de Jong (2002) and shows the asymptotic behavior of this sample quantity.

Lemma 1. Suppose Assumption A holds. For $m \geq 1$, the sample covariance between X_t^m and u_t satisfies

$$\frac{1}{n^{(m+1)/2}} \sum_{t=1}^{n} X_t^m u_t \Rightarrow^d \int B_x^m \, dB_{u.x} + \Omega_{uv}\Omega_{vv}^{-1} \int B_x^m \, dB_x + m\Lambda_{vu} \int B_x^{m-1}, \quad (3)$$

where $\Lambda_{vu} = \sum_{h=1}^{\infty} E(v_0 u_h)$ and the Brownian motion $B_{u.x} = B_u - \Omega_{uv}\Omega_{vv}^{-1} B_x$ is independent of B_x and has variance $\Omega_{uu.v} = \Omega_{uu} - \Omega_{uv}\Omega_{vv}^{-1}\Omega_{vu}$.

The limit of the sample covariance (3) has two components—$\int B_x^m \, dB_{u.x}$, a mean zero Gaussian mixture, and the remaining two terms that correspond to the so-called endogeneity bias and serial correlation bias of linear cointegration theory (Phillips and Hansen 1990). These second-order bias terms stem from the nonstationarity of X_t and shift the mean of the limit distribution away from zero. In the simplest case of strictly exogenous X_t, $E(v_t u_s) = 0$ for all t, s so that the two bias terms are zero with $\Omega_{uv} = \Lambda_{uv} = 0$. In related work, de Jong (2002) examined nonlinear sample covariance asymptotics under conditions that are less strict on the innovation processes, but more restrictive in terms of functional forms; and a general semimartingale approach to establishing limit results of this type was developed in Ibragimov and Phillips (2004).

The effects of the two bias terms in (3) can be substantial on the distribution of the RESET test statistic. As is well known (e.g., Muirhead 1982, theorem 1.4.5), a quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ in the Gaussian random vector $\mathbf{x} \sim \mathrm{N}(\boldsymbol{\xi}, \mathbf{V})$ follows a noncentral χ^2 distribution, $\chi^2(k, \nu)$, where $k = \mathrm{rank}(\mathbf{AV})$ is the degrees of freedom and $\nu = \boldsymbol{\xi}'\mathbf{A}\boldsymbol{\xi}$ is the noncentrality parameter. Letting $\mathbf{x}$ be the limit of $\sum_{t=1}^{n} \hat{u}_t \mathbf{F}_t$ after appropriate normalization, we can show that the test statistic R_n in (4) follows a mixture of noncentral χ^2 distributions under suitable conditioning. Due to the presence of the fitted residual $\hat{u}_t$ instead of u_t, some additional bias terms appear in the limit, in addition to the two terms shown in (3). The following theorem summarizes this result.

Theorem 2. Under Assumption A, the RESET test statistic R_n asymptotically has a mixture of noncentral $\chi^2(k, \nu)$ distributions with k degrees of freedom and the random noncentrality parameter $\nu = \boldsymbol{\xi}'\mathbf{A}\boldsymbol{\xi}$. That is, the RESET test statistic

$$R_n = \left(\sum_{t=1}^{n} \hat{u}_t \mathbf{F}_t\right)' \left(\hat{\Omega}_{uu.v} \sum_{t=1}^{n} \tilde{\mathbf{F}}_t \tilde{\mathbf{F}}_t'\right)^{-1} \left(\sum_{t=1}^{n} \hat{u}_t \mathbf{F}_t\right) = \hat{\mathbf{u}}'\mathbf{F}(\hat{\Omega}_{uu.v}\tilde{\mathbf{F}}'\tilde{\mathbf{F}})^{-1}\mathbf{F}'\hat{\mathbf{u}} \overset{a}{\sim} \chi^2(k, \nu) \quad (4)$$

for the auxiliary regressors $\mathbf{F}_t = [X_t^2 \;\; \cdots \;\; X_t^{k+1}]'$, $\mathbf{F} = [\mathbf{F}_1, \ldots, \mathbf{F}_n]'$, and the regression residuals $\tilde{\mathbf{F}} = [\tilde{\mathbf{F}}_1, \ldots, \tilde{\mathbf{F}}_n]'$ with $\tilde{\mathbf{F}}_t = \mathbf{F}_t - X_t(\sum_t X_t X_t')^{-1} \sum_t X_t \mathbf{F}_t$. The random vector $\boldsymbol{\xi}$ is $k \times 1$ with $(m-1)$th element defined as, for $m \geq 2$,

$$\boldsymbol{\xi}(m-1) = \Omega_{uv}\Omega_{vv}^{-1} \int \widetilde{B_x^m} \, dB_x + m\Lambda_{vu} \int B_x^{m-1} - \Lambda_{vu}\left(\int B_x^2\right)^{-1} \int B_x^{m+1}, \quad (5)$$

with $\widetilde{B_x^m} = B_x^m - B_x(\int B_x^2)^{-1} \int B_x^{m+1}$ and $\mathbf{A}$ is the inverse of a $k \times k$ limit variance matrix, vis-a-vis,

$$\mathbf{A} = \left[\Omega_{uu.v} \begin{pmatrix} \int \widetilde{B_x^2}^2 & \cdots & \int \widetilde{B_x^2}\widetilde{B_x^{k+1}} \\ \vdots & \ddots & \vdots \\ \int \widetilde{B_x^{k+1}}\widetilde{B_x^2} & \cdots & \int \widetilde{B_x^{k+1}}^2 \end{pmatrix}\right]^{-1}. \quad (6)$$

When X_t is strictly exogenous, all bias terms disappear with $\mathbb{E}(u_t v_s) = 0, \forall t, s$, and the test statistic R_n asymptotically has a mixture of central χ^2 distributions conditional on $\mathcal{F}_x = \sigma(B_x(r), 0 \leq r \leq 1)$. Since the limit distribution is independent of $\mathcal{F}_x$, we deduce that R_n converges to $\chi^2(k)$ unconditionally.

In general, $R_n \sim \chi^2(k, \nu)$ asymptotically and this noncentral distribution can be approximated by a multiple of the central χ^2 distribution, $a\chi^2(b)$, where the two constants are given by (Johnson and Kotz 1970)

$$a = 1 + \frac{\nu}{k+\nu} \geq 1 \qquad \text{and} \qquad b = k + \frac{\nu^2}{k+2\nu} \geq k.$$

Therefore, conditional on $\mathcal{F}_x$, the probability of rejecting the linear null hypothesis can be shown to be at least as great as the nominal size α asymptotically, vis-a-vis,

$$\mathbb{P}[R_n > \chi_\alpha^2] \sim \mathbb{P}\left[\left(1 + \frac{\nu}{k+\nu}\right)\chi^2(b) > \chi_\alpha^2\right] \geq \mathbb{P}\left[\chi^2\left(k + \frac{\nu^2}{k+2\nu}\right) > \chi_\alpha^2\right] \geq \alpha,$$

and this explains the large size distortions in Porter and Kashyap (1984).

3. BIAS CORRECTION & MODIFIED TEST

The previous section shows that nonstationarity of X_t introduces bias terms in the limit distribution of the sample covariance between X_t^m and $\hat{u}_t$, leading to the noncentral limit distribution of the RESET statistic R_n. These bias terms are the main source of the large size distortions of the test and the following theorem presents a method to remove them similar to the direct nonparametric correction method in fully modified (FM) regression (Phillips and Hansen 1990; Phillips 1995).

Theorem 3. Suppose Assumption A holds. If $\{X_t, Y_t\}$ are linearly cointegrated, the following modified RESET statistic has a limiting central $\chi^2(k)$ distribution

$$\begin{aligned} MR_n &= \{\hat{\mathbf{u}}'\mathbf{F}\mathbf{D}_n - \mathbf{E}_n' - \mathbf{S}_n'\}(\hat{\Omega}_{uu.v}\mathbf{D}_n'\tilde{\mathbf{F}}'\tilde{\mathbf{F}}\mathbf{D}_n)^{-1} \\ &\quad \times \{\mathbf{D}_n\mathbf{F}'\hat{\mathbf{u}} - \mathbf{E}_n - \mathbf{S}_n\} \\ &\overset{a}{\sim} \chi^2(k), \end{aligned}$$

where $\hat{\mathbf{u}}$ is an $n \times 1$ vector of residuals from the linear cointegration regression (2) with $\mathbf{F}$ and $\tilde{\mathbf{F}}$ as in Theorem 2. The $k \times k$ normalization matrix $\mathbf{D}_n$ and the $(m-1)$th elements of the two $k \times 1$ correction vectors $\mathbf{E}_n = [E_n(1), \ldots, E_n(k)]'$ and $\mathbf{S}_n = [S_n(1), \ldots, S_n(k)]'$ are defined as

$$\mathbf{D}_n = \operatorname{diag}\big(n^{-3/2}, n^{-4/2}, \ldots, n^{-(k+2)/2}\big),$$

$$\begin{aligned} E_n(m-1) &= \hat{\Omega}_{uv}\hat{\Omega}_{vv}^{-1} \\ &\quad \times \left[\left\{\sum_{t=1}^{n}\left(\frac{X_t}{\sqrt{n}}\right)^m \frac{v_t}{\sqrt{n}} - \hat{\Delta}_{vv}\frac{m}{n}\sum_{t=1}^{n}\left(\frac{X_t}{\sqrt{n}}\right)^{m-1}\right\}\right. \\ &\quad - \left(\frac{1}{n}\sum_{t=1}^{n}X_t v_t - \hat{\Delta}_{vv}\right)\left(\frac{1}{n^2}\sum_{t=1}^{n}X_t^2\right)^{-1} \\ &\quad \times \left.\left(\frac{1}{n}\sum_{t=1}^{n}\left(\frac{X_t}{\sqrt{n}}\right)^{m+1}\right)\right], \end{aligned} \tag{7}$$

$$\begin{aligned} S_n(m-1) &= \hat{\Lambda}_{vu}\frac{m}{n}\sum_{t=1}^{n}\left(\frac{X_t}{\sqrt{n}}\right)^{m-1} \\ &\quad - \hat{\Lambda}_{vu}\left(\frac{1}{n^2}\sum_{t=1}^{n}X_t^2\right)^{-1}\frac{1}{n}\sum_{t=1}^{n}\left(\frac{X_t}{\sqrt{n}}\right)^{m+1}. \end{aligned} \tag{8}$$

Since this type of test is based on a finite approximation, its power naturally depends on the adequacy of the approximation under the alternative specification. The goodness of approximation depends on the given nonlinear functional form that is approximated and two components that can be controlled—the type and the number of basis functions included in the augmented regressors. A good approximation will definitely help in detecting nonlinearity when it is present, but even poor approximations can be effective for testing purposes. This is because the null hypothesis requires that all coefficients be zero, $\beta_j = 0$ for $j = 1, \ldots, k$, and the test will reject the null hypothesis if at least one coefficient deviates enough from zero, that is, if at least one basis function is able to catch some "part" of the nonlinearity.

Although the RESET test is usually thought of as a general linearity test without specific alternatives, it also can be interpreted as an LM test, where the basis functions are treated as possible alternative nonlinear specifications. By construction, the test has highest power against such alternatives. Furthermore, if the test rejects linearity, the estimated nonlinear cointegration relationship provides a possible alternative nonlinear model, or more specifically a partial approximation to an alternative nonlinear model for the data, at least when the relationship is not spurious.

When the linear model is rejected and the alternative polynomial model is estimated, the correction methods in Theorem 3 can be applied again to correct the biases in the least-squares (LS) coefficient estimators. For example, suppose we estimate the following nonlinear cointegration model

$$Y_t = \theta X_t^m + u_t, \qquad t = 1, \ldots, n.$$

Then the corresponding FM estimator of θ is

$$\tilde{\theta}_m = \left(\sum X_t^{2m}\right)^{-1}\left\{\sum X_t^m Y_t - n^{(m+1)/2}[E_m + S_m]\right\}$$

with the correction terms

$$E_m \equiv \hat{\Lambda}_{vu}\frac{m}{n}\sum_{t=1}^{n}\left(\frac{X_t}{\sqrt{n}}\right)^{m-1},$$

$$S_m \equiv \hat{\Omega}_{uv}\hat{\Omega}_{vv}^{-1}\left\{\sum_{t=1}^{n}\left(\frac{X_t}{\sqrt{n}}\right)^m \frac{v_t}{\sqrt{n}} - \hat{\Delta}_{vv}\frac{m}{n}\sum_{t=1}^{n}\left(\frac{X_t}{\sqrt{n}}\right)^{m-1}\right\}$$

and we can show that the modified estimator has a mixed Gaussian asymptotic distribution about the true value, that is, $n^{(m+1)/2}(\tilde{\theta}_m - \theta) \Rightarrow^d (\int B_x^{2m})^{-1}\int B_x^m\, dB_{u.x}$. When $m = 1$, $\tilde{\theta}_m$ is simply the FM estimator in a typical linear cointegration model.

With stationary time series, the auxiliary regressor set $\{X_t^{j+1}\}_{j=1}^k$ often suffers from multicollinearity, in which case principal components can be used instead. If this is the case, the bias correction terms also need to be adjusted accordingly, and the modified test statistic using principal components can be constructed as follows:

$$\begin{aligned} &\{\hat{\mathbf{u}}'\mathbf{F}_n^* - \mathbf{E}_n'\mathbf{G} - \mathbf{S}_n'\mathbf{G}\}(\hat{\Omega}_{uu.v}\tilde{\mathbf{F}}_n^{*\prime}\tilde{\mathbf{F}}_n^*)^{-1} \\ &\qquad\qquad \times \{\mathbf{F}_n^{*\prime}\hat{\mathbf{u}} - \mathbf{G}'\mathbf{E}_n - \mathbf{G}'\mathbf{S}_n\} \overset{a}{\sim} \chi^2(\tilde{k}), \end{aligned}$$

where $\mathbf{G}$ is the $k \times \tilde{k}$ matrix whose columns are the eigenvectors of $\mathbf{F}'F$ divided by the corresponding eigenvalues, and $\mathbf{F}_n^* = \mathbf{F}\mathbf{D}_n\mathbf{G}$ is $n \times \tilde{k}$ normalized matrix with the jth principal component in the jth column. The $\tilde{k}$ eigenvectors are chosen according to the $\tilde{k}$ largest eigenvalues.

This multicollinearity problem is mainly due to the mean-reversion property of stationary time series for which the variation of X_t around zero is dampened by polynomial transformations. However, this is not the case for integrated X_t, which spends little time around the origin and whose variations are typically magnified by polynomial transformations as n increases.

4. MODIFIED TEST UNDER ALTERNATIVES

As discussed earlier, considering nonlinearity together with nonstationarity gives rise to three possible scenarios. Our modified test tests the null hypothesis of linear cointegration against both *nonlinear cointegration* and the *absence of cointegration*, the latter incorporating both the conventional *spurious regression* case and *omitted variable* cases. This section examines test power in these alternative scenarios.

4.1 No Cointegration Case

Lee, Kim, and Newbold (2005) examined six widely used linearity tests and find that evidence of spurious nonlinearity increases with the sample size. The following theorem shows that our modified test statistic also diverges when it is applied to two independent $I(1)$ processes. However, divergence of the test statistic should not be interpreted as evidence of spurious nonlinearity but rather simply as a rejection of the linear cointegration specification with two possible alternative cases. Therefore, the diverging test statistics in the no-cointegration case *correctly* point out the absence of linear cointegration. To determine if the rejection is due to nonlinearity, a further specification test is required.

Theorem 4. Suppose X_t and Y_t are not cointegrated so that

$$Y_t = \theta X_t + u_t, \qquad t = 1, \ldots, n$$

with the $I(1)$ process u_t satisfying $n^{-1/2} u_{t=[n\cdot]} \Rightarrow^d B_u(\cdot)$. In this case the modified RESET statistic diverges at the rate of n/M, where M is the bandwidth parameter used in kernel estimation of the long-run (co)variances.

This result is of some practical interest. The RESET test was originally developed for testing linearity of the model but, when applied to cointegrating relations, the test has power against lack of cointegration as well. Thus, the modified RESET test can serve as an omnibus test for the linear cointegration specification that has power against both no cointegration and nonlinear cointegration.

A similar idea in the context of detecting unit roots is present in Park's (1990) unit root test by variable addition. This test uses deterministic polynomials to detect the presence of leftover stochastic trend(s); the RESET test uses polynomials of the stochastic regressors instead, which have a natural advantage when there is nonlinear cointegration involving these variables.

Since the rate of divergence depends on the relative size of the bandwidth parameter and the number of observations, the choice of M can greatly affect the power of the test against the lack of cointegration. Similar issues arise in other tests that rely on nonparametric estimates, such as the KPSS test for stationarity. We will discuss this issue in the next section together with other practical issues related to applying the modified RESET test.

4.2 Nonlinear Cointegration Case

Among the many types of possible nonlinearities in cointegrated systems, we consider here models in the following nonlinear form

$$Y_t = f(X_t, \theta) + u_t, \qquad t = 1, \ldots, n \tag{9}$$

with $f(\cdot)$ being either H-regular or I-regular.

Theorem 5. Suppose the true model has the nonlinear form (9) and $\{X_t, u_t\}$ satisfy the conditions of Theorem 3. For given M, the modified test statistic MR_n diverges at the rate n/M in the H-regular nonlinear case, but does not diverge in the I-regular nonlinear case.

Obviously, the power of the modified RESET test depends on the true nonlinear functional form. For H-regular nonlinearities, the test statistic diverges at the rate $O_p(\frac{n}{M})$, just as in the case of no cointegration. Note that this result includes the case of a threshold model alternative, where the H-regular transformation is based on indicator functions. The asymptotic order in this case is $\kappa = 1$, as in the case of linear cointegration, but the test statistic still diverges in this case at the rate n/M.

Contrary to the H-regular case, the modified test has particularly low power against I-regular-type nonlinearity. This is because the variations from the I-regular-type nonlinear transformation of X_t that remain in the linear cointegration residuals $\{\hat{u}_t\}$ become negligible relative to the variations of X_t as n increases.

5. SIMULATIONS

Monte Carlo results are presented in this section to show the size distortion of the original RESET test and to investigate how satisfactory the suggested modifications are in achieving the nominal asymptotic size in finite samples. We also report some simulations on the power of the modified RESET test against some specific nonlinear models, choosing the following seven nonlinear models in addition to the linear cointegration model as the reference case:

(1): $Y_t = 1.1 X_t + u_t$,

(2): $Y_t = \log(|X_t| + 1) + u_t$,

(3): $Y_t = X_t^2 + u_t$,

(4): $Y_t = 1.2 \exp(-X_t^2) + u_t$,

(5): $Y_t = 1.1 X_t I_{\{|n^{-1/2} X_t| \geq 0.6\}} - 0.8 X_t I_{\{|n^{-1/2} X_t| < 0.6\}} + u_t$,

(6): $Y_t = 1/(|X_t| + 1) + u_t$,

(7): $Y_t = 1.1(|X_t| + 1)^{1/2} + u_t$,

(8): $Y_t = 1.1(|X_t| + 1)^{3/2} + u_t$.

The regression error $\{u_t\}_{t=1}^n$ and the integrated regressor X_t are generated from the design

$$\Delta X_t = v_t = e_{2,t-1} + 0.4 e_{2,t-2},$$
$$u_t = \rho u_{t-1} + \frac{1}{\sqrt{2}}(e_{1,t} + e_{2,t}),$$

where $\rho \in [0.2, 0.4, 0.6, 0.8]$ controls the level of serial correlation in the error term, and $(e_{1,t}, e_{2,t})' \sim \mathcal{N}(0, I_2)$. Note that the innovation processes are constructed in such a way that X_t is predetermined, as specified in Assumption A. Samples of five different sizes ($n = 50, 100, 250, 500, 1000$) are drawn with 10,000 replications to examine both small sample properties and rate of convergence to the limit

5.1 Size of the Test

Figure 1 compares two RESET tests—before and after bias corrections—when X_t and Y_t are linearly cointegrated. The four graphs summarize the test performance under H_0 from Table 1 with (a) a varying number of observations for a given level of serial correlation ($\rho = 0.6$) and (b) a varying level of serial correlation for a given number of observations. As shown in the upper panels (a), with a moderate level of serial correlation in

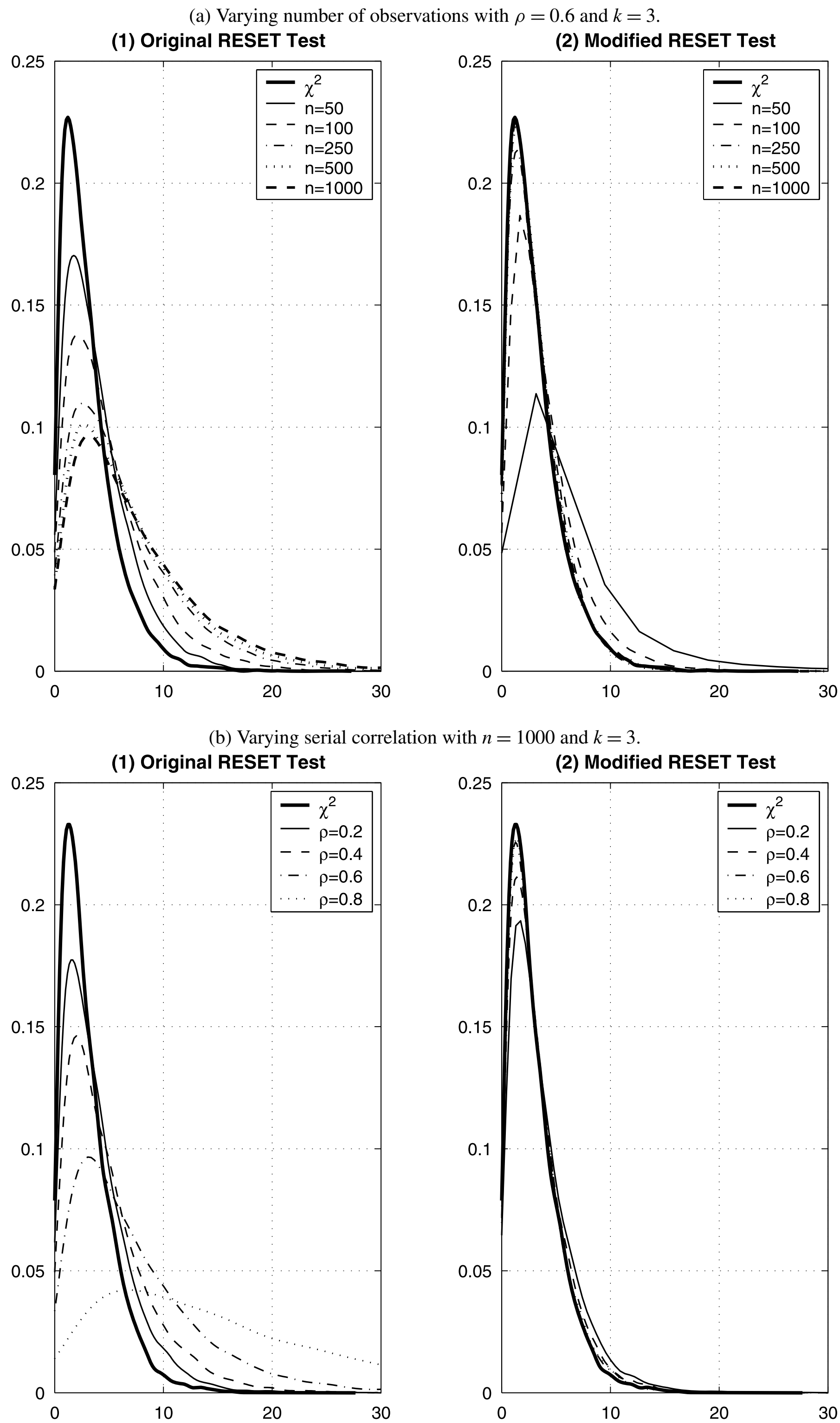

Figure 1. The RESET test statistics before and after modification under H_0. Empirical distributions of the test statistic shown above are from 10,000 simulated samples with $k = 3$. The bandwidth for the kernel estimator of long-run (co)variance is chosen automatically following Andrews (1991). χ^2-distribution in a thick solid line represents the limit distribution of test statistics from a central $\chi^2(k)$ distribution.

the regression error, the RESET test without correction terms shows severe size distortions that become even worse as the sample size increases. For a nominal asymptotic 5% size, the probability of a Type I error rises up to 0.357 with $n = 1000$. This result may be regarded as an extreme version of the earlier findings in Porter and Kashyap (1984). Contrary to the severe size distortions in the original test, the modified test in the right panel of Figure 1(a) exhibits only a minor size distortion, which vanishes as n increases and, at the same time, shows a relatively fast convergence to the limit distribution.

Figure 1(b) shows how the bias correction terms work for different ρ values. The left panel confirms the severe size distortions due to the serially correlated errors. For a nominal asymptotic 5% size, the probability of a Type I error reaches up to 70% for $\rho = 0.8$, while including two correction terms brings it back to 4.99%.

5.2 Power of the Test

Table 1 also reports the power of the modified RESET test against some specific nonlinear models. With linear cointegration as the reference case in (1), simulation results show that the modified RESET test is quite sensitive to many nonlinear possibilities for a wide range of ρ values. The probabilities of rejecting the linearity null are over 90% in most cases except for (4) and (7). As expected, the modified RESET test is most powerful against polynomial type nonlinearity (3) but also shows good powers against logarithmic (2), threshold (5), and reciprocal (6) nonlinearities and a small deviation from linear model (8) as well. Note also that the original RESET test in the second part also shows a similar pattern.

The low power against (4) and (7) is due to fact that in these cases, the nonlinear transforms tend to suppress the variations of X_t, while the polynomials of the RESET test tend to magnify the variations. Therefore, the asymptotic forms of the function $e^{-X_t^2}$ and $(|X_t| + 1)^{1/2}$ when $X_t = O_p(t^{1/2})$ for large t are not well captured by the asymptotic form of the polynomial terms $X_t^m = O_p(t^{m/2})$ for $m \geq 2$.

Table 2 shows another direction for the alternative case of 2 independent $I(1)$ variables. As discussed in Theorem 4, the modified test statistic diverges at the rate n/M so that the rejection rate is sensitive to the choice of the bandwidth parameter M. We report five cases, corresponding to $M = n^{1/5}$, $n^{1/4}$, $n^{1/3}$, $n^{1/2}$ and the usual data-dependent automatic bandwidth (Andrews 1991) for a Parzen kernel. Two aspects of the results in Table 2 confirm Theorem 4. First, the rejection probability tends to be higher for the smaller bandwidth choices for given k and n. Second, the rejection probability increases with n as well as with the number of augmented regressors k in general, especially for smaller bandwidths. For $M = n^{1/3}$, the effect of increasing k on the rejection probability is not as large as in the case of $M = n^{1/5}$, and even decreases for $M = n^{1/2}$. When an automatic bandwidth rule is employed, increasing k has a more significant effect on power for a given n than increasing n for a given k.

5.3 Limitations and Practical Issues

The limitations of the modified RESET test are related to the approximation method that the test is based on, and the nature of the cointegration functional forms. Once the cointegrating function is given, the size of the approximation error is determined by the "type" and "number" of the basis functions $\{F_j\}_{j=1}^k$. These choices determine how well a linear combination of the basis functions can approximate unknown nonlinear cointegrating function $f(X_t)$. If there exists a set of coefficients $\{\beta_j\}_{j=1}^k$ such that $\sum_{j=1}^k \beta_j F_j(X_t)$ is close to $f(X_t)$ over a wide enough domain, then it is clear that we can expect the test to reject linear cointegration in favor of some form of nonlinear cointegration, corresponding to the nonzero $\{\beta_j\}$ estimates.

Once the type of basis functions $\{F_j\}$ is selected, the number of them, k, needs to be chosen. Although larger k may produce an improved approximation to $f(\cdot)$, in a finite sample testing framework, there exist some trade-offs. On the one hand, larger values of k will, at least to a certain point, generally increase the power of the test by virtue of their improved approximation capability. On the other hand, larger k increases the risk of spurious nonlinearity resulting in a higher probability of a Type I error under the null as well as a decrease in degrees of freedom in the regression. Moreover, to reject the null hypothesis $H_0 : \beta_1 = \cdots = \beta_k = 0$, at least one significant coefficient will suffice, a condition that is less restrictive than requiring a good fit to $f(X_t)$ by $\sum_{j=1}^k \hat{\beta}_j F_j(X_t)$. Simulations (not reported here) suggest that the use of $k = 2$ or 3 generally produces good size and reasonable power, while increasing k to $k = 3$ or 4 adds power without too much compromise in size.

Another important factor that is not shown explicitly in the regression Equation (2) is the choice of bandwidth parameter M for kernel estimation. As discussed in Theorem 4 and shown in Table 2, the power against the no-cointegration alternative depends on n/M. The test statistic under the same alternatives diverges faster as M/n becomes smaller, but this makes the test statistic under the null converge to the asymptotic distribution at a slower rate. Therefore, in addition to the choice of k, it is recommended to apply the test with different combinations of k and M to get a more concrete result. A popular choice for bandwidth selection is the data dependent method of Andrews (1991). The Parzen kernel is used in the simulations shown but, while not reported here, other kernels with their automatic bandwidths gave similar power and size properites.

Finally, an important but uncontrollable factor that affects the power of the test is the actual nonlinear functional form under the alternative. Although general approximation methods, including the power series approximations that underlie the RESET test, can provide reasonable approximations for a wide class of nonlinear functions, there are nonlinear transformations that are not well approximated by these methods. Low power of the modified RESET test against I-regular-type nonlinearity can be understood in this context. This problem can be alleviated by unit root testing, which can sometimes provide information about the type of nonlinearity. For example, if the dependent variable is a trigonometic function of an $I(1)$ process, this function will behave like a stationary AR(1) process (e.g., Ermini and Granger 1993) and unit root tests prior to cointegration analysis will help to identify the dependent variable as stationary while the independent variables are nonstationary. Certain extensions to polynomial (or rational) approximants are needed in order to produce global approximations for such integrable functions over the whole real line. Phillips (1983) suggested a class of extended rational approximants that have good global approximant performance over the whole real line to integrable functions, which may therefore be useful in this context.

One useful feature of the approximation-based linearity test is that the estimated linear combinations of the basis functions can suggest possible nonlinear alternatives when the linear specification is rejected due to nonlinearity. In this case, the modified RESET test can be interpreted as an LM test that compares a linear cointegration model against an estimated approximation to some unknown nonlinear cointegration model. So, if the null is rejected, we can write down an alternative nonlinear model with additional basis functions until the approximation errors become $I(0)$ and reestimate this model using the

Table 1. Probability of rejecting H_0 of linear cointegration

		Modified RESET test								Original RESET test							
Function	Type	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$\rho = 0.2$:	$n = 50$	17.86	49.84	99.80	32.70	91.98	99.60	23.26	48.78	8.26	50.40	100.00	29.88	94.48	100.00	14.90	47.90
	$n = 100$	11.68	70.82	99.66	31.58	94.98	99.80	23.46	70.94	8.32	74.64	100.00	37.32	96.90	100.00	24.04	72.92
	$n = 250$	8.60	89.26	99.80	31.52	96.72	99.78	31.40	88.02	9.70	93.26	100.00	41.28	98.18	100.00	37.52	90.14
	$n = 500$	8.32	98.52	99.60	34.80	97.34	99.78	45.38	97.62	11.30	99.24	100.00	45.28	98.30	100.00	51.82	98.60
	$n = 1000$	7.20	99.92	99.66	37.62	98.46	99.74	60.18	99.94	9.82	100.00	100.00	46.96	99.08	100.00	65.92	100.00
$\rho = 0.4$:	$n = 50$	15.96	41.08	99.74	28.00	91.58	99.54	20.12	40.40	9.40	47.88	100.00	29.80	94.22	99.92	16.04	45.06
	$n = 100$	9.48	58.64	99.68	23.14	94.58	99.86	17.50	60.92	12.12	72.76	100.00	35.48	96.68	100.00	23.94	71.22
	$n = 250$	7.04	84.36	99.68	22.98	96.44	99.80	21.34	83.86	14.54	92.28	100.00	41.24	98.12	100.00	37.86	89.28
	$n = 500$	6.40	95.16	99.74	23.68	97.64	99.66	30.70	94.40	15.10	98.68	100.00	44.36	98.58	100.00	50.22	97.48
	$n = 1000$	6.34	99.42	99.76	26.06	98.06	99.72	41.42	99.56	16.40	99.92	100.00	44.84	98.86	100.00	60.64	99.88
$\rho = 0.6$:	$n = 50$	15.86	33.00	99.78	24.54	90.94	99.08	18.82	32.86	13.16	43.44	100.00	30.28	93.74	99.96	18.80	41.20
	$n = 100$	8.38	41.20	99.76	16.60	94.40	99.84	12.76	45.16	20.40	68.00	100.00	36.96	97.14	100.00	28.84	67.90
	$n = 250$	5.26	69.18	99.74	13.26	96.02	99.74	11.80	73.24	24.76	88.26	100.00	43.28	97.96	100.00	40.68	87.80
	$n = 500$	5.60	85.58	99.68	13.28	97.22	99.76	16.10	87.26	28.72	96.88	100.00	46.78	98.78	100.00	50.58	96.04
	$n = 1000$	4.96	95.62	99.84	13.90	98.24	99.76	21.72	96.64	28.62	99.68	100.00	47.96	99.02	100.00	59.00	99.58
$\rho = 0.8$:	$n = 50$	21.12	31.12	99.78	24.78	89.08	97.58	22.52	30.92	19.76	40.50	100.00	32.14	94.10	99.70	23.86	38.82
	$n = 100$	10.46	25.32	99.74	12.68	92.54	99.48	11.82	27.90	33.72	61.64	100.00	42.74	97.06	100.00	38.06	62.66
	$n = 250$	4.84	32.86	99.90	7.32	96.20	99.88	6.88	45.78	49.24	83.66	100.00	55.32	98.46	100.00	54.44	85.74
	$n = 500$	5.04	54.48	99.70	6.84	96.80	99.80	7.34	69.14	55.36	92.34	100.00	60.80	98.98	100.00	62.02	92.94
	$n = 1000$	5.06	75.78	99.80	6.96	97.70	99.76	8.88	84.10	60.84	97.40	100.00	65.08	99.40	100.00	68.10	97.84
$\rho = 0.9$:	$n = 50$	26.28	32.90	99.60	27.42	85.08	94.70	26.74	32.52	25.90	41.36	100.00	33.76	93.84	99.26	28.46	40.24
	$n = 100$	14.86	23.24	99.66	15.26	91.18	98.94	15.92	25.64	42.78	59.88	100.00	48.20	97.00	100.00	45.42	60.62
	$n = 250$	5.92	16.66	99.78	6.12	94.32	99.70	6.24	23.46	62.54	80.38	100.00	63.94	98.78	100.00	63.54	83.72
	$n = 500$	4.86	22.64	99.86	5.26	97.00	99.84	5.30	38.46	74.48	88.22	100.00	75.42	99.50	100.00	76.16	91.52
	$n = 1000$	5.14	38.72	99.80	5.80	97.66	99.76	6.22	64.50	78.66	94.34	100.00	79.38	99.62	100.00	79.80	95.92
44.8	62.8	100.0	48.1	98.2	100.0	44.7	64.8										

NOTE: (1)–(8) denote the functional forms defined in the beginning of simulation. The probabilities are calculated from 10,000 simulated samples with $k = 3$ and the bandwidth is chosen automatically following Andrews (1991) for Parzen window.

Table 2. Probability of rejecting linearity/cointegration when X_t and Y_t are not cointegrated

	Number of basis functions (k)				
Bandwidth	1	2	3	4	5
$M = n^{1/5}$					
$n = 50$	15.94	13.02	10.64	8.92	6.92
$n = 100$	25.62	25.06	22.68	19.88	16.88
$n = 500$	54.00	64.96	67.58	67.92	67.26
$n = 1000$	64.40	78.88	82.46	84.12	84.54
$M = n^{1/4}$					
$n = 50$	13.84	10.14	8.10	6.52	4.98
$n = 100$	22.16	19.94	17.24	14.46	11.96
$n = 500$	48.72	57.90	59.38	58.00	56.76
$n = 1000$	59.40	73.08	76.26	76.72	76.98
$M = n^{1/3}$					
$n = 50$	10.10	5.91	4.35	3.99	3.96
$n = 100$	14.52	10.50	8.07	5.73	4.13
$n = 500$	37.53	42.76	41.06	39.05	36.54
$n = 1000$	49.16	58.31	59.39	58.85	57.05
$M = n^{1/2}$					
$n = 50$	6.09	3.81	3.41	5.33	6.31
$n = 100$	6.73	3.05	2.10	1.54	1.41
$n = 500$	16.59	12.98	9.84	7.68	5.25
$n = 1000$	23.68	21.70	18.14	15.13	11.93
Automatic					
$n = 50$	26.02	36.87	42.42	54.43	57.14
$n = 100$	23.08	30.36	33.89	42.31	45.01
$n = 500$	17.77	21.40	23.60	27.23	28.63
$n = 1000$	17.30	19.30	20.65	22.90	24.10

NOTE: The rejection probabilities are calculated from 10,000 replications for the nominal 5% test. The automatic data-determined bandwidth choice in the bottom panel is based on Andrews (1991).

FM regression method presented in the previous section. This approach has clear advantages in empirical research over other residual-based general specification tests like the CUSUM type test, which indicate only whether the null hypothesis is rejected or not. Of course, the alternative nonlinear model is valid only if the rejection of the null hypothesis is due to nonlinearity. If the rejection is due to complete lack of cointegration, then the approximation error will not become $I(0)$. Also, finding a satisfactory nonlinear alternative specification by way of approximation involves choosing a suitable value of k for the regression so that the approximation error is reduced while not attempting to overfit the data. Such complex issues necessarily involve model selection and are beyond the scope of the present article.

6. EMPIRICAL APPLICATION

The introduction of unit root limit theory and cointegration methods has led to a vast number of empirical studies with nonstationary time series, many of them conducted without further attention to specification testing beyond what is implied by unit root and cointegration tests. This section considers the PPP relationship between nominal exchange rates and the foreign–domestic price ratio and applies the modified RESET linearity test to check whether the traditional linear cointegration specification is appropriate in this context.

6.1 PPP Models

PPP is a simple, intuitively appealing empirical proposition dated at least to the 16th Century in Spain (Dornbusch 1987). The theory postulates that once converted to a common currency, the price level of traded goods should be equalized across countries due to arbitrage. In its strict sense, the idea is sometimes understood as an extension of the law of one price (LOP),

$$P_{i,t} = S_t \cdot P^*_{i,t},$$

with a nominal exchange rate, S_t, a domestic price of a traded good i at time t, $P_{i,t}$, and the foreign price for the same good, $P^*_{i,t}$. Aggregating this relationship over traded goods, PPP states that

$$\sum_i P_{i,t} = S_t \cdot \sum_i P^*_{i,t}.$$

For a variety of reasons, this exact form of PPP, the so-called *absolute* PPP, does not hold and a weaker version of PPP is commonly used to provide a definition of the real exchange rate as

$$q_t = s_t + p^*_t - p_t,$$

where q_t and s_t are log transforms of real and nominal exchange rates, and p^*_t and p_t are log transforms of foreign and domestic price levels.

Intuitively accepted as providing a long-run equilibrium relationship among price levels and exchange rates, traditional unit root/cointegration approaches have been the most widely used method in PPP empirical studies, but these methods have often failed to find any strong empirical support for PPP. These failures have led to the use of many new methods in searching for evidence of PPP, including longer datasets, panel unit root evaluations, and the use of nonlinear models. Noticing the low power of unit root tests in small samples, researchers have tested PPP using long-horizon data, finding stronger support for PPP (e.g., Lothian and Taylor 1996) by this method. Using cross-country data to improve the power of unit root tests has also tended to produce stronger support for PPP, but with some criticism for neglecting cross-country dependence (e.g., O'Connell 1998). While these methods have involved the use of different datasets to improve tests of PPP, the last approach takes into account the possibility of different model specifications.

Nonlinear specifications are often obtained from market frictions like transaction/transportation costs or trade barriers (e.g., Sercu, Uppal, and van Hulle 1995 and Michael, Nobay, and Peel 1997). These market frictions are usually formulated in terms of nonlinear adjustments to parity, and some variants of threshold models are being suggested to find stronger empirical evidence in support of these models (Saikkonen and Choi 2004). In addition to the nonlinear short-run adjustment terms associated with the long-run linear equilibrium, Haug and Basher (2003) posited a nonlinear PPP relationship and apply a simple nonlinear cointegration test developed by Breitung (2001), but failed to find any linear and nonlinear cointegration relationship among the G10 countries. We use their model,

$$S_t = \alpha + f\left(\frac{P^*_t}{P_t}\right) + u_t, \qquad (10)$$

and test for linearity in this cointegrating relationship directly.

Not having a specific functional form for $f(\cdot)$ offers some advantages. First, even if the threshold model had strong theoretical justification for *one* tradable good, aggregating over all goods and using a general price level inevitably obscures the form of the implied nonlinearity for the aggregate relationship (for instance, because of the manifold threshold points that appear in the aggregation). Second, setting a regression equation in the general form of (10) allows for a more flexible interpretation. Apart from providing a testable form of PPP, (10) can be thought of as a general model of nominal exchange rate determination in terms of economic fundamentals. Although Meese and Rogoff (1983) found that no existing structural model outperforms a simple random walk model in prediction, the monetary model has been the standard model for exchange rate determination. This model's main implication is that the nominal exchange rate is determined by some economic fundamentals like money and output of the two countries, and the risk premium. Using the price ratio to reflect the economic fundamentals, (10) can be regarded as expressing nominal exchange rates as some unknown function of underlying fundamentals.

In addition to the PPP in levels (or *absolute* PPP), we also test *relative* PPP which can be written as (Rogoff 1996)

$$\frac{P_t}{P_{t-1}} = \left(\frac{S_t}{S_{t-1}}\right) \cdot \frac{P_t^*}{P_{t-1}^*}.$$

Since the price index is the relative value to a base year and we do not know how big the deviation from absolute PPP was at the base year, this relative version of PPP requires the relationship to hold only in terms of changes. In this case, since the logarithms of the price and exchange rate ratios are stationary, we need to interpret empirical results appropriately. Also, note that our modified test becomes equivalent to the traditional RESET test as both the bias and the correction terms vanish asymptotically for stationary time series.

6.2 Data

We consider five countries (U.S., Japan, Canada, Mexico, and U.K.) forming the four pairs: U.S.–Japan, U.S.–Canada, U.S.–Mexico and U.S.–U.K. Both bordering the U.S., Canada and Mexico had strong economic ties to the U.S. even before the North American Free Trade Act (NAFTA) came into effect in January 1994. According to WTO statistics, in the year 2005, 36.7% of exports and 26.8% of imports of the total merchandise trade of the U.S. are with these two countries. Those proportions are as high as 83.9% of exports and 56.5% of imports in Canada with the U.S., and 85.8% of exports and 53.6% of imports in Mexico with the U.S. While both countries depend heavily on trading with the U.S., their experiences with the U.S. are quite different in our sample period, which will be discussed later. Due to the geographic proximity as well as previous trade agreements including NAFTA, we expect that the market frictions—transportation cost, trade barriers, and so on—hampering the international arbitrage are at the lowest level among these countries. The U.K. is one of the biggest economies in the EU as well as in the world and has a long history of close connection with the U.S. Although about half of its merchandise imports and exports are with other EU countries, the next largest trading partner is the U.S., accounting for 14.7% of exports and 8% of imports in U.K. merchandise trade in 2005. Another interesting country is Japan, which used to be the second largest economy in the world excluding the EU. Like the U.K., it's still one of the biggest players in world trade and its biggest trade partner is the U.S. (22.9% of exports and 12.7% of imports). Japan alone takes 6.1% of exports and 8.2% of imports in U.S. merchandise trade. Both the U.K. and Japan are geographically far from the U.S. compared with Canada and Mexico, but Japan is in general very different from the other four countries socioeconomically, so that we expect such differences will cause movements in relative price levels as well as differences in exchange rates.

Our dataset is taken from the IMF's *International Financial Statistics* (IFS) CD-ROM and contains nominal exchange rates, the consumer price index (CPI), and producer price index (PPI)/wholesale price index (WPI) at a monthly frequency. The data span the period from 1971:1 to 2004:12, yielding 34 years or 408 monthly observations except Mexico's PPI series which starts from 1981:1. A monthly average market rate is used for the nominal exchange rate and both the CPI and PPI/WPI are used to calculate price ratios. The data are plotted against time in Figure 2. The left column shows the absolute PPP (in levels)—nominal exchange rate (solid), CPI ratio (dashed), and PPI ratio (dash-dotted)—and the right column shows the relative PPP in the same manner—in changes calculated by year-to-year ratios, that is, for the nominal exchange rate (S_t), S_t/S_{t-12} and for the CPI or PPI (P_t), $(P_t/P_{t-12})/(P_t^*/P_{t-12}^*)$.

6.3 Variations and Two Sample Periods

The fact that exchange rates are much more volatile than the price measures has been posited as one of the reasons why it is hard to find empirical supports for PPP, often leading to the models of factionally integrated real exchange rate series, or other nonlinear models for PPP. We first calculate the standardized variations. For an arbitrary monthly series $\{X_t\}_{t=1}^n$, define a three-year rolling standardized variation of X_t at t by

$$V_t = \frac{\sqrt{\frac{1}{35}\sum_{j=-17}^{18}(X_{t+j} - \bar{X}_t)^2}}{\bar{X}_t},$$

where

$$\bar{X}_t = \frac{1}{36}\sum_{j=-17}^{18} X_{t+j} \quad \text{for } t = 17, \ldots, n-18,$$

which is the ratio of a standard deviation in nearby three-year period (36 months) to its local mean for that period. Therefore, V_t is a unit-free measure of the size of variations in X_t during the three years in the neighborhood of t, proportional to its level. Figure 3 shows these proportional standardized variations of exchange rates, CPIs and PPIs both in levels (in left column) and in ratios (in right column). The right y-axis scale is used for Mexico and the rest of the countries follow the left y-axis scale.

One thing that is clear from these plots is a declining volatility in the case of price measures, especially from early 1980s, but the exchange rates do not show any clear pattern. Only Mexico is an exceptional case, where price levels become much more volatile during 1980s and then, only after early 1990s, they become stabilized at a lower level but still considerably

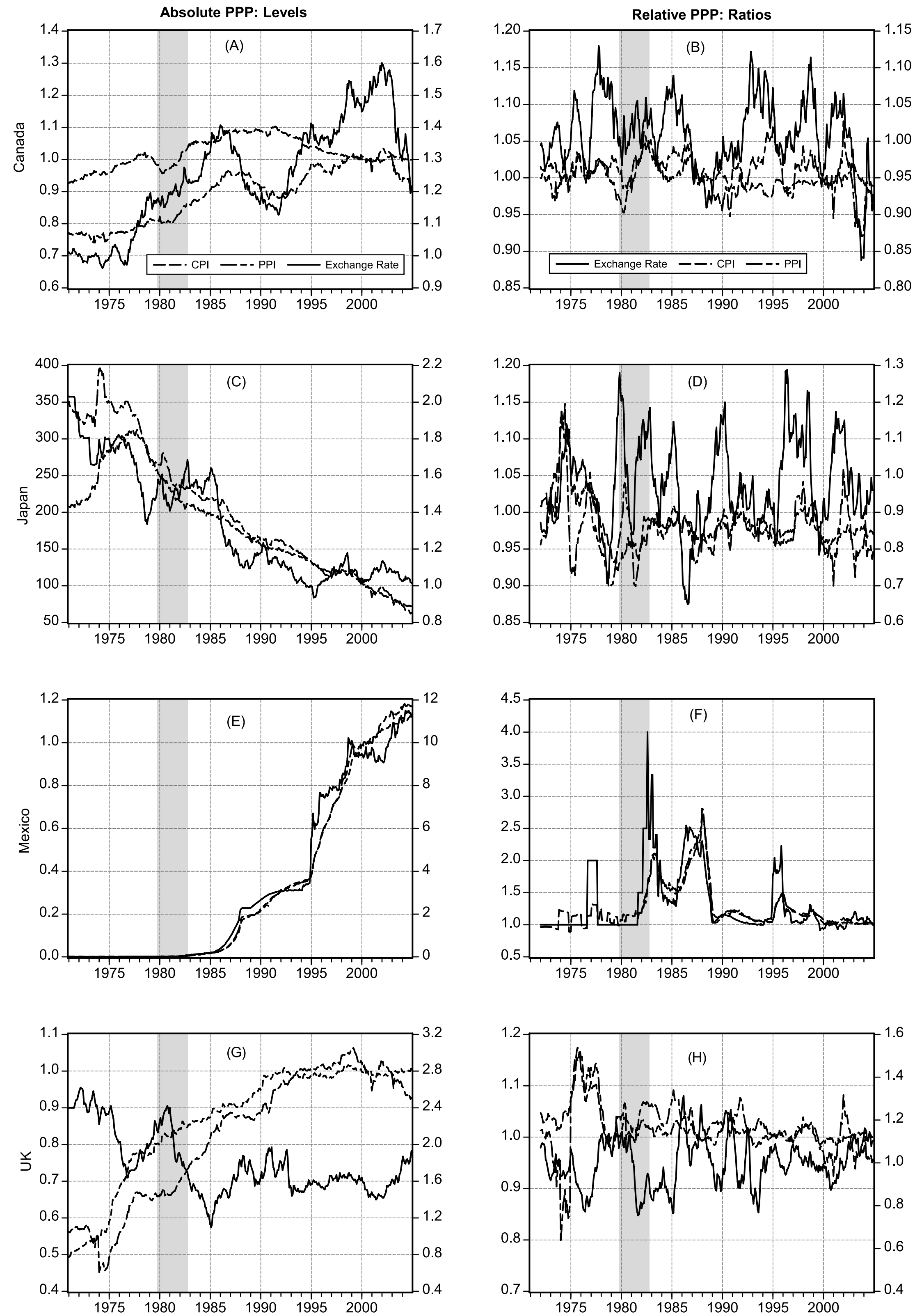

Figure 2. Nominal exchange rates and price ratios: U.S.–Canada, U.S.–Japan, U.S.–Mexico, and U.S.–U.K. The sample spans from 1971:1 to 2004:12. Left figures plot dataset in levels and right figures plot in the changes, S_t/S_{t-12} and $(P_t/P_{t-12})/(P_t^*/P_{t-12}^*)$ where the nominal exchange rate is plotted in the solid line, CPI ratio in the dashed line, and PPI ratio in the dash-dotted line.

higher than those of the other countries. This period of early 1980s roughly coincides with the so-called Volcker period during which the Fed strongly fought for the worldwide high inflation rates, and we consider this a subsample period where the volatilities of price measures are significantly lower than those from the other period.

In our analysis, we first consider the whole sample period (1971M1–2004M12: "Period 1" hereafter) with 408 monthly

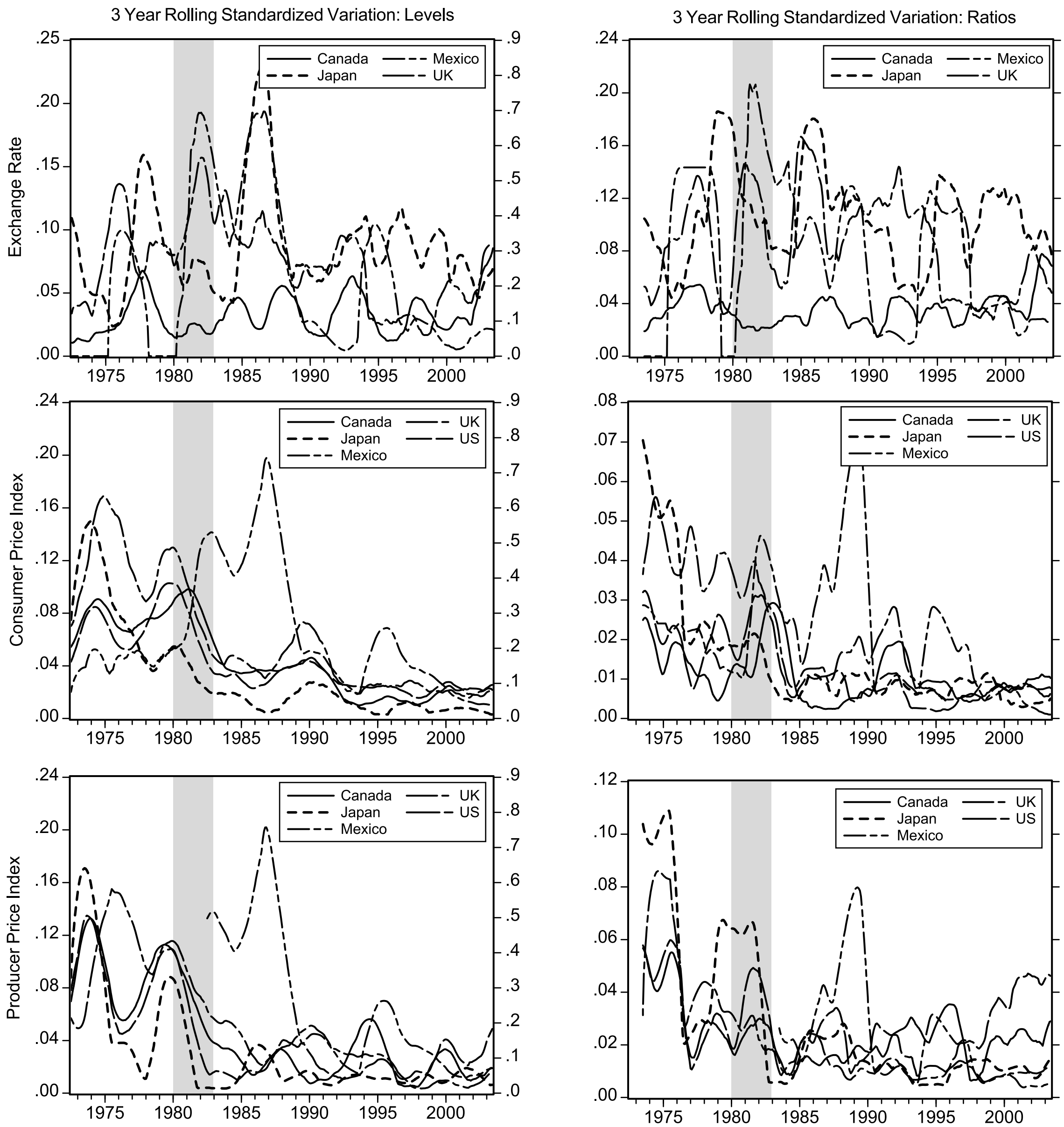

Figure 3. Changes in standarized variations. Graphs show the changes in the standardized variations in exchange rates, CPI ratios, and PPI ratios for a three-year rolling window. Left figures plot the variations of variables in the levels, that is, variables for the absolute PPP, and right figures plot the same for the relative PPP.

observations and then the post-Volcker period (1983M1–2004M12: "Period 2" hereafter) with 264 monthly observations. These two sample sizes are roughly equal to the two sample sizes (250 and 500) we considered in the simulation, so that we expect our test statistic will show a similar performance as shown in the simulation study. As we have seen already, with a moderate level of serial correlation in the error, our modified test is significantly better than the original test in the test size and relatively good powers as well. In Figures 2 and 3, this Volcker period is shown in shade, and our Period 2 covers the sample after this period.

6.4 Traditional Cointegration Analysis

We start with the traditional cointegration analysis and will compare the results with our modified RESET test results. First, we apply augmented Dickey–Fuller (ADF) tests to determine whether our dataset contains integrated processes. Test results, not reported here, indicate that the nominal exchange rate, CPI, and PPI are all unit root nonstationary in levels (for absolute PPP) and stationary in changes (for relative PPP). The Phillips–Perron test gives similar results. Second, we apply ADF and KPSS tests to the regression residuals from regressing the exchange rate (S_t) on a constant and the price ratio (P_t/P_t^*), with varying sample periods, to check whether these residual-based cointegration tests find any meaningful (linear) cointegration relationship (see Table 3). For the ADF test, various specifications of Dickey–Fuller regression are used with different lagged terms and both (1) constant or (2) constant and linear trends. Note that the null hypotheses of the two tests are different: no linear cointegration for the ADF test and linear cointegration for

Table 3. Residual-based cointegration tests of absolute PPP: ADF and KPSS

Number of lags		ADF test 1	2	3	4	KPSS test
		(A) PPP with consumer price index				
Period 1:						
U.S.–Canada	(1)	−0.9850	−0.8228	−0.9342	−1.0758	0.1295
	(2)	−0.9468	−0.7698	−0.8726	−1.0081	
U.S.–Japan	(1)	−2.4526	−2.2983	−2.4672	−2.6947	0.3389***
	(2)	−2.4486	−2.2951	−2.4646	−2.6927	
U.S.–Mexico	(1)	−3.4293**	−3.4236**	−3.0710**	−2.9329**	0.0866
	(2)	−3.4251**	−3.4193*	−3.0672	−2.9292	
U.S.–U.K.	(1)	−2.9394**	−2.6062*	−2.7581*	−2.7298*	0.0655
	(2)	−2.9323	−2.5983	−2.7486	−2.7192	
Period 2:						
U.S.–Canada	(1)	−1.7996	−1.3970	−1.5988	−1.7630	0.0732
	(2)	−1.7400	−1.3175	−1.5032	−1.6614	
U.S.–Japan	(1)	−2.5091	−2.1480	−1.9787	−1.8106	0.3227***
	(2)	−2.5051	−2.1459	−1.9786	−1.8106	
U.S.–Mexico	(1)	−2.7934*	−2.8001*	−2.5178	−2.4025	0.1198
	(2)	−2.7879	−2.7946	−2.5130	−2.3980	
U.S.–U.K.	(1)	−2.9379**	−2.2689	−2.5902*	−2.5077	0.1265
	(2)	−2.9182	−2.2439	−2.5591	−2.4683	
		(B) PPP with wholesale/producer price index				
Period 1:						
U.S.–Canada	(1)	−2.3966	−2.2435	−2.1414	−2.1075	0.1895
	(2)	−2.3910	−2.2365	−2.1335	−2.0994	
U.S.–Japan	(1)	−3.0016	−2.9066	−3.0173	−3.2448	0.1853
	(2)	−3.0020	−2.9119	−3.0809	−3.2589	
U.S.–Mexico[a]						
U.S.–UK	(1)	−2.8720**	−2.5746*	−2.6094*	−2.6623*	0.1745
	(2)	−2.8689	−2.5736	−2.6103	−2.6648	
Period 2:						
U.S.–Canada	(1)	−2.3776	−2.2044	−2.1072	−1.9968	0.2354***
	(2)	−2.3778	−2.2051	−2.1098	−2.0010	
U.S.–Japan	(1)	−2.3365	−2.2941	−2.5456	−2.5605	0.2610***
	(2)	−2.3271	−2.2834	−2.5344	−2.5496	
U.S.–Mexico	(1)	−2.7442*	−2.6802*	−2.4586	−2.3487	0.1446***
	(2)	−2.7388	−2.6748	−2.4536	−2.3438	
U.S.–U.K.	(1)	−2.7148*	−2.1968	−2.5061	−2.3882	0.1687
	(2)	−2.7012	−2.1813	−2.4850	−2.3630	

NOTE: The cointegration regression is estimated for Period 1 (1971M1–2004M12) and Period 2 (1983M1–2004M12) with a constant and a linear trend. The number of lags in the column shows the number of lagged terms in the Dickey–Fuller regression for the regression residuals. The ADF test statistics with a constant term are reported in (1) and statistics with both a constant and a linear time trend are tabulated in (2). *'s show the null hypothesis rejected. One asterisk means rejection at a 10% significance level, 2 and 3 asterisks imply 5% and 1%, respectively.

[a]Since PPI series for Mexico is available only after 1981, cointegration is tested only for the second period.

the KPSS test. As much previous research has reported, conventional linear cointegration tests show somewhat mixed results.

1. The ADF test on U.S.–Canada and U.S.–Japan does not find evidence of any linear cointegration relationship between nominal exchange rate and the ratio of price levels (absolute PPP) with either CPI or PPI. However, tests find significant linear cointegration for U.S.–Mexico and U.S.–U.K. with CPI for the whole sample period and these cointegration relationships become less significant if we look at Period 2. While the stylized fact from the existing empirical studies shows more favorable evidence with PPI, the ADF test with our sample does not show such pattern.
2. Unlike the ADF test, which finds only a few cases of linear cointegration, the KPSS test finds many linear cointegration relations for the whole sample period, but some of these are not supported by tests for Period 2, especially with PPI, and this is exactly opposite to the empirical stylized fact. Although U.S.–Mexico is the most strongly supported linear cointegration by the ADF test, the KPSS test

shows stronger support for U.S.–U.K., finding linear cointegration in all four cases. For U.S.–Japan, however, except the whole period with PPI, the KPSS does not find any linear cointegration at all.

6.5 The Modified RESET Test

When these two popular residual-based cointegration tests produce ambiguous findings, we now apply our modified RESET test and compare the results with the original RESET test. Table 4 summarizes the results from the modified test as well as the original RESET for both absolute PPP (left) and relative PPP (right) with varying bandwidths and numbers of polynomials k. The upper part (A) is the case with CPI and the lower part (B) is the case with PPI, and each part is divided into two sample periods. While the original RESET test tends to find that most relationships are linear, the modified RESET test shows little support for a linear cointegration specification except one special case of U.S.–Mexico where the linear cointegration is strongly supported for all cases. One interesting point is that this tends to be opposite to the original RESET test, which finds little or no evidence for a linear relationship between the U.S. and Mexico compared with other country pairs where the modified test cannot find a linear cointegration relationship. There are a few other cases where our modified RESET test found some evidence for a linear relationship such as U.S.–U.K. with CPI, U.S.–Japan with PPI, or U.S.–Canada with PPI. These findings look consistent with Figure 2, but it seems that their cointegration relationships are not as stable as the U.S.–Mexico case.

Table 4 shows a few additional interesting results. Although many empirical studies find that PPP works better with PPI than CPI (Froot and Rogoff 1995), there seems to be no significant difference between CPI and PPI in our modified test result, and two traditional tests even find CPI is more supportive of linear specification than PPI. Regarding the two sample periods, two traditional tests show considerable differences between these two periods while our modified test does not show such differences. In the case of the relative PPP, where our test becomes a linearity test instead of a linear cointegration test, our modified test does not find any significant linear relationship while the original RESET test found that *all* the relationships are linear. Our results are also more consistent with Figure 2 than the original test results.

7. CONCLUSION

Using some recently developed asymptotic tools in Park and Phillips (1999, 2001), this article presents how nonstationarity combined with nonlinearity interferes with the RESET test and we analyze the resulting severe size distortion that makes the test unsuitable for empirical application. The appropriate modifications to the RESET test are proposed to eliminate the biases that cause these size distortions and the proposed modifications are shown to lead to a corrected test statistic that has a limiting central χ^2 distribution. The proposed modified test statistic has good power against both nonlinear cointegration and no cointegration alternatives so that it can be used to assess the adequacy of a linear cointegrating relation against certain forms of nonlinear cointegration and the alternative of no cointegration.

Some related work is in progress. Since the power of the test depends on the choice of basis functions, we are developing a set of linearity tests using different basis functions. This seems particularly appropriate when we want to allow for functions whose behavior is poorly approximated by polynomials, such as integrable functions that attenuate the influence of integrated regressors. At the same time, there is scope for developing a linearity test that is not based directly on an approximating family, so that the power and the size of the test do not depend on so many choices, such as the basis functions, the number of basis functions, and a bandwidth parameter.

APPENDIX

The following proofs sketch the main steps in the arguments and details are provided in an earlier version of the article (Hong and Phillips 2007). The proofs frequently use standard limit theorems for nonlinearly transformed integrated processes. These are based on lemma 5 of Chang, Park, and Phillips (2001), unless specified otherwise.

Proof of Lemma 1

See de Jong (2002) or Ibragimov and Phillips (2004).

Proof of Theorem 2

The proof is the same as Theorem 3 except that the second-order bias terms are not corrected but are collected together to form the noncentrality parameter.

Proof of Theorem 3

The test statistic is a quadratic form in $\mathbf{D}_n\mathbf{F}'\hat{\mathbf{u}}$ and two bias correction terms, with the weight matrix $(\hat{\Omega}_{uu.v}\mathbf{D}_n'\tilde{\mathbf{F}}'\tilde{\mathbf{F}}\mathbf{D}_n)^{-1}$ as metric in the form. We prove this theorem in two steps. First, conditional on $\mathcal{F}_x = \sigma(B_x(r), 0 \le r \le 1)$, we show that $\mathbf{D}_n\mathbf{F}'\hat{\mathbf{u}}$ becomes a zero mean Gaussian vector after bias corrections in the limit; and second, that its variance matrix is the limit of the weight matrix.

The $(m-1)$th element in $\mathbf{D}_n\mathbf{F}'\hat{u}$ is

$$
\begin{aligned}
&\frac{1}{n^{(m+1)/2}}\sum_{t=1}^{n} X_t^m \hat{u}_t \\
&\quad = \frac{1}{n^{(m+1)/2}}\sum_{t=1}^{n} X_t^m u_t \\
&\qquad - \frac{1}{n^{(m+1)/2}}\sum_{t=1}^{n} X_t u_t \left(\sum_{t=1}^{n} X_t^2\right)^{-1} \sum_{t=1}^{n} X_t^{m+1} \\
&\quad \Rightarrow \int B_x^m \, dB_\varepsilon C(1) + m\Lambda_{vu}\int B_x^{m-1} \\
&\qquad - \left(\int B_x\, dB_u + \Lambda_{vu}\right)\left(\int B_x^2\right)^{-1}\int B_x^{m+1} \\
&\quad = \int \widetilde{B_x^m}\, dB_u + m\Lambda_{vu}\int B_x^{m-1} - \Lambda_{vu}\left(\int B_x^2\right)^{-1}\int B_x^{m+1}
\end{aligned}
$$

Table 4. p-values of the modified and original RESET tests

	Bandwidth	Choice of k:	Absolute PPP: Modified RESET 2	3	4	Absolute PPP: Original RESET 2	3	4	Relative PPP: Modified RESET 2	3	4	Relative PPP: Original RESET 2	3	4
(1) Period 1: 1971M1–2004M12			(A) PPP with consumer price index											
U.S.–Canada	$M=n^{1/3}$		0.000	0.000	0.000	0.089	0.179	0.298	0.000	0.000	0.000	0.612	0.806	0.809
	$M=n^{2/3}$		0.000	0.000	0.000	0.280	0.460	0.629	0.000	0.000	0.000	0.696	0.867	0.881
	Auto		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.604	0.799	0.801
U.S.–Japan	$M=n^{1/3}$		0.000	0.000	0.000	0.006	0.011	0.006	0.000	0.000	1.000	0.205	0.336	0.514
	$M=n^{2/3}$		0.000	0.000	0.000	0.073	0.126	0.112	0.000	0.000	1.000	0.214	0.348	0.527
	Auto		0.000	0.000	0.000	0.039	0.069	0.055	0.000	0.000	1.000	0.023	0.044	0.099
U.S.–Mexico	$M=n^{1/3}$		0.455	0.274	0.041	0.015	0.012	0.002	0.000	0.000	0.000	0.119	0.188	0.306
	$M=n^{2/3}$		0.482	0.406	0.146	0.079	0.087	0.038	0.000	0.000	0.000	0.066	0.106	0.187
	Auto		0.035	0.027	0.000	0.008	0.006	0.001	0.000	0.000	0.000	0.054	0.088	0.159
U.S.–U.K.	$M=n^{1/3}$		0.481	0.561	0.329	0.454	0.465	0.532	0.000	0.000	1.000	0.358	0.559	0.727
	$M=n^{2/3}$		0.720	0.000	0.000	0.625	0.677	0.758	0.000	0.000	1.000	0.433	0.641	0.796
	Auto		0.000	0.000	0.000	0.007	0.001	0.001	0.000	0.000	1.000	0.090	0.184	0.308
(2) Period 2: 1983M1–2004M12														
U.S.–Canada	$M=n^{1/3}$		0.000	0.000	0.000	0.796	0.908	0.938	0.000	0.000	0.000	0.623	0.814	0.906
	$M=n^{2/3}$		0.000	0.000	0.000	0.860	0.948	0.970	0.000	0.000	0.000	0.733	0.892	0.955
	Auto		1.000	1.000	1.000	1.000	1.000	1.000	0.000	0.000	0.000	0.670	0.849	0.929
U.S.–Japan	$M=n^{1/3}$		0.000	0.000	0.000	0.002	0.006	0.012	0.000	0.000	0.000	0.906	0.946	0.960
	$M=n^{2/3}$		0.000	0.000	0.000	0.037	0.085	0.143	0.000	0.000	0.000	0.921	0.959	0.971
	Auto		0.000	0.000	0.000	0.034	0.080	0.136	0.000	0.000	0.000	0.768	0.803	0.794
U.S.–Mexico	$M=n^{1/3}$		0.943	0.962	0.046	0.038	0.051	0.010	0.000	0.000	0.000	0.088	0.156	0.209
	$M=n^{2/3}$		0.981	0.993	0.232	0.156	0.220	0.107	0.000	0.000	0.000	0.092	0.163	0.217
	Auto		0.194	0.252	0.002	0.024	0.032	0.004	0.000	0.000	0.000	0.084	0.150	0.200
U.S.–U.K.	$M=n^{1/3}$		0.000	0.000	0.000	0.636	0.323	0.430	0.000	1.000	0.000	0.300	0.501	0.645
	$M=n^{2/3}$		0.000	0.000	0.000	0.748	0.526	0.653	0.000	1.000	0.000	0.356	0.566	0.709
	Auto		0.000	0.000	0.000	0.582	0.244	0.333	0.000	1.000	0.000	0.228	0.407	0.546

Table 4. (Continued)

	Bandwidth	Choice of k:	Absolute PPP: Modified RESET 2	3	4	Absolute PPP: Original RESET 2	3	4	Relative PPP: Modified RESET 2	3	4	Relative PPP: Original RESET 2	3	4
(B) PPP with wholesale/producer price index														
(1) Period 1: 1971M1–2004M12														
U.S.–Canada	$M = n^{1/3}$		0.000	0.000	0.000	0.020	0.048	0.059	0.000	0.000	1.000	0.477	0.464	0.684
	$M = n^{2/3}$		0.000	0.000	0.000	0.150	0.279	0.354	0.000	0.000	1.000	0.539	0.544	0.753
	Auto		0.000	0.000	0.000	0.059	0.125	0.160	0.000	0.000	1.000	0.493	0.485	0.703
U.S.–Japan	$M = n^{1/3}$		0.000	0.000	0.000	0.034	0.064	0.122	0.000	0.000	0.000	0.821	0.940	0.721
	$M = n^{2/3}$		0.000	0.000	0.000	0.153	0.258	0.399	0.000	0.000	0.000	0.827	0.943	0.737
	Auto		0.123	0.061	0.005	0.065	0.117	0.205	0.000	0.000	0.000	0.690	0.860	0.418
U.S.–Mexico[a]														
U.S.–U.K.	$M = n^{1/3}$		0.000	0.000	0.000	0.073	0.041	0.079	0.000	0.000	0.000	0.299	0.358	0.512
	$M = n^{2/3}$		0.000	0.000	0.000	0.236	0.209	0.330	0.003	0.000	0.000	0.368	0.445	0.607
	Auto		0.000	0.000	0.000	0.114	0.078	0.140	0.000	0.000	0.000	0.103	0.108	0.186
(2) Period 2: 1983M1–2004M12														
U.S.–Canada	$M = n^{1/3}$		0.000	0.000	1.000	0.220	0.387	0.562	0.000	0.000	0.000	0.515	0.571	0.704
	$M = n^{2/3}$		0.000	0.000	1.000	0.460	0.670	0.822	0.000	0.000	0.000	0.609	0.682	0.804
	Auto		0.000	0.000	1.000	0.297	0.488	0.666	0.000	0.000	0.000	0.534	0.594	0.726
U.S.–Japan	$M = n^{1/3}$		0.000	0.000	0.000	0.004	0.011	0.022	0.000	0.000	1.000	0.742	0.896	0.963
	$M = n^{2/3}$		0.000	0.000	0.000	0.040	0.088	0.149	0.000	0.000	1.000	0.769	0.913	0.971
	Auto		0.000	0.000	0.000	0.018	0.044	0.079	0.000	0.000	1.000	0.436	0.644	0.797
U.S.–Mexico	$M = n^{1/3}$		0.402	0.507	0.083	0.028	0.025	0.009	0.000	0.000	0.000	0.064	0.121	0.128
	$M = n^{2/3}$		0.777	0.901	0.351	0.137	0.157	0.111	0.000	0.000	0.000	0.049	0.095	0.097
	Auto		0.190	0.261	0.012	0.040	0.038	0.016	0.000	0.000	0.000	0.060	0.115	0.121
U.S.–U.K.	$M = n^{1/3}$		0.000	0.000	0.000	0.199	0.006	0.013	0.000	0.000	1.000	0.517	0.687	0.831
	$M = n^{2/3}$		0.000	0.000	0.000	0.394	0.065	0.122	0.000	0.000	1.000	0.568	0.737	0.867
	Auto		0.000	0.000	0.000	0.201	0.006	0.014	0.000	0.000	1.000	0.377	0.534	0.702

NOTE: The modified RESET test results with bandwidths $M = n^{1/3}$ and $M = n^{2/3}$ and automatic bandwidth are reported. The p-values from the original RESET test without bias corrections are reported in the right panel for comparison.
[a] Since PPI series for Mexico is available only after 1981, cointegration is tested only for the second period.

$$= \int \widetilde{B_x^m}\, dB_{u.x} + \Omega_{uv}\Omega_{vv}^{-1} \int \widetilde{B_x^m}\, dB_x$$
$$+ \Lambda_{vu}\left(m \int B_x^{m-1} - \left(\int B_x^2 \right)^{-1} \int B_x^{m+1} \right)$$

with $\widetilde{B_x^m} = B_x^m - B_x(\int B_x^2)^{-1} \int B_x^{m+1}$. Note that this limit is the sum of three elements—a zero mean Gaussian mixture, the endogeneity bias, and the serial correlation bias. With consistent estimators of Ω_{vv}, Ω_{vu}, and Λ_{vu}, 2 bias correction terms (7) and (8) converge to the corresponding bias terms in the above limit, so that the $(m-1)$th element of the sample covariance $\{\mathbf{D}_n\mathbf{F}'\hat{\mathbf{u}} - \mathbf{E}_n - \mathbf{S}_n\}$ becomes

$$\frac{1}{n^{(m+1)/2}} \sum_{t=1}^{n} \widetilde{X_t^m} u_t - E_n(m-1) - S_n(m-1)$$
$$\Rightarrow^d \int \widetilde{B_x^m}\, dB_{u.x}, \quad (11)$$

which follows $\mathcal{N}(0, \Omega_{uu.x} \int \widetilde{B_x^m}\widetilde{B_x^m}')$, conditional on $\mathcal{F}_x$.

For the weight matrix, the (i,j) element of $(\mathbf{D}_n'\tilde{\mathbf{F}}'\tilde{\mathbf{F}}\mathbf{D}_n)$ has the following limit

$$\frac{1}{n}\frac{1}{n^{(i+j+2)/2}} \sum_{t=1}^{n} \widetilde{X_t^{i+1}}\widetilde{X_t^{j+1}}' \Rightarrow^d \int \widetilde{B_x^{i+1}}\widetilde{B_x^{j+1}}'. \quad (12)$$

From (11) and (12), it follows that the modified RESET statistic is a quadratic form with a limiting χ^2 distribution as

$$\{\hat{u}'F\mathbf{D}_n - \mathbf{E}_n' - \mathbf{S}_n'\}(\hat{\Omega}_{uu.v}\mathbf{D}_n'\tilde{\mathbf{F}}'\tilde{\mathbf{F}}\mathbf{D}_n)^{-1}\{\mathbf{D}_n\mathbf{F}'\hat{\mathbf{u}} - \mathbf{E}_n - \mathbf{S}_n\}$$
$$\Rightarrow^d \left(\int \tilde{\mathbf{B}}_{x,k}\, dB_{u.x} \right)' \mathbf{A} \left(\int \tilde{\mathbf{B}}_{x,k}\, dB_{u.x} \right) \overset{a}{\sim} \chi^2(k),$$

where $\int \tilde{\mathbf{B}}_{x,k}\, dB_{u.x} = [\int \widetilde{B_x^2}\, dB_{u.x}, \ldots, \int \widetilde{B_x^{k+1}}\, dB_{u.x}]$ and an inverse covariance matrix $\mathbf{A}$ as defined in (6) of Theorem 2. Note that the test statistic follows a central $\chi^2(k)$ unconditionally.

Proof of Theorem 4

Note that if u_t is $I(1)$, then $\mathbf{D}_n\mathbf{F}'\mathbf{u} = O_p(n)$ since

$$\frac{1}{n}\left[\sum_{t=1}^{n} \left(\frac{X_t}{\sqrt{n}} \right)^m \frac{u_t}{\sqrt{n}} \right] \quad \Rightarrow^d \quad \int B_x^m B_u.$$

Also, since $\hat{\Omega}_{vu}$ and $\hat{\Lambda}_{vu}$ are $O_p(M)$ (see Xiao and Phillips 2002, lemma 1), the two bias correction terms in (7) and (8) diverge at the rate of $O_p(M)$ as well. Therefore, the sample covariance, augmented by the correction terms, diverges at the rate of n, that is, $(\mathbf{D}_n\mathbf{F}'\hat{\mathbf{u}} - \mathbf{E}_n - \mathbf{S}_n) = O_p(n)$, and the variance matrix term diverges at the rate of nM, that is, $(\hat{\Omega}_{uu.v}\mathbf{D}_n\tilde{\mathbf{F}}'\tilde{\mathbf{F}}\mathbf{D}_n) = O_p(nM)$ from Xiao and Phillips (2002, lemma 1). Combining these two, the modified RESET test statistic diverges at the rate of n/M.

Proof of Theorem 5

Under the alternative specification of nonlinear cointegration, the modified test statistic changes only through $\hat{u}$, that is, through $D_nF'\hat{u}$, $\hat{\Lambda}_{vu}$, $\hat{\Omega}_{uv}$ and $\hat{\Omega}_{uu.v}$. We will examine the changes in the statistic by checking the orders of each of these terms.

First, if we estimate the following misspecified linear regression by LS

$$Y_t = \theta X_t + u_t,$$

where the true relationship is nonlinear (9), the coefficient estimate $\hat{\theta}$ can be shown to be either convergent to zero in the I-regular case (assuming $x^k f(x)$ is integrable) or of the order of $n^{-1/2}\kappa_n$ for the H-regular case using Chang, Park, and Phillips (2001). With affixes I and H to designate these cases, we have

$$\hat{\theta}^{(I)} \Rightarrow \frac{1}{n}\left(\int B_x^2 \right)^{-1} \left(o_p(1) + \int B_x\, dB_u + \Lambda_{vu} \right) \equiv \theta^{(I)},$$
$$\hat{\theta}^{(H)} \Rightarrow \frac{\kappa_n}{\sqrt{n}}\left(\int B_x^2 \right)^{-1}$$
$$\times \left(\int H(B_x)B_x + \frac{1}{\kappa_n\sqrt{n}} \int B_x\, dB_u + \frac{\Lambda_{vu}}{\kappa_n\sqrt{n}} \right)$$
$$\equiv \theta^{(H)},$$

Note that when the true model is a linear cointegration we have $H(B_x) = \theta B_x$ and then $\hat{\theta}^{(H)} - \theta = O_p(n^{-1})$, as usual.

We first consider the H-regular case. The $(m-1)$th element of the normalized nonlinear sample covariance $\mathbf{D}_n\mathbf{F}'\hat{\mathbf{u}}$ is $O_p(\kappa_n\sqrt{n})$ since

$$\frac{1}{\kappa_n\sqrt{n}}\left\{ \sum_{t=1}^{n} \left(\frac{X_t}{\sqrt{n}} \right)^m \frac{\hat{u}_t}{\sqrt{n}} \right\}$$
$$= \frac{1}{\kappa_n\sqrt{n}}\left\{ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left(\frac{X_t}{\sqrt{n}} \right)^m \left[f(X_t) + u_t - \hat{\theta}^{(H)}X_t \right] \right\}$$
$$\Rightarrow \int B_x^m h(B_x) + O_p\left(n^{-1/2}\kappa_n^{-1}\right) - \theta^{(H)} \int B_x^{m+1}. \quad (13)$$

The orders of the two bias correction terms (7) and (8) depend on the asymptotic order of the kernel estimators $\hat{\Lambda}_{vu}$ and $\hat{\Omega}_{uv}$. Letting $K(j/M)$ be the lag kernel, we may decompose each of these estimates as follows:

$$\hat{\Omega}_{uv} = \sum_{j=-M}^{M} K\left(\frac{j}{M}\right)\left\{ \frac{1}{n}\sum_t \hat{u}_{t+j}v_t \right\}$$
$$= \sum_{j=-M}^{M} K\left(\frac{j}{M}\right)\left\{ \frac{1}{n}\sum u_{t+j}v_t \right.$$
$$\left. + \frac{1}{n}\sum f(X_{t+j})v_t - \hat{\theta}^{(H)}\frac{1}{n}\sum X_{t+j}v_t \right\}. \quad (14)$$

The first term in braces in (14) is $O_p(1)$ and the other two are $O_p(n^{-1/2}\kappa_n)$, so that the overall maximum asymptotic orders of

$\hat{\Omega}_{uv}$ and $\hat{\Lambda}_{vu}$ are all $n^{-1/2}M\kappa_n$. Thus, combining (14) and (13) we find that

$$\mathbf{D}_n\mathbf{F}'\hat{\mathbf{u}} - \mathbf{E}_n - \mathbf{S}_n = O_p\big(n^{1/2}\kappa_n\big) - O_p\big(n^{-1/2}M\kappa_n\big) - O_p\big(n^{-1/2}M\kappa_n\big)$$

has order $O_p(n^{1/2}\kappa_n)$ since $M/n \to 0$.

For the variance term $\hat{\Omega}_{uu.v}\mathbf{D}_n\tilde{\mathbf{F}}'\tilde{\mathbf{F}}\mathbf{D}_n$, the order now depends on the order of $\hat{\Omega}_{uu.v}$, since the remaining factor is of order $O_p(1)$ under both the null and the alternative hypotheses. The kernel estimator $\hat{\Omega}_{uu}$ can be shown to be of the maximum order of $M\kappa_n^2$ (but $O_p(1)$ as usual under the null hypothesis). In particular, with

$$\begin{aligned}\hat{\Omega}_{uu} &= \sum_{j=-M}^{M} K\Big(\frac{j}{M}\Big)\Big[\frac{1}{n}\sum_t \hat{u}_t\hat{u}_{t+j}\Big] \\ &= \sum_{j=-M}^{M} K\Big(\frac{j}{M}\Big)\Big[\frac{1}{n}\sum_t f(X_t)f(X_{t+j}) + \frac{1}{n}\sum_t f(X_t)u_{t+j} \\ &\quad - \hat{\theta}^{(H)}\frac{1}{n}\sum_t f(X_t)X_{t+j} + \frac{1}{n}\sum_t f(X_{t+j})u_t \\ &\quad + \frac{1}{n}\sum_t u_tu_{t+j} - \hat{\theta}^{(H)}\frac{1}{n}\sum_t X_{t+j}u_t \\ &\quad - \hat{\theta}^{(H)}\frac{1}{n}\sum_t f(X_{t+j})X_t - \hat{\theta}^{(H)}\frac{1}{n}\sum_t X_tu_{t+j} \\ &\quad + \hat{\theta}^{(H)2}\frac{1}{n}\sum_t X_tX_{t+j}\Big]\end{aligned}$$

the maximum order of each term in the square bracket can be determined as follows.

1. By virtue of the Cauchy inequality, the maximum orders of following terms are $O_p(\kappa_n^2)$:

$$\frac{1}{n}\sum f(X_t)f(X_{t+j}), \qquad \hat{\theta}^{(H)}\frac{1}{n}\sum f(X_t)X_{t+j},$$
$$\hat{\theta}^{(H)}\frac{1}{n}\sum f(X_{t+j})X_t.$$

2. The following terms are all of the same order $O_p(n^{-1/2}\kappa_n)$:

$$\frac{1}{n}\sum f(X_t)u_{t+j}, \qquad \frac{1}{n}\sum f(X_{t+j})u_t,$$
$$\hat{\theta}^{(H)}\frac{1}{n}\sum X_{t+j}u_t, \qquad \hat{\theta}^{(H)}\frac{1}{n}\sum X_tu_{t+j}.$$

3. $\frac{1}{n}\sum_t u_tu_{t+j} = O_p(1)$ and $\hat{\theta}^{(H)2}\frac{1}{n}\sum_t X_tX_{t+j} = O_p(n^{-1}\kappa_n^2)\cdot O_p(n) = O_p(\kappa_n^2)$.

Combining these results, the modified RESET test statistic is a quadratic form in a vector of $O_p(n^{1/2}\kappa_n)$ elements with a weight matrix of order $O_p(M^{-1}\kappa_n^{-2})$, so that the overall order of the test statistic is at most $O_p(n/M)$.

For the I-regular case, we can show that the sample covariance does not diverge

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\Big(\frac{X_t}{\sqrt{n}}\Big)^m\big[f(X_t) + u_t - \hat{\theta}^{(I)}X_t\big] \\ &\quad\sim O_p(1) + \int B_x^m\, dB_u - \theta^{(I)}\int B_x^{m+1} \\ &\quad= O_p(1).\end{aligned}$$

The kernel estimator of the long-run (co)variance is

$$\begin{aligned}\hat{\Omega}_{uv} &= \sum_{j=-M}^{M} K\Big(\frac{j}{M}\Big)\Big[\frac{1}{n}\sum u_{t+j}v_t \\ &\qquad + \frac{1}{n}\sum f(X_{t+j})v_t - \hat{\theta}^{(I)}\frac{1}{n}\sum X_{t+j}v_t\Big] \\ &\sim \Omega_{uv} + O_p\big(M/n^{3/4}\big) - O_p(M/n) \\ &= O_p\big(\max\{1, M/n^{3/4}\}\big),\end{aligned}$$

so that the two correction terms $\mathbf{E}_n$ and $\mathbf{S}_n$ do not diverge either, as long as $M/n^{3/4} \to 0$. Therefore,

$$\mathbf{D}_n\mathbf{F}'\hat{\mathbf{u}} - \mathbf{E}_n - \mathbf{S}_n = O_p\big(\max\{1, M/n^{3/4}\}\big).$$

The variance term, as in the H-regular case, can be shown to have the following order:

$$\hat{\Omega}_{uu} \approx \Omega_{uu} + O_p\big(M/n^{3/4}\big) + O_p(M/n) = O_p\big(\max\{1, M/n^{3/4}\}\big),$$

so that with $\hat{\Omega}_{uu.v} = \hat{\Omega}_{uu} - \hat{\Omega}_{uv}\hat{\Omega}_{vv}^{-1}\hat{\Omega}_{vu}$, the test statistic is $O_p(1)$.

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