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Citation

Job market signaling with human capital investment[*]

Gea M. Lee[†] and Seung Han Yoo[‡]

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Abstract

In this paper, we consider the social value of signaling by recasting the Spence's (1973) signaling model in a causal relationship: human capital investment is necessary to reduce the marginal cost of signaling. Our model contains distinct features: (i) the choice of signaling affects the level of human capital investment and (ii) the proportion of high and low type in the entire workers is endogenously determined. From the perspective of welfare, we compare two contrasting forms of signaling, separating and pooling, and find that the choice of a proper form of signaling is dependent on how each signaling induces the human capital investment. We identify circumstances where it is socially beneficial to stay with a separating signaling and focus on promoting the human capital investment, and where it is socially beneficial to switch from a separating signaling to a pooling signaling and moderate the investment level.

Keywords and Phrases: Education, Human capital, Signaling, Social welfare
JEL Classification Numbers: **D63, I21, J24**

1 Introduction

Since the birth of the signaling model by Spence (1973), the role of education has been extensively tested to identify whether its signaling effect exists and is greater than the other competing effect, human capital augmenting (see Wolpin (1977), Riley (1979), Lang and Kropp (1986), Tyler, Murnane and Willett (2000), Bedard (2001), and section 5.1 in Riley (2001) among others).[1] Importantly, the existing literature has paid attention to the two *separate* aspects of education: education level measured by a length of schooling acts to signal individual ability to firms in one aspect, while it enhances the productivity valued by firms in the other aspect. Despite such extensive studies, theoretical papers have rarely considered a potential relationship between the two aspects.

This paper proceeds from the following long-standing questions in economics. What is the *social* value of signaling? Is signaling socially beneficial or wasteful? It is well-known that it may be socially harmful to convey private information in the form of costly signaling. In this sense, it is rather unsurprising that standard signaling models often contradict the social value of signaling: the signaling aspect of education is socially wasteful though it may well increase private earnings. At the same time, however, it is quite surprising that theoretical models commonly ignore the potential contribution of signaling to the other aspect of education, human capital investment: the signaling aspect of education may promote the human capital investment by delivering the benefit of the investment. The social value of signaling may be unduly underrated if its effect on the human capital investment is not taken into account. This possibility thus raises important questions. What is a proper form or a proper level of signaling when signaling induces the human capital investment? Is it socially beneficial or harmful to promote the human capital investment by increasing costly signaling?

Motivated by these questions, this paper recasts the classic signaling model by Spence (1973) in a dynamic framework where human capital and signaling have a *causal relationship*: human capital investment is necessary to reduce the marginal cost of signaling.

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[1]For example, using a unique data set containing the General Educational Development (GED) test scores, Tyler, Murnane and Willett (2000) identify the signaling value of the GED, net of human capital effects. They observe that there are substantial signaling effects for young white dropouts, estimated at about 20% earnings gain after 5 years.

Two aspects of education are thus sequentially decomposed in the model: education level acts solely as signaling and the marginal cost of education (signaling) is conditional on whether the worker has undertaken the human capital investment. We consider a two-type model in which a worker's high or low type is determined according to the presence or absence of the human capital investment. When a worker contemplates whether to make a human capital investment, he compares the cost and benefit of the investment. The cost occurs together with the investment. The cost level, exogenously drawn from an interval, depends on the worker's aggregate endowment such as innate intellect, maturity, initial wealth, and parental environment. The benefit of the investment is delivered later when signaling is less costly and wage is determined based on signaling. Consequently, our model contains two distinct features: (i) the choice of signaling affects the level of human capital investment and (ii) the proportion of high and low type in the entire workers is endogenously determined.[2]

We are primarily interested in separating and pooling equilibria. These two kinds of equilibria contain two contrasting forms of signaling: separating signaling and pooling signaling.[3] Each equilibrium is classified as an interior equilibrium or a boundary equilibrium: in an interior equilibrium, a positive fraction of workers are motivated to make the human capital investment and become high type, whereas in a boundary equilibrium, workers select no costly signaling and make no human capital investment. Under some mild assumptions, we characterize an interior separating equilibrium and an interior pooling equilibrium, and then establish conditions under which each type of interior equilibrium is sure to exist and fails to exist. We associate a separating or pooling signaling with an interior separating or pooling equilibrium that exists under the government's market design: we assume that the government acts as a market maker and implements a certain signaling through the equilibrium that exists under its market design. A rationale for this intervention is that it can significantly increase the set of implementable signaling under the Cho-Kreps' intuitive criterion (see Cho and Kreps (1987)) by expanding the

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[2]The features shown in this paper might provide some clues as to why the existing literature that attempted to test the two aspects of education, including the aforementioned papers, faces the difficulty of identifying one aspect of education from the other aspect.

[3]Two different natures of signaling may also be referred to as separating education and pooling education given that education acts as signaling in the model.

scope of parameters in which there exists an interior separating or pooling equilibrium that satisfies the criterion.

With these analyses in hand, we present welfare analyses of two kinds of signaling. Indeed, the social value of signaling has long been challenged: the signaling aspect of education is wasteful from the perspective of welfare. We confirm that this view is valid in a *benchmark model* where signaling choice has no effect on the human capital investment so that the proportion of two types is exogenously determined. In contrast with this view, we use each interior equilibrium in our original model and show that the signaling aspect of education is justifiable from the perspective of welfare: this seemingly wasteful action becomes socially beneficial when it induces any human capital investment by delivering the benefits of the investment.

We next characterize the signaling that maximizes the social welfare in each interior equilibrium. In an interior separating equilibrium, it is socially beneficial to maximize the human capital investment: since the investment can be promoted by increasing the utility gain that a worker expects from undertaking the investment and becoming high type, the least costly signaling for high type maximizes the social welfare. In an interior pooling equilibrium, however, it may be socially harmful to maximize the human capital investment: an increase in pooling signaling improves the social welfare by promoting the human capital investment, but it also worsens the welfare by increasing the signaling cost of a worker with low type. We find that an interior pooling equilibrium suffers from an oversignaling if an additional human capital (marginal increase in the fraction of high type) resulting from an increase in the signaling is sufficiently small.

We then compare two contrasting forms of signaling by associating a separating signaling with an interior separating equilibrium and associating a pooling signaling with an interior pooling equilibrium. We begin by observing that an interior separating equilibrium uses a higher signaling level than does an interior pooling equilibrium: the separating signaling that ensures high type must be higher than any pooling signaling that ensures the expected type of all workers.[4] We find that a separating signaling, despite its use of a higher signaling level, does not necessarily generate the human capital investment more

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[4]A pooling signaling can approximate a separating signaling for high type only when the fraction of workers with high type approches one.

than a pooling signaling: if the signaling for high type grows higher, then the signaling cost for high type also becomes larger and thus workers become less encouraged from making the human capital investment and becoming high type. The signaling aspect of education is socially beneficial in an interior equilibrium, as we state above, and it may be better represented by a separating signaling than by a pooling signaling in that private information about individual types is conveyed by the separating signaling. From the perspective of welfare, however, we find that a separating signaling may be inferior to a pooling signaling, without a larger inducement of the human capital investment.

We further formalize this argument and specify conditions under which we can rank two types of signaling in terms of social welfare. Our starting point of comparison is based on the condition that ensures the existence of an interior pooling equilibrium that approximates the full investment and makes the fraction of high type close to one.[5] We then establish our findings in two logical steps. First, we observe that the very best of all possible interior separating equilibria, by motivating most of workers to have only one type (high type), approaches an interior pooling equilibrium that approximates the full investment. Second, we show that a pooling signaling is strictly better than a separating signaling in terms of social welfare under two circumstances: (i) an additional separating signaling reduces human capital (the fraction of high type) by making the signaling cost for high type too high, so that an interior separating equilibrium fails to approximate the full investment, and (ii) even when an interior separating equilibrium can approximate the full investment, it approaches an interior pooling equilibrium that is too costly to implement and suffers from an overinvestment.

It seems socially beneficial to motivate more workers to undertake the human capital investment; in fact, an interior separating equilibrium maximizes the human capital investment to maximize the social welfare. We find, however, that it may be socially harmful to stay with the separating signaling: before an interior separating equilibrium approaches an interior pooling equilibrium that approximates the full investment and suffers from an overinvestment, then it is socially beneficial to switch from the separating signaling to a pooling signaling and moderate the investment level. On the other hand,

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[5]It is impossible that every worker becomes high type in equilibrium: if every worker has high type, the use of costly signaling has no gain, and with no use of signaling, there will be no human capital investment since its benefit cannot be delivered by signaling.

we also find that, if a separating signaling still generates a sufficiently small incentive to make the human capital investment, then it is socially beneficial to continue to use the separating signaling and focus on promoting the investment: a pooling signaling may then suffer from an underinvestment by failing to provide any incentive to make the human capital investment or by inducing too little incentive.

The relationship between signaling and human capital investment has not received a well-deserved attention from the existing literature. Despite extensive studies of human capital investment and prevalent uses of signaling models, theoretical papers have rarely considered a potential relationship between the two aspects.[6] We find that the causal relationship assumed in the model, a seemingly natural and yet surprisingly rare extension of the Spence's (1973) model, provides a new insight into the choice of a proper form of signaling from the perspective of social welfare: whether it is socially beneficial or wasteful to convey private information about individual types in the form of a separating signaling, as opposed to a pooling signaling, is dependent on how each form of signaling promotes the human capital investment. This explanatory variable has long been missing in the literature.

This paper is organized as follows. We introduce the model in Section 2. We then characterize interior equilibria and present the existence of each interior equilibrium in Section 3. In Section 4, we offer welfare analyses of signaling in the benchmark model and in our original model, and characterize the welfare-maximizing signaling in each interior equilibrium. In Section 5, we present circumstances under which we can rank two forms of signaling in terms of social welfare. We conclude in Section 6.

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[6]In broad terms, our model is related to Daley and Green (2013) in their working paper version that includes the pre-investment stage. They introduce an additional noisy signal, grade, which enables them to apply an equilibrium selection criterion, and they examine how the selected equilibrium changes with the noisy signal. Our model has no noisy signal, and inherits Spence's framework with "minimal assumptions" by adding the pre-invesment stage. Under government intervention, we study both separating and pooling equilibria, and furthermore, our main focus is market design; conditions that induce more human capital investment and higher social welfare given a population distribution. We also use the Cho-Kreps' intuitive criterion for an equilibrium selection under government intervention.

2 Model

We consider a unit mass of ex ante identical workers. Each worker endogenously decides whether to make a human capital investment with which to determine his own type $q \in \{H, L\}$ that will be appraised in the future. If a worker makes the investment, he incurs the cost c. The level of c represents a composite cost and depends on the worker's aggregate endowment such as innate intellect, maturity, initial wealth, and parental environment. We assume that the cost c is exogenously drawn from an absolutely continuous distribution function G with the support $[\underline{c}, \overline{c}]$, where $\overline{c} > \underline{c} \geq 0$. The density $g \equiv G'$ is everywhere positive. After making the choice of type $q \in \{H, L\}$ that is private information and is conditional on the investment decision, a worker selects education $e \in \mathbb{R}_+$ that is public information and acts solely as signaling. The worker then participates in the labor market where two risk-neutral firms engage in a Bertrand-style competition and offer wages simultaneously. The worker earns wage $w \in \mathbb{R}_+$ if he is hired by one of two firms. The worker has utility 0 from outside options. Each firm obtains the value $y_q \in \mathbb{R}_+$ if it employs a worker with type $q \in \{H, L\}$, where $y_H > y_L$.

A worker with type $q \in \{H, L\}$ has a continuous utility function $u_q(w, e)$ that is strictly increasing in w and strictly decreasing in e. We assume that a worker with type H has a lower marginal cost of signaling (education) than does a worker with type L: if $e' > e$, then

$$u_H(w, e') - u_H(w, e) > u_L(w, e') - u_L(w, e). \tag{1}$$

This inequality means that $u_q(w, e)$ satisfies the Spence-Mirrlees property (SMP). We also assume that type q is relevant only for education e so that $u_q(w, e)$ does not have any "cross effect" between q and w. In other words, the utility gain associated with any wage increase is type-irrelevant:[7] if $w' > w$, then

$$u_H(w', e) - u_H(w, e) = u_L(w', e) - u_L(w, e). \tag{2}$$

In addition, for no education $e = 0$, it is reasonable to assume that the level of utility is type-irrelvant: $u_H(w, 0) = u_L(w, 0)$. This assumption and SMP imply $u_H(w, e) > u_L(w, e)$ for all $e > 0$.

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[7]This assumption is satisfied for all separable utility functions such that $u_q(w, e) = v(w) - c_q(e)$ for any increasing function $v(\cdot)$, which is widely used in many applications and textbooks.

The time line is described as follows:

Time 1. Nature chooses c.

Time 2. Each worker chooses type q.

Time 3. Each worker chooses signal e.

Time 4. The two firms simultaneously make wage offers.

Time 5. Each worker accepts the highest wage and produces. If indifferent, he chooses each firm with equal probability.

The worker's investment strategy at time 2, $Q(c)$, is a mapping $Q : [\underline{c}, \overline{c}] \rightarrow \{H, L\}$, and the worker's education strategy at time 3, $E(q)$, is a mapping $E : \{H, L\} \rightarrow \mathbb{R}_+$. Time 2 represents the stage of the human capital investment and time 3 represents the signaling stage. When a worker contemplates undertaking a human capital investment at time 2 to determine $q \in \{H, L\}$, he compares the cost and benefit of the investment. As we present below, the equilibrium strategy Q takes the form of a "cutoff" strategy: there exists a threshold cost level k such that workers with cost c make the investment (no investment) if $c < k$ (if $c > k$). An equilibrium is called an *interior* equilibrium when it has the threshold k on an interior point of the support $[\underline{c}, \overline{c}]$, $k \in (\underline{c}, \overline{c})$, so that some fraction of the population interval $[\underline{c}, \overline{c}]$ is motivated to make the investment and become type H. An equilibrium is called a *boundary* equilibrium when it has the threshold $k \notin (\underline{c}, \overline{c})$. Each firm's strategy at time 4, $w_i(e)$, is a mapping $w_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for $i = 1, 2$. The firms form (common posterior) beliefs $\mu(e)$, the probability of $q = H$, after observing e in equilibrium. Given the labor-market competition assumed above, in equilibrium, each firm's strategy must satisfy $w(e) = w_i(e) = \mu(e) y_H + (1 - \mu(e)) y_L$ for all i.

A strategy profile $\{(Q(c), E(q)), w(e)\}$ is a *perfect Bayesian equilibrium* if in each time line, the strategy of each player is the best response to the other players' strategies, and firms' beliefs about the worker's quality are updated by the Bayes' rule whenever possible.[8]

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[8]Formally, a set of strategies $\{(Q(c), E(q)), (w_i(e))_{i=1}^2\}$ and a belief function $\mu(e)$ constitute a perfect Bayesian equilibrium if:

(i) $(Q(c), E(q))$ is optimal for the worker given $(w_i(e))_{i=1}^2$

(ii) $\mu(e)$ is derived from $E(q)$ via the Bayes' rule where possible

(iii) $(w_i(e))_{i=1}^2$ is a Nash equilibrium of the simultaneous move game in which both firms make wage offers to the worker knowing that $q = H$ with probability $\mu(e)$.

We are primarily interested in separating and pooling equilibria. Each equilibrium is classified as an interior equilibrium or a boundary equilibrium. For now, we focus on interior equilibria and characterize these two kinds of interior equilibria. Consider first the signaling stage. For separating equilibria, let $e_H \equiv E(H) \neq e_L \equiv E(L)$. The Bayes' rule entails that $\mu(e_H) = 1$ and $\mu(e_L) = 0$ on the equilibrium path; thus, y_H (y_L) becomes the wage for type H (type L) and the worker with type L maximizes his utility by selecting $e_L = 0$. For pooling equilibria, let $e \equiv E(H) = E(L)$. The Bayes' rule entails that $\mu(e) = \lambda$ on the equilibrium path, where λ denotes the proportion of type H; thus, $\mathbb{E}^\lambda[y] = \lambda y_H + (1 - \lambda) y_L$ becomes an expected wage of both types. An important feature of our model is that λ is not exogenously given but *endogenously determined* by the investment decision in equilibrium.

Incentive compatibility conditions for separating and pooling equilibria are respectively described by

$$u_H(y_H, e_H) \geq u_H(y_L, 0) \text{ and } u_L(y_L, 0) \geq u_L(y_H, e_H), \tag{3}$$

and

$$u_H(\mathbb{E}^\lambda[y], e) \geq u_H(y_L, 0) \text{ and } u_L(\mathbb{E}^\lambda[y], e) \geq u_L(y_L, 0). \tag{4}$$

For separating equilibria, we can obtain the boundaries of e_H when two constraints in (3) are binding. Define $\underline{e}_H$ by $u_L(y_L, 0) = u_L(y_H, \underline{e}_H)$ and $\overline{e}_H$ by $u_H(y_H, \overline{e}_H) = u_H(y_L, 0)$. From the assumption $u_L(y_L, 0) = u_H(y_L, 0)$, we find that $\overline{e}_H > \underline{e}_H > 0$ and interior separating equilibria have the range of e_H:

$$e_H \in [\underline{e}_H, \overline{e}_H]. \tag{5}$$

For pooling equilibria, we use the binding constraint for type L and define the upper bound $\overline{e}(\lambda)$ by

$$u_L(\mathbb{E}^\lambda[y], \overline{e}(\lambda)) = u_L(y_L, 0), \tag{6}$$

and find that interior pooling equilibria have the range of e:

$$e \in [0, \overline{e}(\lambda)]. \tag{7}$$

If $\lambda < 1$ in a pooling equilibrium, then the intervals for e_H and e in (5) and (7) do not overlap: $\underline{e}_H > \overline{e}(\lambda)$ since $u_L(y_H, \underline{e}_H) = u_L(y_L, 0) = u_L(\mathbb{E}^\lambda[y], \overline{e}(\lambda))$ and $y_H > \mathbb{E}^\lambda[y]$ for $\lambda < 1$.

Consider next the stage of the human capital investment. For separating equilibria, if a worker selects $q = H$ by making the investment and incurring the cost c, then he has utility $u_H(y_H, e_H) - c$ and if a worker selects $q = L$ by making no investment, then he has utility $u_L(y_L, 0)$. Hence, an interior separating equilibrium has the threshold of undertaking the investment:

$$k_s = u_H(y_H, e_H) - u_L(y_L, 0). \tag{8}$$

This value k_s represents the utility gain that a worker enjoys when he makes the investment in an interior separating equilibrium where $k_s \in (\underline{c}, \overline{c})$ and $G(k_s) \in (0, 1)$. This value may thus be called the worker's incentive to make the investment in an interior separating equilibrium. The following lemma reports a distinct feature of our model: the separating signaling e_H affects the human capital investment. It shows that an increase in the signaling $e_H \in [\underline{e}_H, \overline{e}_H]$ reduces the worker's incentive to make the investment since it increases the signaling cost of e_H and thus decreases the utility gain from becoming type H.

Lemma 1 *In an interior separating equilibrium, the threshold k_s is a strictly decreasing function of $e_H \in [\underline{e}_H, \overline{e}_H]$.*

Similarly, for pooling equilibria, if a worker selects $q = H$ by making the investment, then he has utility $u_H(\mathbb{E}^\lambda[y], e) - c$ and if a worker selects $q = L$ by making no investment, then he has utility $u_L(\mathbb{E}^\lambda[y], e)$. Hence, an interior pooling equilibrium has the threshold of undertaking the investment:

$$k_p = u_H(\mathbb{E}^\lambda[y], e) - u_L(\mathbb{E}^\lambda[y], e). \tag{9}$$

This value k_p represents the utility gain that a worker enjoys when he makes the investment in an interior pooling equilibrium where $k_p \in (\underline{c}, \overline{c})$ and $G(k_p) \in (0, 1)$. This value may thus be called the worker's incentive to make the investment in an interior pooling equilibrium. Since the utility gain from any wage increase is type-irrelevant by assumption in (2),

$$u_H(\mathbb{E}^\lambda[y], e) - u_L(\mathbb{E}^\lambda[y], e) = u_H(0, e) - u_L(0, e),$$

the value k_p becomes the signaling cost advantage for type H:

$$k_p = u_H(0, e) - u_L(0, e). \tag{10}$$

In the following lemma, we report that the pooling signaling e also affects the human capital investment: an increase in e enlarges the signaling cost advantage for type H, which encourages workers to make the investment and become type H.

Lemma 2 *In an interior pooling equilibrium, the threshold k_p is a strictly increasing function of $e \in [0, \overline{e}(\lambda)]$.*

In an interior separating or pooling equilibrium, the respective fraction of type H is endogenously determined by the respective distribution function:

$$G(k_s) = G(u_H(y_H, e_H) - u_L(y_L, 0)) \tag{11}$$

$$G(k_p) = G(u_H(0, e) - u_L(0, e)). \tag{12}$$

We now define interior equilibria. An interior separating equilibrium is defined as a pair (k_s^*, e_H^*) that satisfies

$$G(k_s^*) = G(u_H(y_H, e_H^*) - u_L(y_L, 0)) \in (0, 1) \text{ and } e_H^* \in [\underline{e}_H, \overline{e}_H]. \tag{13}$$

An interior pooling equilibrium, involving an endogenously determined λ, is defined as a pair (k_p^*, e^*) that satisfies

$$G(k_p^*) = G(u_H(0, e^*) - u_L(0, e^*)) \in (0, 1) \text{ and } e^* \in [0, \overline{e}(G(k_p^*))]. \tag{14}$$

A worker, when contemplating whether to make the human capital investment, compares the consequent cost and gain. While the cost is directly incurred, the gain is delivered later by the use of signaling. In each interior equilibrium, there are some fraction of workers who find it profitable to undertake the investment.

We finally consider boundary equilibria.[9] We proceed to present a boundary equilibrium in which no human capital investment is made for any $c \in [\underline{c}, \overline{c}]$. If this no-investment equilibrium exists, then a worker has type L and enjoys utility $u_L(y_L, e_0)$ for any signaling level e_0; the worker thus selects no costly signaling, $e_0 = 0$. Further, in the absence of

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[9]If there is no interior separating equilibrium, then $u_H(y_H, e_H) - u_L(y_L, 0) \leq \underline{c}$ or $u_H(y_H, e_H) - u_L(y_L, 0) \geq \underline{c}$ is necessary. If there is no interior pooling equilibrium, then $u_H(0, e) - u_L(0, e) \leq \underline{c}$ or $u_H(0, e) - u_L(0, e) \geq \underline{c}$ is necessary. Thus, if there is no interior equilibrium, or if there is a boundary equilibrium, then the equation (11) or (12) is still satisfied.

signaling ($e_0 = 0$), the worker makes no human capital investment since its benefit is not delivered. Hence, the boundary equilibrium with no human capital and no signaling, denoted by (k_0^*, e_0^*), always exists and satisfies $G(k_0^*) = 0$ and $e_0^* = 0$. We next consider a boundary equilibrium in which the human capital investment is made for all $c \in [\underline{c}, \overline{c}]$. If this full-investment equilibrium exists, then a worker has type H and has utility $u_H(y_H, e_1)$ for any signaling level e_1; the worker thus selects no costly signaling, $e_1 = 0$. With no signaling ($e_1 = 0$), however, the worker makes no human capital investment, which causes a contradiction. Hence, the boundary equilibrium with the full investment does not exist. We now report the existence and uniqueness of boundary equilibrium.

Proposition 1 *A boundary equilibrium always exists and is unique. This boundary equilibrium (k_0^*, e_0^*) satisfies $G(k_0^*) = 0$ and $e_0^* = 0$.*

Due to this proposition, the analysis of boundary equilibrium is greatly simplified: the boundary equilibrium has no signaling and no investment. The nonexistence of the full investment equilibrium shows that the fraction of type H must be below 1 in any separating or pooling equilibrium.

3   Interior equilibria

The existence of boundary equilibrium was established in the previous section. In this section, we establish the existence of interior equilibria and also associate an interior equilibrium with the equilibrium that exists under a government intervention. To simplify analysis, we henceforth assume that u_q is differentiable.

3.1   Existence of interior equilibria

The existence of an interior separating equilibrium involves two conditions in (13): an interior separating equilibrium exists if and only if $k_s \in (\underline{c}, \overline{c})$ and $e_H \in [\underline{e}_H, \overline{e}_H]$. For now, we are interested in the condition under which there exists an interior separating equilibrium with the *least costly* education level $\underline{e}_H$ that satisfies Cho-Kreps' criterion (Cho and Kreps (1987)). To search for the condition, recall that the incentive to make

the human capital investment, represented by k_s, is strictly decreasing in $e_H \in [\underline{e}_H, \overline{e}_H]$. The threshold k_s thus has the maximum $\overline{k}_s$ when $e_H = \underline{e}_H$,

$$\overline{k}_s = u_H\left(y_H, \underline{e}_H\right) - u_L\left(y_L, 0\right) = u_H(y_H, \underline{e}_H) - u_L(y_H, \underline{e}_H),$$

where the second equality follows from the definition of $\underline{e}_H$, $u_L\left(y_L, 0\right) = u_L\left(y_H, \underline{e}_H\right)$. Since the utility gain from wage increase is type-irrelevant, the investment gain becomes the signaling cost advantage for type H for the education level $\underline{e}_H$,

$$\overline{k}_s = u_H\left(0, \underline{e}_H\right) - u_L\left(0, \underline{e}_H\right), \tag{15}$$

where $\overline{k}_s > 0$ since $\underline{e}_H > 0$. We now formally present a necessary and sufficient condition for the existence of an interior separating equilibrium with $\underline{e}_H$.

Proposition 2 *There exists an interior separating equilibrium with $\underline{e}_H$ if and only if $[u_H\left(0, \underline{e}_H\right) - u_L\left(0, \underline{e}_H\right)] \in (\underline{c}, \overline{c})$.*

The existence of an interior pooling equilibrium involves two conditions in (14). To simplify notations, we define the distribution function:

$$D(e) \equiv G(u_H\left(0, e\right) - u_L\left(0, e\right)). \tag{16}$$

Observing that this function $D(e)$ is strictly increasing in $e \in [0, \overline{e}\left(\lambda\right)]$ for any $D(e) \in (0, 1)$, the following proposition provides a necessary and sufficient condition for the existence of an interior pooling equilibrium.

Proposition 3 *There exists an interior pooling equilibrium if and only if $D\left(\overline{e}\left(\lambda\right)\right) \geq \lambda$ for some $\lambda \in (0, 1)$.*

Proof. Suppose first that there exists $\lambda \in (0, 1)$ such that $D\left(\overline{e}\left(\lambda\right)\right) \geq \lambda$. Define a correspondence $\Psi : [0, 1] \rightrightarrows [0, 1]$ using (14) such that

$$\Psi\left(\lambda\right) \equiv \{x \in [0, 1] : x = D\left(e\right) \text{ for } e \in [0, \overline{e}\left(\lambda\right)]\}.$$

Thus, an equilibrium fraction of type H, λ^*, is a fixed point of Ψ, $\lambda^* \in \Psi\left(\lambda^*\right)$. Since $D\left(e\right) \in (0, 1)$ is an increasing function of e, the correspondence can be rewritten as $\Psi\left(\lambda\right) = [0, D\left(\overline{e}\left(\lambda\right)\right)]$, and the condition implies the existence of $\lambda^* \in (0, 1)$ such that
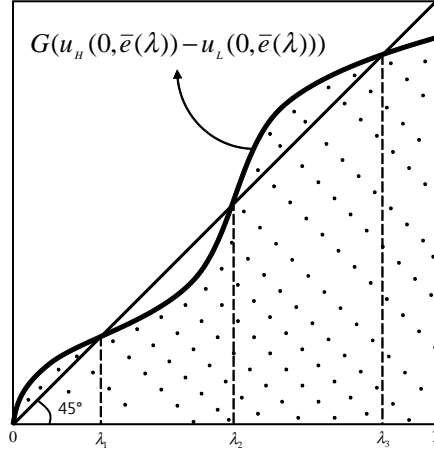
Figure 1: Two intervals of equilirium proportions

$\lambda^* \in \Psi\left(\lambda^*\right)$ and $\left(k_p^*, e^*\right)$ is derived from $G\left(k_p^*\right) = D\left(e^*\right) = \lambda^*$. Suppose next that there exists an interior pooling equilibrium and $D\left(\overline{e}\left(\lambda\right)\right) < \lambda$ for all $\lambda \in (0,1)$. Then only a boundary pooling equilibrium with $\lambda = 0$ or $\lambda = 1$ exists, which causes a contradiction. ∎

Figure 1 depicts the case with two sets of equilibrium proportions, $[0, \lambda_1]$ and $[\lambda_2, \lambda_3]$, where the dotted area below the curved line,

$$D(\overline{e}(\lambda)) = G(u_H\left(0, \overline{e}\left(\lambda\right)\right) - u_L\left(0, \overline{e}\left(\lambda\right)\right)),$$

represents the correspondence $\Psi\left(\lambda\right)$. The function $D\left(\overline{e}\left(\lambda\right)\right)$ in Figure 1 is strictly increasing in λ for all $\lambda \in (0,1)$. As we show later, however, this function may be constant for some λ. For now, we can show that $D(\overline{e}(\lambda))$ is strictly increasing in λ in the range of λ where $D(\overline{e}(\lambda)) \in (0,1)$, or equivalently $[u_H\left(0, \overline{e}\left(\lambda\right)\right) - u_L\left(0, \overline{e}\left(\lambda\right)\right)] \in (\underline{c}, \overline{c})$. From (6), we know that $\overline{e}\left(\lambda\right)$ is defined by

$$u_L(y_L + \lambda B, \overline{e}\left(\lambda\right)) = u_L(y_L, 0), \tag{17}$$

where $B \equiv y_H - y_L$ and $y_L + \lambda B = \mathbb{E}^\lambda\left[y\right]$. It follows from (17) that the upper bound $\overline{e}(\lambda)$ must be strictly increasing in λ for all $\lambda \in (0,1)$ with the endpoints, $\overline{e}\left(0\right) = 0$ and $\overline{e}\left(1\right) = \underline{e}_H$. This strictly increasing $\overline{e}(\lambda)$ in turn implies that the value $u_H(0, \overline{e}(\lambda)) - u_L(0, \overline{e}(\lambda))$ is also strictly increasing in λ for all $\lambda \in (0,1)$. Thus, in the range of λ where $D(\overline{e}(\lambda)) \in (0,1)$, we can be sure that $D(\overline{e}(\lambda))$ is strictly increasing in λ. In addition,

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the function $D\left(\overline{e}\left(\lambda\right)\right)$ shifts up if the productivity y_H becomes greater: an increase in y_H raises $\mathbb{E}^{\lambda}\left[y\right]$ and $\overline{e}\left(\lambda\right)$ given λ. For an increase in y_H, this function shifts more if the signaling cost gap, $u_H\left(0,e\right) - u_L\left(0,e\right)$, is larger.

We now identify some sufficient conditions under which an interior pooling equilibrium exists.

Corollary 1 *If y_H is sufficiently large given y_L, if $\lim_{\lambda \to 0} dD\left(\overline{e}\left(\lambda\right)\right)/d\lambda > 1$, or if $D\left(\overline{e}\left(1\right)\right) = 1$ and $\lim_{\lambda \to 1} dD\left(\overline{e}\left(\lambda\right)\right)/d\lambda < 1$, then there exists an interior pooling equilibrium.*

Proof. If y_H increases given y_L, then $y_L + \lambda B$ increases in (17) and, to satisfy (17), $\overline{e}\left(\lambda\right)$ increases for $\lambda \in \left(0,1\right)$ and $\overline{e}\left(1\right) = \underline{e}_H$ also increases.[10] Hence, for any $\lambda \in \left(0,1\right)$, we can find y_H such that $u_H\left(0,\overline{e}\left(\lambda\right)\right) - u_L\left(0,\overline{e}\left(\lambda\right)\right)$ is sufficiently large and $D\left(\overline{e}\left(\lambda\right)\right) \geq \lambda$ holds. For the second condition, we observe $D\left(\overline{e}\left(0\right)\right) = 0$ since $\overline{e}\left(0\right) = 0$. Now, $D\left(\overline{e}\left(0\right)\right) = 0$ and $\lim_{\lambda \to 0} dD\left(\overline{e}\left(\lambda\right)\right)/d\lambda > 1$ imply $D\left(\overline{e}\left(\lambda\right)\right) > \lambda$ for λ sufficiently small. Similarly, $D\left(\overline{e}\left(1\right)\right) = 1$ and $\lim_{\lambda \to 1} dD\left(\overline{e}\left(\lambda\right)\right)/d\lambda < 1$ imply $D\left(\overline{e}\left(\lambda\right)\right) > \lambda$ for λ sufficiently large. ∎

We next use Proposition 2 and 3 to present a necessary and sufficient condition under which an interior separating equilibrium with $\underline{e}_H$ and an interior pooling equilibrium exist at the same time.

Corollary 2 *There exist an interior pooling equilibrium and an interior separating equilibrium with $\underline{e}_H$ if and only if $D\left(\overline{e}\left(\lambda\right)\right) \geq \lambda$ for some $\lambda > 0$ and $D\left(\overline{e}\left(1\right)\right) < 1$.*

Proof. We first rewrite the condition $\overline{k}_s \in \left(\underline{c},\overline{c}\right)$ in Proposition 2 as $D\left(\overline{e}\left(1\right)\right) \in \left(0,1\right)$ since $D(\overline{e}(1)) = D(\underline{e}_H) = G(\overline{k}_s)$ from $u_L(y_H,\underline{e}_H) = u_L(y_L,0) = u_L(\mathbb{E}^{\lambda}[y],\overline{e}(\lambda))$. Now, the conditions in Proposition 2 and 3 become $D\left(\overline{e}\left(\lambda\right)\right) \geq \lambda$ for some $\lambda \in \left(0,1\right)$ and $D\left(\overline{e}\left(1\right)\right) \in \left(0,1\right)$. We next observe that the condition, $D\left(\overline{e}\left(\lambda\right)\right) \geq \lambda$ for some $\lambda > 0$, already includes the condition $D\left(\overline{e}\left(1\right)\right) > 0$ since $D(\overline{e}(1)) \geq D(\overline{e}(\lambda)) \geq \lambda > 0$. In addition, $D\left(\overline{e}\left(1\right)\right) < 1$ implies that λ satisfying the condition, $D\left(\overline{e}\left(\lambda\right)\right) \geq \lambda$ for some $\lambda > 0$, must be less than 1. Hence, the conditions in Proposition 2 and 3 are equivalent to the conditions stated in this corollary. ∎

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[10] Similarly, if $\mathbb{E}^{\lambda}\left[y\right]$ is sufficiently large given λ, then there exists an interior pooling equilibrium.

3.2 Exclusivity of interior equilibria

In this subsection, we examine the existence of interior equilibria conditional on the level of $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$. We also show that the existence of interior equilibria is mutually exclusive to some degree: in some parameter range, one type of interior equilibrium fails to exist while the other type exists.

First, suppose that

$$u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) \leq \underline{c}. \tag{18}$$

Proposition 2 then implies that there is no interior separating equilibrium with $\underline{e}_H$: if $e_H = \underline{e}_H$, then $k_s = u_H(y_H, \underline{e}_H) - u_L(y_L, 0) = u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ which is lower than any interior point of $[\underline{c}, \overline{c}]$ under the inequality. This nonexistence extends for any separating equilibrium: since $k_s = u_H(y_H, e_H) - u_L(y_L, 0)$ is strictly decreasing in $e_H \in [\underline{e}_H, \overline{e}_H]$, if there is no separating equilibrium with $\underline{e}_H$, there is no interior separating equilibrium with $e_H \in (\underline{e}_H, \overline{e}_H]$. Further, there is no interior pooling equilibrium: if an interior pooling equilibrium with e exists, then $k_p = u_H(0, e) - u_L(0, e) \in (\underline{c}, \overline{c})$ and $e < \underline{e}_H$ are necessary but impossible to satisfy since $u_H(0, e) - u_L(0, e)$ is strictly increasing in e. Thus, under the condition (18), there is no interior separating or pooling equilibrium. Instead, there exists the boundary equilibrium (k_0^*, e_0^*) with no signaling and no human capital investment.

Second, suppose that

$$u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) > \overline{c}. \tag{19}$$

Then the separating equilibrium with $\underline{e}_H$ cannot exist: if $e_H = \underline{e}_H$, then $k_s = u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ which is above any interior point of $[\underline{c}, \overline{c}]$ under the inequality. For the existence of a pooling equilibrium, we characterize the function:

$$D(\overline{e}(\lambda)) = G(u_H(0, \overline{e}(\lambda)) - u_L(0, \overline{e}(\lambda))).$$

Since $\overline{e}(\lambda)$ is strictly increasing in λ with $\overline{e}(0) = 0$ and $\overline{e}(1) = \underline{e}_H$, the value $u_H(0, \overline{e}(\lambda)) - u_L(0, \overline{e}(\lambda))$ is also strictly increasing in λ with

$$
\begin{aligned}
u_H(0, \overline{e}(0)) - u_L(0, \overline{e}(0)) &= 0 \\
u_H(0, \overline{e}(1)) - u_L(0, \overline{e}(1)) &> \overline{c}.
\end{aligned}
$$

Thus, there exists a unique $\overline{\lambda} \in (0,1)$ such that

$$u_H(0, \overline{e}(\overline{\lambda})) - u_L(0, \overline{e}(\overline{\lambda})) = \overline{c}, \tag{20}$$

where $\overline{e}(\overline{\lambda}) < \overline{e}(1) = \underline{e}_H$. The use of a pooling signaling e is restricted to the relevant range $e < \overline{e}(\overline{\lambda})$; since no pooling equilibrium uses the signaling $e \in [\overline{e}(\overline{\lambda}), \overline{e}(1)]$, the distribution function $D\left(\overline{e}\left(\lambda\right)\right)$ must be constant, $D\left(\overline{e}\left(\lambda\right)\right) = 1$, for all $\lambda \in [\overline{\lambda}, 1]$. We can similarly find that there exists a unique $\underline{\lambda} \in [0, \overline{\lambda})$ such that

$$u_H(0, \overline{e}(\underline{\lambda})) - u_L(0, \overline{e}(\underline{\lambda})) = \underline{c}. \tag{21}$$

We now have a complete characterization of $D\left(\overline{e}\left(\lambda\right)\right)$ under (19): $D\left(\overline{e}\left(\lambda\right)\right) = 0$ for all $\lambda \leq \underline{\lambda}$, the function $D\left(\overline{e}\left(\lambda\right)\right)$ is strictly increasing in λ for all $\lambda \in (\underline{\lambda}, \overline{\lambda})$, and $D\left(\overline{e}\left(\lambda\right)\right) = 1$ for all $\lambda \in [\overline{\lambda}, 1]$. This characterization implies $D\left(\overline{e}\left(\lambda\right)\right) \geq \lambda$ for λ sufficiently large. Thus, under the condition (19), there exists no separating equilibrium with $\underline{e}_H$, but there exists an interior pooling equilibrium.

Third, suppose that

$$u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) = \overline{c}, \tag{22}$$

which corresponds to $\overline{\lambda} = 1$ in (20): $D\left(\overline{e}\left(\lambda\right)\right) = 0$ for all $\lambda \leq \underline{\lambda}$, the function $D\left(\overline{e}\left(\lambda\right)\right)$ strictly increases in λ for all $\lambda \in (\underline{\lambda}, 1)$ and $D\left(\overline{e}\left(1\right)\right) = 1$. We know from Proposition 2 that the separating equilibrium with $\underline{e}_H$ cannot exist, but we are not sure whether an interior pooling equilibrium exists. Under (22), given $D\left(\overline{e}\left(1\right)\right) = 1$, if $\lim_{\lambda \to 1} dD\left(\overline{e}\left(\lambda\right)\right)/d\lambda < 1$, then $D\left(\overline{e}\left(\lambda\right)\right) \geq \lambda$ for λ sufficiently large, which ensures the existence of an interior pooling equilibrium. For here and later use, we characterize the slope:

$$\lim_{\lambda \to 1} \frac{dD\left(\overline{e}\left(\lambda\right)\right)}{d\lambda} = \lim_{\lambda \to 1} \frac{dG\left(\overline{k}_p(\lambda)\right)}{d\lambda} = \lim_{\lambda \to 1} g(\overline{k}_p) \frac{d\overline{k}_p(\lambda)}{d\overline{e}\left(\lambda\right)} \frac{d\overline{e}\left(\lambda\right)}{d\lambda}, \tag{23}$$

where $\overline{k}_p(\lambda) \equiv u_H\left(0, \overline{e}\left(\lambda\right)\right) - u_L\left(0, \overline{e}\left(\lambda\right)\right)$ and the term,

$$\frac{d\overline{e}\left(\lambda\right)}{d\lambda} = -\frac{B\partial u_L/\partial w}{\partial u_L/\partial e}\bigg|_{e=\overline{e}(\lambda)} > 0,$$

follows from the definition of $\overline{e}\left(\lambda\right)$, $u_L(y_L + \lambda B, \overline{e}\left(\lambda\right)) = u_L(y_L, 0)$. The slope in (23) is sufficiently small if the density at the top $g(\overline{c})$ is sufficiently small since $\lim_{\lambda \to 1} g(\overline{k}_p(\lambda)) = g(\overline{c})$. Thus, under the condition (22), if $g(\overline{c})$ is sufficiently small, then there exists no separating equilibrium with $\underline{e}_H$, but there exists an interior pooling equilibrium.

Lastly, suppose that

$$[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \overline{c}). \tag{24}$$

Then there exists $\underline{\lambda} \in [0, 1)$ such that $D(\overline{e}(\lambda)) = 0$ for $\lambda \in [0, \underline{\lambda}]$, the function $D(\overline{e}(\lambda))$ is strictly increasing in λ for $\lambda \in (\underline{\lambda}, 1]$ and $D(\overline{e}(1)) \in (0, 1)$. We know from Proposition 2 that the separating equilibrium with $\underline{e}_H$ exists, but we are not sure whether an interior pooling equilibrium exists. Proposition 3 shows that there is no interior pooling equilibrium if $D(\overline{e}(\lambda)) < \lambda$ for all $\lambda \in (0, 1)$. If the value $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ gets sufficiently close to $\underline{c}$, then $D(\overline{e}(1))$ approaches zero. Thus, under the condition (24), if $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ is sufficiently close to $\underline{c}$ and $\lim_{\lambda \to \underline{\lambda}} dD(\overline{e}(\lambda))/d\lambda < 1$, then there exists no interior pooling equilibrium, but there exists the separating equilibrium with $\underline{e}_H$. The slope $\lim_{\lambda \to \underline{\lambda}} dD(\overline{e}(\lambda))/d\lambda < 1$ if $g(\underline{c})$ is sufficiently small since $\lim_{\lambda \to \underline{\lambda}} g(\overline{k}_p) = g(\underline{c})$.[11]

We now summarize our findings.

Proposition 4 *(i) If $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) \le \underline{c}$, then there exists no interior separating or pooling equilibrium. (ii) If $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) > \overline{c}$, or if $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) = \overline{c}$ and $g(\overline{c})$ is sufficiently small, then there exists no interior separating equilibrium with $\underline{e}_H$ while there exists an interior pooling equilibrium. (iii) If $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \widehat{c})$ for some $\widehat{c}$ sufficiently close to $\underline{c}$ and $g(\underline{c})$ is sufficiently small, then there exists no interior pooling equilibrium while there exists the interior separating equilibrium with $\underline{e}_H$.*

In this and following sections, we occasionally illustrate our findings in association with the productivity gap, $B = y_H - y_L$. For this particular comparative static analysis, we focus on the case in which y_H increases given y_L.[12] From the definition of $\underline{e}_H$, $u_L(y_H, \underline{e}_H) = u_L(y_L, 0)$, we find that, if $B \to 0$, then $\underline{e}_H \to 0$, and if B increases, then $\underline{e}_H$ also increases. This relationship between B and $\underline{e}_H$ indicates that $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ is a strictly increasing function of B and that there exists a unique critical value $\underline{B}$ such that

$$u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) = \underline{c},$$

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[11]If $\underline{c}$ is sufficiently large, then this additional assumption on the density $g(\underline{c})$ is not necessary. We make the assumption on $g(\underline{c})$ given the possibility that $\underline{c}$ is sufficiently small.

[12]There are many ways in which B increases. For example, (i) y_H increases given y_L, and (ii) y_L decreases given y_H. We can greatly simplify our analysis by focusing on (i), not mixing (i) and (ii), since $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ is then monotonically increasing in B. Notice also that this monotonic property directly holds, regardless of the way B changes, if u_q is a linear function of w.

and there exists a unique value $\overline{B}$ such that

$$u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) = \overline{c}.$$

Therefore, when y_H increase given y_L, the results in Proposition 4 can be presented conditional on B. For example, if $B \leq \underline{B}$, then $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) \leq \underline{c}$, if $B > \overline{B}$, then $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) > \overline{c}$, and if $B = \overline{B}$, then $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) = \overline{c}$.

The following example describes the relationship between $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ and B. Suppose that the worker's utility function is separable and represented by

$$u_q(w, e) = w - c_q(e) \text{ for } q \in \{L, H\}, \tag{25}$$

where the educational cost $c_q(e)$ takes quadratic forms, $c_L(e) = e^2$ and $c_H(e) = ae^2$, with $a \in (0, 1)$. Suppose also that G is a uniform distribution $G(c) = c$ with its support $[0, 1]$. From $u_L(y_L, 0) = u_L(y_H, \underline{e}_H)$, we find $\underline{e}_H = \sqrt{B}$ and $\overline{k}_s$ is strictly increasing in B:

$$\overline{k}_s = u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) = (1 - a)(\underline{e}_H)^2 = (1 - a)B.$$

From $(1 - a)B = \underline{c} = 0$ and $(1 - a)B = \overline{c} = 1$, we find the critical values $\underline{B} = 0$ and $\overline{B} = 1/(1 - a)$.

We conclude this subsection by making some remarks. First, the value $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ is affected not only by the productivity gap B but by the signaling cost gap. In the example shown above, since $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) = (1-a)B$, the existence of interior equilibria presented above may also be conditional on $(1 - a)B$.[13] Second, the interior separating equilibrium with the least costly signaling for type H, $\underline{e}_H$, that satisfies Cho-Kreps' criterion exists in a limited parameter range. As two propositions show above, the interior separating equilibrium with $\underline{e}_H$ exists only when $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \overline{c})$. In the example, the interior separating equilibrium with $\underline{e}_H$ exists only in the range $(1 - a)B < \overline{c} = 1$. Third, the existence of the interior separating equilibrium with $\underline{e}_H$ and an interior pooling equilibrium is mutually exclusive to some degree. A sufficiently large $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ excludes the existence of the interior separating equilibrium with $\underline{e}_H$, whereas a small $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ close to $\underline{c}$ may exclude the existence of an

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[13] As noted above, we illustrate our findings conditional on the level of B. Under the untility function in (25), those same findings may also be conditional on the level of $(1 - a)B$.
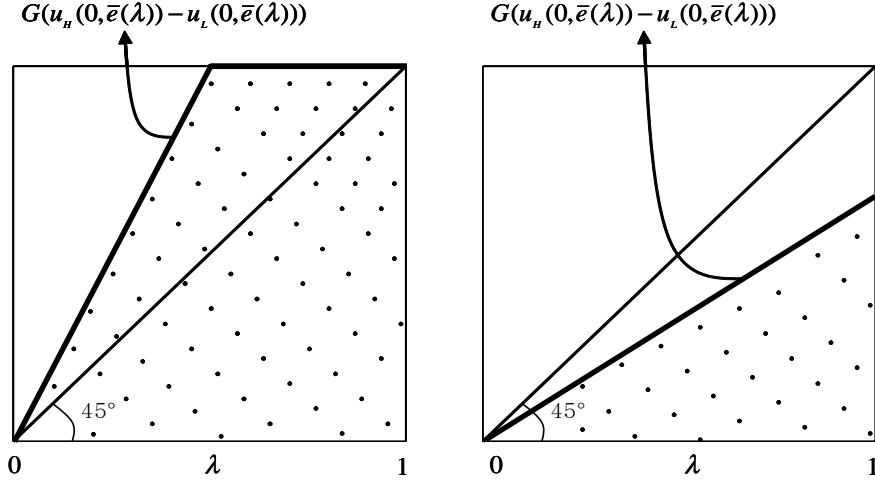
Figure 2: $(1-a)B > 1$ and $(1-a)B < 1$ under uniform G.

interior pooling equilibrium. In the example, the existence of these two interior equilibria is entirely exclusive. For an interior pooling equilibrium, we find the value:

$$k_p = u_H(0, e) - u_L(0, e) = c_L(e) - c_H(e) = (1-a)e^2.$$

The signaling e in an interior pooling equilibrium is bounded and satisfies $k_p = (1-a)e^2 < \bar{c} = 1$. Under the uniform distribution of G, the function $D(e)$ becomes

$$D(e) = G(u_H(0, e) - u_L(0, e)) = \min\{(1-a)e^2, 1\}.$$

Using $u_L(y_L + \lambda B, \bar{e}(\lambda)) = u_L(y_L, 0)$, we find the upper bound $\bar{e}(\lambda) = \sqrt{B\lambda}$ and the function:

$$D(\bar{e}(\lambda)) = \min\{(1-a)B\lambda, 1\}.$$

As Figure 2 shows, the condition $D(\bar{e}(\lambda)) \geq \lambda$ holds for any $\lambda \in (0,1)$ only in the parameter range $(1-a)B \geq 1$ where the interior separating equilibrium with $\underline{e}_H$ cannot exist. In this parameter range, any combination (k_p^*, e^*) that satisfies $k_p^* = (1-a)e^{*2}$ and $k_p^* \in (0,1)$ is an interior pooling equilibrium.

3.3 Interior equilibria under market maker

We associate a separating or pooling signaling with an interior separating or pooling equilibrium that exists under the government's market design: we assume that the government

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acts as a market maker and implements a certain signaling through the equilibrium that exists under its market design. A rationale for this intervention is that it can significantly increase the set of implementable signaling under the Cho-Kreps' intuitive criterion by expanding the scope of parameters in which there exists an interior separating or pooling equilibrium that satisfies the criterion. In this subsection, we formalize this argument under the assumption: the government's market design takes the form of restricting signaling choices to a set,

$$\{0\} \cup [\underline{E}, \overline{E}], \tag{26}$$

where $\underline{E}, \overline{E} \in \mathbb{R}_+$ and $\overline{E} \geq \underline{E}$.

We first broaden our attention beyond the interior separating equilibrium with $\underline{e}_H$ by showing that, if $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) > \underline{c}$, then an interior separating equilibrium always exists. If $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \overline{c})$, then the interior separating equilibrium with $\underline{e}_H$ exists. If $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) \geq \overline{c}$, then the interior separating equilibrium with $\underline{e}_H$ cannot exist, but an interior separating equilibrium with $e_H \in (\underline{e}_H, \overline{e}_H]$ exists: since $\overline{e}_H$ is defined by $u_H(y_H, \overline{e}_H) = u_H(y_L, 0)$, if e_H increases above $\underline{e}_H$ and approaches $\overline{e}_H$, then $k_s = u_H(y_H, e_H) - u_L(y_L, 0)$ monotonically decreases and approaches $u_H(y_L, 0) - u_L(y_L, 0) = 0$ where $0 \leq \underline{c}$. Intuitively, an increase in the signaling cost of e_H decreases the incentive to make the human capital investment, and if $e_H \to \overline{e}_H$, then the incentive diminishes to zero since workers then find it indifferent to become type H and L.[14]

We now associate an interior separating equilibrium with $e_H \in [\underline{e}_H, \overline{e}_H]$ with the equilibrium that exists under the government's market design (26): $\{0\} \cup [\underline{E}, \overline{E}]$ is hereafter called the market design (S) when a point $(k_s^*, e_H^*) = (u_H(y_H, \underline{E}) - u_L(y_L, 0), \underline{E})$ satisfies (13) and thus constitutes an interior separating equilibrium under $\{0\} \cup [\underline{E}, \overline{E}]$. The market design (S) selects $\underline{E}$ such that (i) the point (k_s^*, e_H^*) is a unique interior separating equilibrium that satisfies the Cho-Kreps' intuitive criterion, since there is no lower

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[14]Consider the previous example where the utility function is (25) and G is a uniform distribution $G(c) = c$ with its support $[0, 1]$. From $u_L(y_L, 0) = u_L(y_H, \underline{e}_H)$ and $u_H(y_H, \overline{e}_H) = u_H(y_L, 0)$, we can derive two bounds: $\underline{e}_H = \sqrt{B}$ and $\overline{e}_H = \sqrt{B/a}$. Using $k_s = u_H(y_H, e_H) - u_L(y_L, 0) = B - a(e_H)^2$, we can derive the corresponding range of the threshold: $k_s \in [0, (1-a)B]$. Thus, if $\overline{k}_s = (1-a)B < \overline{c} = 1$, there exists the interior separating equilibrium with $\underline{e}_H$, and if $\overline{k}_s = (1-a)B \geq \overline{c} = 1$, there exists no interior separating equilibrium with $\underline{e}_H$, but there exists an interior separating equilibrium with $e_H \in (\underline{e}_H, \overline{e}_H]$; the threshold k_s decreases below $\overline{c}$ when e_H increases above $\underline{e}_H$.

education level below $\underline{E}$ to which a worker with type H can gain by deviating, and (ii) there exists no interior pooling equilibrium, since any interior pooling equilibrium with e satisfies $e < \underline{e}_H \le \underline{E} = e_H^*$.

We next associate an interior pooling equilibrium with the equilibrium that exists under the government's market design (26): $\{0\} \cup [\underline{E}, \overline{E}]$ is hereafter called the market design (P) when a point $(k_p^*, e^*) = (u_H(0, \overline{E}) - u_L(0, \overline{E}), \overline{E})$ satisfies (14) and thus constitutes an interior pooling equilibrium under $\{0\} \cup [\underline{E}, \overline{E}]$. The market design (P) selects $\overline{E}$ such that (i) the point (k_p^*, e^*) is an interior pooling equilibrium that satisfies the Cho-Kreps' intuitive criterion, since there is no higher education level above $\overline{E}$ to which a worker with type H can deviate, and (ii) there exists no interior separating equilibrium, since any interior separating equilibrium with e_H satisfies $e_H > \overline{e}(\lambda^*) \ge e^* = \overline{E}$.[15]

The following proposition summarizes the uniqueness results.

Proposition 5 *Under the market design (S), the selected interior separating equilibrium is a unique interior equilibrium that satisfies the Cho-Kreps' intuitive criterion. Under the market design (P), the selected interior pooling equilibrium is a unique interior equilibrium that satisfies the Cho-Kreps' intuitive criterion for u_L that has a sufficiently large marginal cost of education.*

4   Signaling and social welfare

The social value of signaling has long been challenged: the signaling aspect of education is wasteful from the perspective of welfare. In this section, we use interior equilibria and show that this seemingly wasteful action becomes socially beneficial with its inducement of the other aspect of education, human capital investment. We then characterize the welfare-maximizing signaling in each interior equilibrium.

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[15]Notice that the *uniqueness* of an interior pooling equilibrium is not reported in the finding (i). As Proposition 5 shows, this uniqueness is ensured by an assumption: u_L has sufficiently large marginal cost of education. A formal proof for our findings is in the Appendix.

4.1 Role of human capital investment

We consider the *benchmark* signaling model in which the aspect of human capital investment is entirely disregarded. An essential feature of this benchmark model is that signaling choice has no effect on the human capital investment: $q \in \{L, H\}$ is determined by the prior proportion $\lambda \in (0,1)$. Other than this feature, we can directly use the previous characterization: a separating equilibrium has $e_L = 0$ and $e_H \in [\underline{e}_H, \overline{e}_H]$, and a pooling equilibrium has $e \in [0, \overline{e}(\lambda)]$.

We use the benchmark model and show that the signaling aspect of education can hardly be justified from the perspective of welfare: it is socially wasteful without its effect on the human capital investment. To present this benchmark result, we describe the social welfare. A separating equilibrium (e_L, e_H) generates the social welfare:

$$\lambda u_H(y_H, e_H) + (1 - \lambda) u_L(y_L, 0).$$

Since $u_H(y_H, e_H)$ is strictly decreasing in $e_H \in [\underline{e}_H, \overline{e}_H]$, the optimal separating equilibrium is $(e_L, e_H) = (0, \underline{e}_H)$ that satisfies the intuitive criterion and generates the social welfare:

$$U_s^B = \lambda u_H(y_H, \underline{e}_H) + (1 - \lambda) u_L(y_L, 0). \tag{27}$$

A pooling equilibrium, $e_H = e_L = e$, generates the social welfare:

$$U_p^B = \lambda u_H\left(\mathbb{E}^\lambda[y], e\right) + (1 - \lambda) u_L\left(\mathbb{E}^\lambda[y], e\right). \tag{28}$$

In comparison, the zero education (no signaling) leads to the same wage $\mathbb{E}^\lambda[y]$ and generates the social welfare:

$$U_0^B = \lambda u_H\left(\mathbb{E}^\lambda[y], 0\right) + (1 - \lambda) u_L\left(\mathbb{E}^\lambda[y], 0\right).$$

We now make the following comparison while providing the proof in the Appendix.

Proposition 6 *In the benchmark model, signaling is socially wasteful; (i) if u_L is concave in w, then the ban on education (no signaling) is strictly better than any separating equilibrium in terms of social welfare, and (ii) the ban on education is strictly better than any pooling equilibrium with $e > 0$ in terms of social welfare.*

In comparison with the separating equilibrium with $\underline{e}_H$, the zero education benefits workers with type L since $u_L\left(\mathbb{E}^\lambda\left[y\right],0\right) > u_L\left(y_L,0\right)$, while it benefits workers with type H only if $\lambda \in (0,1)$ is sufficiently large to satisfy $u_H\left(\mathbb{E}^\lambda\left[y\right],0\right) > u_H\left(y_H,\underline{e}_H\right)$. The concavity assumption of u_L ensures that the benefit of type L is greater than the loss of type H for small λ. In any pooling equilibrium with $e \in [0,\overline{e}\left(\lambda\right)]$, it is socially optimal to ban education since it clearly benefits workers with both types.

We now return to our original model and consider any interior equilibrium in which some fraction of workers are motivated to make the investment and become type H. The ban on education (no signaling) leads to the boundary equilibrium in which every worker has type L and the social welfare is

$$U_0 = u_L\left(y_L,0\right). \tag{29}$$

We next show that the social welfare is higher in any interior separating or pooling equilibrium than in the boundary equilibrium. In an interior separating equilibrium, a worker with $c \in (k_s,\overline{c})$ has type L and obtains utility $u_L\left(y_L,0\right)$, while a worker with $c \in (\underline{c},k_s)$ has type H and obtains utility $u_H\left(y_H,e_H\right) - c > u_L\left(y_L,0\right)$, where the inequality is given by $k_s = u_H\left(y_H,e_H\right) - u_L\left(y_L,0\right) > c$. In an interior pooling equilibrium, a worker with $c \in (k_p,\overline{c})$ has type L and obtains utility $u_L\left(\mathbb{E}^\lambda\left[y\right],e\right) \geq u_L\left(y_L,0\right)$ for $e \in [0,\overline{e}\left(\lambda\right)]$, while a worker with $c \in (\underline{c},k_p)$ has type H and obtains utility $u_H\left(\mathbb{E}^\lambda\left[y\right],e\right) - c > u_L\left(y_L,0\right)$, where the inequality is given by $k_p = u_H(\mathbb{E}^\lambda\left[y\right],e) - u_L(\mathbb{E}^\lambda\left[y\right],e) > c$. The following proposition reports this comparison.

Proposition 7 *In any interior separating or pooling equilibrium, signaling is socially beneficial; the ban on education (no signaling) reduces social welfare from the level that would obtain under any interior separating or pooling equilibrium.*

This result shows that the signaling aspect of education is justifiable from the perspective of welfare: this seemingly wasteful action becomes socially beneficial when it induces any human capital investment by delivering the benefits of the investment. The result also implies that the social value of signaling may be unduly underrated if its effect on the human capital investment is not taken into account.

4.2 Welfare-maximizing signaling

In this subsection, we characterize the welfare-maximizing signaling in each interior equilibrium.[16] We say that an interior equilibrium is optimal when it uses the welfare-maximizing signaling.

We assume that $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) > \underline{c}$. Then an interior separating equilibrium exists and has the social welfare:

$$
\begin{aligned}
U_s &= \int_{\underline{c}}^{k_s} [u_H(y_H, e_H) - c]dG(c) + \int_{k_s}^{\overline{c}} u_L(y_L, 0)\, dG(c) \\
&= u_L(y_L, 0) + \int_{\underline{c}}^{k_s} [k_s - c]dG(c).
\end{aligned}
$$

The second equality follows from $k_s = u_H(y_H, e_H) - u_L(y_L, 0)$. The social welfare U_s consists of two terms: the welfare for type L and the expected gain from the human capital investment. By integrating by parts, we can rewrite U_s as

$$
U_s = u_L(y_L, 0) + \int_{\underline{c}}^{k_s} G(c)dc. \tag{30}
$$

Hence, in an interior separating equilibrium, it is socially beneficial to maximize the human capital investment (k_s): an interior separating equilibrium maximizes the investment to maximize the social welfare. Since it follows from Lemma 1 that a decrease in e_H increases the threshold k_s by increasing the utility gain from becoming type H, an optimal interior separating equilibrium selects the least costly signaling for type H that maximizes the human capital investment (k_s).

We assume that $D(\overline{e}(\lambda)) \geq \lambda$ for some $\lambda \in (0, 1)$. Then an interior pooling equilibrium exists and has the social welfare:

$$
U_p = \int_{\underline{c}}^{k_p} [u_H(\mathbb{E}^\lambda[y], e) - c]dG(c) + \int_{k_p}^{\overline{c}} u_L(\mathbb{E}^\lambda[y], e)dG(c).
$$

Using $k_p = u_H(\mathbb{E}^\lambda[y], e) - u_L(\mathbb{E}^\lambda[y], e)$ and integration by parts, we find that the social welfare U_p consists of the welfare for type L and the expected gain from the human capital

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[16]In the Appendix, we also examine how an optimal interior signaling is affected when there exists an environmental improvement that reduces the expected cost of the human capital investment: the distribution function G shifts up to F in terms of first-order stochastic dominance (FOSD) such that F has lower expected value, $\int_{\underline{c}}^{\overline{c}} cdF(c) < \int_{\underline{c}}^{\overline{c}} cdG(c)$.

investment:

$$U_p = u_L(\mathbb{E}^\lambda [y], e) + \int_{\underline{c}}^{k_p} G(c)dc. \tag{31}$$

The choice of $e = D^{-1}(\lambda)$ is made under the constraint $\{\lambda \in (0, 1) : D(\bar{e}(\lambda)) \geq \lambda\}$. An optimal pooling equilibrium is not straightforward to find: to increase the worker's incentive to make the investment (k_p), it is necessary to increase the signaling cost for type L. Since u_q is assumed to be differentiable and the expected wage is $\mathbb{E}^\lambda [y] = y_L + G(k_p) B$, we have the marginal welfare of signaling e:

$$\frac{dU_p}{de} = \frac{\partial U_p}{\partial k_p}\frac{dk_p}{de} + \frac{\partial U_p}{\partial e} = \left(\frac{\partial u_L}{\partial w} \cdot B \cdot g(k_p) + G(k_p)\right)\frac{dk_p}{de} + \frac{\partial u_L}{\partial e}. \tag{32}$$

Hence, in an interior pooling equilibrium, it is not necessarily socially beneficial to maximize the human capital investment (k_p) by increasing the costly signaling: an increase in e improves the welfare by promoting the human capital investment, but it also worsens the welfare directly by increasing the signaling cost of type L. The following lemma highlights this point.

Lemma 3 *For any interior pooling equilibrium with e and $k_p = u_H(0, e) - u_L(0, e)$, if the density $g(u_H(0, e) - u_L(0, e))$ is sufficiently small, then an increase in e causes the welfare loss.*

Proof. The last two terms in (32) are negative,

$$G(k_p)\frac{dk_p}{de} + \frac{\partial u_L}{\partial e} < \frac{dk_p}{de} + \frac{\partial u_L}{\partial e} = \frac{\partial u_H}{\partial e} < 0.$$

Thus, if $g(k_p) = g(u_H(0, e) - u_L(0, e))$ is sufficiently small in (32), then an increase in e causes the welfare loss, $dU_p/de < 0$. ∎

This result holds for any interior pooling equilibrium: for any interior pooling equilibrium, an increase in the signaling e incurs the welfare loss whenever the corresponding density $g(k_p)$ is sufficiently low at the margin where it is indifferent between making the human capital investment and making no investment. The result shows that an interior pooling equilibrium with e suffers from an oversignaling if an additional signaling e increases a sufficiently small investment: an increase in e raises the worker's incentive to make the investment, $u_H(0, e) - u_L(0, e)$, but it leads to only a small increase in human capital (increase in the fraction of type H) given the low density at the margin, $g(k_p) = g(u_H(0, e) - u_L(0, e))$.

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5   Comparison between separating and pooling

Thus far, we have separately analyzed two types of signaling by associating a separating signaling with an interior separating equilibrium and associating a pooling signaling with an interior pooling equilibrium. It might be interesting to compare two contrasting forms of signaling. In this section, we identify circumstances under which two types of signaling can be clearly ranked in terms of social welfare. We provide some insights into a proper form of signaling: whether it is socially beneficial to stay with a separating signaling or switch to a pooling signaling is dependent on how each signaling induces the human capital investment.

5.1   Signaling form and human capital investment

As our previous analysis shows, regardless of signaling types, any signaling is socially beneficial whenever it induces the human capital investment by delivering the benefit of the investment. In this subsection, we show that a separating signaling may be socially inferior to a pooling signaling, without a larger inducement of the human capital investment.

A difficulty of making this comparison is that the existence of two interior equilibria is partially or entirely exclusive. We thus broaden the scope of parameters in which two interior equilibria coexist by considering a separating signaling for all $e_H \in [\underline{e}_H, \overline{e}_H]$. Allowing for the signaling $e_H \in (\underline{e}_H, \overline{e}_H]$ has the following consequences. First, an interior separating equilibrium with $e_H \in (\underline{e}_H, \overline{e}_H]$ exists under the condition, $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) \geq \overline{c}$, where the interior separating equilibrium with $\underline{e}_H$ cannot exist. Second, if any interior pooling equilibrium exists, then an interior separating equilibrium always exists: if an interior pooling equilibrium with e exists, then $u_H(0, e) - u_L(0, e) > \underline{c}$ is necessary, and $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) > \underline{c}$ is also necessary, which confirms that an interior separating equilibrium always exists since it still exists even when $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) \geq \overline{c}$.

We begin by recalling that the intervals for $e_H \in [\underline{e}_H, \overline{e}_H]$ and $e \in [0, \overline{e}(\lambda)]$ are exclusive and satisfy $\underline{e}_H > \overline{e}(\lambda)$ for any $\lambda \in (0, 1)$. We show that a separating signaling, despite its use of a higher signaling level e_H, does not necessarily generate the human capital investment more than a pooling signaling: as we show in the Appendix, we define a

function that is strictly decreasing in e_H,

$$\Phi\left(e_H,e\right) \equiv k_s - k_p = u_H\left(y_H,e_H\right) - u_L\left(y_L,0\right) - \left[u_H\left(0,e\right) - u_L\left(0,e\right)\right],$$

and find that there is a unique $\phi\left(e\right) \in \left(\underline{e}_H,\overline{e}_H\right)$ such that $\Phi\left(\phi\left(e\right),e\right) = 0$. Intuitively, if $e_H \in \left[\underline{e}_H,\overline{e}_H\right]$ increases above $\underline{e}_H$ and gets closer to the upper bound $\overline{e}_H$, then the signaling cost of e_H becomes larger and thus workers become less encouraged to make the human capital investment and become type H.

Proposition 8 *Assume that there exists an interior pooling equilibrium with e. There exists a unique $\phi\left(e\right) \in \left(\underline{e}_H,\overline{e}_H\right)$ such that $k_s > k_p$ for $e_H \in \left[\underline{e}_H,\phi\left(e\right)\right)$ and $k_s < k_p$ for $e_H \in \left(\phi\left(e\right),\overline{e}_H\right]$.*

This result is perhaps surprising and has immediate welfare implications: (i) if $k_p > k_s$, then $U_p > U_s$ since

$$U_p - U_s = u_L(\mathbb{E}^\lambda\left[y\right],e) - u_L\left(y_L,0\right) + \int_{k_s}^{k_p} G(c)dc, \tag{33}$$

where $u_L(\mathbb{E}^\lambda\left[y\right],e) - u_L\left(y_L,0\right) \geq 0$ (with equality only if $e = \overline{e}(\lambda)$) and (ii) if $k_p = k_s$, then $U_p \geq U_s$ (with equality only if $e = \overline{e}(\lambda)$). This comparison of social welfare based on (33) leads us to the following proposition.

Proposition 9 *Assume that there exists an interior pooling equilibrium with e. (i) If $k_p > k_s$, then $U_p > U_s$, and if $k_p = k_s$, then $U_p \geq U_s$ (= only if $e = \overline{e}\left(\lambda\right)$). (ii) If $k_p < k_s$, then $U_p < U_s$ for $e = \overline{e}\left(\lambda\right)$, but the relative magnitude of welfare depends on the model's specification for $e < \overline{e}\left(\lambda\right)$.*

As we state in Proposition 7, the signaling aspect of education is socially beneficial in an interior equilibrium where the signaling induces the human capital investment. This signaling aspect of education may be better represented by a separating signaling than by a pooling signaling in that private information about individual types is conveyed by the separating signaling. From the perspective of welfare, however, Proposition 9 shows that a separating signaling may be inferior to a pooling signaling, without a larger inducement of the human capital investment.

5.2 Pooling signaling and moderation of investment

In this subsection, we specify the conditions under which an interior pooling equilibrium is sure to exist and is strictly better than any separating equilibrium in terms of social welfare.

We first rank two types of signaling in terms of social welfare under the condition, $u_H(0, \bar{e}(1)) - u_L(0, \bar{e}(1)) \geq \bar{c}$, where there is no separating equilibrium with $\underline{e}_H$, but there is an interior separating equilibrium with $e_H > \underline{e}_H$. This interior separating equilibrium with $e_H > \underline{e}_H$ has the social welfare:

$$U_s = u_L(y_L, 0) + \int_{\underline{c}}^{k_s} G(c) dc,$$

where $k_s = u_H(y_H, e_H) - u_L(y_L, 0)$ decreases when the separating signaling e_H increases. Thus, under the condition, an increase in e_H reduces the human capital investment by increasing the signaling cost of e_H too much. An additional signaling e_H is then socially harmful.

Our first finding is that a pooling signaling is strictly better than any separating signaling under the condition:

$$u_H(0, \bar{e}(1)) - u_L(0, \bar{e}(1)) > \bar{c}, \tag{34}$$

where $\bar{e}(1) = \underline{e}_H$. This condition implies that there exists a unique $\overline{\lambda} \in (0, 1)$ such that

$$u_H(0, \bar{e}(\overline{\lambda})) - u_L(0, \bar{e}(\overline{\lambda})) = \bar{c} \tag{35}$$

and $D(\bar{e}(\lambda)) = 1$ for all $\lambda \in [\overline{\lambda}, 1]$. Then there is an interior pooling equilibrium with e such that $e \to \bar{e}(\overline{\lambda}) < \bar{e}(1) = \underline{e}_H < e_H$, $k_p \to \bar{c}$, $G(k_p) \to 1$ and its welfare

$$U_p \to u_L(y_H, \bar{e}(\overline{\lambda})) + \int_{\underline{c}}^{\bar{c}} G(c) dc. \tag{36}$$

Since $u_L(y_H, \bar{e}(\overline{\lambda})) > u_L(y_H, \underline{e}_H) = u_L(y_L, 0)$, it follows that

$$u_L(y_H, \bar{e}(\overline{\lambda})) + \int_{\underline{c}}^{\bar{c}} G(c) dc > \overline{U}_s \equiv u_L(y_L, 0) + \int_{\underline{c}}^{\bar{c}} G(c) dc, \tag{37}$$

where $\overline{U}_s$ is the upper bound of U_s. Hence, under the condition (34), an interior pooling equilibrium with signaling in moderation below $\underline{e}_H$ can generate the welfare U_p above $\overline{U}_s$.

Our second finding is that a pooling signaling may be strictly better than any separating signaling under the condition:

$$u_H(0, \overline{e}(1)) - u_L(0, \overline{e}(1)) = \overline{c}. \tag{38}$$

This condition implies that $D(\overline{e}(\lambda)) = 1$ only for $\lambda = 1$, which corresponds to $\overline{\lambda} = 1$ in (35). It is useful to know that the condition (34) and (38) have the difference in terms of the density $g(\overline{k}_p(\lambda))$ for $\lambda = 1$, where

$$\overline{k}_p(\lambda) \equiv u_H(0, \overline{e}(\lambda)) - u_L(0, \overline{e}(\lambda)).$$

(i) If $\overline{\lambda} \in (0, 1)$, then $g(\overline{k}_p(1)) = 0$ given $D(\overline{e}(\lambda)) = G(\overline{k}_p(\lambda)) = 1$ for all $\lambda \in [\overline{\lambda}, 1]$ and (ii) if $\overline{\lambda} = 1$, then $g(\overline{k}_p(1)) = g(\overline{c})$ given $\overline{k}_p(1) = \overline{c}$.

Now, our second finding obtains in two steps. First, under (38), if the density at the top $g(\overline{c})$ is sufficiently small, then there exists an interior pooling equilibrium with the limiting value e that satisfies $e \to \overline{e}(1)$, $k_p \to \overline{c}$ and $G(k_p) \to 1$. This interior pooling equilibrium with the limiting value is sure to exist: under (38), given $D(\overline{e}(1)) = 1$, if $g(\overline{c})$ is sufficiently small, then the slope $\lim_{\lambda \to 1} dD(\overline{e}(\lambda))/d\lambda$ in (23) is sufficiently small since $\lim_{\lambda \to 1} g(\overline{k}_p(\lambda)) = g(\overline{c})$. Second, as we show in Lemma 3, for any interior pooling equilibrium with the limiting value e that satisfies $e \to \overline{e}(1)$, $k_p \to \overline{c}$ and $G(k_p) \to 1$, if $g(\overline{c})$ is sufficiently small, then an increase in e causes the welfare loss. This interior pooling equilibrium with the limiting value e suffers from an oversignaling: a decrease in e increases the social welfare. The interior pooling equilibrium with the limiting value has its welfare U_p that approaches $\overline{U}_s$,

$$U_p \to u_L(y_H, \underline{e}_H) + \int_{\underline{c}}^{\overline{c}} G(c)dc = \overline{U}_s,$$

where the last equality follows from $u_L(y_H, \underline{e}_H) = u_L(y_L, 0)$, but this second step shows that a decrease in e below the limiting value increases the welfare U_p above $\overline{U}_s$. Hence, under the condition (38), if $g(\overline{c})$ is sufficiently small, then there exists a pooling equilibrium that is strictly better than any separating equilibrium.

We next rank two types of signaling in terms of social welfare under the condition:

$$[u_H(0, \overline{e}(1)) - u_L(0, \overline{e}(1))] \in (\underline{c}, \overline{c}), \tag{39}$$

where there exists the separating equilibrium with $\underline{e}_H$. Our third finding is that, under the condition (39), if $g(\overline{c})$ is sufficiently small and $u_H(0, \overline{e}(1)) - u_L(0, \overline{e}(1))$ is sufficiently close to $\overline{c}$, then a pooling signaling is strictly better than the separating signaling $\underline{e}_H$. Since $u_H(0, \overline{e}(1)) - u_L(0, \overline{e}(1))$ is sufficiently close to $\overline{c}$, there exists the interior separating equilibrium with $\underline{e}_H$ such that $\overline{k}_s \to \overline{c}$ and its welfare $U_s \to \overline{U}_s$. If there exists the separating equilibrium with $\underline{e}_H$ such that $\overline{k}_s \to \overline{c}$ and $G(\overline{k}_s) \to 1$, then for $g(\overline{c})$ sufficiently small, there also exists an interior pooling equilibrium with the limiting value e such that $e \to \underline{e}_H$, $k_p \to \overline{c}$, $G(k_p) \to 1$ and its welfare

$$U_p \to u_L\left(y_H, \underline{e}_H\right) + \int_{\underline{c}}^{\overline{c}} G(c)dc = \overline{U}_s,$$

since $D\left(\overline{e}\left(1\right)\right) \to 1$ and $\lim_{\lambda \to 1} dD\left(\overline{e}\left(\lambda\right)\right)/d\lambda$ in (23) is sufficiently small. For $g(\overline{c})$ sufficiently small, a decrease in e below the limiting value increases the welfare U_p above $\overline{U}_s$.

We can summarize our findings as follows: if $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) > \overline{c}$, or if $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\widehat{c}, \overline{c}]$ for some $\widehat{c}$ sufficiently close to $\overline{c}$ and $g(\overline{c})$ is sufficiently small, then there exists an interior pooling equilibrium that is strictly better than any separating equilibrium in terms of social welfare. We now use Proposition 5 and associate the socially preferred interior pooling equilibrium with the equilibrium that exists under the government's market design $\{0\} \cup [\underline{E}, \overline{E}]$.

Proposition 10 *If $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) > \overline{c}$, or if $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\widehat{c}, \overline{c}]$ for some $\widehat{c}$ sufficiently close to $\overline{c}$ and $g(\overline{c})$ is sufficiently small, then the market design (P) maximizes the social welfare.*

Proposition 10 consists of two parts: the uniqueness and advantage of an interior pooling equilibrium. The result is based on the condition that ensures the existence of an interior pooling equilibrium that approximates the full investment $G(k_p) \to 1$. We establish the result in two logical steps. First, we observe that the very best of all possible interior separating equilibria, by motivating most of workers to become one type (type H), approaches an interior pooling equilibrium that approximates the full investment. Second, we show that a pooling signaling is strictly better than a separating signaling under two circumstances: (i) an additional separating signaling e_H reduces human capital (the

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fraction of type H) by making the signaling cost for type H too high, so that any interior separating equilibrium fails to approximate the full investment, and (ii) even when an interior separating equilibrium can approximate the full investment, it approaches an interior pooling equilibrium that is too costly to implement and suffers from an overinvestment.

The following corollary presents the same result based on the productivity gap, $B = y_H - y_L$.[17]

Corollary 3 *Suppose that y_H increases given y_L. If $B > \overline{B}$, or if $B \in (\widehat{B}, \overline{B}]$ for some $\widehat{B}$ sufficiently close to $\overline{B}$ and $g(\overline{c})$ is sufficiently small, then the market design (P) maximizes the social welfare.*

It seems socially beneficial to motivate more workers to undertake the human capital investment; indeed, an interior separating equilibrium maximizes the investment to maximize the social welfare. Our finding shows, however, that it may be socially harmful to stay with the separating signaling: before an interior separating equilibrium approaches an interior pooling equilibrium that approximates the full investment and suffers from an overinvestment, then it is socially beneficial to switch from the separating signaling to a pooling signaling and moderate the investment level. Specifically, if $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ becomes sufficiently close to $\overline{c}$ and $g(\overline{c})$ is sufficiently small, then an interior separating equilibrium approaches an interior pooling equilibrium that approximates the full investment and suffers from an overinvestment.

For a concrete example, we consider the separating equilibrium with $\underline{e}_H$ under the circumstances where y_H is sufficiently large given y_L. The consequent enlargement of B is socially beneficial within the class of the interior separating equilibrium with $\underline{e}_H$.[18] As we show above, however, if $B > \overline{B}$, or if $B = \overline{B}$ with $g(\overline{c})$ sufficiently small, then a pooling signaling is strictly better than any separating signaling. Further, even before B reaches the critical point $\overline{B}$, if B becomes closer to $\overline{B}$ and the density at the margin

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[17]Recall that this comparative static analysis focuses on the case in which y_H increases given y_L; $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ is then a strictly increasing function of B. This focus is not necessary when u_q is a linear function of w.

[18]An increase in the wage gap B raises the signaling $\underline{e}_H$, which promotes the human capital investment $(\overline{k}_s)$ and the social welfare.

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$g(u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H))$ becomes sufficiently small, then it is socially beneficial to switch from the separating signaling $\underline{e}_H$ to a pooling signaling in moderation.

5.3 Separating signaling and promotion of investment

In this subsection, we show that a separating signaling may still be a socially preferred option when a pooling signaling causes an underinvestment.

We first report an immediate result that follows from Proposition 4 (iii): if $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \overline{c})$ is sufficiently close to $\underline{c}$ and $g(\underline{c})$ is sufficiently small, then there exists no interior pooling equilibrium, but there exists the interior separating equilibrium with $\underline{e}_H$.[19] A pooling signaling then suffers from the welfare loss associated with undersignaling by failing to induce any human capital investment: the unique pooling equilibrium is the boundary equilibrium, and this boundary equilibrium is inferior to any interior equilibrium in terms of social welfare.

To summarize our findings, for $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \overline{c})$, if $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ is sufficiently close to $\underline{c}$ and $g(\underline{c})$ is sufficiently small, then the unique pooling equilibrium is the inferior boundary equilibrium, and if $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ is sufficiently close to $\overline{c}$ and $g(\overline{c})$ is sufficiently small, then there exists an interior pooling equilibrium that is strictly better than the interior separating equilibrium with $\underline{e}_H$. We find it elusive to compare two types of equilibria, other than in the parameter ranges presented above. We conclude this subsection by showing that the separating signaling $\underline{e}_H$ continues to be socially preferred even when an interior pooling equilibrium exists, if this interior pooling equilibrium generates a sufficiently small human capital investment. To obtain this result, we make the following assumption for the rest of this subsection: $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \overline{c})$, and $g(\underline{c})$ and $g(\overline{c})$ are sufficiently small. Under the assumption, the function $D(\overline{e}(\lambda))$ is strictly increasing in λ for all $\lambda \in [\underline{\lambda}, 1]$ with the intercept $D(\overline{e}(1)) = D(\underline{e}_H) \in (0, 1)$, while the slopes of $D(\overline{e}(\lambda))$ when $\lambda \to \underline{\lambda}$ and $\lambda \to 1$ remain sufficiently flat.

We next consider an interior pooling equilibrium that exists when the function $D(\overline{e}(\lambda))$ has a tangent to the 45 degree line and thus $D(\overline{e}(\lambda')) = \lambda'$ for some $\lambda' \in (0, 1)$. For the existence of this interior pooling equilibrium, suppose that y_H increases given y_L. From

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[19] The conditions imply that $D(\overline{e}(\lambda)) < \lambda$ for all $\lambda \in (0, 1)$ and $D(\overline{e}(1)) \in (0, 1)$ hold.

the definitions of $\underline{e}_H$ and $\overline{e}(\lambda)$, it follows that $\underline{e}_H$ increases and $\overline{e}(\lambda)$ increases for all $\lambda > 0$. Since the function $D(\overline{e}(\lambda))$ monotonically shifts up all $\lambda \geq \underline{\lambda}$ together with the intercept $D(\overline{e}(1)) = D(\underline{e}_H)$, we can select the function $D(\overline{e}(\lambda))$ that satisfies $D(\overline{e}(\lambda')) = \lambda'$ for some λ'. This interior pooling equilibrium has the social welfare:

$$U'_p = u_L(y_L + \lambda'B, \overline{e}(\lambda')) + \int_{\underline{c}}^{k'_p} G(c)dc, \tag{40}$$

where $\lambda' = G(k'_p)$. Note that this equilibrium, given $D(\overline{e}(\lambda')) = \lambda'$, uses the signaling $\overline{e}(\lambda')$ since any $e < \overline{e}(\lambda')$ implies $D(e) < D(\overline{e}(\widehat{\lambda}))$. Further, from the definition of $\overline{e}(\lambda')$, we find

$$u_L(y_L + \lambda'B, \overline{e}(\lambda')) = u_L(y_L, 0).$$

We now compare U'_p to U_s.[20] Since there exists the interior separating equilibrium with $\underline{e}_H$ under the assumption and since $\overline{e}(\lambda') < \overline{e}(1) = \underline{e}_H$, we have $k'_p < \overline{k}_s$ and

$$U'_p < U_s = u_L(y_L, 0) + \int_{\underline{c}}^{\overline{k}_s} G(c)dc.$$

Thus, this pooling equilibrium, while generating some human capital investment, is inferior to the separating equilibrium with $\underline{e}_H$. We emphasize that an interior pooling equilibrium remains inferior, unless the function $D(\overline{e}(\lambda))$ is sufficiently high above the 45 degree line.[21]

In summary, under the assumption, if $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ is sufficiently close to $\underline{c}$ and $g(\underline{c})$ is sufficiently small, then we have two possibilities: (i) the unique pooling equilibrium induces no investment and has the welfare U_0 or (ii) an interior pooling equilibrium exists, but it is inferior to the interior separating equilibrium with $\underline{e}_H$. For an interior pooling equilibrium to remain inferior, $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ must be sufficiently below $\overline{c}$ given $g(\overline{c})$ sufficiently small; if $u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$ is sufficiently close to $\overline{c}$ for $g(\overline{c})$ sufficiently small, then we have the result in Proposition 10.

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[20]If λ' is not unique, we select the largest λ'; the first term in U'_p is the same, $u_L(y_L + \lambda'B, \overline{e}(\lambda')) = u_L(y_L, 0)$ for all λ', but the second term is larger when $\lambda' = G(k'_p)$ is larger.

[21]An interior pooling equilibrium becomes superior to any separating equilibrium if the function $D(\overline{e}(\lambda))$ is sufficiently high above the 45 degree line for sufficiently large λ. If $D(\overline{e}(\lambda)) > \lambda$ only for λ sufficiently small, then $\overline{e}(\lambda)$ and k_p are close to zero; an interior pooling equilibrum has the welfare close to U_0.

We now use Proposition 5 and associate the socially preferred interior separating equilibrium with the equilibrium that exists under the government's market design.

Proposition 11 *Assume that $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \overline{c})$, and $g(\underline{c})$ and $g(\overline{c})$ are sufficiently small. There exists $\widehat{c} \in (\underline{c}, \overline{c})$ such that, if $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \widehat{c})$, then the market design (S) maximizes the social welfare.*

This result shows that, if the separating signaling $\underline{e}_H$ still generates a sufficiently small incentive to make the investment, then it is socially beneficial to stay with the separating signaling and focus on promoting the human capital investment: a pooling signaling may then suffer from an underinvestment by failing to provide any incentive to make the human capital investment or by inducing too little incentive.

We illustrate our finding based on the productivity gap B under the same assumption as above. Suppose that y_H increases given y_L. An increase in B then raises $\overline{e}(\lambda)$ for all $\lambda > 0$, and the function $D(\overline{e}(\lambda))$ monotonically shifts up all $\lambda \geq \underline{\lambda}$ together with the intercept $D(\overline{e}(1)) = D(\underline{e}_H)$.[22] There is a critical value $\widehat{B}$ such that $D(\overline{e}(\widehat{\lambda})) = \widehat{\lambda}$ for some $\widehat{\lambda}$: if $B > \widehat{B}$, then $D(\overline{e}(\lambda)) > \lambda$ for some $\lambda \in (0, 1)$, and if $B < \widehat{B}$, then $D(\overline{e}(\lambda)) < \lambda$ for all $\lambda \in (0, 1)$.

Corollary 4 *Assume that $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \overline{c})$, and $g(\underline{c})$ and $g(\overline{c})$ are sufficiently small. If y_H increases given y_L, then there exists a unique $\widehat{B} \in (\underline{B}, \overline{B})$ such that, for $B \in (\underline{B}, \widehat{B})$, there exists no interior pooling equilibrium, and for $B \in [\widehat{B}, \overline{B})$, there exists an interior pooling equilibrium.*

We build on this lemma and compare two interior equilibria in terms of welfare under the same assumption. For all $B \in (\underline{B}, \overline{B})$, the interior separating equilibrium with $\underline{e}_H$ exists and has the social welfare U_s in (30) with $\overline{k}_s = u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)$. The welfare

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[22]The value $\overline{e}(\lambda)$ is sensitive to the way B increases. For example, (i) y_H increases given y_L, and (ii) y_L decreases given y_H. For (i), we know from $u_L(y_L + \lambda B, \overline{e}(\lambda)) = u_L(y_L, 0)$ that $\overline{e}(\lambda)$ for $\lambda \in (0, 1)$ and $\overline{e}(1) = \underline{e}_H$ increases. For (ii), we find that $\overline{e}(\lambda)$ decreases for $\lambda \in (0, 1)$ and $\overline{e}(1) = \underline{e}_H$ remains constant. Again, we can greatly simplify our comparative static analysis by focusing on (i), not mixing (i) and (ii), since when B increases, the function $D(\overline{e}(\lambda))$ monotonically shifts up for all $\lambda \geq \underline{\lambda}$. Notice also that this monotonic property directly holds, regardless of the way B changes, if u_q is a linear function with w.

is strictly increasing in B,

$$\frac{dU_s}{dB} = G(\bar{k}_s)\frac{d\bar{k}_s}{dB} > 0,$$

with the boundary condition: if $B \to \underline{B}$, then $\bar{k}_s \to \underline{c}$ and $U_s \to U_0$. For $B \in (\underline{B}, \widehat{B})$, the unique pooling equilibrium is the boundary equilibrium with the welfare U_0. For $B \in [\widehat{B}, \overline{B})$, an interior pooling equilibrium exists with the welfare U_p in (31). We focus on the optimal interior pooling equilibrium with e^o and k_p^o which generates the welfare U_p^o. We use the envelope theorem and find that U_p^o is strictly increasing in B,

$$\frac{dU_p^o}{dB} = \frac{\partial u_L}{\partial w}G(k_p^o) > 0,$$

with the boundary condition: if $B = \widehat{B}$, then $U_p^o = \widehat{U}_p < U_s$. Hence, if $B \in (\underline{B}, \widehat{B})$, then $U_p^o = U_0 < U_s$, and if $B = \widehat{B}$, then $U_p^o = \widehat{U}_p < U_s$. Since U_p^o is strictly increasing in B for $B \in [\widehat{B}, \overline{B})$ and $U_p^o > U_s$ for B sufficiently close to $\overline{B}$, there exists $\widehat{B}' \in (\widehat{B}, \overline{B})$ such that, for $B \in (\underline{B}, \widehat{B}')$, the separating signaling with $\underline{e}_H$ is strictly better than any pooling signaling in terms of social welfare.

Corollary 5 *Assume that $[u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H)] \in (\underline{c}, \bar{c})$, and $g(\underline{c})$ and $g(\bar{c})$ are sufficiently small. If y_H increases given y_L, then there exists $\widehat{B}' \in (\widehat{B}, \overline{B})$ such that, for all $B \in (\underline{B}, \widehat{B}')$, then the market design (S) maximizes the social welfare.*

Together with Corollary 4, this result demonstrates the circumstance under which the separating signaling $\underline{e}_H$ continues to be socially preferred when a pooling signaling generates an insufficient incentive to make the human capital investment.

5.4 Numerical examples

For a numerical work, we use the previous utility function, $u_q(w, e) = w - c_q(e)$, where $c_L(e) = e^2$ and $c_H(e) = ae^2$. In the previous example where G is a uniform distribution function on $[0, 1]$, we showed that the separating equilibrium with $\underline{e}_H$ and an interior pooling equilibrium exists only in entirely separate parameter ranges. To present the existence of both types of interior equilibria, we here consider a truncated normal distribution function on an interval $[0, 1]$,

$$G(x) = \frac{\int_0^x f(t)dt}{\int_0^1 f(x)dx}, \tag{41}$$

where $f(x)$ is the density function of a normal random variable with mean and variance, μ and σ^2: $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2)$, where $-\infty < x < \infty$. The separating signaling remains the same as above and satisfies

$$\underline{e}_H = \sqrt{B} \text{ and } \overline{k}_s = (1-a)B.$$

To ensure the existence of this separating equilibrium with $\underline{e}_H$, we restrict attention to $(1-a)B < 1$. A pooling signaling e has the threshold $k_p = (1-a)e^2$. For an interior equilibrium, e must be curtailed to satisfy $(1-a)e^2 < 1$. Since $\overline{e}(\lambda) = \sqrt{B\lambda}$, we have

$$\overline{k}_p(\lambda) = (1-a)(\overline{e}(\lambda))^2 = (1-a)B\lambda.$$

Using the distribution function G in (41), we can now derive the function:

$$D(\overline{e}(\lambda)) = G\left(\overline{k}_p(\lambda)\right) = G((1-a)B\lambda).$$

We have numerically confirmed that an interior pooling equilibrium exists in a wide range of (μ, σ). In particular, an increase in $(1-a)B = \overline{k}_s$ expands the parameter range in which an interior pooling equilibrium exists: if the fraction of type H increases in the separating equilibrium with $\underline{e}_H$, then an interior pooling equilibrium becomes more likely to exist. If $(1-a)B$ increases, then $\overline{k}_p(\lambda)$ increases for all $\lambda > 0$, which means that the function $G((1-a)B\lambda)$ shifts up for all $\lambda > 0$. In extreme, if $(1-a)B$ approaches 1, then the condition $G((1-a)B\lambda) \geq \lambda$ holds for λ sufficiently large in a wide range of (μ, σ). For this case of $(1-a)B \to 1$, if $\mu = 0.5$, then $G((1-a)B\lambda) \geq \lambda$ continues to hold approximately for $\lambda \in (0.5, 1)$ even when σ changes in a relevant range: given $\mu = 0.5$, if σ decreases, then $G((1-a)B\lambda)$ shifts up (down) for higher (lower) λ with the same values of $G(0)$ and $G(1)$.[23] On the other hand, if $(1-a)B$ decreases, then $\overline{k}_p(\lambda)$ decreases for all $\lambda > 0$, which means that $D(\overline{e}(\lambda))$ shifts down for all $\lambda > 0$; thus, a decrease in $(1-a)B$ diminishes the parameter range in which an interior pooling equilibrium exists. For example, if $(1-a)B$ decreases to 0.5 given $\mu = 0.5$ as above, then $G((1-a)B\lambda) \geq \lambda$ fails for all $\lambda \in (0, 1)$. When $(1-a)B$ decreases, however, if μ becomes small, then the condition $G((1-a)B\lambda) \geq \lambda$ may still hold for some $\lambda \in (0, 1)$. This decrease in μ represents that there is an environmental improvement that reduces the expected cost of

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[23]If σ becomes sufficiently large, then $G(x)$ approximates the uniform distribution.

the human capital investment. For this case of $(1-a)B = 0.5$, an increase in σ reduces the potential parameter range in which an interior pooling equilibrium exists: given $\mu = 0.25$, if $\sigma = 0.1$, then $G((1-a)B\lambda) \geq \lambda$ holds approximately for $\lambda \in (0.5, 1)$, if $\sigma = 0.15$, then the condition holds approximately for $\lambda \in (0.57, 0.91)$ and if $\sigma = 0.25$, then the condition fails for all $\lambda \in (0,1)$. In summary, if $(1-a)B$ increases, or if μ becomes smaller, then the distribution function $G((1-a)B\lambda)$ tends to shift in favor of expanding the parameter range in which an interior pooling equilibrium exists.

In the Appendix, Table 1 selects a few results from extensive numerical examples. The comparison of two types of interior equilibria is based on the restriction, $(1-a)B < 1$, which ensures the existence of the separating equilibrium with $\underline{e}_H$. Recall that this restriction is not necessary for the existence of an interior pooling equilibrium; if $(1-a)B > 1$, or if $(1-a)B = 1$ and $g(\bar{c})$ is sufficiently small, then an interior pooling equilibrium is sure to exist and is strictly better than any separating signaling. The table reports 6 different sets of parameters in which two types of equilibrium exist at the same time. The pooling signaling e, reported in the table, is optimal: it is selected from the set of interior pooling equilibria and it maximizes U_p in (31). The first 5 columns report the parameter sets in which the pooling signaling is strictly better than the separating signaling with $\underline{e}_H$, whereas the last column reports the parameter set in which the separating is strictly better.[24]

We can summarize the features found in the numerical work. First, if $(1-a)B = \bar{k}_s$ is small, then the separating signaling tends to dominate a pooling signaling in terms of social welfare: if $(1-a)B = \bar{k}_s$ is small, then a pooling signaling often leads to undersignaling by failing to induce any human capital investment, or by inducing only an insufficient level of investment. Second, if $(1-a)B = \bar{k}_s$ becomes larger and an interior pooling equilibrium begins to exist, then the dominance of one type of signaling over the other becomes unclear, and if $(1-a)B = \bar{k}_s$ gets sufficiently large, then oversignaling of a separating signaling becomes evident and a pooling signaling tends to dominate the separating signaling in terms of social welfare.

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[24]The table shows that the proportions of type H, $G(\bar{k}_s)$ and $G(k_p)$, are quite high. This outcome, however, depends on the model specification: if the cost of education becomes greater in the utility function, the proportions may be reduced.

6 Conclusions

The relationship between signaling and human capital investment has rarely received attention from the analytical literature. In this paper, we modify the Spence's (1973) classic signaling model with mild assumptions and develop a signaling model that has a causal relationship between the two. We then ask some essential questions. First, we ask whether signaling is socially beneficial or wasteful, and find that the social value of signaling may be significantly underrated if its inducement of the human capital investment is not taken into account: this seemingly wasteful action becomes socially beneficial whenever it induces the human capital investment by delivering the benefits of the investment. Second, we ask whether it is socially beneficial or harmful to promote the human capital investment, and find that the human capital investment may be overly emphasized if the cost of signaling that induces the investment is not taken into account: it may be socially harmful to maximize the human capital investment by increasing the costly signaling. Third, we ask whether it is socially beneficial or harmful to convey private information about individual types in the form of a separating signaling, as opposed to a pooling signaling, we find that the choice of a proper form of signaling may be dependent on how each signaling promotes the human capital investment.

7 Appendix I

Proof of Proposition 5. (i) Since $\underline{E}$ satisfies (13), $\underline{E} \in [\underline{e}_H, e_H]$. We show that any $e > \underline{E}$ does not satisfy the intuitive criterion. Suppose a separating equilibrium in which type H chooses $e_H > \underline{E}$. It is sufficient to demonstrate that type H can attain a higher payoff by deviating from the separating equilibrium, and type L cannot imitate the action of type H. In other words, we show that given any $e' \in (\underline{E}, e_H)$,

$$u_H\left(y_H, e'\right) > u_H\left(y_H, e_H\right) \text{ and } u_L\left(y_L, 0\right) > u_L\left(y_H, e'\right).$$

The former inequality follows from $e' < e_H$, and the latter is from $u_L\left(y_L, 0\right) = u_H\left(y_H, \underline{e}_H\right) > u_L\left(y_H, e'\right)$ for $e' > \underline{E} \geq \underline{e}_H$.

    (ii) Suppose that there is an interior pooling equilibrium satisfying the intuitive criterion. Then, the wage that each type can obtain is $w \equiv \theta y_H + (1 - \theta) y_L$. It is sufficient

to demonstrate that a worker with type H can attain a higher payoff by deviating from a pooling equilibrium $e < \overline{E}$, and a worker with type L cannot imitate the action of the worker with type H. In other words, we show that given any $e < \overline{E}$, there exists $e' \leq \overline{E}$ such that

$$u_H(y_H, e') > u_H(w, e) \text{ and } u_L(w, e) > u_L(y_H, e') \tag{42}$$

Since $y_H > w$, there exists $e'' > e$ such that $u_L(w, e) = u_L(y_H, e'')$. It follows from SMP that

$$u_H(y_H, e'') - u_H(y_H, e) > u_L(y_H, e'') - u_L(y_H, e),$$

which can be rewritten as

$$u_H(y_H, e'') - u_L(y_H, e'') > u_H(y_H, e) - u_L(y_H, e) = u_H(w, e) - u_L(w, e),$$

where the last equality follows from (2). By $u_L(w, e) = u_L(y_H, e'')$,

$$u_H(y_H, e'') - u_H(w, e) > u_L(y_H, e'') - u_L(w, e) = 0.$$

Since e is continuous, there exists $e' > e''$ such that e' satisfies (42). Furthermore, if u_L's marginal cost is sufficiently high, e' is close to e such that $e' \leq \overline{E}$. ∎

Proof of Proposition 6. For a separating equilibrium, since $u_H(y_H, 0) > u_H(y_H, \underline{e}_H)$, we have

$$\lambda u_H(y_H, 0) + (1 - \lambda)u_L(y_L, 0) > U_s^B.$$

Thus, to verify the result $U_0^B > U_s^B$, it suffices to show that

$$\lambda u_H\left(\mathbb{E}^\lambda[y], 0\right) + (1 - \lambda)u_L\left(\mathbb{E}^\lambda[y], 0\right) - [\lambda u_H(y_H, 0) + (1 - \lambda)u_L(y_L, 0)] \geq 0.$$

The LHS of this inequality becomes

$$\begin{aligned}
&\lambda[u_H\left(\mathbb{E}^\lambda[y], 0\right) - u_H(y_H, 0)] + (1 - \lambda)[u_L\left(\mathbb{E}^\lambda[y], 0\right) - u_L(y_L, 0)] \\
&= \lambda[u_L\left(\mathbb{E}^\lambda[y], 0\right) - u_L(y_H, 0)] + (1 - \lambda)[u_L\left(\mathbb{E}^\lambda[y], 0\right) - u_L(y_L, 0)] \\
&= u_L\left(\mathbb{E}^\lambda[y], 0\right) - [\lambda u_L(y_H, 0) + (1 - \lambda)u_L(y_L, 0)] \geq 0.
\end{aligned}$$

The first equality follows from the assumption that the utility gain from any wage increase is type-irrelevant, and the last inequality is given by concavity of u_L in w. For a pooling equilibrium, for any $e > 0$, it is immediate from (28) that $U_0^B > U_p^B$. ∎

Proof of Proposition 8. Since $u_L(y_H, \underline{e}_H) = u_L(\mathbb{E}^\lambda[y], \bar{e}(\lambda))$, so we have $\bar{e}(\lambda) < \underline{e}_H$ such as in the standard case. Define

$$\Phi(e_H, e) \equiv k_s - k_p = u_H(y_H, e_H) - u_L(y_L, 0) - (u_H(0, e) - u_L(0, e)).$$

Given any $e \in (0, \bar{e}(\lambda)]$, by the assumption (2) and the definitions of $\underline{e}_H$ and $\bar{e}_H$ in (5),

$$\begin{aligned}
\Phi(\underline{e}_H, e) &= u_H(y_H, \underline{e}_H) - u_L(y_L, 0) - (u_H(0, e) - u_L(0, e)) \\
&= u_H(y_H, \underline{e}_H) - u_L(y_H, \underline{e}_H) - (u_H(0, e) - u_L(0, e)) \\
&= u_H(0, \underline{e}_H) - u_L(0, \underline{e}_H) - (u_H(0, e) - u_L(0, e)) \\
&= u_H(0, \underline{e}_H) - u_H(0, e) - (u_L(0, \underline{e}_H) - u_L(0, e)) > 0 \text{ from SMP,}
\end{aligned}$$

and

$$\begin{aligned}
\Phi(\bar{e}_H, e) &= u_H(y_H, \bar{e}_H) - u_L(y_L, 0) - (u_H(0, e) - u_L(0, e)) \\
&= u_H(y_L, 0) - u_L(y_L, 0) - (u_H(0, e) - u_L(0, e)) \\
&= -(u_H(0, e) - u_L(0, e)) < 0.
\end{aligned}$$

In addition, $\Phi(e_H, e)$ is a strictly decreasing function of e_H. Hence, given each e, there exists a unique implicit function $\phi(e) \in (\underline{e}_H, \bar{e}_H)$ such that $\Phi(\phi(e), e) = 0$. Hence, for $e_H \in [\underline{e}_H, \phi(e))]$, $\Phi(e_H, e) > 0$, so $k_s > k_p$. For $e_H \in [\phi(e), \bar{e}_H]$, $\Phi(e_H, e) < 0$, so $k_s < k_p$. ∎

Environmental Improvement in Investment. We examine the effect of an environmental improvement that reduces the expected cost of the human capital investment: the distribution function G shifts up to F in terms of first-order stochastic dominance (FOSD) such that F has lower expected value, $\int_{\underline{c}}^{\bar{c}} c \, dF(c) < \int_{\underline{c}}^{\bar{c}} c \, dG(c)$. In an interior separating equilibrium with $e_H \in [\underline{e}_H, \bar{e}_H]$, this shift from G to F has no effect on the least costly signaling e_H and $k_s = u_H(y_H, e_H) - u_L(y_L, 0)$; thus, the social welfare U_s increases directly from the direct cost reduction.

In an interior pooling equilibrium, since $\mathbb{E}^\lambda[y] = y_L + G(k_p)B$ increases for each e and the choice set of $e \in [0, \bar{e}(\lambda)]$ expands, the social welfare U_p unambiguously increases. The effect on the signaling e is less clear. We consider the derivative $\frac{dU_p}{de}$ in (32) and make the following assumptions: (i) $u_L(w, e)$ takes a separable form,

$$u_L(w, e) = v(w) - c_L(e) \text{ for any linear function } v(\cdot), \tag{43}$$

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and U_p in (31) is strictly concave (at least locally) at the welfare-maximizing signaling e, and (ii) G shifts up to F in terms of the likelihood ratio (LR): density functions, $g(c) \equiv G'(c)$ and $f(c) \equiv F'(c)$, satisfy

$$\frac{g(c)}{f(c)} \leq \frac{g(\widehat{c})}{f(\widehat{c})} \text{ for all } c < \widehat{c}, \tag{44}$$

such that f crosses g only once from above at the point k^* where $\frac{g(k^*)}{f(k^*)} = 1$.[25] Under the assumptions, by selecting e^* and $k^* = u_H(0, e^*) - u_L(0, e^*)$, we can find that, if the existing signaling is sufficiently small such that $e \leq e^*$ and $k_p \leq k^*$, then the marginal welfare benefit $\frac{dU_p}{de}$ clearly increases for each e since both $g(k_p)$ and $G(k_p)$ shift up in (32). The shift from G to F then generates robust force in favor of increasing the signaling and the fraction of type H. On the other hand, if the existing signaling is sufficiently large such that $e > e^*$ and $k_p > k^*$, then it becomes difficult to determine whether the marginal welfare benefit $\frac{dU_p}{de}$ increases or decreases, since $g(k_p)$ falls while $G(k_p)$ rises. For example, under G and F, if e becomes sufficiently large such that $k_p \to \overline{c}$, then $\frac{dU_p}{de}$ respectively approaches

$$\frac{\partial u_L}{\partial w} \cdot B \cdot g(\overline{c}) + \frac{\partial u_H}{\partial e} \text{ and } \frac{\partial u_L}{\partial w} \cdot B \cdot f(\overline{c}) + \frac{\partial u_H}{\partial e}.$$

If $g(\overline{c})$ and $f(\overline{c})$ are sufficiently close to zero, or if $g(\overline{c}) \geq f(\overline{c})$, then the signaling e cannot increase. In summary, if G shifts up to F in terms of LR, and u_L satisfies (43), then the pooling signaling e increases for sufficiently small e. ∎

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[25] Note that the LR dominance of G over F is more restrictive than the FOSD of G over F: $\frac{g(c)}{f(c)} \leq \frac{g(\widehat{c})}{f(\widehat{c})}$ for all $c < \widehat{c}$ implies $G(c) \leq F(c)$ for all c.

8 Appendix II: Table 1

(μ, σ)	$(0.5, 0.25)$	$(0.5, 0.15)$	$(0.5, 0.1)$	$(0.5, 0.1)$	$(0.25, 0.1)$	$(0.25, 0.15)$
(a, B)	$(0.6, 2.4)$	$(0.7, 3)$	$(0.5, 1.5)$	$(0.6, 2)$	$(0.5, 1.2)$	$(0.5, 1)$
$\underline{e}_H = \sqrt{B}$	1.54919	1.73205	1.22474	1.41421	1.09545	1.00000
$\overline{k}_s = (1-a)B$	0.96000	0.90000	0.75000	0.80000	0.60000	0.50000
$G(\overline{k}_s)$	0.98938	0.99660	0.99379	0.99865	0.99977	0.94981
e	1.45175	1.56297	1.17267	1.30534	0.92205	0.95374
k_p	0.84303	0.73286	0.68757	0.68157	0.42509	0.45481
$G(k_p)$	0.93477	0.94010	0.96966	0.96530	0.95977	0.90961

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