5-2011

Technical Appendix to "Stochastic Capacity Investment and Flexible vs. Dedicated Technology Choice in Imperfect Capital Markets"

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TECHNICAL APPENDIX TO: STOCHASTIC CAPACITY INVESTMENT AND FLEXIBLE VERSUS DEDICATED TECHNOLOGY CHOICE IN IMPERFECT CAPITAL MARKETS

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July 2007, Revised 24 June and 25 October 2010, 19 March and 19 May 2011

§A contains the proofs for our technical statements in the paper. We present the proofs for technical statements that we develop in this Appendix in §B. We use the following identities for the standard normal random variable with cdf $\Phi(.)$ and pdf $\phi(.)$ throughout the Appendix: $\phi'(z) = -z\phi(z)$, $\int_{-\infty}^{v} z\phi(z)dz = -\phi(v)$ and $1 > \left[\frac{\phi(v)}{1-\Phi(v)}\right]^2 - \frac{v\phi(v)}{1-\Phi(v)} > 0$, where the last two inequalities are proven in Sampford (1953).

A Main Proofs

Proof of Proposition 1: If the firm does not have the limited liability option, then $\pi_D(K_D)$ is strictly concave in K_D and the unique optimal capacity investment level K_D^* and the optimal expected equity value π_D^* are given by

$$K_D^* = \begin{cases} K_D^0 \doteq \left(\frac{\bar{\xi}(1+\frac{1}{b})}{(1-\gamma_D)c_D}\right)^{-b} & \text{if } B \geq 2c_D K_D^0 \\ \frac{B}{2c_D} & \text{if } 2c_D K_D^1 \leq B < 2c_D K_D^0 \\ K_D^1 \doteq \left(\frac{\bar{\xi}(1+\frac{1}{b})}{(1+a_D-\gamma_D)c_D}\right)^{-b} & \text{if } B < 2c_D K_D^1, \end{cases} \tag{5}$$

$$\pi_D^* = \begin{cases} \frac{2c_D K_D^0(1-\gamma_D)}{-(b+1)} + B + P & \text{if } B \geq 2c_D K_D^0 \\ 2\bar{\xi}\frac{B}{2c_D} + \gamma_D B + P & \text{if } 2c_D K_D^1 \leq B < 2c_D K_D^0 \\ \frac{2c_D K_D^1(1+a_D-\gamma_D)}{-(b+1)} + B(1+a_D) + P & \text{if } B < 2c_D K_D^1 \end{cases}$$

With limited liability, when the firm borrows ($K_D > \frac{B}{2c_D}$), we have $l_D(K_D) \doteq K_D^{\frac{-1}{b}}(1+a_D-\gamma_D)2c_D - K_D^{\left(-1-\frac{1}{b}\right)}[B(1+a_D)+P]$ such that the firm is able to pay back the face value of the loan if and only if $\tilde{\xi}_1+\tilde{\xi}_2$ is no less than $l_D(K_D)$. For $\tilde{\xi}_1+\tilde{\xi}_2 > l_D(K_D)$, the optimal equity value $\Pi_D^* > 0$, and for $\tilde{\xi}_1+\tilde{\xi}_2 \leq l_D(K_D)$, $\Pi_D^* = 0$. We obtain

$$\frac{\partial l_D(K_D)}{\partial K_D} = -\frac{1}{b}K_D^{\left(\frac{-1}{b}-1\right)}\left(1+a_D-\gamma_D\right)2c_D + \left(1+\frac{1}{b}\right)K_D{}^{\left(-2-\frac{1}{b}\right)}[B\left(1+a_D\right)+P] > 0. \quad (6)$$

Therefore, we can identify the unique $K_D^l < K_D^u$ such that $l_D(K_D^l) \doteq 2\xi^l$ and $l_D(K_D^u) \doteq 2\xi^u$. Since $l_D(K_D)$ is strictly increasing in K_D, we have $l_D(K_D) \geq 2\xi^u$ for $K_D \geq K_D^u$; hence $\Pi_D^* = 0$ at each $\tilde{\xi}_1+\tilde{\xi}_2$ and $\pi_D^* = 0$ for $K_D \in [K_D^u, \infty)$. Therefore, it is sufficient to analyze the problem for $K_D \in [0, K_D^u)$. We have three separate cases to consider:

Case 1: For $K_D \in \left[0, \frac{B}{2c_D}\right]$, similar to the no limited liability case, the firm does not borrow, and the expected equity value of the firm is $\pi_D^* = \max_{K_D} 2\overline{\xi}K_D^{\left(1+\frac{1}{b}\right)} + B + P - 2c_D\left(1-\gamma_D\right)K_D$.

Case 2: For $K_D \in \left[\frac{B}{2c_D}, K_D^l\right]$, similar to the no limited liability case, the firm optimally borrows, and is always able to pay back the face value of the loan.[1] The expected equity value of the firm is $\pi_D^* = \max_{K_D} 2\overline{\xi}K_D^{\left(1+\frac{1}{b}\right)} + B(1+a_D) + P - 2c_D\left(1+a_D-\gamma_D\right)K_D$.

Case 3: For $K_D \in \left(K_D^l, K_D^u\right)$ the firm always borrows, and for some demand realization, is not able to pay back the face value of the loan; hence the expected equity value of the firm is

$$\pi_D^* = \max_{K_D} \int\int_{\Upsilon_D(\boldsymbol{\xi};K_D)} \left[(\tilde{\xi}_1+\tilde{\xi}_2)K_D{}^{\left(1+\frac{1}{b}\right)} - 2c_DK_D(1+a_D-\gamma_D) + B(1+a_D) + P\right] f(\xi_1,\xi_2)d\xi_1 d\xi_2,$$

where $\Upsilon_D(\boldsymbol{\xi};K_D) \doteq \{\boldsymbol{\xi} : \xi_1+\xi_2 \geq l_D(K_D)\}$ and $f(\xi_1,\xi_2)$ is the joint pdf of $\boldsymbol{\xi}$.

Let $g_D(K_D)$ denote the objective function in the overall optimization problem and $g_D^i(K_D)$ denote the objective function in case i. It is easy to establish that $g_D(K_D)$ is continuous at the boundaries $K_D = \frac{B}{2c_D}$ and $K_D = K_D^l$; and hence $g_D(K_D)$ is continuous in K_D. It follows from (5) that $g_D(K_D)$ is strictly concave in K_D for $K_D \in \left[0, K_D^l\right]$ and has a kink at $K_D = \frac{B}{2c_D}$. We obtain

$$\frac{\partial g_D^3(K_D)}{\partial K_D} = \int\int_{\Upsilon_D(\boldsymbol{\xi};K_D)} \left[\left(1+\frac{1}{b}\right)(\xi_1+\xi_2)K_D^{\left(\frac{1}{b}\right)} - 2\left(1+a_D-\gamma_D\right)c_D\right] f(\xi_1,\xi_2)d\xi_1 d\xi_2. \quad (7)$$

It is easy to verify that $\frac{\partial g_D^2(K_D)}{\partial K_D}\Big|_{K_D^l{}^-} = \frac{\partial g_D^3(K_D)}{\partial K_D}\Big|_{K_D^l{}^+}$; hence $g_D(K_D)$ does not have a kink at $K_D = K_D^l$. Define $G(K_D,\boldsymbol{\xi}) \doteq \left(1+\frac{1}{b}\right)(\xi_1+\xi_2)K_D^{\left(\frac{1}{b}\right)} - 2\left(1+a_D-\gamma_D\right)c_D$ as the integrand of (7) (without the density function). Note that $G(K_D,\boldsymbol{\xi})$ is increasing in

[1] It can be shown that for $\xi^l \geq 0$ and $\gamma_D \geq 0$, $K_D^l \geq \frac{B}{2c_D}$, where the equality only holds if $\xi^l = 0$ and $\gamma_D = 0$.

ξ_i for $i = 1, 2$, and decreasing in K_D. We define $\hat{K}_D \doteq \left(\frac{\xi^u\left(1+\frac{1}{b}\right)}{(1+a_D-\gamma_D)c_D}\right)^{-b}$. We have $l_D(\hat{K}_D) = 2\left(1+\frac{1}{b}\right)\xi^u\left[1-\frac{B(1+a_D)+P}{2\hat{K}_D(1+a_D-\gamma_D)c_D}\right] < 2\xi^u$, thus $\hat{K}_D < K_D^u$ and is in the feasible region of K_D. Note that for $\boldsymbol{\xi^{u\prime}} \doteq (\xi^u, \xi^u)$, $G_D\left(\hat{K}_D, \boldsymbol{\xi^u}\right) = 0$. Since $(\xi_1 + \xi_2)$ takes its maximum value at $\boldsymbol{\xi} = \boldsymbol{\xi^u}$, and $G_D(K_D, \boldsymbol{\xi})$ is strictly increasing in ξ_i for $i \in \{1, 2\}$, we have $G_D(K_D, \boldsymbol{\xi}) < 0$ for $\boldsymbol{\xi} \in \Upsilon_D(\boldsymbol{\xi}, \hat{K}_D)$. Therefore $\frac{\partial g_D^3(K_D)}{\partial K_D}|_{\hat{K}_D} < 0$. Since $G_D(K_D, \boldsymbol{\xi})$ is strictly decreasing in K_D, $\frac{\partial g_D^3(K_D)}{\partial K_D} < 0$ for $K_D \in \left[\hat{K}_D, K_D^u\right)$.

In summary, $g_D(K_D)$ is strictly concave in K_D for $K_D \in \left[0, K_D^l\right]$ (with a kink at $K_D = \frac{B}{2c_D}$), and is strictly decreasing in K_D for $K_D \in \left[\hat{K}_D, K_D^u\right)$. It follows that $g_D(K_D)$ will be unimodal if $K_D^l \geq \hat{K}_D$. Since $\frac{\partial l_D(K_D)}{\partial K_D} > 0$ (from (6)), this is equivalent to $l_D(\hat{K}_D) \leq 2\xi^l$, which gives us $B \geq B_D^h$. In this case, K_D^* is in the strictly concave part and is unique. K_D^* is identical to (5). ■

Proof of Proposition 2: In the proof of Proposition 1, we already established that the stage-1 objective function $g_D(K_D)$ is strictly concave in K_D for $K_D \in \left[0, K_D^l\right]$ and strictly decreasing in K_D for $K_D \in \left[\hat{K}_D, K_D^u\right)$. We obtain $\frac{\partial g_D(K_D)}{\partial K_D}\Big|_{K_D^l} = \frac{(1+\frac{1}{b})2\bar{\xi}}{\left(K_D^l\right)^{-1/b}}\left[1-\left(\frac{K_D^l}{K_D^1}\right)^{-\frac{1}{b}}\right]$ where $K_D^1 = \left(\frac{\bar{\xi}\left(1+\frac{1}{b}\right)}{(1+a_D-\gamma_D)c_D}\right)^{-b}$. It follows that $\frac{\partial g_D(K_D)}{\partial K_D}|_{K_D^l} > 0$ if and only if $K_D^l < K_D^1$. In this case $g_D(K_D)$ is strictly increasing for $K_D \in \left[0, K_D^l\right]$ and strictly decreasing in K_D for $K_D \in \left[\hat{K}_D, K_D^u\right)$. Since $g_D(K_D)$ is continuous in K_D, there exists at least one $K_D^* \in \left(K_D^l, K_D^u\right)$ such that $\frac{\partial g_D(K_D)}{\partial K_D}\Big|_{K_D^*} = 0$. $MP_D(K_D)$ characterizes this first-order-condition. Since $\frac{\partial l_D(K_D)}{\partial K_D} > 0$ (from (6)), $K_D^l < K_D^1$ is equivalent to $l_D(K_D^1) > 2\xi^l$, which gives $B < B_D^l$.

To prove that $K_D^* \in \left(K_D^1, K_D^u\right)$, it is sufficient to show that $\frac{\partial g_D(K_D)}{\partial K_D} > 0$ for $K_D \in \left(K_D^l, K_D^1\right]$. For $K_D > K_D^l$, as follows from (7), we have

$$\frac{\partial g_D(K_D)}{\partial K_D} = \left(1+\frac{1}{b}\right)K_D^{\frac{1}{b}}\int\int_{\Upsilon_D(\boldsymbol{\xi};K_D)}\left[\xi_1+\xi_2-2\bar{\xi}\left(\frac{K}{K^1}\right)^{-\frac{1}{b}}\right]f(\xi_1,\xi_2)d\xi_1 d\xi_2$$

Let $H_D(K_D) \doteq \int\int_{\Upsilon_D(\boldsymbol{\xi};K_D)}\left[\xi_1+\xi_2-2\bar{\xi}\left(\frac{K}{K^1}\right)^{-\frac{1}{b}}\right] f(\xi_1,\xi_2)d\xi_1 d\xi_2$. Note that, for $K_D > K_D^l$, $H_D(K_D)$ and $\frac{\partial g_D(K_D)}{\partial K_D}$ have the same sign, so we can use $H_D(K_D)$ to characterize the sign of $\frac{\partial g_D^3(K_D)}{\partial K_D}$. Define $M_D(K_D, \boldsymbol{\xi}) \doteq \xi_1 + \xi_2 - 2\bar{\xi}\left(\frac{K}{K^1}\right)^{-\frac{1}{b}}$ as the integrand in $H_D(K_D)$ (without the density function). For $\boldsymbol{\xi}$ such that $\xi_1 + \xi_2 = l_D(K_D)$, we obtain $M(K_D, \boldsymbol{\xi}) = K_D^{-\frac{1}{b}}\left[\frac{2(1+a_D-\gamma_D)c_D}{(b+1)} - \frac{B(1+a_D)+P}{K_D}\right] < 0$ since $b < -1$. As $M_D(K_D, \boldsymbol{\xi})$ is strictly increasing in ξ_i for $i \in \{1, 2\}$, $M_D(K_D, \boldsymbol{\xi}) < 0$ for $\boldsymbol{\xi}$ such that $\xi_1 + \xi_2 < l_D(K_D)$. Therefore, we have $H_D(K_D) > \int\int\left[\xi_1+\xi_2-2\bar{\xi}\left(\frac{K_D}{K_D^1}\right)^{-\frac{1}{b}}\right] f(\xi_1,\xi_2)d\xi_1 d\xi_2 = 2\bar{\xi}\left[1-\left(\frac{K_D}{K_D^1}\right)^{-\frac{1}{b}}\right]$. For $K_D \leq$

K_D^1, we have $2\overline{\xi}\left[1-\left(\frac{K_D}{K_D^1}\right)^{-\frac{1}{b}}\right] \geq 0$; and hence $H_D(K_D) > 0$ for $K_D \in \left(K_D^l, K_D^1\right]$. ■

Proof of Proposition 3: The stage-1 objective function $g_D(K_D)$ is strictly concave in K_D for $K_D \in \left[0, K_D^l\right]$ (and strictly increasing at $K_D = 0$) and strictly decreasing in K_D for $K \in \left[\hat{K}_D, K_D^u\right)$. If π_D is unimodal in K_D, thus $g_D(K_D)$ is unimodal in K_D, it follows that

1. If $\frac{\partial g_D^2(K_D)}{\partial K_D}|_{K_D^l} \leq 0$, then the unique K_D^* is characterized by the strictly concave part (similar to Proposition 1).
2. If $\frac{\partial g_D^2(K_D)}{\partial K_D}|_{K_D^l} > 0$, then the unique K_D^* is characterized by $MP_D(K_D^*) = 0$ as defined in Proposition 2. Let $\overline{K}_D$ denote the optimal solution in this case. From Proposition 2, we have $\overline{K}_D \geq K_D^1$.

As it follows from the proof of Proposition 2, $\left.\frac{\partial g_D^2(K_D)}{\partial K_D}\right|_{K_D^l} > 0$ is equivalent to $B < B_D^l$. The optimal expected equity value of the firm, π_D^*, follows directly.

We now prove that π_D^* decreases in a_D. We have two cases to consider:

Case 1: $2c_D K_D^1\left[1-\frac{\xi^l}{\overline{\xi}\left(1+\frac{1}{b}\right)}\right]\left[1-\frac{\gamma_D}{1+a_D}\right] - \frac{P}{1+a_D} \leq B < 2c_D K_D^1$

The firm's optimal expected equity value is given by $\pi_D^* = \frac{2(1+a_D-\gamma_D)c_D K_D^1(a_D)}{-(b+1)} + B(1+a_D) + P$ where $K_D^1(a_D) = \left(\frac{\overline{\xi}(1+\frac{1}{b})}{(1+a_D-\gamma_D)c_D}\right)^{-b}$. We obtain $\frac{\partial \pi_D^*}{\partial a_D} = -2c_D K_D^1(a_D) + B < 0$ as follows from the definition of Case 1.

Case 2: $0 \leq B < 2c_D K_D^1\left[1-\frac{\xi^l}{\overline{\xi}\left(1+\frac{1}{b}\right)}\right]\left[1-\frac{\gamma_D}{1+a_D}\right] - \frac{P}{1+a_D}$

The firm's optimal expected equity value is given by

$$\pi_D^* = \int\int_{\Upsilon_D(\boldsymbol{\xi};\overline{K}_D)} \left[(\tilde{\xi}_1+\tilde{\xi}_2)\overline{K}_D^{(1+\frac{1}{b})} - 2c_D\overline{K}_D(1+a_D-\gamma_D) + B(1+a_D) + P\right] f(\xi_1,\xi_2)d\xi_1 d\xi_2.$$

Note that $\frac{\partial \pi_D^*(\overline{K}_D)}{\partial a_D} = \left.\frac{\partial \pi_D(K_D)}{\partial a_D}\right|_{\overline{K}_D} + \left.\frac{\partial \pi_D(K_D)}{\partial K_D}\right|_{\overline{K}_D}\frac{\partial \overline{K}_D}{\partial a_D}$. Since $\left.\frac{\partial \pi_D(K_D)}{\partial K_D}\right|_{\overline{K}_D} = 0$, we obtain

$$\left.\frac{\partial \pi_D(K_D)}{\partial a_D}\right|_{\overline{K}_D} = \int\int_{\Upsilon_D(\boldsymbol{\xi};\overline{K}_D)} \left[-2c_D\overline{K}_D + B\right] f(\xi_1,\xi_2)d\xi_1 d\xi_2 < 0$$

as follows from $K_D^1 < \overline{K}_D$ and the definition of Case 2. ■

Lemma A.1 *If $b \geq -2$ and $\boldsymbol{\xi}$ has a bivariate normal distribution, then π_D is unimodal in K_D.*

Proof of Proposition 4: We define $S^1(a_D) \doteq 2c_D K_D^0 (1-\gamma_D)^{-b}\left[1-\frac{\xi^l}{\overline{\xi}\left(1+\frac{1}{b}\right)}\right]\frac{(1+a_D-\gamma_D)^{(b+1)}}{(1+a_D)}$ such that for a given a_D, for $B \geq S^1(a_D)$, the firm uses a secured loan (and invests in $K_D^*(a_D) = K_D^1(a_D)$) without default possibility. $B \geq S^1(a_D)$ is equivalent to $d_D(K_D^1) \leq 2\xi^l$. Hence, both the default cost and the expected loss due to the unsecured part of the loan are 0 in (4). We define $S^2(a_D) \doteq S^1(a_D) - \frac{P}{(1+a_D)}$ such that for $S^1(a_D) > B \geq S^2(a_D)$, the firm uses a secured loan (and invests in $K_D^*(a_D) = K_D^1(a_D)$) with default possibility. $B \geq S^2(a_D)$ is equivalent to $l_D(K_D^1) \leq 2\xi^l$. Hence, the default cost is strictly positive but the expected loss due to the unsecured part of the loan is 0 in (4). For $B < S^2(a_D)$, the firm optimally borrows to invest in $K_D^*(a) = \overline{K}_D(a_D)$. In this case, the firm uses an unsecured loan and both the default cost and the expected loss due to the unsecured part of the loan are strictly positive in (4).

In summary, for any given a_D, the ordering of B and thresholds $S^1(a_D)$ and $S^2(a_D)$ determine the optimal borrowing level of the firm, and hence the form of $\Lambda_D(a_D)$. We obtain $\frac{\partial S^1(a_D)}{\partial a_D} < 0$ for $a_D \in [0, a_D^{max})$ thus, we can analyze the problem in two cases.

Case 1: $B \geq S^1(0) = 2c_D K_D^0(1-\gamma_D)\left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})}\right]$.

As $S^1(a_D)$ is strictly decreasing, we have $B \geq S^1(a_D)$ (and hence $B > S^2(a_D)$) $\forall a_D \in [0, a_D^{max})$. Therefore, we have $\Lambda_D(a_D) = (2c_D K_D^1(a_D) - B)a_D$ for $0 \leq a_D < a_D^{max}$.

Case 2: $B < S^1(0) = 2c_D K_D^0(1-\gamma_D)\left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})}\right]$.

In this case, the ordering of B and $S^2(a_D)$ is important in characterizing $\Lambda_D(a_D)$. We have $S^2(0) = 2c_D K_D^0(1-\gamma_D)\left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})}\right] - P$ and we obtain

$$\frac{\partial S^2(a_D)}{\partial a_D} = \frac{1}{(1+a_D)^2}\left[P - 2c_D K^0 (1-\gamma_D)^{-b}\left(1 - \frac{\xi^l}{\bar{\xi}\left(1+\frac{1}{b}\right)}\right)(1+a_D-\gamma_D)^b\left[-b(1+a_D)-\gamma_D\right]\right]$$

Notice that $S^2(0)$ is positive (negative) if P is less (greater) than $2c_D K_D^0(1-\gamma_D)\left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})}\right]$. Since $(1+a_D-\gamma_D)^b\left[-b(1+a_D)-\gamma_D\right]$ is strictly decreasing in a_D, for $P \geq 2c_D K_D^0(-b-\gamma_D)\left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})}\right]$, we have $\frac{\partial S^2(a_D)}{\partial a_D} \geq 0$ for $a_D \geq 0$. For $P < 2c_D K_D^0(-b-\gamma_D)\left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})}\right]$, there exists a unique $\underline{a}_D$ such that $\frac{\partial S^2(a_D)}{\partial a_D} \leq 0$ for $a_D \leq \underline{a}_D$ and $\frac{\partial S^2(a_D)}{\partial a_D} > 0$ for $a_D > \underline{a}_D$. Since the signs of $S^2(0)$ and $\frac{\partial S^2(a_D)}{\partial a_D}$ depend on P, we have three subcases. Before analyzing them, we first present a Lemma that we will use throughout the rest of the proof.

Lemma A.2 *We have* $B \geq S^1\left(a_D^{max}\right) > S^2\left(a_D^{max}\right)$, $\forall B \geq 0$.

Subcase 2.1: $P \geq 2c_D K_D^0\left[1 - \frac{\xi^l}{\bar{\xi}(1+\frac{1}{b})}\right](-b-\gamma_D)$.

In this case, we have $S^2(0) < 0$ and $\frac{\partial S^2(a_D)}{\partial a_D} \geq 0, \forall a_D$. For $a_D^{max} = \left[\left(\frac{2c_D K_D^0}{B}\right)^{-\frac{1}{b}} - 1\right](1-\gamma_D)$, we obtain $S^2(a_D^{max}) < 0$. Hence, for $a_D \in [0, a_D^{max})$, we have $S^2(a_D) < 0 < B$. It follows that the firm always uses a secured loan (and invests in $K_D^1(a_D)$). For $B < S^1(0)$ (which follows from the definition of Case 2), since $S^1(a_D)$ is strictly decreasing in a_D and $B \geq S^1(a_D^{max})$ (from Lemma A.2), it follows that there exists a unique a_D^d, as defined by $S^1(a_D^d) \doteq B$

(where the superscript d refers to "default"). We have $B < S^1(a_D)$ for $a_D < a_D^d$, and the firm uses a secured loan with default possibility, and $B \geq S^1(a_D)$ for $a_D \geq a_D^d$, the firm uses a secured loan without default possibility. Therefore, $\Lambda_D(a_D)$ is characterized by

$$\Lambda_D(a_D) = \begin{cases} (2c_D K_D^1(a_D) - B)a_D - F(d_D(K_D^1(a_D)))S & \text{if} \quad 0 \leq a_D < a_D^d \\ (2c_D K_D^1(a_D) - B)a_D & \text{if} \quad a_D^d \leq a_D < a_D^{max}. \end{cases},$$

Subcase 2.2: $2c_D K_D^0 \left[1 - \frac{\xi^l}{\overline{\xi}(1+\frac{1}{b})}\right](1-\gamma_D) \leq P < 2c_D K_D^0 \left[1 - \frac{\xi^l}{\overline{\xi}(1+\frac{1}{b})}\right](-b-\gamma_D)$.
We have $S^2(0) \leq 0$, and $S^2(a_D)$ is first strictly decreasing, and then strictly increasing in a. We obtain $S^2(a_D^{max}) < 0$; hence $S^2(a_D) < 0$ for $a_D \in [0, a_D^{max})$ in this case. Therefore $\Lambda_D(a_D)$ is identical to subcase 2.1.
Subcase 2.3: $2c_D K_D^0 \left[1 - \frac{\xi^l}{\overline{\xi}(1+\frac{1}{b})}\right](1-\gamma_D) > P$
We have $S^2(0) > 0$, and $S^2(a_D)$ is first strictly decreasing, and then strictly increasing in a_D.
If $B \geq S^2(0)$ (and $B < S^1(0)$ by definition of Case 2), since $B \geq S^1(a_D^{max}) > S^2(a_D^{max})$ (from Lemma A.2), $\Lambda_D(a_D)$ is characterized in a similar fashion to the other two subcases. If $B < S^2(0)$, as $S^2(a_D)$ is first strictly decreasing, and then strictly increasing in a_D and $B \geq S^1(a_D^{max}) > S^2(a_D^{max})$ (from Lemma A.2), there exists a unique $a_D^l \in [0, a_D^{max})$, as defined in $S^2(a_D^l) \doteq B$ (where the superscript l refers to "limited liability"). We have $B < S^2(a_D)$ for $a_D < a_D^l$ and $B \geq S^2(a_D)$ for $a_D \geq a_D^l$. Since $S^2(a_D) = S^1(a_D) - \frac{P}{1+a_D}$, it follows that $a_D^l \leq a_D^d$, with equality only holding for $P = 0$. Therefore, we have the following three regions: For $a_D < a_D^l (< a_D^d)$, we have $B < S^2(a_D)$(and $B < S^1(a_D)$), the firm uses an unsecured loan; for $a_D^l \leq a_D < a_D^d$, we have $S^2(a_D) \leq B < S^1(a_D)$, and the firm uses a secured loan with default possibility; and for $a_D \geq a_D^d$, we have $S^2(a_D) < S^1(a_D) \leq B$, and the firm uses a secured loan without default possibility. ■

Proof of Proposition 5: Since this equilibrium is relevant for firms that may default but use a secured loan (Case *ii* of Proposition 4) and firms that may use an unsecured loan (Case *iii* of Proposition 4); we will analyze these two cases separately. At equilibria where the firm uses a secured loan with default possibility, the creditor's expected return with the dedicated technology is given by

$$\Lambda_D(a_D) = \left(2c_D \dot{K}_D^1 - B\right)\dot{a}_D - S \times Pr\left(\xi_1 + \xi_2 < d_D(\dot{K}_D^1)\right),$$

where $d_D(\dot{K}_D^1) = 2\overline{\xi}\left(1 + \frac{1}{b}\right)\left[1 - \frac{B(1+\dot{a}_D)}{2c_D \dot{K}_D^1(1+\dot{a}_D-\gamma_D)}\right]$. Since $\boldsymbol{\xi}$ has a bivariate normal distribution, $\xi_1 + \xi_2$ is normally distributed with mean $\overline{\mu} = 2\overline{\xi}$ and standard deviation $\overline{\sigma} = \sigma\sqrt{2(1+\rho)}$. Since $b < -1$ and $B < 2c_D \dot{K}_D^1[1-\gamma_D]$, we obtain $d_D(\dot{K}_D^1) < \overline{\mu}$. We have $Pr\left(\xi_1+\xi_2<d_D(\dot{K}_D^1)\right)=\Phi\left(\frac{d_D(\dot{K}_D^1)-\overline{\mu}}{\overline{\sigma}}\right)$ where $\Phi(.)$ is the cdf of the standard normal random variable.

For firms that may default but use a secured loan (Case *ii* of Proposition 4), for any $a_D\in\left[0,a_D^d\right)$, we obtain

$$\begin{aligned}\frac{\partial\Lambda_D(a_D)}{\rho} &= -BC\,\phi\left(\frac{d_D(K_D^1)-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{\overline{\mu}-d_D(K_D^1)}{\overline{\sigma}^2}\right)\frac{\partial\overline{\sigma}}{\partial\rho}<0,\\ \frac{\partial\Lambda_D(a_D)}{\sigma} &= -BC\,\phi\left(\frac{d_D(K_D^1)-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{\overline{\mu}-d_D(K_D^1)}{\overline{\sigma}^2}\right)\frac{\partial\overline{\sigma}}{\partial\sigma}<0\end{aligned}$$

where $\phi(.)$ is the density function of the standard normal random variable, as follows from $\frac{\partial}{\partial\rho}\overline{\sigma}=\frac{\sigma}{\overline{\sigma}}>0$, $\frac{\partial}{\partial\sigma}\overline{\sigma}=\sqrt{2(1+\rho)}>0$, and $d_D(K_D^1)<\overline{\mu}$. From the Pareto-optimality of the equilibrium, i.e. $\dot{a}_D$ is the minimum a_D that satisfies $\Lambda_D(a_D)=0$, it follows that with an increase in σ or ρ, $\dot{a}_D$ increases.

For firms that may use an unsecured loan, since $\dot{a}_D\in[a_D^l,a_D^d)$, it follows from above that $\frac{\partial}{\partial\tau}\Lambda_D(a_D)\big|_{\dot{a}_D}<0$ for $\tau\in\{\sigma,\rho\}$. In fact, $\Lambda_D(a_D)$ is decreasing in σ or ρ for any $\dot{a}_D\in[a_D^l,a_D^d)$, but we cannot characterize the effect of σ or ρ on $\Lambda_D(a_D)$ for $a_D\in\left[0,a_D^l\right)$. Let $\dot{a}_D(\tau)$ denote the equilibrium financing cost for a given $\tau\in\{\sigma,\rho\}$. With a small increment in τ from τ_0 to τ_1, we can guarantee that $\Lambda_D(a_D;\tau_1)<0$ for $\forall\ \ a_D<\dot{a}_D(\tau_0)$ because i) $\Lambda_D(a_D;\tau_0)<0$ for $\forall\ \ a_D<\dot{a}_D(\tau_0)$ from the definition of the equilibrium, and ii) $\left|\frac{\partial}{\partial\tau}\Lambda_D(a_D)\right|$ and $\left|\frac{\partial}{\partial a_D}\Lambda_D(a_D)\right|$ are bounded. Therefore $\dot{a}_D$ increases (locally) in $\tau\in\{\sigma,\rho\}$.

Since for a given a_D, π_D^* is independent of $\tau\in\{\sigma,\rho\}$, we have $\frac{\partial\dot{\pi}_D}{\partial\tau}=\frac{\partial\pi_D^*}{\partial a_D}\Big|_{\dot{a}_D}\frac{\partial\dot{a}_D}{\partial\tau}$. From Proposition 3, we have $\frac{\partial\pi_D^*}{\partial a_D}<0$, hence $\frac{\partial\dot{\pi}_D}{\partial\tau}<0$. Similarly, $\dot{K}_D=K_D^1(\dot{a}_D)$ is independent of $\tau\in\{\sigma,\rho\}$; hence we have $\frac{\partial\dot{K}_D}{\partial\tau}=\frac{\partial K_D^1(a_D)}{\partial a_D}\Big|_{\dot{a}_D}\frac{\partial\dot{a}_D}{\partial\tau}$. Since $K_D^1(a_D)$ decreases in a_D, we have $\frac{\partial\dot{K}_D}{\partial\tau}<0$ for $\tau\in\{\sigma,\rho\}$. ■

Lemma A.3 *If $b\geq-2$ and $\boldsymbol{\xi}$ has a bivariate normal distribution, then for a given financing cost a_D with the dedicated technology, for the firm that uses an unsecured loan, K_D^* and π_D^* increase in σ and ρ, and decrease in a_D.*

Lemma A.4 *If $b\geq-2$ and $\boldsymbol{\xi}$ has a bivariate normal distribution, when the firm uses an unsecured loan with the dedicated technology, the creditor's net gain from secured lending and its expected loss due to the unsecured part of the loan increase in σ and ρ. Its expected default cost increases in σ and ρ if $d_D(\overline{K}_D(a_D))\leq2\overline{\xi}$.*

Proof of Remark 2 The form of $\overline{c}_D^P(c_F)$ follows from a direct comparison of $\dot{\pi}_D$ and $\dot{\pi}_F$ in perfect capital markets. Since $\gamma_F\geq\gamma_D$ by assumption, to prove $\overline{c}_D^P(c_F)\leq c_F$ it is sufficient to show $\mathbb{E}^{-b}\left[\left(\xi_1^{-b}+\xi_2^{-b}\right)^{-\frac{1}{b}}\right] \geq 2\overline{\xi}^{-b}$. From Hardy et. al (1988, p.146), if $d \in (0,1)$ and X, Y are non-negative random variables then the following is true: $\mathbb{E}^{1/d}\left[(X+Y)^d\right] \geq \mathbb{E}^{1/d}\left[X^d\right]+\mathbb{E}^{1/d}\left[Y^d\right]$ where equality only holds when X and Y are effectively proportional, i.e. $X = \lambda Y$. In $\overline{c}_D^P(c_F)$, we have $d = -\frac{1}{b} \in (0,1)$ and $\boldsymbol{\xi} \geq \boldsymbol{\xi^l} \geq \mathbf{0}$, replacing X with ξ_1^{-b} and Y with ξ_2^{-b} gives the desired result. Notice that $\overline{c}_D^P(c_F) = c_F$ only if $\xi_1 = \xi_2$ (since we focus on the symmetric bivariate distribution) and $\gamma_F = \gamma_D$. $\xi_1 = \xi_2$ is only possible if either $\boldsymbol{\xi}$ is deterministic or $\rho = 1$. ■

B Proofs for Supporting Lemmas

Proof of Lemma A.1: Since $\boldsymbol{\xi}$ has a bivariate normal distribution, $\psi \doteq \xi_1 + \xi_2$ is normally distributed with mean $\overline{\mu} = 2\overline{\xi}$ and standard deviation $\overline{\sigma} = \sigma\sqrt{2(1+\rho)}$. Let $F(.)$ denote the cdf of ψ, and $\overline{F}(.) = 1 - F(.)$. By using ψ, as follows from the proof of Proposition 2; for $K_D \geq K_D^l$, we have $sgn\left(\frac{\partial g_D^3(K_D)}{\partial K_D}\right) = sgn(H_D(K_D))$ where $H_D(K_D) = \int_{l_D(K_D)}^{\xi^u}\left[\psi - 2\overline{\xi}\left(\frac{K_D}{K_D^1}\right)^{-\frac{1}{b}}\right] f(\psi)d\psi$. Therefore we will focus on $H_D(K_D)$ to prove the unimodality of $g_D(K_D)$. From integration by parts, we obtain

$$H_D(K_D) = \int_{l_D(K_D)}^{2\xi^u} \overline{F}(\psi)d\psi - \overline{F}(l_D(K_D))\left[K_D^{-\frac{1}{b}}\left(\frac{2\left(1+a_D-\gamma_D\right)c_D}{-\left(b+1\right)} + \frac{B(1+a_D)+P}{K_D}\right)\right].$$

Define $\Delta(K_D) \doteq K_D^{-\frac{1}{b}}\left(\frac{2(1+a_D-\gamma_D)c_D}{-(b+1)} + \frac{B(1+a_D)+P}{K_D}\right)$. We obtain

$$\frac{\partial \Delta(K_D)}{\partial K_D} = \left(1+\frac{1}{b}\right)K_D^{-1}\left(l_D(K_D) + \frac{-b(b+2)}{(b+1)^2}2\left(1+a_D-\gamma_D\right)c_D K_D^{-\frac{1}{b}}\right).$$

Note that for $K_D > K_D^l$, $l_D(K_D) > 2\xi^l \geq 0$; hence for $b \geq -2$ the second term is positive and $\frac{\partial \Delta(K_D)}{\partial K_D} > 0$ for $K_D > K_D^l$. We obtain $H_D(K_D) = \overline{F}(l_D(K_D))\left[\frac{\int_{l_D(K_D)}^{2\xi^u}\overline{F}(\psi)d\psi}{\overline{F}(l_D(K_D))} - \Delta(K_D)\right]$. As $\Delta(K_D)$ is increasing in K_D, if we can show that $\frac{\int_{l_D(K_D)}^{2\xi^u}\overline{F}(\psi)d\psi}{\overline{F}(l_D(K_D))}$ is decreasing in K_D, then for $K_D > K_D^l$, $H_D(K_D)$ can only change sign once, which is from positive to negative.

We now show that $\frac{\int_{l_D(K_D)}^{2\xi^u}\overline{F}(\psi)d\psi}{\overline{F}(l_D(K_D))}$ is decreasing in K_D. Since ψ is normally distributed with mean $\overline{\mu}$ and standard deviation $\overline{\sigma}$, by using the standard normal random variable, this expression can be written as

$$\frac{-1}{\left[1-\Phi\left(\frac{l_D(K_D)-\overline{\mu}}{\overline{\sigma}}\right)\right]^2}\frac{\partial l_D(K_D)}{\partial K_D}\left[\left[1-\Phi\left(\frac{l_D(K_D)-\overline{\mu}}{\overline{\sigma}}\right)\right]^2 - \phi\left(\frac{l_D(K_D)-\overline{\mu}}{\overline{\sigma}}\right)\int_{\frac{l_D(K_D)-\overline{\mu}}{\overline{\sigma}}}^{\infty}(1-\Phi(z))dz\right] \quad (8)$$

where $\Phi(.)$ and $\phi(.)$ are the cdf and pdf of the standard normal random variable respectively. Since $\frac{\partial l_D(K_D)}{\partial K_D} > 0$, it is sufficient to show that the last term in parenthesis is positive. Let

$v = \frac{l_D(K_D)-\overline{\mu}}{\overline{\sigma}}$. Using integration by parts, we obtain $\int_v^\infty (1-\Phi(z))dz = \phi(v) - v(1-\Phi(v))$. Substituting this in (8), it is sufficient to show that $1 > \left[\frac{\phi(v)}{1-\Phi(v)}\right]^2 - \frac{v\phi(v)}{1-\Phi(v)}$ which directly follows from Sampford (1953). ■

Proof of Lemma A.3: For a given a_D, the optimal expected equity value π_D^* is given by

$$\left[1-\Phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)\right]\left[\overline{\mu}\overline{K}_D^{(1+\frac{1}{b})} + B(1+a_D) + P - 2(1+a_D)c_D\overline{K}_D\right] + \overline{\sigma}\overline{K}_D^{(1+\frac{1}{b})}\phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right).$$

where $\overline{\mu} = 2\xi$ and $\overline{\sigma} = \sigma\sqrt{2(1+\rho)}$. Since $\overline{\sigma}$ is increasing in σ or ρ, it is sufficient to analyze the impact of $\overline{\sigma}$. We have $\frac{\partial\pi_D^*}{\partial\overline{\sigma}} = \frac{\partial\pi_D}{\partial K_D}\Big|_{\overline{K}_D}\frac{\partial K_D}{\partial\overline{\sigma}} + \frac{\partial\pi_D}{\partial\overline{\sigma}}\Big|_{\overline{K}_D}$ where the first term is zero from the optimality of $\overline{K}$. We obtain $\frac{\partial\pi_D}{\partial\overline{\sigma}}\Big|_{\overline{K}_D} = \overline{K}_D^{(1+\frac{1}{b})}\phi\left(\frac{l(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right) > 0$.

For $K_D^* = \overline{K}_D$, since $\overline{K}_D$ is the unique maximizer, we have $sgn\left(\frac{\partial\overline{K}_D}{\partial\overline{\sigma}}\right) = sgn\left(\frac{\partial MP_D(K_D)}{\partial\overline{\sigma}}\Big|_{\overline{K}_D}\right)$. Using the optimality condition

$$\left[1-\Phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)\right]\left[(1+\frac{1}{b})\overline{\mu}\overline{K}_D^{\frac{1}{b}} - 2(1+a_D)c_D\right] = -(1+\frac{1}{b})\overline{\sigma}\overline{K}_D^{\frac{1}{b}}\phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right), \quad (9)$$

we obtain $\frac{\partial MP_D(K_D)}{\partial\overline{\sigma}}\Big|_{\overline{K}_D} = (1+\frac{1}{b})\overline{K}_D^{\frac{1}{b}}\phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)\left[\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)^2 + 1 - \frac{\phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)}{1-\Phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)}\right]$.

Let $z = \left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)$. We need to show that $1 > z\left[\frac{\phi(z)}{1-\Phi(z)} - z\right]$. It follows from Sampford (1953) that $\left[\frac{\phi(z)}{1-\Phi(z)} - z\right] < \frac{1-\Phi(z)}{\phi(z)}$; therefore it is sufficient to show $1 > \frac{z(1-\Phi(z))}{\phi(z)}$ which also follows from Sampford (1953).

For the impact of $\dot{a}_D$ on $\overline{K}_D$, we have $sgn\left(\frac{\partial\overline{K}_D}{\partial a_D}\right) = sgn\left(\frac{\partial MP_D(K_D)}{\partial a_D}\Big|_{\overline{K}_D}\right)$. Using the optimality condition in (9), we obtain $\frac{\partial MP_D(K_D)}{\partial a_D}\Big|_{\overline{K}_D} =$

$$\left[1-\Phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)\right]$$
$$\times\left[-2c_D + (1+\frac{1}{b})\overline{K}_D^{\frac{1}{b}}\frac{\partial l_D(\overline{K}_D)}{\partial a_D}\left[\left(\frac{\phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)}{1-\Phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)}\right)^2 - \frac{\phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)}{1-\Phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)}\right]\right].$$

Denoting Y as the last expression in brackets and using $\frac{\partial l_D(\overline{K}_D)}{\partial a_D} = K_D^{\frac{-1}{b}}\left[2c_D - \frac{B}{\overline{K}_D}\right]$, the desired result follows because $-2c_D + (1+\frac{1}{b})\left[2c_D - \frac{B}{\overline{K}_D}\right]Y < 0$ as $Y < 1$ from Sampford (1953). ■

Proof of Lemma A.4: We only provide the proof for the expected loss due to the unsecured part of the loan. The proofs for the default risk and the net gain from secured lending can be obtained in a similar fashion, and are omitted. Since $\overline{\sigma}$ is increasing in σ or ρ, it is sufficient to analyze the impact of $\overline{\sigma}$. By using the standard normal random variable, the expected loss due to the unsecured part of the loan can be written as

$$\Phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)\left[\overline{K}_D(1+a_D)2c_D-\overline{\mu}\overline{K}_D^{(1+\frac{1}{b})}-B(1+a_D)-P\right]+\overline{\sigma}\phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)\overline{K}_D^{(1+\frac{1}{b})}.$$

Taking the derivative with respect to $\overline{\sigma}$, and using the optimality condition in (9), the derivative with respect to $\overline{\sigma}$ is given by $\phi\left(\frac{l_D(\overline{K}_D)-\overline{\mu}}{\overline{\sigma}}\right)\overline{K}_D^{(1+\frac{1}{b})}+\frac{\partial\overline{K}_D}{\partial\overline{\sigma}}\left[2(1+a_D)c_D-\overline{\mu}\overline{K}_D^{\frac{1}{b}}\right]$. This term is positive because $\frac{\partial\overline{K}_D}{\partial\overline{\sigma}}>0$ from Lemma A.3 and the last expression is positive from the optimality condition in (9). ■

C References

Hardy, G., J.E. Littlewood, G. Polya. 1988. *Inequalities.* Cambridge Press, NY.

Sampford, M.R. 1953. Some inequalities on Mill's ratio and related functions. *The Annals of Mathematical Statistics* **24** 130–132.