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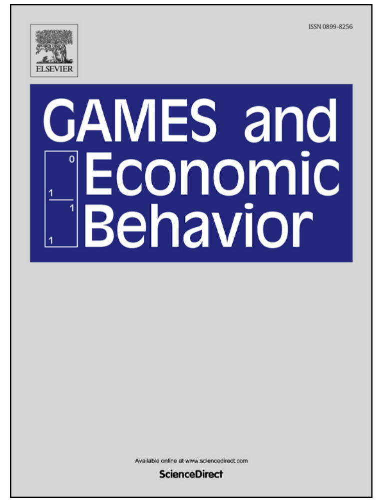

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On Random Social Choice Functions with the Tops-only Property*

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Abstract

We study the standard voting model with randomization. A Random Social Choice Function (or RSCF) satisfies the tops-only property if the social lottery under each preference profile depends only on the peaks of voters' preferences. We identify a general condition on domains of preferences (the Interior Property and the Exterior Property) which ensures that *every* strategy-proof RSCF satisfying unanimity has the tops-only property. We show that our condition applies to important classes of voting domains which include restricted connected domains (Sato, 2013) and the multi-dimensional single-peaked domain (Barberà et al., 1993). As an application of our result, we show that every ex-post efficient and strategy-proof RSCF defined on the multi-dimensional single-peaked domain is a random dictatorship.

Keywords: Random Social Choice Functions; Unanimity; Strategy-proofness; The Tops-only Property; The Interior Property; The Exterior Property

JEL Classification: D71.

1 Introduction

Randomization is a natural device which is ubiquitous in economic environments. It is, for instance, used to bring fairness to the ex-ante consideration of collective decision making problems, object assignment problems, etc. Recently, randomization has been shown to

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significantly enlarge the scope of designing "well-behaved" voting mechanisms (Chatterji et al., 2014). In this paper, we study randomization in the classic voting environment where each voter submits an *ordinal strict preference order* over a finite set of alternatives; a "desirable" *social lottery* over all alternatives is chosen, and no money transfer is allowed. Each voter's preference order is her *private information*. A *Random Social Choice Function* (or RSCF) determines the social lottery under every profile of reported preferences. In particular, if a *degenerate lottery*, i.e., one where an alternative receives probability one, is chosen under each preference profile, the RSCF is referred to as a *Deterministic Social Choice Function* (or DSCF).

Starting from the seminal work of Gibbard (1977), RSCFs have increasingly received attention in a growing literature,[1] where the incentives for truthfully revealing private information are prominently at the forefront. Fixing an ordinal preference and a utility function representing the ordinal preference, we assume that each voter evaluates lotteries according to the von Neumann-Morgenstern expected utility hypothesis. We then adopt the notion of *strategy-proofness* established by Gibbard (1977) which requires that no voter can obtain a strictly higher expected utility by misreporting her preferences for *any* utility representing her true ordinal preference and *any* belief regarding the reports of other voters. Equivalently, this notion of strategy-proofness can be reformulated in terms of *(strong) first-order stochastic dominance* which says that for each voter, the social lottery induced by truthtelling first-order stochastically dominates (according to her true ordinal preference) any lottery obtained via a unilateral misrepresentation.

If strategy-proofness is the only concern, one can construct a *constant* RSCF which ignores all information of voters' preferences and fixes a lottery as the social outcome for every preference profile. However, such an RSCF is clearly not desirable. On the other hand, while allowing the social lottery to vary with preference profiles is desirable, maintaining strategy-proofness becomes correspondingly harder as the social lottery begins to depend more intricately on preferences. In this paper, we study preference domains where strategy-proof RSCFs use only the peaks of voters' preferences to calculate the social lottery. This class of RSCFs is said to satisfy *the tops-only property*, which implies that if the peaks of each voter across two preference profiles are identical, the social lottery remains the same; RSCFs satisfying this property are pervasive in the literature. There however remains the possibility that by insisting on the tops-only property, one may constrain significantly the scope for designing strategy-proof RSCFs. Indeed, there exist other intuitive RSCFs that use some non-top information and have nice incentive properties, e.g., the point voting schemes

[1]There are numerous papers (e.g., Barberà, 1979; Hylland, 1980; Duggan, 1996; Dutta et al., 2002; Ehlers et al., 2002; Dutta et al., 2007; Sen, 2011; Picot and Sen, 2012; Chatterji et al., 2012; Aziz et al., 2014; Aziz and Stursberg, 2014; Chatterji et al., 2014; Peters et al., 2014; Pycia and Ünver, 2015; Brandl et al., 2016) that study RSCFs.

of Barberà (1979).[2] In view of this possibility, establishing the tops-only property as a consequence of strategy-proofness is invariably a critical step in the literature that provides characterization results for strategy-proof voting rules and allocation rules.[3] This is typically accomplished by verifying explicitly that the domain in question is a *tops-only domain*, i.e., one where *every* strategy-proof and *unanimous* RSCF must be tops-only.[4] For tops-only domains, there is thus no loss of design possibilities in restricting attention to tops-only RSCFs.

It is well-known that appropriate richness conditions are required on domains in order for them to be tops-only domains. We observe in this paper that it is possible for a domain to be tops-only for DSCFs without being tops-only for RSCFs (see Example 1). We provide a new sufficient condition for a domain to be tops-only for RSCFs (see the Theorem). Importantly, we emphasize that under our sufficient condition, the tops-only property emerges endogenously: Our methodology allows us to assert this property without requiring us to explicitly characterize the class of all unanimous and strategy-proof RSCFs. Our condition is context free, and should be useful in delineating the possibilities for designing simple and desirable strategy-proof RSCFs by facilitating characterization results that are based on the tops-only property in a variety of settings.

Before describing our condition, we note that RSCFs satisfying the tops-only property afford additional conveniences to a planner confronted with the task of designing an RSCF on a particular restricted domain of preferences. One the one hand, such rules are easier to operationalize as they have to be defined for a much smaller number of preference profiles, while on the other hand, the actual act of agents reporting their preferences simplifies as each agent merely reports her top ranked alternative. The truthful reporting of one's top alternative is of course predicated upon there being no gainful manipulations of prefer-

[2] A point voting scheme is a randomized scoring rule. Assume that there are m alternatives and N voters, and all ordinal preferences on alternatives are strict. A non-negative real number α_k is the score associated to the kth ranked alternative according to a preference. A higher ranked alternative naturally receives a higher score, i.e., $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$, and moreover, $\sum_{k=1}^{m} \alpha_k = \frac{1}{N}$. After all voters submit their preferences, the probability assigned to an alternative is the sum of scores it receives from each preference.

[3] In the voting environment, see the results on dictatorship (Gibbard, 1973; Satterthwaite, 1975), random dictatorship (Gibbard, 1977), voting by committees (Barberà et al., 1991), phantom voter rules and generalized median voter rules (Moulin, 1980; Border and Jordan, 1983; Barberà et al., 1993; Ching, 1997; Reffgen, 2015), fixed-probabilistic-ballots rules (Ehlers et al., 2002), voting by issues (Nehring and Puppe, 2007), meet social choice functions (Mishra and Roy, 2012) and generalized random dictatorship (Chatterji et al., 2012). In economic environments, see the results on dictatorship (Barberà and Peleg, 1990; Zhou, 1991), random dictatorship (Dutta et al., 2002), and minimax rules (Barberà and Jackson, 1994). In the fair division literature, see the results on uniform rules (Sprumont, 1991) and sequential allotment rules (Barberà et al., 1997). For more related literature, please refer to the survey paper of Sprumont (1995).

[4] In a unanimous RSCF, if all voters share the same peak in a preference profile, this peak is chosen with probability one. This is a natural condition to impose on the social lottery. The point voting schemes mentioned earlier need not satisfy unanimity.

ences. This task of verifying that there are no gainful manipulations too is much simpler with tops-only RSCFs as any manipulation using a preference with the same top as the true preference does not affect the social lottery and hence can never be beneficial.[5] In models with many agents, many alternatives and with large variation in preferences with the same top, these informational and computational gains afforded by tops-only RSCFs may be considerable. Indeed, calculating the computational costs associated to eliciting preferences and finding gainful manipulations is at the forefront of a recent and growing literature (e.g., Ailon, 2010; Faliszewski and Procaccia, 2010; Vaish et al., 2016) which studies, for instance, recommendation systems and other internet related design problems.[6]

Our condition requires that a particular *Interior Property* and an *Exterior Property*, respectively, hold. Both the Interior Property and the Exterior Property are variations of the notion of connectedness that is well studied in preference domains (e.g., Monjardet, 2009; Sato, 2013; Cho, 2016). A preference domain is *connected* if every pair of distinct preferences is connected via a path of preferences in the domain, where each consecutive pair of preferences is *adjacent*, in other words, differs in the ranking of exactly one contiguous pair of alternatives. Connectedness thus implies that the differences between two preferences can be reconciled via a trackable successive evolution process. The Interior Property implements the connectedness idea on each sub-domain of preferences with a common peak. In order to verify that an RSCF satisfies the tops-only property, we need to check that the social lottery remains unchanged when an arbitrary agent switches to a different preference with the same peak. Since such a check requires one to consider only pairs of preferences with the same peak, one might expect that the relevant restriction on domains to render them tops-only domains be one that, like the Interior Property, applies to each sub-domain of preferences with the same peak. However, it does not suffice to restrict attention to each sub-domain of preferences with the same peak (see Example 2); we need to augment the Interior Property by the Exterior Property which is a restriction that holds across sub-domains with different peaks. To describe the Exterior Property, we introduce the notion of isolation. A pair of alternatives, say x and y, is termed *isolated* in a pair of preferences if we can partition the

[5]For instance, assume that there are m alternatives and the domain of preferences contains all linear orders. Accordingly, under each preference profile, each agent has $m! - 1$ possible manipulations in an RSCF, while the degree of possible manipulations is significantly reduced to $m - 1$ in a tops-only RSCF.

[6]In a voting system, it might be too demanding to elicit a voter's full ranking on a large set of alternatives. Instead, a voter is assumed to focus on a few alternatives which are ranked above all others (e.g., Ailon, 2010; Reffgen, 2011). Similarly, detecting a manipulation in a mechanism that depends too much on information on preferences would be computationally hard, e.g., see the second-order Copeland Schemes of Bartholdi III et al. (1989). In this context, note that imposing the tops-only property to voting schemes substantially simplifies and shortens the testing time of the *Algorithm Greedy-Manipulation* of Bartholdi III et al. (1989) which is adopted to calculate or to claim the non-existence of a gainful manipulation in worst-case polynomial time. Recently, robustness of rules to manipulation has been used as a criteria for comparing generalized median voter rules (see Arribillaga and Massó, 2016).

alternative set into two disjoint subsets, one containing x and the other containing y, where both preferences agree to rank one subset above the other. A sequence of preferences is an (x, y) isolation path if alternatives x and y remain isolated across each pair of successive preferences. The Exterior Property requires that for any two preferences that rank x above y and have distinct peaks, there be an (x, y) isolation path that starts at one preference and ends at the other.[7] Finally, we note that one may use appropriate graphs to verify our condition in an arbitrary preference domain. The Interior Property immediately holds whenever the graph of adjacencies over each sub-domain with the same peak is a connected graph.[8] An analogous graph based on the notion of isolation can be used to verify the Exterior Property.

We now turn to applications of our Theorem. It is immediate that our condition holds whenever a domain is connected in the sense of Sato (2013) (see Proposition 1). This class of domains covers many well studied domains that include the complete domain (Gibbard, 1973), the single-peaked domain (Moulin, 1980; Demange, 1982), the single-dipped domain (Barberà et al., 2012) and single-crossing domains (Saporiti, 2009; Carroll, 2012). Therefore, our result can be used to characterize strategy-proof rules in all these domains by restricting attention to tops-only RSCFs. The verification of our condition can be less straightforward in a multi-dimensional setting. While it is possible to write a matrix based algorithm to verify our condition on a given domain, we do not pursue this approach here, but provide instead a direct verification of our condition on the multi-dimensional single-peaked domain introduced by Barberà et al. (1993) (see Proposition 2). We then use this result to derive a new characterization: Every *ex-post efficient* and strategy-proof RSCF on the multi-dimensional single-peaked domain is a *random dictatorship* (see Proposition 3).[9]

Our model uses an ordinal formulation of strategy-proofness introduced by Gibbard (1977). An alternative formulation of strategy-proofness uses cardinal information on preferences (e.g., Hylland, 1980; Duggan, 1996; Dutta et al., 2007). Here too the tops-only property plays an important role in characterizing random strategy-proof voting rules. We conjecture

[7]Formulating the Exterior Property using a path of adjacent preferences (as in the Interior Property) to connect two preferences with distinct peaks turns out to be too demanding, and in particular narrows the scope of studying preferences in the multi-dimensional setting (see Section 4.2).

[8]Given a domain, we construct a graph where vertices are preferences, and a pair of preferences constitutes an edge if and only if they are adjacent. The graph is termed a "connected" graph if we can move from one vertex to another via a path of edges.

[9]Ex-post efficiency implies that an alternative that is *Pareto dominated* by another alternative in a preference profile should receive zero probability in the corresponding social lottery. One important class of ex-post efficient and strategy-proof RSCFs is random dictatorships. Assume that there are N voters, and consider, for simplicity, a preference profile where all peaks of preferences are distinct. A particular formulation of a random dictatorship determines the corresponding social lottery by choosing each voter's peak of preference with probability $\frac{1}{N}$. The formal definition of a random dictatorship can be found in Section 4.3.

that a version of our richness condition would allow us to endogenize the tops-only property in these cardinal models. Earlier work has studied the tops-only property for DSCFs. Weymark (2008) initiated the study of the tops-only property with single-peaked preferences on a real line and continuous preferences on a metric space. Subsequent work focuses on the case of finite alternatives and strict preferences, e.g., generalized single-peaked domains (Nehring and Puppe, 2007) and two general richness conditions (Chatterji and Sen, 2011) for tops-only domains. Their sufficient conditions are only valid for DSCFs, and cannot directly be applied to RSCFs.

The remainder of the paper is organized as follows. Section 2 introduces the model and definitions. Section 3 presents the main result. Section 4 provides three applications while Section 5 elaborates on the relation to earlier literature, and briefly discusses the necessity of our condition. The Appendix gathers all omitted proofs and some additional material.

2 Preliminaries

Let $A = \{a, b, c, \dots\}$ be a finite set of alternatives with $|A| = m \geq 3$, and $\Delta(A)$ denote the lottery space on A. An element of $\Delta(A)$ is a lottery or a probability distribution over alternatives. In particular, $e_a \in \Delta(A)$ is a degenerate lottery where alternative a is chosen with probability one. Let $I = \{1, \dots, N\}$ be a finite set of voters with $|I| = N \geq 1$.[10] Each voter i has a (strict preference) order P_i over A which is antisymmetric, complete and transitive, i.e., a linear order. For any $a, b \in A$, aP_ib is interpreted as "a is strictly preferred to b according to P_i".[11] Let $\mathbb{P}$ denote the set containing *all* linear orders over A. The set of all admissible orders is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as the **preference domain**.[12] Let $r_k(P_i)$ denote the kth ranked alternative in P_i, $k = 1, \dots, m$. A pair of alternatives $a, b \in A$ is *contiguous* in P_i if $\{a, b\} = \{r_k(P_i), r_{k+1}(P_i)\}$ for some $1 \leq k \leq m-1$. Accordingly, let $aP_i!b$ denote that a and b are contiguous in P_i, and aP_ib. Given $1 \leq k \leq m$ and $P_i \in \mathbb{D}$, $B^k(P_i) = \cup_{t=1}^{k}\{r_t(P_i)\}$ is the set of top-k ranked alternatives. For notational convenience, let $\mathbb{D}^a = \{P_i \in \mathbb{D} | r_1(P_i) = a\}$ denote the set of preferences with peak a. Correspondingly, a domain $\mathbb{D}$ is **minimally rich** if $\mathbb{D}^a \neq \emptyset$ for every $a \in A$. A preference profile $P \equiv (P_1, \dots, P_N) \equiv (P_i, P_{-i}) \in \mathbb{D}^N$ is an N-tuple of orders where P_{-i} represents a collection of $N-1$ voters' preferences without considering voter i's preference.

A **Random Social Choice Function** (or RSCF) is a map $\varphi : \mathbb{D}^N \to \Delta(A)$. At every profile $P \in \mathbb{D}^N$, $\varphi(P)$ is referred to as the "socially desirable" lottery associated to this preference profile. For any $a \in A$, $\varphi_a(P)$ is the probability with which alternative a will be

[10] The case of a single voter is included to simplify the proofs (see Chatterji and Sen, 2011).

[11] In a table, we specify a preference "vertically". In a sentence, we specify a preference "horizontally". For instance, P_i: $a\,b\,c \cdots$ signifies that a is the top, b is the second best, c is the third ranked alternative while the rest of the rankings in P_i are arbitrary.

[12] We refer to $\mathbb{P}$ as *the complete domain*. When $\mathbb{D} \neq \mathbb{P}$, $\mathbb{D}$ is referred to as a *restricted domain*.

chosen in the social lottery $\varphi(P)$. Thus, $\varphi_a(P) \geq 0$ for all $a \in A$ and $\sum_{a \in A} \varphi_a(P) = 1$. A **Deterministic Social Choice Function** (or DSCF) is a particular RSCF where a degenerate lottery is chosen under each preference profile, i.e., $\varphi(P) = e_a$ for some $a \in A$ at profile P.

An RSCF satisfies unanimity if it assigns probability one to any alternative that is top ranked by all voters, i.e., RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is **unanimous** if $[r_1(P_i) = a$ for all $i \in I] \Rightarrow [\varphi_a(P) = 1]$ for all $a \in A$ and $P \in \mathbb{D}^N$.

An axiom stronger than unanimity is ex-post efficiency which requires that every Pareto dominated alternative in a preference profile must receive zero probability in the associated social lottery. Formally, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is **ex-post efficient** if for all $a, b \in A$ and $P \in \mathbb{D}^N$, $[aP_ib$ for all $i \in I] \Rightarrow [\varphi_b(P) = 0]$.

An RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is **strategy-proof** if for all $i \in I$, $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$, the lottery $\varphi(P_i, P_{-i})$ (strongly) first-order stochastically dominates $\varphi(P_i', P_{-i})$ according to P_i, i.e., $\sum_{x \in B^t(P_i)} \varphi_x(P_i, P_{-i}) \geq \sum_{x \in B^t(P_i)} \varphi_x(P_i', P_{-i})$, $t = 1, \ldots, m$.

A prominent class of RSCFs is the class of tops-only RSCFs. The social lottery selected by these RSCFs at every preference profile depends only on voters' peaks. Formally, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies **the tops-only property** if $[r_1(P_i) = r_1(P_i')$ for all $i \in I] \Rightarrow [\varphi(P) = \varphi(P')]$ for all $P, P' \in \mathbb{D}^N$. Accordingly, a domain is referred to as a **tops-only domain** if every unanimous and strategy-proof RSCF satisfies the tops-only property.[13]

3 The main result

In this section, we introduce a condition on domains under which every unanimous and strategy-proof RSCF satisfies the tops-only property. We begin by observing that a tops-only domain for DSCFs need not be tops-only for RSCFs. We provide the following example to illustrate.

Example **1** Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and consider the domain $\mathbb{D}$, containing fourteen preferences, specified below.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
a_1	a_1	a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_4	a_4	a_4	a_5	a_5
a_2	a_3	a_5	a_1	a_3	a_4	a_1	a_2	a_4	a_2	a_3	a_5	a_1	a_4
a_3	a_2	a_3	a_3	a_1	a_3	a_2	a_1	a_2	a_3	a_2	a_3	a_4	a_1
a_5	a_5	a_4	a_5	a_5	a_5	a_4	a_5	a_1	a_5	a_5	a_2	a_2	a_2
a_4	a_4	a_2	a_4	a_4	a_1	a_5	a_4	a_5	a_1	a_1	a_1	a_3	a_3

Table 1: Domain $\mathbb{D}$

[13]Note that to specify tops-only domains, we must restrict attention to the class of unanimous RSCFs. Otherwise, for instance, we can construct a particular point voting scheme (recall footnote 2) with $\alpha_1 < \frac{1}{N}$ which satisfies strategy-proofness, but avoids unanimity and the tops-only property.

It is easy to verify that domain $\mathbb{D}$ is *linked* (Aswal et al., 2003).[14] By Theorem 3.1 of Aswal et al. (2003), every unanimous and strategy-proof DSCF is a *dictatorship* and therefore satisfies the tops-only property.[15] However, domain $\mathbb{D}$ admits the following unanimous and strategy-proof RSCF

$$\varphi(P_i, P_j) = \begin{cases} \frac{1}{2}e_{r_1(P_i)} + \frac{1}{2}e_{r_1(P_j)} & \text{if either } P_i \notin \mathbb{D}^{a_3} \text{ or } P_j \notin \mathbb{D}^{a_5} \\ \frac{1}{4}e_{a_3} + \frac{1}{4}e_{a_2} + \frac{1}{2}e_{a_5} & \text{if } P_i = P_8 \text{ and } P_j \in \mathbb{D}^{a_5} \\ \frac{1}{4}e_{a_3} + \frac{1}{4}e_{a_1} + \frac{1}{4}e_{a_4} + \frac{1}{4}e_{a_5} & \text{if } P_i \in \{P_7, P_9\} \text{ and } P_j \in \mathbb{D}^{a_5} \end{cases}$$

which violates the tops-only property, e.g., $r_1(P_7) = r_1(P_8) = a_3$ and $\varphi_{a_2}(P_7, P_{13}) = 0 \neq \frac{1}{4} = \varphi_{a_2}(P_8, P_{13})$. The verification of the strategy-proofness of φ is available in Appendix A. $\square$

We identify a richness condition on domains that renders them tops-only domains. Our condition requires two properties, which are referred to as the Interior Property and the Exterior Property, respectively.

We partition the domain into sub-domains where all preferences in a sub-domain have an identical peak. The Interior Property refers to a requirement across any two preferences within a given sub-domain, while the Exterior Property refers to a requirement that applies to any two preferences belonging to two distinct sub-domains. To describe the Interior Property, we adopt the notion of adjacency (Sato, 2013), while to describe the Exterior Property, we use a more general notion called isolation.

A pair of distinct preferences $P_i, P_i' \in \mathbb{D}$ is **adjacent**, denoted $P_i \sim^A P_i'$, if there exists $1 \leq k \leq m-1$ such that the following two conditions are satisfied

(i) $r_k(P_i) = r_{k+1}(P_i')$ and $r_{k+1}(P_i) = r_k(P_i')$;

(ii) $r_t(P_i) = r_t(P_i')$ for all $t \neq k, k+1$.

In other words, two preferences are adjacent if exactly one pair of contiguous alternatives locally switches their relative rankings. We refer to this pair of alternatives as a **local switching pair**. Given distinct $P_i, P_i' \in \mathbb{D}$, an **Ad-path** connecting P_i and P_i' is a sequence $\{P_i^k\}_{k=1}^l$ such that $P_i^1 = P_i$, $P_i^l = P_i'$ and $P_i^k \sim^A P_i^{k+1}$, $k = 1, \ldots, l-1$. Accordingly, we say that a domain is **connected** if every pair of distinct preferences is connected via an Ad-path in the domain.

The Interior Property requires that given two distinct preferences with *the same peak*, there is an Ad-path connecting them such that every preference on the path *shares that peak*.

[14] A pair of alternatives a, b is said to be *linked*, denoted $a \sim b$, if there exist $P_i, P_i' \in \mathbb{D}$ such that $r_1(P_i) = r_2(P_i') = a$ and $r_2(P_i) = r_1(P_i') = b$. A domain $\mathbb{D}$ is **linked** if the alternative set can be labeled as $A = \{a_1, a_2, \ldots, a_m\}$ such that (i) $a_1 \sim a_2$ and (ii) for every $3 \leq k \leq m$, $a_k \sim a_s$ and $a_k \sim a_t$ for some $1 \leq s < t \leq k-1$. In Example 1, $a_1 \sim a_2$; $a_3 \sim a_1$, $a_3 \sim a_2$; $a_4 \sim a_2$, $a_4 \sim a_3$; $a_5 \sim a_1$ and $a_5 \sim a_4$.

[15] A DSCF $f : \mathbb{D}^N \to \Delta(A)$ is a **dictatorship** if there exists $i \in I$ such that for all $P \in \mathbb{D}^N$, $f(P) = e_{r_1(P_i)}$.

Definition 1 *Domain $\mathbb{D}$ satisfies the **Interior Property** if for all $a \in A$ and distinct $P_i, P_i' \in \mathbb{D}^a$, there exists an Ad-path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}^a$ connecting P_i and P_i'.*

Note that the Interior Property does not hold for the domain of Example 1, e.g., preferences P_7, P_8 and P_9 form the sub-domain with peak a_3, but no pair of them is adjacent.

We next present an example of a non-tops-only domain that satisfies the Interior Property.

Example 2 Let $A = \{a, b, c\}$ and consider the domain $\mathbb{D}$, containing three preferences, specified below.

P_1	P_2	P_3
a	b	b
c	a	c
b	c	a

Table 2: Domain $\mathbb{D}$

Evidently, domain $\mathbb{D}$ satisfies the Interior Property, i.e., $P_2 \sim^A P_3$. Moreover, domain $\mathbb{D}$ admits the following two-voter unanimous and strategy-proof DSCF:

(i) $f(P_1, P_1) = e_a$ and $f(P_i, P_j) = e_b$ for all $P_i, P_j \in \{P_2, P_3\}$.

(ii) $f(P_1, P_2) = f(P_2, P_1) = e_a$ and $f(P_1, P_3) = f(P_3, P_1) = e_c$,

Since social lotteries vary at profiles (P_1, P_2) and (P_1, P_3) in favour of the second voter's preference over a and c, the DSCF f does not satisfy the tops-only property. $\square$

To ensure that a domain is a tops-only domain, the Interior Property has to be augmented by a condition imposed on preferences with *distinct peaks,* so that *all* sub-domains (within each of which all preferences have the same peak) are in a sense "well-organized" with respect to each other. We refer to this condition as the Exterior Property. As a first step towards describing the Exterior Property, we establish the notion of isolation. Given distinct $P_i, P_i' \in \mathbb{D}$, alternatives $x, y \in A$ are **isolated** in (P_i, P_i') if there exists $1 \leq k \leq m-1$ such that

(i) $B^k(P_i) = B^k(P_i')$,

(ii) either $x \in B^k(P_i)$ and $y \notin B^k(P_i)$, or $x \notin B^k(P_i)$ and $y \in B^k(P_i)$.

In an isolation, the two sets of top-k ranked alternatives in P_i and P_i' are identical, include one alternative in $\{x, y\}$ and exclude the other. Note that if x and y are isolated in (P_i, P_i'), the relative rankings of x and y are identical in P_i and P_i', i.e., $[xP_iy] \Leftrightarrow [xP_i'y]$.

Remark 1 An isolation is independent of an adjacency since the preferences in the definition of an isolation are not necessarily adjacent. Given $P_i, P_i' \in \mathbb{D}$ with $P_i \sim^A P_i'$, two alternatives $x, y \in A$ are isolated in (P_i, P_i') if and only if the relative rankings of x and y are identical in P_i and P_i'. $\square$

Now, we present a result (Lemma 1 below) which plays an important role in endogenously establishing the tops-only property. For simplicity, consider a two-voter strategy-proof RSCF. Voter i has two adjacent preferences P_i and P_i' where the local switching pair is x and y. Voter j also has two preferences P_j and P_j' where x and y are isolated. In other words, voter i disagrees exactly on the relative ranking of x and y while voter j happens to agree on the relative ranking of x and y in the sense of an isolation (for instance, one subset of alternatives containing x is considered better than the complementary subset containing y). Lemma 1 asserts that if the social lottery does not vary according to voter i's report on her preference when voter j reports P_j, i.e., $\varphi(P_i, P_j) = \varphi(P_i', P_j)$, then the social lottery should not be affected by voter i's report in the situation voter j reports P_j' either, i.e., $\varphi(P_i, P_j') = \varphi(P_i', P_j')$. Note that this result is independent of the Interior Property and is generated simply by the combination of an adjacency and an isolation.

Lemma **1** *Let $\varphi : \mathbb{D}^N \to \Delta(A)$ be a strategy-proof RSCF. Given $P_i, P_i' \in \mathbb{D}$ with $P_i \sim^A P_i'$, assume $xP_i!y$ and $yP_i'!x$. Given $P_j, P_j' \in \mathbb{D}$, if x and y are isolated in (P_j, P_j'), then for all $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, we have*

$$\big[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})\big] \Rightarrow \big[\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})\big].$$

Proof: Given P_i and P_i', strategy-proofness implies

Statement (1) $\varphi_z(P_i, P_j', P_{-\{i,j\}}) = \varphi_z(P_i', P_j', P_{-\{i,j\}})$ for all $z \notin \{x, y\}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$.[16]

Therefore, to verify $\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})$, it suffices to show either $\varphi_x(P_i, P_j', P_{-\{i,j\}}) = \varphi_x(P_i', P_j', P_{-\{i,j\}})$ or $\varphi_y(P_i, P_j', P_{-\{i,j\}}) = \varphi_y(P_i', P_j', P_{-\{i,j\}})$.

Next, since x and y are isolated in (P_j, P_j'), there exists $1 \leq t \leq m-1$ such that either $x \in B^t(P_j) = B^t(P_j')$ and $y \notin B^t(P_j) = B^t(P_j')$, or $x \notin B^t(P_j) = B^t(P_j')$ and $y \in B^t(P_j) = B^t(P_j')$. We assume $x \in B^t(P_j) = B^t(P_j')$ and $y \notin B^t(P_j) = B^t(P_j')$. The verification related to the other case is symmetric and we hence omit it. Consequently, strategy-proofness implies that for all $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$,

Statement (2) $\sum_{z \in B^t(P_j)} \varphi_z(P_i, P_j, P_{-\{i,j\}}) = \sum_{z \in B^t(P_j')} \varphi_z(P_i, P_j', P_{-\{i,j\}})$;

Statement (3) $\sum_{z \in B^t(P_j)} \varphi_z(P_i', P_j, P_{-\{i,j\}}) = \sum_{z \in B^t(P_j')} \varphi_z(P_i', P_j', P_{-\{i,j\}})$.

[16]Lemma 2 of Gibbard (1977) shows that if $B^k(P_i) = B^k(P_i') \equiv B$, then strategy-proofness implies $\sum_{a \in B} \varphi_a(P_i, P_{-i}) = \sum_{a \in B} \varphi_a(P_i', P_{-i})$ for all $P_{-i} \in \mathbb{D}^{N-1}$. The verification of this lemma can be adapted to verify Statement (1) and Statements (2) and (3) below as well.

Finally we have

$$\begin{aligned}
\varphi_x(P_i, P'_j, P_{-\{i,j\}}) &= \sum_{z\in B^t(P'_j)} \varphi_z(P_i, P'_j, P_{-\{i,j\}}) - \sum_{z\in B^t(P'_j)\backslash\{x\}} \varphi_z(P_i, P'_j, P_{-\{i,j\}}) \\
&= \sum_{z\in B^t(P_j)} \varphi_z(P_i, P_j, P_{-\{i,j\}}) - \sum_{z\in B^t(P'_j)\backslash\{x\}} \varphi_z(P_i, P'_j, P_{-\{i,j\}}) \quad \text{by Statement (2)} \\
&= \sum_{z\in B^t(P_j)} \varphi_z(P'_i, P_j, P_{-\{i,j\}}) - \sum_{z\in B^t(P'_j)\backslash\{x\}} \varphi_z(P_i, P'_j, P_{-\{i,j\}}) \quad \text{by the hypothesis of Lemma 1} \\
&= \sum_{z\in B^t(P_j)} \varphi_z(P'_i, P_j, P_{-\{i,j\}}) - \sum_{z\in B^t(P'_j)\backslash\{x\}} \varphi_z(P'_i, P'_j, P_{-\{i,j\}}) \quad \text{by Statement (1)} \\
&= \sum_{z\in B^t(P'_j)} \varphi_z(P'_i, P'_j, P_{-\{i,j\}}) - \sum_{z\in B^t(P'_j)\backslash\{x\}} \varphi_z(P'_i, P'_j, P_{-\{i,j\}}) \quad \text{by Statement (3)} \\
&= \varphi_x(P'_i, P'_j, P_{-\{i,j\}}).
\end{aligned}$$

Therefore, $\varphi(P_i, P'_j, P_{-\{i,j\}}) = \varphi(P'_i, P'_j, P_{-\{i,j\}})$. ■

Note that in Example 2, aP_1c and aP_2c, but a and c are not isolated in (P_1, P_2). Next we slightly modify Example 2 to restore the isolation of a and c in the two preferences, and then show how Lemma 1 forces a two-voter unanimous and strategy-proof RSCF to satisfy the tops-only property.

Example 2 [continued] We retain preferences P_2 and P_3 in Example 2 (and so the Interior Property continues to hold), and replace preference P_1 in Example 2 by $\bar{P}_1 : a\,b\,c$. Thus, we have the domain $\bar{\mathbb{D}} = \{\bar{P}_1, P_2, P_3\}$. To show that every two-voter unanimous and strategy-proof RSCF $\varphi : \bar{\mathbb{D}}^2 \to \Delta(A)$ satisfies the tops-only property, it suffices to show $\varphi(\bar{P}_1, P_2) = \varphi(\bar{P}_1, P_3)$ (symmetrically, $\varphi(P_2, \bar{P}_1) = \varphi(P_3, \bar{P}_1)$). First, we have $P_2 \sim^A P_3$, $aP_2!c$ and $cP_3!a$. Second, a and c are isolated in $(\bar{P}_1, P_2)$. Third, since both P_2 and P_3 have peak b, unanimity implies $\varphi(P_2, P_2) = \varphi(P_2, P_3)$. Thus, all hypotheses of Lemma 1 are met, and hence we assert $\varphi(\bar{P}_1, P_2) = \varphi(\bar{P}_1, P_3)$, as required. A similar argument applies to the general case of an arbitrary number of voters. Therefore, $\bar{\mathbb{D}}$ is a tops-only domain. □

We next generalize the notion of isolation between two preferences to the notion of isolation along a path of preferences. This will allow us to extend the applicability of Lemma 1 to a broader class of domains than the rudimentary one considered in the example above, as Lemma 1 can then be applied to every successive pair of preferences along the path. Given distinct $P_i, P'_i \in \mathbb{D}$ and $x, y \in A$, let $\{P_i^k\}_{k=1}^l$ be a sequence of preferences (not necessarily an Ad-path) such that $P_i^1 = P_i$, $P_i^l = P'_i$, and x and y are isolated in (P_i^k, P_i^{k+1}), $k = 1, \ldots, l-1$. Then, $\{P_i^k\}_{k=1}^l$ is referred to as an (x, y)-Is-path connecting P_i and P'_i. Note that either x is preferred to y in every preference of the (x, y)-Is-path, or vice versa.

Finally we define the Exterior Property to specify the relation between preferences with distinct peaks. The Exterior Property will say that fixing a pair of preferences with distinct peaks and a pair of alternatives with the same relative ranking across these two preferences, we can construct an Is-path with respect to this pair of alternatives to connect the pair of fixed preferences.

Definition 2 *Domain $\mathbb{D}$ satisfies the* ***Exterior Property*** *if given $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P_i')$ and $x, y \in A$ with xP_iy and $xP_i'y$, there exists an (x, y)-Is-path connecting P_i and P_i'.*

We now state our main result.

Theorem 1 *A domain satisfying the Interior Property and the Exterior Property is a tops-only domain.*

Proof: Let domain $\mathbb{D}$ satisfy the Interior Property and the Exterior Property.

If $N = 1$, unanimity implies the tops-only property. Now, we provide an induction argument on the number of voters.

Induction Hypothesis: Given $N \geq 2$, for all $1 \leq n < N$, every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ satisfies the tops-only property.

Given a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$, we show that φ satisfies the tops-only property. It is easy to verify that φ satisfies the tops-only property if and only if for all $i \in I$, $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = r_1(P_i')$ and $P_{-i} \in \mathbb{D}^{N-1}$, $\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})$. Given distinct $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = r_1(P_i') \equiv a$, the Interior Property implies that there exists an Ad-path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}^a$ connecting P_i and P_i'. Then, it suffices to show that for each $1 \leq k \leq l-1$, $\varphi(P_i^k, P_{-i}) = \varphi(P_i^{k+1}, P_{-i})$ for all $P_{-i} \in \mathbb{D}^{N-1}$. Equivalently, we show that for all $i \in I$, $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = r_1(P_i')$ and $P_i \sim^A P_i'$, and $P_{-i} \in \mathbb{D}^{N-1}$, $\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})$.

Fixing two voters $i, j \in I$, we induce a function $\psi : \mathbb{D}^{N-1} \to \Delta(A)$ such that $\psi(P_i, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$ for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. Evidently, ψ is a well-defined RSCF satisfying unanimity and strategy-proofness.[17] Hence the induction hypothesis implies that ψ satisfies the tops-only property. Thus, for all $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = r_1(P_i')$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, we have $\varphi(P_i, P_i, P_{-\{i,j\}}) = \psi(P_i, P_{-\{i,j\}}) = \psi(P_i', P_{-\{i,j\}}) = \varphi(P_i', P_i', P_{-\{i,j\}})$.

Fixing $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = r_1(P_i')$ and $P_i \sim^A P_i'$, we assume $xP_i!y$ and $yP_i'!x$. Given $P_j \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, we prove $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$.

Claim 1: If $r_1(P_j) = r_1(P_i)$, then $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$.

Proof of Claim 1: First, by strategy-proofness, we have that for all $t = 1, \ldots, m$,

$$\sum_{x \in B^t(P_i)} \varphi_x(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{x \in B^t(P_i)} \varphi_x(P_i, P_j, P_{-\{i,j\}}) \leq \sum_{x \in B^t(P_i)} \varphi_x(P_i, P_i, P_{-\{i,j\}})$$

$$\sum_{x \in B^t(P_i')} \varphi_x(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{x \in B^t(P_i')} \varphi_x(P_i', P_j, P_{-\{i,j\}}) \leq \sum_{x \in B^t(P_i')} \varphi_x(P_i', P_i', P_{-\{i,j\}})$$

[17]The proof of Lemma 3 of Sen (2011) provides a clear verification. We omit the details here.

Moreover, since $r_1(P_j) = r_1(P_i) = r_1(P_i')$, we have $\varphi(P_j, P_j, P_{-\{i,j\}}) = \psi(P_j, P_{-\{i,j\}}) = \psi(P_i, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$ and $\varphi(P_j, P_j, P_{-\{i,j\}}) = \psi(P_j, P_{-\{i,j\}}) = \psi(P_i', P_{-\{i,j\}}) = \varphi(P_i', P_i', P_{-\{i,j\}})$. Consequently, for all $t = 1, \ldots, m$, we have $\sum_{x \in B^t(P_i)} \varphi_x(P_i, P_j, P_{-\{i,j\}}) = \sum_{x \in B^t(P_i)} \varphi_x(P_i, P_i, P_{-\{i,j\}})$ and $\sum_{x \in B^t(P_i')} \varphi_x(P_i', P_j, P_{-\{i,j\}}) = \sum_{x \in B^t(P_i')} \varphi_x(P_i', P_i', P_{-\{i,j\}})$. Therefore, $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$ and $\varphi(P_i', P_j, P_{-\{i,j\}}) = \varphi(P_i', P_i', P_{-\{i,j\}})$, which imply $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$. □

Next, assume $r_1(P_j) \neq r_1(P_i)$. Evidently, either xP_jy or yP_jx. We assume xP_jy. The verification related to yP_jx is symmetric and we hence omit it. Since xP_iy and xP_jy, the Exterior Property implies that there exists an (x, y)-Is-path $\{P_j^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting P_i and P_j. First, since $P_j^1 = P_i$, Claim 1 implies $\varphi(P_i, P_j^1, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}}) = \varphi(P_i', P_i, P_{-\{i,j\}}) = \varphi(P_i', P_j^1, P_{-\{i,j\}})$. Next, following the Is-path $\{P_j^k\}_{k=1}^l$, since $P_i \sim^A P_i'$; $xP_i!y$, $yP_i'!x$; and x and y are isolated in (P_j^k, P_j^{k+1}), $k = 1, \ldots, l-1$, we can repeatedly apply Lemma 1, which eventually implies $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$. This completes the verification of the induction hypothesis and the proof of the Theorem. ■

4 Applications

In this section, we first study two important classes of restricted domains in the literature: Connected domains in the sense of Sato (2013), and the multi-dimensional single-peaked domain introduced by Barberà et al. (1993). We show that these two classes of domains satisfy the Interior Property and the Exterior Property, and are therefore tops-only domains. After establishing the tops-only property for all unanimous and strategy-proof RSCFs over the multi-dimensional single-peaked domain, we further demonstrate that every ex-post efficient and strategy-proof RSCF defined on this domain is a random dictatorship.

4.1 Restricted connected domains

Sato (2013) introduced the property of *weak non-restoration* which is imposed on the class of connected domains and is satisfied by many voting domains in the literature.

Definition **3** *Domain $\mathbb{D}$ is* ***connected with weak non-restoration*** *if given $P_i, P_i' \in \mathbb{D}$ and $x, y \in A$, there exists an Ad-path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting P_i and P_i', and moreover, the Ad-path satisfies the non-restoration property with respect to x and y, i.e.,*

$$[xP_i^ky \text{ and } yP_i^{k+1}x \text{ for some } 1 \leq k \leq l-1] \Rightarrow [xP_i^ty,\ 1 \leq t \leq k, \text{ and } yP_i^{t'}x,\ k+1 \leq t' \leq l].$$

Evidently, connectedness with weak non-restoration implies the Exterior Property. However, the inverse argument does not hold since the Exterior Property only considers two preferences with distinct peaks, and the related Is-path need not be an Ad-path.

In this paper, a domain satisfying the Interior Property and connectedness with weak non-restoration is referred to as a **restricted connected domain**.[18] By the Theorem, every unanimous and strategy-proof RSCF on a restricted connected domain must satisfy the tops-only property.

PROPOSITION 1 *A restricted connected domain is a tops-only domain.*

REMARK 2 The complete domain, the single-peaked domain (Moulin, 1980; Demange, 1982), the single-dipped domain (Barberà et al., 2012) and single-crossing domains (Saporiti, 2009; Carroll, 2012) are all restricted connected domains, and hence tops-only domains. □

4.2 The multi-dimensional single-peaked domain

In many political and economic settings, the restriction of multi-dimensional single-peakedness arises naturally. For instance, in a political election, each candidate can be described as a combination of positions on various political issues, e.g., expenditure on education, health, etc. Normally, the preference of a voter over all candidates is formulated according to the criteria of "closeness", i.e., a candidate with positions "closer" to the voter's ideal political attitude is preferred to another candidate with "more distant" positions. Hence, multi-dimensional single-peakedness is embedded in the formulation of a voter's preference.

To study multi-dimensional single-peaked preferences, we assume that the alternative set can be represented as a Cartesian product of a *finite* number of sets each of which contains a *finite* cardinality of elements, i.e., $A = \times_{s\in M} A^s$ where $M = \{1, 2, \ldots, q\}$ is finite with $q \geq 2$, and A^s, referred to as a **component set**, is finite with $|A^s| \geq 2$ for each $s \in M$. An element in a component set A^s can be denoted as a^s. A q-tuple $a \equiv (a^1, a^2, \ldots, a^q) \equiv (a^s)_{s\in M}$ describes an alternative by specifying the element in each component set.[19] Given a *nonempty strict* subset $S \subseteq M$, let $A^S \equiv \times_{s\in S} A^s$ denote the Cartesian product of all component sets A^s, $s \in S$, and $a^S \equiv (a^s)_{s\in S} \in A^S$ denote a combination of elements in A^s, $s \in S$. Similarly, let $A^{-S} \equiv \times_{s\notin S} A^s$ and $a^{-S} \equiv (a^s)_{s\notin S} \in A^{-S}$. Accordingly, we can write an alternative

[18] Sato (2013) shows that connectedness with weak non-restoration is necessary for the equivalence in DSCFs of strategy-proofness and *adjacent manipulation-proofness,* a weakening of strategy-proofness where only a manipulation via a preference adjacent to the sincere one is required to be non-profitable. We note that the Interior Property is not implied by connectedness with weak non-restoration (for instance, one can refer to Example 3.2 of Sato (2013)). For our purposes, it is appropriate to combine the Interior Property with connectedness with weak non-restoration to formulate the class of restricted connected domains.

[19] Recall the political election example above. All four candidates can be represented by a Cartesian product of two sets of political issues, A^1: expenditure on education, and A^2: expenditure on health; and each component set contains two elements 0 and 1 where 0 represents "low" and 1 represents "high". Accordingly, for instance, $(0, 1)$ represents a candidate who favors the low position on education expenditure and the high position on health expenditure .

$a \equiv (a^s, a^{-s}) \equiv (a^S, a^{-S})$. For notational convenience, given $s \in M$ and $a^s \in A^s$, let $(a^s, A^{-s}) = \{x \in A | x^s = a^s\}$.

We assume moreover that for each $s \in M$, all elements in A^s are located on a **tree**, denoted $G(A^s)$.[20] Let $\langle a^s, b^s \rangle$ denote the *unique* path between a^s and b^s in $G(A^s)$.[21] Combining all trees $G(A^s)$, $s \in M$, we generate a **product of trees** $\times_{s \in M} G(A^s)$ where the set of vertices is A, and two distinct alternatives a and b constitute an edge if and only if $a^{-s} = b^{-s}$ for some $s \in M$, and a^s and b^s constitute an edge in $G(A^s)$. Given $a, b \in A$, let $\langle a, b \rangle = \{x \in A | x^s \in \langle a^s, b^s \rangle$ for each $s \in M\}$ denote the "minimal box" containing all alternatives located between a and b in each dimension.

Definition 4 *Given a product of trees $\times_{s \in M} G(A^s)$, a preference P_i is **multi-dimensional single-peaked** on $\times_{s \in M} G(A^s)$ if for all $a, b \in A$, $\big[a \in \langle r_1(P_i), b \rangle \backslash \{b\}\big] \Rightarrow [a P_i b]$.*

Given a product of trees $\times_{s \in M} G(A^s)$, let $\mathbb{D}_{MSP}$ denote *the* multi-dimensional single-peaked domain on $\times_{s \in M} G(A^s)$ containing *all* admissible preferences.[22]

Remark 3 Our formulation of multi-dimensional single-peakedness is one where all elements in each component set are located on a tree. This generalizes the earlier notion introduced by Barberà et al. (1993) where all elements in each component set must be arranged on a line. □

The multi-dimensional single-peaked domain satisfies both the Interior Property and the Exterior Property. We provide a simple example to illustrate.

Example 3 Let $A \equiv A^1 \times A^2 = \{0, 1\} \times \{0, 1\}$. The product of lines $G(A^1) \times G(A^2)$ and domain $\mathbb{D}_{MSP}$ are specified in the following diagram and table respectively.

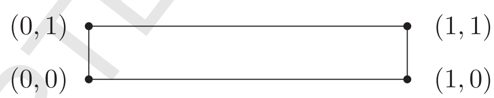

Figure 1: The product of lines $G(A^1) \times G(A^2)$

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$(0,0)$	$(0,0)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(1,1)$	$(1,1)$
$(1,0)$	$(0,1)$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(1,0)$	$(0,1)$
$(0,1)$	$(1,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(0,0)$	$(0,1)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,0)$	$(0,0)$

Table 3: Domain $\mathbb{D}_{MSP}$

[20] A *graph* is a combination of vertices and edges. A *path* in a graph is a sequence of vertices where each contiguous pair of vertices forms an edge. A *tree* is a particular graph where between each pair of vertices, there exists a unique path.

[21] If $a^s = b^s$, $\langle a^s, b^s \rangle = \{a^s\}$ is a singleton set.

[22] Henceforth, any strict subset of the multi-dimensional single-peaked domain is just referred to as "a multi-dimensional single-peaked domain".

The Interior Property is satisfied since $P_1 \sim^A P_2$, $P_3 \sim^A P_4$, $P_5 \sim^A P_6$ and $P_7 \sim^A P_8$. We use an instance to illustrate how the requirement of the Exterior Property is met. Note that $(1,0)P_1(0,1)$ and $(1,0)P_7(0,1)$. Correspondingly, $\{P_1, P_3, P_4, P_7\}$ is a $\big((1,0),(0,1)\big)$-Is-path connecting P_1 and P_7, i.e., all $B^2(P_1) = B^2(P_3) = \{(0,0),(1,0)\}$, $B^1(P_3) = B^1(P_4) = \{(1,0)\}$ and $B^2(P_4) = B^2(P_7) = \{(1,0),(1,1)\}$ include $(1,0)$ and exclude $(0,1)$. □

Now, we state the formal result.

Proposition 2 *The multi-dimensional single-peaked domain satisfies the Interior Property and the Exterior Property, and is hence a tops-only domain.*

The proof of Proposition 2 is available in Appendix B.

4.3 A characterization of strategy-proof RSCFs on the multi-dimensional single-peaked domain

Deterministic strategy-proof voting rules have been widely explored over the multi-dimensional single-peaked domain, e.g., voting by committee (Barberà et al., 1991; Barberà et al., 2005), generalized median voter rules (Barberà et al., 1993, 1997), decomposable rules (Le Breton and Sen, 1999) and voting by issues (Nehring and Puppe, 2007). In the randomized setting, Dutta et al. (2002) show that every unanimous and strategy-proof RSCF is a *random dictatorship* when preferences are single-peaked, strictly convex and continuous on a convex subset of the Euclidean space.

In characterizing strategy-proof DSCFs and RSCFs in the literature mentioned above, the tops-only property is always established in advance, and this simplifies the rest of the characterization significantly. In particular, in the deterministic setting, the tops-only property is used to establish *the decomposability property* (see Barberà et al., 1993; Le Breton and Sen, 1999).[23] The subsequent characterization can then be simplified to a consideration of each component set. In the randomized environment, the characterization is significantly more subtle since a unanimous and strategy-proof RSCF usually fails *the independent decomposability property* (see Chatterji et al., 2012).[24] Consequently, any characterization must be directly derived from the tops-only property, endogenously established in Proposition 2.

[23]For notational convenience we write a DSCF as $f : \mathbb{D}^N \to A$. A tops-only DSCF $f : \mathbb{D}^N \to A$ satisfies the **decomposability property** if there exists a marginal voting rule $f^s : [A^s]^N \to A^s$ for each $s \in M$, such that for every $P \equiv (P_1, \ldots, P_N) \in \mathbb{D}^N$, say $r_1(P_i) \equiv a_i \equiv (a_i^s)_{s\in M}$, $i \in I$, the social outcome $f(P)$ is simply a combination of all marginal outcomes $f^s(a_1^s, \ldots, a_N^s)$, $s \in M$, i.e., $f(P) = \big(f^s(a_1^s, \ldots, a_N^s)\big)_{s\in M}$.

[24]A tops-only RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies the **independent decomposability property** if there exists a marginal random voting rule $\varphi^s : [A^s]^N \to \Delta(A^s)$ for each $s \in M$, such that for every $P \equiv (P_1, \ldots, P_N) \in \mathbb{D}^N$, say $r_1(P_i) \equiv a_i \equiv (a_i^s)_{s\in M}$, $i \in I$, the probability assigned to $x \equiv (x^s)_{s\in M} \in A$ equals to the product of all marginal probabilities $\varphi^s_{x^s}(a_1^s, \ldots, a_N^s)$, $s \in M$, i.e., $\varphi_x(P) = \Pi_{s\in M}\varphi^s_{x^s}(a_1^s, \ldots, a_N^s)$.

In this section, we demonstrate that every ex-post efficient and strategy-proof RSCF over the multi-dimensional single-peaked domain is a *random dictatorship*. This generalizes the impossibility result of Barberà et al. (1991) to the randomized environment.[25]

An RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a **random dictatorship** if there exists a sequence $[\varepsilon_i]_{i\in I} \in \mathbb{R}^N_+$ with $\sum_{i\in I} \varepsilon_i = 1$ such that for all $P \in \mathbb{D}^N$, $\varphi(P) = \sum_{i\in I} \varepsilon_i e_{r_1(P_i)}$. It is evident that a random dictatorship satisfies ex-post efficiency, the tops-only property and strategy-proofness.

Proposition **3** *Assume $|M| \geq 3$. An ex-post efficient RSCF over $\mathbb{D}_{MSP}$ is strategy-proof if and only if it is a random dictatorship.*

The proof of Proposition 3 is available in Appendix C. The proof relies heavily on the tops-only property. For instance, given a preference profile $P \equiv (P_1, P_2) \in \mathbb{D}^2_{MSP}$ where the peaks of two preferences disagree on at least two components, fixing an arbitrary alternative a other than the two peaks, we pin down the probability assigned to a under profile P by the following method. We construct another profile $\bar{P} \equiv (\bar{P}_1, \bar{P}_2) \in \mathbb{D}^2_{MSP}$ that is *top-equivalent* to P, i.e., $r_1(\bar{P}_1) = r_1(P_1)$, and $r_1(\bar{P}_2) = r_1(P_2)$, such that a is Pareto dominated in $\bar{P}$. Then, ex-post efficiency implies that a gets probability zero under $\bar{P}$, and finally the tops-only property implies that a also receives probability zero under profile P.

Remark **4** The random dictatorship characterization result in Proposition 3 is an instance of the "extreme-point property", i.e., every strategy-proof RSCF satisfying some additional axiom (e.g., unanimity or ex-post efficiency) is a convex combination of the counterpart DSCFs. The extreme-point property is valid over several voting domains, e.g., the complete domain (Gibbard, 1977; Sen, 2011), the binary domain (Picot and Sen, 2012), the single-peaked domain (Ehlers et al., 2002; Peters et al., 2014; Pycia and Ünver, 2015), the lexicographically separable domain (Chatterji et al., 2012) and random dictatorship domains (Chatterji et al., 2014). On the one hand, the tops-only property is always the key step in establishing the extreme-point property (see all of the literature mentioned above), while on the other hand, the violation of the tops-only property is used to illustrate the failure of the extreme-point property (see Example 1 above). We conjecture that the extreme-point property remains valid on the multi-dimensional single-peaked domain when ex-post efficiency is weakened to unanimity.[26] □

[25]The separable domain introduced by Barberà et al. (1991) can be reinterpreted in the Cartesian product setting and viewed as a particular formulation of the multi-dimensional single-peaked domain where each component set contains exactly two elements. Theorem 4 of Barberà et al. (1991) implies that every efficient DSCF is strategy-proof if and only if it is a dictatorship, provided that the alternative set can be decomposed in at least three dimensions.

[26]The tops-only domain result established in Proposition 2 could facilitate the resolution of this long-term conjecture since on the multi-dimensional single-peaked domain, the study of a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N_{MSP} \to \Delta(A)$ has been significantly, but without loss of generality, simplified to the investigation of the corresponding random voting rule $\varphi : A^N \to \Delta(A)$.

5 Discussion

In this section we discuss related literature, comment on the necessity of our condition and provide some final remarks.

5.1 Relation to the literature

In DSCFs, general sufficient conditions which are used to establish tops-only domains usually imply that the domains are minimally rich (see for instance Weymark, 2008; Nehring and Puppe, 2007; Chatterji and Sen, 2011). Our condition is independent of minimal richness, and therefore includes some non-minimally rich domains, e.g., the single-dipped domain (Barberà et al., 2012) and maximal single-crossing domains (Saporiti, 2009).

Chatterji and Sen (2011) also study two non-minimally rich domains: The domain of in-between preferences (Gravel et al., 2008) and Kelly's domain (Kelly, 1989), and show that they are tops-only domains for DSCFs. These two domains do not satisfy our condition directly. However, observe that for instance, in the domain of in-between preferences, an alternative which is never the peak of any preference is *irrelevant*, i.e., it is never a social choice under any preference profile in any unanimous and strategy-proof DSCF (a similar argument holds in Kelly's domain). After inducing new preferences by removing all irrelevant alternatives, the refined domains satisfy the Interior Property and the Exterior Property.

In the class of minimally rich domains, Chatterji and Sen (2011) propose two general sufficient conditions, *Property T* and *Property T**, for tops-only domains for DSCFs.[27] All *commonly studied* minimally rich restricted domains that satisfy our condition also satisfy Property T. However we have not been able to prove that our condition with minimal richness implies Property T. More importantly, we observe that Property T is no longer sufficient for guaranteeing the tops-only property in RSCFs; domain $\mathbb{D}$ of Example 1 satisfies Property T but admits a unanimous and strategy-proof RSCF violating the tops-only property.[28] We note that while our condition covers some domains, like maximal single-crossing domains (Saporiti, 2009) that are excluded by Property T*, we have not been able to ascertain whether Property T* remains sufficient for tops-only domains for RSCFs.

[27] Property T can only be applied to establish the tops-only property in every two-voter unanimous and strategy-proof DSCF. Recall the notion of linkedness between two alternatives in footnote 14. A domain $\mathbb{D}$ satisfies **Property T** if for every $P_i \in \mathbb{D}$ and $a \in A\backslash\{r_1(P_i)\}$, there exists $b \in A$ such that bP_ia and $b \sim a$. Property T* is more sophisticated and is sufficient for tops-only domains for DSCFs for an arbitrary number of voters. The formal definition of Property T* can be found in Definition 9 of Chatterji and Sen (2011).

[28] Recall the notion of linkedness between two alternatives in footnote 14. To verify Property T in domain $\mathbb{D}$ of Example 1, consider for instance preference P_1, where we have $a_1P_1a_2$, $a_1 \sim a_2$; $a_2P_1a_3$, $a_2 \sim a_3$; $a_1P_1a_5$, $a_1 \sim a_5$; $a_3P_1a_4$ and $a_3 \sim a_4$.

5.2 Necessity

We observe that the Interior Property and the Exterior Property are *not* necessary for tops-only domains. This is not altogether surprising as every *random dictatorship domain* ensures the tops-only property.[29] While the complete domain is an instance of a random dictatorship domain that satisfies the Interior Property and the Exterior Property, we can use Theorem 3 of Chatterji et al. (2014) to construct a random dictatorship domain violating both the Interior Property and the Exterior Property and where the tops-only property prevails via a random dictatorship characterization result (see Example 4 below).

Example **4** Let $A = \{a_1, a_2, a_3, a_4\}$ and consider the domain $\mathbb{D}$, containing ten preferences, specified below.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
a_1	a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_4	a_4
a_2	a_3	a_1	a_3	a_4	a_1	a_2	a_4	a_2	a_3
a_3	a_2	a_3	a_1	a_1	a_2	a_1	a_2	a_1	a_1
a_4	a_4	a_4	a_4	a_3	a_4	a_4	a_1	a_3	a_2

Table 4: Domain $\mathbb{D}$

First, domain $\mathbb{D}$ violates the Interior Property, e.g., $\mathbb{D}^{a_2} = \{P_3, P_4, P_5\}$, but P_5 is not adjacent to either P_3 or P_4. Second, domain $\mathbb{D}$ violates the Exterior Property, e.g., there exists no (a_1, a_3)-Is-path connecting P_3 and P_9.[30] However, domain $\mathbb{D}$ is linked (recall footnote 14) and satisfies *Condition H* of Chatterji et al. (2014) which implies that $\mathbb{D}$ is a random dictatorship domain, and hence a tops-only domain.[31] □

5.3 Final remarks

In Appendix D.1 we briefly discuss the domain of separable preferences (Le Breton and Sen, 1999), while in Appendix D.2 we show that the Exterior Property can be replaced by a weaker version in our Theorem. Finally, we turn to some issues that remain unresolved. While Example 2 is an instance of a non-tops-only domain that satisfies the Interior Property and violates the Exterior Property, we have been unable to construct an example of a non-tops-only domain which violates the Interior Property and satisfies the Exterior Property.

[29] A domain is a **random dictatorship domain** if every unanimous and strategy-proof RSCF is a random dictatorship. Characterizing the necessary and sufficient conditions for random dictatorship domains is an important open question in the literature.

[30] Note that a_1 is ranked above a_3 in all preferences $\{P_1, P_2, P_3, P_5, P_9\}$, while a_3 is preferred to a_1 in all other preferences. We cannot construct an (a_1, a_3)-Is-path connecting P_3 and P_9 in $\{P_1, P_2, P_3, P_5, P_9\}$.

[31] Recall the notion of linkedness between two alternatives in footnote 14. A domain $\mathbb{D}$ satisfies **Condition H** if there exists $x \in A$, referred to as a *hub*, such that $a \sim x$ for all $a \in A \backslash \{x\}$. Theorem 3 of Chatterji et al. (2014) shows that a linked domain satisfying Condition H is a random dictatorship domain.

Second, we have also been unable to establish that the Exterior Property is by itself sufficient for tops-only domains.

Appendix

A Strategy-proofness of RSCF φ in Example 1

RSCF φ follows three distinct functional forms according to preference profiles. Evidently, if both voters share the same peak of preferences, by unanimity, no one has the incentive to deviate. Next, it is easy to show that if two social lotteries, which are induced by truthtelling and misrepresentation respectively of some voter, are both generated by the same functional form, the one under truthtelling always stochastically dominates the other one according to the true preference. Therefore, we only need to consider possible manipulations where the corresponding social lotteries are generated by distinct functional forms. In these possible manipulations (16 situations specified below), we assert that probabilities are always transferred systematically from the preferred alternatives to less preferred alternatives according to the true preference, which thereby indicates the required stochastic dominance.

We first consider voter i's possible manipulations.[32]

1. In $(P_i, P_j) \leftrightarrow (P'_i, P_j)$, where $P_i \in \mathbb{D}^{a_1}$, $P'_i \in \{P_7, P_9\}$ and $P_j \in \mathbb{D}^{a_5}$,

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_1 P_i a_3}, \xrightarrow[1/4]{a_5 P_i a_4} \varphi(P'_i, P_j), \text{ and } \varphi(P'_i, P_j) \xrightarrow[1/4]{a_3 P'_i a_1}, \xrightarrow[1/4]{a_4 P'_i a_5} \varphi(P_i, P_j).$$ [33]

2. In $(P_i, P_j) \leftrightarrow (P'_i, P_j)$, where $P_i \in \mathbb{D}^{a_2} = \{P_4, P_5, P_6\}$, $P'_i \in \{P_7, P_9\}$ and $P_j \in \mathbb{D}^{a_5}$,

$$\begin{aligned}
&\varphi(P_i, P_j) \xrightarrow[1/4]{a_2 P_i a_3}, \xrightarrow[1/4]{a_2 P_i a_1}, \xrightarrow[1/4]{a_5 P_i a_4} \varphi(P'_i, P_j) && \text{if } P_i \in \{P_4, P_5\};\\
&\varphi(P_i, P_j) \xrightarrow[1/4]{a_2 P_i a_3}, \xrightarrow[1/4]{a_2 P_i a_4}, \xrightarrow[1/4]{a_5 P_i a_1} \varphi(P'_i, P_j) && \text{if } P_i = P_6;\\
&\varphi(P'_i, P_j) \xrightarrow[1/4]{a_3 P'_i a_2}, \xrightarrow[1/4]{a_1 P'_i a_2}, \xrightarrow[1/4]{a_4 P'_i a_5} \varphi(P_i, P_j) && \text{if } P'_i = P_7;\\
&\varphi(P'_i, P_j) \xrightarrow[1/4]{a_3 P'_i a_2}, \xrightarrow[1/4]{a_4 P'_i a_2}, \xrightarrow[1/4]{a_1 P'_i a_5} \varphi(P_i, P_j) && \text{if } P'_i = P_9.
\end{aligned}$$

3. In $(P_i, P_j) \leftrightarrow (P'_i, P_j)$ where $P_i \in \mathbb{D}^{a_2}$, $P'_i = P_8$ and $P_j \in \mathbb{D}^{a_5}$,

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_2 P_i a_3} \varphi(P'_i, P_j), \text{ and } \varphi(P'_i, P_j) \xrightarrow[1/4]{a_3 P'_i a_2} \varphi(P_i, P_j).$$

[32] The notation $(P_i, P_j) \to (P'_i, P_j)$ represents a possible manipulation of voter i at (P_i, P_j) via P'_i. The notation $(P_i, P_j) \leftrightarrow (P'_i, P_j)$ represents two possible manipulations of voter i: (i) at (P_i, P_j) via P'_i, and (ii) at (P'_i, P_j) via P_i.

[33] The notation $\varphi(P_i, P_j) \xrightarrow[1/4]{a_1 P_i a_3}, \xrightarrow[1/4]{a_5 P_i a_4} \varphi(P'_i, P_j)$ represents that (i) $a_1 P_i a_3$ and $a_5 P_i a_4$, and (ii) from $\varphi(P_i, P_j)$ to $\varphi(P'_i, P_j)$, probabilities $\frac{1}{4}$ and $\frac{1}{4}$ are transferred from a_1 to a_3, and from a_5 to a_4 respectively.

4. In $(P_i, P_j) \leftrightarrow (P_i', P_j)$, where $P_i \in \mathbb{D}^{a_4}$, $P_i' \in \{P_7, P_9\}$ and $P_j \in \mathbb{D}^{a_5}$,

$$\varphi(P_i,P_j) \xrightarrow[1/4]{a_4 P_i a_3}, \xrightarrow[1/4]{a_5 P_i a_1} \varphi(P_i',P_j), \text{ and } \varphi(P_i',P_j) \xrightarrow[1/4]{a_3 P_i' a_4}, \xrightarrow[1/4]{a_1 P_i' a_5} \varphi(P_i,P_j).$$

5. In $(P_i, P_j) \leftrightarrow (P_i', P_j)$ where $P_i \in \mathbb{D}^{a_1} \cup \mathbb{D}^{a_4}$, $P_i' = P_8$ and $P_j \in \mathbb{D}^{a_5}$,

$$\varphi(P_i,P_j) \xrightarrow[1/4]{r_1(P_i) P_i a_3}, \xrightarrow[1/4]{r_1(P_i) P_i a_2} \varphi(P_i',P_j), \text{ and } \varphi(P_i',P_j) \xrightarrow[1/4]{a_3 P_i' r_1(P_i)}, \xrightarrow[1/4]{a_2 P_i' r_1(P_i)} \varphi(P_i,P_j).$$

6. In $(P_i, P_j) \leftrightarrow (P_i', P_j)$ where $P_i \in \{P_7, P_9\}$, $P_i' = P_8$ and $P_j \in \mathbb{D}^{a_5}$,

$$\begin{aligned} &\varphi(P_i,P_j) \xrightarrow[1/4]{a_1 P_i a_2}, \xrightarrow[1/4]{a_4 P_i a_5} \varphi(P_i',P_j) && \text{if } P_i = P_7; \\ &\varphi(P_i,P_j) \xrightarrow[1/4]{a_4 P_i a_2}, \xrightarrow[1/4]{a_1 P_i a_5} \varphi(P_i',P_j) && \text{if } P_i = P_9; \\ &\varphi(P_i',P_j) \xrightarrow[1/4]{a_2 P_i' a_1}, \xrightarrow[1/4]{a_5 P_i' a_4} \varphi(P_i,P_j). \end{aligned}$$

7. In $(P_i, P_j) \rightarrow (P_i', P_j)$ where $P_i \in \{P_7, P_9\}$, $P_i' \in \mathbb{D}^{a_5}$ and $P_j \in \mathbb{D}^{a_5}$.

$$\varphi(P_i,P_j) \xrightarrow[1/4]{a_3 P_i a_5}, \xrightarrow[1/4]{a_1 P_i a_5}, \xrightarrow[1/4]{a_4 P_i a_5} \varphi(P_i',P_j).$$

8. In $(P_i, P_j) \rightarrow (P_i', P_j)$ where $P_i = P_8$, $P_i' \in \mathbb{D}^{a_5}$ and $P_j \in \mathbb{D}^{a_5}$,

$$\varphi(P_i,P_j) \xrightarrow[1/4]{a_2 P_i a_5}, \xrightarrow[1/4]{a_3 P_i a_5} \varphi(P_i',P_j).$$

Next, we consider voter j's possible manipulations.

9. In $(P_i, P_j) \leftrightarrow (P_i, P_j')$, where $P_i \in \{P_7, P_9\}$, $P_j \in \mathbb{D}^{a_1}$ and $P_j' \in \mathbb{D}^{a_5}$,

$$\varphi(P_i,P_j) \xrightarrow[1/4]{a_1 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_4} \varphi(P_i,P_j'), \text{ and } \varphi(P_i,P_j') \xrightarrow[1/4]{a_5 P_j' a_1}, \xrightarrow[1/4]{a_4 P_j' a_3} \varphi(P_i,P_j).$$

10. In $(P_i, P_j) \leftrightarrow (P_i, P_j')$ where $P_i = P_8$, $P_j \in \mathbb{D}^{a_1} = \{P_1, P_2, P_3\}$ and $P_j' \in \mathbb{D}^{a_5}$,

$$\begin{aligned} &\varphi(P_i,P_j) \xrightarrow[1/4]{a_1 P_j a_2}, \xrightarrow[1/4]{a_1 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_5} \varphi(P_i,P_j') && \text{if } P_j = P_1; \\ &\varphi(P_i,P_j) \xrightarrow[1/2]{a_1 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_2} \varphi(P_i,P_j') && \text{if } P_j \in \{P_2, P_3\}; \\ &\varphi(P_i,P_j') \xrightarrow[1/2]{a_5 P_j' a_1}, \xrightarrow[1/4]{a_2 P_j' a_3} \varphi(P_i,P_j). \end{aligned}$$

11. In $(P_i, P_j) \leftrightarrow (P_i, P_j')$, where $P_i \in \{P_7, P_9\}$, $P_j \in \mathbb{D}^{a_2}$ and $P_j' \in \mathbb{D}^{a_5}$,

$$\varphi(P_i,P_j) \xrightarrow[1/4]{a_2 P_j a_1}, \xrightarrow[1/4]{a_2 P_j a_4}, \xrightarrow[1/4]{a_3 P_j a_5} \varphi(P_i,P_j'), \text{ and } \varphi(P_i,P_j') \xrightarrow[1/4]{a_5 P_j' a_3}, \xrightarrow[1/4]{a_1 P_j' a_2}, \xrightarrow[1/4]{a_4 P_j' a_2} \varphi(P_i,P_j).$$

12. In $(P_i, P_j) \leftrightarrow (P_i, P'_j)$ where $P_i = P_8$, $P_j \in \mathbb{D}^{a_2}$ and $P'_j \in \mathbb{D}^{a_5}$,

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_2 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_5} \varphi(P_i, P'_j), \text{ and } \varphi(P_i, P'_j) \xrightarrow[1/4]{a_5 P'_j a_2}, \xrightarrow[1/4]{a_5 P'_j a_3} \varphi(P_i, P_j).$$

13. In $(P_i, P_j) \leftrightarrow (P_i, P'_j)$, where $P_i \in \{P_7, P_9\}$, $P_j \in \mathbb{D}^{a_4}$ and $P'_j \in \mathbb{D}^{a_5}$,

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_4 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_1} \varphi(P_i, P'_j), \text{ and } \varphi(P_i, P'_j) \xrightarrow[1/4]{a_5 P'_j a_4}, \xrightarrow[1/4]{a_1 P'_j a_3} \varphi(P_i, P_j).$$

14. In $(P_i, P_j) \leftrightarrow (P_i, P'_j)$ where $P_i = P_8$, $P_j \in \mathbb{D}^{a_4} = \{P_{10}, P_{11}, P_{12}\}$ and $P'_j \in \mathbb{D}^{a_5}$,

$$\begin{aligned} \varphi(P_i, P_j) \xrightarrow[1/4]{a_4 P_j a_2}, \xrightarrow[1/4]{a_4 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_5} \varphi(P_i, P'_j) \qquad & \text{if } P_j = P_{10}; \\ \varphi(P_i, P_j) \xrightarrow[1/2]{a_4 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_2} \varphi(P_i, P'_j) \qquad & \text{if } P_j \in \{P_{11}, P_{12}\}; \\ \varphi(P_i, P'_j) \xrightarrow[1/2]{a_5 P'_j a_4}, \xrightarrow[1/4]{a_2 P'_j a_3} \varphi(P_i, P_j). & \end{aligned}$$

15. In $(P_i, P_j) \rightarrow (P_i, P'_j)$ where $P_i \in \{P_7, P_9\}$, $P_j \in \mathbb{D}^{a_5}$ and $P'_j \in \mathbb{D}^{a_3}$,

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_5 P_j a_3}, \xrightarrow[1/4]{a_1 P_j a_3}, \xrightarrow[1/4]{a_4 P_j a_3} \varphi(P_i, P'_j).$$

16. In $(P_i, P_j) \rightarrow (P_i, P'_j)$ where $P_i = P_8$, $P_j \in \mathbb{D}^{a_5}$ and $P'_j \in \mathbb{D}^{a_3}$,

$$\varphi(P_i, P_j) \xrightarrow[1/2]{a_5 P_j a_3}, \xrightarrow[1/4]{a_2 P_j a_3} \varphi(P_i, P'_j).$$

B Proof of Proposition 2

The proof of Proposition 2 consists of three steps.

Step 1 includes Lemmas 2 - 7. Each lemma shows the existence of a multi-dimensional single-peaked preference satisfying some particular properties. Step 1 serves as a preparation for the verifications in Steps 2 and 3.

Step 2 includes Lemmas 8 and 9. Lemma 8 shows that when two distinct multi-dimensional single-peaked preferences P_i and P'_i share the same peak, there exists an Ad-path connecting them such that for every pair of alternatives with the same relative rankings across P_i and P'_i, the relative ranking of the pair is fixed along the whole Ad-path. The proof of Lemma 8 is a repeated application of Lemma 3. We provide a simple example to illustrate before Lemma 8. Lemma 9 shows that when two multi-dimensional single-peaked preferences P_i and P'_i disagree on peaks in *exactly* one component, and agree on the relative rankings on some pair of alternatives $x, y \in A$, there exists an (x, y)-Is-path connecting them. The construction of the (x, y)-Is-path in the proof of Lemma 9 relies completely on the existence of the particular multi-dimensional single-peaked preferences specified in Lemmas 5 and 7, and the Ad-path constructed in Lemma 8.

Step 3 shows that $\mathbb{D}_{MSP}$ satisfies the Interior Property and the Exterior Property.

We begin Step 1.

Lemma **2** *Given a pair of distinct alternatives $a, b \in A$, if $\langle a, b\rangle \neq A$, there exists $P_i \in \mathbb{D}^a_{MSP}$ such that xP_iy for all $x \in \langle a, b\rangle$ and $y \notin \langle a, b\rangle$.*

Proof: We can construct an admissible preference in two steps.[34] First, pick an arbitrary $\bar{P}_i \in \mathbb{D}^a_{MSP}$ and check whether it satisfies the requirement of this lemma. If yes, it is an admissible preference. Otherwise, we move to the second step. According to $\bar{P}_i$, we induce two preferences over $\langle a, b\rangle$ and $A\backslash\langle a, b\rangle$ respectively, i.e., $(\bar{P}_i, \langle a, b\rangle)$ and $(\bar{P}_i, A\backslash\langle a, b\rangle)$, and then construct a new preference P_i over A which combines these two induced preferences such that all alternatives of $\langle a, b\rangle$ are ranked above others, i.e., $(P_i, \langle a, b\rangle) = (\bar{P}_i, \langle a, b\rangle)$, $(P_i, A\backslash\langle a, b\rangle) = (\bar{P}_i, A\backslash\langle a, b\rangle)$ and $[x \in \langle a, b\rangle$ and $y \notin \langle a, b\rangle] \Rightarrow [xP_iy]$. It is evident that P_i is a linear order and $r_1(P_i) = a$. To complete the verification, we show that P_i is multidimensional single-peaked. Suppose not, i.e., there exist $\bar{x}, \bar{y} \in A$ such that $\bar{x} \in \langle a, \bar{y}\rangle$ and $\bar{y}P_i\bar{x}$. Since $\bar{x} \in \langle a, \bar{y}\rangle$, multi-dimensional single-peakedness of $\bar{P}_i$ implies $\bar{x}\bar{P}_i\bar{y}$. Thus, P_i and $\bar{P}_i$ disagree on the relative ranking of $\bar{x}$ and $\bar{y}$. There are four possible cases: (1) $\bar{x}, \bar{y} \in \langle a, b\rangle$, (2) $\bar{x}, \bar{y} \notin \langle a, b\rangle$, (3) $\bar{x} \in \langle a, b\rangle$ and $\bar{y} \notin \langle a, b\rangle$, and (4) $\bar{x} \notin \langle a, b\rangle$ and $\bar{y} \in \langle a, b\rangle$. The first two cases are not valid since $(P_i, \langle a, b\rangle) = (\bar{P}_i, \langle a, b\rangle)$ and $(P_i, A\backslash\langle a, b\rangle) = (\bar{P}_i, A\backslash\langle a, b\rangle)$. In case (3), the construction of P_i implies $\bar{x}P_i\bar{y}$. Contradiction! In the last case, since $\bar{y} \in \langle a, b\rangle$, the hypothesis $\bar{x} \in \langle a, \bar{y}\rangle$ implies $\bar{x} \in \langle a, b\rangle$. Contradiction! Therefore, $P_i \in \mathbb{D}_{MSP}$. ■

Lemma **3** *Given $P_i, P'_i \in \mathbb{D}^a_{MSP}$, assume $xP_i!y$ and yP'_ix. There exists $P''_i \in \mathbb{D}^a_{MSP}$ such that $P''_i \sim^A P_i$ and $yP''_i!x$ (equivalently, (x, y) is the local switching pair in P_i and P''_i.).*

Proof: Since $r_1(P_i) = r_1(P'_i) = a$, it is evident that $a \notin \{x, y\}$. Let P''_i be a preference induced by locally switching x and y in P_i. Thus, $r_1(P''_i) = a$, $P''_i \sim^A P_i$ and $yP''_i!x$. We show $P''_i \in \mathbb{D}_{MSP}$.

Suppose not, i.e., there exist $x', y' \in A$ such that $x' \in \langle a, y'\rangle$ and $y'P''_ix'$. Since $x' \in \langle a, y'\rangle$, we know $x'P_iy'$. Since $P_i \sim^A P''_i$, $xP_i!y$ and $yP''_i!x$, it must be the case that $x' = x$ and $y' = y$. Consequently, $x \in \langle a, y\rangle$ and hence xP'_iy. Contradiction! Therefore, $P''_i \in \mathbb{D}_{MSP}$. ■

Lemma **4** *Given $P_i \in \mathbb{D}^a_{MSP}$, $s \in M$ and $c^s \in A^s$ with $\langle a^s, c^s\rangle = \{a^s, c^s\}$, there exists $P'_i \in \mathbb{D}^a_{MSP}$ satisfying the following two conditions:*

(1) *For all $x, y \notin (c^s, A^{-s})$, $[xP_iy] \Leftrightarrow [xP'_iy]$.*

[34] We thank an anonymous referee for suggesting this proof.

(2) *For all* $z^{-s} \in A^{-s}$, $(a^s, z^{-s}) P_i' ! (c^s, z^{-s})$.

Proof: We first construct a preference P_i' satisfying conditions (1) and (2) by the following method. First, we remove all alternatives in (c^s, A^{-s}) from P_i, and thus have an induced preference $\big(P_i, A \backslash (c^s, A^{-s})\big)$. Next, we construct preference P_i' over A by plugging all alternatives of (c^s, A^{-s}) back into the induced preference $\big(P_i, A \backslash (c^s, A^{-s})\big)$ in a particular way: $(a^s, z^{-s}) P_i' ! (c^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. Evidently, $r_1(P_i') = a$. In the rest of the proof, we show $P_i' \in \mathbb{D}_{MSP}$.

Given $x, y \in A$ with $x \in \langle a, y \rangle \backslash \{y\}$, we show $x P_i' y$. Note that $x P_i y$ and $(a^s, z^{-s}) P_i (c^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. We consider four cases: (i) $x, y \notin (c^s, A^{-s})$, (ii) $x \notin (c^s, A^{-s})$ and $y \in (c^s, A^{-s})$, (iii) $x \in (c^s, A^{-s})$ and $y \notin (c^s, A^{-s})$ and (iv) $x, y \in (c^s, A^{-s})$.

In case (i), $x P_i y$ implies $x P_i' y$ by condition (1).

In case (ii), $y = (c^s, y^{-s})$. Since $x \in \langle a, y \rangle = \langle a, (c^s, y^{-s}) \rangle$ and $\langle a^s, c^s \rangle = \{a^s, c^s\}$, we know $x^s \in \{a^s, c^s\}$ and $x^{-s} \in \langle a^{-s}, y^{-s} \rangle$. Moreover, $x \notin (c^s, A^{-s})$ implies $x^s = a^s$. Hence $x \in \langle a, (a^s, y^{-s}) \rangle$. Now, either $x = (a^s, y^{-s})$ or $x P_i (a^s, y^{-s})$. If $x = (a^s, y^{-s})$, then $x P_i' y$ by condition (2). If $x P_i (a^s, y^{-s})$, condition (1) first implies $x P_i' (a^s, y^{-s})$. Next, since $(a^s, y^{-s}) P_i' y$ by condition (2), we have $x P_i' y$.

In case (iii), $x = (c^s, x^{-s})$. Evidently, since $(a^s, x^{-s}) P_i x$ and $x P_i y$, we have $(a^s, x^{-s}) P_i y$. Then, by condition (1), $(a^s, x^{-s}) P_i' y$. Furthermore, since $(a^s, x^{-s}) P_i' ! x$ by condition (2), it must be the case that $x P_i' y$.

In case (iv), $x = (c^s, x^{-s})$ and $y = (c^s, y^{-s})$ where $x^{-s} \neq y^{-s}$. Since $x \in \langle a, y \rangle$, it is true that $x^{-s} \in \langle a^{-s}, y^{-s} \rangle$ and hence $(a^s, x^{-s}) \in \langle a, (a^s, y^{-s}) \rangle$. Consequently, $(a^s, x^{-s}) P_i (a^s, y^{-s})$. Then, condition (1) implies $(a^s, x^{-s}) P_i' (a^s, y^{-s})$. Furthermore, since $(a^s, x^{-s}) P_i' ! x$ and $(a^s, y^{-s}) P_i' ! y$ by condition (2), we have $x P_i' y$. In conclusion, $P_i' \in \mathbb{D}_{MSP}$. ■

Lemma 5 *Given* $P_i \in \mathbb{D}^a_{MSP}$ *and* $P_i' \in \mathbb{D}^{(b^s, a^{-s})}_{MSP}$ *with* $a^s \neq b^s$, *assume* $x P_i y$ *and* $x P_i' y$. *There exists* $P_i'' \in \mathbb{D}^a_{MSP}$ *satisfying the following two conditions:*

(1) *For every* $z^{-s} \in A^{-s}$, $(a^s, z^{-s}) P_i'' ! (c^s, z^{-s})$ *where* $c^s \in \langle a^s, b^s \rangle$ *and* $\langle a^s, c^s \rangle = \{a^s, c^s\}$.

(2) $x P_i'' y$.

Proof: We consider two situations: (i) $y \notin (c^s, A^{-s})$ and (ii) $y \in (c^s, A^{-s})$.

Assume that situation (i) occurs. Let $P_i'' \in \mathbb{D}^a_{MSP}$ be a preference induced by P_i satisfying conditions (1) and (2) in Lemma 4. Hence, condition (1) of this lemma is satisfied. Evidently, either $x \notin (c^s, A^{-s})$ or $x \in (c^s, A^{-s})$. If $x \notin (c^s, A^{-s})$, by condition (1) of Lemma 4, $x P_i y$ implies $x P_i'' y$. Next, if $x \in (c^s, A^{-s})$, then $x = (c^s, x^{-s})$. Since $(a^s, x^{-s}) \in \langle a, x \rangle$ and $x P_i y$, we have $(a^s, x^{-s}) P_i x$ and hence $(a^s, x^{-s}) P_i y$. Then, condition (1) of Lemma 4 implies $(a^s, x^{-s}) P_i'' y$. Furthermore, since $(a^s, x^{-s}) P_i'' ! x$ by condition (1) of this lemma, it must be the case that $x P_i'' y$. This completes the verification of situation (i).

Next, assume that situation (ii) occurs. Thus, $y = (c^s, y^{-s})$. Evidently, either $x \in (c^s, A^{-s})$ or $x \notin (c^s, A^{-s})$. First, assume $x \in (c^s, A^{-s})$. Thus, $x = (c^s, x^{-s})$. Since xP_iy, it is true that $(c^s, y^{-s}) = y \notin \langle a, x\rangle = \langle a, (c^s, x^{-s})\rangle$. Consequently, $y^{-s} \notin \langle a^{-s}, x^{-s}\rangle$ and hence $(a^s, y^{-s}) \notin \langle a, (a^s, x^{-s})\rangle$. By Lemma 2, there exists $\bar{P}_i \in \mathbb{D}^a_{MSP}$ such that $(a^s, x^{-s})\bar{P}_i(a^s, y^{-s})$. Let $P''_i \in \mathbb{D}^a_{MSP}$ be a preference induced by $\bar{P}_i$ satisfying conditions (1) and (2) of Lemma 4. Hence, condition (1) of this lemma is satisfied. Since $(a^s, x^{-s})\bar{P}_i(a^s, y^{-s})$, condition (1) of Lemma 4 implies $(a^s, x^{-s})P''_i(a^s, y^{-s})$. Since $(a^s, x^{-s})P''_i!x$ and $(a^s, y^{-s})P''_i!y$ by condition (1) of this Lemma, we have xP''_iy.

Last, assume $x \notin (c^s, A^{-s})$. We claim $(a^s, y^{-s}) \notin \langle a, x\rangle$. Suppose not, i.e., $(a^s, y^{-s}) \in \langle a, x\rangle$. Thus, $y^{-s} \in \langle a^{-s}, x^{-s}\rangle$. Since $c^s \in \langle a^s, b^s\rangle$, it is true that either $c^s \in \langle a^s, x^s\rangle$ or $c^s \in \langle b^s, x^s\rangle$. Consequently, either $y = (c^s, y^{-s}) \in \langle (a^s, a^{-s}), (x^s, x^{-s})\rangle = \langle a, x\rangle$, or $y = (c^s, y^{-s}) \in \langle (b^s, a^{-s}), (x^s, x^{-s})\rangle = \langle (b^s, a^{-s}), x\rangle$, and hence either yP_ix or yP'_ix. Contradiction! Therefore, $(a^s, y^{-s}) \notin \langle a, x\rangle$. By Lemma 2, there exists $\bar{P}_i \in \mathbb{D}^a_{MSP}$ such that $x\bar{P}_i(a^s, y^{-s})$. Now, let $P''_i \in \mathbb{D}^a_{MSP}$ be a preference induced by $\bar{P}_i$ satisfying conditions (1) and (2) of Lemma 4. Hence, condition (1) of this lemma is satisfied. By condition (1) of Lemma 4, $x\bar{P}_i(a^s, y^{-s})$ implies $xP''_i(a^s, y^{-s})$. Next, since $(a^s, y^{-s})P''_i!y$ by condition (1) of this lemma, we have xP''_iy. This completes the verification of situation (ii) and hence the lemma. ■

LEMMA **6** *Given $P_i \in \mathbb{D}^a_{MSP}$ and $P'_i \in \mathbb{D}^b_{MSP}$, assume $a^s \neq b^s$ for all $s \in S$ where $S \subseteq M$ and $|S| \geq 2$, and $a^{-S} = b^{-S}$. Given $x, y \in A$, assume xP_iy and xP'_iy. There exist $s \in S$ and $\bar{P}_i \in \mathbb{D}^{(b^s, a^{-s})}_{MSP}$ such that $x\bar{P}_iy$.*

Proof: Suppose that it is not true. Then, for all $s \in S$ and $\bar{P}_i \in \mathbb{D}^{(b^s, a^{-s})}_{MSP}$, $y\bar{P}_ix$ which implies $y \in \langle (b^s, a^{-s}), x\rangle$ for every $s \in S$. Thus, $y^s \in \langle b^s, x^s\rangle$ for all $s \in S$, and $y^{-S} \in \langle a^{-S}, x^{-S}\rangle$. Consequently, $y \in \langle (b^S, a^{-S}), x\rangle = \langle b, x\rangle$, and hence yP'_ix. Contradiction! ■

We define a variant of adjacency, called *multiple adjacency*, for establishing the next lemma. A pair of preferences P_i and P'_i is **multiple adjacent**, denoted $P_i \sim^{MA} P'_i$, if there exist multiple pairs of alternatives $\{(a_t, a'_t)\}^s_{t=1}$ such that

(1) for each pair (a_t, a'_t), there exists $1 \leq k \leq m-1$ such that $r_k(P_i) = r_{k+1}(P'_i) = a_t$ and $r_{k+1}(P_i) = r_k(P'_i) = a'_t$;

(2) for every $x \notin \{a_t, a'_t\}^s_{t=1}$, $[x = r_k(P_i)] \Leftrightarrow [x = r_k(P'_i)]$.

In the definition of multiple adjacency, $\{(a_t, a'_t)\}^s_{t=1}$ are referred to as the **multiple local switching pairs** in P_i and P'_i. Multiple adjacency generalizes adjacency by allowing the co-existence of multiple local switching pairs. Note that multiple adjacency is independent of multi-dimensional single-peakedness.

Remark 5 Given a pair of multiple adjacent preferences P_i and P_i', let $\{(a_t, a_t')\}_{t=1}^s$ be the corresponding multiple local switching pairs. If a pair of alternatives $(x, y) \notin \{(a_t, a_t')\}_{t=1}^s$, then the relative rankings of x and y remains identical in P_i and P_i', i.e., $[xP_iy] \Leftrightarrow [xP_i'y]$, and more importantly, x and y are isolated in (P_i, P_i'). $\square$

Lemma 7 *Given $P_i \in \mathbb{D}_{MSP}^a$, $s \in M$ and $c^s \in A^s$, assume $(a^s, z^{-s})P_i!(c^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. There exists $P_i' \in \mathbb{D}_{MSP}$ such that $P_i \sim^{MA} P_i'$ and $\big\{\big((a^s, z^{-s}), (c^s, z^{-s})\big)\big\}_{z^{-s}\in A^{-s}}$ are the multiple local switching pairs in P_i and P_i'.*

Proof: First, by flipping the relative ranking of (a^s, z^{-s}) and (c^s, z^{-s}) in P_i for each $z^{-s} \in A^{-s}$, and keeping the rankings of all other alternatives fixed, we can construct preference P_i' such that $P_i \sim^{MA} P_i'$ and the corresponding multiple local switching pairs are $\big\{\big((a^s, z^{-s}), (c^s, z^{-s})\big)\big\}_{z^{-s}\in A^{-s}}$. In the rest of the proof, we show $P_i' \in \mathbb{D}_{MSP}$. Note that since $r_1(P_i) = a$ and $aP_i!(c^s, a^{-s})$, it is true that $r_2(P_i) = (c^s, a^{-s})$ and hence $r_1(P_i') = (c^s, a^{-s})$.

Suppose $P_i' \notin \mathbb{D}_{MSP}$. Then, there exist $x, y \in A$ such that $x \in \langle (c^s, a^{-s}), y\rangle$ and $yP_i'x$. We know that either xP_iy or yP_ix. If xP_iy, then $yP_i'x$ implies that (x, y) is one local switching pair. Thus, $x = (a^s, z^{-s})$ and $y = (c^s, z^{-s})$. Consequently, $x = (a^s, z^{-s}) \notin \langle (c^s, a^{-s}), (c^s, z^{-s})\rangle = \langle (c^s, a^{-s}), y\rangle$. Contradiction!

Next, assume yP_ix. Then, it is true that $x \notin \langle a, y\rangle$. Since $x \in \langle (c^s, a^{-s}), y\rangle$, we know $x^s \in \langle c^s, y^s\rangle$ and $x^{-s} \in \langle a^{-s}, y^{-s}\rangle$. Furthermore, $x \notin \langle a, y\rangle$ implies $x^s \notin \langle a^s, y^s\rangle$. Since $a = r_1(P_i)$ and $(c^s, a^{-s}) = r_2(P_i)$, it is true that $\langle a^s, c^s\rangle = \{a^s, c^s\}$. Since $x^s \in \langle c^s, y^s\rangle$, $\langle a^s, c^s\rangle = \{a^s, c^s\}$ and $x^s \notin \langle a^s, y^s\rangle$, it must be the case that $a^s \in \langle c^s, y^s\rangle$ and $x^s = c^s$. Thus, $x = (c^s, x^{-s})$. Since $x^{-s} \in \langle a^{-s}, y^{-s}\rangle$, we have $(a^s, x^{-s}) \in \langle a, y\rangle$. Thus, either $(a^s, x^{-s})P_iy$ or $(a^s, x^{-s}) = y$. If $(a^s, x^{-s})P_iy$, then $(a^s, x^{-s})P_i!x$ implies xP_iy. Contradiction! Therefore, $(a^s, x^{-s}) = y$ and hence $yP_i!x$ and (y, x) is one local switching pair in P_i and P_i'. Consequently, $xP_i'!y$ by the construction of P_i', a contradiction to the hypothesis $yP_i'x$. Therefore, $P_i' \in \mathbb{D}_{MSP}$. ■

This completes the verification of Step 1. We turn to Step 2.

We first provide a simple example to illustrate Lemma 8 below. Given $P_i, P_i' \in \mathbb{D}_{MSP}^a$ specified below, we construct a *particular* Ad-path connecting P_i and P_i' in $\mathbb{D}_{MSP}^a$.

$$\begin{aligned} P_i &: \; a\, b\, c\, y\, x_2\, x_1\, x \cdots \\ P_i' &: \; a\, b\, c\, x \cdots\cdots y \cdots \end{aligned}$$

Observe that P_i and P_i' agree on the top-three alternatives and disagree subsequently. There are exactly two alternatives x_2 and x_1 ranked between y and x in P_i. Then, by Lemma 3, we can identify the following three preferences $\bar{P}_i, \hat{P}_i, \tilde{P}_i \in \mathbb{D}_{MSP}^a$.

$$\begin{aligned} \bar{P}_i &: \; a\, b\, c\, y\, x_2\, x\;\; x_1 \cdots \\ \hat{P}_i &: \; a\, b\, c\, y\, x\;\; x_2\, x_1 \cdots \\ \tilde{P}_i &: \; a\, b\, c\, x\, y\;\; x_2\, x_1 \cdots \end{aligned}$$

where (i) $P_i \sim^A \bar{P}_i$, $x_1 P_i ! x$ and $x \bar{P}_i ! x_1$; (ii) $\bar{P}_i \sim^A \hat{P}_i$, $x_2 \bar{P}_i ! x$ and $x \hat{P}_i ! x_2$; and (iii) $\hat{P}_i \sim^A \tilde{P}_i$, $y \hat{P}_i ! x$ and $x \tilde{P}_i ! y$. Now, $\tilde{P}_i$ is "closer" to P'_i than P_i, since $\tilde{P}_i$ and P'_i agree on the top-four alternatives. Next, we identify another ranking position $k > 4$ such that $\tilde{P}_i$ and P'_i disagree on the kth ranked alternatives, but agree on all alternatives ranked above k, i.e., $r_k(\tilde{P}_i) \neq r_k(P'_i)$ and $r_{k'}(\tilde{P}_i) = r_{k'}(P'_i)$ for all $1 \leq k' < k$. Then, applying the same argument, we can construct another Ad-path in $\mathbb{D}^a_{MSP}$ starting from $\tilde{P}_i$ and reaching some preference P''_i "closer" to P'_i. Eventually, we have an Ad-path in $\mathbb{D}^a_{MSP}$ connecting P_i and P'_i.

Lemma **8** *Given distinct $P_i, P'_i \in \mathbb{D}_{MSP}$, assume $r_1(P_i) = r_1(P'_i) \equiv a$. There exists an Ad-path $\{P^k_i\}^l_{k=1} \subseteq \mathbb{D}^a_{MSP}$ connecting P_i and P'_i such that for all $x, y \in A$, $[xP_i y$ and $xP'_i y] \Rightarrow [xP^k_i y, 1 < k < l]$.*

Proof: By the algorithm below, we generate an Ad-path in $\mathbb{D}^a_{MSP}$ connecting P_i and P'_i.

Algorithm:

Step 1 : Identify the minimal $k \in \{1, \ldots, m\}$ such that $r_k(P_i) \neq r_k(P'_i)$ (evidently, $k > 1$). For notational convenience, let $r_k(P'_i) = x$. Assume $x = r_{\bar{k}}(P_i)$ (evidently, $\bar{k} > k$). Moreover, for notational convenience, let $r_\nu(P_i) = x_{\bar{k}-\nu}$, $k \leq \nu \leq \bar{k} - 1$. By Lemma 3, we construct a sequence $\{P^{(1,\nu)}_i\}^{l_1}_{\nu=1} \subseteq \mathbb{D}^a_{MSP}$, where $l_1 = \bar{k} - k$, such that

$$P^{(1,\nu-1)}_i \sim^A P^{(1,\nu)}_i,\ x_\nu P^{(1,\nu-1)}_i ! x \text{ and } x P^{(1,\nu)}_i ! x_\nu,\ \nu = 1, \ldots, l_1, \text{ where } P^{(1,0)}_i = P_i.$$

Step $t \geq 2$: According to $P^{(t-1,l_{t-1})}_i$ generated in Step $t-1$, identify the minimal $k \in \{1, \ldots, m\}$ such that $r_k(P^{(t-1,l_{t-1})}_i) \neq r_k(P'_i)$. For notational convenience, let $r_k(P'_i) = x$. Assume $x = r_{\bar{k}}(P^{(t-1,l_{t-1})}_i)$ (evidently, $\bar{k} > k$). Moreover, for notational convenience, let $r_\nu(P^{(t-1,l_{t-1})}_i) = x_{\bar{k}-\nu}$, $k \leq \nu \leq \bar{k} - 1$. By Lemma 3, we construct a sequence $\{P^{(t,\nu)}_i\}^{l_t}_{\nu=1} \subseteq \mathbb{D}^a_{MSP}$, where $l_t = \bar{k} - k$, such that

$$P^{(t,\nu-1)}_i \sim^A P^{(t,\nu)}_i,\ x_\nu P^{(t,\nu-1)}_i ! x \text{ and } x P^{(t,\nu)}_i ! x_\nu,\ \nu = 1, \ldots, l_t, \text{ where } P^{(t,0)}_i = P^{(t-1,l_{t-1})}_i.$$

If $r_k(P^{(t-1,l_{t-1})}_i) = r_k(P'_i)$, $k = 1, \ldots, m$, (in other words, $P^{(t-1,l_{t-1})}_i = P'_i$), the algorithm terminates.

Evidently, this algorithm terminates in finite steps. Assume that the algorithm terminates at *Step* $t+1$. Then, we have sequences of preferences $\{P_i\}$, $\{P^{(1,\nu)}_i\}^{l_1}_{\nu=1}$, $\ldots$, $\{P^{(t,\nu)}_i\}^{l_t}_{\nu=1}$. Combining these sequences, we have an Ad-path

$$\{P^k_i\}^l_{k=1} \equiv \{P_i; P^{(1,1)}_i, \ldots, P^{(1,l_1)}_i; \ldots; P^{(t,1)}_i, \ldots, P^{(t,l_t)}_i\} \subseteq \mathbb{D}^a_{MSP}$$

connecting P_i and P'_i.

Next, given $x, y \in A$ with xP_iy and xP'_iy, we show xP_i^ky, $1 < k < l$. Suppose not, i.e., there exists $1 < k < l$ such that yP_i^kx. Assume w.l.o.g. that $xP_i^{k'}y$ for all $1 \leq k' < k$. Thus, $xP_i^{k-1}!y$ and $yP_i^k!x$. Moreover, we can assume that P_i^k is generated in *Step s* of the algorithm, i.e., $P_i^k = P_i^{(s,\nu)}$ and $P_i^{k-1} = P_i^{(s,\nu-1)}$ for some $1 \leq s \leq t$ and some $1 \leq \nu \leq l_s$. Thus, $P_i^{(s,\nu-1)} \sim^A P_i^{(s,\nu)}$, $xP_i^{(s,\nu-1)}!y$ and $yP_i^{(s,\nu)}!x$. Then, according to the algorithm, it must be the case that yP'_ix. Contradiction! ■

Note that according to Remark 1, for all $x, y \in A$ with xP_iy and xP'_iy, the Ad-path $\{P_i^k\}_{k=1}^l$ in Lemma 8 is also an (x, y)-Is-path connecting P_i and P'_i.

Lemma 9 *Given $P_i, P'_i \in \mathbb{D}_{MSP}$, assume $r_1(P_i) = a$ and $r_1(P'_i) = (b^s, a^{-s})$ where $b^s \neq a^s$ for some $s \in M$. Given $x, y \in A$, assume xP_iy and xP'_iy. There exists an (x, y)-Is-path in $\mathbb{D}_{MSP}$ connecting P_i and P'_i.*

Proof: We relabel the interval $\langle a^s, b^s\rangle = \{a_k^s\}_{k=1}^t$ where $t \geq 2$, $a_1^s = a^s$, $a_t^s = b^s$, and $a_k^s \in \langle a_1^s, a_{k+1}^s\rangle$, $k = 1, \ldots, t-1$. Accordingly, $a_{k+1}^s \in \langle a_k^s, a_t^s\rangle$, $k = 1, \ldots, t-1$.

Claim 1: For every $z^{-s} \in A^{-s}$ and $1 \leq k \leq t-1$, $\{x, y\} \neq \{(a_k^s, z^{-s}), (a_{k+1}^s, z^{-s})\}$.

Proof of Claim 1: Given $z^{-s} \in A^{-s}$ and $1 \leq k \leq t-1$, since $(a_k^s, z^{-s}) \in \langle(a_1^s, a^{-s}), (a_{k+1}^s, z^{-s})\rangle = \langle a, (a_{k+1}^s, z^{-s})\rangle$ and $(a_{k+1}^s, z^{-s}) \in \langle(a_t^s, a^{-s}), (a_k^s, z^{-s})\rangle = \langle(b^s, a^{-s}), (a_k^s, z^{-s})\rangle$, it is true that $(a_k^s, z^{-s})P_i(a_{k+1}^s, z^{-s})$ and $(a_{k+1}^s, z^{-s})P'_i(a_k^s, z^{-s})$. Consequently, xP_iy and xP'_iy imply $\{x, y\} \neq \{(a_k^s, z^{-s}), (a_{k+1}^s, z^{-s})\}$. □

Now, we identify $t-1$ pairs of multi-dimensional single-peaked preferences $\{(\bar{P}_i^k, \hat{P}_i^k)\}_{k=1}^{t-1}$ specified below by the repeated application of Lemmas 5 and 7.

$$
\begin{array}{rlr}
P_i: & (a_1^s, a^{-s}) \cdots x \cdots y \cdots & \\
 & \vdots & \\
\bar{P}_i^1: & (a_1^s, a^{-s})\,(a_2^s, a^{-s}) \cdots (a_1^s, z^{-s})\,(a_2^s, z^{-s}) \cdots & \text{with } x\bar{P}_i^1y \\
\hat{P}_i^1: & (a_2^s, a^{-s})\,(a_1^s, a^{-s}) \cdots (a_2^s, z^{-s})\,(a_1^s, z^{-s}) \cdots & \text{with } x\hat{P}_i^1y \\
 & \vdots & \\
\bar{P}_i^k: & (a_k^s, a^{-s})\,(a_{k+1}^s, a^{-s}) \cdots (a_k^s, z^{-s})\,(a_{k+1}^s, z^{-s}) \cdots & \text{with } x\bar{P}_i^ky \\
\hat{P}_i^k: & (a_{k+1}^s, a^{-s})\,(a_k^s, a^{-s}) \cdots (a_{k+1}^s, z^{-s})\,(a_k^s, z^{-s}) \cdots & \text{with } x\hat{P}_i^ky \\
 & \vdots & \\
\bar{P}_i^{t-1}: & (a_{t-1}^s, a^{-s})\,(a_t^s, a^{-s}) \cdots (a_{t-1}^s, z^{-s})\,(a_t^s, z^{-s}) \cdots & \text{with } x\bar{P}_i^{t-1}y \\
\hat{P}_i^{t-1}: & (a_t^s, a^{-s})\,(a_{t-1}^s, a^{-s}) \cdots (a_t^s, z^{-s})\,(a_{t-1}^s, z^{-s}) \cdots & \text{with } x\hat{P}_i^{t-1}y \\
 & \vdots & \\
P'_i: & (a_t^s, a^{-s}) \cdots x \cdots y \cdots &
\end{array}
$$

According to Lemma 5, $r_1(\bar{P}_i^1) = r_1(P_i) = a$, $(a_1^s, z^{-s})\bar{P}_i^1!(a_2^s, z^{-s})$ for every $z^{-s} \in A^{-s}$, and $x\bar{P}_i^1 y$. Next, according to Lemma 7, we can identify $\hat{P}_i^1 \in \mathbb{D}_{MSP}$ such that $\bar{P}_i^1 \sim^{MA} \hat{P}_i^1$ and $\big\{\big((a_1^s, z^{-s}), (a_2^s, z^{-s})\big)\big\}_{z^{-s}\in A^{-s}}$ is the corresponding multiple local switching pairs. Furthermore, by Claim 1 and Remark 5, we know that $x\bar{P}_i^1 y$ implies $x\hat{P}_i^1 y$, and moreover, x and y are isolated in $(\bar{P}_i^1, \hat{P}_i^1)$. By a similar argument, for all $k = 2, \dots, t-1$, we have the pair of multi-dimensional single-peaked preferences $\bar{P}_i^k$ and $\hat{P}_i^k$, where $r_1(\hat{P}_i^{k-1}) = r_1(\bar{P}_i^k)$, $x\bar{P}_i^k y$, $x\hat{P}_i^k y$, and x and y are isolated in $(\bar{P}_i^k, \hat{P}_i^k)$.

For notational convenience, let $\hat{P}_i^0 = P_i$ and $\bar{P}_i^t = P_i'$. For every $1 \le k \le t$, since $r_1(\hat{P}_i^{k-1}) = r_1(\bar{P}_i^k) = (a_k^s, a^{-s})$, $x\hat{P}_i^{k-1}y$ and $x\bar{P}_i^k y$, Lemma 8 implies that there exists an (x, y)-Is-path in $\mathbb{D}_{MSP}$ connecting $\hat{P}_i^{k-1}$ and $\bar{P}_i^k$. Combining all (x, y)-Is-paths, we eventually have an (x, y)-Is-path in $\mathbb{D}_{MSP}$ connecting P_i and P_i'. ■

This completes the verification of Step 2. Now, we turn to Step 3.

Lemma **10** *Domain $\mathbb{D}_{MSP}$ satisfies the Interior Property.*

Proof: This lemma follows from Lemma 8. ■

Lemma **11** *Domain $\mathbb{D}_{MSP}$ satisfies the Exterior Property.*

Proof: We fix $P_i, P_i \in \mathbb{D}_{MSP}$ with $r_1(P_i) \neq r_1(P_i')$ and $x, y \in A$ with $xP_i y$ and $xP_i' y$. We consider two situations: (i) $r_1(P_i)$ and $r_1(P_i')$ disagree on exactly one component, and (ii) $r_1(P_i)$ and $r_1(P_i')$ disagree on at least two components.

In situation (i), the requirement of the Exterior Property follows from Lemma 9.

In situation (ii), we assume $r_1(P_i) = a$ and $r_1(P_i') = (b^S, a^{-S})$ where $a^s \neq b^s$ for all $s \in S$, $S \subseteq M$ and $|S| \ge 2$. By a repeated application of Lemma 6, we can relabel $S = \{1, \dots, s\}$ such that for each $1 \le k \le s-1$, there exists $\bar{P}_i^k \in \mathbb{D}_{MSP}$ such that $r_1(\bar{P}_i^k) = (b^1, \dots, b^k, a^{k+1}, \dots, a^s, a^{-S})$ and $x\bar{P}_i^k y$.

Let $\bar{P}_i^0 = P_i$ and $\bar{P}_i^s = P_i'$. Thus, (i) for all $0 \le k \le s$, $x\bar{P}_i^k y$; and (ii) for all $0 \le k \le s-1$, $r_1(\bar{P}_i^k)$ and $r_1(\bar{P}_i^{k+1})$ disagree on exactly one component. Now, for each $0 \le k \le s-1$, by Lemma 9, there exists an (x, y)-Is-path in $\mathbb{D}_{MSP}$ connecting $\bar{P}_i^k$ and $\bar{P}_i^{k+1}$. Finally, combining these (x, y)-Is-paths, we have an (x, y)-Is-path in $\mathbb{D}_{MSP}$ connecting P_i and P_i'. ■

This completes the verification of Step 3 and hence proves Proposition 2. We observe that Proposition 2 remains valid for any subset of the multi-dimensional single-peaked domain that satisfies Lemmas 2 - 7 above since any such subset satisfies both the Interior Property and the Exterior Property. For instance, let $B^s \subseteq A^s$, $s \in M$, be such that $G(B^s)$ is a connected sub-graph of $G(A^s)$, and $B = \times_{s\in M} B^s$. Sub-domain $\mathbb{D}_{MSP}^B = \{P_i \in \mathbb{D}_{MSP} | r_1(P_i) \in B\}$ satisfies Lemmas 2 - 7, and therefore satisfies the Interior Property and the Exterior Property.

C Proof of Proposition 3

It is evident that a random dictatorship is ex-post efficient and strategy-proof since it is a convex combination of dictatorships. We focus on showing the necessity part of Proposition 3. We first show that every two-voter ex-post efficient and strategy-proof RSCF $\varphi : \mathbb{D}^2_{MSP} \to \Delta(A)$ is a random dictatorship.[35] Proposition 2 implies that φ satisfies the tops-only property. For notational convenience, we can represent a profile $P \in \mathbb{D}^2$ by a pair of alternatives (a, b) where $r_1(P_1) = a$ and $r_1(P_2) = b$. We shall also occasionally let (a, P_2) denote a profile (P_1, P_2) where $r_1(P_1) = a$.

Lemma **12** *For all $a, b \in A$ with $a \neq b$, $\varphi_a(a, b) + \varphi_b(a, b) = 1$.*

Proof: *Claim 1*: Given $a, b \in A$ with $a \neq b$, and $x \notin \langle a, b\rangle$, $\varphi_x(a, b) = 0$.

Proof of Claim 1: Since $x \notin \langle a, b\rangle$, there exists unique $x' \in \langle a, b\rangle$ such that $\langle a, x\rangle \cap \langle b, x\rangle = \langle x', x\rangle$. Accordingly, there exist $P_1 \in \mathbb{D}^a_{MSP}$ and $P_2 \in \mathbb{D}^b_{MSP}$ such that $x'P_1x$ and $x'P_2x$. Thus, by tops-onlyness and ex-post efficiency, we have $\varphi_x(a, b) = \varphi_x(P_1, P_2) = 0$. □

Claim 2: Given $a, b \in A$, assume $a^s \neq b^s$ and $a^\tau \neq b^\tau$ for some $s, \tau \in M$. Given $x \in \langle a, b\rangle \backslash \{a, b\}$, $\varphi_x(a, b) = 0$.

Proof of Claim 2: Since a and b disagree on at least two components, and $x \in \langle a, b\rangle\backslash\{a, b\}$, it is true that $\langle a, b\rangle\backslash[\langle a, x\rangle \cup \langle b, x\rangle] \neq \emptyset$. Fixing $x' \in \langle a, b\rangle\backslash[\langle a, x\rangle \cup \langle b, x\rangle]$, we know $x \notin \langle a, x'\rangle$ and $x \notin \langle b, x'\rangle$. By Lemma 2, there exist $P_1 \in \mathbb{D}^a_{MSP}$ and $P_2 \in \mathbb{D}^b_{MSP}$ such that $x'P_1x$ and $x'P_2x$. Then, by tops-onlyness and ex-post efficiency, $\varphi_x(a, b) = \varphi_x(P_1, P_2) = 0$. □

According to Claims 1 and 2, we know that for all $a, b \in A$ with $a^s \neq b^s$ and $a^\tau \neq b^\tau$ for some $s, \tau \in M$, $\varphi_a(a, b) + \varphi_b(a, b) = 1$.

Claim 3: Given $a, b \in A$, assume $a^s \neq b^s$ and $a^{-s} = b^{-s}$ for some $s \in M$. Given $x \in \langle a, b\rangle\backslash\{a, b\}$, $\varphi_x(a, b) = 0$.

Proof of Claim 3: We assume $a = (a^s, x^{-s})$, $b = (b^s, x^{-s})$ and $x = (x^s, x^{-s})$ where $x^s \in \langle a^s, b^s\rangle\backslash\{a^s, b^s\}$. We identify two other alternatives $\bar{b} = (b^s, y^\tau, x^{-s,\tau})$ and $\bar{x} = (x^s, y^\tau, x^{-s,\tau})$ where (x^τ, y^τ) is an edge in $G(A^\tau)$.[36] Since $a^s \neq b^s = \bar{b}^s$ and $a^\tau = x^\tau \neq y^\tau = \bar{b}^\tau$, Claims 1 and 2 imply $\varphi_x(a, \bar{b}) = 0$ and $\varphi_{\bar{x}}(a, \bar{b}) = 0$. Given b and $\bar{b}$, by Lemmas 4 and 7, we have $P_2 \in \mathbb{D}^b_{MSP}$ and $P_2' \in \mathbb{D}^{\bar{b}}_{MSP}$ such that $P_2 \sim^{MA} P_2'$; $(x^\tau, z^{-\tau})P_2!(y^\tau, z^{-\tau})$ and $(y^\tau, z^{-\tau})P_2'!(x^\tau, z^{-\tau})$ for all $z^{-\tau} \in A^{-\tau}$. By tops-onlyness and strategy-proofness, $\varphi_x(a, b) + \varphi_{\bar{x}}(a, b) = \varphi_x(a, P_2) + \varphi_{\bar{x}}(a, P_2) = \varphi_x(a, P_2') + \varphi_{\bar{x}}(a, P_2') = \varphi_x(a, \bar{b}) + \varphi_{\bar{x}}(a, \bar{b}) = 0$. Hence, $\varphi_x(a, b) = 0$. □

[35] In the case of two voters, we *do not* require $|M| \geq 3$. When the number of voters increases to at least 3, the restriction $|M| \geq 3$ must be imposed.

[36] Alternatives $\bar{b}$ and $\bar{x}$ exist since $A = \times_{q\in M} A^q$ and $|M| \geq 2$.

As a consequence of Claims 1 and 3, we know that for all $a, b \in A$ with $a^s \neq b^s$ and $a^{-s} = b^{-s}$ for some $s \in M$, $\varphi_a(a,b) + \varphi_b(a,b) = 1$. Therefore, by Claims 1, 2 and 3, $\varphi_a(a,b) + \varphi_b(a,b) = 1$ for all $a, b \in A$ with $a \neq b$. ■

LEMMA **13** *Given $a, b; x, y \in A$ with $a \neq b$ and $x \neq y$, $\varphi_a(a,b) = \varphi_x(x,y)$.*

Proof: Assume $\varphi_a(a,b) = \lambda$. We consider two cases: (i) either $x \notin \{a,b\}$ or $y \notin \{a,b\}$, and (ii) $x \in \{a,b\}$ and $y \in \{a,b\}$.

In case (i), we assume w.l.o.g. that $x \notin \{a,b\}$. The verification related to $y \notin \{a,b\}$ is symmetric and we hence omit it. Since $|M| \geq 2$, there exists a sequence $\{a_k\}_{k=1}^{t} \subseteq A$ such that $a_1 = a$, $a_t = x$, (a_k, a_{k+1}) is an edge in $\times_{s\in M} G(A^s)$, $k = 1, \ldots, t-1$, and $b \notin \{a_k\}_{k=1}^{t}$. Given a_1 and a_2, we have $P_1 \in \mathbb{D}_{MSP}^{a_1}$ and $P_1' \in \mathbb{D}_{MSP}^{a_2}$ such that $r_2(P_1) = a_2$ and $r_2(P_1') = a_1$. By tops-onlyness and strategy-proofness, $\varphi_{a_1}(a_1,b) + \varphi_{a_2}(a_1,b) = \varphi_{a_1}(P_1,b) + \varphi_{a_2}(P_1,b) = \varphi_{a_1}(P_1',b) + \varphi_{a_2}(P_1',b) = \varphi_{a_1}(a_2,b) + \varphi_{a_2}(a_2,b)$. Then, Lemma 12 implies $\varphi_{a_2}(a_2,b) = \varphi_{a_1}(a_1,b) = \lambda$. Following the sequence $\{a_k\}_{k=1}^{t}$ and repeatedly applying the symmetric argument step by step, we have $\varphi_x(x,b) = \varphi_{a_t}(a_t,b) = \lambda$. Hence, $\varphi_b(x,b) = 1-\lambda$ by Lemma 12. If $y = b$, the verification is completed. We assume $y \neq b$. Then, there exists a sequence $\{b_k\}_{k=1}^{t'} \subseteq A$ such that $b_1 = b$, $b_{t'} = y$, (b_k, b_{k+1}) is an edge in $\times_{s\in M} G(A^s)$, $k = 1, \ldots, t'-1$, and $x \notin \{b_k\}_{k=1}^{t'}$. Following the sequence $\{b_k\}_{k=1}^{t'}$, by a symmetric argument, we have $\varphi_y(x,y) = 1-\lambda$. Then, by Lemma 12, $\varphi_x(x,y) = \lambda = \varphi_a(a,b)$.

In case (ii), since $x \neq y$, it must be either $(x,y) = (a,b)$ or $(x,y) = (b,a)$. The lemma evidently holds if $(x,y) = (a,b)$. Assume $(x,y) = (b,a)$. Fix $x' \notin \{a,b\}$. Between (a,b) and (x',b), since $x' \notin \{a,b\}$, the verification of case (i) implies $\varphi_{x'}(x',b) = \lambda$. Similarly, between (b,a) and (x',b), since $x' \notin \{b,a\}$, the verification of case (i) implies $\varphi_b(b,a) = \varphi_{x'}(x',b) = \lambda$. Thus, $\varphi_a(a,b) = \varphi_b(b,a)$. This completes the verification of the lemma. ■

Fixing arbitrary $a, b \in A$ with $a \neq b$, let $\varphi_a(a,b) = \lambda$. We show that for all $x, y \in A$, $\varphi(x,y) = \lambda e_x + (1-\lambda)e_y$. If $x = y$, it evidently holds. If $x \neq y$, it holds by Lemmas 12 and 13. Therefore, φ is a random dictatorship.

Next, we modify the Ramification Theorem of Chatterji et al. (2014) so that the random dictatorship result over $\mathbb{D}_{MSP}$ (henceforth, assume $|M| \geq 3$) can be extended to the case of an arbitrary number of voters. We first introduce the primary induction hypothesis.

The Primary Induction Hypothesis: Given $N > 2$, for all $2 \leq n < N$, we have

$$[\varphi : \mathbb{D}_{MSP}^{n} \to \Delta(A) \text{ is ex-post efficient and strategy-proof}] \Rightarrow [\varphi \text{ is a random dictatorship}].$$

Fixing an ex-post efficient and strategy-proof RSCF $\varphi : \mathbb{D}_{MSP}^{N} \to \Delta(A)$, we show that φ is a random dictatorship. By Proposition 2, RSCF φ satisfies the tops-only property.

Lemma 14 *RSCF φ is a* ***quasi random dictatorship****, i.e., there exists $[\varepsilon_i]_{i\in I} \in \mathbb{R}^N_+$ with $\sum_{i\in I} \varepsilon_i = 1$ such that for all $P \in \mathbb{D}^N_{MSP}$ with $P_i = P_j$ for some $i, j \in I$, $\varphi(P) = \sum_{i\in I} \varepsilon_i e_{r_1(P_i)}$.*

Proof: We consider two cases: $N > 3$ and $N = 3$. If $N > 3$, the verification is exactly identical to the verification of Proposition 5 of Chatterji et al. (2014) by simply changing "unanimity" to "ex-post efficiency". Thus, we focus on the case $N = 3$.[37]

According to RSCF $\varphi : \mathbb{D}^3 \to \Delta(A)$, we define three RSCFs as follows: $g^{(2,3)}(P_1, P_2) = \varphi(P_1, P_2, P_2)$, $g^{(1,3)}(P_1, P_2) = \varphi(P_1, P_2, P_1)$ and $g^{(1,2)}(P_1, P_3) = \varphi(P_1, P_1, P_3)$ for all $P_1, P_2, P_3 \in \mathbb{D}$. Evidently, $g^{(2,3)}$, $g^{(1,3)}$ and $g^{(1,2)}$ are ex-post efficient and strategy-proof. Then, $g^{(2,3)}$, $g^{(1,3)}$ and $g^{(1,2)}$ are random dictatorships by the primary induction hypothesis. Thus, there exist ε_1, ε_2, $\varepsilon_3 \geq 0$ such that for all $P_1, P_2, P_3 \in \mathbb{D}$,

$$\begin{aligned}
\varphi(P_1, P_2, P_2) &= g^{(2,3)}(P_1, P_2) = \varepsilon_1 \, e_{r_1(P_1)} + (1 - \varepsilon_1) e_{r_1(P_2)} \\
\varphi(P_1, P_2, P_1) &= g^{(1,3)}(P_1, P_2) = (1 - \varepsilon_2) e_{r_1(P_1)} + \varepsilon_2 \, e_{r_1(P_2)} \\
\varphi(P_1, P_1, P_3) &= g^{(1,2)}(P_1, P_3) = (1 - \varepsilon_3) e_{r_1(P_1)} + \varepsilon_3 \, e_{r_1(P_3)}
\end{aligned}$$

To establish that φ is a quasi random dictatorship, it suffices to show $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$. Fixing $\{1, 2, 3\} \subseteq M$; $\{x^s, y^s\} \subseteq A^s$ where (x^s, y^s) is an edge in $G(A^s)$, $s = 1, 2, 3$, and $z^{-\{1,2,3\}} \in A^{-\{1,2,3\}}$, we identify the following eight alternatives (see the diagram below):

$$\begin{aligned}
&a = (x^1, x^2, x^3, z^{-\{1,2,3\}}),\ b = (y^1, y^2, x^3, z^{-\{1,2,3\}}),\ c = (y^1, x^2, y^3, z^{-\{1,2,3\}}); \\
&\bar{a} = (x^1, y^2, x^3, z^{-\{1,2,3\}}),\ \bar{b} = (y^1, y^2, y^3, z^{-\{1,2,3\}}),\ \bar{c} = (x^1, x^2, y^3, z^{-\{1,2,3\}}); \\
&\bar{x} = (y^1, x^2, x^3, z^{-\{1,2,3\}}),\ \bar{y} = (x^1, y^2, y^3, z^{-\{1,2,3\}}).
\end{aligned}$$

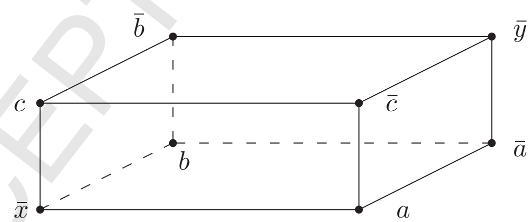

Figure 2: The geometric relations among $a, b, c, \bar{a}, \bar{b}, \bar{c}, \bar{x}$ and $\bar{y}$

By Lemma 2, we can construct two preference profiles: $P = (P_1, P_2, P_3) \in \mathbb{D}^3_{MSP}$ and $P' = (P'_1, P'_2, P'_3) \in \mathbb{D}^3_{MSP}$ such that the following five conditions are satisfied

[37] Proposition 4 of Chatterji et al. (2014) characterizes quasi random dictatorship in the case of three voters. Their verification relies on an additional condition called *Richness Condition α*. However, $\mathbb{D}_{MSP}$ violates Richness Condition α. Instead, the proof of Lemma 14 relies on the restriction of multi-dimensional single-peakedness and the tops-only property.

(i) $r_1(P_1) = r_1(P'_1) = a$, $r_1(P_2) = r_1(P'_2) = b$ and $r_1(P_3) = r_1(P'_3) = c$;

(ii) $r_2(P_1) = \bar{x}$, $r_3(P_1) = \bar{a}$ and $r_4(P_1) = b$;

(iii) $r_2(P_2) = \bar{x}$, $r_3(P_2) = \bar{b}$ and $r_4(P_2) = c$;

(iv) $r_2(P_3) = \bar{x}$, $r_3(P_3) = \bar{c}$ and $r_4(P_3) = a$;

(v) $\bar{y}P'_i\bar{x}$, $i = 1, 2, 3$

By a similar argument to the one in the proof of Proposition 4 of Chatterji et al. (2014), we first have $\varphi_a(P) = \varepsilon_1$, $\varphi_b(P) = \varepsilon_2$, $\varphi_c(P) = \varepsilon_3$ and $\varphi_x(P) = 0$ for all $x \notin \{a, b, c, \bar{x}\}$. Moreover, since $\varphi_{\bar{x}}(P) = \varphi_{\bar{x}}(P') = 0$ by tops-onlyness and ex-post efficiency, we have $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varphi_a(P) + \varphi_b(P) + \varphi_c(P) = \sum_{x\in A} \varphi_x(P) = 1$, as required. ■

Furthermore, following the argument of Lemma 14 of Chatterji et al. (2014), we know that for all $P \in \mathbb{D}^N_{MSP}$ with $r_1(P_i) = r_1(P_j)$ for some $i, j \in I$, $\varphi(P) = \sum_{i\in I} \varepsilon_i e_{r_1(P_i)}$. Therefore, to complete the verification of the primary induction hypothesis, we show in Lemmas 15 and 16 below that for all $P \in \mathbb{D}^N$ with $r_1(P_i) \neq r_1(P_j)$ for all $i, j \in I$, $\varphi(P) = \sum_{i\in I} \varepsilon_i e_{r_1(P_i)}$.

We first introduce new notation. Given a nonempty subset $\hat{I} \subseteq I$ and $P_{\hat{I}} \in \mathbb{D}^{|\hat{I}|}$, let $\tau(P_{\hat{I}}) = \cup_{i\in\hat{I}}\big\{r_1(P_i)\big\}$ denote the set of peaks in $P_{\hat{I}}$. Given $P_i \in \mathbb{D}$ and $a \in A$, let $W(P_i, a) = \{x \in A|aP_ix\}$ denote the (strict) lower contour set of a at P_i. Given $P \in \mathbb{D}^N$ with $|\tau(P)| = N$, let $\overline{W}(P) = \cup_{i\in I}W\big(P_i, \max(P_i, \tau(P_{-i}))\big)$.

Lemma **15** *For all* $P \in \mathbb{D}^N_{MSP}$ *with* $|\tau(P)| = N$ *and* $x \in \overline{W}(P)$, $\varphi_x(P) = \sum_{i\in I:r_1(P_i)=x} \varepsilon_i$.

Proof: This lemma follows from Lemma 16 of Chatterji et al. (2014). ■

Lemma **16** *For all* $P \in \mathbb{D}^N_{MSP}$ *with* $|\tau(P)| = N$, $\varphi(P) = \sum_{i\in I} \varepsilon_i e_{r_1(P_i)}$.

Proof: Fix $P \in \mathbb{D}^N_{MSP}$ with $|\tau(P)| = N$. For notational convenience, let $a_i = r_1(P_i)$ for all $i \in I$. We can identify two voters $i, j \in I$ such that the minimal box $\langle a_i, a_j\rangle$ contains no other voter's peak, i.e., $\langle a_i, a_j\rangle \cap \tau(P_{-\{i,j\}}) = \emptyset$. By Lemma 2, we have two preferences $\bar{P}_i \in \mathbb{D}^{a_i}_{MSP}$ and $\bar{P}_j \in \mathbb{D}^{a_j}_{MSP}$ such that for all $x \in \langle a_i, a_j\rangle$ and $y \notin \langle a_i, a_j\rangle$, $x\bar{P}_iy$ and $x\bar{P}_jy$. Thus, for all $l \notin \{i, j\}$, $a_l \in \overline{W}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$ and hence $\varphi_{a_l}(P) = \varphi_{a_l}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}}) = \varepsilon_l$ by tops-onlyness and Lemma 15. Moreover, by tops-onlyness, strategy-proofness and quasi random dictatorship, we have $\sum_{x\in\langle a_i,a_j\rangle} \varphi_x(P) = \sum_{x\in\langle a_i,a_j\rangle} \varphi_x(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}}) = \sum_{x\in\langle a_i,a_j\rangle} \varphi_x(\bar{P}_i, \bar{P}_i, P_{-\{i,j\}}) = \varphi_{a_i}(\bar{P}_i, \bar{P}_i, P_{-\{i,j\}}) = \varepsilon_i + \varepsilon_j$.

Choose arbitrary $l \in I\backslash\{i, j\}$. We know that either $a_iP_la_j$ or $a_jP_la_i$. Thus, either $a_j \in \overline{W}(P)$ or $a_i \in \overline{W}(P)$. Hence, either $\varphi_{a_j}(P) = \varepsilon_j$ or $\varphi_{a_i}(P) = \varepsilon_i$ by Lemma 15. Assume $a_jP_la_i$. Thus, $a_i \in \overline{W}(P)$ and $\varphi_{a_i}(P) = \varepsilon_i$. The verification related to the other case is symmetric and we hence omit it. To complete the verification, it suffices to show either $\varphi_{a_j}(P) = \varepsilon_j$ or

$\varphi_{a_i}(P)+\varphi_{a_j}(P)=\varepsilon_i+\varepsilon_j$. If there exists $\bar{P}_l \in \mathbb{D}^{a_l}_{MSP}$ such that $a_i\bar{P}_l a_j$, then $a_j \in \overline{W}(\bar{P}_l, P_{-l})$ and hence, by tops-onlyness and Lemma 15, $\varphi_{a_j}(P)=\varphi_{a_j}(\bar{P}_l, P_{-l})=\varepsilon_j$, as required. If $a_j\bar{P}_l a_i$ for all $\bar{P}_l \in \mathbb{D}^{a_l}_{MSP}$, then it must be the case that $a_j \in \langle a_l, a_i\rangle$. Hence, $\langle a_j, a_i\rangle \subseteq \langle a_l, a_i\rangle$ and $a_j P_l x$ for all $x \in \langle a_i, a_j\rangle\backslash\{a_i, a_j\}$ by multi-dimensional single-peakedness. Consequently, for all $x \in \langle a_i, a_j\rangle\backslash\{a_i, a_j\}$, $x \in \overline{W}(P)$, and hence, $\varphi_x(P)=0$ by Lemma 15. Therefore, $\varphi_{a_i}(P)+\varphi_{a_j}(P)=\sum_{x\in\langle a_i,a_j\rangle}\varphi_x(P)=\varepsilon_i+\varepsilon_j$, as required. ■

Now, we assert that RSCF $\varphi : \mathbb{D}^N_{MSP} \to \Delta(A)$ is a random dictatorship. This completes the verification of the primary induction hypothesis and hence proves Proposition 3.

D Some additional material

D.1 Separable preferences

In this section, we briefly discuss the domain of separable preferences (Le Breton and Sen, 1999). Recall the Cartesian product setting in Section 4.2.

Definition 5 *A preference P_i is **separable** if for all $s \in M$ and $a^s, b^s \in A^s$, we have*

$$\big[(a^s, x^{-s})P_i(b^s, x^{-s}) \text{ for some } x^{-s} \in A^{-s}\big] \Rightarrow \big[(a^s, y^{-s})P_i(b^s, y^{-s}) \text{ for all } y^{-s} \in A^{-s}\big].$$

Let $\mathbb{D}_S$ denote *the* separable domain containing all separable preferences. We provide an example to illustrate how the Interior Property is violated by the separable domain when one component set contains at least 3 elements.

Example 5 Let $A=\{0,1,2\}\times\{0,1\}$ and consider the sub-domain of all ten separable preferences with the peak $(0,0)$, specified below.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(2,0)$	$(2,0)$	$(2,0)$
$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(2,0)$	$(2,0)$	$(2,0)$	$(0,1)$	$(0,1)$	$(1,0)$
$(2,0)$	$(1,1)$	$(1,1)$	$(2,0)$	$(0,1)$	$(1,0)$	$(2,1)$	$(2,1)$	$(1,0)$	$(0,1)$
$(1,1)$	$(2,0)$	$(2,0)$	$(1,1)$	$(1,1)$	$(2,1)$	$(1,0)$	$(1,0)$	$(2,1)$	$(2,1)$
$(2,1)$	$(2,1)$	$(2,1)$	$(2,1)$	$(2,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$

Table 5: The sub-domain of all separable preferences with the peak $(0,0)$

We can separate all ten preferences of Table 5 into two groups: $\{P_1,P_2,P_3,P_4,P_5\}$ and $\{P_6,P_7,P_8,P_9,P_{10}\}$. In each group, preferences are consecutively adjacent, i.e., $P_k \sim^A P_{k+1}$, $1 \le k \le 4$ and $6 \le k \le 9$. However, between two groups, there exists no pair of adjacent preferences. Therefore, the Interior Property fails. From the first group to the second group, we have to adopt the notion of multiple adjacency introduced in the proof of Proposition 2, e.g., $P_1 \sim^{MA} P_6$ and $P_5 \sim^{MA} P_{10}$. □

One way to show that the separable domain is a tops-only domain might be to weaken the Interior Property so that the co-existence of adjacencies and multiple adjacencies in each sub-domain with the same peak is allowed, and correspondingly strengthen the Exterior Property so that the multiple local switching pairs in two multiple adjacent preferences can be covered. We leave a formal treatment for future work.

D.2 A weakening of the Exterior Property

In this section, we provide a weaker sufficient condition for tops-only domains. We weaken the Exterior Property by defining it with respect to the local switching pairs involved in the Interior Property. This weakening is referred to as the *weak Exterior Property*, and helps eliminate redundant Is-paths in the domain. The proof of the Theorem can be easily modified to show that a domain satisfying the Interior Property and the weak Exterior Property is also a tops-only domain.

Definition **6** *A domain $\mathbb{D}$ satisfies the* ***weak Exterior Property*** *if given $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P_i')$, and $x, y \in A$ with $xP_i!y$ and $xP_i'y$, we have*

$$[\text{there exists } \bar{P}_i \in \mathbb{D} \text{ such that } r_1(P_i) = r_1(\bar{P}_i), P_i \sim^A \bar{P}_i \text{ and } y\bar{P}_i!x]$$
$$\Rightarrow [\text{there exists an } (x,y)\text{-Is-path connecting } P_i \text{ and } P_i'].$$

In the definition of the weak Exterior Property, preference $\bar{P}_i$ can be viewed as a *benchmark* preference which tests whether P_i and (x, y) are *critical* in the sense that P_i and $\bar{P}_i$ share the same peak, and are adjacent to each other with the local switching pair (x, y). Once the criticality is verified, the weaker Exterior Property requires the existence of a (x, y)-Is-path connecting P_i and P_i'. The following corollary shows that the combination of the Interior Property and the weak Exterior Property is sufficient for tops-only domains.

Corollary **1** *A domain satisfying the Interior Property and the weaker Exterior Property is a tops-only domain.*

Proof: The verification of Corollary 1 follows from a slight modification of the proof of the Theorem. Replace the fifth sentence of the last paragraph in the proof of the Theorem by the following sentence: Since (i) $r_1(P_i) \neq r_1(P_j)$, $xP_i!y$ and xP_jy, and (ii) $r_1(P_i) = r_1(P_i')$; $P_i \sim^A P_i'$ and $yP_i'!x$, the weaker Exterior Property implies that there exists an (x, y)-Is-path $\{P_j^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting P_i and P_j. ■

We provide the following example to illustrate the weak Exterior Property, and show thereby that more tops-only domains can be covered by the weaker sufficient condition.

Example **6** Let $A = \{a_1, a_2, a_3, a_4\}$ and consider the domain $\mathbb{D}$, containing five preferences, specified below.

P_1	P_2	P_3	P_4	P_5
a_1	a_2	a_2	a_3	a_4
a_2	a_3	a_3	a_2	a_3
a_3	a_1	a_4	a_4	a_2
a_4	a_4	a_1	a_1	a_1

Table 6: Domain $\mathbb{D}$

Domain $\mathbb{D}$ satisfies the Interior Property, i.e., $P_2 \sim^A P_3$. To verify whether the Exterior Property is met, the (a_3, a_4)-Is-path $\{P_1, P_2, P_3, P_4\}$ which connects P_1 and P_4 must be considered. However, the Exterior Property fails in domain $\mathbb{D}$, as there exists no (a_2, a_3)-Is-path connecting P_1 and P_2. Consequently, the Theorem cannot be applied to verify whether $\mathbb{D}$ is a tops-only domain. Note that neither P_1 and (a_3, a_4), nor P_2 and (a_2, a_3) are critical. Therefore, the (a_3, a_4)-Is-path $\{P_1, P_2, P_3, P_4\}$ is redundant, and the non-existence of (a_2, a_3)-Is-path connecting P_1 and P_2 does not matter. On the contrary, P_3 and (a_4, a_1) are critical according to the bench-mark preference P_2. Correspondingly, we have $\{P_3, P_4, P_5\}$ as an (a_4, a_1)-Is-path connecting P_3 and P_5. One can easily verify that domain $\mathbb{D}$ satisfies the weak Exterior Property and therefore domain $\mathbb{D}$ is a tops-only domain. $\square$

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