ISSN 1063-7761, Journal of Experimental and Theoretical Physics, 2018, Vol. 127, No. 3, pp. 398–421. © Pleiades Publishing, Inc., 2018.
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ATOMS, MOLECULES, OPTICS

Unitary Quantization and Para-Fermi Statistics of Order 2

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Received March 29, 2018

Abstract—We consider the relationship between the unitary quantization scheme and the para-Fermi statistics of order 2. We propose an appropriate generalization of Green's ansatz, which has made it possible to transform bilinear and trilinear commutation relations for the creation and annihilation operators for two different para-Fermi fields ϕ_a and ϕ_b into identities. We also propose a method for incorporating para-Grassmann numbers ξ_k into the general unitary quantization scheme. For the parastatistics of order 2, a new fact has been revealed: the trilinear relations containing both para-Grassmann variables ξ_k and field operators a_k and b_m are transformed under a certain reversible mapping into unitary equivalent relations in which commutators are replaced by anticommutators, and vice versa. It is shown that this leads to the existence of two alternative definitions of the coherent state for para-Fermi oscillators. The Klein transformation for Green's components of operators a_k and b_m is constructed in explicit form, which enabled us to reduce the initial commutation rules for the components to the normal commutation relations for ordinary Fermi fields. We have analyzed a nontrivial relationship between the trilinear commutation relations of the unitary quantization scheme and the so-called Lie supertriple system. The possibility of incorporating the Duffin–Kemmer–Petiau theory into the unitary quantization scheme is discussed briefly.

DOI: 10.1134/S1063776118090054

1. INTRODUCTION

The problem of quantization of finite-dimensional physical systems has permanently attracted the attention of theoretical physicists. From the recent publications in this field, the work by Assirati and Gitman [1] is worth mentioning. In this work, we consider another approach to the investigation of quantization in finite-dimensional classical theories, in which major attention is paid to the Lie-algebra aspects of physical systems in question. An approach to the quantization of fields based on the Lie algebra relations for unitary group $SU(2M+1)$ was proposed in [2–4] and independently in [5–8].[1] The quantization scheme proposed in [2] was called uniquantization, while in [5] it was referred to as "*A*-quantization." In this work, we investigate in greater detail some properties of the relations obtained in [2] and, in particular, establish the relationship between unitary quantization and the para-Fermi statistics of order 2. Several basic formulas from [2], which will be repeatedly referred to in further analysis, will be given below. A brief scheme of the derivation of these expressions is given in Appendix A.

Let $a_k^\dagger$ and a_k be the creation and annihilation operators that obey the Green trilinear commutation relations [12] (we confine our analysis to only para-Fermi statistics)

$$[[\hat{a}_k, \hat{a}_l], \hat{a}_m] = 2\hat{\delta}_{lm}\hat{a}_k - 2\hat{\delta}_{km}\hat{a}_l, \tag{1.1}$$

where $k, l, m = 1, 2, \ldots, M$ and [,] denotes a commutator. Operator $\hat{a}_k$ denotes a_k or $a_k^\dagger$ and $\hat{\delta}_{kl} = \delta_{kl}$ when $\hat{a}_k = a_k(a_k^\dagger)$ and $\hat{a}_l = a_l^\dagger(a_l)$; otherwise, $\hat{\delta}_{kl} = 0$. For unitary quantization, operators $\hat{a}_k$ are supplemented with another set of creation and annihilation operators $\hat{b}_k$ obeying the same commutation relations

$$[[\hat{b}_k, \hat{b}_l], \hat{b}_m] = 2\hat{\delta}_{lm}\hat{b}_k - 2\hat{\delta}_{km}\hat{b}_l. \tag{1.2}$$

In addition to trilinear Green's relation (1.1) and (1.2), the given quantization scheme unambiguously leads to

[1] It should be noted that some aspects of special cases of quantization based on $SU(2)$ and $SU(3)$ algebras were considered earlier in [9–11].

the two types of mutual commutation relations between operators $\hat{a}_k$ and $\hat{b}_k$:

(i) trilinear relations

$$[[\hat{b}_m, \hat{a}_k], \hat{a}_l] = 4\hat{\delta}_{km}\hat{b}_l + 2\hat{\delta}_{lk}\hat{b}_m + 2\hat{\delta}_{lm}\hat{b}_k, \quad (1.3)$$

$$[[\hat{a}_m, \hat{b}_k], \hat{b}_l] = 4\hat{\delta}_{km}\hat{a}_l + 2\hat{\delta}_{lk}\hat{a}_m + 2\hat{\delta}_{lm}\hat{a}_k, \quad (1.4)$$

(ii) bilinear relations

$$[\hat{a}_k, \hat{b}_m] = [\hat{a}_m, \hat{b}_k], \quad (1.5)$$

$$[\hat{a}_k, \hat{a}_m] = [\hat{b}_k, \hat{b}_m]. \quad (1.6)$$

Thus, we have two para-Fermi fields of order p, which must satisfy not only trilinear, but also bilinear relations. It is well known [13, 14] that commutation relations (1.1) and (1.2) generate an algebra that is isomorphic to the algebra of orthogonal group $SO(2M+1)$. Remaining relations (1.3)–(1.6) supplement this algebra to the algebra of unitary group $SU(2M + 1)$. The particle number operator

$$N = \frac{1}{2}\sum_{k=1}^{M}([a_k^{\dagger}, a_k] + p)\left(\equiv \frac{1}{2}\sum_{k=1}^{M}([b_k^{\dagger}, b_k] + p)\right) \quad (1.7)$$

together with algebra (1.1)–(1.6) unambiguously define the unitary quantization scheme.

In addition to the above arguments, it should be noted that in Govorkov's construction [2] for the $SU(2M + 1)$ group, there exists another important operator denoted by ζ_0. In view of relations (A.10) and (A.18), this operator can be expressed in terms of operators $\hat{a}_k$ and $\hat{b}_k$ as follows:

$$\zeta_0 = \frac{i}{2(2M+1)}\sum_{k=1}^{M}([a_k^{\dagger}, b_k] + [b_k^{\dagger}, a_k]). \quad (1.8)$$

Operator ζ_0 possesses the commutation properties:

$$[\hat{a}_k, \zeta_0] = 2i\hat{b}_k, \quad [\hat{b}_k, \zeta_0] = -2i\hat{a}_k. \quad (1.9)$$

This article is organized as follows. In Section 2, a brief review of Greenberg and Messiah's article [15] is given for the case of different para-Fermi fields. In Sections 3 and 4, the set of the commutation rules is generalized for the Green components of the creation and annihilation operators of two parafields ϕ_a and ϕ_b. A detailed proof of the fact that for parastatistics of order $p = 2$, this system converts the bilinear and trilinear Govorkov relations into identities is presented. Section 5 is devoted to the inclusion of para-Grassmann numbers ξ_k into the general scheme of unitary quantization. Section 6 deals with the construction of the commutation relations between operators a_k and b_m, para-Grassmann numbers ξ_k, and operator $e^{\alpha i\tilde{N}}$, where $\tilde{N}$ is defined by formula (5.9) and α is an arbitrary real number. Two important particular cases of the general relations, in which $\alpha = \pm\pi$ and $\alpha = \pm\pi/2$ are considered. A certain invertible mapping of trilinear relations, which include both para-Grassmann numbers and field operators, is also considered. Nontrivial peculiarities of this mapping are revealed. In Section 7, the action of some operators, which emerge in the unitary quantization scheme, on the vacuum state is defined.

Section 8 is devoted to the discussion of the properties of coherent states. In particular, an interesting fact of the existence of another state for para-Fermi statistics of order 2 is discovered, which possesses the same properties as those for the commonly used coherent states. In Section 9, the possibility of deriving the trilinear Govorkov relations from the requirement of the invariance of the commutation relations between operators a_k, b_m, and $\tilde{N}$ under unitary transformation of operators a_k and b_m is analyzed. It is shown that in contrast to the case of a single parafield, this requirement of invariance alone is insufficient for reconstructing all trilinear Govorkov's relations. In Section 10, the so-called Klein transformation is constructed for the Green components of the creation and annihilation operators of parafields. In Section 11, the relation between trilinear Govorkov relations and the Lie supertriple system is considered. Section 12 deals with the inclusion of the Duffin–Kemmer–Petiau formalism into the general unitary quantization scheme. It is shown that these two approaches are incompatible in the long run. In concluding Section 13, the possible relation between the unitary quantization scheme based on the Lie algebra of unitary group $SU(2M)$ and the para-Bose statistics is discussed briefly. In the same section, some unusual properties inherent only in the parastatistics of order 2 are accentuated.

In Appendix A, all basic relations of the Lie algebra for unitary group $SU(2M + 1)$ are given. Some inaccuracies we noticed in Govorkov's publications [2, 4] are also indicated. In Appendix B, various operator identities which are used throughout the work are written. In Appendix C, all basic commutation relations involving operator $e^{\alpha i\tilde{N}}$ are collected.

2. REVIEW OF THE GREENBERG AND MESSIAH WORK

Let us write general relation (1.1) in a more detailed form:

$$[[a_k^{\dagger}, a_l], a_m] = -2\delta_{km}a_l, \quad (2.1)$$

$$[[a_k, a_l], a_m] = 0. \quad (2.2)$$

By virtue of Jacobi identity (B.1) this gives

$$[[a_k, a_l], a_m^{\dagger}] = 2\delta_{lm}a_k - 2\delta_{km}a_l. \quad (2.3)$$

Greenberg and Messiah [15] proposed a generalization of relations (2.1)–(2.3) to the case of several different parafields. For determining the corresponding commutation rules between different parafields, it was required by the authors that the desired relations satisfy the following three conditions:

(i) the left-hand side of these relations must have the trilinear form[2]

$$[[A, B], C],$$

while the right-hand sides must be linear;

(ii) when the internal pair $[A, B]$ is formed by the operators of the same field, it must commute with C if C corresponds to the other field;

(iii) ordinary Bose and Fermi fields must satisfy these relations.

In the case when these conditions are imposed to two para-Fermi fields ϕ_a and ϕ_b, Greenberg and Messiah obtained the following system of trilinear relations involving field ϕ_a twice and field ϕ_b once:

$$[[a_k^\dagger, a_l], b_m] = 0, \tag{2.4}$$

$$[[a_k, a_l], b_m] = 0, \tag{2.5}$$

$$[[a_k^\dagger, a_l^\dagger], b_m] = 0. \tag{2.6}$$

If Jacobi identity (B.1) together with conditions (i) and (iii) is employed, relation (2.4) leads to two more trilinear relations:

$$[[b_m, a_k^\dagger], a_l] = 2\delta_{kl} b_m, \tag{2.7}$$

$$[[a_l, b_m], a_k^\dagger] = -2\delta_{kl} b_m. \tag{2.8}$$

The derivation of these relations will be considered in Section 9 in greater detail. Relations (2.4)–(2.8) are supplemented with their Hermitian conjugate and 18 more trilinear relations involving field ϕ_b twice and field ϕ_a once.

The authors of [15] also proposed a direct generalization of the Green ansatz [12]. Each field operator is written in the form of an expansion in Green's components:

$$a_k = \sum_{\alpha=1}^{p} a_k^{(\alpha)}, \quad b_m = \sum_{\alpha=1}^{p} b_m^{(\alpha)}, \tag{2.9}$$

where p is the order of the parastatistics. Each pair of components corresponding to the same field satisfies the commutation relations

$$\begin{gathered} \{a_k^{(\alpha)}, a_l^{\dagger(\alpha)}\} = \delta_{kl}, \quad \{a_k^{(\alpha)}, a_l^{(\alpha)}\} = 0, \\ [a_k^{(\alpha)}, a_l^{(\beta)}] = [a_k^{(\alpha)}, a_l^{\dagger(\beta)}] = 0, \quad \alpha \neq \beta, \end{gathered} \tag{2.10}$$

analogous relations can be written for field ϕ_b. Here, braces { , } denote an anticommutator. For each pair of Green's components of different fields, Greenberg and Messiah postulated the following rules:

$$\begin{gathered} \{a_k^{\dagger(\alpha)}, b_m^{(\alpha)}\} = \{a_k^{(\alpha)}, b_m^{\dagger(\alpha)}\} = 0, \\ \{a_k^{(\alpha)}, b_m^{(\alpha)}\} = \{a_k^{\dagger(\alpha)}, b_m^{\dagger(\alpha)}\} = 0, \end{gathered} \tag{2.11}$$

and

$$\begin{gathered} [a_k^{(\alpha)}, b_m^{(\beta)}] = [a_k^{\dagger(\alpha)}, b_m^{\dagger(\beta)}] = 0, \\ [a_k^{\dagger(\alpha)}, b_m^{(\beta)}] = [a_k^{(\alpha)}, b_m^{\dagger(\beta)}] = 0, \quad \alpha \neq \beta. \end{gathered} \tag{2.12}$$

The fields obeying rules (2.11) and (2.12) turn trilinear relations (2.4)–(2.8) into identities.

Finally, the uniqueness conditions of vacuum state $|0\rangle$,

$$a_k|0\rangle = b_k|0\rangle = 0, \quad \text{for all } k \tag{2.13}$$

and

$$\begin{gathered} a_k a_l^\dagger|0\rangle = p\delta_{kl}|0\rangle, \quad \text{for all } k, l, \\ b_m b_n^\dagger|0\rangle = p\delta_{mn}|0\rangle, \quad \text{for all } m, n \end{gathered} \tag{2.14}$$

were supplemented by Greenberg and Messiah with two more conditions:

$$\begin{gathered} b_m a_k^\dagger|0\rangle = 0, \quad \text{for all } m, k, \\ a_k b_m^\dagger|0\rangle = 0. \end{gathered} \tag{2.15}$$

These additional conditions can be obtained from commutation relations (2.4)–(2.8) and from the uniqueness of vacuum state $|0\rangle$. This derivation will be considered in more detail in Section 7 in the context of our problem.

3. GREEN'S ANSATZ FOR GOVORKOV'S RELATIONS

In Introduction, the trilinear and bilinear commutation relations emerging in the Govorkov unitary quantization scheme were written out. As the first step, we consider trilinear relations for two different parafields. The following expressions are the special case of general formula (1.3):

$$[[b_m, a_k^\dagger], a_l] = 2\delta_{kl} b_m + 4\delta_{km} b_l, \tag{3.1}$$

$$[[a_l, b_m], a_k^\dagger] = -2\delta_{kl} b_m - 2\delta_{km} b_l. \tag{3.2}$$

These relations differ from analogous relations (2.7) and (2.8) of the Greenberg–Messiah quantization scheme in the presence of the last two terms on the right-hand sides. Summing expressions (3.1) and (3.2) and using the Jacobi identity, we obtain an analog of trilinear relation (2.4):

$$[[a_k^\dagger, a_l], b_m] = -2\delta_{km} b_l. \tag{3.3}$$

Here, we also observe the nonzero term emerging on the right-hand side.

Let us write the a and b operators in the form of Green expansion (2.9). The following question arises: what must be the form of commutation rules for Green's components $a_k^{(\alpha)}$ and $b_m^{(\beta)}$ for trilinear Govo-

[2] However, the authors themselves did not rule out the existence of bilinear commutation and anticommutation relations between different parafields. Nevertheless, they concentrated attention only on trilinear relations. In the unitary quantization scheme, bilinear relations (see Eqs. (1.5) and (1.6)) appear inevitably.

rkov relations (3.1) and (3.3) to be satisfied identically? Clearly, commutation rules (2.10) (and analogously for the ϕ_b field) must hold in this case also since the sets of the a and b operators separately satisfy the standard trilinear relations (1.1) and (1.2) for para-Fermi fields. Therefore, we need to generalize relations (2.11) and (2.12) for Green's components of different fields. It should be noted that these commutation rules are quite trivial in a certain sense.

Let us consider specifically relation (3.3). The left-hand side can be written in terms of Green's components. For commutator $[a_k^\dagger, a_l]$, we have

$$[a_k^\dagger, a_l] = \sum_{\alpha=1}^{p}[a_k^{\dagger(\alpha)}, a_l^{(\alpha)}] + \sum_{\alpha\neq\beta}[a_k^{\dagger(\alpha)}, a_l^{(\beta)}].$$

The last term on the right-hand side is equal to zero by virtue of relations (2.10). For the double commutator, we can write

$$[[a_k^\dagger, a_l], b_m] = \sum_{\alpha=1}^{p}[[a_k^{\dagger(\alpha)}, a_l^{(\alpha)}], b_m^{(\alpha)}] + \sum_{\alpha\neq\beta}[[a_k^{\dagger(\alpha)}, a_l^{(\alpha)}], b_m^{(\beta)}]. \tag{3.4}$$

Symbol $\sum_{\alpha\neq\beta}$ denotes summation over both α and β indices with a single constraint $\alpha \neq \beta$. For the first expression in the summand on the right-hand side of relation (3.4), we use operator identity (B.2), while for the second expression, usual Jacobi identity (B.1) must be used:

$$[[a_k^{\dagger(\alpha)}, a_l^{(\alpha)}], b_m^{(\alpha)}] = \{\alpha_k^{\dagger(\alpha)}, \{b_m^{(\alpha)}, a_l^{(\alpha)}\}\} - \{a_l^{(\alpha)}, \{b_m^{(\alpha)}, a_k^{\dagger(\alpha)}\}\}, \tag{3.5}$$

$$[[a_k^{\dagger(\alpha)}, a_l^{(\alpha)}], b_m^{(\beta)}] = -[[a_l^{(\alpha)}, b_m^{(\beta)}], a_k^{\dagger(\alpha)}] - [[b_m^{(\beta)}, a_k^{\dagger(\alpha)}], a_l^{(\alpha)}], \quad \alpha \neq \beta. \tag{3.6}$$

In view of Greenberg–Messiah commutation rules (2.11) and (2.12), these expressions vanish, and we arrive at relation (2.4). Let us modify the first two relations in (2.11), leaving the remaining terms unchanged (in this case, the second double commutator (3.6) vanishes). For this purpose, we introduce a new operator Ω as a certain additional algebraic element that satisfies the relations

$$\begin{aligned}\{a_k^{\dagger(\alpha)}, b_m^{(\alpha)}\} &= 2\delta_{mk}\Omega, \quad \{a_k^{(\alpha)}, \Omega\} = b_k^{(\alpha)},\\ \{b_m^{\dagger(\alpha)}, a_k^{(\alpha)}\} &= 2\delta_{mk}\Omega^\dagger, \quad \{b_m^{(\alpha)}, \Omega\} = -a_m^{(\alpha)}.\end{aligned} \tag{3.7}$$

It can easily be seen that expression (3.5) in this case leads to

$$[[a_k^{\dagger(\alpha)}, a_l^{(\alpha)}], b_m^{(\alpha)}] = -2\delta_{mk}b_l^{(\alpha)}$$

and, hence, by virtue of relation (3.4), we reproduce (3.3). If, however, we try to apply commutation rules (3.7) to trilinear relation (3.1), it can be seen that the last term on the right-hand side of relation (3.1) is not reproduced. In this case, a more radical modification of expressions (2.11) and (2.12) is required. We will postulate below a new system of bilinear relations for Green's components $a_k^{(\alpha)}$ and $b_m^{(\beta)}$. Further, we verify that these commutation rules turn bilinear (1.5) and (1.6) and trilinear (3.1) and (3.3) Govorkov's relations into identities. This, however, will occur only for the special case of parastatistics of order 2.

Let us require that Green's components $a_k^{(\alpha)}$ and $b_m^{(\beta)}$ and additional operator Ω satisfy the following set of commutation rules:

$$[b_m^{(\alpha)}, a_k^{\dagger(\alpha)}] = 2\delta_{mk}\Omega, \quad [a_k^{(\alpha)}, b_m^{\dagger(\alpha)}] = 2\delta_{mk}\Omega^\dagger, \tag{3.8}$$

$$[a_k^{(\alpha)}, b_m^{(\alpha)}] = [a_k^{\dagger(\alpha)}, b_m^{\dagger(\alpha)}] = 0, \tag{3.9}$$

$$[\Omega, a_k^{(\alpha)}] = b_k^{(\alpha)}, \quad [\Omega, b_m^{(\alpha)}] = -a_m^{(\alpha)}, \tag{3.10}$$

$$\begin{aligned}\{a_k^{(\alpha)}, b_m^{(\beta)}\} &= \{a_k^{\dagger(\alpha)}, b_m^{(\beta)}\} = \{a_k^{(\alpha)}, b_m^{\dagger(\beta)}\}\\ &= \{a_k^{\dagger(\alpha)}, b_m^{\dagger(\beta)}\} = 0, \quad \alpha \neq \beta.\end{aligned} \tag{3.11}$$

It should be noted that not all of these relations are independent. It will be shown at the end of this section that relations (3.10) are a consequence of relations (3.8), (3.11) and bilinear relations (1.5). Comparing relations (3.8) and (3.10) with relations (3.7), we see that the latter relations contain commutators instead of anticommutators. The same is true for Greenberg–Messiah relations (2.12) also, in which commutators are replaced by anticommutators (3.11).

Let us first consider the simplest relations from Gorvorkov's commutation rules, namely, bilinear relations (1.5) and (1.6). In particular, relation (1.5) implies that

$$[a_k, b_m] = [a_m, b_k].$$

Substituting expansion (2.9) into the left-hand side of this relation and taking into account relations (3.9), (3.10), and the Jacobi identity, we obtain the following chain of equalities:

$$\begin{aligned}[a_k, b_m] &= \sum_{\alpha\neq\beta}[a_k^{(\alpha)}, b_m^{(\beta)}] = \sum_{\alpha\neq\beta}[a_k^{(\alpha)}, [\Omega, a_m^{(\beta)}]]\\ &= -\sum_{\alpha\neq\beta}([\Omega, [a_m^{(\beta)}, a_k^{(\alpha)}]] + [a_m^{(\beta)}, [a_k^{(\alpha)}, \Omega]])\\ &= \sum_{\alpha\neq\beta}[a_m^{(\beta)}, b_k^{(\alpha)}] \equiv [a_m, b_k].\end{aligned}$$

In deriving these relations, we have also used the commutation rules for Green's components of the ϕ_a field (2.10).

Further, we consider bilinear relation (1.6), which implies, in particular, that

$$[a_k, a_m] = [b_k, b_m].$$

Using commutation rules (2.10), relations (3.9), (3.10), and Jacobi identity (B.1), we obtain the chain of equalities

$$[a_k, a_m] = \sum_{\alpha=1}^{p} [a_k^{(\alpha)}, a_m^{(\alpha)}] = \sum_{\alpha=1}^{p} [a_k^{(\alpha)}, [\Omega, b_m^{(\alpha)}]]$$
$$= -\sum_{\alpha} ([b_m^{(\alpha)}, [a_k^{(\alpha)}, \Omega]] + [\Omega, [b_m^{(\alpha)}, a_k^{(\alpha)}]])$$
$$= -\sum_{\alpha} [b_m^{(\alpha)}, b_k^{(\alpha)}] \equiv [b_k, b_m].$$

Let us return again to bilinear relation (1.5) and analyze a slightly more complicated case when one operator is a creation operator and the other is an annihilation operator:

$$[a_k^{\dagger}, b_m] = [a_m, b_k^{\dagger}]. \tag{3.12}$$

Using relations (3.8) and (3.10), we obtain for the left-hand side of (3.12)

$$\begin{aligned}[a_k^{\dagger}, b_m] &= \sum_{\alpha} [a_k^{\dagger(\alpha)}, b_m^{(\alpha)}] + \sum_{\alpha\neq\beta} [a_k^{\dagger(\alpha)}, b_m^{(\beta)}] \\ &= -2p\delta_{km}\Omega + \sum_{\alpha\neq\beta} [a_k^{\dagger(\alpha)}, [\Omega, a_m^{(\beta)}]].\end{aligned} \tag{3.13}$$

By virtue of the Jacobi identity and commutation rules (2.10) and (3.10), the expression in the summand in the last term assumes the form

$$[a_k^{\dagger(\alpha)}, [\Omega, a_m^{(\beta)}]] = -[\Omega, [a_m^{(\beta)}, a_k^{\dagger(\alpha)}]]$$
$$- [a_m^{(\beta)}, [a_k^{\dagger(\alpha)}, \Omega]] = [a_m^{(\beta)}, b_k^{\dagger(\alpha)}].$$

Adding and subtracting the sum

$$\sum_{\alpha} [a_m^{(\alpha)}, b_k^{\dagger(\alpha)}] (\equiv 2p\delta_{mk}\Omega^{\dagger})$$

to the right-hand side of relation (3.13), we obtain

$$[a_k^{\dagger}, b_m] = -2p\delta_{km}\Omega - \sum_{\alpha} [a_m^{(\alpha)}, b_k^{\dagger(\alpha)}]$$
$$+ \left(\sum_{\alpha} [a_m^{(\alpha)}, b_k^{\dagger(\alpha)}] + \sum_{\alpha\neq\beta} [a_m^{(\beta)}, b_k^{\dagger(\alpha)}]\right)$$
$$\equiv -2p\delta_{km}(\Omega + \Omega^{\dagger}) + [a_m, b_k^{\dagger}].$$

Thus, bilinear relation (3.12) holds if operator Ω satisfies the following condition:

$$\Omega + \Omega^{\dagger} = 0. \tag{3.14}$$

The examples considered here are sufficient to state that bilinear relations (1.5) and (1.6) are turned into identities using system of commutation relations (2.10), (3.8)–(3.11) and the additional condition imposed on operator Ω (3.14).

Concluding the section, we will show that for a specific case of parastatistics (namely, for $p = 2$), commutation rules (3.10) are consequences of relations (3.8), (3.11) and bilinear relation (3.12). In other words, if we postulate the validity of relations (3.8), (3.11), and (3.12), their inevitable consequence will be relations (3.10). For this purpose, we write relation (3.12) in terms of Green's components:

$$[a_k^{\dagger(1)}, b_m^{(2)}] + [a_k^{\dagger(2)}, b_m^{(1)}] = -([b_k^{\dagger(1)}, a_m^{(2)}] + [b_k^{\dagger(2)}, a_m^{(1)}]). \tag{3.15}$$

Let us now calculate the commutator between this relation and operator $a_l^{(1)}$. In this case, we have two nontrivial trilinear commutators:

$$\begin{aligned}[[a_k^{\dagger(1)}, b_m^{(2)}], a_l^{(1)}] &= \{a_k^{\dagger(1)}, \{a_l^{(1)}, b_m^{(2)}\}\} \\ &- \{b_m^{(2)}, \{a_l^{(1)}, a_k^{\dagger(1)}\}\} = -2\delta_{kl} b_m^{(2)}, \\ [[b_k^{\dagger(1)}, a_m^{(2)}], a_l^{(1)}] &= -[[a_m^{(2)}, a_l^{(1)}], b_k^{\dagger(1)}] \\ &- [[a_l^{(1)}, b_k^{\dagger(1)}], a_m^{(2)}] = 2\delta_{kl}[\Omega, a_m^{(2)}].\end{aligned} \tag{3.16}$$

In the former case, we used identity (B.2). As a result, the required commutator of $a_l^{(1)}$ with (3.15) leads to relation $[\Omega, a_m^{(2)}] = b_m^{(2)}$, and the analogous commutator with $a_l^{(2)}$ gives $[\Omega, a_m^{(1)}] = b_m^{(1)}$, and we reproduce the first relation in (3.10). For obtaining the second relation, we must take the commutator between $b_l^{(\alpha)}$ and (3.15). For $\alpha = 1$, nonzero commutators are

$$\begin{aligned}[[a_k^{\dagger(1)}, b_m^{(2)}], b_l^{(1)}] &= -[[b_m^{(2)}, b_l^{(1)}], a_k^{\dagger(1)}] \\ &- [[b_l^{(1)}, a_k^{\dagger(1)}], b_m^{(2)}] = -2\delta_{kl}[\Omega, b_m^{(2)}], \\ [[b_k^{\dagger(1)}, a_m^{(2)}], b_l^{(1)}] &= \{b_k^{\dagger(1)}, \{b_l^{(1)}, a_m^{(2)}\}\} \\ &- \{a_m^{(2)}, \{b_l^{(1)}, b_k^{\dagger(1)}\}\} = -2\delta_{kl} a_m^{(2)}.\end{aligned} \tag{3.17}$$

This gives $[\Omega, b_m^{(2)}] = -a_m^{(2)}$. The commutator containing $b_l^{(2)}$ leads to an analogous expression with the replacement of Green's index $2 \to 1$; in this way, we reproduce the second relation in (3.10).

4. TRILINEAR GOVORKOV'S RELATIONS

Let us now analyze trilinear Govorkov's relation (1.3) and (1.4). In this case, it is sufficient to consider only particular cases (3.1)–(3.3). We have already analyzed relation (3.3) in the previous section, but we will now proceed in a different way. We will use the Jacobi identity for the first expression on the right-hand side of relation (3.4) and identity (B.2) for the second expression. In accordance with commutation rules (3.8)–(3.11), instead of relations (3.5) and (3.6), we obtain

$$\sum_{\alpha=1}^{p}[[a_k^{\dagger(\alpha)}, a_l^{(\alpha)}], b_m^{(\alpha)}]$$
$$= -\sum_{\alpha}([[a_l^{(\alpha)}, b_m^{(\alpha)}], a_k^{\dagger(\alpha)}] + [[b_m^{(\alpha)}, a_k^{\dagger(\alpha)}], a_l^{(\alpha)}])$$
$$= -2\delta_{km}\sum_{\alpha}[\Omega, a_l^{(\alpha)}] = -2\delta_{km}b_l,$$
$$\sum_{\alpha\neq\beta}[[a_k^{\dagger(\alpha)}, a_l^{(\alpha)}], b_m^{(\beta)}]$$
$$= \sum_{\alpha\neq\beta}(\{a_k^{\dagger(\alpha)}, \{b_m^{(\beta)}, a_l^{(\alpha)}\}\} - \{a_l^{(\alpha)}, \{b_m^{(\beta)}, a_k^{\dagger(\alpha)}\}\}) = 0.$$

Here, we also reproduce relation (3.3) as was done for rules (3.7).

Let us now consider more complicated trilinear relation (3.1), which we write again for convenience:

$$[[b_m, a_k^{\dagger}], a_l] = 2\delta_{kl}b_m + 4\delta_{km}b_l. \tag{4.1}$$

For the “internal” commutator we can use the result (3.13):

$$[b_m, a_k^{\dagger}] = 2p\delta_{km}\Omega + \sum_{\alpha\neq\beta}[b_m^{(\alpha)}, a_k^{\dagger(\beta)}].$$

Then the initial expression for analyzing the left-hand side of (4.1) takes the form

$$\begin{aligned}&[[b_m, a_k^{\dagger}], a_l]\\ &= 2p\delta_{mk}[\Omega, a_l] + \sum_{\alpha\neq\beta}\sum_{\gamma}[[b_m^{(\alpha)}, a_k^{\dagger(\beta)}], a_l^{(\gamma)}].\end{aligned} \tag{4.2}$$

Using identity (B.2), we can write the double commutator in the summand in the form

$$[[b_m^{(\alpha)}, a_k^{\dagger(\beta)}], a_l^{(\gamma)}]$$
$$= \{b_m^{(\alpha)}, \{a_l^{(\gamma)}, a_k^{\dagger(\beta)}\}\} - \{a_k^{\dagger(\beta)}, \{a_l^{(\gamma)}, b_m^{(\alpha)}\}\},$$

and present the triple sum as

$$\sum_{\alpha\neq\beta}\sum_{\gamma} = \sum_{\alpha=\gamma\neq\beta} + \sum_{\alpha\neq\beta=\gamma} + \sum_{\alpha\neq\beta\neq\gamma}. \tag{4.3}$$

Taking into account relations (2.10) and (3.11), we obtain the following nonzero terms:

$$\sum_{\alpha\neq\beta}\sum_{\gamma}[[b_m^{(\alpha)}, a_k^{\dagger(\beta)}], a_l^{(\gamma)}]$$
$$= 2(p-1)\delta_{lk}b_m + \sum_{\alpha=\gamma\neq\beta}(\{b_m^{(\alpha)}, \{a_l^{(\alpha)}, a_k^{\dagger(\beta)}\}\}$$
$$-\{a_k^{\dagger(\beta)}, \{a_l^{(\alpha)}, b_m^{(\alpha)}\}\}) + \sum_{\alpha\neq\beta\neq\gamma}\{b_m^{(\alpha)}, \{a_l^{(\gamma)}, a_k^{\dagger(\beta)}\}\}.$$

The second contribution on the right-hand side of the last expression can be presented with the help of identities (B.2) and (B.1) as

$$\sum_{\alpha=\gamma\neq\beta}(\{b_m^{(\alpha)}, \{a_l^{(\alpha)}, a_k^{\dagger(\beta)}\}\} - \{a_k^{\dagger(\beta)}, \{a_l^{(\alpha)}, b_m^{(\alpha)}\}\})$$
$$= -\sum_{\alpha=\gamma\neq\beta}([[a_m^{\dagger(\beta)}, a_l^{(\alpha)}], b_m^{(\alpha)}] + [[a_l^{(\alpha)}, b_m^{(\alpha)}], a_k^{\dagger(\beta)}]).$$

This contribution is zero by virtue of relations (2.10) and (3.9). Finally, we obtain instead of relation (4.2)

$$\begin{aligned}[[b_m, a_k^{\dagger}], a_l] &= 2(p-1)\delta_{lk}b_m + 2p\delta_{mk}b_l\\ &+ \sum_{\alpha\neq\beta\neq\gamma}\{b_m^{(\alpha)}, \{a_l^{(\gamma)}, a_k^{\dagger(\beta)}\}\}.\end{aligned} \tag{4.4}$$

It can be seen that this expression reproduces (4.1) only when $p = 2$. In this particular case, the last term on the right-hand side of relation (4.4) is just absent, and numerical coefficients of the remaining terms take correct values.

Trilinear relation (3.2) holds automatically by virtue of the Jacobi identity. Nevertheless, it is instructively to demonstrate this directly. In view of relation (3.9), the following equality holds:

$$[a_l, b_m] = \sum_{\alpha\neq\beta}[a_l^{(\alpha)}, b_m^{(\beta)}],$$

therefore, we can write

$$\begin{aligned}&[[a_l, b_m], a_k^{\dagger}] = \sum_{\alpha\neq\beta}\sum_{\gamma}[[a_l^{(\alpha)}, b_m^{(\beta)}], a_k^{\dagger(\gamma)}]\\ &= \sum_{\alpha\neq\beta}\sum_{\gamma}(\{a_l^{(\alpha)}, \{a_k^{\dagger(\gamma)}, b_m^{(\beta)}\}\} - \{b_m^{(\beta)}, \{a_k^{\dagger(\gamma)}, a_l^{(\alpha)}\}\}).\end{aligned} \tag{4.5}$$

We represent the triple sum on the right-hand side of this relation again in the form of decomposition (4.3). With allowance for commutation rules (3.11) and (2.10), expression (4.5) takes the form

$$[[a_l, b_m], a_k^{\dagger}] = -2(p-1)\delta_{kl}b_m$$
$$+ \sum_{\alpha\neq\beta=\gamma}(\{a_l^{(\alpha)}, \{a_k^{\dagger(\beta)}, b_m^{(\beta)}\}\} - \{b_m^{(\beta)}, \{a_k^{\dagger(\beta)}, a_l^{(\alpha)}\}\})$$
$$- \sum_{\alpha\neq\beta\neq\gamma}\{b_m^{(\beta)}, \{a_k^{\dagger(\gamma)}, a_l^{(\alpha)}\}\}.$$

For the second contribution on the right-hand side, we have a chain of equalities

$$\sum_{\alpha\neq\beta=\gamma}(\{a_l^{(\alpha)}, \{a_k^{\dagger(\beta)}, b_m^{(\beta)}\}\} - \{b_m^{(\beta)}, \{a_k^{\dagger(\beta)}, a_l^{(\alpha)}\}\})$$
$$= -\sum_{\alpha\neq\beta=\gamma}([[b_m^{(\beta)}, a_k^{\dagger(\beta)}], a_l^{(\alpha)}] + [[a_k^{\dagger(\beta)}, a_l^{(\alpha)}], b_m^{(\beta)}])$$
$$= 2\delta_{mk}\sum_{\alpha\neq\beta=\gamma}[\Omega, a_l^{(\alpha)}] = 2(p-1)\delta_{mk}b_l,$$

which gives, instead of (4.5),

$$[[a_l, b_m], a_k^\dagger] = -2(p-1)\delta_{kl} b_m - 2(p-1)\delta_{mk} b_l$$
$$- \sum_{\alpha\neq\beta\neq\gamma} \{b_m^{(\beta)}, \{a_k^{\dagger(\gamma)}, a_l^{(\alpha)}\}\}.$$

It can be seen that this expression reproduces trilinear relation (3.2) only for $p = 2$.

5. INCLUSION OF PARA-GRASSMANN NUMBERS

In this section, we intend to include in the general scheme of uniquantization the para-Grassmann numbers that will be denoted by ξ_k, $k = 1, \ldots, M$. Our task is the formulation of the commutation rules including simultaneously ξ_k and operators a_k and b_m. In the case of a single para-Fermi field (e.g., ϕ_a), such commutation rules were proposed in [16]:

$$[a_k, [a_l, \xi_m]] = 0, \quad [a_k, [a_l^\dagger, \xi_m]] = 2\delta_{kl}\xi_m,$$
$$[\xi_k, [\xi_l, a_m]] = 0, \quad [\xi_k, [\xi_l, \xi_m]] = 0. \tag{5.1}$$

For the special case of parastatistics $p = 2$, we can use instead of the last relation in (5.1) the simpler expression

$$\xi_k\xi_l\xi_m + \xi_m\xi_l\xi_k = 0.$$

The remaining relations can be obtained from (5.1) by the Hermitian conjugation. We assume that analogous commutation rules exist for the second ϕ_b field also. For para-Grassmann numbers ξ_k, the Green representation

$$\xi_k = \sum_{\alpha=1}^{p} \xi_k^{(\alpha)}$$

also holds. The bilinear commutation relations for Green's components $a_k^{(\alpha)}$, $b_m^{(\alpha)}$, and $\xi_l^{(\alpha)}$ were given in [17]:

$$\{a_k^{(\alpha)}, \xi_l^{(\alpha)}\} = 0, \quad \{b_m^{(\alpha)}, \xi_l^{(\alpha)}\} = 0,$$
$$\{\xi_k^{(\alpha)}, \xi_l^{(\alpha)}\} = 0, \quad [a_k^{(\alpha)}, \xi_l^{(\beta)}] = 0, \tag{5.2}$$
$$[b_m^{(\alpha)}, \xi_l^{(\beta)}] = 0, \quad [\xi_k^{(\alpha)}, \xi_l^{(\beta)}] = 0, \quad \alpha \neq \beta,$$

including their Hermitian conjugates. They turn relation (5.1) into identity.

As noted above, we are interested in the trilinear relations including simultaneously operators a_k, b_m, and para-Grassmann numbers ξ_k. It is natural to begin our analysis with the following expressions:

$$[b_m, [a_k^\dagger, \xi_l]], \quad [a_k, [b_m^\dagger, \xi_l]].$$

Preliminary analysis of these relations using (3.8)–(3.11) and (5.2) has shown, however, that the double commutators are ultimately reduced to tangled expressions. For this reason, keeping in mind the above analysis, we consider slightly different trilinear relations, namely,

$$\{b_m, [a_k^\dagger, \xi_l]\}, \quad \{a_k, [b_m^\dagger, \xi_l]\}. \tag{5.3}$$

In terms of the Green components and with allowance for relations (5.2), the first relation takes the form

$$\{b_m, [a_k^\dagger, \xi_l]\} = \sum_\alpha \{b_m^{(\alpha)}, [a_k^{\dagger(\alpha)}, \xi_l^{(\alpha)}]\}$$
$$+ \sum_{\alpha\neq\beta} \{b_m^{(\beta)}, [a_k^{\dagger(\alpha)}, \xi_l^{(\alpha)}]\}. \tag{5.4}$$

Using identity (B.3) and rules (3.8) and (5.2), we can write the first term on the right-hand side of this relation in the form

$$\{b_m^{(\alpha)}, [a_k^{\dagger(\alpha)}, \xi_l^{(\alpha)}]\} = \{\xi_l^{(\alpha)}, [b_m^{(\alpha)}, a_k^{\dagger(\alpha)}]\}$$
$$+ [a_k^{\dagger(\alpha)}, \{\xi_l^{(\alpha)}, b_m^{(\alpha)}\}] = 2\delta_{mk}\{\xi_l^{(\alpha)}, \Omega\}.$$

It can easily be seen using the same identity that the second term in expression (5.4) vanishes and, hence, we obtain instead of (5.4)

$$\{b_m, [a_k^\dagger, \xi_l]\} = 2\delta_{mk}\{\xi_l, \Omega\}. \tag{5.5}$$

Analogous line of reasoning for the second expression in (5.3) leads to

$$\{a_k, [b_m^\dagger, \xi_l]\} = -2\delta_{mk}\{\xi_l, \Omega\}. \tag{5.6}$$

As a direct consequence of relations (5.5), (5.6), and identity (B.3), we obtain the following equalities:

$$[\xi_l, \{b_m, a_k^\dagger\}] = 0, \quad [\xi_l, \{a_k, b_m^\dagger\}] = 0.$$

It should be noted that relations (5.5) and (5.6) are a direct consequence of commutation rules (3.8)–(3.11) and (5.2) for the Green components and contain no new information. However, we can proceed further and postulate the following condition:

$$\{\xi_l, \Omega\} = \Lambda\xi_l, \tag{5.7}$$

where Λ is a certain constant satisfying (by virtue of relation (3.14)) the condition

$$\Lambda = -\Lambda^*,$$

(asterisk denotes complex conjugation). Therefore, instead of relations (5.5) and (5.6), we now have the desired trilinear relations

$$\{b_m, [a_k^\dagger, \xi_l]\} = 2\Lambda\delta_{mk}\xi_l,$$
$$\{a_k, [b_m^\dagger, \xi_l]\} = 2\Lambda^*\delta_{mk}\xi_l. \tag{5.8}$$

Govorkov has introduced in [2] an important operator $\tilde{N}$:

$$\tilde{N} = \frac{i}{2(2M+1)} \sum_{k=1}^{M} ([a_k^\dagger, b_k] + \lambda), \tag{5.9}$$

where λ is a real-valued nonzero constant.[3] In terms of operator ζ_0 (1.8), expression (5.9) with allowance for (3.12) can be written in the form

[3] Note that number λ was fixed neither in [2] nor in review [4].

$$\tilde{N} = \frac{1}{2}\zeta_0 + \frac{iM}{2(2M+1)}\lambda. \quad (5.10)$$

Operator $\tilde{N}$ possesses the following properties:

$$[i\tilde{N}, a_k] = b_k, \quad [i\tilde{N}, b_k] = -a_k. \quad (5.11)$$

These relations exactly coincide with relations (3.10) for Green's components $a_k^{(\alpha)}$ and $b_k^{(\alpha)}$. Nevertheless, operator Ω in relation (3.10) cannot be literally identified with operator $i\tilde{N}$, since this would lead to contradictions in further analysis. A certain nontrivial relation between $i\tilde{N}$ and Ω can be seen at a simple level if we derive a relation of type (5.7) for operator $i\tilde{N}$. For this purpose, we consider the anticommutator

$$\left\{\xi_l, \frac{1}{2}\zeta_0\right\} = \frac{i}{2(2M+1)}\sum_{k=1}^{M}\{\xi_l, [a_k^\dagger, b_k]\}$$
$$= \frac{i}{2(2M+1)}\sum_{k=1}^{M}(\{b_k, [\xi_l, a_k^\dagger]\} + [a_k^\dagger, \{b_k, \xi_l\}]).$$

The first term in the summand is defined by the first relation in (5.8), while the second term is given by

$$[a_k^\dagger, \{b_k, \xi_l\}] = 2\Lambda^*\xi_l$$

(validity of this expression will be shown in the next section). Taking into account the above arguments, we can write

$$\left\{\xi_l, \frac{1}{2}\zeta_0\right\} = -\frac{iM}{2M+1}(\Lambda - \Lambda^*)\xi_l,$$

then expression (5.10) leads to

$$\{\xi_l, i\tilde{N}\} = \tilde{\Lambda}\xi_l, \quad (5.12)$$

where

$$\tilde{\Lambda} = \frac{M}{2M+1}(2\Lambda - \lambda). \quad (5.13)$$

We can derive the explicit expressions for the commutators between operator $i\tilde{N}$ and Green components $a_k^{(\alpha)}$ and $b_k^{(\alpha)}$. For this purpose, we substitute relation (3.13) for $p = 2$ into the right-hand side of relation (5.9), which gives

$$i\tilde{N} = \frac{2M}{2M+1}\Omega - \frac{1}{2(2M+1)}$$
$$\times\sum_{k=1}^{M}([a_k^{\dagger(1)}, b_k^{(2)}] + [a_k^{\dagger(2)}, b_k^{(1)}]) - \frac{M}{2(2M+1)}\lambda. \quad (5.14)$$

Using relations (3.10) and equality (3.16), we obtain

$$[i\tilde{N}, a_k^{(1)}] = \frac{1}{2M+1}(2Mb_k^{(1)} + b_k^{(2)}),$$
$$[i\tilde{N}, a_k^{(2)}] = \frac{1}{2M+1}(2Mb_k^{(2)} + b_k^{(1)}). \quad (5.15)$$

Analogous arguments for commutators with Green components $b_m^{(\alpha)}$ taking into account relations (3.10) and (3.17) lead to

$$[i\tilde{N}, b_m^{(1)}] = -\frac{1}{2M+1}(2Ma_m^{(1)} + a_m^{(2)}),$$
$$[i\tilde{N}, b_m^{(2)}] = -\frac{1}{2M+1}(2Ma_m^{(2)} + a_m^{(1)}). \quad (5.16)$$

In spite of the somewhat unusual form of commutation relations (5.15) and (5.16), these relations correctly reproduce (5.11), which can easily be verified by simply summing the relations in (5.15) and in (5.16) and taking into account the fact that

$$a_k = a_k^{(1)} + a_k^{(2)}, \quad b_m = b_m^{(1)} + b_m^{(2)}.$$

6. COMMUTATION RELATIONS WITH OPERATOR $e^{\alpha i\tilde{N}}$

Let us define a set of commutation relations between operator $e^{\alpha i\tilde{N}}$, operators a_k, b_m, and para-Grassmann numbers ξ_l. Here, α is an arbitrary real number. For this purpose, we note above all that operator a_k satisfies the following equality:

$$e^{\alpha i\tilde{N}}a_k e^{-\alpha i\tilde{N}} = a_k + \alpha[i\tilde{N}, a_k] + \frac{1}{2!}\alpha^2[i\tilde{N}, [i\tilde{N}, a_k]] + \dots$$
$$= a_k\left(1 - \frac{1}{2!}\alpha^2 + \frac{1}{4!}\alpha^4 - \dots\right) + b_k\left(\alpha - \frac{1}{3!}\alpha^3 + \dots\right) \quad (6.1)$$
$$\equiv a_k\cos\alpha + b_k\sin\alpha.$$

In deriving this relation, we have taken into account identity (B.7) and relations (5.11). A similar expression can also be obtained for operator b_m. Using equality (6.1), we can write basic relations determining the rules of permutation between $e^{\alpha i\tilde{N}}$ and a_k, b_m:

$$e^{\alpha i\tilde{N}}a_k = (a_k\cos\alpha + b_k\sin\alpha)e^{\alpha i\tilde{N}},$$
$$e^{\alpha i\tilde{N}}b_m = (b_m\cos\alpha - a_m\sin\alpha)e^{\alpha i\tilde{N}}. \quad (6.2)$$

Here, we are mainly interested in two important special cases of these formulas:

(i) when $\alpha = \pm\pi$, we have

$$\{e^{\pm\pi i\tilde{N}}, a_k\} = 0, \quad \{e^{\pm\pi i\tilde{N}}, b_m\} = 0; \quad (6.3)$$

(ii) when $\alpha = \pm\pi/2$, we get

$$e^{\pm\pi i\tilde{N}/2}a_k = \pm b_k e^{\pm\pi i\tilde{N}/2}, \quad (6.4)$$

$$e^{\pm\pi i\tilde{N}/2}b_m = \mp a_m e^{\pm\pi i\tilde{N}/2}. \quad (6.5)$$

It should be emphasized that relations (6.4) and (6.5) indicate the possibility of two equivalent "mappings" of operator b_k into operator a_k:

$$a_k = \pm e^{\mp\pi i\tilde{N}/2}b_k e^{\pm\pi i\tilde{N}/2},$$
$$a_k = \mp e^{\pm\pi i\tilde{N}/2}b_k e^{\mp\pi i\tilde{N}/2}. \quad (6.6)$$

This circumstance is convenient, in particular, in analysis of specific expressions. Anticommutation relations (6.3) coincide with analogous relations for operator $(-1)^N = e^{\pm\pi iN}$:

$$\{e^{\pm\pi iN}, a_k\} = 0, \quad \{e^{\pm\pi iN}, b_m\} = 0, \tag{6.7}$$

where N is the particle number operator (1.7). Relations (6.7) hold, since

$$\begin{aligned} [a_k, N] &= a_k, \quad [a_k^\dagger, N] = -a_k^\dagger, \\ [b_m, N] &= b_m, \quad [b_m^\dagger, N] = -b_m^\dagger. \end{aligned} \tag{6.8}$$

As regards relations (6.4) and (6.5), we can mention an interesting formal relationship with Schwinger's publications [18, 19] (see also [20]) devoted to the construction and analysis of unitary operator bases. The relation

$$\hat{X}(\alpha)\hat{U} = \hat{U}\hat{Y}(\alpha)$$

from [18, 19] is analogous to (6.4), (6.5). Here, $\hat{X}(\alpha)$ and $\hat{Y}(\alpha)$ are two orthonormal operator bases in a given space, which are connected by unitary operator $\hat{U} = (\hat{U}_{ab})$:

$$\hat{U}_{ab} = \sum_{k=1}^{N} |a_k\rangle\langle b_k|,$$

where $|a_k\rangle$, $|b_k\rangle$, and their adjoints form two ordered sets of vectors. In our case, operator $e^{\pm i\tilde{N}/2}$ plays the role of operator $\hat{U}$. A number of other coincidences between two formalisms can be indicated, but we will not dwell on detailed analysis of this relationship.

Let us introduce para-Grassmann numbers ξ_k. Since we now have anticommutation relation (5.7), we must consider the following expression instead of (6.1):

$$\begin{aligned} e^{\alpha\Omega}\xi_k e^{\alpha\Omega} &= \xi_k + \alpha\{\Omega, \xi_k\} + \frac{1}{2!}\alpha^2\{\Omega, \{\Omega, \xi_K\}\} + \ldots \\ &= \xi_k\left(1 + \Lambda\alpha + \frac{1}{2!}(\Lambda\alpha)^2 + \ldots\right) \equiv \xi_k e^{\alpha\Lambda}. \end{aligned}$$

Here, we have used identity (B.8). Thus, we have the following commutation rule between operator $e^{\alpha\Omega}$ and para-Grassmann numbers ξ_k:

$$e^{\alpha\Omega}\xi_k = \xi_k e^{\alpha\Lambda} e^{-\alpha\Omega}.$$

Analogously, with allowance for relation (5.12), we obtain the following expression for operator $e^{\alpha i\tilde{N}}$:

$$e^{\alpha i\tilde{N}}\xi_k = \xi_k e^{\alpha\tilde{\Lambda}} e^{-\alpha i\tilde{N}}. \tag{6.9}$$

On the basis of permutation relations (6.4), (6.5), and (6.9), the following question arises: what is the form acquired by various trilinear relations under mapping (6.6)? The answer to this question is quite unexpected: this mapping is reduced to simple replacement $a_k \rightleftarrows b_k$ not in all cases.

Let us consider the mapping of the simplest trilinear relation from (5.1), which involves only one operator a_m:

$$[\xi_k, [\xi_l, a_m]] = 0. \tag{6.10}$$

We consider first the commutator $[\xi_l, a_m]$. Using relations (6.6) and (6.9), we arrive at the following expression:

$$[\xi_l, a_m] = \mp e^{\pm\pi\tilde{\Lambda}/2} e^{\mp\pi i\tilde{N}/2}\{\xi_l, b_m\} e^{\mp\pi i\tilde{N}/2}. \tag{6.11}$$

Pay attention to the fact that an *anticommutator* appears on its right-hand side. The substitution of relation (6.11) into (6.10) gives

$$\xi_k\{\xi_l, b_m\} - e^{\mp\pi i\tilde{N}}\{\xi_l, b_m\}\xi_k e^{\pm\pi i\tilde{N}} = 0. \tag{6.12}$$

Further, using relations (6.3) and (6.9), we obtain the following expression for the last term in (6.12):

$$\begin{gathered} e^{\mp\pi i\tilde{N}}\{\xi_l, b_m\}\xi_k e^{\pm\pi i\tilde{N}} \\ = -e^{\mp\pi\tilde{\Lambda}}\{\xi_l, b_m\}(e^{\pm\pi i\tilde{N}}\xi_k e^{\pm\pi i\tilde{N}}) = -\{\xi_l, b_m\} \end{gathered}$$

which gives, instead of (6.12),

$$\{\xi_k, \{\xi_l, b_m\}\} = 0. \tag{6.13}$$

Contrary to the expectation, relation (6.10) under mapping (6.6) is not transformed into an analogous expression differing only in replacement $a_m \to b_m$. It can be seen that in addition to this replacement, all commutators are replaced by anticommutators. This situation can take place only for parastatistics of order 2.

We can directly verify the correctness of trilinear relation (6.13) using the Green ansatz. Indeed, in view of commutation rules (5.2), the following equality holds:

$$\{\xi_l, b_m\} = \sum_{\alpha\neq\beta}\{\xi_l^{(\alpha)}, b_m^{(\beta)}\}.$$

Using now the decomposition of triple sum (4.3), we obtain

$$\begin{aligned} \{\xi_k, \{\xi_k, b_m\}\} &= \sum_{\alpha=\gamma\neq\beta}\{\xi_k^{(\alpha)}, \{\xi_l^{(\alpha)}, b_m^{(\beta)}\}\} \\ &+ \sum_{\alpha\neq\beta=\gamma}\{\xi_k^{(\beta)}, \{\xi_l^{(\alpha)}, b_m^{(\beta)}\}\} + \sum_{\alpha\neq\beta\neq\gamma}\{\xi_k^{(\gamma)}, \{\xi_l^{(\alpha)}, b_m^{(\beta)}\}\}. \end{aligned} \tag{6.14}$$

Taking into account identity (B.2) and relations (5.2), we can obtain the following expressions for the first two terms on the right-hand side of relation (6.14):

$$\begin{gathered} \sum_{\alpha=\gamma\neq\beta}\{\xi_k^{(\alpha)}, \{\xi_l^{(\alpha)}, b_m^{(\beta)}\}\} \\ = \sum_{\alpha=\gamma\neq\beta}(\{b_m^{(\beta)}, \{\xi_k^{(\alpha)}, \xi_l^{(\alpha)}\}\} + [\xi_l^{(\alpha)}, [b_m^{(\beta)}, \xi_k^{(\alpha)}]]) = 0, \end{gathered}$$

$$\sum_{\alpha\neq\beta=\gamma}\{\xi_k^{(\beta)},\{b_m^{(\beta)},\xi_l^{(\alpha)}\}\}$$
$$=\sum_{\alpha\neq\beta=\gamma}([b_m^{(\beta)},[\xi_l^{(\alpha)},\xi_k^{(\beta)}]]+\{\xi_l^{(\alpha)},\{b_m^{(\beta)},\xi_k^{(\beta)}\}\})=0.$$

The third term in relation (6.14) is absent for $p = 2$.

Let us consider the mapping of more nontrivial trilinear relations (5.8). Specifically, we consider the second of these relations:

$$\{a_k,[\xi_l,b_m^\dagger]\}=2\Lambda\delta_{mk}\xi_l. \tag{6.15}$$

Using the second formula in (6.6), we obtain the initial expression for our analysis:

$$\{a_k,[\xi_l,b_m^\dagger]\}=\mp(e^{\pm\pi i\tilde{N}/2}b_k e^{\mp\pi i\tilde{N}/2}[\xi_l,b_m^\dagger]$$
$$+[\xi_l,b_m^\dagger]e^{\pm\pi i\tilde{N}/2}b_k e^{\mp\pi i\tilde{N}/2}).$$

For the commutator on the right-hand side, we use the expression analogous to (6.11):

$$[\xi_l,b_m^\dagger]=\pm e^{\pm\pi\tilde{\Lambda}/2}e^{\mp\pi i\tilde{N}/2}\{\xi_l,a_m^\dagger\}e^{\mp\pi i\tilde{N}/2}.$$

Multiplying both sides of expression (6.15) by operator $e^{\pm\pi i\tilde{N}/2}$ and taking into account the above arguments, we obtain

$$e^{\mp\pi\tilde{\Lambda}/2}(e^{\pm\pi i\tilde{N}}b_k\{\xi_l,a_m^\dagger\}e^{\pm\pi i\tilde{N}})-e^{\pm\pi\tilde{\Lambda}/2}\{\xi_l,a_m^\dagger\}b_k$$
$$=2\Lambda\delta_{mk}(e^{\pm\pi i\tilde{N}/2}\xi_k e^{\pm\pi i\tilde{N}/2}).$$

The expression in the parentheses on the left-hand side is equal to $e^{\pm\pi\tilde{\Lambda}}b_k\{\xi_l,\ a_m^\dagger\}$, while the expression in the parentheses on the right-hand side is $e^{\pm\pi\tilde{\Lambda}/2}\xi_k$. Cancelling out the common factor $e^{\pm\pi\tilde{\Lambda}/2}$ on the left- and right-hand sides, we finally obtain

$$[b_k,\{\xi_l,a_m^\dagger\}]=2\Lambda\delta_{mk}\xi_l. \tag{6.16}$$

Here, we again observe that under the mapping of expression (6.6), not only the replacement of operators $a \rightleftharpoons b$ occurs in trilinear relation (6.15), but the commutator is replaced by the anticommutator, and vice versa. Similarly to the previous case with (6.13), we can verify relation (6.16) using the Green representation for operators and para-Grassmann numbers.

Let us consider the mapping of the trilinear relation from (5.1), which contains operators a_k and $a_l^\dagger$:

$$[a_k,[a_l^\dagger,\xi_m]]=2\delta_{kl}\xi_m.$$

The arguments completely analogous to those in the previous case lead to the relation

$$\{b_k,\{b_l^\dagger,\xi_m\}\}=2\delta_{kl}\xi_m, \tag{6.17}$$

the validity of which can be verified using the Green ansatz.

The peculiarity of all examples considered above is that para-Grassmann numbers ξ_k always appear in the commutator or anticommutator together with operators a_k or b_m (or with their Hermitian conjugates). Let us consider the mapping of the relations for which this is not true, for example, the relations of the form

$$\begin{gathered}\{\xi_l,[b_m^\dagger,a_k]\}=2(\Lambda-\Lambda^*)\delta_{mk}\xi_l,\\ [\xi_l,\{a_k^\dagger,b_m\}]=0.\end{gathered} \tag{6.18}$$

Clearly, under mapping (6.6), these two relations can never be converted into each other, since their right-hand sides are different. Repeating the above arguments, we obtain

$$\{\xi_l,[a_m^\dagger,b_k]\}=2(\Lambda^*-\Lambda)\delta_{mk}\xi_l,$$
$$[\xi_l,\{b_k^\dagger,a_m\}]=0,$$

i.e., the structure of trilinear relations remains unchanged, and these relations in the given case are just Hermitian conjugation of (6.18). The same also holds for trilinear relations that do not contain variable ξ_k at all, for example,

$$[[a_k^\dagger,a_l],b_m]=-2\delta_{km}b_l.$$

Under mapping (6.6), this relation is transformed to

$$[[b_k^\dagger,b_l],a_m]=-2\delta_{km}a_l.$$

In this case also, the structure is preserved completely. Therefore, all trilinear commutation relations can be divided into two sets, in one of which their structure changes under mapping (6.6), while in the other, the structure is preserved. It depends on how para-Grassmann variable ξ_k enters into a specific trilinear relation. All above arguments are obviously also valid for the mapping inverse to (6.6), i.e.,

$$\begin{aligned}b_m&=\pm e^{\pm\pi i\tilde{N}/2}a_m e^{\mp\pi i\tilde{N}/2},\\ b_m&=\mp e^{\mp\pi i\tilde{N}/2}a_m e^{\pm\pi i\tilde{N}/2}.\end{aligned} \tag{6.19}$$

We must only replace operators a_k in initial relations (6.10), (6.16), etc., by b_k (and vice versa) and do the same in final formulas (6.13), (6.16), etc.

7. ACTION OF OPERATORS Ω AND $i\tilde{N}$ ON THE VACUUM STATE

Let us consider the action of operators Ω and $\tilde{N}$ on vacuum state $|0\rangle$. For operator $\tilde{N}$ (5.10), we can write

$$\tilde{N}|0\rangle=\frac{1}{2}\zeta_0|0\rangle+\frac{iM}{2(2M+1)}\lambda|0\rangle. \tag{7.1}$$

In accordance with the definition of operator ζ_0 (1.8), taking into account relation (2.13), we obtain

$$\zeta_0|0\rangle=-\frac{i}{2M+1}\sum_{k=1}^{M}b_k a_k^\dagger|0\rangle. \tag{7.2}$$

Imposing the additional Greenberg—Messiah condition (2.15), we find

$$\zeta_0|0\rangle = 0,$$

therefore, it follows from expression (7.1) that

$$\tilde{N}|0\rangle = \lambda\frac{iM}{2(2M+1)}|0\rangle.$$

If we required that the condition

$$\tilde{N}|0\rangle = 0 \tag{7.3}$$

be satisfied analogously to the similar condition for the particle number operator,

$$N|0\rangle = 0,$$

we would arrive at the trivial requirement $\lambda = 0$. However, the latter condition actually leads to degeneracy of the theory under consideration. The only way to avoid this is to reject conditions (2.15).

To find out how relations (2.15) should be changed in this case, we consider in detail the derivation of conditions (2.15) as presented in [15]. However, we will now proceed from trilinear Govorkov relations. At the first step, we act by relation (4.1) on the vacuum state:

$$a_l(b_m a_k^\dagger)|0\rangle = 0 \quad \text{for all } k, l, m.$$

The uniqueness condition for vacuum state $|0\rangle$ implies that

$$b_m a_k^\dagger|0\rangle = c_{mk}|0\rangle, \tag{7.4}$$

where c_{mk} are certain numbers. It should be noted that at this stage of analysis, additional term $4\delta_{km}b_l$ on the right-hand side of relation (4.1) is immaterial. Further, we consider a commutator of the form

$$[[b_l, b_m^\dagger], b_m a_k^\dagger] = [[b_l, b_m^\dagger], b_m]a_k^\dagger + b_m[[b_l, b_m^\dagger], a_k^\dagger]$$
(no summation!).

Using trilinear relations (2.1) and (3.3) with the interchange of a and b, we obtain

$$[[b_l, b_m^\dagger], b_m a_k^\dagger] = 2\delta_{mm}b_l a_k^\dagger - 2\delta_{kl}b_m a_m^\dagger.$$

Precisely at this stage, a new term appears on the right-hand side as compared to the Greenberg and Messiah case. Acting with the last expression on the vacuum state and using Eqs. (2.14) and (7.4), we obtain

$$0 = 2b_l a_k^\dagger|0\rangle - 2\delta_{kl}b_m a_m^\dagger|0\rangle = 2c_{lk}|0\rangle - 2\delta_{lk}c_{mm}|0\rangle$$

or

$$c_{lk} = \delta_{lk}c_{mm}.$$

We set

$$c_{mm} \equiv c \quad \text{for all } m,$$

where c is an arbitrary generally speaking complex-valued constant. Thus, using the unitary quantization scheme, we arrive at the following additional conditions instead of (2.15):

$$\begin{aligned} b_m a_m^\dagger|0\rangle &= c\delta_{mk}|0\rangle, \\ a_k b_m^\dagger|0\rangle &= c^*\delta_{mk}|0\rangle. \end{aligned} \tag{7.5}$$

In this case, relation (7.2) implies that

$$\zeta_0|0\rangle = -c\frac{iM}{2M+1}|0\rangle \tag{7.6}$$

and, hence,

$$\tilde{N}|0\rangle = -(c-\lambda)\frac{iM}{2(2M+1)}|0\rangle. \tag{7.7}$$

Acting further with relation (5.12) on the vacuum state and taking into account the rule [17]

$$\xi_l|0\rangle = |0\rangle\xi_l$$

and Eq. (7.7), we obtain the equation connecting constants Λ and c:

$$(c-\lambda)\frac{M}{2M+1} = (2\Lambda-\lambda)\frac{M}{2M+1}$$

or

$$\Lambda = \frac{1}{2}c. \tag{7.8}$$

As a direct consequence of relations (7.8) and (5.7), we obtain the rule of action of operator Ω on the vacuum state:

$$\Omega|0\rangle = \frac{1}{4}c|0\rangle.$$

If we required the fulfillment of condition (7.3), Eqs. (7.7) and (7.8) would result in unambiguous fixation of constants Λ and c in terms of parameter λ:

$$c = \lambda, \quad \Lambda = \frac{1}{2}\lambda. \tag{7.9}$$

It is the only parameter that remains undefined in the theory considered here. It should be noted that in the case of (7.9), constant $\tilde{\Lambda}$ vanishes, and we obtain instead of relation (5.12)

$$\{\xi_l, \tilde{N}\} = 0.$$

8. COHERENT STATES

The coherent states of para-Fermi operators were constructed and studied in [16] based on para-Grassmann algebra. The coherent state of the set of para-Fermi oscillators a_k was defined as

$$|(\xi)_p; a\rangle = \exp\left(-\frac{1}{2}\sum_{l=1}^{M}[\xi_l, a_l^\dagger]\right)|0\rangle, \tag{8.1}$$

so that

$$a_k|(\xi)_p; a\rangle = \xi_k|(\xi)_p; a\rangle. \tag{8.2}$$

In the notation of coherent state $|(\xi)_p\rangle$ adopted in [16], we have used additional symbol a to emphasize that this state is associated with field ϕ_a. Expression (8.2) is a consequence of the operator relation

$$a_k \exp\left\{-\frac{1}{2}\sum_l [\xi_l, a_l^\dagger]\right\} = \exp\left\{-\frac{1}{2}\sum_l [\xi_l, a_l^\dagger]\right\} a_k + \xi_k \exp\left\{-\frac{1}{2}\sum_l [\xi_l, a_l^\dagger]\right\}, \tag{8.3}$$

which can easily be obtained using identity (B.7) and trilinear relations (5.1). It should only be noted in this connection that the simple form of the second term on the right-hand side of relation (8.3) is due to exact truncation of the series

$$\begin{aligned}&\exp\left\{-\frac{1}{2}[\xi_l, a_l^\dagger]\right\} a_k \exp\left\{\frac{1}{2}[\xi_{l'}, a_{l'}^\dagger]\right\}\\ &= a_k + \left(-\frac{1}{2}\right)[[\xi_l, a_l^\dagger], a_k]\\ &+\frac{1}{2!}\left(-\frac{1}{2}\right)^2 [[\xi_l, a_l^\dagger],[[\xi_{l'}, a_{l'}^\dagger], a_k]] + \ldots = a_k - \xi_k\end{aligned}$$

after the second term of the expansion.

In a similar way, we can define the coherent state for a set of para-Fermi oscillators b_k:

$$|(\xi)_p; b\rangle = \exp\left(-\frac{1}{2}\sum_{l=1}^{M}[\xi_l, b_l^\dagger]\right)|0\rangle, \tag{8.4}$$

so that

$$b_m|(\xi)_p; b\rangle = \xi_m|(\xi)_p; b\rangle. \tag{8.5}$$

In the general case, coherent state (8.4) for b-operators is not at all a coherent state for a-operators. However, the situation with uniquantization for a parastatistics of order 2 is somewhat different. Indeed, let us consider the following operator identity:

$$\begin{aligned}&a_k \exp\left\{-\frac{1}{2}[\xi_l, b_l^\dagger]\right\} - \exp\left\{\frac{1}{2}[\xi_l, b_l^\dagger]\right\} a_k\\ &\equiv \left(a_k - \exp\left\{\frac{1}{2}[\xi_l, b_l^\dagger]\right\} a_k \exp\left\{\frac{1}{2}[\xi_{l'}, b_{l'}^\dagger]\right\}\right)\\ &\times\exp\left\{-\frac{1}{2}[\xi_{l''}, b_{l''}^\dagger]\right\}.\end{aligned} \tag{8.6}$$

Here and below, we are using for simplicity of notation the conventional rule of summation over two repeated indices. By virtue of identity (B.8) and relations (5.8), we can write

$$\begin{aligned}&\exp\left\{\frac{1}{2}[\xi_l, b_l^\dagger]\right\} a_k \exp\left\{\frac{1}{2}[\xi_{l'}, b_{l'}^\dagger]\right\}\\ &= a_k + \frac{1}{2}\{[\xi_l, b_l^\dagger], a_k\}\\ &+\frac{1}{2!}\left(\frac{1}{2}\right)^2\{[\xi_l, b_l^\dagger],\{[\xi_{l'}, b_{l'}^\dagger], a_k\}\} + \ldots\\ &= a_k + \Lambda\xi_k + \frac{1}{2!}\Lambda\xi_k[\xi_l, b_l^\dagger] + \ldots\end{aligned} \tag{8.7}$$

consequently, identity (8.6) leads to an operator equality of the form

$$\begin{aligned}&a_k \exp\left\{-\frac{1}{2}[\xi_l, b_l^\dagger]\right\} = \exp\left\{\frac{1}{2}[\xi_l, b_l^\dagger]\right\} a_k\\ &-\Lambda\xi_k\left(\sum_s \frac{1}{(s+1)!}[\xi_m, b_m^\dagger]^s\right)\exp\left\{-\frac{1}{2}[\xi_l, b_l^\dagger]\right\}.\end{aligned} \tag{8.8}$$

In contrast to equality (8.3), the sign in the exponential function of first term on the right-hand side of this relation is reversed. This, however, is immaterial since under the action of operator (8.8) on the vacuum state, this term vanishes. More serious changes have occurred in the second term; as compared to relation (8.3), it acquired the additional factor

$$\begin{aligned}&\Lambda\sum_s \frac{1}{(s+1)!}[\xi_m, b_m^\dagger]^s\\ &= \Lambda\left(1 + \frac{1}{2!}[\xi_m, b_m^\dagger] + \frac{1}{3!}[\xi_m, b_m^\dagger]^2 + \ldots\right).\end{aligned} \tag{8.9}$$

This is due to the fact that series in expression (8.7) is not terminated after the second term $\Lambda\xi_k$ as in the derivation of relation (8.3). The only positive aspect is the finiteness of series (8.9). In particular, for the most important case from the physical point of view, when $M = 2$ (and $p = 2$), this series contains only the terms appearing in (8.9). Acting with operator relation (8.8) on the vacuum state, we obtain

$$a_k|(\xi)_2; b\rangle = \Lambda\xi_k\left(\sum_s \frac{1}{(s+1)!}[\xi_m, b_m^\dagger]^s\right)|(\xi)_2; b\rangle. \tag{8.10}$$

If we introduce the conjugate coherent state for the ϕ_b field,

$$\langle(\bar{\xi}')_2; b| \equiv \langle 0|\exp\left(\frac{1}{2}\sum_{l=1}^{M}[\bar{\xi}'_l, b_l]\right),$$

we can write the matrix element of operator a_k for special case $M = 2$ in the basis of coherent states for b-operators:

$$\begin{aligned}&\langle(\bar{\xi}')_2; b|a_k|(\xi)_2; b\rangle\\ &= -\Lambda\left(1 + \frac{1}{2!}[\xi_l, \bar{\xi}'_l] + \frac{1}{3!}[\xi_l, \bar{\xi}'_l]^2\right)\langle(\bar{\xi}')_2; b|(\xi)_2; b\rangle,\end{aligned}$$

where the overlap function has the standard form [16]

$$\langle(\bar{\xi}')_2; b|(\xi)_2; b\rangle = \exp\left\{\frac{1}{2}[\bar{\xi}'_l, \xi_l]\right\}. \tag{8.11}$$

The expression on the right-hand side of relation (8.10) is inevitably cumbersome in this approach. This is ultimately a consequence of "implicating" the coherent state with the opposite sign of para-Grassmann variable ξ_k. Indeed, by acting with operator $[\xi_l, b_l^\dagger]$ on relation (8.10), we obtain

$$[\xi_l, b_l^\dagger]a_k|(\xi)_2; b\rangle = \Lambda\xi_k(|(\xi)_2; b\rangle + |(-\xi)_2; b\rangle).$$

By virtue of relation (5.8), this expression can also be written as

$$a_k[\xi_l, b_l^\dagger]|(\xi)_2; b\rangle = \Lambda\xi_k(|(\xi)_2; b\rangle - |(-\xi)_2; b\rangle)$$

(it should be recalled that the summation over repeated indices is implied). In turn, the state $|(-\xi)_2;\, b\rangle$ can be represented as a result of action of the parafermion operator of the so-called G-parity $(-1)^N$ with particle number operator (1.7) on the initial coherent state $|(\xi)_2;\, b\rangle$; i.e.,

$$(-1)^N|(\xi)_2; b\rangle = |(-\xi)_2; b\rangle.$$

It should be noted that such states were also considered in [21] using usual Fermi statistics in the context of construction of the worldline path integral for the imaginary part of the effective action, i.e., of the phase of the fermion functional determinant. It is also noteworthy that the G-operator appears in the so-called deformed Heisenberg algebra (Calogero–Vasiliev operator) [22–24] involving reflection operator $R = (-1)^N$ and deformation parameter $\nu \in \mathbb{R}$. It was shown in [25] that this single-mode algebra has finite-dimensional representations of a certain deformed parafermion algebra that can be reduced to the standard parafermion algebra of order 2 for deformation parameter $\nu = -3$. This may indicate a certain relationship between the Govorkov unitary quantization and the deformed Heisenberg algebra.

In Section 6, we analyzed the mappings of various trilinear commutation relations. Here, we consider the mapping of coherent states (8.1) and (8.4). It would be interesting to find out whether the coherent states are interrelated by a transformation of the type (6.6) (or (6.19)). To be specific, we take (8.1) as the starting expression and choose the following relation as a transformation connecting operators a_k and b_k:

$$a_k = -e^{\pi i\tilde{N}/2} b_k e^{-\pi i\tilde{N}/2}. \tag{8.12}$$

The final expression has a simple form, but its derivation is slightly laborious.

Taking into account relation (8.12), we can write formula (8.2) in the form (for $p = 2$)

$$-\exp\left\{\frac{\pi i\tilde{N}}{2}\right\} b_k \exp\left\{-\frac{\pi i\tilde{N}}{2}\right\} \times\exp\left\{-\frac{1}{2}[\xi_l, a_l^\dagger]\right\}|0\rangle = \xi_k \exp\left\{-\frac{1}{2}[\xi_l, a_l^\dagger]\right\}|0\rangle.$$

By acting with operator $e^{-\pi i\tilde{N}/2}$ on the left and inserting unit operator

$$I \equiv e^{\pm\pi i\tilde{N}/2} e^{\mp\pi i\tilde{N}/2},$$

in front of vacuum state $|0\rangle$ on the left-hand side and, in addition, the unit operator into the right-hand side between ξ_k and the exponential function, we obtain

$$\begin{aligned} &b_k\left(\exp\left\{-\frac{\pi i\tilde{N}}{2}\right\}\exp\left\{-\frac{1}{2}[\xi_l, a_l^\dagger]\right\}\exp\left\{\frac{\pi i\tilde{N}}{2}\right\}\right) \\ &\times\exp\left\{-\frac{\pi i\tilde{N}}{2}\right\}|0\rangle = -\left(\exp\left\{-\frac{\pi i\tilde{N}}{2}\right\}\xi_k \right.\\ &\left.\times\exp\left\{-\frac{\pi i\tilde{N}}{2}\right\}\right)\left(\exp\left\{\frac{\pi i\tilde{N}}{2}\right\}\exp\left\{-\frac{1}{2}[\xi_l, a_l^\dagger]\right\}\right.\\ &\left.\times\exp\left\{-\frac{\pi i\tilde{N}}{2}\right\}\right)\exp\left\{\frac{\pi i\tilde{N}}{2}\right\}|0\rangle. \end{aligned} \tag{8.13}$$

Using operator identity (B.9), we get

$$\begin{aligned} &\exp\left\{\pm\frac{\pi i\tilde{N}}{2}\right\}\exp\left\{-\frac{1}{2}[\xi_l, a_l^\dagger]\right\}\exp\left\{\mp\frac{\pi i\tilde{N}}{2}\right\} \\ &= \exp\left[-\frac{1}{2}\left(\exp\left(\pm\frac{\pi i\tilde{N}}{2}\right)[\xi_l, a_l^\dagger]\right.\right.\\ &\left.\left.\times\exp\left(\pm\frac{\pi i\tilde{N}}{2}\right)\right)\exp(\mp\pi i\tilde{N})\right] \\ &= \exp\left(\pm\frac{1}{2}\exp\left(\pm\frac{\pi\tilde{\Lambda}}{2}\right)\{\xi_l, b_l^\dagger\}\exp(\mp\pi i\tilde{N})\right). \end{aligned}$$

At the last step, we have used relation (6.11). By virtue of relation (6.9), the expression in the first parentheses on the right-hand side of (8.13) has the form

$$e^{-\pi i\tilde{N}/2}\xi_k e^{-\pi i\tilde{N}/2} = e^{-\pi\tilde{\Lambda}/2}\xi_k.$$

Taking into account the above arguments, we obtain instead of relation (8.13)

$$\begin{aligned} &b_k \exp\left(-\frac{1}{2}\exp\left(-\frac{\pi\tilde{\Lambda}}{2}\right)\{\xi_l, b_l^\dagger\}\exp(\pi i\tilde{N})\right) \\ &\times\exp\left(-\frac{\pi i\tilde{N}}{2}\right)|0\rangle = -\exp\left(-\frac{\pi\tilde{\Lambda}}{2}\right)\xi_k \\ &\times\exp\left(\frac{1}{2}\exp\left(\frac{\pi\tilde{\Lambda}}{2}\right)\{\xi_l, b_l^\dagger\}\exp(-\pi i\tilde{N})\right)\exp\left(\frac{\pi i\tilde{N}}{2}\right)|0\rangle. \end{aligned} \tag{8.14}$$

In Section 7, we have derived the rule of action of operator $i\tilde{N}$ on the vacuum state:

$$i\tilde{N}|0\rangle = \frac{1}{2}\tilde{\Lambda}|0\rangle. \tag{8.15}$$

Using this rule, we obtain

$$e^{\mp\pi i\tilde{N}/2}|0\rangle = e^{\mp\pi\tilde{\Lambda}/2}|0\rangle.$$

It remains for us to analyze the exponential operator

$$\exp\left(\mp\frac{1}{2}\exp\left(\mp\frac{\pi\tilde{\Lambda}}{2}\right)\{\xi_l, b_l^\dagger\}\exp(\pm\pi i\tilde{N})\right) \equiv e^A. \tag{8.16}$$

Let us consider the following expansion:

$$e^A = \cosh A + \sinh A = \sum_{s=0}^{\infty}\frac{A^{2s+1}}{(2s+1)!} + \sum_{s=0}^{\infty}\frac{A^{2s}}{(2s)!}. \tag{8.17}$$

At first we define the explicit form of operator A^2:

$$A^2 = \left(\mp\frac{1}{2}\right)^2 e^{\mp\pi\tilde{\Lambda}}\{\xi_l, b_l^\dagger\}(e^{\pm\pi i\tilde{N}}\{\xi_{l'}, b_{l'}^\dagger\}e^{\pm\pi i\tilde{N}}) = (-1)\left(\mp\frac{1}{2}\right)^2\{\xi_l, b_l^\dagger\}^2.$$

In deriving this expression, we have taken into account the fact that by virtue of relations (6.3) and (6.9), the following equality holds:

$$e^{\pm\pi i\tilde{N}}\{\xi_{l'}, b_{l'}^\dagger\}e^{\pm\pi i\tilde{N}} = (-1)e^{\pm\pi\tilde{\Lambda}}\{\xi_{l'}, b_{l'}^\dagger\}.$$

Therefore, we have

$$A^{2s} = (-1)^s\left(\mp\frac{1}{2}\right)^{2s}\{\xi_l, b_l^\dagger\}^{2s}$$

and

$$A^{2s+1} = (-1)^s\left(\mp\frac{1}{2}\right)^{2s+1}\{\xi_l, b_l^\dagger\}^{2s+1}e^{\mp\pi\tilde{\Lambda}/2}e^{\pm\pi i\tilde{N}}.$$

Substituting the resulting expressions into (8.17), we find that exponential operator (8.16) can be written in the form

$$\exp\left(\mp\frac{1}{2}\exp\left(\mp\frac{\pi\tilde{\Lambda}}{2}\right)\{\xi_l, b_l^\dagger\}\exp(\pm\pi i\tilde{N})\right) = \cos\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right) \mp \sin\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right) \times\exp\left(\mp\frac{\pi\tilde{\Lambda}}{2}\right)\exp(\pm\pi i\tilde{N}).$$

On the right-hand side of this expression, the action of operator $e^{\pm\pi i\tilde{N}}$ on the vacuum state is defined by formula (8.15). In view of the above arguments, the basic expression (8.14) takes the form

$$b_k\left[\cos\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right) - \sin\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right)\right]|0\rangle = -\xi_k\left[\cos\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right) + \sin\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right)\right]|0\rangle. \quad (8.18)$$

It should be emphasized that all exponential factors containing constant $\tilde{\Lambda}$ have been cancelled out exactly. This is an indirect proof of the correctness of our line of reasoning. Slightly cumbersome expression (8.18) becomes an identity if the following relation holds[4]

$$b_k\exp\left(\pm\frac{1}{2}i\{\xi_l, b_l^\dagger\}\right)|0\rangle = (\pm i)\xi_k\exp\left(\pm\frac{1}{2}i\{\xi_l, b_l^\dagger\}\right)|0\rangle. \quad (8.19)$$

Thus, the mapping of coherent state (8.1) with (8.2) leads not to coherent state (8.4) with (8.5), but to expression (8.19). This is in full agreement with the rule established in Section 6: if para-Grassmann number ξ_k appears in a commutator (anticommutator) together with operator a_k or b_k (or their conjugates), in addition to the replacement $a_k \rightleftharpoons b_k$ in mapping (6.6) or (6.19), we must replace the commutator (anticommutator) by the anticommutator (commutator). Clearly, factor ($\pm i$) in expression (8.19) is immaterial in this case.

Let us now prove relation (8.19) by direct calculation. Omitting factor ($\pm i$) on the left- and right-hand sides, we can write the relation in the form

$$b_k\exp\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right)|0\rangle = \xi_k\exp\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right)|0\rangle. \quad (8.20)$$

To prove this relation, it is sufficient to consider the following expression:

$$\begin{aligned}\exp\left(-\frac{1}{2}\{\xi_l, b_l^\dagger\}\right)b_k\exp\left(-\frac{1}{2}\{\xi_{l'}, b_{l'}^\dagger\}\right) &= b_k + \left(-\frac{1}{2}\right)\{\{\xi_l, b_l^\dagger\}, b_k\}\\ &+ \frac{1}{2!}\left(-\frac{1}{2}\right)^2\{\{\xi_l, b_l^\dagger\},\{\{\xi_{l'}, b_{l'}^\dagger\}, b_k\}\} + \dots \quad (8.21)\\ &= b_k + \left(-\frac{1}{2}\right)2\xi_k + \frac{1}{2!}\left(-\frac{1}{2}\right)^2\times 2\{\{\xi_l, b_l^\dagger\},\xi_k\} + \dots\\ &= b_k - \xi_k.\end{aligned}$$

Here, we have used trilinear commutation rules (6.17) and (6.13), which hold for $p = 2$. The series in expression (8.21) is truncated precisely after the second term of the expansion as in the calculation of operator relation (8.3) for the standard definition of coherent state (8.1), (8.2). An analog of expression (8.3) is now

$$b_k\exp\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right) = \exp\left(-\frac{1}{2}\{\xi_l, b_l^\dagger\}\right)b_k + \xi_k\exp\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right).$$

Acting on vacuum state $|0\rangle$ with the above expression, we arrive at formula (8.20). Further, we can prove that instead of expression (8.8), we obtain

$$a_k\exp\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right) = \exp\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right)a_k - \Lambda\xi_k\left(\sum_s\frac{(-1)^s}{(s+1)!}\{\xi_m, b_m^\dagger\}^s\right)\exp\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right).$$

Finally, we can obtain the overlap function for the "coherent" state $\exp\left(\frac{1}{2}\{\xi_l, b_l^\dagger\}\right)|0\rangle$. After cumbersome calculations that are omitted here, we get

[4] The relation of form (8.19) is naturally not unique. For example, the relation $b_k\exp\left(\pm\frac{1}{2}i\{\xi_l, b_l^\dagger\}\right)|0\rangle = -\xi_k\exp\left(\mp\frac{1}{2}i\{\xi_l, b_l^\dagger\}\right)|0\rangle$ also converts relation (8.18) into an identity. However, the sign in the exponential function on the right-hand side has changed, and there is no factor i in front of ξ_k. In addition, it can be verified by direct calculations that in contrast to (8.19), this relation does not hold.

$$\langle 0|\exp\left(\frac{1}{2}\{\bar{\xi}'_l, b_l^\dagger\}\right)\exp\left(\frac{1}{2}\{\xi_{l'}, b_{l'}^\dagger\}\right)|0\rangle = \exp\left(\frac{1}{2}[\bar{\xi}'_l, \xi_l]\right).$$

This should be expected because the right-hand side in expression (8.11) is invariant under mapping (6.6) (or (6.19)).

9. UNITARY TRANSFORMATIONS

It was shown in [26] that trilinear relations (2.1)–(2.3) for single para-Fermi field ϕ_a can be derived from the requirement that the equations

$$[a_k, N] = a_k, \quad [a_k^\dagger, N] = -a_k^\dagger, \tag{9.1}$$

where N is the particle number operator (1.7), be invariant under unitary transformation of a_k operators:

$$a'_k = a_k + \sum_{l=1}^{M}\alpha_{kl}a_l.$$

Here, infinitesimal transformation parameters α_{kl} obey the condition

$$\alpha_{kl} + \alpha_{lk}^* = 0. \tag{9.2}$$

The following question arises: can trilinear Govorkov relations (3.1)–(3.3) containing operators of two different para-Fermi fields ϕ_a and ϕ_b be obtained proceeding from the requirement of the invariance of the equations

$$[i\tilde{N}, a_k] = b_k, \quad [i\tilde{N}, b_k] = -a_k \tag{9.3}$$

under an infinitesimal linear transformation of operators a_k and b_k? For convenience of further analysis, we write once again the explicit form of operator $i\tilde{N}$:

$$i\tilde{N} = \varrho\sum_{k=1}^{M}([a_k^\dagger, b_k] + \lambda), \quad \varrho \equiv -\frac{1}{2(2M+1)}. \tag{9.4}$$

Clearly, the required transformation leaving Eqs. (9.3) invariant must "mix" operators a_k and b_k, i.e., must have the form

$$\begin{aligned} a'_k &= a_k + \alpha_{kl}b_l, \quad a'^\dagger_k = a_k^\dagger - \alpha_{lk}b_l^\dagger, \\ b'_k &= b_k + \beta_{kl}a_l, \quad b'^\dagger_k = b_k^\dagger - \beta_{lk}a_l^\dagger. \end{aligned} \tag{9.5}$$

Here, we have omitted for brevity the summation symbols and required that infinitesimal transformation parameters α_{kl} and β_{kl} satisfy condition (9.2). The commutator in expression (9.4) can be written in the form

$$[a'^\dagger_k, b'_k] = [a_k^\dagger, b_k] - \alpha_{lk}[b_l^\dagger, b_k] + \beta_{kl}[a_k^\dagger, a_l].$$

The requirement of invariance of the first equation in (9.3) leads to the following relation:

$$\alpha_{ms}[b_s, i\tilde{N}] - \varrho\,\alpha_{lk}[a_m, [b_l^\dagger, b_k]] + \varrho\,\beta_{kl}[a_m, [a_k^\dagger, a_l]] = \beta_{ms}a_s.$$

The commutator in the first term on the left-hand side is equal to $-a_s$ by virtue of Eqs. (9.3), while the double commutator in the third term is equal to $2\delta_{mk}a_l$ in view of (2.1). Therefore, we can represent the above expression as

$$-\varrho\,\alpha_{lk}[a_m, [b_l^\dagger, b_k]] = \delta_{ml}(\beta_{lk}(1-2\varrho) + \alpha_{lk})a_k.$$

We now take the next step and require the fulfillment of the additional condition connecting parameters α_{kl} and β_{kl}:

$$\alpha_{kl} = -\beta_{kl}.$$

Only in this particular case, parameter ϱ on the left- and right-hand sides is cancelled out exactly, and we arrive at

$$[a_m, [b_l^\dagger, b_k]] = -2\delta_{ml}a_k.$$

The requirements of the invariance of the second equation in (9.3) leads to the analogous expression

$$[[a_k^\dagger, a_l], b_m] = -2\delta_{km}b_l. \tag{9.6}$$

Therefore, we have reproduced Govorkov's relation (3.3).

The form of transform (9.5) is more visual in matrix notation

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix},$$

where $a = (a_1, \ldots, a_M)^T$, $b = (b_1, \ldots, b_M)^T$ (T is the sign of the transposition), and $\alpha = (\alpha_{kl})$. The matrix

$$X = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

satisfies the condition $X^\dagger = X$ and, hence, belongs to the algebra of unimodular group $SU(2M)$, which is quite natural since Govorkov's relations were obtained using the field quantization based on the relations of the Lie algebra of group $SU(2M+1)$.

However, it remains unclear whether one trilinear relation (9.6) can be used for obtaining other relations (3.1) or (3.2). In the case of single para-Fermi field ϕ_a, the answer is positive [26]. Indeed, the requirement of invariance of equations (9.1) leads to relation (2.1). The use of the Jacobi identity and relation (2.1) is sufficient for restoring the other trilinear relation (2.3). In the case of (9.6), the Jacobi identity gives

$$[[b_m, a_k^\dagger], a_l] + [[a_l, b_m], a_k^\dagger] = 2\delta_{km}b_l \tag{9.7}$$

and in contrast to the case of a single field, here we obtain two different expressions on the left-hand side, and it is not clear a priory how these expressions can be "uncoupled."

Greenberg and Messiah [15] proposed their own approach to uncoupling relations of type (9.7). This was mentioned in Section 2. Let us consider this approach in more detail. In this case, (2.4) is the initial relation. The application of the Jacobi identity leads to

$$[[b_m, a_k^\dagger], a_l] + [[a_l, b_m], a_k^\dagger] = 0. \quad (9.8)$$

In uncoupling of this relation, Greenberg and Messiah used conditions (i) and (iii) from Section 2. In particular, condition (iii) requires that the desired trilinear relation be satisfied for the ordinary Bose and Fermi fields.

Let operator a_k and b_k be operators of the Fermi oscillators, i.e., satisfy the usual commutation relations

$$\{a_k^\dagger, a_l\} = \delta_{kl}, \quad \{b_m^\dagger, b_n\} = \delta_{mn}, \quad (9.9)$$

$$\{a_k, a_l\} = 0, \quad \{b_m, b_n\} = 0, \ldots, \quad (9.10)$$

$$\{a_k^\dagger, b_m\} = 0, \quad \{a_k, b_m\} = 0, \ldots. \quad (9.11)$$

Using identity (B.2) for the first term on the left-hand side of relation (9.8), we obtain

$$[[b_m, a_k^\dagger], a_l] = \{b_m, \{a_l, a_k^\dagger\}\} - \{a_k^\dagger, \{a_l, b_m\}\} = 2\delta_{lk} b_m. \quad (9.12)$$

accordingly, for the second term, we get

$$[[a_l, b_m], a_k^\dagger] = \{a_l, \{a_k^\dagger, b_m\}\} - \{b_m, \{a_k^\dagger, a_l\}\} = -2\delta_{lk} b_m. \quad (9.13)$$

The expressions on the right-hand sides of relations (9.12) and (9.13) are simply postulated as the right-hand sides in trilinear relations for para-Fermi oscillators a_k and b_k as was done in relations (2.7) and (2.8).

Using this approach for relation (9.7), we clearly see that the standard system of commutation rules (9.9)–(9.11) is inapplicable in this case. This system must be somehow modified for reproducing the right-hand sides of trilinear relations (3.1) and (3.2). Relations (9.9) and (9.10) must remain unchanged, and instead of the first relation in (9.11), we should consider, for example,

$$\{a_k^\dagger, b_m\} = -2\delta_{km}\mathrm{G}.$$

Here, we have introduced an additional algebraic quantity G such that

$$\{a_l, \mathrm{G}\} = b_l.$$

In this case, we find that the following relation holds instead of (9.13):

$$[[a_l, b_m], a_k^\dagger] = -2\delta_{km} b_l - 2\delta_{kl} b_m.$$

However, relation (9.12) still remains unchanged. This means that in contrast to the system of the Greenberg–Messiah trilinear relations, Govorkov's trilinear relations cannot in principle be reduced to simpler bilinear relations of the usual Fermi statistics even after a modification of bilinear relations containing different fields. Therefore, the rule for uncoupling relations (9.7) still remains unclear.

10. KLEIN TRANSFORMATION

The so-called Klein transformation [29–34] of Green's components of a para-Fermi field of arbitrary order p was constructed in [27] (see also [28]). This transformation has made it possible to reduce initial relations (2.10) that contain both commutators and anticommutators to the normal commutation relations for p ordinary Fermi fields. It was shown in particular in [27] that for the reduction to normal commutation rules, it is necessary to consider $p/2$ Klein's operator H_{2j}, $j = 1, \ldots, p/2$ for even p and $(p - 1)/2$ operators for odd p. Therefore, for a parastatistics of order $p = 2, 3$, one Klein's operator H_2 is required, while for $p = 4$, two operators H_2 and H_4 need to be used.

In the problem with two parafields ϕ_a and ϕ_b of order $p = 2$ considered here, we can assume that at least two Klein's operators denoted by $H_A^{(2)}$ and $H_B^{(1)}$ are required (the meaning of these notations will be explained below). We must define the Klein transformation of Green's components $a_k^{(\alpha)}$ and $b_m^{(\alpha)}$ so that both commutation relations (2.10) separately for each set $\{a_k^{(\alpha)}\}$ and $\{b_m^{(\alpha)}\}$ and commutation relations (3.8)–(3.11) of the mixed type could be simultaneously reduced to the normal form. We assume that the required Klein transformation has the form

$$\begin{aligned} a_k^{(1)} &= A_k^{(1)} H_A^{(2)}, \quad & b_m^{(1)} &= -iB_m^{(1)} H_B^{(1)}, \\ a_k^{(2)} &= iA_k^{(2)} H_A^{(2)}, \quad & b_m^{(2)} &= B_m^{(2)} H_B^{(1)}, \end{aligned} \quad (10.1)$$

where $A_k^{(\alpha)}$ and $B_m^{(\alpha)}$ are the new Green components satisfying the following commutation rules with Klein operators $(H_A^{(2)}, H_B^{(1)})$:

$$\begin{cases} [A_k^{(1)}, H_A^{(2)}] = 0, \quad \{A_k^{(1)}, H_B^{(1)}\} = 0, \\ \{A_k^{(2)}, H_A^{(2)}\} = 0, \quad [A_k^{(2)}, H_B^{(1)}] = 0, \end{cases}$$
$$\begin{cases} \{B_m^{(1)}, H_B^{(1)}\} = 0, \quad [B_m^{(1)}, H_A^{(2)}] = 0, \\ [B_m^{(2)}, H_B^{(1)}] = 0, \quad \{B_m^{(2)}, H_A^{(2)}\} = 0. \end{cases} \quad (10.2)$$

At the same time, Klein's operators themselves satisfy the conditions

$$(H_A^{(2)})^2 = (H_B^{(1)})^2 = I, \quad [H_A^{(2)}, H_B^{(1)}] = 0. \quad (10.3)$$

The Klein operators will be written in explicit form later. We will just show now that Klein's transformation (10.1) with rules (10.2) and (10.3) indeed gives the required result.

It can easily be verified that commutation relations (2.10) are reduced by transformation (10.1) to the normal form. This enables us to write these relations as

$$\begin{gathered}\{A_k^{(\alpha)}, A_l^{\dagger(\alpha)}\} = \delta_{kl}, \quad \{A_k^{(\alpha)}, A_l^{\dagger(\beta)}\} = 0, \\ \{B_m^{(\alpha)}, B_n^{\dagger(\alpha)}\} = \delta_{mn}, \quad \{B_m^{(\alpha)}, B_n^{\dagger(\beta)}\} = 0, \\ \{A_k^{(\alpha)}, A_l^{(\beta)}\} = 0, \quad \alpha \neq \beta, \\ \{B_m^{(\alpha)}, B_n^{(\beta)}\} = 0.\end{gathered} \tag{10.4}$$

Therefore, we concentrate our attention on analysis of system (3.8)–(3.11). Let us consider relation (3.8) for $\alpha = 1$. Direct substitution of transformation (10.1) into (3.8) with allowance for (10.2) and (10.3) gives

$$[b_m^{(1)}, a_k^{\dagger(1)}] = iH_A^{(2)}\{B_m^{(1)}, A_k^{\dagger(1)}\}H_B^{(1)} = 2\delta_{mk}\Omega. \tag{10.5}$$

This expression suggests that operator Ω should also be transformed. Therefore, Klein's transformation (10.1) need to be supplemented with the following rule:

$$\Omega = H_A^{(2)}\tilde{\Omega}H_B^{(1)}, \tag{10.6}$$

where $\tilde{\Omega}$ is a new operator. A relation analogous to (10.5) also holds for $\alpha = 2$. Therefore, instead of (3.8), we now have

$$\{B_m^{(\alpha)}, A_k^{\dagger(\alpha)}\} = \frac{2}{i}\delta_{mk}\tilde{\Omega}. \tag{10.7}$$

It can now be easily verified that instead of relations (3.9), we obtain

$$\{A_k^{(\alpha)}, B_m^{(\alpha)}\} = 0, \quad \{A_k^{\dagger(\alpha)}, B_m^{\dagger(\alpha)}\} = 0, \tag{10.8}$$

and relations (3.10) combined with (10.6) are transformed into

$$\{A_k^{(\alpha)}, \tilde{\Omega}\} = iB_k^{(\alpha)}, \quad \{B_m^{(\alpha)}, \tilde{\Omega}\} = iA_m^{(\alpha)}. \tag{10.9}$$

Pay attention to the fact that the signs on the right-hand sides of these expressions are identical in contrast to relations (3.10). Finally, anticommutation relations (3.11) remain unchanged under the Klein transformation, and we simply perform the substitution

$$a_k^{(\alpha)} \to A_k^{(\alpha)} \quad \text{and} \quad b_m^{(\alpha)} \to B_m^{(\alpha)}.$$

Let us now define the explicit form of Klein operators $H_A^{(2)}$ and $H_B^{(1)}$. For this purpose, we write the parafermion number operators (1.7) in terms of new Green's components $A_k^{(\alpha)}$ and $B_m^{(\alpha)}$. To be specific, we consider the particle number operator for b para-Fermi oscillators. For $p = 2$, we have

$$N_b = \frac{1}{2}\sum_{m=1}^{M}([b_m^{\dagger(1)}, b_m^{(1)}] + [b_m^{\dagger(2)}, b_m^{(2)}]) + M.$$

Substituting Klein's transformation (10.1) into the previous expression and taking into account relations (10.2)–(10.4), we find that this parafermion particle number operator can be written as

$$N_b = N_B^{(1)} + N_B^{(2)},$$

where

$$N_B^{(\alpha)} \equiv \frac{1}{2}\sum_{m=1}^{M}[B_m^{\dagger(\alpha)}, B_m^{(\alpha)}] + \frac{1}{2}M, \quad \alpha = 1, 2 \tag{10.10}$$

are the particle number operators for ordinary fermions. The explicit form of Klein operators $H_A^{(2)}$ and $H_B^{(1)}$ is defined by the following expressions:

$$H_A^{(2)} = (-1)^{N_A^{(2)}}, \quad H_B^{(1)} = (-1)^{N_B^{(1)}}.$$

Most commutation relations in (10.2) obviously hold. Only two of them require special consideration, namely,

$$\{A_k^{(1)}, H_B^{(1)}\} = 0, \quad \{B_m^{(2)}, H_A^{(2)}\} = 0. \tag{10.11}$$

Let us consider the first of these relations. Above all, we write Klein operator $H_B^{(1)}$ in a slightly different form:

$$H_B^{(1)} = e^{i\pi N_B^{(1)}}.$$

Then the considered expression can be written as

$$\{A_k^{(1)}, H_B^{(1)}\} = (e^{i\pi N_B^{(1)}}A_k^{(1)}e^{-i\pi N_B^{(1)}} + A_k^{(1)})e^{i\pi N_B^{(1)}}. \tag{10.12}$$

Further, using identity (B.7), we obtain

$$\begin{aligned}e^{i\pi N_B^{(1)}}A_k^{(1)}e^{-i\pi N_B^{(1)}} &= A_k^{(1)} + i\pi[N_B^{(1)}, A_k^{(1)}] \\ &+ \frac{1}{2!}(i\pi)^2[N_B^{(1)}, [N_B^{(1)}, A_k^{(1)}]] + \dots.\end{aligned} \tag{10.13}$$

Therefore, the proof boils down to the calculation of commutator $[N_B^{(1)}, A_k^{(1)}]$. Using the definition of fermion number operator (10.10) and identity (B.2), we get

$$\begin{aligned}[N_B^{(1)}, A_k^{(1)}] &= \frac{1}{2}\sum_{m=1}^{M}[[B_m^{\dagger(1)}, B_m^{(1)}], A_k^{(1)}] \\ &= \frac{1}{2}\sum_{m=1}^{M}(\{B_m^{\dagger(1)}, \{A_k^{(1)}, B_m^{(1)}\}\} - \{B_m^{(1)}, \{A_k^{(1)}, B_m^{\dagger(1)}\}\}).\end{aligned}$$

The first term on the right-hand side is equal to zero by virtue of relation (10.8), while in view of relations (10.7) and (10.9), for the second term we obtain

$$\{B_m^{(1)}, \{A_k^{(1)}, B_m^{\dagger(1)}\}\} = \frac{2}{i}\delta_{km}\{B_m^{(1)}, \tilde{\Omega}\} = 2\delta_{km}A_m^{(1)}.$$

Thus, the desired commutator is given by

$$[N_B^{(1)}, A_k^{(1)}] = -A_k^{(1)}.$$

Substituting this relation into (10.13), we get

$$\begin{aligned}e^{i\pi N_B^{(1)}}A_k^{(1)}e^{-i\pi N_B^{(1)}} &= A_k^{(1)}\left(1 - i\pi + \frac{1}{2!}(i\pi)^2 - \frac{1}{3!}(i\pi)^3 + \dots\right) \\ &\equiv A_k^{(1)}e^{-i\pi} = -A_k^{(1)}.\end{aligned}$$

Thus, the right-hand side of equality (10.12) vanishes indeed. The second relation in (10.11) can be proved analogously.

11. LIE SUPERTRIPLE SYSTEM

In this section, we will consider an interesting link between trilinear Govorkov relations (3.1)–(3.3) and the so-called Lie supertriple system. Such a system, which is a generalization of the standard Lie triple system [35–38], was studied in detail in [39–42]. Our analysis will be based on publication [39], in which the author reformulated the parastatistics as a Lie supertriple system. A number of examples of such a reformulation were also given in [39]. We are especially interested in Example 3 from [39] (the explicit formulation of this example is given below). We will require a few definitions (in the notation used in [39]).

Let V be a vector superspace, which has the form of direct sum

$$V = V_B \oplus V_F.$$

In this superspace, we introduce the grade

$$\sigma(x) = \begin{cases} 0, & \text{for} \quad x \in V_B \\ 1, & \text{for} \quad x \in V_F, \end{cases} \tag{11.1}$$

and triple superproduct [..., ..., ...] is defined as a trilinear mapping

$$[\ldots,\ldots,\ldots]; \quad V \otimes V \otimes V \to V.$$

The triple superproduct obeys three conditions that can be found in [39]. If $V_f = 0$ (i.e., $V = V_B$), triple product $[x, y, z]$ leads to the standard Lie triple system.

In addition, we assume that vector superspace V always possesses bilinear form $\langle x|y\rangle$ satisfying the conditions

$$\begin{gathered} \langle x|y\rangle = (-1)^{\sigma(x)\sigma(y)}\langle y|x\rangle, \\ \langle x|y\rangle = 0 \quad \text{if} \quad \sigma(x) \neq \sigma(y). \end{gathered} \tag{11.2}$$

We now consider the formulation of Example 3 from [39]. Let $P : V \to V$ be a grade-preserving linear map in V, i.e.,

$$\sigma(Px) = \sigma(x) \quad \text{for any} \quad x \in V \tag{11.3}$$

we also assume the validity of relations

$$P^2 = \lambda I, \tag{11.4}$$

$$\langle x|Py\rangle = -\langle Px|y\rangle, \tag{11.5}$$

where I is the identity mapping in V and λ is a nonzero constant. The expression

$$\begin{aligned} [x, y, z] &= \langle y|Pz\rangle Px - (-1)^{\sigma(x)\sigma(y)}\langle x|Pz\rangle Py \\ &- 2\langle x|Py\rangle Pz + \lambda\langle y|z\rangle x - (-1)^{\sigma(x)\sigma(y)}\lambda\langle x|z\rangle y \end{aligned} \tag{11.6}$$

for the triple product transforms V into a Lie supertriple system. It should be noted that the same constant for λ appears in condition (11.4) as well as in the definition (11.6) of the triple product.

Let us prove that the Govorkov trilinear relations (3.1)–(3.3) are particular cases of general expression (11.6). In addition, the triple product also contains the standard trilinear relations for single field ϕ_a (and ϕ_b) (2.1)–(2.3). At the first step, we fix two sets of operators

$$(a_k, a_k^\dagger) \quad \text{and} \quad (b_k, b_k^\dagger), \quad k = 1,\ldots,M,$$

between which we specify a map P in accordance with the rule (cf. (5.11))

$$\begin{aligned} Pa_k &= b_k, \quad Pb_k = -a_k, \\ Pa_k^\dagger &= b_k^\dagger, \quad Pb_k^\dagger = -a_k^\dagger. \end{aligned} \tag{11.7}$$

It follows, hence, that

$$P^2 a_k = -a_k, \quad P^2 b_k = -b_k,$$

and, in view of condition (11.4), constant λ is fixed uniquely:

$$\lambda = -1. \tag{11.8}$$

Let us consider the second condition in (11.5). We set $x = a_k^\dagger$ and $y = b_m$; by virtue of relations (11.7), the condition for bilinear form $\langle\cdot|\cdot\rangle$ can be reduced to

$$\langle a_k^\dagger|a_m\rangle = \langle b_k^\dagger|b_m\rangle. \tag{11.9}$$

We fix the grade

$$\sigma(a_k) = \sigma(a_k^\dagger) = 0, \quad \sigma(b_m) = \sigma(b_m^\dagger) = 0,$$

then the first condition in (11.2) gives

$$\langle x|y\rangle = \langle y|x\rangle \quad \text{for any} \quad x, y \in V.$$

Thus, $V_F = 0$ and $V = V_B$. We choose bilinear form $\langle x|y\rangle$ so that the following conditions are satisfied:

$$\begin{aligned} \langle a_k^\dagger|a_m\rangle &= \langle a_m|a_k^\dagger\rangle = -2\delta_{km}, \\ \langle b_k^\dagger|b_m\rangle &= \langle b_m|b_k^\dagger\rangle = -2\delta_{km}, \\ \langle a_k^\dagger|a_m^\dagger\rangle &= \langle a_k|a_m\rangle = 0, \\ \langle b_k^\dagger|b_m^\dagger\rangle &= \langle b_k|b_m\rangle = 0, \\ \langle a_k^\dagger|b_m\rangle &= \langle a_k|b_m\rangle = 0, \\ \langle b_k^\dagger|a_m\rangle &= \langle b_k^\dagger|a_m^\dagger\rangle = 0. \end{aligned} \tag{11.10}$$

Condition (11.9) holds automatically.

We now return to basic relation (11.6) and set $x = a_k^\dagger$, $y = a_l$, and $z = b_m$ in it. By virtue of relations (11.7), (11.8), and (11.10), we then get

$$\begin{aligned} [a_k^\dagger, a_l, b_m] &= -\langle a_l|a_m\rangle b_k^\dagger + \langle a_k^\dagger|a_m\rangle b_l + 2\langle a_k^\dagger|b_l\rangle a_m \\ &- \langle a_l|b_m\rangle a_k^\dagger + \langle a_k^\dagger|b_m\rangle a_l = -2\delta_{km}b_l. \end{aligned}$$

Therefore, we have reproduced trilinear relation (3.3). Further, if we set $x = b_m$, $y = a_k^\dagger$, and $z = a_l$, triple product (11.6) takes the form

$$\begin{aligned} [b_m, a_k^\dagger, a_l] &= -\langle a_k^\dagger|b_l\rangle a_m - \langle b_m|b_l\rangle b_k^\dagger \\ - 2\langle b_m|b_k^\dagger\rangle b_l - \langle a_k^\dagger|a_l\rangle b_m &+ \langle b_m|a_l\rangle a_k^\dagger = 4\delta_{mk}b_l + 2\delta_{kl}b_m, \end{aligned}$$

which gives relation (3.1). It is not difficult to verify that for $x = a_l$, $y = b_m$, and $z = a_k^\dagger$, we reproduce relation (3.2). It can be stated that with rules (11.7), (11.8), and (11.10), all trilinear Govorkov's relations are contained in the single expression (11.6).

To complete our analysis, let us consider the triple product for one set of operators, e.g., for $(a_k, a_k^\dagger)$. Let us suppose that $x = a_k^\dagger$, $y = a_l$, and $z = a_m$; then we obtain from relation (11.6)

$$[a_k^\dagger, a_l, a_m] = -\langle a_l|a_m\rangle a_k^\dagger + \langle a_k^\dagger|a_m\rangle a_l = -2\delta_{km}a_l.$$

We see that the triple product with rules (11.7), (11.8), and (11.10) correctly reproduces the standard trilinear relations of the para-Fermi statistics. This particular case was considered in [39] as Example 2 for the triple product

$$[x, y, z] = \lambda\{\langle y|z\rangle x - (-1)^{\sigma(x)\sigma(y)}\lambda\langle x|z\rangle y\}. \tag{11.11}$$

In fact, this triple product represents the last two terms in relation (11.6). Our case differs, however, from that in [39]. We fix constant λ and the bilinear form as follows:

$$\lambda = -1, \quad \langle a_k^\dagger|a_l\rangle = \langle a_l|a_k^\dagger\rangle = -2\delta_{kl},$$

while this was done in [39] in a different manner:

$$\lambda = 2, \quad \langle a_k^\dagger|a_l\rangle = \langle a_l|a_k^\dagger\rangle = \delta_{kl}.$$

In both cases, triple product (11.11) correctly reproduces the relations for the para-Fermi statistics, but the Govorkov relations are not reproduced in the latter case.

12. RELATIONSHIP WITH THE DUFFIN–KEMMER–PETIAU FORMALISM

In our earlier work [43], we obtained the Fock–Schwinger proper-time representation for inverse operator $\hat{\mathcal{L}}^{-1}$:

$$\frac{1}{\hat{\mathcal{L}}} \equiv \frac{\hat{\mathcal{L}}^2}{\hat{\mathcal{L}}^3} = i\int_0^\infty dT \int \frac{d^2\chi}{T^2}\exp\left\{-iT(\hat{H}(z) - i\epsilon) + \frac{1}{2}\left(T[\chi, \hat{\mathcal{L}}] + \frac{1}{4}T^2[\chi, \hat{\mathcal{L}}]^2\right)\right\}, \quad \epsilon \to +0, \tag{12.1}$$

where

$$\hat{\mathcal{L}} \equiv \hat{\mathcal{L}}(z, D) = A\left(\frac{i}{\varepsilon^{1/3}(z)}\eta_\mu(z)D_\mu + mI\right),$$

and

$$\hat{H}(z) \equiv \hat{\mathcal{L}}^3(z, D)$$

is the Hamilton operator, $D_\mu = \partial_\mu + ieA_\mu(x)$ is the covariant derivative, and χ is the para-Grassmann variable of order $p = 2$ (i.e., $\chi^3 = 0$) with the following integration rules [16]:

$$\int d^2\chi = 0 = \int d^2\chi[\chi, \hat{\mathcal{L}}], \quad \int d^2\chi[\chi, \hat{\mathcal{L}}]^2 = 4\hat{\mathcal{L}}^2.$$

Operator $\hat{\mathcal{L}}(z, D)$ is the cubic root of a certain third-order wave operator in an external electromagnetic field. Matrices $\eta_\mu(z)$ are defined in terms of matrices β_μ of the Duffin–Kemmer–Petiau (DKP) algebra and complex deformation parameter z:

$$\eta_\mu(z) \equiv \left(1 + \frac{1}{2}z\right)\beta_\mu - z\left(\frac{\sqrt{3}}{2}\right)\zeta_\mu,$$

where

$$\zeta_\mu = i[\beta_\mu, \omega] \tag{12.2}$$

and

$$\omega = \frac{1}{(M!)^2}\epsilon_{\mu_1\mu_2\ldots\mu_{2M}}\beta_{\mu_1}\beta_{\mu_2}\ldots\beta_{\mu_{2M}}. \tag{12.3}$$

Completing the calculations, we must proceed to the limit $z \to q$ (q is a primitive cubic root of unity).

One of the main goals of this study was the development of a convenient mathematical technique that would enable us to construct the path integral representation for the inverse operator $\hat{\mathcal{L}}^{-1}(z, D)$ (12.1) in a certain parasuperspace using the DKP approach. The matrix element of operator $\hat{\mathcal{L}}^{-1}(z, D)$ in the appropriate basis of states can be treated as a propagator of a massive vector particle in an external gauge field. Unfortunately, Govorkov's unitary quantization formalism turned out to be unsuitable for this purpose. This problem will be considered below in greater detail.

At the first step, we calculate commutator $[\zeta_\mu, \omega]$, where ζ_μ is defined by Eq. (12.2). Using the algebraic relations[5]

$$\omega^2\beta_\mu + \beta_\mu\omega^2 = \beta_\mu, \quad \omega\beta_\mu\omega = 0, \tag{12.4}$$

we can easily find that

$$[\zeta_\mu, \omega] = i\beta_\mu. \tag{12.5}$$

Comparing expressions (12.2) and (12.5) with (A.17), we find that the simplest way for establishing the relationship between the DKP theory and the unitary quantization is the literal identification of matrices β_μ and ζ_μ from the DKP approach with quantities β_μ and ζ_μ emerging in uniquantization (Eqs. (A.6)). In particular, it follows, hence, that

$$\omega \equiv -\frac{1}{2}\zeta_0. \tag{12.6}$$

We must now verify whether or not relations (A.11)–(A.16) hold if analysis is based only on the DKP formalism. Taking into account relations (12.4), we

[5] All basic formulas for the $\omega - \beta_\mu$ matrix algebra for spin 1 are given in Appendix A of our earlier paper [43].

obtain the following expression for the right-hand side of relation (A.15):

$$[\zeta_\mu, \zeta_\nu] = -[[\beta_\mu, \omega], [\beta_\nu, \omega]] = [\omega, [\beta_\mu, \beta_\nu]]\omega + [\beta_\mu, \beta_\nu].$$

In the $\omega - \beta_\mu$ algebra, the following equality holds:

$$[\omega, [\beta_\mu, \beta_\nu]] = 0 \tag{12.7}$$

and we therefore arrive at relation (A.15). Taking into account the Jacobi identity and relation (12.7), we obtain for bilinear relation (A.16)

$$\begin{aligned}[\zeta_\mu, \beta_\nu] &= -i[[\omega, \beta_\mu], \beta_\nu] \\ &= i([[\beta_\mu, \beta_\nu], \omega] + [[\beta_\nu, \omega], \beta_\mu]) = [\zeta_\nu, \beta_\mu],\end{aligned} \tag{12.8}$$

where the complete coincidence is also observed. However, relation (12.8) in the DKP theory is in fact less stringent. Indeed, let us consider again bilinear relation (A.16) without resorting now to the Jacobi identity. Using the relations

$$\omega\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\omega = \omega\delta_{\mu\nu},$$
$$\beta_\mu\omega\beta_\nu + \beta_\nu\omega\beta_\mu = 0,$$

we obtain

$$\begin{aligned}[\zeta_\mu, \beta_\nu] &= -i[[\omega, \beta_\mu], \beta_\nu] \\ &\equiv -i(\omega\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\omega - \beta_\mu\omega\beta_\nu - \beta_\nu\omega\beta_\mu) = -i\omega\delta_{\mu\nu}\end{aligned}$$

and, hence, can write instead of (A.16)

$$[\zeta_\mu, \beta_\nu] = [\zeta_\nu, \beta_\mu] = -i\omega\delta_{\mu\nu}. \tag{12.9}$$

It is this circumstance that has negative consequences for trilinear relations that will be considered below.

Trilinear relation (A.11) obviously holds in accordance with the DKP algebra:

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu. \tag{12.10}$$

Relation (A.12) also holds because algebra (12.10) is also valid for ζ_μ matrices. Let us now consider the mutual commutation relations between ζ_μ and β_μ. By virtue of relations (12.9) and (12.5), we obtain the following relation for (A.13):

$$[\zeta_\lambda, [\zeta_\mu, \beta_\nu]] = -i\delta_{\mu\nu}[\zeta_\lambda, \omega] \equiv \delta_{\mu\nu}\beta_\lambda,$$

while there should be

$$[\zeta_\lambda, [\zeta_\mu, \beta_\nu]] = 2\delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu + \delta_{\lambda\mu}\beta_\nu.$$

It can be seen that the right-hand sides of the last two relations are different. Trilinear relation (A.13), as well as (A.14), is not valid.

The discrepancy between the unitary quantization scheme and the DKP theory can also be seen if we consider relation (A.10) in which ζ_0 is replaced by ω in accordance with rule (12.6). Taking into account relation (12.9), we get

$$\begin{aligned}\omega &= \frac{i}{2(2M+1)}\sum_{\mu=1}^{2M}[\zeta_\mu, \beta_\mu] \\ &= \frac{1}{2(2M+1)}\omega\sum_{\mu=1}^{2M}\delta_{\mu\mu} = \frac{M}{2M+1}\omega.\end{aligned} \tag{12.11}$$

This relation demonstrates obvious contradiction. Summarizing the above arguments, we can state that in spite of certain similarity between two formalisms, the quantization scheme based on the Duffin–Kemmer–Petiau theory cannot be embedded into the unitary quantization scheme proposed by Govorkov.

However, there is one more possibility associated with the parafermion quantization in accordance with the Lie algebra of orthogonal group $SO(2M+2)$. Such a quantization was considered earlier in [44] (see also [45]). It is important for our analysis that in the case of the $SO(2M+2)$ group, there also appears a certain additional operator denoted in [44] by a_0. This operator should be treated as an analog of operator ζ_0 (1.8). Unfortunately, in contrast to the unitary quantization scheme, we have for the $SO(2M+2)$ group only one set of operators $(a_k, a_k^\dagger)$, which are connected with initial quantities β_μ by relations (A.18). Nevertheless, in this situation we can simply introduce the second set of operators $(b_k, b_k^\dagger)$, setting by definition[6]

$$b_k \equiv [a_0, a_k], \quad b_k^\dagger \equiv [a_0, a_k^\dagger], \quad a_0^\dagger = -a_0.$$

In this case, the trilinear relations in [44], which contain operator a_0, assume the form

$$[a_k^\dagger, b_m] = -2\delta_{km}a_0, \quad [b_m^\dagger, a_k] = 2\delta_{mk}a_0,$$
$$[a_0, b_m] = -a_m, \quad [a_0, b_m^\dagger] = -a_m^\dagger,$$
$$[a_k, b_m] = 0, \quad [a_k^\dagger, b_m^\dagger] = 0$$

and, in particular, we have

$$a_0 = -\frac{1}{4M}\sum_{k=1}^{M}([a_k^\dagger, b_k] + [b_k^\dagger, a_k]). \tag{12.12}$$

In addition, the action of operator a_0 on the vacuum state (cf. (7.6)) was also defined in [44]:

$$a_0|0\rangle = \pm\frac{i}{2}p|0\rangle.$$

Pay attention to the fact that the expression on the right-hand side of relation (12.12) contains a different factor in front of the summation symbol as compared to (1.8). This enables us to eliminate the contradiction in relation (12.11) when operator a_0 was identified with operator ω from the DKP theory. It should also be noted that all quantities ζ_μ are connected with $(b_k, b_k^\dagger)$

[6] We have redefined the operators from [44] for our case as follows: $a_0 \to 2ia_0$, $a_k \to \sqrt{2}a_k$, $b_m \to 2\sqrt{2}ib_m$.

by relation (A.18). In this case, a direct consequence of expression (12.2) is

$$a_0 \equiv -i\omega.$$

All questions associated with parafermion quantization based on the orthogonal $SO(2M + 2)$ group, as well as the relationship between the quantization scheme and the Duffin–Kemmer–Petiau theory, are the subject of separate investigation.

13. CONCLUSIONS

In this article, we have considered various aspects of the relationship between the unitary quantization theory and the parastatistics. In analysis of this relationship, the main emphasis has been laid on the application of the Green decomposition of the creation and annihilation operators as well as para-Grassmann numbers. It turns out that the set of commutation relations derived by Govorkov using uniquantization is quite rigid because it can be related only with a particular case of the parastatistics (namely, para-Fermi statistics of order 2). However, the introduction of a number of additional assumptions and a new operator Ω (see Section 3) was required even in this case. It should also be noted that the case of odd number of dimensions, i.e., unitary group $SU(2M)$, was also considered in [2, 3]. Govorkov proved that the Lie algebra of this unitary group contains the Lie algebra of the simplectic $Sp(2M)$ group, as well as other operators that supplement it to the Lie algebra of the initial $SU(2M)$ group. It is well known [13] that the quantization in accordance with the Lie algebra of simplectic group $Sp(2M)$ corresponds to the paraboson quantization. Therefore, we can formulate an analogous problem of the relationship between the unitary quantization scheme based on the Lie algebra of unitary group $SU(2M)$ and the para-Bose statistics.

In this concluding section, however, we would like to consider a little more detailed one of the consequences of the constructions described in this work (see Section 6), which remained slightly shaded because of the large number of formulas. This consequence is associated with the para-Fermi statistics of order 2 as such and is not a specific feature of the unitary quantization scheme. It turns out that certain trilinear relations containing both a_k (or b_m) and para-Grassmann numbers ξ_k have another equivalent (dual?) representation. This can be illustrated by the following two relations:

$$[\xi_k, [\xi_l, a\ ]] = 0, \quad [a_k, [a_l^\dagger, \xi_m]] = 2\delta_{kl}\xi_m, \tag{13.1}$$

for which the dual relations have the form

$$\{\xi_k, \{\xi_l, a_m\}\} = 0, \quad \{a_k, \{a_l^\dagger, \xi_m\}\} = 2\delta_{kl}\xi_m. \tag{13.2}$$

All these relations turn into identities when the commonly used commutation rules for the Green components of operators a_k and para-Grassmann numbers ξ_k (Eqs. (2.10) and (5.2)) are taken into account.

It was shown in Section 8 that these two formulations of trilinear relations (13.1) and (13.2) lead to the existence of two alternative definitions of the parafermion coherent state, namely,

$$|(\xi)_2\rangle = \exp\left(-\frac{1}{2}\sum_{l=1}^{M}[\xi_l, a_l^\dagger]\right)|0\rangle,$$

$$|(\xi)_2\rangle = \exp\left(\frac{1}{2}\sum_{l=1}^{M}\{\xi_l, a_l^\dagger\}\right)|0\rangle.$$

In both cases, the main property of the coherent state is fulfilled :

$$a_k|(\xi)_2\rangle = \xi_k|(\xi)_2\rangle$$

in addition, the overlap function in both cases has the usual form

$$\langle(\bar{\xi}')_2|(\xi)_2\rangle = e^{(1/2)[\bar{\xi}_l', \xi_l]}.$$

The exact meaning of the emergence of such "twins" remains unclear for us. One of possible reasons of a purely algebraic origin is that only two of main identities (B.1)–(B.4) (namely, (B.2) and (B.3)) are independent. This circumstance and its consequences were analyzed in detail by Lavrov et al. [46]. In particular, Jacobi's identity (B.1) is a consequence of generalized identity (B.2). The latter contains double anticommutators on the right-hand side as in (13.2). This means that relations (13.1) and (13.2) follow from each other for $p = 2$. In any case, we can state that the para-Fermi statistics of order 2 (as well as the ordinary Fermi statistics with $p = 1$) is a very specific case of parastatistics because it possesses the properties that are completely absent for para-Fermi statistics of higher orders ($p \geq 3$).

APPENDIX A

Lie Algebra of $SU(2M + 1)$

The Lie algebra of unitary group $SU(2M + 1)$ has the form [47]

$$[X_{\mu\nu}, X_{\sigma\lambda}] = \delta_{\nu\sigma}X_{\mu\lambda} - \delta_{\mu\lambda}X_{\sigma\nu}, \quad \sum_{\mu=0}^{2M} X_{\mu\mu} = 0, \tag{A.1}$$

where indices μ, ν, ... run through values 0, 1, 2, ..., $2M$. If we introduce a new set of operators

$$F_{\mu\nu} = X_{\mu\nu} - X_{\nu\mu}, \quad F_{\mu\nu} = -F_{\nu\mu},$$

$$\tilde{F}_{\mu\nu} = X_{\mu\nu} + X_{\nu\mu}, \quad \tilde{F}_{\mu\nu} = +\tilde{F}_{\nu\mu},$$

Lie algebra (A.1) assumes a somewhat different form:

$$[F_{\mu\nu}, F_{\sigma\lambda}] = \delta_{\nu\sigma}F_{\mu\lambda} + \delta_{\mu\lambda}F_{\nu\sigma} - \delta_{\mu\sigma}F_{\nu\lambda} - \delta_{\nu\lambda}F_{\mu\sigma}, \tag{A.2}$$

$$[\tilde{F}_{\mu\nu}, \tilde{F}_{\sigma\lambda}] = \delta_{\nu\sigma}F_{\mu\lambda} + \delta_{\mu\lambda}F_{\nu\sigma} + \delta_{\mu\sigma}F_{\nu\lambda} + \delta_{\nu\lambda}F_{\mu\sigma}, \tag{A.3}$$

$$[F_{\mu\nu}, \tilde{F}_{\sigma\lambda}] = \delta_{\nu\sigma}\tilde{F}_{\mu\lambda} - \delta_{\mu\lambda}\tilde{F}_{\nu\sigma} - \delta_{\mu\sigma}\tilde{F}_{\nu\lambda} + \delta_{\nu\lambda}\tilde{F}_{\mu\sigma}, \quad (A.4)$$

and the condition of speciality is transformed to

$$\sum_{\mu=0}^{2M} \tilde{F}_{\mu\mu} = 0. \quad (A.5)$$

Operators $F_{\mu\nu}$ form the Lie algebra of orthogonal group $SO(2M+1)$, while operators $\tilde{F}_{\mu\nu}$ supplement this algebra to the algebra of unitary group $SU(2M+1)$.

The unitary quantization procedure is based on the choice of the Lie algebra of group $SO(2M+1)$ as the basic algebra. Govorkov [2] has introduced the following quantities:

$$\begin{aligned} \beta_\mu &\equiv iF_{\mu 0}, \quad \beta_0 = iF_{00} = 0, \\ \zeta_\mu &\equiv \tilde{F}_{\mu 0}, \quad \zeta_0 = \tilde{F}_{00} \neq 0. \end{aligned} \quad (A.6)$$

The relations

$$F_{\mu\nu} = [\beta_\nu, \beta_\mu] - i(\delta_{0\nu}\beta_\mu - \delta_{0\mu}\beta_\nu), \quad (A.7)$$

$$[\zeta_\mu, \zeta_\nu] = [\beta_\mu, \beta_\nu] - 2i(\delta_{0\nu}\beta_\mu - \delta_{0\mu}\beta_\nu), \quad (A.8)$$

$$[\zeta_\mu, \beta_\nu] = -i\tilde{F}_{\mu\nu} + i\delta_{\mu\nu}\zeta_0 + i(\delta_{0\nu}\zeta_\mu - \delta_{0\mu}\zeta_\nu) \quad (A.9)$$

are the consequences of algebra (A.2)–(A.4). In view of the equality $\tilde{F}_{\mu\nu} = \tilde{F}_{\nu\mu}$, relation (A.9) implies that

$$[\zeta_\mu, \beta_\nu] - [\zeta_\nu, \beta_\mu] = 2i(\delta_{0\nu}\zeta_\mu - \delta_{0\mu}\zeta_\nu).$$

This relation defines the antisymmetric part of commutator $[\zeta_\mu, \beta_\nu]$. In [2], formula (A.7) contains no terms in the parentheses, there is no factor 2 in formula (A.8), and the last but one term is absent in formula (A.9). All these terms and the factor are important when the consistency of various expressions is verified (see below).

Further, assuming that $\mu = \nu$ in (A.9) and summing over μ with allowance for relation (A.5), we obtain one more important relation

$$\zeta_0 = -\frac{i}{2M+1}\sum_{\mu=1}^{2M}[\zeta_\mu, \beta_\mu], \quad (A.10)$$

i.e., operator ζ_0 is not independent, but is determined by other operators.

In terms of variables (A.6) we can write algebra (A.2)–(A.4) in an equivalent form of trilinear relations

$$[\beta_\lambda, [\beta_\mu, \beta_\nu]] = \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu, \quad (A.11)$$

$$[\zeta_\lambda, [\zeta_\mu, \zeta_\nu]] = \delta_{\lambda\mu}\zeta_\nu - \delta_{\lambda\nu}\zeta_\mu, \quad (A.12)$$

$$[\zeta_\lambda, [\zeta_\mu, \beta_\nu]] = 2\delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu + \delta_{\lambda\mu}\beta_\nu, \quad (A.13)$$

$$[\beta_\lambda, [\zeta_\mu, \beta_\nu]] = -2\delta_{\mu\nu}\zeta_\lambda - \delta_{\lambda\nu}\zeta_\mu - \delta_{\lambda\mu}\zeta_\nu, \quad (A.14)$$

and bilinear relations

$$[\beta_\mu, \beta_\nu] = [\zeta_\mu, \zeta_\nu], \quad (A.15)$$

$$[\zeta_\mu, \beta_\nu] = [\zeta_\nu, \beta_\mu]. \quad (A.16)$$

The indices here run through the values 1, 2, ..., $2M$. It is stated, however, in Govorkov's review [4] that relations (A.11)–(A.16) "*...are satisfied for ζ_0 automatically due to their fulfillment for other ζ_μ. Therefore, we can assume that indices μ, ν, and λ in these relations run through the values* 0, 1, 2, ... , $2M$." The incorrectness of this statement follows, for example, from comparison of bilinear relations (A.15) and (A.16) with relations (A.8) and (A.9). In the former case, for $\nu = 0$, we have (it should be recalled that $\beta_0 = 0$)

$$[\zeta_\mu, \zeta_0] = 0, \quad [\beta_\mu, \zeta_0] = 0,$$

while in the latter case, the commutators assume the form

$$[\zeta_\mu, \zeta_0] = -2i\beta_\mu, \quad [\beta_\mu, \zeta_0] = 2i\zeta_\mu. \quad (A.17)$$

The generalization of trilinear relations (A.11) and (A.12), which hold for any values of indices, has the form

$$\begin{aligned} [\beta_\lambda, [\beta_\mu, \beta_\nu]] &= \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu \\ &+ \delta_{0\nu}(\delta_{0\lambda}\beta_\mu - \delta_{0\mu}\beta_\lambda) - \delta_{0\mu}(\delta_{0\lambda}\beta_\nu - \delta_{0\nu}\beta_\lambda), \end{aligned}$$

$$\begin{aligned} [\zeta_\lambda, [\zeta_\mu, \zeta_\nu]] &= \delta_{\lambda\mu}\zeta_\nu - \delta_{\lambda\nu}\zeta_\mu \\ &+ \delta_{0\nu}(\delta_{0\lambda}\zeta_\mu - \delta_{0\mu}\zeta_\lambda - \delta_{\mu\lambda}\zeta_0) \\ &- \delta_{0\mu}(\delta_{0\lambda}\zeta_\nu - \delta_{0\nu}\zeta_\lambda - \delta_{\nu\lambda}\zeta_0) + 2i(\delta_{0\nu}[\beta_\mu, \zeta_\lambda] - \delta_{0\mu}[\beta_\nu, \zeta_\lambda]), \end{aligned}$$

respectively. The characteristic feature of the last expression is the emergence of terms which are bilinear in the β and ζ operators and cannot be eliminated in principle.

Finally, a more general expression for (A.13) has the form

$$\begin{aligned} [\zeta_\lambda, [\zeta_\mu, \beta_\nu]] &= 2\delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu + \delta_{\lambda\mu}\beta_\nu \\ &- \delta_{0\nu}(\delta_{0\lambda}\beta_\mu - \delta_{0\mu}\beta_\lambda) - \delta_{0\mu}(\delta_{0\lambda}\beta_\nu - \delta_{0\nu}\beta_\lambda) + 2i\delta_{0\mu}[\zeta_\nu, \zeta_\lambda]. \end{aligned}$$

Here, the right-hand side also contains a term bilinear in ζ.

For the unitary representation of the algebra under consideration, quantities β_μ and ζ_μ are Hermitian:

$$\beta_\mu^\dagger = \beta_\mu, \quad \zeta_\mu^\dagger = \zeta_\mu.$$

This enables us to introduce the Hermitian conjugate operators

$$\begin{aligned} a_k &= \beta_{2k-1} - i\beta_{2k}, \quad b_k = \zeta_{2k-1} - i\zeta_{2k}, \\ a_k^\dagger &= \beta_{2k-1} + i\beta_{2k}, \quad b_k^\dagger = \zeta_{2k-1} + i\zeta_{2k}, \end{aligned} \quad (A.18)$$

where $k = 1, 2, ..., M$. Algebra (1.1)–(1.6) and (1.9) for operators a_k, b_k, and ζ_0 is a direct consequence of relations (A.11)– (A.16) and (A.17).

APPENDIX B

Operator Identities

In this appendix, we consider a number of operator identities that have been repeatedly used in the above text:

$$[A,[B,C]] = -[B,[C,A]] - [C,[A,B]], \tag{B.1}$$
$$[A,[B,C]] = \{C,\{A,B\}\} - \{B,\{C,A\}\}, \tag{B.2}$$
$$\{A,[B,C]\} = \{B,[C,A]\} - [C,\{A,B\}], \tag{B.3}$$
$$[A,\{B,C\}] = -[B,\{C,A\}] - [C,\{A,B\}], \tag{B.4}$$

where [,] and { , } denote the commutators and anticommutators, respectively. In addition to identities (B.1)–(B.4), the following simple relations are also useful:

$$[A,BC] = \{A,B\}C - B\{A,C\} = B[A,C] + [A,B]C, \tag{B.5}$$
$$\{A,BC\} = \{A,B\}C - B[A,C] = B\{A,C\} + [A,B]C. \tag{B.6}$$

Finally, the operator identities involving the exponential functions have the form [48–51]

$$\mathrm{e}^X Y \mathrm{e}^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \ldots, \tag{B.7}$$

$$\mathrm{e}^X Y \mathrm{e}^{X} = Y + \{X,Y\} + \frac{1}{2!}\{X,\{X,Y\}\} + \frac{1}{3!}\{X\{X,\{X,Y\}\}\} + \ldots, \tag{B.8}$$

$$\mathrm{e}^X \mathrm{e}^Y \mathrm{e}^{-X} = \exp(\mathrm{e}^X Y \mathrm{e}^{-X}). \tag{B.9}$$

APPENDIX C

Commutation Relations with Operator $\mathrm{e}^{\alpha i\tilde{N}}$

Here, we collect all (anti)commutation relations containing operator $\mathrm{e}^{\alpha i\tilde{N}}$, which appeared in Sections 6 and 8 for different reasons. For values of parameter $\alpha = \pm\pi$, we have

$$\{\mathrm{e}^{\pm\pi i\tilde{N}}, a_k\} = 0, \quad \{\mathrm{e}^{\pm\pi i\tilde{N}}, b_m\} = 0,$$

for $\alpha = \pm\pi/2$

$$a_k \mathrm{e}^{\pm\pi i\tilde{N}/2} = \mp \mathrm{e}^{\pm\pi i\tilde{N}/2} b_k, \quad b_m \mathrm{e}^{\pm\pi i\tilde{N}/2} = \pm \mathrm{e}^{\pm\pi i\tilde{N}/2} a_m,$$
$$\mathrm{e}^{\pm\pi i\tilde{N}/2} a_k = \pm b_k \mathrm{e}^{\pm\pi i\tilde{N}/2}, \quad \mathrm{e}^{\pm\pi i\tilde{N}/2} b_m = \mp a_m \mathrm{e}^{\pm\pi i\tilde{N}/2},$$

or, in equivalent form,

$$a_k = \mp \mathrm{e}^{\pm\pi i\tilde{N}/2} b_k \mathrm{e}^{\mp\pi i\tilde{N}/2}, \quad b_m = \pm \mathrm{e}^{\pm\pi i\tilde{N}/2} a_m \mathrm{e}^{\mp\pi i\tilde{N}/2},$$
$$a_k = \mp \mathrm{e}^{\mp\pi i\tilde{N}/2} b_k \mathrm{e}^{\pm\pi i\tilde{N}/2}, \quad b_m = \mp \mathrm{e}^{\mp\pi i\tilde{N}/2} a_m \mathrm{e}^{\pm\pi i\tilde{N}/2}.$$

The commutation relations including para-Grassmann numbers ξ_k have the form for $\alpha = \pm\pi$,

$$\mathrm{e}^{\pm\pi i\tilde{N}}[\xi_l, a_k] = -\mathrm{e}^{\pm\pi\tilde{\Lambda}}[\xi_l, a_k]\mathrm{e}^{\mp\pi i\tilde{N}},$$
$$\mathrm{e}^{\pm\pi i\tilde{N}}\{\xi_l, a_k\} = -\mathrm{e}^{\pm\pi\tilde{\Lambda}}\{\xi_l, a_k\}\mathrm{e}^{\mp\pi i\tilde{N}},$$
$$\mathrm{e}^{\pm\pi i\tilde{N}}[\xi_l, b_m] = -\mathrm{e}^{\pm\pi\tilde{\Lambda}}[\xi_l, b_m]\mathrm{e}^{\mp\pi i\tilde{N}},$$
$$\mathrm{e}^{\pm\pi i\tilde{N}}\{\xi_l, b_m\} = -\mathrm{e}^{\pm\pi\tilde{\Lambda}}\{\xi_l, b_m\}\mathrm{e}^{\mp\pi i\tilde{N}},$$

while for $\alpha = \pm\pi/2$, we have

$$\mathrm{e}^{\pm\pi i\tilde{N}/2}[\xi_l, a_k] = \mp \mathrm{e}^{\pm\pi\tilde{\Lambda}/2}\{\xi_l, b_k\}\mathrm{e}^{\mp\pi i\tilde{N}/2},$$
$$\mathrm{e}^{\pm\pi i\tilde{N}/2}[\xi_l, b_m] = \pm \mathrm{e}^{\pm\pi\tilde{\Lambda}/2}\{\xi_l, a_m\}\mathrm{e}^{\mp\pi i\tilde{N}/2}.$$

ACKNOWLEDGMENTS

The work of D.M. G. was supported by the Russian Foundation for Basic Research (project no. 18-02-00149), Sãn Paulo Research Foundation (FAPESP, grant no. 2016/03319-6), and the National Council for Research (CNPq), as well as within the program for improving competitiveness of the Tomsk National Research University among leading world scientific and educational centers.

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