Department of Economics
Working Paper No. 197

Games with the Total Bandwagon Property

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July 2015

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Games with the Total Bandwagon Property[*]

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May 15, 2015

Abstract

We consider the class of two-player symmetric $n \times n$ games with the total bandwagon property (TBP) introduced by Kandori and Rob (1998). We show that a game has TBP if and only if the game has $2^n - 1$ symmetric Nash equilibria. We extend this result to bimatrix games by introducing the generalized TBP. This sheds light on the (wrong) conjecture of Quint and Shubik (1997) that any $n \times n$ bimatrix game has at most $2^n - 1$ Nash equilibria. As for an equilibrium selection criterion, I show the existence of a $1/2$–dominant equilibrium for two subclasses of games with TBP: (i) supermodular games; (ii) potential games. As an application, we consider the minimum-effort game, which does not satisfy TBP, but is a limit case of TBP.

JEL Classification: C62; C72; C73

Keywords: Bandwagon; Nash Equilibrium; Number of Equilibria; Coordination Game; Equilibrium Selection.

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1 Introduction

Bandwagon effect is a form of groupthinking in social psychology which says that as more people adopt a belief or an action, others are more likely to do the same thing. We can observe it in markets through fads or trends, such as consumer product choices and customs.[1] This concept is explicitly introduced by Leibenstein (1950) into economics for consumer demand theory, and has been further investigated theoretically and empirically.[2] Here we consider a related but stronger concept, the total bandwagon property (TBP), which is introduced by Kandori and Rob (1998) where TBP is used as a bandwagon effect regarding consumer technology adoptions in an evolutionary context as follows. There is a society consisting of a single population, and each consumer in the society observes a product choice profile taken by all other consumers in the last period and chooses one of products used by some other people as a best response. Formally, TBP is the property imposed on the class of symmetric two-player games under which all best responses against any mixed strategy are in the support of this mixed strategy. Our main purpose is to unveil hidden sides of the above mentioned bandwagon effect on games.

We first show a characterization of games with TBP via the number of Nash equilibria: A symmetric $n \times n$ game has TBP if and only if the game has $2^n - 1$ symmetric Nash equilibria. Furthermore, by considering the generalized TBP to allow for asymmetric games, we extend the characterization to bimatrix games. This characterization result suggests that a game with the bandwagon property may have so many Nash equilibria that it is hard to select a single equilibrium. Given no observed history of actions taken in society, when choosing an action, agents cannot have clear selection criteria on which equilibrium is chosen.[3] With this in mind, the second objective of our

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[1] The bandwagon effect can be interpreted as a network externality and is related to conformity, herd behavior, information cascade, and so on. Note that we can also see the opposite effect known as snob effect that when many people adopt something, a person avoids to have or be associated with the same thing. Exclusive products, such as designer clothing and rare artworks, are typical examples.

[2] For instance, see Granovetter and Soong (1986), Becker (1991), Bikhchandani et al. (1992), Karni and Levin (1994), and Pesendorfer (1995) as related theoretical research; Biddle (1991) and McAllister and Studlar (1991) as empirical research; Plott and Smith (1999) as experimental research on markets. See Rohlfs (2001) for a comprehensive analysis on bandwagon effect in high-tech industries.

[3] In the late 1970s and the 1980s, consumers struggled to choose between videotape formats of VHS by Matsushita as JVC and Betamax sold by Sony, but VHS prevailed in the end. The striking force behind this market outcome is that consumers tend to adopt the more popular technology. Similar examples are observed for the high definition optical disc formats between Blu-ray Disc by Sony and HD-DVD by Toshiba (Fackler, 2008) and browsers between Internet Explorer by Microsoft and Netscape by Navigator in the late 1990s and between Google Chrome, Mozilla Firefox, Internet

paper is to provide a simple equilibrium selection criterion. Such a simple but strong equilibrium selection criterion is the solution concept of $1/2$–*dominant equilibrium* proposed by Morris, Rob and Shin (1995), which is a generalization of *risk dominant equilibrium* (Harsanyi and Selten, 1988). It is chosen by various equilibrium selection methods including the "evolutionary learning method" based on the best response dynamics with mutation (Kandori et al., 1993; Young, 1993); the "global game method" (Carlsson and van Damme, 1993); the "incomplete information game method" (Kajii and Morris, 1997); the "perfect foresight dynamics method" (Matsui and Matsuyama, 1995; Hofbauer and Sorger, 1999); the "spatial dominance method" (Hofbauer, 1999). A $1/2$–dominant equilibrium needs not to exist in games with TBP. I show the existence of a $1/2$–dominant equilibrium for two subclasses of games with TBP.

One of them is the class of supermodular games. Supermodularity (strategic complementarity) has been considered to be important in economics (Milgrom and Roberts, 1990; Milgrom and Shannon, 1994; Topkis, 1998; Vives, 1990, 2001). We show that a (generic) symmtric two-player supermodular game with TBP has a unique $1/2$–dominant equilibrium, and the equilibrium is either the lowest or the highest strategy profile. This implies that the various equilibrium selection methods consistently predict either the lowest or the highest strategy profile to be chosen in this subclass of games with TBP.

The other is the class of potential games (Hofbauer and Sigmund, 1988; Monderer and Shapley, 1996). We show that if a game with TBP has a potential function with a unique potential maximizer, the potential maximizer is a $1/2$–dominant equilibrium. More generally, when considering a *local potential maximizer* (Morris, 1999), which is a generalization of *potential maximizer* and chosen by the evolutionary learning method based on the log-linear dynamics (Blume, 1993; Okada and Tercieux, 2012),[4] we can show that if a local potential maximizer (with constant weights) exists in a game, it is a $1/2$–dominant equilibrium.

Lastly, we apply our results to a classical experimental game—the minimum-effort game—introduced by Van Huyck, Battalio and Beil (1990) where subjects choose their individual effort levels with incurring constant per-unit effort cost while their benefits are commonly determined by the minimum level of efforts chosen by all subjects, and therefore every subject has no incentive to choose a higher effort level than the other(s). This game is a (symmetric) supermodular coordination game with multiple Pareto-

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Explorer, Safari by Apple, and Opera in recent years (The Economist, Aug 10, 2013).

[4]See also Alós-Ferrer and Netzer (2010, 2015).

ranked Nash equilibria. We show that the two-player minimum-effort game does not satisfy TBP but is a limit case of TBP.[5] Then we examine the result of their experiment in the two-player case by the selection criterion of the 1/2–dominant equilibrium.

This article contributes to two strands of the literature. First of all, this article is related to the literature on the number of Nash equilibria in games. To the best of my knowledge, this is the first paper to provide a characterization of a class of games via the number of Nash equilibria. Interestingly, this characterization sheds light on the (wrong) conjecture of Quint and Shubik (1997) that any (nondegenerate) $n \times n$ bimatrix game has at most $2^n - 1$ Nash equilibria.[6] Our result implies that the number of Nash equilibria given by our class of games with the bandwagon property is exactly the same as the maximum number given by the Quint-Shubik conjecture. Secondly, we provide new insights on equilibrium selection methods.

The rest of the paper is organized as follows. The next section presents the underlying game considered in this paper. Section 3 first gives the characterization result of symmetric games with TBP via the number of Nash equilibria and then extend the charaterization to bimatrix games. Section 4 focuses on the equilibrium selection problem for the class of games with TBP, thereby providing the simple selection criterion consistently chosen by various methods. Section 5 applies the above obtained results to the experimental game. Section 6 concludes.

2   The Underlying Game

We consider a symmetric two-player game $\mathbf{g} = (A, g)$ where $A = \{1, 2, \ldots, n\}$ with $|A| = n \geq 2$ is the finite set of pure strategies and $g : A^2 \to \mathbb{R}$ is the symmetric payoff function. We write the set of mixed strategies by the $(n-1)$–dimensional simplex $\Delta = \{x \in \mathbb{R}^n \mid \forall i \in A, x_i \geq 0, \sum_{i \in A} x_i = 1\}$. For any $x \in \Delta$, let $\text{supp}(x) = \{i \in A \mid x_i > 0\}$ be the support of x and $\text{br}(x) = \text{argmax}_{i \in A} \sum_{j \in A} g_{ij} x_j$ be the set of pure strategy best

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[5]The two-player minimum-effort game used in their experiment of Van Huyck et al. (1990) is also the knife-edge case for a potential maximizer (Monderer and Shapley, 1996) and a logit equilibrium (Anderson et al., 2001).

[6]Keiding (1997) and McLennan and Park (1999) prove the conjecture in the case of $n \leq 4$ and Quint and Shubik (2002) for games where payoff matrices are identical between two players, while von Stengel (1999) shows that it does not hold in general. In fact, von Stengel (1999) constructs a general lower bound on the maximal number of Nash equilibria based on a technique of polytope theory, and then as its application, he provides a counterexample of an asymmetric 6×6 game with 75 Nash equilibria, which is larger than $2^6 - 1 = 63$. Nonetheless, the case of $n = 5$ is still unknown to the best of our knowledge.

responses against x. When $\text{supp}(x) = S \subseteq A$, we write $x \in \mathring{\Delta}(S)$ instead of $x \in \Delta$. A game is *nondegenerate* if $|\text{br}(x)| \leq |\text{supp}(x)|$ for any $x \in \Delta$, it is a *coordination game* if any symmetric pure strategy profile is a strict Nash equilibrium, and it is a *pure coordination game* if for any $i, j \in A$, $g_{ij} > 0$ if $i = j$, otherwise $g_{ij} = 0$.

Kandori and Rob (1998) introduce the following concept capturing a bandwagon effect regarding consumer technology adoptions in an evolutionary context.

Definition 1. A game $\mathbf{g} = (A, g)$ has the total bandwagon property (TBP) if $\text{br}(x) \subseteq \text{supp}(x)$ for any $x \in \Delta$.

TBP is a strong condition when just considering the symmetric game itself, but it is meaningful when considering the following evolutionary model usually adopted in the literature. There is a society made of a single population. Each period each agent of the population is randomly matched with one other agent in the society to play a game $\mathbf{g}$. We assume that each agent is a myopic decision maker and observes an action distribution $x \in \Delta$ taken in the last period, and then believes that the opponent with whom he is randomly matched chooses the same mixed strategy as the observed action distribution x, thereby choosing a best response against the belief x. In this situation, TBP requires that it is the best for the agent to take one of the actions taken in the society instead of taking an action not taken in the society. Note that any game with TBP is a nondegenerate coordination game and in addition that any pure coordination game and slightly perturbed ones satisfy TBP.

TBP is related to the set-valued solution concept, *curb set*, introduced by Basu and Weibull (1991) and further investigated by Ritzberger and Weibull (1995) in an evolutionary context. Since we here focus on symmetric (coordination) games, we simply introduce the definition of curb sets by using subset of strategies instead of subset of strategy profiles in the following way.

Definition 2. A subset of strategies $S \subseteq A$ is a *curb set* if for any $x \in \mathring{\Delta}(S)$,

$$\text{br}(x) \ \subseteq \ S \ \left(= \bigcup_{x \in \Delta(S)} \text{supp}(x) \right)$$

From this definition, it is easy to see that TBP is equivalent to the condition that any $S \subseteq A$ is a curb set.[7]

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[7]The curb set is not a condition for a class of games but is used for a solution concept, while TBP is a condition for games. Formally, a curb set is defined in an N-person normal-form game and a product

3 Characterization

We provide the characterization of games with TBP via the number of NE as follows.

Theorem 1. *Let $\mathbf{g} = (A, g)$ be a symmetric $n \times n$ game. The game $\mathbf{g}$ has TBP if and only if $\mathbf{g}$ has $2^n - 1$ symmetric Nash equilibria.*

The proof of Theorem 1 is given in the Appendix. Theorem 1 gives us the following interesting points. First of all, the class of games with the bandwagon property is characterized by the number of NE. To the best of our knowledge, this is the first paper to provide a characterization of games via the number of Nash equilibria. Secondly, Theorem 1 is related to the conjecture of Quint and Shubik (1997) for the number of NE that any nondegenerate $n \times n$ bimatrix game has at most $2^n - 1$ NE including asymmetric ones. In general however, von Stengel (1999) shows that the Quint-Shubik conjecture does not hold. But we can say that the conjecture holds for any symmetric $n \times n$ game with TBP.[8] More precisely, the class of symmetric games with TBP obtains the maximum number of NE given by the Quint-Shubik conjecture, and conversely, the game with the maximum number of NE must have TBP as long as the game is symmetric.

3.1 Extension to Bimatrix Games

It is natural to extend the notion of TBP to allow for asymmetric games as follows.

We write a bimatrix game as $\mathbf{g} = (\{1,2\}, (A_i)_{i=1,2}, (g^i)_{i=1,2})$ where $A_1 = A_2 = \{1, 2, \ldots, n\}$ is the linearly ordered set of strategies, and $g^i : A^2 \to \mathbb{R}$ the payoff function of player $i = 1, 2$.[9] Since we consider the common set of strategies $A_1 = A_2 = \{1, 2, \ldots, n\}$, we simply denote by $\mathbf{g} = (\{1,2\}, A, (g^i)_{i=1,2})$ a bimatrix game and use notations in almost the same way as in symmetric games. For clarity, we write by $x^i \in \Delta$ a mixed strategy of player i. We denote by $\mathrm{br}_i(x^j) = \mathrm{argmax}_{k \in A} \sum_{h \in A} g^i(k, h) x_h^j$

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set of pure strategies. There are two related set-valued concepts. One of them is *retract* defined by Kalai and Samet (1984), which is a product set of mixed strategies. This concept is used to generalize the concept of NE and give a refinement of NE as in trembling hand perfect equilibrium (Selten, 1975) and proper equilibrium (Myerson, 1978). The other is *pre set* defined by Voorneveld (2004), which is a product set of pure strateiges as in curb set. For a relation among the three concepts, see van Damme (2002) and Voorneveld (2005).

[8]We can show that the game has no asymmetric NE in addition to Theorem 1. For proof showing that any symmetric game with TBP has no asymmetric equilibrium, see Remark 2 in the Appendix.

[9]We consider the linearly ordered set of strategies for simple notations. We assume that the size of set of strategies is common for both players by $|A_1| = |A_2| = n$ in order to extend TBP to asymmetric games (see the Appendix for an example to explain why we need the same size of set of strategies).

the set of pure strategy best responses of player i against the opponent j's mixed strategy $x^j \in \Delta$ as $j \neq i$. A game $\mathbf{g}$ is *nondegenerate* if for any $i, j = 1, 2$ and any $x^j \in \Delta$ as $i \neq j$, $|\mathrm{br}_i(x^j)| \leq |\mathrm{supp}(x^j)|$, and a (pure) coordination game is defined in the same way as in symmetric games.

We define the naturally extended TBP to bimatrix games as follows.

Definition 3. The game $\mathbf{g} = (\{1,2\}, A, (g^i)_{i=1,2})$ has the generalized TBP (GTBP) if for $i, j = 1, 2$ with $i \neq j$ and any $x^j \in \Delta$, $\mathrm{br}_i(x^j) \subseteq \mathrm{supp}(x^j)$.

Note that a game with GTBP is a coordination game and nondegenerate, and in addition that by allowing for permutations or reordering of strategies, we can consider a slightly larger class of games than those with GTBP, such as a class of anti-coordination games including Hawk-Dove games.

We extend the characterization of Theorem 1 to bimatrix games by incorporating restrictions on support of NE as follows.

Theorem 2. *Let $\mathbf{g} = (\{1,2\}, A, (g^i)_{i=1,2})$ be an $n \times n$ bimatrix game. The game $\mathbf{g}$ has GTBP if and only if $\mathbf{g}$ has $2^n - 1$ Nash equilibria, each of which gives the same support for both players that is distinct from those of other Nash equilibria.*

See the Appendix for the proof of Theorem 2. It turns out that $\mathbf{g}$ has GTBP if and only if both g_1 and g_2 (viewed as symmetric $n \times n$ games) have TBP. This tells us that the conjecture of Quint and Shubik (1997) still holds for the class of bimatrix games with the bandwagon property. The reason why we restrict support of NE is given in the Appendix where we show that the game without the restriction may not have GTBP.

4 The Equilibrium Selection Problem

So far we have shown that any two-player game with the bandwagon property has many NE, so that it seems hard to select a single equilibrium. With this in mind, the objective of this section is to provide a simple equilibrium selection criterion. Such a simple but strong equilibrium selection criterion is the solution concept of $1/2$–*dominant equilibrium* proposed by Morris, Rob and Shin (1995), which is a generalization of *risk dominant equilibrium* (Harsanyi and Selten, 1988). It is chosen by various equilibrium selection methods mentioned below. A $1/2$–dominant equilibrium needs not to exist in games with TBP. We will show the existence of a $1/2$–dominant equilibrium for two subclasses of games with TBP: (i) supermodular games; (ii) potential games.

4.1 Half–Dominant Equilibrium

There are various equilibrium selection methods: (1) the evolutionary learning method of best-response dynamics with mutations (Kandori et al., 1993; Young, 1993); (2) the global game method (Carlsson and van Damme, 1993); (3) the incomplete information method (Kajii and Morris, 1997); (4) the perfect foresight dynamics method (Matsui and Matsuyama, 1995; Hofbauer and Sorger, 1999); (5) the spatial dominance method (Hofbauer, 1999), among others. One common property among those methods is that if a two-player game has a $1/2$–dominant equilibrium, then it is uniquely selected by all above methods. The $1/2$–dominant equilibrium (Morris et al., 1995) is defined as follows.[10]

Definition 4. A strategy profile $(i^*, i^*) \in A^2$ is a $1/2$–*dominant equilibrium* if for any $x \in \Delta$ with $x_{i^*} \geq 1/2$,

$$\text{br}(x) = \{i^*\}.$$

Note that if a strategy profile is a $1/2$–dominant equilibrium, then it is a strict NE. To make the above condition more clear, we can rewrite it by the following equivalent condition: for strategy profile (i^*, i^*) and any strategy profile $(i, j) \in A^2$ with $(i, j) \neq (i^*, i^*)$,

$$\frac{1}{2}g_{i^*i^*} + \frac{1}{2}g_{i^*j} > \frac{1}{2}g_{ii^*} + \frac{1}{2}g_{ij}. \tag{1}$$

Also, note that a game can have at most one $1/2$–dominant equilibrium. A game with TBP may have no $1/2$–dominant equilibrium.

As another generalization of risk-dominant equilibrium, Kandori and Rob (1998) consider a risk–dominant concept for strategies based on pairwise comparison. Consider a coordination game and two distinct pure strategy NE (i, i) and (j, j). Then, strategy i *pairwise risk dominates* j (i PRD j) if

$$g_{ii} - g_{ji} > g_{jj} - g_{ij}.$$

If i PRD j for any strategy $j \neq i$, then i is *globally pairwise risk dominant* (GPRD). We call a symmetric pure strategy NE in which the strategy is GPRD a *GPRD-equilibrium*.

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[10]Note that the following definition is for symmetric (two-player) coordination games but the formal definition is for all games.

Note that if a game has a GPRD-equilibrium, by definition, the GPRD-equilibrium is unique and also that since a GPRD-equilibrium is a natural extension of risk-dominant equilibrium and a weaker concept than a 1/2–dominant equilibrium, a game may have a GPRD-equilibrium but not a 1/2–dominant equilibrium. But it is easily shown:

Lemma 1. *If a GPRD-equilibrium exists in a game with TBP, it is a 1/2–dominant equilibrium.*

See the Appendix for the proof of Lemma 1.

4.2 Supermodularity

In economics, strategic complementality (supermodularity) has been considered to be important (Milgrom and Roberts, 1990), and is defined as follows.

Definition 5. A game $\mathbf{g} = (A, g)$ with $A = \{1, 2, \ldots, n\}$ is *supermodular* if for any $i, i', j, j' \in A$ with $i > i'$ and $j \geq j'$,

$$g_{ij} - g_{i'j} \geq g_{ij'} - g_{i'j'}. \tag{2}$$

By definition, if a game $\mathbf{g} = (A, g)$ is supermodular, each person's best response is non-decreasing in his opponent's strategies.

Adding supermodularity to games with TBP, we can provide a simple equilibrium selection criterion based on the 1/2–dominance as follows.

Proposition 1. *We consider a symmetric $n \times n$ game $\mathbf{g} = (A, g)$ where $A = \{1, \ldots, n\}$ and $g_{11} - g_{n1} \neq g_{nn} - g_{1n}$. If $\mathbf{g}$ has TBP and is supermodular, the game always has a 1/2–dominant equilibrium. The 1/2–dominant equilibrium is either the lowest or the highest strategy profile, $(1, 1)$ or (n, n), and given by*

$$\begin{cases} (1, 1), & if \quad g_{11} - g_{1n} > g_{nn} - g_{n1}, \\ (n, n), & if \quad g_{11} - g_{1n} < g_{nn} - g_{n1}. \end{cases}$$

The proof of Proposition 1 is given in the Appendix. It turns out that a GPRD–equilibrium in a supermodular game with TBP is a 1/2–dominant equilibrium. Proposition 1 basically tells us that we can guarantee the existence of a 1/2–dominant equilibrium in a subclass of games with TBP and then provide a simple prediction to select a single equilibrium. Note that the condition $g_{11} - g_{n1} \neq g_{nn} - g_{1n}$ is very mild and holds in all generic games.

4.3 Potential Games

We consider a potential game (Monderer and Shapley, 1996) or partnership game (Hofbauer and Sigmund, 1988) in order to show a connection of equilibrium selection methods between the potential game method and the other methods introduced in Section 4.1, given that the game has TBP. In the literature, it has been shown that the potential game method is consistent with other equilibrium selection methods including the incomplete information method (Ui, 2001; Morris and Ui, 2005; Oyama and Tercieux, 2009), the global game method (Frankel, Morris and Pauzner, 2003), and the perfect foresight dynamics method (Hofbauer and Sorger, 1999, 2002; Oyama, Takahashi and Hofbauer, 2008).[11]

We introduce a potential game (Monderer and Shapley, 1996) for a symmetric two-player game as follows. Given a symmetric game $\mathbf{g} = (A, g)$, a symmetric function $v : A^2 \to \mathbb{R}$ with $v_{ij} = v_{ji}$ for any $i, j \in A$ is a *potential function* of $\mathbf{g}$ if for any $i, i', j \in A$,[12]

$$g_{i'j} - g_{ij} = v_{i'j} - v_{ij}.$$

Definition 6. A pure strategy profile $(i^*, j^*) \in A^2$ is a *potential maximizer* if there exists a potential function $v : A^2 \to \mathbb{R}$ such that $(i^*, j^*) \in \arg\max_{(i,j) \in A^2} v_{ij}$. We call $\mathbf{g}$ a *potential game* if there exists a potential function $v : A^2 \to \mathbb{R}$.

Any potential maximizer is a NE. If a game has TBP, since we know by Theorem 1 that every symmetric pure strategy profile is a NE while there is no asymmetric NE, only a symmetric pure strategy profile can be a potential maximizer. It is shown by Monderer and Shapley (1996, Lemma 2.7) that if a game $\mathbf{g} = (A, g)$ has a potential function v, it is unique up to constant, meaning that when taking v and v' as potential functions of $\mathbf{g}$, there exists a constant c such that for any $(i, j) \in A^2$, $v_{ij} - v'_{ij} = c$. This implies that it is enough to focus on a potential function when considering the equilibrium selection criterion of potential games.

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[11]For the relation between the potential game method, the incomplete information method, and the global game method, see Basteck and Daniëls (2011), Honda (2011), and Oyama and Takahashi (2011).

[12]In fact, Monderer and Shapley (1996) define a potential function in an (possibly asymmetric) N-person game where the symmetry of potential functions does not necessarily hold. For instance, a two-player bimatrix game $\mathbf{g} = (A, (g^i)_{i=1,2})$ as defined in previous section has a potential function $v : A^2 \to \mathbb{R}$ of $\mathbf{g}$ if for any $h = 1, 2$ and any $i, i', j \in A$, $g^h_{ij} - g^h_{i'j} = v_{ij} - v_{i'j}$ If a game is symmetric, by definition, we obtain the symmetry of $v_{ij} = v_{ji}$. Hofbauer and Sigmund (1988) call such a game a (rescaled) partnership game.

We show the existence of a 1/2–dominant equilibrium in a subclass of games with TBP as follows.

Proposition 2. *We consider a two-player symmetric game* $\mathbf{g} = (A, g)$ *with TBP. Suppose that* $\mathbf{g}$ *has a potential function with a unique potential maximizer. Then, the potential maximizer is a* 1/2*–dominant equilibrium.*

The proof of Proposition 2 is given in the Appendix where we show that the potential maximizer is a GPRD–equilibrium, and then use Lemma 1 to show that the potential maximizer is 1/2–dominant equilibrium. Actually we can extend Proposition 2 to a generalized potential maximizer introduced below.

4.3.1 Generalized Potential Games and Log-Linear Dynamics

As an equilibrium selection method, we consider the log-linear dynamics introduced by Blume (1993).[13] We briefly explain what is the log-linear dynamics. Let us consider a single population consisting of N players who interact in a normal-form game. The log-linear dynamic is a stochastic process in discrete time and its state space is the set of all pure strategy profiles. An initial strategy profile is chosen according to a distribution. At each subsequent period, only one of players is randomly selected and gets an opportunity to revise his or her strategy. The strategy revisions follow the log-linear stochastic choice rule under which the log likelihood ratio between two strategies is proportional to the difference between payoffs of those actions. The (common) factor of proportionality in the choice rule is exogenously given and is interpreted as noise of payoff information. The log-linear choice rule generates a time-homogeneous Markov chain on the set of pure strategy profiles, which is irreducible and aperiodic. As the long-run outcome in this dynamic, we consider a unique invariant distribution of the Markov chain when the noise level goes to zero. It is known that if an exact potential function exists in a game, the unique invariant distribution in the log-linear dynamic is explicitly given by a simple closed form.[14] This gives us a powerful tool when using an explicit stationary distribution in an application.[15]

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[13]See also Blume (1997) and Young (1998). Note that this paper takes into account a discrete time version of log-linear dynamics (Blume, 1997) instead of its continuous time version (Blume, 1993).

[14]This is because the Markov chain satisfies reversibility, the detailed balance conditions hold for an invariant distribution, and the Gibbs representation of an invariant distribution is applied.

[15]As a suitable application, Young and Burke (2001) consider agricultural contracts of crops in Illinois as a case study to investigate an evolutionary process for the contracts and then provide an explanation why currently adopted contracts according to regions are established and stable.

In the following, we introduce a solution concept used in the study of the log-linear dynamics, local potential maximizer (Morris, 1999), which is a generalization of potential maximizer (Monderer and Shapley, 1996). Consider a symmetric game $\mathbf{g} = (A, g)$ with $A = \{1, , \ldots, n\}$. Then, we define a simplified local potential maximizer (with constant weights) as in Okada and Tercieux (2012, Definition 1).[16]

Definition 7. A pure strategy profile $s^* = (i^*, j^*) \in A^2$ is a *local potential maximizer* (LP-maximizer) of $\mathbf{g}$ if there exists a local potential function $v : A^2 \to \mathbb{R}$ with $v_{s^*} > v_s$ for all $s \in A^2 \backslash \{s^*\}$ such that any $i, j \in A$,

$$v_{i+1j} - v_{ij} \leq g_{i+1j} - g_{ij}, \quad \text{if } i < i^*,$$
$$v_{i-1j} - v_{ij} \leq g_{i-1j} - g_{ij}, \quad \text{if } i > i^*,$$

and similarly

$$v_{j+1i} - v_{ji} \leq g_{j+1i} - g_{ji}, \quad \text{if } j < j^*,$$
$$v_{j-1i} - v_{ji} \leq g_{j-1i} - g_{ji}, \quad \text{if } j > j^*.$$

The notion of LP-maximizer relaxes the equality condition of potential-maximizer by an inequality under a certain requirement on the local relation in terms of order of strategies between v and g. Note that since LP-maximizer is a generalization of the potential maximizer, there is a class of games where an LP-maximizer exists but no potential maximizer does, and also note that a game may have multiple LP-maximizers.[17] A $1/2$–dominant equilibrium is in general irrelevant for the selection criterion under the log-linear dynamics. It is known that if a local potential maximizer exists in a supermodular game, it is selected in the log-linear dynamics (Okada and Tercieux, 2012).

We provide a relation between LP-maximizer and $1/2$–dominant equilibrium as follows.

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[16]See Morris (1999) for the detail and Okada and Tercieux (2012, Definition 3) for the simplified version with non-constant weights.

[17]Although Frankel et al. (2003) claim that an LP-maximizer of a supermodular game that satisfies own-action *quasi-concavity* is unique in terms of noise-independent selection in global games, Oyama and Takahashi (2009, Example 1) provide a counterexample that this claim does not hold. In fact, Oyama et al. (2008) show that an LP-maximizer of a supermodular game that satisfies diminishing marginal return (or own-action *concavity*) is at most one, and so if an LP-maximizer exists in such a game, it is unique and no multiplicity happens.

Proposition 3. *Let us consider a supermodular game* $\mathbf{g} = (A, g)$ *with TBP. Suppose that there exists an LP-maximizer* (i, i) *for* $i \in A$. *Then,* (i, i) *is a* $1/2$*–dominant equilibrium.*

For the proof of Proposition 3, see the Appendix. Together with Proposition 1, the above considered LP-maximizer must be either the lowest strategy profile or the highest strategy profile. In a subclass of games with the bandwagon property, the log-linear dynamic selects the $1/2$–dominant equilibrium chosen out of many equilibria as well as the best response dynamics with mutation (Kandori et al., 1993; Young, 1993) and others.

5 Application to the Minimum-Effort Game

We introduce the minimum-effort (or the weak-link) game defined by Van Huyck, Battalio and Beil (1990) and then examine one of their experimental results showing that no stable outcome obtains in their experiment if subjects repeatedly play the two-player minimum-effort game with the *random matching*. To this aim, we first show that the two-player minimum-effort game does not satisfy TBP but is a limit case of TBP. Then we examine the result of their experiment by the selection criterion of the $1/2$–dominant equilibrium.

The Minimum-Effort Game

The two-player minimum-effort game is a symmetric $n \times n$ coordination game $\mathbf{g} = (A, g)$ such that each strategy $i \in A = \{1, 2, \ldots, n\}$ represents an effort level and the payoff of player who takes i given the opponent's effort level j is determined by

$$g_{ij} = a \min\{i, j\} - bi + c \tag{3}$$

where a, b, and c are positive constants such that $a > b > 0$ and $c > 0$ guarantees positive payoffs for all subjects in the experiment for any given effort profile. This payoff function entails an interesting feature in capturing coordination problems that the effort cost of each subject depends only on the individual effort choice while the benefit depends upon the minimum of effort levels chosen by both subjects, and therefore each subject has no incentive to choose a higher effort level than that of the other subject. The minimum-effort game is a symmetric $n \times n$ supermodular coordination game. The lowest strategy

	1	2	3
1	0.70	0.70	0.70
2	0.60	0.80	0.80
3	0.50	0.70	0.90

Table 1: The 3×3 minimum-effort game where $(a, b, c) = (0.20, 0.10, 0.60)$.
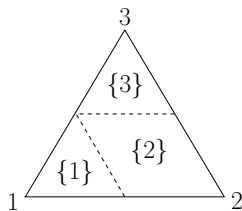

Figure 1: The best response regions of the 3×3 minimum-effort game.

profile $(1, 1)$ and the highest strategy profile (n, n) are the most inefficient and the most efficient equilibria in the game, respectively.

Property of the Minimum-Effort Game

We first illustrate that the minimum-effort game does not satisfy TBP but is a limit case of TBP. Consider a symmetric 3×3 minimum-effort game given by Table 1. The best response regions of the game are given by Figure 1 where we can see that the 3×3 minimum-effort game "almost" satisfies TBP in the sense that $\text{br}(x) \subseteq \text{supp}(x)$ holds for "almost" all $x \in \Delta$ except for one specific point $x^{13} \in \mathring{\Delta}(\{1, 3\})$ violating TBP where $\text{supp}(x^{13}) \subsetneq \text{br}(x^{13})$ holds. Actually we can show that the two-player minimum-effort game does not satisfies TBP but is a limit case of TBP in the following sense.

Lemma 2. *For any $l, m, h \in A$ with $l < m < h$, there exists a unique point $x^* \in \mathring{\Delta}(\{l, h\})$ such that $x_l^* = (a - b)/a$, $x_h^* = 1 - x_l^*$, and*

$$supp(x^*) \subsetneq br(x^*), \tag{4}$$

$$\forall x \in \Delta(\{l, h\}) \setminus \{x^*\}, \quad br(x) \subseteq supp(x). \tag{5}$$

The proof of Lemma 2 is given in the Appendix.

From above, the minimum-effort game contains the knife-edge case for the $1/2$–dominant equilibrium in the minimum effort game $\mathbf{g} = (A, g)$ with $|A| = n$ because the

14

	1	2	3	4	5	6	7
1	0.70	0.70	0.70	0.70	0.70	0.70	0.70
2	0.60	0.80	0.80	0.80	0.80	0.80	0.80
3	0.50	0.70	0.90	0.90	0.90	0.90	0.90
4	0.40	0.60	0.80	1.00	1.00	1.00	1.00
5	0.30	0.50	070	0.90	1.10	1.10	1.10
6	0.20	0.40	0.60	0.80	1.00	1.20	1.20
7	0.10	0.30	0.50	0.70	0.90	1.10	1.30

Table 2: The minimum-effort game used in the experiment.

$1/2$–dominant equilibrium is given by

$$
\begin{cases}
\{(1,1)\}, & \text{if } a < 2b, \\
\{(n,n)\}, & \text{if } a > 2b.
\end{cases}
\tag{6}
$$

Similarly, it is easily shown (Monderer and Shapley, 1996) that the minimum-effort game is a potential game where a potential function $v : A^2 \to \mathbb{R}$ is given by $v_{ij} = a\min\{i,j\} - b(i+j)$ for any $(i,j) \in A^2$ and the potential maximizer by (6) as well as the $1/2$–dominant equilibrium except for the knife-edge case where all symmetric pure strategy profiles are potential maximizers.

Examination of Experimental Result

Van Huyck et al. (1990) run the experiments for the two-subject case together with the many-subject case by specifying the payoff matrix of the minimum-effort game in such a way that $A = \{1, 2, \ldots, 7\}$ and $(a, b, c) = (0.20, 0.10, 0.60)$, which is described by Table 2. At each session in each case, one group of subjects repeatedly play the game under the fix-pair while the other under the random-pair. Their experimental result in the two-subject case shows that the most efficient equilibrium is selected under the fix-pair, while no stable outcome obtains under the random-pair.[18] For the equilibrium selection result in case of the fixed-pair, Van Huyck et al. (1990) apply the repeated game argument to

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[18]In the many-subject case, the clear convergence to the most inefficient outcome obtains in their experiment, which is one of well known coordination failure problems. This can be simply explained in a way that, when choosing a higher effort level than the minimum level, a player thinks that all subjects are less likely to coordinate to choose high effort levels as there are more subjects. For details, see Van Huyck et al. (1990).

justify the result, which makes sense. For the result in case of the random-pair, however, they do not give a justification. When $a = 2b$ used in the experiment, by (6), this is the knife-edge case where there is no $1/2$–dominant equilibrium,[19] and therefore we cannot apply the theoretical prediction obtained in the previous sections to the experimental game. The equilibrium selection methods considered in this paper are based on random matching where no convergence is obtained in their experiment.

6 Conclusion

This paper considered two-player games with the bandwagon property and then pinned down the underlying characteristic of those games. In doing so, we provided two main results. One of them is a characterization of a class of games via the number of Nash equilibria. To the best of our knowledge, this is the first paper to characterize a class of games by the number of Nash equilibria. It is of interest that the class of games with the maximum number of Nash equilibria given by the Quint-Shubik conjecture is equivalent to that of games with the bandwagon property. Secondly, taking into account that the games with the bandwagon property has too many equilibria to select a single equilibrium, we gave it a simple equilibrium selection criterion that is commonly chosen by various methods. Applying our results to the minimum-effort game, we clarified the property of the game.

Appendix: Proofs

Proof of Theorem 1

We provide proofs for the if part and the only if part of separately.

Remark 1. Any game with TBP is nondegenerate, and so the game has the oddness property regarding the number of NE. For this fact, see Shapley (1974, Theorem 2) and Quint and Shubik (1997, Lemma 2.2), among others. Note that any nondegenerate game $\mathbf{g}$ has at most symmetric $2^n - 1$ NE.

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[19]Similarly, Goeree and Holt (2005) point out that this is the knife-edge case for a logit-equilibrium (Anderson et al., 2001).

	1	2	3	4
1	a	c	b	c
2	c	a	c	b
3	b	c	a	c
4	c	b	c	a

Table 3: A symmetric 4×4 coordination game where $a > b$, $a > c$, and $a + b > 2c$.

Proof of the if part

Proof. Consider a symmetric game $\mathbf{g} = (A, g)$ with $2^n - 1$ symmetric NE. Assume to the contrary that TBP does not hold. Taking into account that TBP is equivalent to the condition that $\mathrm{br}(x)\backslash\mathrm{supp}(x) = \emptyset$ for any $x \in \Delta$, if TBP does not hold, there must exist some $\tilde{x} \in \Delta$ such that

$$\mathrm{br}(\tilde{x})\backslash\mathrm{supp}(\tilde{x}) \neq \emptyset. \tag{A.1}$$

If $\tilde{x}$ is in the standard basis denoted by $\cup_{i \in A}\mathbf{e}_i$, $\mathrm{br}(\tilde{x}) = \mathrm{supp}(\tilde{x}) = j$ holds for some $j \in A$ because all symmetric pure strategy profile are NE, a contradiction to (A.1). Also, if $\tilde{x} \in \mathring{\Delta}(A)$, $\mathrm{br}(\tilde{x}) \subseteq A = \mathrm{supp}(\tilde{x})$ holds, a contradiction to (A.1). Next, we consider the remaining case of $\tilde{x} \in \Delta\backslash((\cup_{i \in A}\mathbf{e}_i) \cup \mathring{\Delta}(A))$ and let $S = \mathrm{supp}(\tilde{x}) \subsetneq A$. Assume that $j \in \mathrm{br}(\tilde{x})\backslash\mathrm{supp}(\tilde{x}) \neq \emptyset$. Given that the game has all symmetric NE, it has a unique symmetric NE (x^*, x^*) such that $x^* \in \mathring{\Delta}(\{j\} \cup S)$. But, since $j \in \mathrm{br}(\tilde{x})\backslash\mathrm{supp}(\tilde{x}) \neq \emptyset$ is assumed to hold, together with the fact that all pure strategy best response sets are convex, the best response region of pure strategy j goes across the entire interior of the face spanned by all pure strategies in $\{j\} \cup \mathrm{supp}(\tilde{x})$ and prevents the NE (x^*, x^*) where the best responses of all strategies in $\{j\} \cup \mathrm{supp}(\tilde{x})$ meet to exist, a contradiction. $\square$

We will illustrate via an example how the above shown proof works.

Example 1. Let us consider a symmetric 4×4 coordination game given by Table 3 where there are three parameters a, b, and c such that $a > b$, $a > c$, and $a + b > 2c$.

We can show that the game has $2^4 - 1 = 15$ symmetric NE. For example, as $(a, b, c) = (10, 5, 7)$, all (symmetric) NE of this game are given by 4 pure strategy NE, 6 completely mixed strategy NE over S with $|S| = 2$ such that $x_i = x_j = 1/2$ for any two distinct $i, j \in A$, 4 completely mixed strategy NE over S with $|S| = 3$ such that $(3/7, 1/7, 3/7, 0), (0, 3/7, 1/7, 3/7), (3/7, 0, 3/7, 1/7), (1/7, 3/7, 0, 3/7)$, and a unique interior NE, $(1/4, 1/4, 1/4, 1/4)$.

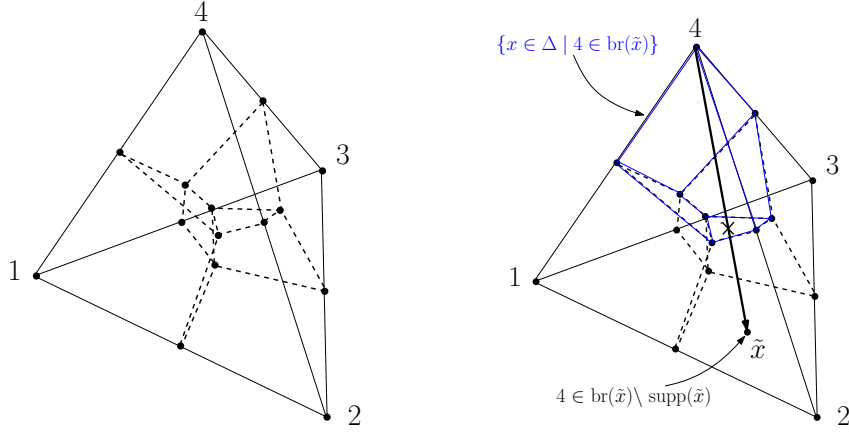Assume to the contrary that TBP does not hold, so that there exists some $\tilde{x} \in$

Figure 2: The best response regions of the game (Left) and an illustration that the condition $4 \in \mathrm{br}(\tilde{x})\backslash\mathrm{supp}(\tilde{x}) \neq \emptyset$ breaks down the best response regions (Right).

$\mathring{\Delta}(\{1,2,3\})$ such that (A.1) holds. This corresponds to the case of $\tilde{x} \in \Delta\backslash((\cup_{i\in A}\mathbf{e}_i) \cup \mathring{\Delta}(A))$ as $A = \{1,2,3,4\}$ where $S = \mathrm{supp}(\tilde{x}) = \{1,2,3\} \subsetneq A$, and there exists the strategy $j = 4 \in \mathrm{br}(\tilde{x})\backslash\mathrm{supp}(\tilde{x}) \neq \emptyset$, which breaks down the best response regions of the game, which is described by Figure 2.

Alternative proof of the if part

Let us consider any symmetric two-player game $\mathbf{g} = (A, g)$ with TBP and $|A| \geq 2$. For any nonempty subset $S \subseteq A$, we define $\mathbf{g}|_S$ by the *restricted game* of $\mathbf{g}$ where the players choose actions only from S.

Proof. We show the if part as follows. If the number of symmetric NE in symmetric game $\mathbf{g} = (A, g)$ with $|A| = n$ is $2^n - 1$, from Remark 1, then the game must have all possible $2^n - 1$ symmetric NE. Let $X^1 = \{x \in \Delta \mid |\mathrm{supp}(x)| = 1\}$. Then, all symmetric pure strategy profiles are NE, and so $\mathrm{br}(x) \subseteq \mathrm{supp}(x)$ holds for all $x \in X^1$. Also, for all $x \in \mathring{\Delta}(A)$, $\mathrm{br}(x) \subseteq \mathrm{supp}(x)$ holds. To show that $\mathbf{g}$ has TBP, letting $X^2 = \Delta\backslash(X^1 \cup \mathring{\Delta}(A))$, we still have to show that $\mathrm{br}(x) \subseteq \mathrm{supp}(x)$ also holds for all $x \in X^2$. Take any given $x \in X^2$. Then, we construct x via a sequence of strategies of symmetric NE in restricted games denoted by $(x^*|_S)_{S\subseteq\mathrm{supp}(x)}$, showing that $\mathrm{br}(x) \subseteq \mathrm{supp}(x)$ in the following.

Let us consider a sequence of pairs of subset of strategies and some positive constants, $\{(S^m, c^m)\}_m$ with $S^m \subseteq A$ and $c^m > 0$ for $m = 0, 1, \ldots, M$ such that the

18

following condition $(*)$ holds:

$$S^0 = \text{supp}(x), \quad c^0 = \min_{i \in S^0} \frac{x_i}{x_i^*|_{S^0}} = \frac{x_{i_0}}{x_{i_0}^*|_{S^0}} \quad (i_0 = \arg\min_{i \in S^0} \frac{x_i}{x_i^*|_{S^0}}),$$

$$S^1 = S^0 \setminus \arg\min_{i \in S^0} \frac{x_i}{x_i^*|_{S^0}} = S^0 \setminus \{i_0\},$$

$$c^1 = \min_{i \in S^1} \frac{x_i - c^0 x_i^*|_{S^0}}{x_i^*|_{S^1}} = \frac{x_{i_1} - c^0 x_{i_1}^*|_{S^0}}{x_{i_1}^*|_{S^1}} \quad (i_1 = \arg\min_{i \in S^1} \frac{x_i - c^0 x_i^*|_{S^0}}{x_i^*|_{S^1}}),$$

$$(1 \le m \le M-1) \quad S^{m+1} = S^m \setminus \arg\min_{i \in S^m} \frac{x_i - \sum_{j=0}^{m-1} c^j x_i^*|_{S^j}}{x_i^*|_{S^m}} = S^m \setminus \{i_m\} = S^0 \setminus \{i_0, i_1, \dots, i_m\},$$

$$c^{m+1} = \min_{i \in S^{m+1}} \frac{x_i - \sum_{j=0}^{m} c^j x_i^*|_{S^j}}{x_i^*|_{S^{m+1}}},$$

$$\emptyset \subsetneq S^M \subsetneq \cdots \subsetneq S^1 \subsetneq S^0.$$

For the above defined sequence $\{(S^m, c^m)\}_m$, we can write x by

$$x = \sum_{m=0}^{M} c^m x^*|_{S^m}.$$

One can show that

$$\text{br}(x) = \bigcap_{m=0}^{M} \text{br}(x^*|_{S^m}). \tag{A.2}$$

Since $\text{br}(x^*|_{S^m}) = \text{supp}(x^*|_{S^m}) = S^m$ and $S^{m+1} \subsetneq S^m$ for $m = 0, 1 \dots, M-1$, together with (A.2), we have

$$\text{br}(x) = \bigcap_{m=0}^{M} \text{br}(x^*|_{S^m}) = \bigcap_{m=0}^{M} S^m = S^M \subsetneq S^0 = \text{supp}(x).$$

Note that if $M = 0$, $\text{br}(x) = \text{supp}(x)$ holds. This implies that $\text{br}(x) \subseteq \text{supp}(x)$ holds for all $x \in X^2$.

Thus, since we have shown that $\text{br}(x) \subseteq \text{supp}(x)$ holds for all $x \in \Delta$, the symmetric game $\mathbf{g}$ with $2^n - 1$ symmetric NE has TBP. $\square$

We will illustrate via an example how the above used iterative construction process works to show (A.2).

Example 2 (Symmetric 4×4 pure coordination game)**.** Let us consider the symmetric

	1	2	3	4
1	1,1	0,0	0,0	0,0
2	0,0	2,2	0,0	0,0
3	0,0	0,0	3,3	0,0
4	0,0	0,0	0,0	4,4

Table 4: A 4×4 pure coordination game.

4×4 pure coordination game of Table 4 where there are $2^4 - 1 = 15$ symmetric NE. All symmetric NE strategies of the game except for pure strategies are given as follows.

$$(\frac{2}{3}, \frac{1}{3}, 0, 0), (\frac{3}{4}, 0, \frac{1}{4}, 0), (0, \frac{3}{5}, \frac{2}{5}, 0), (0, 0, \frac{4}{7}, \frac{3}{7}), (\frac{4}{5}, 0, 0, \frac{1}{5}), (0, \frac{2}{3}, 0, \frac{1}{3}),$$
$$(\frac{6}{11}, \frac{3}{11}, \frac{2}{11}, 0), (0, \frac{6}{13}, \frac{4}{13}, \frac{3}{13}), (\frac{4}{7}, \frac{2}{7}, 0, \frac{1}{7}), (\frac{12}{19}, 0, \frac{4}{19}, \frac{3}{19}),$$
$$(\frac{12}{25}, \frac{6}{25}, \frac{4}{25}, \frac{3}{25})$$

For all $x \in \mathring{\Delta}(A)$, it is obvious that $\mathrm{br}(x) \subseteq \mathrm{supp}(x)$ holds because $\mathrm{br}(x) \subseteq A = \{1, 2, 3, 4\} = \mathrm{supp}(x)$.

For instance, take two specific points, $x = (\frac{3}{11}, \frac{6}{11}, \frac{2}{11}, 0)$ and $x' = (\frac{1}{3}, \frac{2}{5}, \frac{4}{15}, 0)$. Then, we demonstrate that both $\mathrm{br}(x) \subseteq \mathrm{supp}(x)$ and $\mathrm{br}(x') \subseteq \mathrm{supp}(x')$ hold by constructing sequences of strategies through Condition $(*)$ used in the proof. Consider a sequence of $\{(S^m, c^m)\}_m$ with $S^m \subseteq A$ and $c^m > 0$ for each $m = 0, 1, \ldots, M$ such that Condition $(*)$ holds. For $x = (\frac{3}{11}, \frac{6}{11}, \frac{2}{11}, 0)$, we have

$$S^0 = \{1, 2, 3\}, \quad c^0 = \min\left\{\frac{x_1}{x_1^*|_{S^0}}, \frac{x_2}{x_2^*|_{S^0}}, \frac{x_3}{x_3^*|_{S^0}}\right\} = \min\left\{\frac{3/11}{6/11}, \frac{6/11}{3/11}, \frac{2/11}{2/11}\right\} = \frac{1}{2} \quad (i_0 = 1),$$

$$S^1 = S^0\backslash\{i_0\} = \{1, 2, 3\}\backslash\{1\} = \{2, 3\},$$

$$c^1 = \min\left\{\frac{x_2 - c^0 x_2^*|_{S^0}}{x_2^*|_{S^1}}, \frac{x_3 - c^0 x_3^*|_{S^0}}{x_3^*|_{S^1}}\right\} = \min\left\{\frac{\frac{6}{11} - \frac{1}{2} \times \frac{3}{11}}{3/5}, \frac{\frac{2}{11} - \frac{1}{2} \times \frac{2}{11}}{2/5}\right\}$$

$$= 5\min\left\{\frac{3}{22}, \frac{1}{22}\right\} = \frac{5}{22} \quad (i_1 = 3),$$

$$S^2 = \{2\},$$

$$c^2 = \min\left\{\frac{x_2 - c^0 x_2^*|_{S^0} - c^1 x_2^*|_{S^1}}{x_2^*|_{S^2}}\right\} = \left\{\frac{\frac{6}{11} - \frac{1}{2} \times \frac{3}{11} - \frac{5}{22} \times \frac{3}{5}}{1}\right\}$$

$$= \frac{12 - 3 - 3}{22} = \frac{6}{22} = \frac{3}{11} \quad (i_2 = 2),$$

20

and as $M = 2$, we can write x by

$$
\begin{aligned}
x &= \sum_{m=0}^{2} c^m x^*|_{S^m} = c^0 x^*|_{S^0} + c^1 x^*|_{S^1} + c^2 x^*|_{S^2} \\
&= \frac{1}{2}\left(\frac{6}{11}, \frac{3}{11}, \frac{2}{11}, 0\right) + \frac{5}{22}\left(0, \frac{3}{5}, \frac{2}{5}, 0\right) + \frac{3}{11}(0, 1, 0, 0) = \left(\frac{3}{11}, \frac{6}{11}, \frac{2}{11}, 0\right).
\end{aligned}
$$

For $x' = (\frac{1}{3}, \frac{2}{5}, \frac{4}{15}, 0)$, similarly we have

$$
\begin{aligned}
\tilde{S}^0 &= \{1, 2, 3\}, \quad \tilde{c}^0 = \min\left\{\frac{1/3}{6/11}, \frac{2/5}{3/11}, \frac{4/15}{2/11}\right\} = \frac{11}{18} \quad (i_0 = 1), \\
\tilde{S}^1 &= \{2, 3\}, \\
\tilde{c}^1 &= \min\left\{\frac{x'_2 - \tilde{c}^0 x^*_2|_{\tilde{S}^0}}{x^*_2|_{\tilde{S}^1}}, \frac{x'_3 - \tilde{c}^0 x^*_3|_{\tilde{S}^0}}{x^*_3|_{\tilde{S}^1}}\right\} = \min\left\{\frac{\frac{2}{5} - \frac{11}{18} \times \frac{3}{11}}{3/5}, \frac{\frac{4}{15} - \frac{11}{18} \times \frac{2}{11}}{2/5}\right\} \\
&= 5 \min\left\{\frac{7}{90}, \frac{7}{90}\right\} = \frac{7}{18} \quad (i_1 = 2, 3),
\end{aligned}
$$

and as $M = 1$, we can write x' by

$$
\begin{aligned}
x' &= \sum_{m=0}^{1} \tilde{c}^m x^*|_{\tilde{S}^m} = \tilde{c}^0 x^*|_{\tilde{S}^0} + \tilde{c}^1 x^*|_{\tilde{S}^1} \\
&= \frac{11}{18}\left(\frac{6}{11}, \frac{3}{11}, \frac{2}{11}, 0\right) + \frac{7}{18}\left(0, \frac{3}{5}, \frac{2}{5}, 0\right) = \left(\frac{1}{3}, \frac{2}{5}, \frac{4}{15}, 0\right).
\end{aligned}
$$

From above, since

$$
\begin{aligned}
\mathrm{br}(x) &= \cap_{m=0}^{2} \mathrm{br}(x^*|_{S^m}) = \mathrm{br}(x^*|_{S^2}) = \{2\}, \\
\mathrm{br}(x') &= \cap_{m=0}^{2} \mathrm{br}(x^*|_{\tilde{S}^m}) = \mathrm{br}(x^*|_{\tilde{S}^2}) = \{2, 3\},
\end{aligned}
$$

and

$$
\mathrm{supp}(x) = \mathrm{supp}(x') = \{1, 2, 3\},
$$

it follows that both $\mathrm{br}(x) \subseteq \mathrm{supp}(x)$ and $\mathrm{br}(x') \subseteq \mathrm{supp}(x')$ hold.

Proof of the only if part

We show the only if part of Theorem 1 by using an induction argument together with the oddness property of NE.

First, we show:

Lemma 3. *Let $\mathbf{g} = (A, g)$ be any symmetric $n \times n$ game with TBP for any $n = 2, 3, \ldots$. Then any restricted game $\mathbf{g}|_S$ with $|S| \leq k(= 2, 3, \ldots, n)$ has a unique symmetric interior Nash equilibrium.*

Any restricted game $\mathbf{g}|_S$ has TBP if $\mathbf{g}$ has TBP.

Proof. We show Lemma 3 by induction as follows.

(I) It is obvious for $k = 2$.

(II) Suppose that any restricted game $\mathbf{g}|_S$ with $|S| \leq k(= 2, 3, \ldots, n-1)$ has a unique symmetric NE, $(x^*|_S, x^*|_S)$, such that $x^*|_S \in \mathring{\Delta}(S)$. This implies that the restricted game $\mathbf{g}|_S$ with $|S| = k' = 1, \ldots, k$ has $2^{k'} - 1$ symmetric NE. Take any (restricted) game $\mathbf{g}|_{S'}$ with $|S'| = k + 1$. Then we consider all combinations of restricted games of $\mathbf{g}|_{S'}$ with at least one strategy and at most $k + 1$ strategies that are subsets of S'. The number of all combinations of restricted games $\mathbf{g}|_S$ except for $\mathbf{g}|_{S'}$ is $2^{k+1} - 2$ and, by the assumption, each restricted game $\mathbf{g}|_S$ has a unique symmetric NE, $(x^*|_S, x^*|_S)$, such that $x^*|_S \in \mathring{\Delta}(S)$. Taking into account that any restricted game $\mathbf{g}|_S$ has TBP if $\mathbf{g}$ has TBP, it follows that $\mathbf{g}|_{S'}$ has at least $2^{k+1} - 2$ symmetric NE. Since $\mathbf{g}|_{S'}$ has at most symmetric $2^{k+1} - 1$ NE due to Remark 1, if $\mathbf{g}|_{S'}$ has more symmetric NE than $2^{k+1} - 2$, the only possibility is to have a unique symmetric NE, $(x^*|_{S'}, x^*|_{S'})$, such that $x^*|_{S'} \in \mathring{\Delta}(S')$. Since the number of NE, $2^{k+1} - 2$, is even, by the oddness property of NE, $\mathbf{g}|_{S'}$ must have $(x^*|_{S'}, x^*|_{S'})$. $\qquad\square$

The only if part of Theorem 1 straightforwardly follows from Lemma 3 as follows.

Proof. Any restricted game $\mathbf{g}|_S$ with $|S| \leq n-1$ has a unique symmetric NE, $(x^*|_S, x^*|_S)$, such that $x^*|_S \in \mathring{\Delta}(S)$ due to Lemma 3, and the number of all restricted games $\mathbf{g}|_S$ with $|S| \leq n - 1$ is $2^n - 2$. This implies that the game $\mathbf{g} = (A, g)$ has at least $2^n - 2$ symmetric NE, which do not include an interior NE in Δ, and, again by Lemma 3, the game must have a unique interior NE. Thus, $\mathbf{g} = (A, g)$ has $(2^n - 2) + 1 = 2^n - 1$ symmetric NE. $\qquad\square$

Remark 2. We can show that any symmetric game with TBP has no asymmetric NE as follows. Assume to the contrary that there is an asymmetric NE, $(x^1, x^2) \in \Delta^2$ with $x^1 \neq x^2$. For (x^1, x^2), there are two cases to consider: (i) $\text{supp}(x^1) = \text{supp}(x^2)$ and (ii) $\text{supp}(x^1) \neq \text{supp}(x^2)$. In Case (i), from the proof of the only if part of Theorem 1, we know that for any $S \subseteq A$, there is a unique symmetric NE, $x^*|_S$, which is

<div align="center">

Player 2

		1	2	3	4
	1	4,4	0,1	2,2	0,3
Player 1	2	0,0	4,11/6	0,3/2	2,1

Table 5: A 2×4 coordination game.

</div>

completely mixed over S. Together with this and TBP, $x^1 \neq x^2$ implies that there exists some $j = 1, 2$ such that $\mathrm{br}(x^j) \subsetneq \mathrm{supp}(x^j)$. Although (x^1, x^2) is an equilibrium, $\mathrm{br}(x^j) \subsetneq \mathrm{supp}(x^j)$ implies that some action in x^j of player j should not be chosen, a contradiction. In Case (ii), without loss of generality, there is some $j \in \mathrm{supp}(x^1)$ such that $j \notin \mathrm{supp}(x^2)$. By TBP, for player 1, strategy j is not a best response to player 2's mixed strategy x^2, a contradiction because x^1 must be a best response to x^2.

Why do we need the same size of set of strategies to extend the characterization?

Suppose that a game has $|A_1| \neq |A_2|$. Then we show that $|A_1| \neq |A_2|$ can violate GTBP by a counterexample, and therefore we assume that $|A_1| = |A_2|$ when considering games with GTBP. The counterexample given here is a slightly modified version of the game given by Quint and Shubik (2002, Remark 1) where the order of strategies for player 2 is changed.

Example 3 (Asymmetric coordination game)**.** Let us consider the 2×4 coordination game given by Table 5 where $|A_1| = 2 < 4 = |A_2|$. We can find 5 NE: two symmetric pure strategy profiles and three asymmetric strategy profiles,

$$((1/3, 2/3), (0, 0, 1/2, 1/2)), ((1/2, 1/2), (1/3, 0, 0, 2/3)),$$
$$((1/4, 3/4), (0, 1/3, 2/3, 0)).$$

One can easily observe that GTBP does not hold. For some player 2's mixed strategy $x^2 \in \Delta(A_2)$, the player 1's pure strategy best responses, $\mathrm{br}_1(x^2) \not\subset \mathrm{supp}(x^2)$. For example, for the equilibrium $((1/3, 2/3), (0, 0, 1/2, 1/2))$, $\mathrm{br}_1(x^2) \cap \mathrm{supp}(x^2) = \emptyset$.

Proof of Theorem 2

Proof of the if part

Proof. Given that an $n \times n$ bimatrix game $\mathbf{g} = (\{1, 2\}, A, (g^i)_{i=1,2})$ has $2^n - 1$ NE, each of which has the same support for both players that is distinct from those of other NE, we can straightforwardly extend the argument used in the proof of the if part of Theorem 1 to the bimatrix game. $\qquad \square$

Proof of the only if part

The main point of the proof is to use each player's payoff function for finding out a symmetric NE in a "fictitious" symmetric game and combine two strategies of two (possibly different) symmetric NE with the same support to construct a NE in the original game.

Proof. Fix any game $\mathbf{g} = (\{1, 2\}, A, (g^i)_{i=1,2})$ where $A = \{1, 2, \ldots, n\}$ and GTBP holds. For any player $i = 1, 2$, we construct the symmetric two-player game by using the player i's set of strategies A and payoff function g^i. We denote it by $\mathbf{g}^i = (A, g^i)$. Since the game $\mathbf{g}^i$ has the same set of strategies A for two players and the common payoff function g^i, by GTBP for player i, $\mathbf{g}^i$ satisfies TBP. From Theorem 1, if $\mathbf{g}^i$ has TBP, it has $2^n - 1$ symmetric NE, those of which are different in terms of supports. Consider the symmetric NE $(x^i|_S, x^i|_S) \in \Delta$ with $\mathrm{supp}(x^i|_S) = S \subseteq A$ in $\mathbf{g}^i$. The strategy profiles $(x^2|_S, x^1|_S)$ is a NE of the game $\mathbf{g}$ such that $\mathrm{supp}(x^1|_S) = \mathrm{supp}(x^2|_S) = S$. Since $\mathbf{g}^i$ has $2^n - 1$ symmetric NE for any $i = 1, 2$, this implies that the bimatrix game $\mathbf{g} = (\{1, 2\}, A, (g^i)_{i=1,2})$ has $2^n - 1$ NE, each of which gives the same support for two players that is distinct from those of other NE. $\qquad \square$

In the following, we give three examples regarding Theorem 2.

Example 4 (Asymmetric pure-coordination game)**.** Let us consider the 3×3 pure-coordination game given by Table 6 where a_i and b_i for $i = 1, 2, 3$ are positive constants. One can easily observe that the game satisfies GTBP and find $2^3 - 1(= 7)$ NE including asymmetric ones: three symmetric pure strategy profiles and four mixed strategy

		Player 2	
	1	**2**	**3**
1	a_1, b_1	0,0	0,0
2	0,0	a_2, b_2	0,0
3	0,0	0,0	a_3, b_3

Player 1 labels rows 1, 2, 3.

Table 6: A 3×3 pure-coordination game.

profiles,

$(1/(b_1 + b_2)(b_2, b_1, 0), 1/(a_1 + a_2)(a_2, a_1, 0)), (1/(b_1 + b_3)(b_3, 0, b_1), 1/(a_1 + a_3)(a_3, 0, a_1))$
$(1/(b_2 + b_3)(0, b_3, b_2), 1/(a_2 + a_3)(0, a_3, a_2)),$
$(1/(b_1 b_2 + b_1 b_3 + b_2 b_3)(b_2 b_3, b_1 b_3, b_1 b_2), 1/(a_1 a_2 + a_1 a_3 + a_2 a_3)(a_2 a_3, a_1 a_3, a_1 a_2)).$

Thus, it is consistent with Theorem 2.

Remark 3. Theorem 2 holds for all $n \times n$ pure coordination games and slightly perturbed ones with respect to payoffs as in Example 4.

Example 5 (Bertrand duopoly market)**.** We consider a Bertrand competition with convex costs, which has been theoretically investigated by Dastidar (1995) and experimentally by Abbink and Brandts (2008) and Argenton and Müller (2012) among others.[20] There are two firms in a market where each firm faces a linear demand under a quadratic per-unit cost function, and then competes in prices, which are assumed to be discrete.[21] We denote by $D^i : A^2 \to \mathbb{R}$ and $C^i : A^2 \to \mathbb{R}$ a linear demand function and quadratic cost function of firm i. Then, this Bertrand duopoly market is described by an $n \times n$ bimatrix game $\mathbf{g} = (\{1, 2\}, A, (g_1, g_2))$ where $A = \{p_1, \ldots, p_n\}$ for $0 \le p_1 < p_2 < \cdots < p_n$ and $g^i : A^2 \to \mathbb{R}$ is the firm i's payoff function such that for any price profile $p = (p^1, p^2) \in A^2$,

$$g^i(p) = p^i D^i(p) - C^i(p)$$

---

[20]For instance, price competitions with convex costs are relevant due to adjustment costs of productions in utilities and telecommunications industries (see, for instance, Green and Newbery, 1992; Green, 1996; Wolfram, 1998; Armstrong and Porter, 2007; Hortaçsu and Puller, 2008; Janssen and Karamychev, 2010).

[21]This framework is used by Argenton and Müller (2012) in their experiment.

where for some $a > p_n$, $c^i > 0$, and $j \neq i$,[22]

$$D^i(p) = \begin{cases} a - p^i & \text{if } p^i < p^j, \\ \frac{1}{2}(a - p^i) & \text{if } p^i = p^j, \\ 0 & \text{if } p^i > p^j, \end{cases} \qquad C^i(p) = c^i(D^i(p))^2.$$

Let us consider the special case of $A = \{1, 2, 3\}$ and then find an equivalent parameter condition to GTBP in the game. To show that $\mathrm{br}_i(x^j) \subseteq \mathrm{supp}(x^j)$ holds for any $i, j = 1, 2$ with $i \neq j$ and any $x^j \in \Delta$, we first consider the condition under which the game is a coordination game, that is, $\mathrm{br}_i(x^j) \subseteq \mathrm{supp}(x^j)$ holds for any standard basis vector $x^j \in \cup_{k \in A} \mathbf{e}_k$. This is equivalent to the conditions of $g_{11}^i > \max\{g_{21}^i, g_{31}^i\} = 0$, $g_{22}^i > \max\{g_{12}^i, g_{32}^i\}$, and $g_{33}^i > \max\{g_{13}^i, g_{23}^i\}$. Given $a > p_3 = 3$, these are reduced to

$$\frac{2(a+1)}{3a^2 - 10a + 7} < c^i < \frac{2}{a - 1}. \tag{A.3}$$

In fact, one can easily show that the above derived condition (A.3) also guarantees the condition under which $\mathrm{br}_i(x^j) \subseteq \mathrm{supp}(x^j)$ holds for any $x^j \in \cup_{k \in A} \mathbf{e}_k$ due to the specific payoff structure of this game. Furthermore this implies that the game has a unique interior NE as well. Thus, the Bertrand duopoly game $\mathbf{g} = (\{1, 2\}, A, (g_1, g_2))$ with $A = \{1, 2, 3\}$ has GTBP if and only if the condition (A.3) holds for each $i = 1, 2$ given $a > 3$.

For instance, as $a = 5$, since (A.3) is reduced to $3/8 < c^i < 1/2$, take $(c^1, c^2) = (3/7, 4/9)$. Then, the game satisfies GTBP, and in addition, one can see that there are $2^3 - 1 = 7$ NE in the game. This is consistent with Theorem 2.

Games with $2^n - 1$ NE without the restriction on support of NE

Let us consider an asymmetric game $\mathbf{g}$ with $2^n - 1$ NE and then will see that $\mathbf{g}$ does not satisfy GTBP in general. In fact, to guarantee GTBP, $\mathbf{g}$ must be nondegenerate even in the case of $n = 3$ and also must satisfy an additional condition in the case of $n \geq 4$. To show this, we give several counterexamples below.

---

[22]We assume that $a > p_n$ holds because each firm's demand under any price profile is non-negative, otherwise a price with $a \leq p_n$ cannot give a positive payoff and so it is never chosen.

Player 2

		1	2	3
	1	2,1	2,0	2,0
Player 1	2	1,0	3,2	3,0
	3	0,2	2,0	4,3

Table 7: An asymmetric 3×3 coordination game.
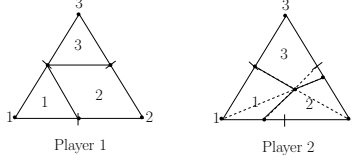

Figure 3: The best response regions of the game.

Case of $n = 3$

First of all we consider the case of three strategies to show that a game must be nondegenerate.

Example 6 (Degenerate 3×3 coordination game)**.** We consider the asymmetric 3×3 coordination game given by Table 7. From Table 7, one can easily see that the game has 3 symmetric pure strategy NE. The best response regions of the game are given by Figure 3. By the computation, we can find 4 mixed strategy NE:

$$((2/3, 1/3, 0), (1/2, 1/2, 0)), ((1/2, 0, 1/2), (1/2, 0, 1/2)), ((0, 3/5, 2/5), (0, 1/2, 1/2)),$$
$$((2/7, 3/7, 2/7), (1/2, 0, 1/2)).$$

So, the game has $7(= 2^3 - 1)$ NE. But the NE $(x^1, x^2) = ((2/7, 3/7, 2/7), (1/2, 0, 1/2))$ satisfies

$$3 = |\text{supp}(x^1)| \neq |\text{supp}(x^2)| = 2.$$

Since $\text{br}_1(x^2) = \{1, 2, 3\}$ and $\text{supp}(x^2) = \{1, 3\}$, this implies that $\text{supp}(x^2) \subsetneq \text{br}_1(x^2)$, violating GTBP. This is also easily seen by Figure 3. Here, since $|\text{br}_1(x^2)| > |\text{supp}(x^2)|$ holds for $x^2 = (1/2, 0, 1/2)$, the game is degenerate.

From this example, if a game with $2^3 - 1$ NE is degenerate, it does not necessarily satisfy GTBP. But we can show that if a game with $2^3 - 1$ NE is nondegenerate, it has

27

	1	2	3	4
1	-81,72	36,36	-126,17	126,-3
2	-180,-180	72,-81	-333,-30	297,20
3	20,297	-3,126	42,42	-33,-33
4	-30,-333	17,-126	-33,-33	42,42

Table 8: An asymmetric 4×4 game.

GTBP.[23]

Case of $n \geq 4$

Next we consider a nondegenerate asymmetric 4×4 game below and then show by an example that the game has $2^4 - 1$ NE but does not satisfy GTBP.

Example 7 (Nondegenerate asymmetric 4×4 game)**.** We consider the asymmetric 4×4 game (Table 8), which is constructed by using a restricted game of the 6×6 bimatrix game given by Savani and Stengel (2006) where there are $75(> 2^6 - 1 = 63)$ NE. We can find the following 15 NE:[24]

$$|\{(x^1, x^2) \in \text{NE} \mid i = 1, 2, |\text{supp}(x^i)| = 1\}| = 2 :$$

$$((0, 1, 0, 0), (0, 0, 0, 1)), ((0, 0, 1, 0), (1, 0, 0, 0)).$$

$$|\{(x^1, x^2) \in \text{NE} \mid i = 1, 2, |\text{supp}(x^i)| = 2\}| = 10 :$$

$$((11/15, 4/15, 0, 0), (4/15, 11/15, 0, 0), \tag{A.4}$$
$$((0, 0, 1/2, 1/2), (0, 0, 1/2, 1/2)), \tag{A.5}$$
$$((51/70, 19/70, 0, 0), (0, 23/27, 4/27, 0), \quad ((23/27, 0, 0, 4/27), (19/70, 51/70, 0, 0)),$$
$$((2/7, 5/7, 0, 0), (0, 0, 19/42, 23/42)), \quad ((93/112, 0, 0, 19/112), (0, 93/112, 19/112, 0)),$$
$$((0, 0, 23/42, 19/42), (2/7, 5/7, 0, 0)), \quad ((0, 0, 31/59, 28/59), (0, 15/19, 4/19, 0)),$$
$$((15/19, 0, 0, 4/19), (0, 0, 28/59, 31/59)), \quad ((0, 33/53, 20/53, 0), (33/53, 0, 0, 20/53)).$$

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[23]The proof is available from the author upon request.

[24]Keiding (1997) and McLennan and Park (1999) show that $2^4 - 1 = 15$ is the maximal number of NE for all 4×4 games.

$$|\{(x^1, x^2) \in \mathrm{NE} \mid i = 1, 2, |\mathrm{supp}(x^i)| = 3\}| = 2 :$$

$$((60/109, 39/109, 10/109, 0), (31/74, 0, 17/111, 95/222)),$$
$$((0, 31/74, 95/222, 17/111), (39/109, 60/109, 0, 10/109)).$$

$$|\{(x^1, x^2) \in \mathrm{NE} \mid i = 1, 2, |\mathrm{supp}(x^i)| = 4\}| = 1 :$$

$$((5/11, 4/11, 5/33, 1/33), (4/11, 5/11, 1/33, 5/33)). \tag{A.6}$$

One can show that the game is nondegenerate, and the game has $15(= 2^4 - 1)$ NE. But the game does not satisfy GTBP because the game is not a coordination game (even if it is allowed to permute strategies). Note that $\mathrm{supp}(x^1) = \mathrm{supp}(x^2)$ does not hold for all NE (x^1, x^2) except for (A.4)–(A.6).

Example 7 implies that a nondegenerate $n \times n$ game with $2^n - 1$ NE may not be a coordination game. So, in order to show that a nongedenerate $n \times n$ game with $2^n - 1$ NE has GTBP, the game must be at least a coordination game. But even under the assumption that a game with $2^n - 1$ NE is a nondegenerate coordination game, the game may not satisfy GTBP. In general, when considering many actions, we must impose the restriction on support of NE as in Theorem 2.

Proof of Lemma 1

Proof. Consider a game with TBP. Suppose that there is a GPRD–equilibrium $(i, i) \in A^2$ in the game. Together with TBP, this implies that $\mathrm{br}(x) = \{i\}$ holds for $x \in \mathring{\Delta}(\{i, j\})$ with $x_i = x_j = 1/2$ given any $j \in A\backslash\{i\}$. Since any pure strategy best response set is convex in Δ, this implies that $\mathrm{br}(x) = \{i\}$ holds for all $x \in \Delta$ with $x_i \geq 1/2$, meaning that (i, i) is a $1/2$–dominant equilibrium. $\square$

Proof of Proposition 1

To show Proposition 1, we introduce two notations for convenience as follows. For $x, x' \in \Delta$, we write $x \succ x'$ if x stochastically dominates x', that is, for any $i \in A$, $\sum_{i \leq j \leq n} x_j \geq \sum_{i \leq j \leq n} x'_j$ with strict inequality for at least some i. When considering opponent's mixed strategies $x \in \mathring{\Delta}(\{i, j\})$ for any $i, j \in A$ with $i \neq j$, we write x^{ij} instead of x for clarity. Given these notations, we show Proposition 1 by using the

property on supermodular games $\mathbf{g} = (A, g)$ that each player's best response corre-
spondence is non-decreasing in opponent's strategies: for any $x, x' \in \Delta$ with $x \succ x'$,
$\min \mathrm{br}(x) \geq \min \mathrm{br}(x')$ and $\max \mathrm{br}(x) \geq \max \mathrm{br}(x')$ where $\min \mathrm{br}(x)$ is the lowest pure
strategy best response against opponent's mixed strategy x and $\max \mathrm{br}(x)$ the highest
one.[25]

Proof. By TBP, it follows that for $x^{1n} \in \mathring{\Delta}(\{1, n\})$ with $x_1^{1n} = x_n^{1n} = 1/2$,

$$\mathrm{br}(x^{1n}) \subseteq \{1, n\}.$$

By the assumption of $g_{11} - g_{n1} \neq g_{nn} - g_{1n}$, $\mathrm{br}(x^{1n}) = \{1\}$ or $\{n\}$. Suppose that
$\mathrm{br}(x^{1n}) = \{1\}$. For any $i \in A \backslash \{1\}$, take $x^{1i} \in \mathring{\Delta}(\{1, i\})$ with $x_1^{1i} = x_i^{1i} = 1/2$. Since
$x^{1n} \succ x^{1i}$ for any $i \in A \backslash \{n\}$, by supermodularity, it follows that

$$\{1\} = \max \mathrm{br}(x^{1n}) \geq \max \mathrm{br}(x^{1i}), \tag{A.7}$$

which implies that for any $i \in A$ and any belief $x^{1i} \in \mathring{\Delta}(\{1, i\})$ with $x_1^{1i} = x_i^{1i} = 1/2$,

$$\mathrm{br}(x^{1i}) = \{1\}. \tag{A.8}$$

The condition (A.8) implies that the strategy profile $(1,1)$ is a GPRD-equilibrium. By
Lemma 1, $(1, 1)$ is a $1/2$–dominant equilibrium.

To show the uniqueness, suppose that there are two $1/2$–dominant equilibria, (i, i)
and (j, j). This gives the following two inequalities,

$$\frac{1}{2} g_{ii} + \frac{1}{2} g_{ij} > \frac{1}{2} g_{jj} + \frac{1}{2} g_{ji},$$
$$\frac{1}{2} g_{jj} + \frac{1}{2} g_{ji} > \frac{1}{2} g_{ii} + \frac{1}{2} g_{ij},$$

a contradiction.

Similarly, if $\mathrm{br}(x^{1n}) = \{n\}$ holds for $x^{1n} \in \mathring{\Delta}(\{1, n\})$ with $x_1^{1n} = x_n^{1n} = 1/2$, we can
show that (n, n) is the unique $1/2$–dominant equilibrium. $\square$

---

[25]For the detail, see, for instance, Milgrom and Roberts (1990); Milgrom and Shannon (1994); Topkis
(1998); Vives (1990, 2001).

Proof of Proposition 2

Proof. Suppose that there is a potential function v such that (i,i) is a unique potential maximizer. For any given $j \neq i$, we have $v_{ii} > v_{jj}$ by definition of potential maximizer and $v_{ij} = v_{ji}$ due to symmetry of potential function, thereby leading to

$$v_{ii} - v_{jj} > 0 = v_{ji} - v_{ij}.$$

Replacing v_{jj} with v_{ij}, the inequality derived above is rewritten by $v_{ii} - v_{ji} > v_{jj} - v_{ij}$. By definition of potential function, this gives $g_{ii} - g_{ji} > g_{jj} - g_{ij}$, in other words,

$$\frac{1}{2}(g_{ii} + g_{ij}) > \frac{1}{2}(g_{ji} + g_{jj})$$

holds for any $j \in A \backslash \{i\}$. Thus, the unique potential maximizer (i,i) is a GPRD–equilibrium. By Lemma 1, (i,i) is a 1/2–dominant equilibrium. $\qquad \square$

Proof of Proposition 3

To show Proposition 3, we first introduce the solution concept of monotone potential maximizer (Morris and Ui, 2005), which is a generalization of potential maximizer as LP-maximizer.[26] Then, we show that an LP-maximizer with constant weights (introduced in the proof of Proposition 3) is an MP-max. Following Morris and Ui (2005), a (strict) MP-max is unique in any generic supermodular game because it is robust to incomplete information in the sense of Kajii and Morris (1997) and the robust equilibrium is unique.[27] While on the other hand, Proposition 1 tells us that any symmetric two-player supermodular game with TBP where $g_{1n} - g_{n1} \neq g_{nn} - g_{1n}$ always has a unique 1/2–dominant equilibrium, which is also robust to incomplete information (Kajii and Morris, 1997). Taken together, if an LP-maximizer with constant weights exists in a symmetric two-player supermodular game with TBP, both of the LP-maximizer and the 1/2–dominant equilibrium must be equivalent.

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[26]We pay attention to a specific LP-maximizer (with constant weights) that is a unique solution if any (Okada and Tercieux, 2012, Proposition 1), but there may exist multiple LP-maximizers in general. See Oyama and Takahashi (2009, Example 1). In fact, they correct the statement of Frankel et al. (2003) in such a way that a (strict) LP-maximizer of a supermodular game with own-action *concavity* instead of own-action *quasiconcavity* is chosen as the noise-independent selection in the global game method (Carlsson and van Damme, 1993).

[27]Similarly, Oyama et al. (2008) show that any "generic" supermodular game has at most one MP-max through the argument of perfect foresight dynamics. They also show that if a supermodular game satisfies own-action concavity, a supermodular game has at most one LP-maximizer.

31

Thus, to prove Proposition 3, it is enough to show that an LP-maximizer with constant weights implies an MP-max. In doing so, below we introduce (i) the definition of MP-max and (ii) the equivalent definition of LP-maximizer, which is useful for the proof.

Fix any symmetric game $\mathbf{g} = (A, g)$ with $A = \{1, , \ldots, n\}$. For a function $f : A^2 \to \mathbb{R}$, a mixed strategy $x \in \Delta$, each player $i = 1, 2$, and a non-empty subset of strategies $S \subseteq A$, let

$$\mathrm{br}_f^i(x \mid S) \quad = \quad \arg\max_{s_i \in S} \sum_{j \in A} x_j f_{s_i j}.$$

MP-max is defined for a general class of games, but our interest lies in a simple class of games, and therefore we use its simplified version and refinement, strict MP-max, following Oyama et al. (2008) below.

Definition 8. A pure strategy profile $s^* = (s_i^*, s_j^*) \in A^2$ is a *monotone potential maximizer* (MP-max) of $\mathbf{g}$ if there exists a function $v : A^2 \to \mathbb{R}$ with $v(s^*) > v(s)$ for all $s \in A^2 \backslash \{s^*\}$ such that for each $i = 1, 2$ and any $x \in \Delta$,

$$\min \mathrm{br}_v^i(x \mid \{1, \ldots, i^*\}) \quad \leq \quad \max \mathrm{br}_g^i(x \mid \{1, \ldots, s_i^*\}),$$
$$\max \mathrm{br}_v^i(x \mid \{s_i^*, \ldots, n\}) \quad \geq \quad \min \mathrm{br}_g^i(x \mid \{s_i^*, \ldots, n\}).$$

Such a function v is called a *monotone potential function* (MP-function) for s^*.

A pure strategy profile $s^* = (s_1^*, s_2^*) \in A^2$ is a *strict MP-max* of $\mathbf{g}$ if there exists a function $v : A^2 \to \mathbb{R}$ with $v(s^*) > v(s)$ for all $s \neq s^*$ such that for each $i = 1, 2$ and any $x \in \Delta$,

$$\min \mathrm{br}_v^i(x \mid \{1, \ldots, s_i^*\}) \quad \leq \quad \min \mathrm{br}_g^i(x \mid \{1, \ldots, s_i^*\}),$$
$$\max \mathrm{br}_v^i(x \mid \{s_i^*, \ldots, n\}) \quad \geq \quad \max \mathrm{br}_g^i(x \mid \{s_i^*, \ldots, n\}).$$

Such a function v is called a *strict MP-function* for s^*.

A (strict) MP-max is a (strict) NE, and a potential maximizer is a strict MP-max. Under a generic choice of payoffs, an MP-max is a strict MP-max. A supermodular game can have at most one strict MP-max (Oyama et al., 2008).

Next, we introduce the following equivalent definition for an LP-maximizer (Morris and Ui, 2005, see Definition 11 and Lemma 9).

Definition 9. A pure strategy profile $s^* = (s_1^*, s_2^*)$ is an LP-maximizer with a local potential function v if

(i) for any player $i = 1, 2$, any strategy $s_i < s_i^*$, and any $x \in \Delta$ such that $\sum_{j \in A} x_j v_{s_i j} < \sum_{j \in A} x_j v_{s_i+1 j}$,

$$\sum_{j \in A} x_j g_{s_i j} \leq \sum_{j \in A} x_j g_{s_i+1 j}. \tag{A.9}$$

(ii) for any player $i = 1, 2$, any strategy $s_i > s_i^*$, and any $x \in \Delta$ such that $\sum_{j \in A} x_j v_{s_i j} < \sum_{j \in A} x_j v_{s_i-1 j}$,

$$\sum_{j \in A} x_j g_{s_i j} \leq \sum_{j \in A} x_j g_{s_i-1 j}. \tag{A.10}$$

We show that an LP-maximizer with constant weights is an MP-max.

Proof. We define $\underline{s}_i \equiv \min \mathrm{br}_v^i(x \mid \{1, \dots, s_i^*\})$ such that $\sum_{j \in A} x_j v_{s_i j} < \sum_{j \in A} x_j v_{\underline{s}_i-1 j}$ for every $1 \leq s_i < \underline{s}_i$ (if any). By (A.9), for any $1 \leq s_i < \underline{s}_i$,

$$\sum_{j \in A} x_j g_{s_i j} \leq \sum_{j \in A} x_j g_{\underline{s}_i j}. \tag{A.11}$$

Thus, (A.11) implies that $\underline{s}_i \leq \min \mathrm{br}_g^i(x \mid \{1, \dots, s_i^*\})$, that is,

$$\min \mathrm{br}_v^i(x \mid \{1, \dots, s_i^*\}) \leq \min \mathrm{br}_g^i(x \mid \{1, \dots, s_i^*\}). \tag{A.12}$$

Similarly, by (A.10), we can show that

$$\max \mathrm{br}_v^i(x \mid \{s_i^*, \dots, n\}) \geq \max \mathrm{br}_g^i(x \mid \{s_i^*, \dots, n\}). \tag{A.13}$$

Since (A.12) and (A.13) satisfy the definition of MP-max, we have shown that an LP-maximizer with constant weights, s^*, is an MP-max. $\qquad \square$

Proof of Lemma 2

Proof. Fix any $l, m, h \in A$ with $l < m < h$ and let $S = \{l, h\}$. For any $x \in \mathring{\Delta}(S)$,

$$\sum_{i=1}^{n} x_i g_{li} - \sum_{i=1}^{n} x_i g_{mi} = a(m-l)\left(x_l - \frac{a-b}{a}\right),$$

implying that for any $x \in \mathring{\Delta}(S)$ with $S = \{l, h\}$,

$$\arg\max_{m \in \{l, l+1, \ldots, h\}} \sum_{i=1}^{n} x_i g_{mi} = \begin{cases} \{l\} & \text{if } x_l > x_l^*, \\ \{l, l+1, \ldots, h\} & \text{if } x_l = x_l^*, \\ \{h\} & \text{if } x_l < x_l^* \end{cases}$$

where $x_l^* = (a-b)/a$. Thus, $m \notin \mathrm{br}(x)$ holds for any $l, m, h \in A$ with $l < m < h$ and all $x \in \Delta(\{l, h\}) \backslash \{x^*\}$. $\qquad \square$

Proof of Supermodularity of the Minimum-Effort Game

For any $i, i', j, j' \in A$ with $i > i'$ and $j > j'$,

$$(g_{ii} - g_{i'j}) - (g_{ij'} - g_{i'j'}) = a((\min\{i, j\} - \min\{i', j\}) - (\min\{i, j'\} - \min\{i', j'\})) \geq 0$$

where the last inequality follows from the property that $\min\{i, j\} - \min\{i', j\}$ is weakly increasing in $j \in A = \{1, \ldots, n\}$. When $j = j'$, $(g_{ii} - g_{i'j}) - (g_{ij'} - g_{i'j'}) = 0$ holds. Thus, that the minimum-effort game is supermodular. Below we show the above used weakly increasing property by considering all six possible cases.

Case 1: $i > i' \geq j > j'$. $\min\{i, j\} - \min\{i', j\} = 0 = \min\{i, j'\} - \min\{i', j'\}$.
Case 2: $i \geq j > i' \geq j'$. $\min\{i, j\} - \min\{i', j\} = j - i' > 0 = \min\{i, j'\} - \min\{i', j'\}$..
Case 3: $i \geq j > j' > i'$. $\min\{i, j\} - \min\{i', j\} = j - i' > j' - i' = \min\{i, j'\} - \min\{i', j'\}$.
Case 4: $j > i \geq j' > i'$. $\min\{i, j\} - \min\{i', j\} = i - i' \geq j' - i' = \min\{i, j'\} - \min\{i', j'\}$.
Case 5: $j > i > i' \geq j'$. $\min\{i, j\} - \min\{i', j\} = i - i' > 0 = \min\{i, j'\} - \min\{i', j'\}$.
Case 6: $j > j' > i > i'$. $\min\{i, j\} - \min\{i', j\} = i - i' = \min\{i, j'\} - \min\{i', j'\}$.

Remark 4. We can straightforwardly extend the proof shown above to the case of multiple players.

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