Factorization of Graphs

A thesis submitted for the degree of Doctor of Philosophy

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Dedication

Every challenging work needs self efforts as well as guidance of elders especially those who were very close to our heart.

I dedicate this thesis to God for his fruitful help to finish the thesis, as well as my family along with all hard working and respected teachers.

Declaration

I, Anitha Rajkumar, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material.

I accept that the College has the right to use plagiarism detection software to check the electronic version of the thesis.

I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university.

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Date:

Details of collaboration and publications:

- A.J.W. Hilton and A Rajkumar "The Pseudograph (r, s, a, t)-threshold number" Discrete Applied Mathematics, 209 (2016), 155-163.
- A.J.W. Hilton and A Rajkumar "The simple graph threshold number-$\sigma(r, s, a, t)$ when $r \geq 3$ is odd and $a \geq 2$ is even" Congressus Numerantium, 223 (2015), 33-44.

Reference to earlier published work is made in the introductory chapter (Chapter 1).

Acknowledgements

I would like to express the deepest appreciation to my advisor, Professor Anthony J.W. Hilton, who has the attitude and the substance of a genius: he continually and convincingly conveyed a spirit of adventure in regard to research and an excitement in regard to teaching. Without his guidance and persistant help this thesis would not have been possible.

I would like to thank Dr. Rainer Klages, Director of Research studies, School of Mathematical Sciences, QMUL for his great support and guidance during my PhD.

In addition, a thank you to all the staff members of my school at QMUL and Prof. Bill Jackson who has been a constant source of knowledge and inspiration.

Abstract

For $d \geq 1, s \geq 0$, a $(d, d+s)$-graph is a graph whose degrees all lie in the interval $\{d, d+1, ..., d+s\}$. For $r \geq 1, a \geq 0$, an $(r, r+a)$-factor of a graph G is a spanning $(r, r+a)$-subgraph of G. An $(r, r+a)$-factorization of a graph G is a decomposition of G into edge-disjoint $(r, r+a)$-factors. A graph is $(r, r+a)$-factorable if it has an $(r, r+a)$-factorization.

For $t \geq 1$, let $\sigma(r, s, a, t)$ be the least integer such that, if $d \geq \sigma(r, s, a, t)$, then every $(d, d+s)$-simple graph G has an $(r, r+a)$-factorization into x $(r, r+a)$-factors for at least t different values of x. Then we show that, for $r \geq 3$ odd and $a \geq 2$ even,

$$\sigma(r,s,a,t) = \begin{cases} r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r + 1 & \text{if } t \geq 2, \text{ or } t = 1 \text{ and } a < r+s+1, \\ r & \text{if } t = 1 \text{ and } a \geq r+s+1; \end{cases}$$

Similarily, we have evaluated $\sigma(r, s, a, t)$ for all other values of r, s, a and t. We call $\sigma(r, s, a, t)$ the *simple graph threshold number*.

A pseudograph is a graph where multiple edges and multiple loops are allowed. A loop counts two towards the degree of the vertex it is on. A multigraph here has no loops.

For $t \geq 1$, let $\pi(r, s, a, t)$ be the least integer such that, if $d \geq \pi(r, s, a, t)$, then every $(d, d+s)$-pseudograph G has an $(r, r+a)$-factorization into x $(r, r+a)$-factors for at least t different values of x. We call $\pi(r, s, a, t)$ as the *pseudograph threshold number*.

We have also evaluated $\pi(r, s, a, t)$ for all values of r, s, a and t. Note that for $r \geq 3$

$$\pi(r, 0, 1, 1) = \infty$$

meaning that $\pi(r, 0, 1, 1)$ cannot be given a finite value.

This study provides various generalisations of Petersen's theorem that "Every 2k-regular graph is 2-factorable".

Preamble

In this second version of my thesis I have tried to respond positively to all the criticisms and comments made by the examiners about the first version. These criticisms were partly about the style, and partly about the corrections of the results, with particular attention being paid to Theorem 4.1.4 in the first version. I felt it best to rewrite the whole thesis.

In this rewritten version, the Introduction includes all the graph theory definitions needed to follow the thesis, and it includes a clear statement of the main results, the evaluation of the pseudograph (r,s,a,t)-threshold numbers, and the partial evaluation of the simple graph (r,s,a,t)-threshold numbers. All earlier results are included as well in the Introduction. I have omitted assorted theorems (by others) on factors in graphs, since these are not pertinent to the main theme, that of factorizations.

Chapter 2 in the first version has been expanded to occupy Chapters 2, 3 and 4 in the current version. Considerable changes have been made, particularly in the content of the new Chapter 4, as a result of the discovery of a mistake in the paper "Degree-bounded factorizations of bipartite subgraphs and of pseudographs" by Prof. Hilton. Thus in Chapter 4, Theorems 4.5, 4.8 and 4.11, as well as other theorems, are different from the corresponding results in Prof. Hilton's paper. There are other changes mainly resulting from additional proofs and explanations being given to try to respond to the criticism that the explanations were too brief "even for a journal paper".

Chapter 5 in the revised thesis corresponds to Chapter 3 in the first thesis. It will be seen that Chapter 5, which is very long, is much longer than Chapter 3 in the first thesis. In Theorem 4.5 in the new thesis, there is an additional case $(r+1)t+1 \not\equiv 3(mod\ a-1)$ which needs its own special treatment. There are similar cases in Theorems 4.8 and 4.11. The additional cases are in fact much more difficult to deal with when they need to be considered in Chapter 5. Moreover while the arguments in Lemma 5.5(1) and 5.7(1), and later in Theorem 5.9, are quite similar (though complicated), the argument arising from Theorem 4.8 in Lemma 5.11 is different.

Chapters 6, 7, 8 and 9 deal with simple graphs, with one chapter each devoted to the cases r odd, a even; r and a both even; r even, and a odd; r and a both odd. Theorem 4.1.4 of the first thesis is now split into two theorems, Theorem 6.2 and Theorem 6.7. Theorem 6.2 now gives an upper bound for $\sigma(r,s,a,t)$, valid in all cases.

Chapter 7 evaluates $\sigma(r,s,a,t)$ when r and a are both even. This is already known from the earlier chapters on pseudographs, but in Chapter 7 a proof is given along the lines of the proofs in the other simple graph cases. Also a lower bound for $\sigma(r,s,a,t)$ is given. This again is known from the earlier chapters, but again a proof like the other proofs for simple graphs seemed not inappropriate.

In Chapter 8 in the case when r is even and a is odd, $\sigma(r,s,a,t)$ is closely bounded-in most cases bounded between two numbers which are just one apart.

In Chapter 9 in the case when r and a are both odd then $\sigma(r,s,a,t)$ is evaluated except in one case.

In Chapter 8 and 9 there is a focus on the inequalities $\frac{d+s}{r+a} < x \leq \frac{d}{r}$ and $\frac{d+s}{r+a} \leq x < \frac{d}{r}$ respectively. In both chapters it is shown that there exist simple graphs which satisfy the equality $\frac{d+s}{r+a} = \frac{d}{r}$ which do not have $(r, r+a)$-factorizations with x factors when $x = \frac{d}{r}$.

The discussion of property z in the first version is omitted in the revised version because it did not lead to the clear-cut numerical evaluations of $\sigma(r, s, a, t)$ which were desired.

Second Preamble

The second version of my thesis had a number of minor mistakes and places where the argument was not presented as clearly as it might have been.

In this third version of my thesis there are two major changes.

The first of these was suggested by the External Examiner, Prof. McDiarmid. He suggested the notation $F_{\{r,a\}}(G)$ to mean the set of integers x such that a graph G has an $(r, r+a)$-factorization with x factors. He informed me that it followed from the theory of unimodular matrices that if G is a bipartite graph, then $F_{\{r,a\}}(G)$ is an interval of integers. He also pointed out that it followed from the methods used in the thesis that, if r and a are even integers, then $F_{\{r,a\}}(G)$ is again an interval of integers. He raised the question of whether $F_{\{r,a\}}(G)$ is an interval of integers in other cases too. The notation $F_{\{r,a\}}(G)$ can be used in various places to give alternative formulations (often more compact) of some of the theorems in the thesis.

The second major change is that Chapters 8 and 9 are changed to give proofs which were previously just conjectures, namely the determination of $\sigma(r, s, a, t)$ when r and a are both odd, or when r is even and a is odd.

In the event of this third version still being considered unsatisfactory (which I hope will not be the case), a further version could either examine Prof. McDiarmid's question about whether $F_{\{r,a\}}(G)$ is an interval of integers in other cases, or could move on to consider the threshold number for multigraphs (for which somewhat different methods might be needed).

Third Preamble

The fourth version of my thesis contains corrections to a few typographical errors in Chapters 1, 2 and 3. Chapter 1 also includes additional discussion concerning the relationship with unimodular matrices, as suggested by Prof. McDiarmid.

text

Fourth Preamble

The fifth and sixth versions of my thesis contain various adjustments as suggested by Prof. McDiarmid.

Theorem numbering

The first number of a theorem number refers to the chapter in which the theorem is stated, and the subsequent numbers refer to the order of the theorems in that chapter. Thus Theorem 4.31 would be the thirty first theorem (including lemmas etc.) in Chapter 4. Occasionally a theorem is stated in more than one chapter. In that case the theorem is given with both numbers, the current chapter being stated first, the other chapter in which the theorem occurs is in brackets afterwards. Thus in Chapter 1 we have Theorem 1.23(5.1) and in Chapter 5 we have Theorem 5.1(1.23), both referring to the same theorem, which it was felt to be convenient and helpful to the reader to repeat.

Contents

Chapter 1

Introduction

I start by giving some graph theory definitions and terminology. So far as I am aware this terminology is standard and does not deviate materially from the terminology detailed le n dozens of textbooks.

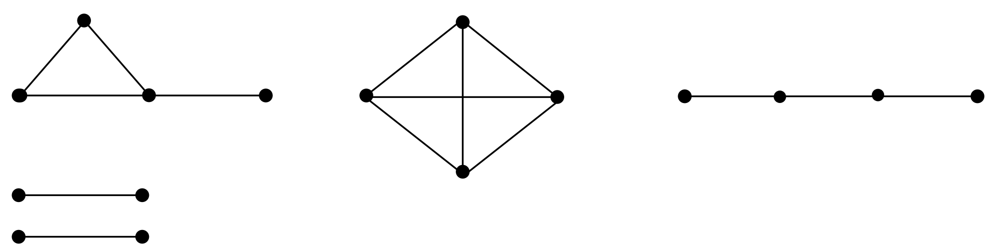
1.1 Definitions and Terminology

In our definitions and in this thesis, all sets are finite; we shall not repeat this.

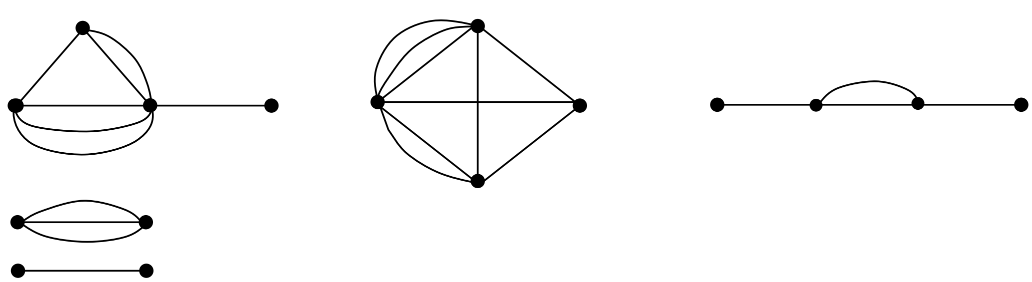
A **graph** G is a set V of **vertices** and a set E of **edges** with the properties that each edge is incident with one or two vertices. Edges which some incident with one vertex are called **loops**. If a graph has no loops and no two edges are incident with the same two vertices, then the graph is a **simple graph**. In many contexts the only interest is in simple graphs. However if a graph has no loops, but pair of two edges may be incident with the same pair of vertices, then the graph is a **multigraph**.

If the vertex set V of a multigraph G is the union of two disjoint sets V_1 and V_2, and if every edge at V is incident with one vertex from V_1 and another vertex from V_2, the G is called **bipartite**.

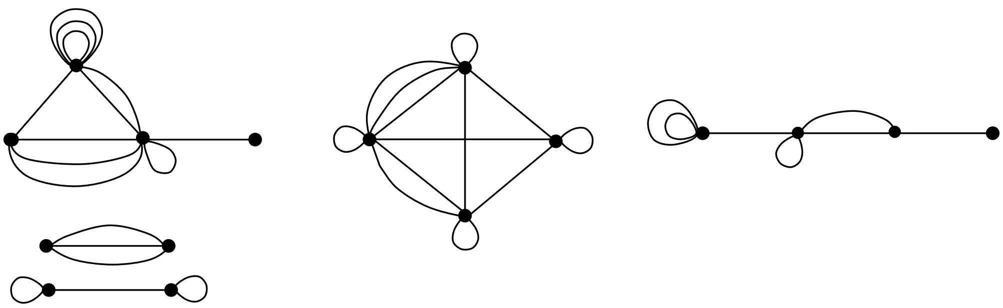
In a multigraph, the set of edges incident with the same pair of vertices is called a **multiedge**. The set of loops incident with a vertex is called a **multiple loop**. If a graph has loops, and may have more than one edge incident with the same pair of vertices, then the graph is a **pseudograph** or **general graph**. These various types of graphs have simple pictorial representations. For example:



Some simple graphs with four vertices.



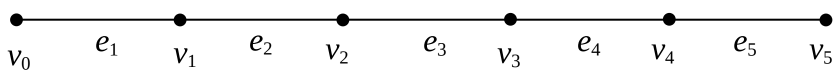
Some multigraphs with four vertices.



Some pseudographs with four vertices.

Figure 1.1: Different kinds of graph.

A **path** is a sequence of distinct vertices and edges $v_0, e_1, v_1, e_2, v_2, ..., v_{p-1}, e_p, v_p$ with e_i incident with v_{i-1} and v_i for $1 \leq i \leq p$.



A **cycle** is a sequence of distinct vertices and edges, except that the first and last vertex is the same: $v_1, e_1, v_2, e_2, v_3, e_3, ..., v_{p-2}, e_{p-2}, a, v_{p-1}, e_{p-1}, v_1$.

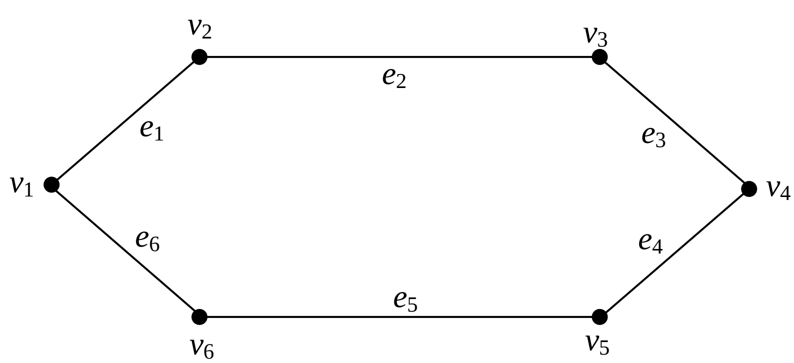


Figure 1.2: A cycle.

A **cycle** with p vertices is often denoted by C_p. By a slight abuse of terminology, A loop with its vertex counts as a cycle, and a double edge with its two end vertices also counts as a cycle.

If $V^{'} \subseteq V$, $E^{'} \subseteq E$, and each edge of $E^{'}$ is incident with two vertices of $V^{'}$ (or one vertex of $V^{'}$ if the edge is a loop), then the graph $G^{'}$ with vertex set $V^{'}$ and edge set $E^{'}$ is called a **subgraph** of G.

If $V^{'} \subseteq V$ and $E^{'} \subseteq E$, then let $V(E^{'})$ be the set of vertices of G which are incident with at least one edge of $E^{'}$. The subgraph of G with vertex set $V^{'} \cup V(E^{'})$ and edge-set consisting of all edges of G which have both end vertices in $V^{'} \cup V(E^{'})$ or are loops incident with a vertex of $V^{'} \cup E(E^{'})$ is said to be **induced** by $V^{'} \cup E^{'}$.

If H is a subgraph of G and $V(H) = V(G)$, then H **spans** G and is a **factor** of G.

The number of edges of a graph G incident with a vertex v is called the **degree** of v, and is written $d(v)$ or $d_G(v)$. Here if a loop is incident with v, then the loop counts two towards the degree of G.

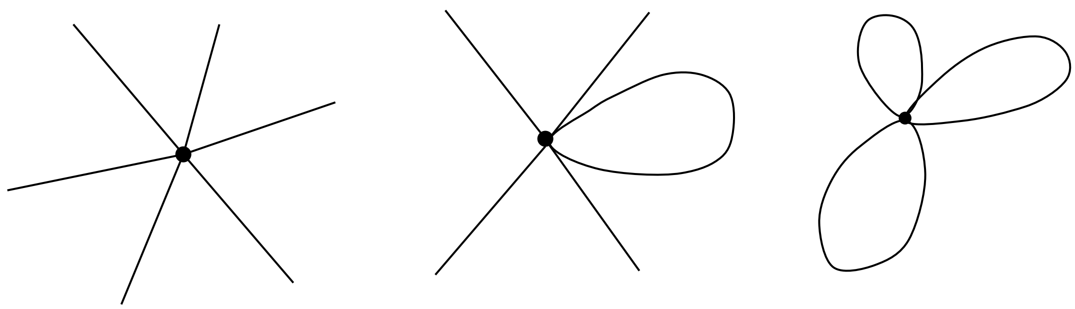


Figure 1.3: Vertices of degree 6.

The maximum degree of G is written $\triangle(G)$ and the minimum degree is written $\delta(G)$.

A factor F of G in which every degree has the same value (within F), say r, is called an r**-factor**. Such a factor is called r**-regular** (or just **regular**). A factor of G in which every vertex has degree (within F) in the set $\{r, r+1, ..., r+a\}$ for some $a \geq 0$, is called an $(r, r+a)$**-factor**.

A **circuit** is a sequence $v_0, e_1, v_1, e_2, v_2, ..., v_{p-1}, e_p, v_p = v_0$, not necessarily distinct, with first and last vertex the same (so $v_0 = v_p$) with e_i incident with v_{i-1} and v_i.

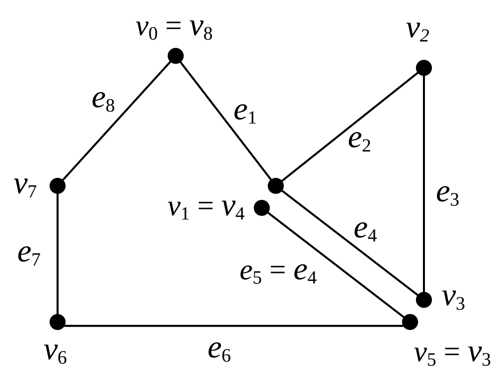


Figure 1.4: A circuit $v_0, e_1, v_1, e_2, v_2, e_3, v_3, e_4, v_4 = v_1, e_5 = e_4, v_5 = v_3, e_6, v_6, e_7, v_7, e_8, v_8 = v_0$

If a graph G has two vertices v_1 and v_2 and contains a path with endvertices v_1 and v_2, then v_1 and v_2 are said to be **connected**.

If, for each pair v_1, v_2 of vertices of a graph G there is a path joining v_1 and v_2, then G is said to be **connected**.

A maximal connected subgraph of G is a **component** of G. If an edge e in a graph G has the property that the removal of e from G (but without removing either of the end vertices of G) increases the number of components of G, then e is called a **bridge**.

A connected graph G is **Eulerian** if each vertex has even degree. In a Eulerian graph, the vertices and edges can be arranged in the form of a circuit, called an **Eulerian circuit**. It is well-known that an Eulerian circuit is the edge-disjoint union of cycles. A circuit can be **oriented** (or **directed**): informally this means that an arrow is placed on each edge, the direction being that in which the circuit is reversed. Less formally, given a circuit $v_0, e_1, v_1, e_2, v_2, ..., v_{p-1}, e_p, v_p$, where each edge e_i is the unordered pair $\{v_i, v_{i+1}\}$, we replace the unordered pair by the ordered pair (v_i, v_{i+1}). We can indicate this by arrows: $\vec{e}$ or $\overrightarrow{v_i v_{i+1}}$. In a directed graph, the number of directed edges directed towards a vertex v is the **in-degree** of v, and the number directed away from v is the **out-degree** of v.

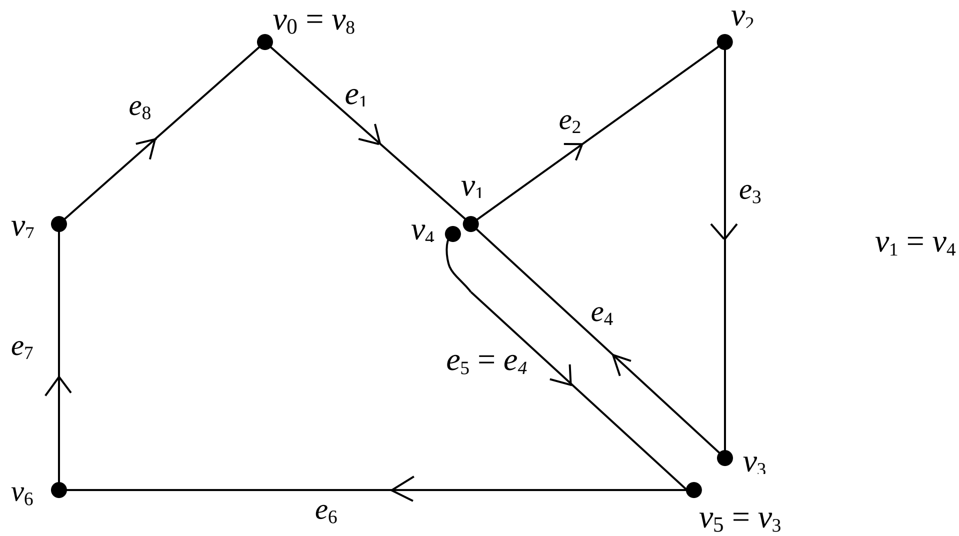


Figure 1.5: A directed circuit $v_0, \vec{e_1}, v_1, \vec{e_2}, v_2, \vec{e_3}, v_3, \vec{e_4}, v_4 = v_1, \vec{e_5}, v_5, \vec{e_6}, v_6, \vec{e_7}, v_7, \vec{e_8}, v_8 = v_0$.

If two graphs G_1 and G_2 have the same vertex set, then the **union** $G_1 \cup G_2$ has the same vertex set and the edge set $E(G_1 \cup G_2)$ is $E(G_1) \cup E(G_2)$. If $E(G_1) \cap E(G_2) = \emptyset$ then $E(G_1) \cup E(G_2)$ may be termed the **edge-disjoint union** of $E(G_1)$ and $E(G_2)$ and denoted by $E(G_1) \dot{\cup} E(G_2)$. If two graphs G_1 and G_2 have disjoint vertex sets then the **union** of G_1 and G_2 has

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$

and edge set $E(G_1) \cup E(G_2)$.

An $(r, r+a)$ - *factorization* of a graph G is a decomposition of G into edge - disjoint $(r, r+a)$ - factors.

This concept is closely related to the idea of edge-colourings of a graphs. An **edge-colouring** ϕ of a graph G is a map $\phi : E(G) \longrightarrow C$, where C is a set of colours. The set of edges with the same colour is called a **colour class**. An edge-colouring in which no vertex is incident with more than one edge of any colour is called a **proper edge-colouring**. The least number of colours needed to properly edge-colour a graph is called the **chromatic index** or **edge-chromatic number** of a graph G, and is written $\chi'(G)$ or $\chi_e(G)$; $\chi'(G)$ is not defined if G has loops. A proper edge-colouring of a graph corresponds to a $(0, 1)$-factorization of a graph. Two well-known theorems concerning the chromatic index are König's Theorem [27] from 1935 and Vizing's Theorem [38] from 1964.

Theorem 1.1 ***König's Theorem.*** *If G is a bipartite multigraph then $\chi'(G) = \Delta(G)$.*

A consequence of König's Theorem is that a regular bipartite multigraph is 1-factorable.

König, in his book, **Theorie der Endlichen and unendlichen Graphen,** gives three proofs of this theorem. The theorem was actually first proved by someone else, possibly van der Waerden. The number of edges joining two vertices v_1, v_2 in a pseudograph is called the **multiplicity** $m(v_1, v_2)$ of the edge $\{v_1, v_2\}$. The maximum multiplicity of a pseudograph G is denoted by $m(G)$.

Theorem 1.2 ***Vizing's Theorem.*** *For a multigraph G,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + m(G).$$

There are more refined versions of Vizing's Theorem than this. König's theorem can be restated as follows; A bipartite multigraph has a $(0, 1)$-factorization with $\Delta(G)$ or more factors. Similarly Vizing's theorem can be restated as follows; A multigraph G has a $(0, 1)$-factorization with $\Delta(G) + m(G)$, or more, factors.

A theorem which is often thought of as a kind of dual to Vizing's Theorem is due to Gupta[15] in 1996. A **proper edge-covering** of a graph G is an edge-colouring with the property that at each vertex there is an incident edge of each colour. The greatest number of colours which can be assigned to the edges so as to yield a proper edge-covering is called the **cover index**, or **edge-cover number** of G, and is written $\kappa^{'}(G)$ or $\kappa_e(G)$.

Theorem 1.3 ***Gupta's Theorem.*** *For a multigraph G,*

$$\kappa^{'}(G) \geq \delta(G) - m(G).$$

Gupta originally just announced his theorem, and for several years did not publish a proof. The proof used very similar notions to those in the proof of Vizing's theorem, but was more complicated. Whereas many textbooks carry a proof of Vizing's theorem, I do not believe that a proof of Gupta's theorem is in any textbook.

A relatively straightforward proof of Gupta's Theorem was published by Hilton in 1975 in [16]. This used another inequality satisfied by $\chi^{'}(G)$ and $\kappa^{'}(G)$.

Theorem 1.4 *For a multigraph G,*

$$\chi^{'}(G) + \kappa^{'}(G) \geq 2\delta(G).$$

This inequality seems to be insignificant beside those of Vizing and Gupta, and is not well known, but it is sufficient to provide a relatively simple deduction of Gupta's Theorem from Vizing's Theorem. The "dual inequality" $2\Delta(G) \geq \chi'(G) + \kappa^{'}(G)$ is not true for multigraphs, although it is for simple graphs.

Equitable edge-colouring:

An edge-colouring concept of great use is furnished by the notion of equitable edge-colouring. Consider a set $C = \{C_1, C_2, ..., C_p\}$ of p colours. Given an edge-colouring ϕ, $E(G) \longrightarrow C$ let $C_i(v)$ be the set of edges of colour C_i incident with the vertex v . Then ϕ is said to be an **equitable edge colouring** if

$$||C_i(v)| - |C_j(v)|| \leq 1$$

for each vertex v and for each pair of indices i, j with $1 \leq i < j \leq p$. The set of edges of colour C_i is sometimes called a colour class. The notation $|C_i(v)|$ is the number of edges in $C_i(v)$.

Various analogous notions are sometimes required in addition to equitability.

One of these is **equalization**. This is that $||C_i| - |C_j|| \leq 1$ for $1 \leq i < j \leq p$ where $|C_i|$ is the number of edges coloured C_i. It is usually the case that if a graph has the property of satisfying one edge-colouring condition, then we can add the requirement that the edge-colouring is equalized without significantly disturbing the original property. For example if a graph has an equitable edge-colouring, then it will usually have an equitable equalized edge-colouring.

Another is balance. For $1 \leq i \leq p$, $v_j, v_k \in V(G)$, let $C_i(v_j, v_k)$ be the number of edges joining v_j and v_k which are in the set C_i (i.e. coloured C_i). An edge-colouring is **balanced** if

$$||C_i(v_k, v_e)| - |C_j(v_k, v_e)|| \leq 1$$

for each pair v_k, v_e of vertices and each pair C_i, C_j of colours. Here $|C_i(v_k, v_e)|$ is the number of edges in the set $C_i(v_j, v_k)$.

There are two useful results on equitable edge-colourings. The first is due independently to de Werra [39] and McDiarmid [30].

Theorem 1.5 *Let $k \geq 1$ be an integer. Let G be a bipartite multigraph. Then G has an equitable, balanced edge-colouring with k colours.*

Taking $k = \Delta(G)$ we recover König's theorem. The second is due to Hilton and de Werra [23].

Theorem 1.6 *(6.3) Let $k \geq 1$ be an integer. Let G be a simple graph with the property that $k \nmid d(v)$ for each vertex v of G. Then G has an equitable edge-colouring with k colours.*

Taking $k = \Delta(G)+1$ we recover Vizing's Theorem for simple graphs. Gupta's Theorem for simple graphs also follows after a short argument.

A link to totally unimodular matrices and bipartite graphs

I am indebted to Prof. McDiarmid for pointing out that Theorem 1.5 is closely related to the study of totally unimodular matrices. In particular, see [31]. A matrix is totally unimodular if the determinant of every square submatrix equals 0 or $\pm$ 1. This can be applied in particular to bipartite multigraphs as their 0, 1 vertex-edge incidence matrices are totally unimodular. Using this connection, from Theorem 4.1 of [31] follows the following very interesting result.

Statement Let $F_{\{r,a\}}(G)$ be the set of integers such that G has an $(r, r+a)$-factorization with x factors. Then for non-negative integers r and a, if G is a bipartite multigraph, then $F_{\{r,a\}}(G)$ is an interval of integers.

A sketch of how this follows from the fact that bipartite graphs are totally unimodular.

Let x, z be integers in $F_{\{r,a\}}(G)$ with $x < z$ and let the integer y satisfy $x \leq y \leq z$. We want to show that $y \in F_{\{r,a\}}(G)$.

Let A be the vertex-edge adjacency matrix of G, which is totally unimodular. Let 1_E be the all ones vector indexed by the edge set E of G, and similarly for 1_V.

Since $x \in F_{\{r,a\}}(G)$, we have $A1_E \leq x(r+a)1_V \leq y(r+a)1_V$, and since $z \in F_{\{r,a\}}(G)$ we have $A1_E \geq zr1_V \geq yr1_V$.

Therefore $yr1_V \leq A1_E \leq y(r+a)1_V$.

Therefore (by Theorem 4.1 of [31], 1_E is the sum of y integral vectors e^i such that $1_V \leq Ae^i \leq (r+a)1_V$; in other words, the edges of G can be partitioned into $y(r, r+a)$-factors.

(Prof. McDiarmid remarked that that the statement above follows easily from Theorem 1.5).

In this connection we mention an almost trivial lemma which is used several times in this thesis, usually without any further remark.

Lemma A Given non-negative integers r and a, if a pseudograph G is the disjoint union of A and B, then

$$F_{\{r,a\}}(G) = F_{\{r,a\}}(A) \cap F_{\{r,a\}}(B). \tag{1.1}$$

Proof If $x \in F_{\{r,a\}}(G)$ then G has an $(r, r+a)$-factorization with x factors, and so therefore do A and B, so $x \in F_{\{r,a\}}(A) \cap F_{\{r,a\}}(B)$. The converse is true by reversing this short argument.

Petersen's Theorem:

One of the oldest theorems in graph theory is due to Petersen [33] in 1891.

Theorem 1.7 *Let G be a regular pseudograph of even degree. Then G is the edge-disjoint union of 2-factors (i.e. G has a 2-factorization).*

This is illustrated by the 4-regular pseudograph in Figure 1.6.

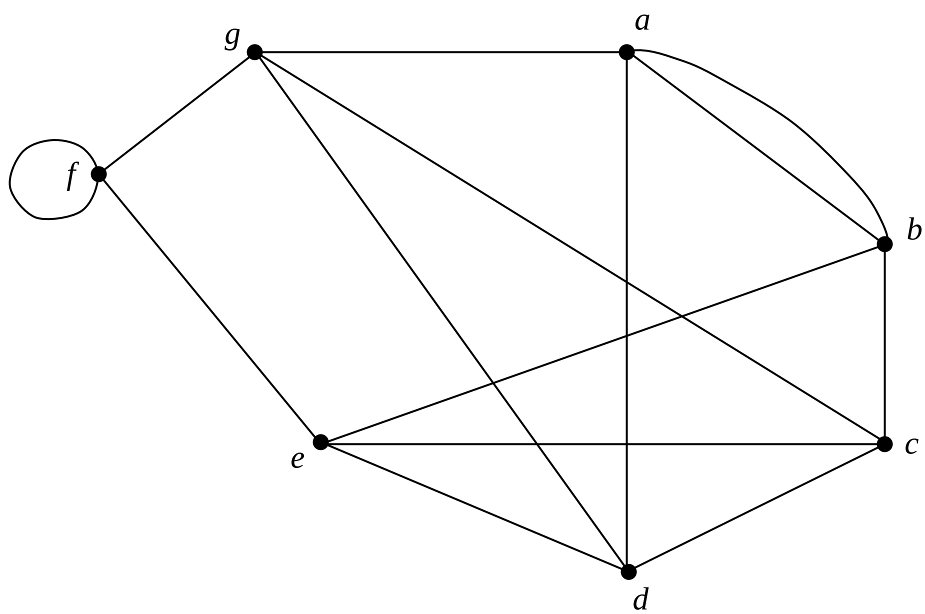


Figure 1.6: A 4-regular pseudograph.

By Petersen's theorem this is the edge-disjoint union of 2-factors. One way of expressing this as the union of 2-factors is shown in Figure 1.7. The solid edges show one 2-factor, the dotted edges show another.

Petersen's original interest was in factorizing polynomials. For example every homogeneous polynomial of degree 4 in several variables, expressed as the product of linear factors, is the product of two such polynomials of degree 2. Thus the polynomial

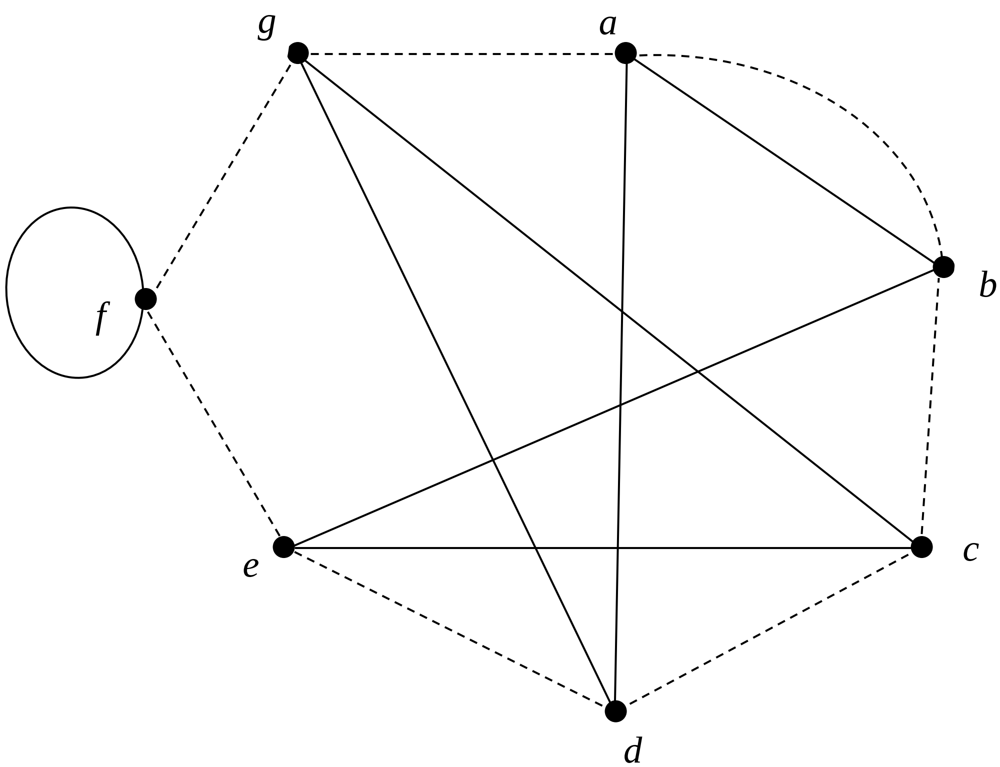


Figure 1.7: A 2-factorization of a 4-regular graph.

$$(a-b)^2(a-d)(a-g)(b-c)(b-e)(c-g)(c-e)(c-d)(d-g)(d-e)(f-e)(f-g)(f-f)$$

which corresponds to the graph of Figure 1.6, is the product of

$$(a-b)(b-c)(c-d)(d-e)(e-f)(f-g)(g-a)$$

and

$$(a-b)(a-d)(b-e)(c-g)(c-e)(d-g)(f-f)$$

which correspond to the 2-factors in the graph of Figure 1.7. ∎

The other theorem that Petersen proved concerned 3-regular graphs (often called cubic graphs).

Theorem 1.8 ***Petersen.*** *A bridgeless cubic graph is the edge-disjoint union of a 1-factor and a 2-factor.*

Not every regular graph has a 1-factor. For degree 2, C_5 is a trivial example. For degree 3, the graph in Figure 1.8 has no 1-factor.

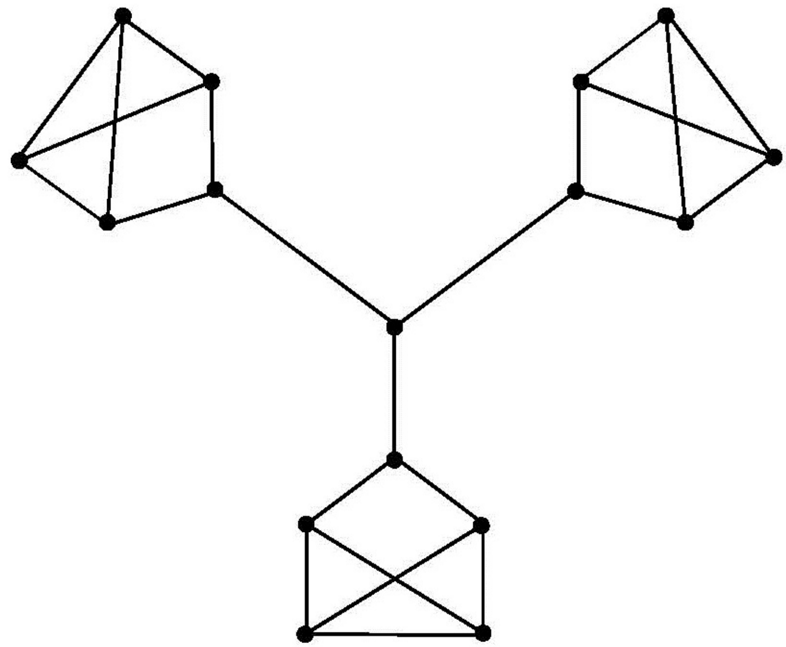


Figure 1.8: A 3-regular graph with no 1-factor.

■

König's theorem that a regular bipartite graph is 1-factorizable provides a nice way to prove Petersen's theorem about 2-factorizing regular graphs of even degree. We explain this connection because it is very flexible and because we use the same idea many times. We call the connection

The Petersen-König's Connection,

A deduction of Petersen's Theorem from König's Theorem.

Let G be a 2r-regular graph (which could have loops or multiple edges). Let $V(G) = v_1, v_2, ..., v_n$. Since G is $2r$-regular, each component of G has an Eulerian circuit. For each component of G, direct the Eulerian circuit. Then each vertex of G has r in-edges and r out-edges. We create an r-regular bipartite graph B with vertex sets $\{u_1, ..., u_n\}$ and $\{w_1, ..., w_n\}$. For each edge v_iv_j of G we place an edge u_iw_j in B. For a loop in G on the vertex v_i we place an edge u_iw_i in B. We then 1-factorize B. The edges of a 1-factor in B then correspond to a 2-factor in G, so we obtain a 2-factorization of G. ■

Another consequence of Petersen's theorem.

Theorem 1.9 *If G is a connected $2r$-regular graph of even order, then G is the edge disjoint union of two r-regular graphs.*

Proof This is true by Petersen's theorem if r is even. If r is odd, then we observe that since $|V(G)|$ is even,

$$\begin{aligned} 2|E(G)| &= \sum_{v \in V(G)} d(v) \\ &= |V(G)|2r, \end{aligned}$$

so $|E(G)|$ is even. Therefore G has an Eulerian circuit, which we can colour alternately red and blue. The red edges give an r-regular subgraph, as do the blue. ■

Chetwynd and Hilton [7] posed the following conjecture, known as the **1-factorization conjecture**.

The 1-factorization conjecture

Let G be a simple regular graph of even order and degree $d(G)$ satisfying $d(G) \geq \frac{1}{2}|V(G)|$. Then G can be 1-factorized. ∎

This conjecture has recently been proved true for very large n by Kuhn [9] and her team at Birmingham University. It was also proved by Chetwynd and Hilton [8], and, independently, by Niessen and Volkmann [32] that it is true if $d(G) \geq 0.823|V(G)|$. A related factorization result was proved by Hilton [17].

Theorem 1.10 *If $r \geq 2$ is a positive integer and G is a simple regular graph of even order and degree $d \geq \frac{1}{2}|V(G)|$, where $r \mid d$, then G is r-factorizable.*

A graph is **planar** if it can be drawn in the plane so that no two edges intersect except at their end vertices.
The 4-colour theorem about planar graphs was proved by Appel and Haken [5] in 1977. A simple consequence is

Theorem 1.11 *Every simple 2-connected planar cubic graph is 1-factorizable.*

As the graph in Figure 1.8 shows, not every regular graph has a 1-factorization. Era and Egawa in 1986 [11], [10] showed that every r-regular graph of sufficiently large degree had a semiregular factorization. A semiregular factor is simply an $(r, r+1)$-factor for some non-negative integer r, and an $(r, r+1)$-factorization is a semiregular factorization. Era and Egawa proved:

Theorem 1.12 *Let $r \geq 0$. If*

$$d \geq \begin{cases} r^2 & \text{if } r \text{ is even,} \\ r^2+1 & \text{if } r \text{ is odd.} \end{cases}$$

then every simple d-regular graph has an $(r, r+1)$-factorization.The numbers r^2 and r^2+1 here are the best possible for $r \geq 3$. ∎

A slight variation on this was provided by Hilton [18] in 2008 who proved.

Theorem 1.13 *If*

$$d \geq \begin{cases} r(r+1) & \text{if } r \text{ is even,} \\ r(r+1)+1 & \text{if } r \text{ is odd.} \end{cases}$$

then every simple $(d, d+1)$-graph has an $(r, r+1)$-factorization.These numbers are also best possible. ∎

Another feature of our study is the following initially rather surprising fact. Consider the example of a 29-regular simple graph. By Vizing's theorem G has a proper edge-colouring with 30 colours. Combining these in 3's, we obtain 10 sets of combined colours. Thus G has a $(2,3)$-factorization with 10 $(2,3)$-factors. By Gupta's theorem, G has an edge-covering with 28 colours. Combining these in 2's, we obtain 14 sets of combined colours. Therefore G has a $(2,3)$-factorization with 14 $(2,3)$-factors. If we let t be the number of values of x for which G has an $(r, r+a)$-factorization, then we see that $t \geq 2$ in the case of a 29-regular simple graph with $r = 2, a = 1$. A deeper analysis shows that in fact $t = 5$ in this case.

These considerations motivate the following definitions.

The threshold numbers:

We let $\sigma(r, s, a, t)$, the **simple graph threshold number**, be the least value of d, say $d = d_0$, such that every $(d, d+s)$-simple graph with $d \geq d_0$ has an $(r, r+a)$-factorization with x factors for at least t values of x. In the case when there is no such d_0, we put $\sigma(r, s, a, t) = \infty$.

The **multigraph threshold number**, $\mu(r,s,a,t)$ is defined similarly except that we consider $(d,d+s)$-multigraphs (without loops), instead of simple graphs.

Similarly we have the **pseudograph threshold number** $\pi(r,s,a,t)$ where we consider $(d,d+s)$-pseudographs (where multiloops are allowed - each loop counts 2 to the degree of its vertex).

We also have the **bipartite graph threshold number** $\beta(r,s,a,t)$ where we consider $(d,d+s)$-bipartite multigraphs, and the **simple bipartite graph threshold number** $\beta_s(r,s,a,t)$, where we consider $(d,d+s)$-simple bipartite graphs.

There are some relationships between these threshold numbers which can be deduced immediately.

Lemma 1.14 *For $r \geq 0, s \geq 0, a \geq 0, t \geq 1$, we have*

$$\beta_s(r,s,a,t) \leq \sigma(r,s,a,t) \leq \mu(r,s,a,t) \leq \pi(r,s,a,t).$$

Proof To see that $\beta_s(r,s,a,t) \leq \sigma(r,s,a,t)$ we first notice that if G is a simple bipartite graph, then it is of course a simple graph. Then we notice that it could well be that a non-bipartite simple graph G does not have an $(r,r+a)$-factorization, whilst every bipartite simple graph of the same minimum and maximum degrees does have an $(r,r+a)$-factorization. Since every simple graph is a special kind of multigraph, and every multigraph to a special kind of pseudograph, it follows similarly that

$$\sigma(r,s,a,t) \leq \mu(r,s,a,t) \leq \pi(r,s,a,t).$$

$\blacksquare$

Lemma 1.14 is almost obvious. It is also almost obvious that if $\rho \leq r$ and $r+a \leq \rho+\alpha$ then any $(r,r+a)$-factor of a pseudograph is a $(\rho,\rho+\alpha)$-factor. From this simple fact we deduce:

Lemma 1.15 *Let ρ,r,s,a,α,t be integers with ρ,r,t positive and a,α,s non-negative. Let $\rho \leq r \leq r+a \leq \rho+\alpha$. Then*

$$\pi(r,s,a,t) \geq \pi(\rho,s,\alpha,t).$$

Proof We note that a graph could have a $(\rho,\rho+\alpha)$-factorization, and yet not have an $(r,r+a)$-factorization. But an $(r,r+a)$-factorization is a $(\rho,\rho+\alpha)$-factorization.

$\blacksquare$

Lemma 1.15 has two useful consequences.

Corollary 1.16 *Let r,s,a,t be integers with r,a,t positive and s non-negative. Then*

(i) $\pi(r,s,a,t) \geq \pi(r,s,a+1,t)$

(ii) $\pi(r,s,a,t) \leq \pi(r+1,s,a-1,t)$.

The same result holds for the same reason in the case of all the other threshold functions: $\beta_s(r,s,a,t)$, $\beta(r,s,a,t), \sigma(r,s,a,t), \mu(r,s,a,t)$.

We illustrate these concepts with a number of examples. Since every $(d,d+s)$-simple graph has a $(0,1)$-factorization it follows that $\sigma(0,s,1,1)=0$. From Gupta's theorem (Theorem 1.3) every d-regular simple graph has a $(1,2)$-factorization into $d-1$ $(1,2)$-factors. It follows that $\sigma(1,0,1,1)=1$. We gave above an example in which any 29-regular simple graph has a $(2,3)$-factorization with x factors for five values of x. It follows that $\sigma(2,0,1,5) \leq 29$. In fact a more detailed analysis shows that $\sigma(2,0,1,5)=28$ (see [24]). As an example of a function which equals ∞, we cite $\pi(r,0,1,1)$. Consider a graph G with one vertex v on which are placed $2x+1$ loops for some $x \geq 1$. Every $(2,3)$-factor of such a graph consists of a single loop. Similarly if $p \geq 2$ is an integer, putting $x=2p$, we see that every $(2p,2p+1)$-factorization consists of p loops and so G does not have a $(2p,2p+1)$-factorization. Thus for indefinitely large values of r, G has no $(r,r+1)$-factorization, so $\pi(r,0,1,1)=\infty$.

The first result on $(r,r+a)$-factorizations seems to be the following one due to Akiyama, Avis and Era [1] in 1980.

Theorem 1.17 *Every regular pseudograph is $(1,2)$-factorable. In particular, if r is an odd integer then every r-regular pseudograph can be decomposed into $\frac{r+1}{2}$ $(1,2)$-factors.*

This led Akiyama to conjecture that for every integer $d \geq 1$, there exists an integer $\sigma(r,0,1,1)$ such that if $d \geq \sigma(r,0,1,1)$ then every d-regular simple graph is $(r,r+1)$-factorable. This led to the Theorem 1.12 quoted earlier by Era and Egawa that for $r \geq 3$,

$$\sigma(r,0,1,1) = \begin{cases} r^2 & \text{if } r \text{ is even,} \\ r^2+1 & \text{if } r \text{ is odd.} \end{cases}$$

Also note that $\sigma(2,0,1,1) = 2$ and $\sigma(1,0,1,1) = 1$ since it is obvious that every 2-regular graph is $(2,3)$ factorable with exactly one factor (a 2-factor) and every 1-regular graph is $(1,2)$-factorable with exactly one factor (a 1-factor). In 2005 in [24], Hilton and Wojciechowski evaluated $\sigma(r,s,1,1)$:

Theorem 1.18 *For integers $r \geq 3$ and $s \geq 0$,*

$$\sigma(r,s,1,1) = \begin{cases} r^2+rs & \text{if } r \text{ is even, } 0 \leq s \leq 1, \\ r^2+rs+1 & \text{if } r \text{ is odd, } 0 \leq s \leq 1, \\ r^2+rs+r+1 & \text{if } s \geq 2. \end{cases}$$

The result was extended further by Hilton [19] in 2009 by bringing the parameter t into a formula, and evaluating $\sigma(r,s,1,t)$:

Theorem 1.19 *For integers $r \geq 3, t \geq 1$ and $s \geq 0$,*

$$\sigma(r,s,1,t) = \begin{cases} tr^2+tr+sr-r & \text{if } r \text{ is even and } s \in \{0,1\}, \\ tr^2+tr+sr-r+1 & \text{if } r \text{ is odd and } s \in \{0,1\}, \\ tr^2+tr+sr+1 & \text{if } s \geq 2. \end{cases}$$

For $(r,s,a,t) = (r,0,1,1)$ in the multigraph case, the best result to date is due to Ferencak and Hilton [12] partly building on earlier work by Era [11]. In this case the result for r odd is very different from the result for r even. Moreover we do not have an exact evaluation of $\mu(r,0,1,1)$ when r is even.

Theorem 1.20 *Let r be a positive integer. Then*

$$\mu(r,0,1,1) = r^2+1 \text{ if } r \text{ is odd,}$$

$$\frac{3}{2}r^2 - 2r - 1 \leq \mu(r,0,1,1) \leq \frac{3}{2}r^2 + 3r + 1 \text{ if } r \text{ is even.}$$

In [13], Ferencak and Hilton examined the gap between $\frac{3}{2}r^2+3r+1$ and $\frac{3}{2}r^2-2r-1$ more closely, obtaining further information.

Though the two types of graphs, pseudographs and simple graphs, might seem to be the most natural, Theorem1.20 (more particularly the difficulty of proving it) seems to suggest that the most difficult threshold function to determine will be $\mu(r,s,a,t)$ concerning loopless multigraphs.

Our main concern in this thesis is to evaluate as many of the threshold functions $\beta(r,s,a,t), \beta_s(r,s,a,t)$, $\sigma(r,s,a,t), \mu(r,s,a,t)$ and $\pi(r,s,a,t)$ as we can.

For bipartite graphs we have the following evaluation.

Theorem 1.21 *Let integers r,a,t be positive and s be non-negative. Then*

$$\beta(r,s,a,t) = \beta_s(r,s,a,t) = r\left\lceil \frac{tr+s-1}{a} \right\rceil + (t-1)r.$$

The result in this theorem may usefully be re-expressed in the following way:

$$\beta(r,s,a,t) = \beta_s(r,s,a,t) = r\frac{tr+s+c}{a} + (t-1)r$$

where c is such that $a \mid tr+s+c$ and $-1 \le c \le a-2$.

This result was mostly obtained by Hilton in [20] but is included here, partly for completeness, and partly because the proof provides a model for many of the rest of the proofs.

In the case when r and a are both even we have the following striking result, due to Hilton[20].

Theorem 1.22 ***(3.15)*** *Let integers r,a,t be positive and s be non-negative and r and a both even and positive. Then*

$$\beta(r,s,a,t) = \beta_s(r,s,a,t) = \sigma(r,s,a,t) = \mu(r,s,a,t) = \pi(r,s,a,t) = r\left\lceil\frac{tr+s-1}{a}\right\rceil + (t-1)r.$$

It is convenient to define

$$N(r,s,a,t) = r\left\lceil\frac{tr+s-1}{a}\right\rceil + (t-1)r$$

when r,s,a,t are integers with r,a,t positive and s non-negative.

The pseudograph threshold numbers are completely determined in the following two theorems. Part (i) of Theorem 1.23 is due to Hilton [20], but the rest of Theorem 1.23 and the whole of Theorem 1.24 is due to me and Prof. Hilton. The published version of Theorem 1.23 [21] contains a mistake in Part (3) concerning the case when $(r+1)t+s \equiv 3(mod\ a-2)$.

Theorem 1.23 ***(5.1)*** *Let r,s,a and t be integers with r and t positive, $a \ge 2$ and s non-negative.*
(1) If r and a are both even, then

$$\pi(r,s,a,t) = N(r,s,a,t).$$

(2) If r and a are both odd, then

$$\pi(r,s,a,t) = \begin{cases} N(r+1,s,a-1,t)-1 & \textit{if } (r+1)t+s \not\equiv 2(mod\ a\text{-}1), \\ N(r+1,s,a-1,t)-(r+1)-1 & \textit{if } (r+1)t+s \equiv 2(mod\ a\text{-}1). \end{cases}$$

(3) If r is odd and a is even, $a \ge 4$, then

$$\pi(r,s,a,t) = \begin{cases} N(r+1,s,a-2,t)-1 & \textit{if } (r+1)t+s \not\equiv 2,3(mod\ a\text{-}2), \\ N(r+1,s,a-2,t)-(r+1)-1 & \textit{if } (r+1)t+s \equiv 2,3(mod\ a\text{-}2). \end{cases}$$

(4) If r is even and a is odd, then

$$\pi(r,s,a,t) = \begin{cases} N(r,s,a-1,t) & \textit{if } rt+s \not\equiv 2(mod\ a\text{-}1), \\ N(r,s,a-1,t)-r & \textit{if } rt+s \equiv 2(mod\ a\text{-}1). \end{cases}$$

For $a=0$ or 1, or $a=2$, r odd we give the results in Theorem 1.24(5.2). Note that we use the notation $\pi(r,s,a,t)=\infty$ when there is no finite threshold number for the given value of r,s,a and t.

If $a=2$ then $\pi(r,s,a,t)$ is given in Theorem 3.14 if r is even, but if r is odd it is given below in Theorem 1.24(5.2).

Theorem 1.24 ***(5.2)*** *Let r,s and t be integers with r and t positive and s non-negative. Then*

$$\pi(r,s,0,t) = \infty$$

and

$$\pi(r,s,1,t)=\begin{cases}2 & \text{if } r=2, s=0 \text{ and } t=1,\\ 1 & \text{if } r=1, s=0 \text{ and } t=1,\\ \infty & \text{otherwise,}\end{cases}$$

and if r is odd, then

$$\pi(r,s,2,t)=\begin{cases}\infty & \text{if } r\geq 1, \text{ and } s>1 \text{ or } t>1,\\ 1 & \text{if } r=1\ ,\ s\in\{0,1\} \text{ and } t=1.\end{cases}$$

It should be noted that these evaluations are slightly different from all earlier attempts at evaluating $\pi(r,s,a,t)$ (which were wrong). Correcting this error has necessitated a considerable amount of additional calculations, and is the cause of the length of Chapter 5. The original error occurred in Theorem 24 of the paper [20] by Hilton, with similar errors in Theorem 27 and 30 of that paper. Corrected versions of those theorems are given in this thesis as Theorem 4.5, 4.8 and 4.11 respectively.

Our final main result is an evaluation of $\sigma(r,s,a,t)$. Theorem 1.19 gives the evaluation when $a=1$. We have the following result valid for $a\geq 1$:

Theorem 1.25 ***(6.1)*** *Let $r\geq 1, s\geq 0, a\geq 2$ and $t\geq 1$ be integers. Then*
(i) If r is even and a is even, then

$$\sigma(r,s,a,t)=r\left\lceil\frac{tr+s-1}{a}\right\rceil+(t-1)r;$$

(ii) If r is odd and a is even, then

$$\sigma(r,s,a,t)=\begin{cases} r\left\lceil\frac{tr+s+1}{a}\right\rceil+(t-1)r+1 & \text{if } t\geq 2, \text{ or if } t=1 \text{ and } a<r+s+1\\ r & \text{if } t=1 \text{ and } a\geq r+s+1;\end{cases}$$

(iii) If r is even and a is odd, then

$$\sigma(r,s,a,t)=r\left\lceil\frac{tr+s}{a}\right\rceil+(t-1)r;$$

(iv) If r and a are both odd, and if $t\geq 2$, or $t=1$ and $a<tr+s$, then

$$\sigma(r,s,a,t)=r\left\lceil\frac{tr+s}{a}\right\rceil+(t-1)r+1$$

However if $t=1$ and $a\geq rt+s$ then $\sigma(r,s,a,t)=r$.

It will be noticed that in the case of simple graphs, the evaluations are all quite close to each other (not far from $r\left\lceil\frac{tr+s}{a}\right\rceil+(t-1)r$) unlike the case of pseudographs where the denominator varies from $a-2$ to a in the various cases.

Another main point of interest in this thesis relates to the following Theorems due to Prof. Hilton and me.

Theorem 1.26 *Let $a\geq 0$ and $r\geq 1$. Let $d\geq 1$ and $s\geq 0$. Every $(d,d+s)$-simple graph G has an $(r,r+a)$-factorization with x factors if and only if*

(i)

$$\frac{d+s}{r+a}\leq x\leq\frac{d}{r}$$

when r and a are both even;

(ii)

$$\frac{d+s}{r+a} < x < \frac{d}{r}$$

when r is odd and a is even;

(iii)

$$\frac{d+s}{r+a} < x \leq \frac{d}{r}$$

when r is even and $a \geq 1$ is odd;

(iv)

$$\frac{d+s}{r+a} \leq x < \frac{d}{r}$$

when r is odd and a is odd.

Part (i) of Theorem 1.26 is Theorem 3.14. Part (ii) of Theorem 1.26 is Theorem 6.8. Part (iii) of Theorem 1.26 is Theorem 8.7. Part (iv) of Theorem 1.26 is Theorem 9.3.

Akiyama and Kano's work.

Finally we remark that the topic of $(r, r+a)$-factorization of graphs has been considered by various Japanese Mathematicians. Akiyama and Kano [4] have recently published a book "Factors and factorizations". The main topic is just factors, and they do not specifically aim to evaluate threshold numbers. They do not draw much distinction between simple graphs, pseudographs and multigraphs, and the parameter t is not mentioned. Generally, where there is a coincidence of topic, their results are weaker than ours.

One result of Kano [25] in 1985 is of particular interest:

Theorem 1.27 *Let a and b be even integers such that $0 \leq a \leq b$, and let $n \geq 1$ be an integer. Then the pseudograph G can be decomposed into n (a,b)-factors if and only if G is an (an, bn)-graph.*

We show that this is a corollary of our Theorem 3.14.

Chapter 2

Bipartite graphs

2.1 Introduction

This chapter is a modification of the first part of a paper by Prof. Hilton [20]. It is included because the actual results are the foundation for our results on pseudographs (specifically Theorem 3.14) and the method of proof provides a template for the proof of our results on simple graphs.

We first define a function $N(r, s, a, t)$.

Definition 2.1 *Let r, a, t be positive integers and let s be non-negative integer. Then*

$$N(r, s, a, t) = r\left\lceil\frac{tr + s - 1}{a}\right\rceil + (t - 1)r.$$

The main result we prove in this chapter is:

Theorem 2.2 ***(1.21)*** *Let r, a, t be positive integers, and let s be a non-negative integer. Then*

$$\beta_s(r, s, a, t) = \beta(r, s, a, t) = N(r, s, a, t).$$

The following lemma follows immediately from the definitions of $\beta(r, s, a, t)$ and $\beta_s(r, s, a, t)$ since each simple graph is just a special kind of multigraph.

Lemma 2.3 *Let r, a, t be positive integers, and let s be a non-negative integer. Then*

$$\beta_s(r, s, a, t) \leq \beta(r, s, a, t).$$

Proof By the definition of $\beta_s(r, s, a, t)$, there is a $(d, d + s)$-simple graph G of degree $\beta_s(r, s, a, t) - 1$ which does have an $(r, r + a)$-factorization with x factors for t different values of x. But G is an example of a $(d, d + s)$-multigraph of degree $\beta_s(r, s, a, t) - 1$ which does not have an $(r, r + a)$-factorization with x factors for t different values of x. Therefore

$$\beta_s(r, s, a, t) \leq \beta(r, s, a, t).$$

■

We do not know from the proof of Lemma 2.3 that there is no larger value of d and a $(d, d+s)$-multigraph M of degree $d-1$ such that M has no $(r, r+a)$-factorization with x factors for t different values of x. Obviously M, if it existed, could not be a simple graph.

To prove Theorem 2.2(1.21) therefore it suffices to show

(i) there is an example of a $(d, d+s)$-simple bipartite graph with $d = N(r, s, a, t) - 1$ which does not have an $(r, r + a)$-factorization with x factors for t different values of x, and

(ii) if M is any $(d, d+s)$-bipartite multigraph with $d \geq N(r, s, a, t)$, then M has an $(r, r+a)$-factorization with x factors for t different values of x.

Theorem 2.2(1.21) is actually a consequence of Theorem 2.8 below.

Before concluding our introduction, let us draw attention to the following lemma about $(r, r + a)$-factorizations of $(d, d + s)$-pseudographs.

Lemma 2.4 *Let r and d be positive integers and s and a be non-negative integers. Let G be a pseudograph with at least one vertex of degree d and at least one vertex of degree $d+s$. Suppose that G is $(r, r+a)$-factorable with exactly $x \geq 1$ factors. Then*

$$\frac{d + s}{r + a} \leq x \leq \frac{d}{r}.$$

Proof Let v be a vertex of degree $d + s$. Then $x(r + a) \geq \mathrm{d}(v) = d + s$, so $x \geq \frac{d+s}{r+a}$. Similarly, if w is a vertex of degree d, then $xr \leq \mathrm{d}(w) = d$, so $x \leq \frac{d}{r}$.

■

2.2 Factorizing bipartite multigraphs

In the first theorem in this section we show that, given d, r, a, s, there is a large interval $I = I(d, r, a, s) = \left[\frac{d}{r+a}, \frac{d+s}{r}\right]$ which has the property that there exist $(d, d+s)$-bipartite multigraphs G which are $(r, r+a)$-factorable with x factors if and only if $x \in I$, and a smaller interval $J = J(d, r, a, s) = \left[\frac{d+s}{r+a}, \frac{d}{r}\right]$ which has the property that all $(d, d+s)$-bipartite multigraphs are $(r, r+a)$-factorable into x factors if and only if $x \in J$. All this is also true for $(d, d+s)$-bipartite simple graphs.

Our first theorem is:

Theorem 2.5 *Let d, r and x be positive integers, and let a, s be non-negative integers.*

(i) If

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$$

then every $(d, d+s)$-bipartite multigraph is $(r, r+a)$-factorable with x factors (so every $(d, d+s)$-bipartite simple graph is $(r, r+a)$-factorable with x factors).

(ii) If $a > 0$ and

$$x \in \left[\frac{d}{r+a}, \frac{d+s}{r+a}\right) \cup \left(\frac{d}{r}, \frac{d+s}{r}\right]$$

then some $(d, d+s)$-bipartite simple graphs are, and some are not, $(r, r+a)$-factorable with x factors (so some $(d, d+s)$-bipartite multigraphs are, and some are not, $(r, r+a)$-factorable with x factors).

(iii) If

$$x \notin \left[\frac{d}{r+a}, \frac{d+s}{r}\right]$$

then no $(d, d+s)$-bipartite multigraph is $(r, r+a)$-factorable with x factors (so no $(d, d+s)$-simple graphs are $(r, r+a)$-factorable with x factors).

Proof (i) Suppose that $\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$. Then $r \leq \frac{d}{x} \leq \frac{d+s}{x} \leq r+a$. Let G be a $(d, d+s)$-bipartite multigraph. By Theorem 1.5, G has an equitable edge-colouring with x colours. Since $r \leq \frac{d}{x} \leq \frac{d+s}{x} \leq r+a$, it follows that each colour class is an $(r, r+a)$-factor of G. Thus G is $(r, r+a)$-factorable with x factors.

(ii) Let $a > 0$ and $x \in \left[\frac{d}{r+a}, \frac{d+s}{r+a}\right) \cup \left(\frac{d}{r}, \frac{d+s}{r}\right]$.

First we show that there are $(d, d+s)$-bipartite simple graphs which are $(r, r+a)$-factorable with x factors. Since $\frac{d}{r+a} \leq x \leq \frac{d+s}{r}$ then, since $\frac{d}{r}$ increases as d increases or as r decreases, there are integers a_1 and s_1 with $0 < a_1 \leq a$ and $0 \leq s_1 \leq s$ such that

$$\frac{d}{r+a} \leq \frac{d+s_1}{r+a_1} \leq x.$$

Since $x \leq \frac{d+s}{r}$, we may suppose that s_1 cannot be increased nor a_1 decreased. Therefore it follows that

$$\frac{d}{r+a} \leq \frac{d+s_1}{r+a_1} \leq x \leq \frac{d+s_1}{r+a_1-1} \leq \frac{d+s}{r},$$

and so $x(r+a_1-1) \leq d+s_1 \leq x(r+a_1)$. Therefore, there are integers $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 + x_2 = x$ such that $x_1(r+a_1) + x_2(r+a_1-1) = d+s_1$, or, putting $a_1 - 1 = a_2$,

$$x_1(r+a_1) + x_2(r+a_2) = d+s_1$$

with $0 \leq a_1, a_2 \leq a$.

Let G be a $(d+s_1)$-regular bipartite simple graph with bipartition (V_1, V_2). Give G an equitable edge-colouring with x colours $F_1, \ldots, F_x$. For $1 \le i \le x$, let F_i also denote the bipartite simple graph with bipartition (V_1, V_2) and edges of colour F_i. Then F_i is an $(r+a_1, r+a_1+1)$-simple bipartite graph with the property that $G = \bigcup_{i=1}^{x} F_i$ where a_2=a_1+1. Then G is a $(d, d+s)$-bipartite simple graph which is $(r, r+a)$-factorable with x factors.

Next we show that there are $(d, d+s)$-bipartite simple graphs which are not $(r, r+a)$-factorable with x factors.

Firstly, let $x \in \left[\frac{d}{r+a}, \frac{d+s}{r+a}\right)$ and let G be a $(d+s)$-regular bipartite simple graph. The average degree over all the factors of the vertices of G in a decomposition of G into x factors is $\frac{d+s}{x}$. But $x < \frac{d+s}{r+a}$ so that $\frac{d+s}{x} > r+a$, so the factors cannot all be $(r, r+a)$-factors.

Secondly, let $x \in \left(\frac{d}{r}, \frac{d+s}{r}\right]$ and let G be a d-regular bipartite simple graph. The average degree over all the factors of the vertices of G in a decomposition of G into x factors is $\frac{d}{x}$. But $x > \frac{d}{r}$ so that $\frac{d}{x} < r$, so the factors cannot all be $(r, r+a)$-factors.

(iii) If $x < \frac{d}{r+a}$ then $x(r+a) < d$. Thus the union of x $(r, r+a)$-bipartite multigraphs has maximum degree less than d, and so no $(d, d+s)$-bipartite multigraph has a decomposition into x $(r, r+a)$-factors. Similarly, if $x > \frac{d+s}{r}$, then $xr > d+s$. Thus the union of x $(r, r+a)$-bipartite multigraphs has minimum degree greater than $d+s$, so no $(d, d+s)$-bipartite multigraph has a decomposition into x $(r, r+a)$-factors.

∎

We note the following two corollaries of Theorem 2.5.

Corollary 2.6 *Let d, r, x and a be positive integers and let s be a non-negative integer. Then every $(d, d+s)$-bipartite multigraph is $(r, r+a)$-factorable with x $(r, r+a)$-factors if and only if*

$$x \in \left[\frac{d+s}{r+a}, \frac{d}{r}\right].$$

Corollary 2.7 *Let d, r, x and a be positive integers and let s be a non-negative integer. Then there is some $(d, d+s)$-bipartite multigraph which is $(r, r+a)$-factorable with x factors if and only if*

$$x \in \left[\frac{d}{r+a}, \frac{d+s}{r}\right].$$

Recall that Prof. McDiarmid has suggested the notation $F_{\{r,a\}}(G)$ to denote the set of integers x for which a pseudograph G has an $(r, r+a)$-factorization with x $(r, r+a)$-factors, valid whenever r and a are non-negative integers. Recall that Prof. McDiarmid has shown that, if G is a bipartite multigraph, then $F_{\{r,a\}}(G)$ is an interval of integers. Using the notation $F_{\{r,a\}}(G)$, we can extend Theorem 2.5 slightly, and change its form slightly, as follows:

Theorem 2.5a Let d, a and r be positive integers and let s be a non-negative integer. Then

(i) For each $(d, d+s)$-bipartite multigraph G,

$$Z \cap \left[\frac{d+s}{r+a}, \frac{d}{r}\right] \subseteq F_{\{r,a\}}(G) \subseteq Z \cap \left[\frac{d}{r+a}, \frac{d+s}{r}\right];$$

(ii) For any $(d, d+s)$-bipartite multigraph G with at least one vertex of degree d and at least one vertex of degree $d+s$,

$$F_{\{r,a\}}(G) = Z \cap \left[\frac{d+s}{r+a}, \frac{d}{r}\right];$$

(iii) For each integer $x \in \left[\frac{d}{r+a}, \frac{d+s}{r}\right]$, there is a $(d, d+s)$-bipartite simple graph G such that $F_{\{r,a\}}(G)$ contains x.

Explanation
Theorem 2.5(i) and Theorem 2.5(iii) $\iff$ Theorem 2.5a(i).
Details for the proof of Theorem 2.5a(ii) are provided in the proof of Theorem 2.5(ii).

A further remark The inequality $\frac{d+s}{r+a} \le x \le \frac{d}{r}$ implies that $\frac{d+s}{r+a} \le \frac{d}{r}$, which is equivalent to $d \ge r(\frac{s}{a})$.

Thus if $d < r(\frac{s}{a})$ and G is a $(d, d+s)$-bipartite graph with a vertex of degree d and a vertex of degree $d+s$ then G has no $(r, r+a)$-factorization. So if $d < r(\frac{s}{a})$, there are $(d, d+s)$-bipartite multigraphs which are not $(r, r+a)$-factorable; depending on the value of d, there may exist $(d, d+s)$-bipartite graphs which are $(r, r+a)$-factorable.

Next we apply Corollary 2.7. Recall that for positive integers r, a, t and a non-negative integer s, $\beta(r, s, a, t)$ is the smallest integer such that, for each integer $d \ge \beta(r, s, a, t)$, each $(d, d+s)$-bipartite multigraph is $(r, r+a)$-factorable with x factors for at least t different values of x. $\beta_s(r, s, a, t)$ is similar, but for simple bipartite graphs. In Theorem 2.8 we evaluate $\beta(r, s, a, t)$.

Theorem 2.8 *Let integers r, a, t be positive and s be non-negative. Then*

$$\beta_s(r, s, a, t) = \beta(r, s, a, t) = \frac{r}{a}(tr + s + c) + (t-1)r,$$

where c is such that $a \mid tr + s + c$ and $-1 \le c \le a - 2$.

The expression

$$\beta(r, s, a, t) = \frac{r}{a}(tr + s + c) + (t-1)r,$$

where c is such that $a \mid tr + s + c$ and $-1 \le c \le a-2$ is another way of saying that

$$\beta(r, s, a, t) = \left\lceil \frac{r}{a}(tr + s - 1) \right\rceil + (t-1)r,$$

i.e.

$$\beta(r, s, a, t) = N(r, s, a, t).$$

Thus Theorem 2.8 is just Theorem 2.2 expressed in a slightly different way.

Proof of Theorem 2.8.

(i) We show that

$$\beta_s(r, s, a, t) \ge \frac{r}{a}(tr + s + c) + (t-1)r$$

where $a \mid tr + s + c$ and $-1 \le c \le a-2$.

Let $d = \frac{r}{a}(tr + s + c) + (t-1)r - 1$. We show that, for this value of d, there do not exist t values of x between $\frac{d+s}{r+a}$ and $\frac{d}{r}$. Then, by Theorem 2.5 a(ii), it follows that there exist $(d, d+s)$-bipartite simple graphs which are not $(r, r+a)$-factorable with x factors for t different values of x.

We have

$$\frac{d}{r} = \frac{1}{a}(tr + s + c) + (t-1) - \frac{1}{r}$$

and

$$d + s = (r+a)\frac{1}{a}(tr + s + c) - c - r - 1$$

so that

$$\frac{d+s}{r+a} = \frac{1}{a}(tr + s + c) - \frac{r+c+1}{r+a}.$$

Since $c+1<a$ it follows that the values of x which satisfy $\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$ are $\frac{1}{a}(tr+s+c)+j$ for $0 \leq j \leq t-2$, so there are indeed fewer than t such values. To complete the proof of (i) we need an example of a bipartite simple graph G which has x $(r, r+a)$-factors only if $x \in \left[\frac{d+s}{r+a}, \frac{d}{r}\right]$. Let G be the disjoint union of A and B, where A is a d-regular bipartite simple graph and B is a $(d+s)$-regular bipartite simple graph. Then A is the union of x $(r, r+a)$-factors only if $x \in \left[\frac{d}{r+a}, \frac{d}{r}\right]$ and B is the union of x $(r, r+a)$-factors only if $x \in \left[\frac{d+s}{r+a}, \frac{d+s}{r}\right]$. Therefore G is the union of x $(r, r+a)$-factors only if $x \in \left[\frac{d+s}{r+a}, \frac{d}{r}\right]$, as required.

(ii) Next we show that $\beta(r,s,a,t) \leq \frac{r}{a}(tr+s+c)+(t-1)r$.

Let $d = \frac{r}{a}(tr+s+c)+(t-1)r+k$, where $k \geq 0$. We show that, in this case, there do exist t values of x between $\frac{d+s}{r+a}$ and $\frac{d}{r}$. Then it follows from Theorem 2.5 that every $(d, d+s)$-bipartite multigraph is $(r, r+a)$-factorable into x factors for at least t values of x.

First note that

$$\frac{d}{r} = \frac{1}{a}(tr+s+c)+t-1+\frac{k}{r}$$

and that

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) - \frac{r+c}{r+a} + \frac{k}{r+a}.$$

Therefore if $r+c \geq k \geq 0$ then, since $r+a > r+a-2 \geq r+c$, the values of x lying between $\frac{d+s}{r+a}$ and $\frac{d}{r}$ include

$$\frac{1}{a}(tr+s+c), \ldots, \frac{1}{a}(tr+s+c)+t-1$$

so there are at least t values of x. We also note that

$$\frac{d}{r} - \frac{d+s}{r+a} = t-1+\frac{r+c}{r+a}+\frac{ak}{r(r+a)}.$$

Therefore if $\frac{r+c}{r+a}+\frac{ak}{r(r+a)} \geq 1$, i.e. $k \geq (1-\frac{c}{a})r$, then $\frac{d}{r} - \frac{d+s}{r+a} \geq t$.

Since c is an integer, if $c \neq -1$ then all values of $k \geq 0$ satisfy one of the inequalities $k \geq (1-\frac{c}{a})r$ and $r+c \geq k \geq 0$, so it follows from Corollary 2.6 that $\beta(r,s,a,t) \leq \frac{r}{a}(tr+s+c)+(t-1)r$.

Now consider further the case when $c=-1$. If $0 \leq k \leq r-1$ then, as we just showed, there are t suitable integral values of x. Now suppose that $2r+a \geq k \geq r$. Then

$$\frac{d}{r} \geq \frac{1}{a}(tr+s+c)+(t-1)+1 = \frac{1}{a}(tr+s+c)+t,$$

while

$$\begin{aligned} \frac{d+s}{r+a} &= \frac{1}{a}(tr+s+c)+1-\frac{2r+a-k+c}{r+a} \\ &\leq \frac{1}{a}(tr+s+c)+1, \end{aligned}$$

since $c=-1<2r+a-k$. So in this case also there are t suitable integral values of x.

The set of inequalities $0 \leq k \leq r-1$ ($r \leq k < 2r+a+c$ when $c=-1$) and $k \geq (1+\frac{1}{a})r$ cover all values of $k \geq 0$. Therefore it follows that

$$\beta(r,s,a,t) \leq \frac{r}{a}(tr+s+c)+(t-1)r$$

in this case also.

It follows that $\beta_s(r,s,a,t) = \beta(r,s,a,t) = N(r,s,a,t)$ asserted. This proves Theorem 2.8 (and Theorem 2.2). ∎

Chapter 3

The pseudograph threshold number $\pi(r, s, a, t)$ with r and a both even

In this chapter we use the König-Petersen connection to evaluate $\pi(r,s,a,t)$ when r and a are both even. We have to assume initially that s is also even, but we are then able to remove this restriction. It should be noted that as multigraphs and simple graphs are special kinds of pseudographs the results apply for them as well. Most of this chapter is also derived from the paper [20] by Prof. Hilton.

We start by deriving an analogue of Theorem 2.5

Theorem 3.1 *Let d, r and a be positive integers, and let s be a non-negative integer.*

(i) If

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r},$$

then every $(2d, 2d+2s)$-pseudograph is $(2r, 2r+2a)$-factorable with x factors.

(ii) If

$$x \in \left[\frac{d}{r+a}, \frac{d+s}{r+a}\right) \cup \left(\frac{d}{r}, \frac{d+s}{r}\right]$$

then some $(2d, 2d+2s)$-pseudographs are and some are not $(2r, 2r+2a)$-factorable with x $(2r, 2r+2a)$-factors.

(iii) If

$$x \notin \left[\frac{d}{r+a}, \frac{d+s}{r}\right]$$

then no $(2d, 2d+2s)$-pseudograph is $(2r, 2r+2a)$-factorable with x factors.

It is convenient to prove Theorem 3.1 by deducing it from Theorem 2.5 using the following well-known connection between pseudographs and bipartite multigraphs, i.e. the König-Petersen connection.

Let G be a pseudograph. Pair off the vertices of G of odd degree, and, for each such pair $\{x, y\}$, introduce an extra edge xy. Call the pseudograph obtained this way G^*. Then each component of G^* is Eulerian. Choose an Eulerian circuit of each component of G^* and orient the edges in one direction round each such Eulerian circuit. If $V = V(G^*) = \{v_1, v_2, \ldots, v_r\}$ then construct a bipartite multigraph $B(G^*)$ with vertex sets $U = \{u_1, \ldots, u_r\}$ and $W = \{w_1, \ldots, w_r\}$. If (v_x, v_y) is an oriented edge of G^* then join u_x to w_y in $B(G^*)$ by an edge. If G^* has a loop on v_x, then join u_x to w_x in $B(G^*)$. Now from $B(G^*)$ construct a bipartite multigraph $B(G)$ by deleting each edge of $B(G^*)$ that corresponds to one of the extra edges introduced above in forming G^* from G. Clearly, given a pseudograph G, the extra edges, the Eulerian circuits of the components, and the orientations can all usually be chosen in many different ways, so there are many possibilities for $B(G)$. They all have the property that

$$|d_{B(G)}(u_i) - d_{B(G)}(w_i)| \leq 1 \text{ for each } i, 1 \leq i \leq r,$$

and the degrees of u_i and w_i sum to $d_G(v_i)$ for each i, $1 \leq i \leq r$.

On the other hand, given a bipartite multigraph B with vertex sets $U = \{u_1, \ldots, u_r\}$ and $W = \{w_1, \ldots, w_r\}$ satisfying the inequality $|d_B(u_i) - d_B(w_i)| \leq 1$, then it is possible to obtain an oriented pseudograph with in and out-degrees differing by at most one. Let $G(B)$ denote this pseudograph with the orientation removed. Given a pseudograph G, although there are many different possibilities for $B(G)$, reversing the construction will always produce the original pseudograph G again. Thus $G(B(G)) = G$.

We now develop this connection in a more specific way for $(2r, 2r+2a)$-factorizations.

Theorem 3.2 *A pseudograph G is $(2r, 2r+2a)$-factorable with x factors if and only if a corresponding bipartite multigraph $B(G)$ is $(r, r+a)$-factorable with x factors, if and only if each corresponding bipartite multigraph $B(G)$ is $(r, r+a)$-factorable with x factors.*

Proof (i) Suppose G has a $(2r,2r+2a)$-factorization into x $(2r,2r+2a)$-factors $F_1,\dots,F_x$. For $1\le i\le x$, construct a bipartite multigraph $B(F_i)$ corresponding to the factor F_i. Then $B(F_i)$ is an $(r,r+a)$-bipartite multigraph and $(B(F_1),\dots,B(F_x))$ is an $(r,r+a)$-factorization of a bipartite multigraph $B(G)$.

(ii) Suppose a bipartite multigraph B has an $(r,r+a)$-factorization into x $(r,r+a)$-factors, say $F_1,\dots,F_x$. For each i, $1\le i\le x$, F_i corresponds to a $(2r,2r+2a)$-pseudograph $G(F_i)$, and $(G(F_1),\dots,G(F_x))$ is a $(2r,2r+2a)$-factorization of $G(B)$.

∎

We now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1.

(i) Let G be a $(2d,2d+2s)$-pseudograph and let $\dfrac{d+s}{r+a}\le x\le \dfrac{d}{r}$. From G we may form a bipartite $(d,d+s)$-multigraph $B(G)$. By Theorem 2.5(i) $B(G)$ has an $(r,r+a)$-factorization into x $(r,r+a)$-factors. By Theorem 3.2, this corresponds to a $(2r,2r+2a)$-factorization of G into x $(2r,2r+2a)$-factors.

(ii) Let $x\in\left[\dfrac{d}{r+a},\dfrac{d+s}{r+a}\right)\cup\left(\dfrac{d}{r},\dfrac{d+s}{r}\right]$. By Theorem 2.5 (ii), some $(d,d+s)$-bipartite multigraphs are and some are not $(r,r+a)$-factorable with x factors. Let B_1 and B_2 be $(d,d+s)$-bipartite multigraphs which do, and do not, respectively, have an $(r,r+a)$-factorization into x $(r,r+a)$-factors. Then, by Theorem 3.2, $G(B_1)$ and $G(B_2)$ are $(2r,2r+2a)$-pseudographs which do, and do not, respectively, have a $(2r,2r+2a)$-factorization into x $(2r,2r+2a)$-factors.

(iii) Let $x\notin\left[\dfrac{d}{r+a},\dfrac{d+s}{r}\right]$. By Theorem 2.5(iii), no $(d,d+s)$-bipartite multigraph is $(r,r+a)$-factorable with x factors. Therefore, by Theorem 3.2, no $(2d,2d+2s)$-pseudograph is $(2r,2r+2a)$-factorable with x factors. ∎

We note the following corollaries to Theorem 3.1.

Corollary 3.3 *Let d,r,x be positive integers and let s and a be a non-negative integers. Then every $(2d,2d+2s)$-pseudograph is $(2r,2r+2a)$-factorable with x factors if and only if*

$$x\in\left[\frac{d+s}{r+a},\frac{d}{r}\right].$$

Corollary 3.4 *Let d,r,x and a be positive integers and let s be a non-negative integer. Then there is some $(2d,2d+2s)$-pseudograph which is $(2r,2r+2a)$-factorable with x $(2r,2r+2a)$-factors if and only if*

$$x\in\left[\frac{d}{r+a},\frac{d+s}{r}\right].$$

We can slightly extend Theorem 3.1 in the same way that we extended Theorem 2.5 to obtain Theorem 2.5a.

<u>Theorem 3.1a</u> Let d, r and a be positive integers and let s be a non-negative integer. Then

(i) For every $(2d,2d+2s)$-pseudograph G,
$Z\cap\left[\frac{d+s}{r+a},\frac{d}{r}\right]\subseteq Z\cap F_{\{r,a\}}(G)\subseteq Z\cap\left[\frac{d}{r+a},\frac{d+s}{r}\right]$.

(ii) There is a $(2d,2d+2s)$-pseudograph G such that
$Z\cap F_{\{r,a\}}(G)=Z\cap\left[\frac{d+s}{r+a},\frac{d}{r}\right]$.

(iii) For each integer $x\in\left[\frac{d+s}{r+a},\frac{d}{r}\right]$, there is a $(2d,2d+2s)$-pseudograph G such that $Z\cap F_{\{r,a\}}(G)$ contains x.

Observation If r and a are even non-negative integers, and if G is a $(2d, 2d+2s)$-pseudograph, then $F_{\{r,a\}}(G)$ is an interval of integers. This follows from the statement on page 7 following Theorem 1.6, the König-Petersen connection and Theorem 3.2.

We are now able to start thinking about evaluating $\pi(r,s,a,t)$ when r and a are both even:

Theorem 3.5 *Let r, a, t be positive integers and s a non-negative integer. Let r, s and a all be even. Let c be an even integer such that $a \mid tr+s+c$ and $0 \le \frac{c}{2} \le \frac{a}{2}-1$. Then*

$$\pi(r,s,a,t) = \frac{r}{a}(tr+s+c)+(t-1)r.$$

In order to prove Theorem 3.5 more easily, we introduce two further functions, $\pi_e(r,s,a,t)$ and $\gamma(r,s,a,t)$. For integers $t \ge 1$, $r \ge 2$, $a \ge 2$, $s \ge 0$ and r, a, s all even, we let $\pi_e(r,s,a,t)$ be the least even integer such that, for each even integer $d \ge \pi_e(r,s,a,t)$, each $(d, d+s)$-pseudograph is $(r, r+a)$-factorable with x factors for at least t different values of x.

For integers $r, a, t \ge 1$ and $s \ge 0$, we let $\gamma(r,s,a,t)$ be the smallest integer such that, for each integer $d \ge \gamma(r,s,a,t)$, each $(2d, 2d+2s)$-pseudograph is $(2r, 2r+2a)$-factorable with x factors for at least t different values of x.

We first determine the value of $\gamma(r,s,a,t)$.

Lemma 3.6 *Let r, s, a, t be integers with r, a and t positive and s non-negative. Then*

$$\gamma(r,s,a,t) = \frac{r}{a}(tr+s+c)+(t-1)r,$$

where c is such that $a \mid tr+s+c$ and $-1 \le c \le a-2$.

Proof It follows from Theorem 3.2 that a $(2d, 2d+2s)$-pseudograph G is $(2r, 2r+2a)$-factorable with x factors if and only if a corresponding $(d, d+s)$-bipartite multigraph $B(G)$ is $(r, r+a)$-factorable with x factors. Therefore $\gamma(r,s,a,t) = \beta(r,s,a,t)$. But, by Theorem 2.8, $\beta(r,s,a,t) = \frac{r}{a}(tr+s+c)+(t-1)r$, where $a \mid tr+s+c$ and $-1 \le c \le a-2$.

∎

From Lemma 3.6 we deduce immediately the following Lemma 3.7. Lemma 3.7 is essentially Lemma 3.6 rephrased.

Lemma 3.7 *Let r, s, a, t be integers with r, a, t positive and s non-negative. Let r, s, and a all be even. Then*

$$\pi_e(r,s,a,t) = \frac{r}{a}(tr+s+c)+(t-1)r,$$

where c is such that $a \mid tr+s+c$ and $-1 \le \frac{c}{2} \le \frac{a}{2}-2$.

Proof From the definitions of $\gamma(r,s,a,t)$ and $\pi_e(r,s,a,t)$ it follows that, if r, s, a are all even, then

$$\pi_e(r,s,a,t) = 2\gamma(\frac{r}{2}, \frac{s}{2}, \frac{a}{2}, t),$$

so by Lemma 3.6,

$$\pi_e(r,s,a,t) = 2\frac{(r/2)}{(a/2)}\left(t\frac{r}{2}+\frac{s}{2}+\frac{c}{2}\right)+2(t-1)\frac{r}{2},$$

where c is such that $(a/2) \mid t(r/2)+(s/2)+(c/2)$ (so that c is also even) and $-1 \le \frac{c}{2} \le \frac{a}{2}-2$. Therefore

$$\pi_e(r,s,a,t) = \frac{r}{a}(tr+s+c)+(t-1)r,$$

where c is such that $a \mid tr+s+c$ (so that c is even) and $-1 \le \frac{c}{2} \le \frac{a}{2}-2$.

■

Lemma 3.8 *Let r,s,a,t be integers with r,a,t all positive and s non-negative. Let r, s, and a all be even. Then*

$$\pi_e(r,s+2,a,t) = \begin{cases} \pi_e(r,s,a,t) & \text{if } a \mid rt+s+c,\ 0 \le \frac{c}{2} \le \frac{a}{2}-2, \\ \pi_e(r,s,a,t)+r & \text{if } a \mid rt+s+c,\ \frac{c}{2}=-1. \end{cases}$$

Proof By Lemma 3.7

$$\pi_e(r,s+2,a,t) = \frac{r}{a}(tr+(s+2)+c')+(t-1)r\,,$$

where $a \mid tr+(s+2)+c'$ and $-1 \le \frac{c'}{2} \le \frac{a}{2}-2$. Put $c^* = c'+2$. Then

$$\begin{aligned}\pi_e(r,s+2,a,t) &= \frac{r}{a}(tr+(s+2)+(c^*-2))+(t-1)r \\ &= \frac{r}{a}(tr+s+c^*)+(t-1)r,\end{aligned}$$

where $a \mid tr+s+c^*$ and $0 \le \frac{c^*}{2} \le \frac{a}{2}-1$. If $0 \le \frac{c^*}{2} \le \frac{a}{2}-2$, then it follows from Lemma 3.7 that

$$\pi_e(r,s+2,a,t) = \pi_e(r,s,a,t).$$

If $\frac{c^*}{2} = \frac{a}{2}-1$, then put $c^+ = c^*-a$. Then

$$\begin{aligned}\pi_e(r,s+2,a,t) &= \frac{r}{a}(tr+s+c^++a)+(t-1)r \\ &= \frac{r}{a}(tr+s+c^+)+(t-1)r+r,\end{aligned}$$

where $a \mid tr+s+c^+$ and $\frac{c^+}{2}=-1$. Therefore, by Lemma 3.7, in this case we have

$$\pi_e(r,s+2,a,t) = \pi_e(r,s,a,t)+r.$$

■

By definition, when r,s,a are all even, if d is EVEN and $d \ge \pi_e(r,s,a,t)$ then each $(d,d+s)$-pseudograph is $(r,r+a)$-factorable with x factors for t different values of x, but $\pi(r,s,a,t)$ has the EXTRA property that if d is ODD and $d \ge \pi(r,s,a,t)$ then each $(d,d+s)$-pseudograph is $(r,r+a)$-factorable with x factors for t different values of x. Thus it is clear that $\pi(r,s,a,t) \ge \pi_e(r,s,a,t)-1$ when r,s,a are all even. We note that Theorem 3.5 tells us that, except when $\frac{c}{2} \ne -1$, $\pi(r,s,a,t) = \pi_e(r,s,a,t)$, but when $\frac{c}{2}=-1$ then $\pi(r,s,a,t) = \pi_e(r,s,a,t)+r$.

Proof of Theorem 3.5.

If $d \ge \pi_e(r,s+2,a,t)$ and if d is even, then any $(d,d+s+2)$-pseudograph is $(r,r+a)$-factorable with x factors for t different values of x. Suppose $d \ge \pi_e(r,s+2,a,t)$ and d is odd. Any $(d,d+s)$-pseudograph G is a $(d-1,d-1+s+2)$-pseudograph, so, since $d-1$ is even and at least $\pi_e(r,s+2,a,t)$, G is $(r,r+a)$-factorable with x factors for t different values of x. Thus $\pi(r,s,a,t) \le \pi_e(r,s+2,a,t)$.

Now let $d = \pi_e(r,s+2,a,t)-1$ and consider a pseudograph $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \emptyset$, G_1 is a regular pseudograph of degree d, and G_2 is a regular pseudograph of degree $d+s$. Any $(r,r+a)$-factorization of G contains an $(r,r+a)$-factorization of G_1 and an $(r,r+a)$-factorization of G_2.

By Lemma 3.8, $\pi_e(r,s+2,a,t) = \pi_e(r,s,a,t)$ or $\pi_e(r,s,a,t)+r$. Suppose first that $\pi_e(r,s+2,a,t) = \pi_e(r,s,a,t)$. Let $a \mid rt+s+c$ where, in accordance with Lemma 3.8, $0 \le \frac{c}{2} \le \frac{a}{2}-2$. Consider G_1. Then

$$\frac{d}{r} = \frac{1}{a}(tr+s+c)+(t-1)-\frac{1}{r},$$

so the number of $(r,r+a)$-factors of G_1 (and therefore G) could have is at most $\frac{1}{a}(tr+s+c)+(t-2)$. Now consider G_2. Then

$$\begin{aligned}\frac{d+s}{r+a} &= \frac{1}{(r+a)}\frac{1}{a}(tr^2+sr+cr)+\frac{(t-1)r}{r+a}-\frac{1}{r+a}+\frac{s}{r+a}\\ &= \frac{1}{(r+a)}\left(\frac{tr(r+a)}{a}+\frac{s(r+a)}{a}+\frac{c(r+a)}{a}-r-1-c\right)\\ &= \frac{1}{a}(tr+s+c)-\frac{r+1+c}{r+a}.\end{aligned}$$

Since $0 \le \frac{c}{2} \le \frac{a}{2}-2$, it follows that $r+1+c < r+a$ so that $\dfrac{r+1+c}{r+a} < 1$. Therefore the number of $(r,r+a)$-factors in any $(r,r+a)$-factorization is at least $\frac{1}{a}(tr+s+c)$. Therefore the number of different values of x for which G has an $(r,r+a)$-factorization with x $(r,r+a)$-factors is at most $t-1<t$.

Now suppose that $\pi_e(r,s+2,a,t)=\pi_e(r,s,a,t)+r$. Let $a \mid rt+s+c$, where, again in accordance with Lemma 3.8, $\frac{c}{2}=-1$. Then

$$\frac{d}{r}=\frac{1}{a}(tr+s+c)+t-\frac{1}{r},$$

so the greatest number of $(r,r+a)$-factors G_1 could have is $\frac{1}{a}(tr+s+c)+t-1$. Now consider G_2. Then

$$\frac{d+s}{r+a}=\frac{1}{a}(tr+s+c)-\frac{1+c}{r+a},$$

where $\frac{c}{2}=-1$. Then $1+c=-1$ so $-\frac{1+c}{r+a}>0$. Therefore the number of $(r,r+a)$-factors G_2 could have is at least $\frac{1}{a}(tr+s+c)+1$. Therefore the number of different values of x for which G has an $(r,r+a)$-factorization with x factors is at most $t-1<t$. Thus

$$\pi(r,s,a,t) \ge \pi_e(r,s+2,a,t).$$

Consequently

$$\pi(r,s,a,t)=\pi_e(r,s+2,a,t),$$

so, by Lemma 3.8,

$$\pi(r,s,a,t)=\begin{cases}\pi_e(r,s,a,t) & \text{if } a \mid rt+s+c,\ 0 \le \frac{c}{2} \le \frac{a}{2}-2,\\ \pi_e(r,s,a,t)+r & \text{if } a \mid rt+s+c,\ \frac{c}{2}=-1.\end{cases}$$

Therefore, by Lemma 3.7,

$$\begin{aligned}\pi(r,s,a,t) &= \begin{cases}\frac{r}{a}(tr+s+c)+(t-1)r & \text{if } a \mid rt+s+c,\ 0 \le \frac{c}{2} \le \frac{a}{2}-2,\\ \frac{r}{a}(tr+s+c)+(t-1)r+r & \text{if } a \mid rt+s+c,\ \frac{c}{2}=-1.\end{cases}\\ &= \frac{r}{a}(tr+s+c)+(t-1)r \qquad \text{if } a \mid rt+s+c,\ 0 \le \frac{c}{2} \le \frac{a}{2}-1.\end{aligned}$$

∎

Corollary 3.9 *Let r,s,a,t be integers with r,a,t all positive and s non-negative. Let r, s and a be even. Then*

$$\pi(r,s,a,t)=\pi_e(r,s+2,a,t).$$

Theorem 3.10 *Let r,s,a,t be integers with r,a,t positive and s non-negative. Then*
(a)

$$\pi(r,s,a,t)=r\left\lceil\frac{tr+s}{a}\right\rceil+(t-1)r,$$

if r,s and a are all even,

(*b*)

$$\pi(r,s,a,t) \le \pi(r,s+1,a,t)$$

otherwise.

Proof (a) follows from Theorem 3.5. To prove (b), let $d \geq \pi(r,s+1,a,t)$. Any $(d,d+s)$-pseudograph is also a $(d,d+s+1)$-pseudograph. Thus if all $(d,d+s+1)$-pseudographs are $(r,r+a)$-factorable with x factors for at least t values of x, then so are all $(d,d+s)$-pseudographs. Therefore $\pi(r,s+1,a,t) \geq \pi(r,s,a,t)$.

∎

It remains to remove the restriction in Theorem 3.10(a) that s be even.

Lemma 3.11 *Let r,s,a,t be integers with r,a,t all positive and s non-negative. Let r,a,s be even. If*

$$\left\lceil \frac{rt+s}{a} \right\rceil = \left\lceil \frac{rt+s+2}{a} \right\rceil$$

then $\pi(r,s,a,t) = \pi(r,s+1,a,t) = \pi(r,s+2,a,t)$.

Proof By Theorem 3.10(b), $\pi(r,s,a,t) \leq \pi(r,s+1,a,t) \leq \pi(r,s+2,a,t)$. By Theorem 3.10(a), since $\left\lceil \frac{rt+s}{a} \right\rceil = \left\lceil \frac{rt+s+2}{a} \right\rceil$, it follows that $\pi(r,s,a,t) = \pi(r,s+2,a,t)$, so Lemma 3.11 follows.

∎

It remains to consider the case when $\left\lceil \frac{rt+s}{a} \right\rceil < \left\lceil \frac{rt+s+2}{a} \right\rceil$. Since r, s and a are even, this occurs when $a \mid rt+s$. Thus we need to evaluate $\pi(r,s+1,a,t)$ when r and a are even, s is odd and $a \mid rt+s-1$. We do this in Lemma 3.12.

Lemma 3.12 *Let r,s,a,t be integers with r,a,t positive and s non-negative. Let r,a be even and s be odd, and let $a \mid rt+s-1$. Then*

$$\pi(r,s,a,t) = r\left(\frac{rt+s-1}{a}\right) + (t-1)r.$$

Proof Let $d^* = r\left(\frac{rt+s-1}{a}\right) + (t-1)r$. First note that

$$\begin{aligned} \pi(r,s,a,t) &\geq \pi(r,s-1,a,t) && \text{by Lemma 3.10,} \\ &= \pi_e(r,s-1,a,t) && \text{by Corollary 3.9,} \\ &= r\left(\tfrac{rt+s-1}{a}\right) + (t-1)r && \text{by Lemma 3.7 with } c=0, \\ &= d^*. \end{aligned}$$

Next notice that, by Lemma 3.7 (with $c=-2$),

$$\pi_e(r,s+1,a,t) = \frac{r}{a}(tr+(s+1)-2) + (t-1)r,$$

and $a \mid tr+(s+1)-2$.

Thus $\pi_e(r,s+1,a,t) = d^*$. Then, for d even, $d \geq d^*$, any $(d,d+s+1)$-pseudograph is $(r,r+a)$-factorable with x factors for t different values of x; therefore any $((d+1),(d+1)+s)$-pseudograph has this property too (since any $((d+1),(d+1)+s)$-pseudograph is a $(d,d+s+1)$-pseudograph), and any $(d,d+s)$-pseudograph has this property (since any $(d,d+s)$-pseudograph is a $(d,d+s+1)$-pseudograph). Therefore, for any integer $d \geq d^*$, any $(d,d+s)$-pseudograph is $(r,r+a)$-factorable with x factors for t different values of x. Thus $d^* \geq \pi(r,s,a,t)$, and so

$$\pi(r,s,a,t) = \frac{r}{a}(tr+s-1) + (t-1)r$$

when $a \mid tr+s-1$.

∎

To sum up our knowledge of $\pi(r,s,a,t)$ when r and a are even, we have:

Theorem 3.13 *Let r,s,a,t be integers with r,a,t positive and s non-negative. Let r and a be even. Then*

$$\pi(r,s,a,t) = r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r.$$

Proof This follows from Theorem 3.10 and Lemma 3.12.

∎

It is convenient to develop Theorem 3.1 (or Corollary 3.3) further to the following:

Theorem 3.14 *Let d, r, x be positive integers and let a and s be non-negative integers. Let r and a be even. Then every $(d, d+s)$-pseudograph is $(r, r+a)$-factorizable with x factors if and only if*

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}.$$

Proof (1). If d and s are even this follows from Corollary 3.3.
(2). If d and s are both odd, if

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$$

then

$$\frac{(d-1)+(s+1)}{r+a} \leq x \leq \frac{d-1}{r}$$

so, by Corollary 3.3, every $(d-1,(d-1)+(s+1))$-pseudograph is $(r, r+a)$-factorizable with x factors. But a $(d, d+s)$-pseudograph is a $(d-1,(d-1)+(s+1)$-pseudograph. Therefore every $(d, d+s)$-pseudograph is $(r, r+a)$-factorizable with x factors.

Conversely, if every $(d, d+s)$-pseudograph is $(r, r+a)$-factorizable with x factors, then by Lemma 2.4,

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}.$$

(3). If d is odd and s is even and $\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$, then

$$\frac{(d-1)+(s+2)}{r+a} \leq x \leq \frac{d-1}{r},$$

so every $(d-1,(d-1)+(s+2))$-pseudograph is $(r, r+a)$-factorable with x factors. But every $(d, d+s)$-pseudograph is a $(d-1,(d-1)+(s+2))$-pseudograph, so every $(d, d+s)$-pseudograph is $(r, r+a)$-factorable with x factors.

The converse is true as in (2).

(4). If d is even and s is odd and $\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$, then

$$\frac{d+(s+1)}{r+a} \leq x \leq \frac{d}{r},$$

so every $(d, d+(s+1))$-pseudograph is $(r, r+a)$-factorable with x factors. But a $(d, d+s)$-pseudograph is a $(d, d+(s+1))$-pseudograph, so every $(d, d+s)$-pseudograph is $(r, r+a)$-factorable with x factors.

The converse is true as in (2) again.

∎

Theorem 3.14 has Theorem 1.27 due to Kano as a corollary. Recall that Kano's theorem says: Let a and b be even integers such that $0 \leq a \leq b$, and let $n \geq 1$ be an integer. Then a pseudograph can be decomposed into n (a, b)-factors if and only if G is an (an, bn)-graph.

To prove this from Theorem 3.14 note that if $\frac{d+s}{r+a} = x = \frac{d}{r}$ then $d = xr$ and $d+s = xr+xa$, so $s = xa$. Thus a pseudograph has an $(r, r+a)$-factorization with x factors if and only if it is an $(xr, xr+xa)$-graph. This is Kano's theorem with our x, a, r, d, s replacing Kano's $n, b-a, a, na, nb-na$ respectively.

From Theorem 2.2, Theorem 3.14 and Theorem 1.14 we obtain a simple proof of Theorem 1.22(3.15) which stated:

Theorem 3.15 *Let r, s, a and t be integers with r and t positive, s non-negative and r and a both even and positive. Then*

$$\beta(r, s, a, t) = \beta_s(r, s, a, t) = \sigma(r, s, a, t) = \mu(r, s, a, t) = \pi(r, s, a, t) =$$
$$N(r, s, a, t) = r\left\lceil \frac{tr + s - 1}{a} \right\rceil + (t-1)r.$$

Proof From Theorem 2.2 we have that

$$\beta(r, s, a, t) = \beta_s(r, s, a, t) = N(r, s, a, t),$$

from Theorem 3.13 we have that

$$\pi(r, s, a, t) = N(r, s, a, t),$$

and from Lemma 1.14 we have that

$$\beta_s(r, s, a, t) \le \sigma(r, s, a, t) \le \mu(r, s, a, t) \le \pi(r, s, a, t).$$

Theorem 3.15 follows immediately.

∎

A further remark. As with the case of bipartite multigraphs, $\frac{d+s}{r+a} \le \frac{d}{r}$ if and only if $d \ge r(\frac{s}{a})$. If $d < r(\frac{s}{a})$ then there is a $(d, d+s)$-pseudograph which does not have an $(r, r+a)$-factorization; depending on the value of d, there may be $(d, d+s)$-pseudographs which do have $(r, r+a)$-factorizations.

Prof. McDiarmid remarked that the following theorem is true. We provide a proof.

Theorem 3.16 *Let $r \ge 0$ and $a \ge 0$ be even integers with $r + a > 0$. Then for each pseudograph G, $F_{\{r,a\}}(G)$ is an interval of integers.*

Proof For any non-negative even integers r and a with $r + a > 0$, and any pseudograph G, and any choice of $B(G)$,

$$F_{\{r,a\}}(G) = F_{\{\frac{r}{2},\frac{a}{2}\}}(B(G))$$

is an interval of integers, as shown by Prof. McDiarmid. Therefore $F_{\{r,a\}}(G)$ is an interval of integers.

∎

Chapter 4

Bounds for $\pi(r, s, a, t)$ when r, a are not both even

Rather surprisingly, we can find reasonable bounds when r and a are not both even. Later, in Chapter 5, we refine those bounds to obtain actual evaluations.

This chapter is, like chapters 2 and 3, largely based on the paper "Degree bounded factorizations of bipartite multigraphs and of pseudographs" by Prof. Hilton[20]. However in the course of discussions with Prof Hilton, it was noticed that there was an oversight in a proof of one of the theorems of that paper [This is manifested here in Theorem 4.5 (Case 3 iii) where originally it was not noticed that the assumption $(r+1)t+s-1 \not\equiv 2(mod\ a-1)$ (or $(r+1)t+s \not\equiv 3(mod\ a-1)$) was needed. It is manifested similarly in Theorem 4.8 (Case 3 iii) and Theorem 4.11(Case 3 iii).] This oversight affects the contents of Chapter 5, making it much longer, and it also affects the final result obtained there, in particular the evaluation of $\pi(r,s,a,t)$ when r is odd and a is even.

We first note the following lemmas.

Lemma 4.1 ***(1.15)*** *Let ρ, r, s, a, α, t be integers with ρ, r, t positive and s, a, α non-negative. Let $\rho \le r \le r+a \le \rho+\alpha$. Then*

$$\pi(r,s,a,t) \ge \pi(\rho,s,\alpha,t).$$

Two special cases of Lemma 4.1(1.15) are of particular importance.

Corollary 4.2 *(1.16)Let r,s,a,t be integers with r,a,t positive and s non-negative. Then*

(i) $\pi(r,s,a,t) \ge \pi(r,s,a+1,t)$.

(ii) $\pi(r,s,a,t) \le \pi(r+1,s,a-1,t)$.

Next we bound $\pi(r,s,a,t)$ when r and a are both odd.

Lemma 4.3 *Let r,s,a,t be integers with r,t positive, $a \ge 3$ and s non-negative. Let r,a be odd and s be even, let $(r+1)t+s \not\equiv 2 \pmod{a-1}$. Then*

$$\pi(r+1,s,a-1,t)-1 \le \pi(r,s,a,t) \le \pi(r+1,s,a-1,t).$$

Note that, as $r+1$ and $a-1$ are both even, $\pi(r+1,s,a-1,t)$ is evaluated in Theorem 3.15.

Proof By Corollary 4.2(1.16), $\pi(r,s,a,t) \le \pi(r+1,s,a-1,t)$.
To prove the other inequality, let $d = \pi(r+1,s,a-1,t)-2$, so that, by the formula in Theorem 3.15, d is even. Let G be the $(d,d+s)$-pseudograph with two components, G_1 and G_2, where G_1 has one vertex on which are placed $\frac{d}{2}$ loops, and G_2 has one vertex on which are placed $\frac{d+s}{2}$ loops. Since r and a are both odd and all the edges of G are in fact loops, any $(r,r+a)$-factor of G is actually an $(r+1,r+a)$-factor, i.e. an $((r+1),(r+1)+(a-1))$-factor.

As in the proof of Theorem 3.14, it follows that for any $((r+1),(r+1)+(a-1))$-factorization of G into x $((r+1),(r+1)+(a-1))$-factors,

$$\frac{d+s}{(r+1)+(a-1)} \le x \le \frac{d}{r+1}.$$

Since $d = \pi(r+1,s,a-1,t)-2$, it follows from Theorem 3.15 (since s, $r+1$ and $a-1$ are even and positive) that

$$d = (r+1)\left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-1)(r+1) - 2,$$

so

$$\frac{d}{r+1} = \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-1) - \frac{2}{r+1}.$$

Therefore

$$x \le \left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-2)\,.$$

We also have that

$$d+s = (r+1)\left\lceil \frac{t(r+1)+s}{a-1} \right\rceil + (t-1)(r+1) + s - 2\,,$$

so that

$$d+s=\frac{(r+1)}{(a-1)}\big(t(r+1)+s+c\big)+(t-1)(r+1)+s-2\,,$$

where $0\le\frac{c}{2}\le\frac{a-1}{2}-1$ and $a-1\mid(r+1)t+s+c$. Therefore

$$\begin{aligned}d+s&=\{(r+1)+(a-1)\}\frac{t(r+1)+s+c}{a-1}-(t(r+1)+s+c)+(t-1)(r+1)+s-2\\&=(r+a)\frac{(t(r+1)+s+c)}{a-1}-(r+1)-c-2\end{aligned}$$

so that

$$\begin{aligned}\frac{d+s}{(r+1)+(a-1)}&=\frac{t(r+1)+s+c}{a-1}-\frac{r+c+3}{r+a}\\&=\left\lceil\frac{t(r+1)+s}{a-1}\right\rceil-\frac{r+c+3}{r+a}.\end{aligned}$$

Since $0\le\frac{c}{2}\le\frac{a-1}{2}-1$ and $(r+1)t+s\not\equiv 2 \pmod{a-1}$, it follows that $r+c+3<r+a$, and so

$$x\ge\left\lceil\frac{t(r+1)+s}{a-1}\right\rceil.$$

There are therefore only $t-1$ values that x can take, so there do not exist t values of x for which G has an $((r+1),(r+1)+(a-1))$-factorization into x $((r+1),(r+1)+(a-1))$-factors. Therefore there do not exist t values of x for which G has an $(r,r+a)$-factorization into x $(r,r+a)$-factors. It follows that $d<\pi(r,s,a,t)$.

We now deduce that $\pi(r+1,s,a-1,t)-1\le\pi(r,s,a,t)$, so that

$$\pi(r+1,s,a-1,t)-1\le\pi(r,s,a,t)\le\pi(r+1,s,a-1,t).$$

∎

The missing case of Lemma 4.3, when $(r+1)t+s\equiv 2 \pmod{a-1}$, is covered less well by Lemma 4.4:

Lemma 4.4 *Let r,s,a,t be integers with r,t positive, $a\ge 3$ and $s\ge 2$. Let r,a be odd and $(r+1)t+s\equiv 2$ $\pmod{a-1}$ (so that s is even). Then*

$$\pi(r+1,s-2,a-1,t)-1\le\pi(r,s,a,t)\le\pi(r+1,s-2,a-1,t)+(r+1).$$

Note that $\pi(r+1,s-2,a-1,t)$ can be written down explicitly using Theorem 3.15.

Proof

$$\begin{array}{lll} & \pi(r+1,s-2,a-1,t)-1 & \\ \le & \pi(r,s-2,a,t) & \text{by Lemma 4.3,}\\ \le & \pi(r,s,a,t) & \text{by Lemma 3.11,}\\ \le & \pi(r,s+2,a,t) & \text{by Lemma 3.11 again,}\\ \le & \pi(r+1,s+2,a-1,t) & \text{by Lemma 4.3,}\\ = & (r+1)\left\lceil\frac{t(r+1)+(s+2)-1}{a-1}\right\rceil+(t-1)(r+1) & \text{by Theorem 3.15,}\\ = & (r+1)\left\lceil\frac{t(r+1)+(s-2)-1}{a-1}\right\rceil+(t-1)(r+1)+(r+1) & \text{since (r+1)t+s}\equiv\text{2(mod a-1),}\\ = & \pi(r+1,s-2,a-1,t)+(r+1) & \text{by Theorem 3.14.}\end{array}$$

∎

Theorem 4.5 *Suppose $r \geq 1$, $a \geq 3$ are odd, and $s \geq 0$, $t \geq 1$. If $(r+1)t+s \not\equiv 1,2,3 \bmod \mathrm{a}-1$ then*

$$\pi(r+1,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq \pi(r+1,s,a-1,t).$$

If $(r+1)t+s \equiv i \in \{1,2\} (\mathrm{mod}\ a-1)$ and $s \geq i$, then

$$\pi(r+1,s-i,a-1,t)-1 \leq \pi(r,s,a,t) \leq \pi(r+1,s-i,a-1,t)+r+1.$$

If $(r+1)t+s \equiv 3 mod\ \ a-1)$ then

$$\pi(r+1,s,a-1,t)-1 \leq \pi(r,s-3,a,t)+(r+1) \leq \pi(r+1,s,a-1,t)$$

and

$$\pi(r+1,s,a-1,t)-1 \leq \pi(r,s+1,a,t) \leq \pi(r+1,s,a-1,t).$$

Note that the outer bounding terms are given explicitly in each case in Theorem 3.15.

Proof We consider various cases.

Case 1: $(r+1)t+s \equiv 2 \ (\mathrm{mod}\ a-1)$.
In this case s is even and the theorem follows from Lemma 4.4.
Case 2: $(r+1)t+s \not\equiv 2 \ (\mathrm{mod}\ a-1)$ *and s is even.*
In this case we also have that $(r+1)t+s \not\equiv 1 \ (\mathrm{mod}\ a-1)$ and so the theorem follows from Lemma 4.3.
Case 3: $(r+1)t+s \not\equiv 2 \ (\mathrm{mod}\ a-1)$ *and s is odd.*
Case 3 (i) $(r+1)t+s \equiv 1 \ (\mathrm{mod}\ a-1)$.
Then

$$\begin{array}{lll} & \pi(r+1,s-1,a-1,t)-1 & \\ \leq & \pi(r,s-1,a,t) & \text{by Lemma 4.3,} \\ \leq & \pi(r,s,a,t) & \text{by Lemma 3.11,} \\ \leq & \pi(r,s+1,a,t) & \text{by Lemma 3.11 again,} \\ \leq & \pi(r+1,s-1,a-1,t)+r+1 & \text{by Lemma 4.4 since (r+1)t+(s+1)}\equiv\text{2(mod a-1).} \end{array}$$

Case 3(ii)
$(r+1)t+s \equiv 3 \ (\mathrm{mod}\ a-1)$. Then

$$\begin{array}{lll} & \pi(r+1,s,a-1,t)-1 & \\ = & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1)-1 & \text{by Theorem 3.15,} \\ = & (r+1)\left\lceil \frac{t(r+1)+(s-3)-1}{a-1} \right\rceil + (t-1)(r+1)-1+(r+1) & \\ = & \pi(r+1,s-3,a-1,t)-1+(r+1) & \text{by Theorem 3.15,} \\ \leq & \pi(r,s-3,a,t)+(r+1) & \text{by Lemma 4.3 since} \\ & & (r+1)t+s-3 \not\equiv 2(mod\ a-1), \\ \leq & \pi(r+1,s-3,a-1,t)+(r+1) & \text{by Lemma 4.3 again,} \\ = & (r+1)\left\lceil \frac{t(r+1)+(s-3)-1}{a-1} \right\rceil + (t-1)(r+1)+r+1 & \text{by Theorem 3.15,} \\ = & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) & \\ = & \pi(r+1,s,a-1,t). & \end{array}$$

Also

$$\begin{array}{rll}
 & \pi(r+1,s,a-1,t)-1 & \\
= & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) - 1 & \text{by Theorem 3.15,} \\
= & (r+1)\left\lceil \frac{t(r+1)+(s+1)-1}{a-1} \right\rceil + (t-1)(r+1) - 1 & \text{since } (r+1)t+s \equiv 3(mod\ a-1), \\
= & \pi(r+1,s+1,a-1,t)-1 & \text{by Theorem 3.15,} \\
\leq & \pi(r,s+1,a,t) & \text{by Lemma 4.3,} \\
\leq & \pi(r+1,s+1,a-1,t) & \text{by Lemma 4.3 again,} \\
= & (r+1)\left\lceil \frac{t(r+1)+(s+1)-1}{a-1} \right\rceil + (t-1)(r+1) & \text{by Theorem 3.15,} \\
= & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) & \text{since } (r+1)t+s \equiv 3(mod\ a-1), \\
= & \pi(r+1,s,a-1,t). &
\end{array}$$

Case 3(iii)
$(r+1)t+s \not\equiv 1 \pmod{a-1}$ *and* $(r+1)t+s \not\equiv 3 \pmod{a-1}$.
Then

$$\begin{array}{rll}
 & \pi(r+1,s,a-1,t)-1 & \\
= & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) - 1 & \text{by Theorem 3.15,} \\
= & (r+1)\left\lceil \frac{t(r+1)+(s-1)-1}{a-1} \right\rceil + (t-1)(r+1) - 1 & \text{since } (r+1)t+s-1 \not\equiv 1(mod\ a-1), \\
= & \pi(r+1,s-1,a-1,t)-1 & \text{by Theorem 3.15,} \\
\leq & \pi(r,s-1,a,t) & \text{by Lemma 4.3 since} \\
 & & (r+1)t+(s-1) \not\equiv 2(mod\ a-1), \\
\leq & \pi(r,s,a,t) & \text{by Lemma 3.11,} \\
\leq & \pi(r,s+1,a,t) & \text{by Lemma 3.11 again,} \\
\leq & \pi(r+1,s+1,a-1,t) & \text{by Lemma 4.3, since} \\
 & & (r+1)t+(s+1) \not\equiv 2(mod\ a-1), \\
= & (r+1)\left\lceil \frac{t(r+1)+(s+1)-1}{a-1} \right\rceil + (t-1)(r+1) & \text{by Theorem 3.15,} \\
= & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) & \text{since } (r+1)t+s \not\equiv 1(mod\ a-1), \\
= & \pi(r+1,s,a-1,t) & \text{by Theorem 3.15.}
\end{array}$$

∎

Our results and proofs in the remaining cases, when one of r and a is even and the other is odd are very similar to the case when both r and a are odd. The reader might feel like breezing through our accounts of these cases, but we include all the details so that proofs can be checked easily.

We look next at the case when r is even and a is odd.

Lemma 4.6 *Let r,s,a,t be integers with r,t positive, $a \geq 3$, and s non-negative. Let r and s be even and a be odd. Let $rt+s \not\equiv 2$ $(mod\ a-1)$. Then*

$$\pi(r,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq \pi(r,s,a-1,t).$$

Proof By Corollary 4.2, $\pi(r,s,a,t) \leq \pi(r,s,a-1,t)$.

To prove the other inequality, let $d = \pi(r,s,a-1,t)-2$, so that d is even. Let G be the $(d,d+s)$-pseudograph with two components, G_1 and G_2, where G_1 has one vertex on which are placed $\frac{d}{2}$ loops, and G_2 has one vertex on which are placed $\frac{d+s}{2}$ loops. Since r is even and a is odd, any $(r,r+a)$-factor of G is actually an $(r,r+(a-1))$-factor.

By Theorem 3.14, it follows that, for any $(r, r+(a-1))$-factorization of G into x $(r, r+(a-1))$-factors,

$$\frac{d+s}{r+(a-1)} \leq x \leq \frac{d}{r}.$$

Using Theorem 3.15 and using the facts that a is odd and r and s are even,

$$\frac{d}{r} = \left\lceil \frac{tr+s}{a-1} \right\rceil + t - 1 - \frac{2}{r},$$

so that

$$x \leq \left\lceil \frac{tr+s}{a-1} \right\rceil + t - 2.$$

We also have that

$$\begin{aligned} d+s &= r\left\lceil \frac{tr+s}{a-1} \right\rceil + (t-1)r - 2 + s \\ &= \frac{r}{a-1}(tr+s+c) + (t-1)r + s - 2. \end{aligned}$$

where $0 \leq \frac{c}{2} \leq \frac{a-1}{2} - 1$, c is even and $a-1 \mid rt+s+c$. Therefore

$$d+s = \frac{r+(a-1)}{a-1}(tr+s+c) - r - c - 2$$

so that

$$\begin{aligned} \frac{d+s}{r+(a-1)} &= \frac{tr+s+c}{a-1} - \frac{r+c+2}{r+(a-1)} \\ &= \left\lceil \frac{tr+s}{a-1} \right\rceil - \frac{c+r+2}{r+(a-1)}. \end{aligned}$$

But $c = a-3$ if and only if $rt+s \equiv 2 \pmod{a-1}$ so that, since $rt+s \not\equiv 2 \pmod{a-1}$,

$$x \geq \left\lceil \frac{tr+s}{a-1} \right\rceil,$$

and so there are at most $t-1$ possible values of x.

Therefore there do not exist t values of x for which G has an $(r, r+a)$-factorization into x $(r, r+a)$-factors. Therefore

$$d < \pi(r,s,a,t)$$

and so

$$\pi(r,s,a-1,t) - 1 \leq \pi(r,s,a,t).$$

∎

The missing case of Lemma 4.6, when $rt+s \equiv 2 \pmod{a-1}$, is covered in Lemma 4.7.

Lemma 4.7 *Let r,s,a,t be integers with r,t positive, $a \geq 3$, a odd and $s \geq 2$. Let r be even and $rt+s \equiv 2$ (mod $a-1$) (so that s is even). Then*

$$\pi(r,s-2,a-1,t) - 1 \leq \pi(r,s,a,t) \leq \pi(r,s-2,a-1,t) + r.$$

Proof

$$\begin{array}{lll} & \pi(r,s-2,a-1,t)-1 & \\ \leq & \pi(r,s-2,a,t) & \text{by Lemma 4.6,} \\ \leq & \pi(r,s,a,t) & \text{by Lemma 3.11,} \\ \leq & \pi(r,s+2,a,t) & \text{by Lemma 3.11 again,} \\ \leq & \pi(r,s+2,a-1,t) & \text{by Lemma 4.6} \\ = & r\left\lceil \frac{tr+(s+2)-1}{a-1} \right\rceil + (t-1)r & \text{by Theorem 3.15,} \\ = & r\left\lceil \frac{tr+(s-2)-1}{a-1} \right\rceil + (t-1)r + r & \text{since } rt+s \equiv 2 (mod\ a-1). \\ = & \pi(r,s-2,a-1,t)+r. & \end{array}$$

∎

Theorem 4.8 *Suppose $r \geq 1$ is even, $a \geq 3$ is odd and $s \geq 0$, $t \geq 1$. If $rt+s \not\equiv 1,2,3 \pmod{a-1}$ then*

$$\pi(r,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq \pi(r,s,a-1,t).$$

If $rt+s \equiv i \in \{1,2\}$ $(mod\ a-1)$ and $s \geq i$, then

$$\pi(r,s-i,a-1,t)-1 \leq \pi(r,s,a,t) \leq \pi(r,s-i,a-1,t)+r.$$

If $rt+s \equiv 3$ $(mod\ a-1)$ then

$$\pi(r,s,a-1,t)-1 \leq \pi(r,s-3,a,t)+r \leq \pi(r,s,a-1,t)$$

and

$$\pi(r,s,a-1,t)-1 \leq \pi(r,s+1,a,t) \leq \pi(r,s,a-1,t).$$

The bounding terms in each case are given explicitly by Theorem 3.15.

The proof of Theorem 4.8 follows that of Theorem 4.5, but uses Lemmas 4.6 and 4.7 instead of 4.3 and 4.4.

Proof We consider various cases.

Case 1:
$rt+s \equiv 2 \pmod{a-1}$. In this case s is even and the theorem follows from Lemma 4.7.
Case 2:
$rt+s \not\equiv 2 \pmod{a-1}$ *and s is even.* In this case we also have that $rt+s \not\equiv 1 \pmod{a-1}$ and so the theorem follows from Lemma 4.6.
Case 3:
$rt+s \not\equiv 2 \pmod{a-1}$ *and s is odd.*
Case 3 (i)
$rt+s \equiv 1 \pmod{a-1}$. Then

$$\begin{array}{lll} & \pi(r,s-1,a-1,t)-1 & \\ \leq & \pi(r,s-1,a,t) & \text{by Lemma 4.6,} \\ \leq & \pi(r,s,a,t) & \text{by Lemma 3.11,} \\ \leq & \pi(r,s+1,a,t) & \text{by Lemma 3.11 again,} \\ \leq & \pi(r,s-1,a-1,t)+r & \text{by Lemma 4.7 since} \\ & & rt+(s+1) \equiv 2 (mod\ a-1). \end{array}$$

Case 3(ii)
$rt+s \equiv 3 \pmod{a-1}$.
Then

$$
\begin{array}{rll}
 & \pi(r,s,a-1,t)-1 & \\
= & r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + (t-1)r - 1 & \text{by Theorem 3.15,} \\
= & r\left\lceil \frac{tr+(s-3)-1}{a-1} \right\rceil + (t-1)r - 1 + r & \\
= & \pi(r,s-3,a-1,t)-1+r & \text{by Theorem 3.13,} \\
\leq & \pi(r,s-3,a,t)+r & \text{by Lemma 4.6 since} \\
 & & rt+s-3 \not\equiv 2(mod\ a-1), \\
\leq & \pi(r,s-3,a-1,t)+r & \text{by Lemma 4.6 again,} \\
= & r\left\lceil \frac{tr+(s-3)-1}{a-1} \right\rceil + (t-1)r + r & \text{by Theorem 3.15,} \\
= & r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + (t-1)r & \\
= & \pi(r,s,a-1,t) & \text{by Theorem 3.15.}
\end{array}
$$

Also

$$
\begin{array}{rll}
 & \pi(r,s,a-1,t)-1 & \\
= & r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + (t-1)r - 1 & \text{by Theorem 3.15,} \\
= & r\left\lceil \frac{tr+(s+1)-1}{a-1} \right\rceil + (t-1)r - 1 & \text{by Lemma 4.6 since} \\
 & & rt+s \equiv 3(mod\ a-1), \\
= & \pi(r,s+1,a-1,t) & \text{by Theorem 3.15,} \\
\leq & \pi(r,s+1,a-1,t) & \text{by Lemma 4.6,} \\
\leq & \pi(r,s+1,a-1,t) & \text{by Lemma 4.6 again,} \\
= & r\left\lceil \frac{tr+(s+1)-1}{a-1} \right\rceil + (t-1)r & \text{by Theorem 3.15,} \\
= & r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + (t-1)r & \text{since } rt+s \equiv 3(mod\ a-1), \\
= & \pi(r,s,a-1,t) & \text{by Theorem 3.15.}
\end{array}
$$

Case 3(iii)

$rt+s \not\equiv 1 \pmod{a-1}$ *and* $rt+s \not\equiv 3 \pmod{a-1}$. Then

$$
\begin{array}{rll}
 & \pi(r,s,a-1,t)-1 & \\
= & r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + (t-1)r - 1 & \text{by Theorem 3.15,} \\
= & r\left\lceil \frac{tr+(s-1)-1}{a-1} \right\rceil + (t-1)r - 1 & \text{since } rt+s \not\equiv 1(mod\ a-1), \\
= & \pi(r,s-1,a-1,t) & \text{by Theorem 3.15,} \\
\leq & \pi(r,s-1,a,t) & \text{by Lemma 4.6 since} \\
 & & rt+(s-1) \not\equiv 2(mod\ a-1), \\
\leq & \pi(r,s,a,t) & \text{by Lemma 3.11,} \\
\leq & \pi(r,s+1,a,t) & \text{by Lemma 3.11 again,} \\
\leq & \pi(r,s+1,a-1,t) & \text{by Lemma 4.6, since} \\
 & & rt+(s+1) \not\equiv 2(mod\ a-1), \\
= & r\left\lceil \frac{tr+(s+1)-1}{a-1} \right\rceil + (t-1)r & \text{by Theorem 3.15,} \\
= & r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + (t-1)r & \text{since } rt+s \not\equiv 1(mod\ a-1), \\
= & \pi(r,s,a-1,t) & \text{by Theorem 3.15.}
\end{array}
$$

∎

Finally we consider the case when r is odd and a is even.

Lemma 4.9 *Let r, s, a, t be integers with r, t positive, $a \geq 3$ and s non-negative. Let r be odd and a, s be even. Let $(r+1)t + s \not\equiv 2 \pmod{a-2}$. Then*

$$\pi(r+1, s, a-2, t) - 1 \leq \pi(r, s, a, t) \leq \pi(r+1, s, a-2, t).$$

Proof By Corollary 4.2,

$$\pi(r, s, a, t) \leq \pi(r+1, s, a-1, t) \leq \pi(r+1, s, a-2, t).$$

To prove the other inequality, let $d = \pi(r+1, s, a-2, t) - 2$. Then d is even. Let G be the $(d, d+s)$-pseudograph with two components, G_1 and G_2, where G_1 has one vertex on which are placed $\frac{d}{2}$ loops, and G_2 has one vertex on which are placed $\frac{d+s}{2}$ loops. Since r is odd and a is even, any $(r, r+a)$-factor of G is actually an $((r+1), (r+1)+(a-2))$-factor.

By Theorem 3.14, it follows that, for any $((r+1), (r+1)+(a-2))$-factorization into x $((r+1), (r+1)+(a-2))$-factors,

$$\frac{d+s}{(r+1)+(a-2)} \leq x \leq \frac{d}{r+1}.$$

By Theorem 3.15,

$$\begin{aligned} d &= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) - 2 \\ &= (r+1)\left\lceil \frac{(t(r+1)+s)}{a-2} \right\rceil - (t-1)(r+1) - 2 \end{aligned}$$

since r is odd and a and s are even. Therefore

$$\frac{d}{r+1} = \left\lceil \frac{t(r+1)+s}{a-2} \right\rceil + (t-1) - \frac{2}{r+1}$$

so

$$x \leq \left\lceil \frac{t(r+1)+s}{a-2} \right\rceil + (t-2).$$

We also have

$$\begin{aligned} d+s &= (r+1)\left\lceil \frac{t(r+1)+s}{a-2} \right\rceil + (t-1)(r+1) + s - 2 \\ &= (r+1)\frac{(t(r+1)+s+c)}{a-2} + (t-1)(r+1) + s - 2 \end{aligned}$$

where $0 \leq \frac{c}{2} \leq \frac{a-2}{2} - 1$, c is even and $(a-2) \mid t(r+1)+s+c$. Therefore

$$\begin{aligned} d+s &= \frac{(r+1)+(a-2)}{a-2}(t(r+1)+s+c) + (t-1)(r+1) + s - 2 - (r+1)t - s - c \\ &= \frac{(r+1)+(a-2)}{a-2}(t(r+1)+s+c) - (r+1) - 2 - c \end{aligned}$$

so

$$\begin{aligned} \frac{d+s}{(r+1)+(a-2)} &= \frac{t(r+1)+s+c}{a-2} - \frac{r+3+c}{(r+1)+(a-2)} \\ &= \left\lceil \frac{t(r+1)+s}{a-2} \right\rceil - \frac{(r+1)+(c+2)}{(r+1)+(a-2)}. \end{aligned}$$

But $c = a-4$ if and only if $(r+1)t + s \equiv 2 \pmod{a-2}$ so that, since $(r+1)t + s \not\equiv 2 \pmod{a-2}$,

$$x \geq \left\lceil \frac{t(r+1)+s}{a-2} \right\rceil.$$

Therefore there do not exist t values of x for which G has an $(r, r+a)$-factorization into x $(r, r+a)$-factors. Therefore $d < \pi(r, s, a, t)$ and so $\pi(r+1, s, a-2, t) - 1 \leq \pi(r, s, a, t)$. ∎

The case when $(r+1)t+s \equiv 2 \pmod{a-2}$, missed by Lemma 4.9, is covered by Lemma 4.10.

Lemma 4.10 *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and $s \geq 2$. Let r be odd, a be even, and $(r+1)t+s \equiv 2 \pmod{a-2}$ (so s is even). Then*

$$\pi(r+1,s-2,a-2,t)-1 \leq \pi(r,s,a,t) \leq \pi(r+1,s-2,a-2,t)+(r+1).$$

Proof

$$\begin{array}{lll} & \pi(r+1,s-2,a-2,t)-1 & \\ \leq & \pi(r,s-2,a,t) & \text{by Lemma 4.9,} \\ \leq & \pi(r,s+2,a,t) & \text{by Lemma 3.11,} \\ \leq & \pi(r+1,s+2,a-2,t) & \text{by Lemma 4.9 again} \\ = & (r+1)\left\lceil \frac{t(r+1)+(s+2)-1}{a-2} \right\rceil + (t-1)(r+1) & \text{by Theorem 3.15,} \\ = & (r+1)\left\lceil \frac{t(r+1)+(s-2)-1}{a-2} \right\rceil + (t-1)(r+1)+(r+1) & \text{since } (r+1)t+s \equiv 2 (mod\ a-2). \\ = & \pi(r+1,s+2,a-2,t)+(r+1). & \end{array}$$

∎

Theorem 4.11 *Suppose $r \geq 1$ is odd, $a \geq 3$ is even, and $s \geq 0$, $t \geq 1$. If $(r+1)t+s \not\equiv i \in \{1,2,3\}$ $(mod\ a-2)$, then*

$$\pi(r+1,s,a-2,t)-1 \leq \pi(r,s,a,t) \leq \pi(r+1,s,a-2,t).$$

If $(r+1)t+s \equiv i \in \{1,2\}$ $(mod\ a-2)$ and $s \geq i$, then

$$\pi(r+1,s-i,a-2,t)-1 \leq \pi(r,s,a,t) \leq \pi(r+1,s-i,a-2,t)+r+1.$$

If $(r+1)t+s \equiv 3 mod\ \ a-2)$ then

$$\pi(r+1,s,a-2,t)-1 \leq \pi(r,s-3,a,t)+r \leq \pi(r+1,s,a-2,t)$$

and

$$\pi(r+1,s,a-2,t)-1 \leq \pi(r,s+1,a,t) \leq \pi(r+1,s,a-2,t).$$

The bounding terms in each case are given explicitly in each case in Theorem 3.15. The proof of Theorem 4.11 follows the proof of Theorem 4.5, but uses Lemmas 4.9 and 4.10 instead of Lemmas 4.3 and 4.4.

Proof We consider various cases.

Case 1:
$(r+1)t+s \equiv 2 \pmod{a-2}$. In this case s is even and the theorem follows from Lemma 4.10.
Case 2:
$(r+1)t+s \not\equiv 2 \pmod{a-2}$ *and s is even.* In this case we also have that $(r+1)t+s \not\equiv 1 \pmod{a-2}$ and the theorem follows from Lemma 4.9.
Case 3:
$(r+1)t+s \not\equiv 2 \pmod{a-2}$ *and s is odd.*
Case 3 (i)
$(r+1)t+s \equiv 1 \pmod{a-2}$. Then

$$\begin{array}{lll} & \pi(r+1,s-1,a-2,t)-1 & \\ \leq & \pi(r,s-1,a,t) & \text{by Lemma 4.9,} \\ \leq & \pi(r,s,a,t) & \text{by Lemma 3.11,} \\ \leq & \pi(r,s+1,a,t) & \text{by Lemma 3.11 again,} \\ \leq & \pi(r+1,s-1,a-2,t)+r+1 & \text{by Lemma 4.10 since } (r+1)t+(s+1) \equiv 2 (mod\ a-2). \end{array}$$

Case 3(ii)
$(r+1)t+s \equiv 3 \pmod{a-2}$. Then

$$\begin{array}{lll}
 & \pi(r+1,s,a-2,t)-1 & \\
= & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1)-1 & \text{by Theorem 3.15,} \\
= & (r+1)\left\lceil \frac{t(r+1)+(s-3)-1}{a-2} \right\rceil + (t-1)(r+1)-1+(r+1) & \\
= & \pi(r+1,s-3,a-2,t)-1+(r+1) & \text{by Theorem 3.15,} \\
\leq & \pi(r,s-3,a,t)-1+(r+1) & \text{by Lemma 4.9 since} \\
 & & (r+1)t+s-3 \not\equiv 2(mod\ a-2), \\
\leq & \pi(r+1,s-3,a-2,t)+(r+1) & \text{by Lemma 4.9 again,} \\
= & (r+1)\left\lceil \frac{t(r+1)+(s-3)-1}{a-2} \right\rceil + (t-1)(r+1)+(r+1) & \text{by Theorem 3.15,} \\
= & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) & \\
= & \pi(r+1,s,a-2,t). &
\end{array}$$

Also

$$\begin{array}{lll}
 & \pi(r+1,s,a-2,t)-1 & \\
= & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1)-1 & \text{by Theorem 3.15,} \\
= & (r+1)\left\lceil \frac{t(r+1)+(s+1)-1}{a-2} \right\rceil + (t-1)(r+1)-1 & \text{since } (r+1)t+s \equiv 3(mod\ a-2), \\
= & \pi(r+1,s+1,a-2,t)-1 & \text{by Theorem 3.15,} \\
\leq & \pi(r,s+1,a,t) & \text{by Lemma 4.8,} \\
\leq & \pi(r+1,s+1,a-2,t) & \text{by Lemma 4.8 again since} \\
 & & (r+1)t+(s+1) \equiv 2(mod\ a-2), \\
= & (r+1)\left\lceil \frac{t(r+1)+(s+1)-1}{a-2} \right\rceil + (t-1)(r+1) & \text{by Theorem 3.15,} \\
= & (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) & \text{since } (r+1)t+s \equiv 3(mod\ a-2), \\
= & \pi(r+1,s,a-2,t). &
\end{array}$$

Case 3(iii)
$(r+1)t+s \not\equiv 1 \pmod{a-2}$ *and* $(r+1)t+s \not\equiv 3 \pmod{a-2}$. Then

$$
\begin{aligned}
& \pi(r+1, s, a-2, t) - 1 & & \\
= \; & (r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) - 1 & & \text{by Theorem 3.15,} \\
= \; & (r+1)\left\lceil \tfrac{t(r+1)+(s-1)-1}{a-2} \right\rceil + (t-1)(r+1) - 1 & & \text{since } (r+1)t + s \not\equiv 1 (mod\ a-2), \\
= \; & \pi(r+1, s-1, a-2, t) - 1 & & \text{by Theorem 3.15,} \\
\leq \; & \pi(r, s-1, a, t) & & \text{by Lemma 4.9 since} \\
& & & (r+1)t + (s-1) \not\equiv 2 (mod\ a-2), \\
\leq \; & \pi(r, s, a, t) & & \text{by Lemma 3.11,} \\
\leq \; & \pi(r, s+1, a, t) & & \text{by Lemma 3.11 again,} \\
\leq \; & \pi(r+1, s+1, a-2, t) & & \text{by Lemma 4.9, since} \\
& & & (r+1)t + (s+1) \not\equiv 2 (mod\ a-2), \\
= \; & (r+1)\left\lceil \tfrac{t(r+1)+(s+1)-1}{a-2} \right\rceil + (t-1)(r+1) & & \text{by Theorem 3.15,} \\
= \; & (r+1)\left\lceil \tfrac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) & & \text{since } (r+1)t + s \not\equiv 1 (mod\ a-2), \\
= \; & \pi(r+1, s, a-2, t). & &
\end{aligned}
$$

∎

Chapter 5

Exact evaluation of the pseudograph threshold numbers $\pi(r, s, a, t)$

5.1 Exact Evaluation of the pseudograph threshold number $\pi(r, s, a, t)$

In this chapter we refine the results in Chapter 4 and obtain exact evaluations of $\pi(r, s, a, t)$ in the cases when r and a are not both even. We recall from Chapter 3 that when r and a are both even and a is positive then

$$\pi(r, s, a, t) = N(r, s, a, t) = r \left\lceil \frac{tr + s - 1}{a} \right\rceil + (t - 1)r.$$

The exact evaluations when $a \geq 2$ are given in Theorem 5.1(1.23).

Theorem 5.1 *(1.23) Let r, s, a and t be integers with r, t positive, $a \geq 2$ and s non-negative.*

1. *If r and a are both even, then*
$$\pi(r, s, a, t) = N(r, s, a, t).$$

2. *If r and a are both odd, then*
$$\pi(r, s, a, t) = \begin{cases} N(r + 1, s, a - 1, t) - 1 & \text{if } (r + 1)t + s \not\equiv 2 (mod\ a - 1), \\ N(r + 1, s, a - 1, t) - (r + 1) - 1 & \text{if } (r + 1)t + s \equiv 2 (mod\ a - 1). \end{cases}$$

3. *If r is odd and a is even, $a \geq 4$, then*
$$\pi(r, s, a, t) = \begin{cases} N(r + 1, s, a - 2, t) - 1 & \text{if } (r + 1)t + s \not\equiv 2, 3 (mod\ a - 2), \\ N(r + 1, s, a - 2, t) - (r + 1) - 1 & \text{if } (r + 1)t + s \equiv 2, 3 (mod\ a - 2). \end{cases}$$

4. *If r is even and a is odd, then*
$$\pi(r, s, a, t) = \begin{cases} N(r, s, a - 1, t) & \text{if } rt + s \not\equiv 2\ (mod\ \ a - 1), \\ N(r, s, a - 1, t) - r & \text{if } rt + s \equiv 2\ (mod\ \ a - 1). \end{cases}$$

For $a = 0$ or 1 or $a = 2$, r odd, we give the results in Theorem 5.2(1.24). Note that we use the notation $\pi(r, s, a, t) = \infty$ when there is no finite threshold number for the given value of r, s, a and t. If $a = 2$ then $\pi(r, s, a, t)$ is given by Theorem 3.15 if r is even, but if r is odd is given below in Theorem 5.2(1.24).

Theorem 5.2 *Let r, s and t be integers with r and t positive and s non-negative. Then*

$$\pi(r, s, 0, t) = \infty$$

and

$$\pi(r, s, 1, t) = \begin{cases} 2 & \text{if } r = 2, s = 0 \text{ and } t = 1, \\ 1 & \text{if } r = 1, s = 0 \text{ and } t = 1, \\ \infty & \text{otherwise}, \end{cases}$$

and if r is odd then

$$\pi(r, s, 2, t) = \begin{cases} \infty & \text{if } r \geq 1, s > 1 \text{ or } t > 1, \\ 1 & \text{if } r = 1, s \in \{0, 1\} \text{ and } t = 1. \end{cases}$$

Chapter 5 is much larger now than it was in earlier drafts. This is a result of the late discovery of the mistake mentioned before in the paper by Prof. Hilton [20], as a consequence of which a lot more careful argument is now needed to establish Theorem 5.1(1.23).

Chapter 5 is very long and so we have divided it into sections. Some of the results needed for Theorem 5.1(1.23) are common to more than one of the cases (2), (3) and (4), but some are particular to just one of the cases. Section 5.2 includes all the results needed for Case(2) of Theorem 5.1(1.23), which is given separately as Theorem 5.7. Sections 5.2 and 5.3 include all the results needed for Case(3) of Theorem 5.1(1.23), which is given separately as Theorem 5.9. Sections 5.2, 5.3 and 5.4 include all the results needed for Case(4) of Theorem 5.1(1.23), which is given separately as Theorem 5.12.

Section 5.5 contains evaluations of $\pi(r, s, 0, t)$, $\pi(r, s, 1, t)$ and $\pi(r, s, 2, t)$, cases which are not covered by Theorem 5.1(1.23).

5.2 Everything related to the case r and a both odd.

We start by lowering the upper bound on $\pi(r,s,a,t)$ in the case $(r+1)t+s \equiv 1(mod\ a-1)$ in Theorem 4.5, raising the lower bound in the case $rt+s \equiv 1(mod\ a-1)$ in Theorem 4.8 and lowering the upper bound in the case $(r+1)t+s \equiv 1(mod\ a-2)$ in Theorem 4.11.

Lemma 5.3 *Let r,s,a and t be integers with r,t positive, $a \geq 3$ and s non-negative.*

1. *If both a and r are odd and $(r+1)t+s \equiv 1(mod\ a-1)$ then* $N(r+1,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq N(r+1,s,a-1,t)$.
2. *If r is odd and a is even and $(r+1)t+s \equiv 1(mod\ a-2)$ then* $N(r+1,s,a-2,t)-1 \leq \pi(r,s,a,t) \leq N(r+1,s,a-2,t)$.
3. *If r is even and a is odd and $rt+s \equiv 1(mod\ a-1)$ then* $N(r,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq N(r,s,a-1,t)$.

Proof: The proofs of (1), (2) and (3) are very similar.

Proof of (1). By Lemma 1.15(4.1), if $\rho \leq r \leq r+a \leq \rho+\alpha$ then $\pi(r,s,a,t) \geq \pi(\rho,s,\alpha,t)$. Since $r \leq r+1 \leq (r+1)+(a-1) \leq r+a$, it follows that

$$\pi(r,s,a,t) \leq \pi(r+1,s,a-1,t).$$

In the case when r and a are both odd and $(r+1)t+s \equiv 1(mod\ a-1)$, it follows from Theorem 4.5 that

$$\pi(r+1,s-1,a-1,t)-1 \leq \pi(r,s,a,t).$$

By Theorem 3.15, since $r+1$ and $a-1$ are both even,

$$\pi(r+1,s-1,a-1,t) = N(r+1,s-1,a-1,t).$$

Also

$$N(r+1,s-1,a-1,t) = (r+1)\left\lceil \frac{(r+1)t+(s-1)-1}{a-1} \right\rceil + (t-1)(r+1).$$

But

$$\left\lceil \frac{(r+1)t+(s-1)-1}{a-1} \right\rceil = \left\lceil \frac{(r+1)t+(s-2)}{a-1} \right\rceil = \left\lceil \frac{(r+1)t+s-1}{a-1} \right\rceil$$

since $a-1$ divides $(r+1)t+s-1$. Therefore

$$\pi(r+1,s-1,a-1,t) = N(r+1,s,a-1,t) = \pi(r+1,s,a-1,t),$$

so

$$\pi(r+1,s,a-1,t)-1 \leq \pi(r,s,a,t).$$

Consequently

$$N(r+1,s,a-1,t)-1 = \pi(r+1,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq \pi(r+1,s,a-1,t) = N(r+1,s,a-1,t).$$

The proofs in cases (2) and (3) are both similar to this.

Proof of (2). Let r be odd and $a \geq 2$ be even. Using Lemma 1.15(4.1) again, we have that since $r \leq r+1 \leq (r+1)+(a-2) = r+(a-1) \leq r+a$, we have in general that

$$\pi(r,s,a,t) \leq \pi(r+1,s,a-2,t).$$

In the present case, when $(r+1)t+s \equiv 1(mod\ a-2)$ we have using Theorem 4.8 that

$$\pi(r+1,s-1,a-2,t)-1 \leq \pi(r,s,a,t).$$

But, as in (1),
$$N(r+1,s-1,a-2,t) = N(r+1,s,a-2,t).$$
Then by Theorem 3.15, since $r+1$ and $a-2$ are both even,
$$N(r+1,s,a-2,t) - 1 \le \pi(r,s,a,t) \le N(r+1,s,a-2,t).$$

Proof of (3). Let r be even and a be odd, and let $rt+s \equiv 1(mod\ a-1)$. Using Lemma 1.15(4.1) again, we have
$r \le r+a-1 \le r+a$, so that
$$\pi(r,s,a,t) \le \pi(r,s,a-1,t).$$
Using Theorem 4.11 we have in this case that
$$\pi(r,s-1,a-1,t) - 1 \le \pi(r,s,a,t).$$
But,
$$N(r,s-1,a-2,t) = N(r,s,a-2,t).$$
Then by Theorem 3.15, since r and $a-1$ are both even,
$$N(r,s,a-1,t) - 1 \le \pi(r,s,a,t) \le N(r,s,a-1,t).$$

∎

Next we deal with the cases in Theorem 4.5, Theorem 4.11 and Theorem 4.8 when $(r+1)t+s \equiv 2(mod\ a-1)$, $(r+1)t+s \equiv 2(mod\ a-2)$ and $rt+s \equiv 2(mod\ a-1)$ respectively, making similar improvements. However in these cases the arguments are more difficult.

Lemma 5.4 *Let r,s,a and t be integers with r,t positive, $a \ge 3$ and s non-negative.*

1. *If both a and r are odd and $(r+1)t+s \equiv 2(mod\ a-1)$ then* $N(r+1,s-2,a-1,t) - 1 \le \pi(r,s,a,t) \le N(r+1,s-2,a-1,t)$.
2. *If r is odd and a is even and $(r+1)t+s \equiv 2(mod\ a-2)$ then* $N(r+1,s-2,a-2,t) - 1 \le \pi(r,s,a,t) \le N(r+1,s-2,a-2,t)$.
3. *If r is even and a is odd and $rt+s \equiv 2(mod\ a-1)$ then* $N(r,s-2,a-1,t) - 1 \le \pi(r,s,a,t) \le N(r,s-2,a-1,t)$.

Remark: In Case 1, since $(r+1)t+s \equiv 2(mod\ a-1)$,
$$\begin{aligned}
N(r+1,s-2,a-1,t) &= (r+1)\left\lceil \frac{t(r+1)+(s-2)-1}{a-1} \right\rceil + (t-1)(r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s-2)}{a-1} \right\rceil + (t-1)(r+1) \\
&= N(r+1,s-1,a-1,t) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s-2)+(a-1)}{a-1} \right\rceil - (r+1) + (t-1)(r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil - (r+1) + (t-1)(r+1) \\
&= N(r+1,s,a-1,t) - (r+1).
\end{aligned}$$

Similar equalities are true in cases 2 and 3.
In Case 2, since $(r+1)t + s \equiv 2(mod\ a-2)$,

$$\begin{aligned}
N(r+1, s-2, a-2, t) &= (r+1)\left\lceil \frac{t(r+1)+(s-2)-1}{a-2} \right\rceil + (t-1)(r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s-2)}{a-2} \right\rceil + (t-1)(r+1) \\
&= N(r+1, s-1, a-2, t) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s-2)+(a-2)}{a-2} \right\rceil - (r+1) + (t-1)(r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil - (r+1), \\
&= N(r+1, s, a-1, t) - (r+1).
\end{aligned}$$

In Case 3, since $rt + s \equiv 2(mod\ a-1)$,

$$\begin{aligned}
N(r, s-2, a-1, t) &= r\left\lceil \frac{tr+(s-2)-1}{a-1} \right\rceil + (t-1)r \\
&= r\left\lceil \frac{tr+(s-2)}{a-1} \right\rceil + (t-1)r \\
&= r\left\lceil \frac{tr+(s-2)+(a-1)}{a-1} \right\rceil - r + (t-1)r \\
&= r\left\lceil \frac{tr+s-1}{a-1} \right\rceil - r + (t-1)r \\
&= N(r, s, a-1, t) - r.
\end{aligned}$$

Proof of Lemma 5.4
Proof of (1) In all cases it follows that s is even. By Lemma 4.4, and Theorem 3.15, since $r+1$ and $a-1$ are even,

$$N(r+1, s-2, a-1, t) - 1 \leq \pi(r, s, a, t).$$

We need to show here that

$$\pi(r, s, a, t) \leq N(r+1, s-2, a-1, t).$$

Let G be a $(d, d+s)$ - pseudograph with $d = d(G) = N(r+1, s, a-1, t) - (r+1) + y = N(r+1, s-2, a-1, t) + y$, where y is a non-negative integer(using the Remark before the proof). By Theorem 3.14, it is enough to show that the inequality $\frac{d+s}{r+a} \leq x \leq \frac{d}{r+1}$ is satisfied by at least t integer values of x. For then G will have an $(r+1, (r+1)+(a-1)) = (r+1, r+a)$ - factorization with x factors for t different values of x.

Since

$$d = N(r+1, s, a-1, t) - (r+1) + y$$

it follows that

$$\begin{aligned}
d &= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) - (r+1) + y \\
&= (r+1)\frac{(t(r+1)+s+a-3)}{a-1} + (t-2)(r+1) + y
\end{aligned}$$

so that

$$\frac{d}{r+1} = \frac{t(r+1)+s+a-3}{a-1} + (t-2) + \frac{y}{r+1}.$$

Also

$$\begin{aligned}
d+s &= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) - (r+1) + y + s \\
&= \frac{r+a}{a-1}(t(r+1)+s+a-3) + (t-2)(r+1) + y + s - t(r+1) - s - a + 3 \\
&= \frac{r+a}{a-1}(t(r+1)+s+a-3) - 2(r+1) - a + 3 + y \\
&= \frac{r+a}{a-1}(t(r+1)+s+a-3) - (r+a) - (r-y) + 1.
\end{aligned}$$

Consequently

$$\frac{d+s}{r+a} = \frac{t(r+1)+s+a-3}{a-1} - 1 - \frac{r-y-1}{r+a}.$$

The inequality

$$\frac{t(r+1)+s+a-3}{a-1} - 1 - \frac{r-y-1}{r+a} \leq x \leq \frac{t(r+1)+s+a-3}{a-1} + (t-2) + \frac{y}{r+1}$$

is satisfied by at least t integer values of x if $0 \leq y \leq r-1$, in particular the following integer values of x:

$$\frac{t(r+1)+s+a-3}{a-1} + i$$

for $i = -1, 0, 1, 2, ..., t-2$.

We defer the argument for the case $y = r$. Suppose now that $y \geq r+1$. Let $y = (r+1)z + w$, where $z \geq 0$ and $0 \leq w \leq r$. Then

$$\begin{aligned}
-1 - \frac{r-y-1}{r+a} &= -1 - \frac{(r+1)-2-(r+1)z-w}{r+a} \\
&= -1 + \frac{(r+1)(z-1)+w+2}{r+a} \\
&\leq -1 + (z-1) + 1 \\
&= z-1
\end{aligned}$$

and

$$(t-2) + \frac{y}{r+1} \geq (t-2) + z,$$

so that it suffices to show that there are at least t integers x satisfying

$$\frac{t(r+1)+s+a-3}{a-1} + z - 1 \leq x \leq \frac{t(r+1)+s+a-3}{a-1} + (t-2) + z.$$

But the following integers satisfy this:

$$\frac{t(r+1)+s+a-3}{a-1} + i$$

for $i = z-1, z, z+1, ..., z+(t-2)$, so there are t integers altogether, as required.

Thus if $d = N(r+1, s, a-1, t) - (r+1) + y, y \geq 0, y \neq r$, G has an $(r, r+a)$-factorization with x factors for at least t integer values of x.

Now consider the case when $y = r$. In this case $d = N(r+1, s, a-1, t) - 1$. If there is an odd number of vertices of minimum degree d in G, take a disjoint further copy of G, and denote the two copies of G by 2G. Now pair off the vertices of minimum degree in G (or in 2G if there is an odd number of such vertices in G). Let G^* denote G (or 2G) with these extra edges added. Then G^* is a $(d+1, d+s)$-pseudograph, where $d+1 = N(r+1, s, a-1, t)$. By Theorem 3.15,

$$\pi(r+1, s, a-1, t) = N(r+1, s, a-1, t),$$

so G^* has an $(r+1, r+a)$-factorization with x factors for t different values of x. Removing the extra edges to revert to G, we see that G has an $(r, r+a)$-factorization with x factors for t different values of G.

Consequently $\pi(r,s,a,t) \leq N(r+1, s-2, a-1, t)$, as asserted. ∎

Proof of (2): The proof of (2) is very similar to the proof of (1). We have that r is odd and a is even, and also that $(r+1)t + s \equiv 2(mod\ a-2)$. By Lemma 4.9 and Theorem 3.15, since $(r+1)$ and $(a-2)$ are even,

$$N(r+1, s-2, a-2, t) - 1 \leq \pi(r,s,a,t).$$

We need to show here that

$$\pi(r,s,a,t) \leq N(r+1, s-2, a-2, t).$$

Let G be a $(d, d+s)$-pseudograph with $d = d(G) = N(r+1, s, a-2, t) - (r+1) + y = N(r+1, s-2, a-2, t) + y$, where y is a non-negative integer. By Theorem 3.14, it is enough to show that the inequality $\frac{d+s}{(r+1)+(a-2)} \leq x \leq \frac{d}{r+1}$ is satisfied by at least t integer values of x. For then G will have an $(r+1, (r+1)+(a-2)) = (r+1, r+a-1)$-factorization with x factors for t different values of x. Clearly an $(r+1, r+a-1)$-factor is an $(r, r+a)$-factor.

Since

$$d = N(r+1, s, a-2, t) - (r+1) + y$$

it follows that

$$\begin{aligned} d &= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) - (r+1) + y \\ &= (r+1)\frac{(t(r+1)+s+a-4)}{a-2} + (t-2)(r+1) + y \end{aligned}$$

so that

$$\frac{d}{r+1} = \frac{t(r+1)+s+a-4}{a-2} + (t-2) + \frac{y}{r+1}.$$

Also

$$\begin{aligned} d+s &= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil + (t-1)(r+1) - (r+1) + y + s \\ &= \frac{r+a-1}{a-2}(t(r+1)+s+a-4) + (t-2)(r+1) + y + s - t(r+1) - s - a + 4 \\ &= \frac{r+a-1}{a-2}(t(r+1)+s+a-4) - 2(r+1) - a + 4 + y \\ &= \frac{r+a-1}{a-2}(t(r+1)+s+a-4) - (r+a+1) - (r-y) + 1. \end{aligned}$$

Consequently

$$\frac{d+s}{r+a-1} = \frac{t(r+1)+s+a-4}{a-2} - 1 - \frac{r-y-1}{r+a-1}.$$

The inequality

$$\frac{t(r+1)+s+a-4}{a-2} - 1 - \frac{r-y-1}{r+a-1} \leq x \leq \frac{t(r+1)+s+a-4}{a-2} + (t-2) + \frac{y}{r+1}$$

is satisfied by at least t integer values of x if $0 \leq y \leq r-1$, in particular the following integer values of x:

$$\frac{t(r+1)+s+a-4}{a-2} + i$$

for $i = -1, 0, 1, 2, ..., t-2$.

We defer the argument for the case $y = r$. Suppose now that $y \geq r+1$. Let $y = (r+1)z + w$, where

$z \geq 0$ and $0 \leq w \leq r$. Then

$$\begin{aligned} -1-\frac{r-y-1}{r+a-1} &= -1-(\frac{(r+1)-2-(r+1)z-w}{r+a-1}) \\ &= -1+\frac{(r+1)(z-1)+w+2}{r+a-1} \\ &\leq -1+(z-1)+1 \\ &= z-1, \end{aligned}$$

and

$$(t-2)+\frac{y}{r+1} \geq (t-2)+z,$$

so that it suffices to show that there are at least t integers x satisfying

$$\frac{t(r+1)+s+a-4}{a-2}+z-1 \leq x \leq \frac{t(r+1)+s+a-4}{a-2}+(t-2)+z.$$

But the following integers satisfy this:

$$\frac{t(r+1)+s+a-4}{a-2}+i$$

for $i=-1,0,1,2,...,(t-2)$, so there are t integers altogether, as required.

Thus if $d = N(r+1,s,a-2,t)-(r+1)+y, y \geq 0, y \neq r$, G has an $(r,r+a)$-factorization with x factors for at least t integer values of x.

Now consider the case when $y=r$. In this case, $d=N(r+1,s,a-2,t)-1$. If there is an odd number of vertices of minimum degree d in G, take a disjoint further copy of G, and denote the two copies of G by 2G. Now pair off the vertices of minimum degree in G (or in 2G if there is an odd number of such vertices in G). Let G^* denote G (or 2G) with these extra edges added. Then G^* is a $(d+1,d+s)$-pseudograph, where $d+1=N(r+1,s,a-2,t)$, so G^* has an $(r+1,r+a-1)=(r+1,(r+1)+(a-2))$-factorization with x factors for t different values of x. Removing the extra edges to revert to G, we find that G has an $(r,r+a)$-factorization with x factors for t different values of G.

Consequently $\pi(r,s,a,t) \leq N(r+1,s-2,a-2,t)$, as asserted. ∎

Proof of (3) The proof of (3) is like that of (1) and (2), but simpler.

We have that r is even and a is odd, and also that $rt+s \equiv 2(mod\ \ a-1)$. By Lemma 4.10 and Theorem 3.15, since r and $(a-1)$ are even,

$$N(r,s-2,a-1,t)-1 \leq \pi(r,s,a,t).$$

We need to show here that

$$\pi(r,s,a,t) \leq N(r,s-2,a-1,t).$$

Let G be a $(d,d+s)$ - pseudograph with $d=d(G)=N(r,s,a-1,t)-r+y=N(r,s-2,a-1,t)+y$, where y is a non-negative integer. By Theorem 3.14, it is enough to show that the inequality $\frac{d+s}{r+(a-1)} \leq x \leq \frac{d}{r}$ is satisfied by at least t integer values of x. For then G will have an $(r,r+(a-1)$-factorization with x factors for t different values of x. Clearly an $(r,r+a-1)$-factorization is an $(r,r+a)$-factorization.

Since

$$d=N(r,s,a-1,t)-(r+1)+y$$

it follows that

$$\begin{aligned} d &= r\left\lceil\frac{tr+s-1}{a-1}\right\rceil+(t-1)r-r+y \\ &= r\frac{(tr+s+a-2)}{a-1}+(t-2)r+y \end{aligned}$$

so that

$$\frac{d}{r} = \frac{tr + s + a - 2}{a - 1} + (t - 2) + \frac{y}{r}.$$

Also

$$\begin{aligned} d + s &= r\left\lceil \frac{tr + s - 1}{a - 2} \right\rceil + (t - 1)r - r + y + s \\ &= \frac{r + a - 1}{a - 1}(tr + s + a - 2) + (t - 2)r + y + s - tr - s - a + 2 \\ &= \frac{r + a - 1}{a - 1}(tr + s + a - 2) - 2r - a + 2 + y \\ &= \frac{r + a - 1}{a - 1}(tr + s + a - 2) - (r + a - 1) - (r - y) + 1. \end{aligned}$$

Consequently

$$\frac{d + s}{r + a - 1} = \frac{tr + s + a - 2}{a - 1} - 1 - \frac{r - y - 1}{r + a - 1}.$$

The inequality

$$\frac{tr + s + a - 2}{a - 1} - 1 - \frac{r - y - 1}{r + a - 1} \leq x \leq \frac{tr + s + a - 2}{a - 1} + (t - 2) + \frac{y}{r}$$

is satisfied by at least t integer values of x if $0 \leq y$. For if $y = rz + w$, where $z \geq 0$ and $0 \leq w \leq r - 1$. Then

$$\begin{aligned} -1 - \frac{r - y - 1}{a - 1} &= -1 - \frac{r - rz - w - 1}{r + a - 1} \\ &= -1 + \frac{(z - 1)r + w + 1}{r + a - 1} \\ &\leq -1 + (z - 1) + 1 \\ &= z - 1, \end{aligned}$$

and

$$(t - 2) + \frac{y}{r} = t - 2 + z + \frac{w}{r} \geq (t - 2) + z,$$

so that the integer values of x include the t integers:

$$\frac{tr + s + a - 2}{a - 1} + i$$

for $i = z - 1, z, z + 1, ..., z + (t - 2)$, so there are t integers altogether, as required.

Thus if $d = N(r, s, a - 1, t) - r + y, y \geq 0$, G has an $(r, r + a)$-factorization with x factors for at least t integer values of x, as required.

This concludes the proof of Lemma 5.4. ∎

Finally we need to deal with the cases in Theorem 4.5, Theorem 4.11 and Theorem 4.8 when $(r+1)t+s \equiv 3(mod\, a-1)$, $(r+1)t+s \equiv 3(mod\, a-2)$ and $rt+s \equiv 3(mod\, a-1)$, respectively, making similar improvements.

Lemma 5.5 *Let r, s, a and t be integers with r, t positive, $a \geq 3$ and s non-negative.*

1. *If both a and r are odd and $(r + 1)t + s \equiv 3(mod\ a - 1)$ then $N(r + 1, s, a - 1, t) - 1 \leq \pi(r, s, a, t) \leq N(r + 1, s, a - 1, t)$.*
2. *If r is odd and a is even and $(r + 1)t + s \equiv 3(mod\ a - 2)$ then $N(r + 1, s, a - 2, t) - 1 \leq \pi(r, s, a, t) \leq N(r + 1, s, a - 2, t)$.*
3. *If r is even and a is odd and $rt + s \equiv 3(mod\ a - 1)$ then $N(r, s, a - 1, t) - 1 \leq \pi(r, s, a, t) \leq N(r, s, a - 1, t)$.*

Remark 1: In the case when both r and a are odd, between the bounding numbers in the case when $(r+1)t+s \equiv 2(mod\ a-1)$ and in the case when $(r+1)t+s \equiv 3(mod\ a-1)$, there is a gap, or jump, of $r+1$. For if $(r+1)t+s \equiv 2(mod\ a-1)$ then

$$\begin{aligned} &N(r+1,s,a-1,t) - N(r+1,s-2,a-1,t) \\ &\qquad = (r+1)\left\lceil\frac{t(r+1)+s-1}{a-1}\right\rceil - (r+1)\left\lceil\frac{t(r+1)+(s-2)-1}{a-1}\right\rceil \\ &\qquad = (r+1)\{\left\lceil\frac{t(r+1)+s-1}{a-1}\right\rceil - \left\lceil\frac{t(r+1)+s-3}{a-1}\right\rceil\} \\ &\qquad = r+1. \end{aligned}$$

There is a similar gap in the other cases.

Remark 2: Lemma 5.5(2) is subsumed by Theorem 5.9 and Lemma 5.5(3) is subsumed by Theorem 5.13. In each case in Lemma 5.5, s is odd. (The possibility that s is even in other cases was considered in Lemma 4.3.) We prove next Lemma 5.5(1).

Lemma 5.6 ***(5.5(1))*** *Let r,s,a,t be integers with r,t positive, $a \geq 3$ and s non-negative. Let a and r be odd and $(r+1)t+s \equiv 3(mod\ a-1)$. Then*

$$N(r+1,s,a-1,t) - 1 \leq \pi(r,s,a,t) \leq N(r+1,s,a-1,t).$$

Proof Recall that in Theorem 4.5 we showed that

$$\pi(r,s+1,a,t) \leq N(r+1,s,a-1,t).$$

But since $\pi(r,s,a,t) \leq \pi(r,s+1,a,t)$ it follows that

$$\pi(r,s,a,t) \leq N(r+1,s,a-1,t).$$

We need to show that $N(r+1,s,a-1,t) - 1 \leq \pi(r,s,a,t)$. We do this by exhibiting a $(d,d+s)$-graph G of degree $d = N(r+1,s,a-1,t) - 2$ which does not have an $(r,r+a)$-factorization with x factors for t values of x. Note that d is even and $d+s$ is odd. We let G have two components, one, G_1, has one vertex and $\frac{d}{2}$ loops. The other, G_2, has two vertices, v_1 and v_2, between which there is an edge, and each of v_1 and v_2 is incident with $\frac{d+s-1}{2}$ loops. In any $(r,r+a)$- factorization of G, all but at most two of the factors must be $(r+1,r+a-1)$-factors, but the remaining factors must be $(r+1,r+a)$-factors. If there are x factors altogether in an $(r,r+a)$-factorization of G, then $(r+a-1)+(x-1)(r+a) \geq d(v_1) = d+s$ so that

$$x(r+a) \geq d+s+1$$

$$x \geq \frac{d+s+1}{r+a}.$$

Similarly all but one of the factors would have minimum degree $r+1$ and one might have minimum degree r. Therefore $(x-1)(r+1)+r \leq d$, so that $x(r+1) \leq d+1$, so $x \leq \frac{d+1}{r+1}$. Therefore x must satisfies the double inequality

$$\frac{d+s+1}{r+a} \leq x \leq \frac{d+1}{r+1}.$$

Since $r+1$ and $a-1$ are even and $d = N(r+1,s,a-1,t) - 2$ it follows that

$$d = (r+1)\left\lceil\frac{(r+1)t+s-1}{a-1}\right\rceil + (t-1)(r+1) - 2$$

so that

$$d+1 = (r+1)\left\lceil\frac{(r+1)t+s-1}{a-1}\right\rceil + (t-1)(r+1) - 1$$

so that, as $(r+1)t + s \equiv 3(mod\ a-1)$

$$\frac{d+1}{r+1} = \frac{(r+1)t+s+a-4}{a-1} + (t-1) - \frac{1}{r+1}.$$

Also

$$\begin{aligned}
d+s+1 &= \frac{(r+1)((r+1)t+s+a-4)}{a-1} + (t-1)(r+1) - 1 + s \\
&= \frac{(r+1)+(a-1)}{a-1}(t(r+1)+s+a-4) + (t-1)(r+1) - 1 \\
&\quad + s - (t(r+1)+s+(a-1)-4) \\
&= \frac{r+a}{a-1}(t(r+1)+s+a-4) - (r+1) - 1 - a + 4 \\
&= \frac{r+a}{a-1}(t(r+1)+s+a-4) - r - a + 2
\end{aligned}$$

so

$$\frac{d+s+1}{r+a} = \frac{1}{a-1}((r+1)t+s+a-4) - 1 + \frac{2}{r+a}.$$

Therefore if G has x $(r, r+a)$-factors then x satisfies

$$\frac{1}{a-1}((r+1)t+s+a-4) - 1 + \frac{2}{r+a} \leq x \leq \frac{1}{a-1}((r+1)t+s+a-4) + (t-1) - \frac{1}{r+1}.$$

The positive integers x which satisfy this double inequality are

$$\frac{1}{a-1}((r+1)t+s+a-4) + i$$

for $i = 0, 1, 2, ..., t-2$ so there are $t-1$ integers. So G does not have an $(r, r+a)$- factorization with x factors for t values of x. Therefore

$$N(r+1, s, a-1, t) - 1 \leq \pi(r, s, a, t).$$

∎

We next determine the value of $\pi(r, s, a, t)$ when r is odd, $a \geq 3$, provided that $(r+1)t + s \not\equiv 3(mod\ a-1)$ when a is odd, or $(r+1)t + s \not\equiv 3(mod\ a-2)$ when a is even.

Lemma 5.7 *Let r, s, a and t be integers with r, t positive, $a \geq 3$ and s non-negative.*

1. *If r and a are both odd, then*

$$\pi(r, s, a, t) = \begin{cases} N(r+1, s, a-1, t) - 1 & \text{if } (r+1)t+s \not\equiv 2 \text{ or } 3(mod\ a-1), \\ N(r+1, s, a-1, t) - (r+1) - 1 & \text{if } (r+1)t+s \equiv 2(mod\ a-1). \end{cases}$$

2. *If r is odd and a is even, then*

$$\pi(r, s, a, t) = \begin{cases} N(r+1, s, a-2, t) - 1 & \text{if } (r+1)t+s \not\equiv 2 \text{ or } 3(mod\ a-2), \\ N(r+1, s, a-2, t) - (r+1) - 1 & \text{if } (r+1)t+s \equiv 2(mod\ a-2). \end{cases}$$

Proof First suppose that r and a are both odd. Then by Theorem 4.5 (and the Remark after Lemma 5.4, and Theorem 3.15),

$$N(r+1, s, a-1, t) - 1 \leq \pi(r, s, a, t) \leq N(r+1, s, a-1, t)$$

if $(r+1)t + s \not\equiv 1, 2, 3(mod\ a-1)$; by Lemma 5.3 this also holds if $(r+1)t + s \equiv 1(mod\ a-1)$. By Lemma 5.4 (and the Remark following Lemma 5.4), if $(r+1)t + s \equiv 2(mod\ a-1)$,

$$N(r+1, s, a-1, t) - (r+1) - 1 \leq \pi(r, s, a, t) \leq N(r+1, s, a-1, t) - (r+1).$$

To prove Lemma 5.7(1), it suffices to show that every $(d, d+s)$- pseudograph of degree $d = N(r+1, s, a-1, t) - 1$ if $(r+1)t + s \not\equiv 2, 3(mod\ a-1)$, or every $(d, d+s)$-psuedograph of degree $d = N(r+1, s, a-1, t) - (r+1) - 1$ if $(r+1)t + s \equiv 2(mod\ a-1)$, has an $(r, r+a)$-factorization with x factors for at least t values of x. So let G be a $(d, d+s)$- pseudograph of degree $N(r+1, s, a-1, t) - 1$ if $(r+1)t + s \not\equiv 2, 3(mod\ a-1)$ or of degree $N(r+1, s, a-1, t) - (r+1) - 1$ if $(r+1)t + s \equiv 2(mod\ a-1)$. In each case d is odd.

Take two copies of G, say G' and G'', and if a vertex $v' \in V(G')$ has minimum degree d, join it by an edge to the corresponding vertex $v'' \in V(G'')$. Let the pseudograph so formed be denoted by G^*.

If $(r+1)t + s \not\equiv 2, 3(mod\ a-1)$ we note that G^* has minimum degree $N(r+1, s, a-1, t)$, and is a $(d, d+s)$ – pseudograph. Therefore G^* has an $((r+1), (r+1)+(a-1)) = (r+1, r+a)$ – factorization with x factors for at least t different values of x. Therefore G has an $(r, r+a)$ – factorization with x factors for t values of x.

If $(r+1)t+s \equiv 2(mod\ a-1)$ then we note that G^* is a $(d+1, (d+1)+(s-1))$-pseudograph, so, putting $d^* = d+1$ $s^* = s-1$, G^* is a (d^*, d^*+s^*)- pseudograph with $(r+1)t + s^* \equiv 1(mod\ a-1)$. Then

$$\begin{aligned}
N(r+1, s, a-1, t) - (r+1) &= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-1} \right\rceil + (t-1)(r+1) - (r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s^*)-1}{a-1} \right\rceil + (t-1)(r+1) \\
&= N(r+1, s^*, a-1, t)
\end{aligned}$$

so G^* has an $(r+1, r+a)$-factorization with x factors for t values of x, and so G has an $(r, r+a)$-factorization with x factors for t values of x.

The argument if r is odd and a is even is more or less the same. Throughout $a-1$ is replaced by $a-2$ and instead of Lemma 4.5 we use Lemma 4.11. In detail it is as follows:

Suppose that r is odd and a is even. Then by Lemma 4.11

$$N(r+1, s, a-2, t) - 1 \leq \pi(r, s, a, t) \leq N(r+1, s, a-2, t)$$

if $(r+1)t + s \not\equiv 1, 2, 3(mod\ a-2)$. By Lemma 5.3(2) this also holds if $(r+1)t + s \equiv 1(mod\ a-2)$, By Lemma 5.4, if $(r+1)t + s \equiv 2(mod\ a-2)$, then

$$N(r+1, s, a-2, t) - (r+1) - 1 \leq \pi(r, s, a, t) \leq N(r+1, s, a-2, t) - (r+1)$$

(using the Remark after Lemma 5.4).

To prove Lemma 5.7(2), it suffices to show that every $(d, d+s)$- pseudograph of degree $d = N(r+1, s, a-1, t) - 1$ if $(r+1)t+s \not\equiv 2$ *or* $(mod\ a-2)$, or every $(d, d+s)$-pseudograph of degree $d = N(r+1, s, a-1, t) - (r+1) - 1$ if $(r+1)t + s \equiv 2(mod\ a-2)$, has an $(r, r+a)$-factorization with x factors for at least t values of x. So let G be a $(d, d+s)$-pseudograph of degree $N(r+1, s, a-2, t) - 1$ if $(r+1)t + s \not\equiv 2, 3(mod\ a-2)$ or of degree $N(r+1, s, a-1, t) - (r+1) - 1$ if $(r+1)t + s \equiv 2(mod\ a-2)$. In each case d is odd.

Take two copies of G, say G' and G'', and if a vertex $v' \in V(G')$ has minimum degree d, join it by an edge to the corresponding vertex $v'' \in V(G'')$. Let the pseudograph so formed be denoted by G^*.

If $(r+1)t+s \not\equiv 2, 3(mod\ a-2)$ we note that G^* has minimum degree $N(r+1, s, a-2, t)$, and so is a $(d, d+s)$ – pseudograph. Since $r+1$ and $a-2$ are both even, G^* has an $((r+1), (r+1)+(a-2)) = (r+1, r+a-1)$ – factorization with x factors for at least t different values of x. Therefore G has an $(r, r+a-1)$ – factorization with x factors for t values of x.

If $(r+1)t+s \equiv 2(mod\ a-2)$ then we note that G^* is a $(d+1, (d+1)+(s-1))$-pseudograph, so, putting $d^* = d+1$ and $s^* = s-1$, G^* is a (d^*, d^*+s^*)-pseudograph with $(r+1)t + s^* \equiv 1(mod\ a-2)$. Then

$$\begin{aligned}
N(r+1, s, a-2, t) - (r+1) &= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil \\
&\quad + (t-1)(r+1) - (r+1) \\
&= (r+1)\left\lceil \frac{t(r+1)+(s^*)-1}{a-2} \right\rceil + (t-1)(r+1) \\
&= N(r+1, s^*, a-2, t)
\end{aligned}$$

so G^* has an $(r+1, r+a-1)$-factorization with x factors for t values of x, and so G has an $(r, r+a)$-factorization with x factors for t values of x.

∎

Now we determine the value of $\pi(r,s,a,t)$ in every case with $a \geq 3$ when r and a are both odd.

Lemma 5.8 *Let r, s, a and t be integers with r, t positive, $a \geq 3$ and s non-negative. If r and a are both odd, then*

$$\pi(r,s,a,t) = \begin{cases} N(r+1,s,a-1,t)-1 & \text{if } (r+1)t+s \not\equiv 2(mod\ a-1), \\ N(r+1,s,a-1,t)-(r+1)-1 & \text{if } (r+1)t+s \equiv 2(mod\ a-1). \end{cases}$$

Proof If $(r+1)t+s \not\equiv 3(mod\ a-1)$, this is part of Lemma 5.7. So we need only consider the case when $(r+1)t+s \equiv 3(mod\ a-1)$. By Lemma 5.6(5.5(1)),

$$N(r+1,s,a-1,t)-1 \leq \pi(r,s,a,t) \leq N(r+1,s,a-1,t).$$

The argument given in the proof of Lemma 5.7(1) for the case when $(r+1)t+s \not\equiv 2, 3(mod\ a-1)$ applies verbatim (except that now we no longer disallow the case when $(r+1)t+s \equiv 3(mod\ a-1)$.

∎

5.3 Everything related to the case r odd and a even that is not in Section 5.2

Theorem 5.9 *Let r, s, a and t be integers with r, t positive, $a \geq 3$ and s non-negative. Let r be odd and a be even, and let $(r+1)t+s \equiv 3(mod\ a-2)$. Then*

$$\pi(r,s,a,t) = N(r+1,s,a-2,t)-(r+1)-1.$$

Proof We first show that

$$\pi(r,s,a,t) \geq N(r+1,s,a-2,t)-(r+1)-1.$$

We do this by exhibiting a $(d, d+s)$-graph G of degree $d = N(r+1,s,a-2,t)-(r+1)-2$ which does not have an $(r, r+a)$-factorization with x factors for t integer values of x. Note that d is even and $d+s$ is odd.

Let G consist of two components G_1 and G_2, where G_1 has one vertex w on which are placed $\frac{d}{2}$ loops, and G_2 has two vertices, v_1 and v_2, which are joined by a single edge, and on each of which are placed $\frac{1}{2}(d+s-1)$ loops.

In any $(r, r+a)$ - factorization of G into x factors, since the vertex w of minimum degree d has $\frac{d}{2}$ loops on it, and since r is odd, it must be that $x(r+1) \leq d$. Similarly since the vertex v_1 of maximum degree $d+s$ has one edge and $\frac{1}{2}(d+s-1)$ loops on it, and since $r+a$ is odd, it must be that $(r+a)+(x-1)(r+a-1) \geq d+s$, so that $x(r+a-1) \geq d+s-1$. Consequently x satisfies

$$\frac{d+s-1}{r+a-1} \leq x \leq \frac{d}{r+1}.$$

Since $r+1$ and $a-2$ are both even, and $d = N(r+1,s,a-2,t)-(r+1)-2$, it follows that

$$d = (r+1)\left\lceil \frac{(r+1)t+s-1}{a-2} \right\rceil + (t-1)(r+1)-(r+1)-2,$$

so that

$$d = \frac{r+1}{a-2}((r+1)t+s+a-5)+(t-2)(r+1)-2$$

so
$$\frac{d}{r+1} = \frac{1}{a-2}((r+1)t+s+a-5)+(t-2)-\frac{2}{r+1}.$$
Also
$$\begin{aligned} d+s-1 &= \frac{r+1}{a-2}((r+1)t+s+a-5)+(t-2)(r+1)+s-3 \\ &= \frac{(r+1)+(a-2)}{a-2}((r+1)t+s+a-5)+(t-2)(r+1) \\ &\quad +s-3-(r+1)t-s-a+5 \\ &= \frac{r+a-1}{a-2}((r+1)t+s+a-5)-2(r+1)-a+2 \\ &= \frac{r+a-1}{a-2}((r+1)t+s+a-5)-2(r+a-1)+a-2. \end{aligned}$$
so that
$$\frac{d+s-1}{r+a-1} = \frac{1}{a-2}((r+1)t+s+a-5)-2+\frac{a-2}{r+a-1}.$$
Therefore if G has an $(r, r+a)$ - factorization with x factors, then
$$\frac{t(r+1)+s+a-5}{a-2}-2+\frac{a-2}{r+a-1} \leq x \leq \frac{1}{a-2}((r+1)t+s+a-5)+(t-2)-\frac{2}{r+1}.$$
The integer values of x satisfying this double inequality are
$$\frac{1}{a-2}((r+1)t+s+a-5)+i$$
for $i=-1,0,1,...,t-3$, so altogether the double inequality is satisfied by exactly $t-1$ integers. Consequently
$$d \geq N(r+1,s,a-2,t)-(r+1)-1.$$
Next we shall show that
$$d \leq N(r+1,s,a-2,t)-(r+1)-1.$$
Let G be a $(d, d+s)$-pseudograph with
$$d = N(r+1,s,a-2,t)-(r+1)-1+y$$
for some $y \geq 0$. Take two copies, G_1 and G_2, of G and join each vertex of lowest degree d in G_1 to the corresponding vertex in G_2. Call the graph so formed G^*. Then G^* has minimum degree $d+1$ and maximum degree $d+s$. An $(r+1, r+a-1)$ factor in G^* corresponds (by leaving out the extra edges joining G_1 to G_2) to an $(r, r+a-1)$-factor in G. The minimum degree of G^* is $d+1$ and we have
$$\begin{aligned} d+1 &= (r+1)\left\lceil \frac{t(r+1)+s-1}{a-2} \right\rceil +(t-1)(r+1)-(r+1)+y \\ &= \frac{r+1}{a-2}((r+1)t+s+a-5)+(t-2)(r+1)+y \end{aligned}$$
since $(r+1)t+s \equiv 3(mod\ a-2)$. Consequently
$$\frac{d+1}{r+1} = \frac{(r+1)t+a+s-5}{a-2}+(t-2)+\frac{y}{r+1}.$$
We also have that
$$\begin{aligned} d+s &= (r+1)\frac{t(r+1)+s+a-5}{a-2}+(t-2)(r+1)-1+y+s \\ &= \frac{(r+1)+(a-2)}{a-2}((r+1)t+s+a-5)+(t-2)(r+1)-1+y+s-(r+1)t-s-a+5 \\ &= \frac{(r+a-1)}{a-2}((r+1)t+s+a-5)-2(r+1)+y-a+4 \\ &= \frac{(r+a-1)}{a-2}((r+1)t+s+a-5)-2(r+a-1)+y+a \end{aligned}$$

so that
$$\frac{d+s}{r+a-1} = \frac{(r+1)t+s+a-5}{a-2} - 2 + \frac{y+a}{r+a-1}.$$
Since $(r+1)$ and $(r+a-1)$ are even, G^* has an $(r+1, r+a-1)$-factorization with x factors if x satisfies the double inequality
$$\frac{t(r+1)+s+a-5}{a-2} - 2 + \frac{y+a}{r+a-1} \leq x \leq \frac{(r+1)t+s+a-5}{a-2} + (t-2) + \frac{y}{r+1}.$$
For $0 \leq y \leq r-1$, the integer values of x satisfying this inequality include
$$\frac{(r+1)t+s+a-5}{a-2} + i$$
for $i = -1, 0, 1, ..., t-2$, so there are t such values of x, so G^* does have an $(r+1, r+a-1)$-factorization with x factors for t values of x.

For $y = r$ we cannot make this deduction without a special argument, which we make later below.

For $r+1 \leq y \leq 2r+1$ we have
$$2 > \frac{r+1+a}{r+a-1} > 1$$
and
$$1 < \frac{2r+1}{r+1} < 2$$
so the integer values of x satisfying the equality above include
$$\frac{(r+1)t+s+a-5}{a-2} + i$$
for $i = 0, 1, ..., t-2, t-1$ so there are t such values of x.

Now let $y = p(r+1) + z$ where $p \geq 2$ and $0 \leq z \leq r$. We have
$$\frac{p(r+1)+a}{r+a-1} < p$$
and
$$p < \frac{p(r+1)+r}{r+1} < p+1,$$
so the integer values of x satisfying the inequality include
$$\frac{(r+1)t+s+a-5}{a-2} + i$$
for $i = p-2, p-1, ..., p+(t-2)$, so there are at least t such values of x.

Now let us consider the case when $y = r$. First select an independent set S of edges of G^* such that each vertex of maximum degree $d+s$ is incident with exactly one edge (and each edge of s is incident with a vertex of degree $d+s$). Let $G^{**} = G^* \setminus S$. Then the maximum degree of G^{**} is $d+s-1$ and the minimum degree is at least $d+1$. For G^{**} to have an $(r+1, r+a-1)$-factorization with x factors for t different values of x, it is necessary that
$$\frac{d+s-1}{r+a-1} \leq x \leq \frac{d+1}{r+1},$$
i.e.,
$$\frac{(r+1)t+s+a-5}{a-2} - 2 + \frac{y+a-1}{r+a-1} \leq x \leq \frac{(r+1)t+s+a-5}{a-2} + (t-2) + \frac{y}{r+1}.$$
In the case when $y = r$, the integers x which satisfy this double inequality include
$$\frac{(r+1)t+s+a-5}{a-2} + i$$

for $i = -1, 0, 1, 2, ..., t-2$ so there are at least t such values of x, so G^{**} does indeed have an $(r+1, r+a-1)$-factorization with x factors for t different values of x. For each such x, choose an $(r+1, r+a-1)$-factorization with x factors. Assign the edges of S to one of the $(r+1, r+a-1)$-factors; then we obtain an $(r+1, r+a)$-factor in G^*. Now remove the edges joining the pairs of corresponding vertices in G_1 and G_2. This then produces an $(r, r+a)$-factorization of G with x factors. Thus in the case $y = r$, G also has an $(r, r+a)$-factorization with x factors, for t values of x.

This completes the proof that when r is odd and a is even,

$$\pi(r, s, a, t) = N(r+1, s, a-2, t) - (r+1) - 1.$$

∎

Now we determine the value of $\pi(r, s, a, t)$ in every case with $a \geq 3$, a even and r odd.

Theorem 5.10 *Let r, s, a, t be integers with r, t positive, $a \geq 3$, a even, r odd and s non-negative. Then*

$$\pi(r,s,a,t) = \begin{cases} N(r+1, s, a-2, t) - 1 & \textit{if } (r+1)t + s \not\equiv 2 \textit{ or } 3(\textit{mod } a-2), \\ N(r+1, s, a-2, t) - (r+1) - 1 & \textit{if } (r+1)t + s \equiv 2(\textit{mod } a-2). \end{cases}$$

Proof If $(r+1)t + s \not\equiv 2$ *or* $3(mod\ a-2)$ then this is part of Lemma 5.7(2).

If $(r+1)t + s \equiv 2(mod\ a-2)$ this is also part of Lemma 5.7(2). If $(r+1)t + s \equiv 3(mod\ a-2)$ this is Theorem 5.9.

∎

5.4 Everything related to the case r even and a odd that is not in Sections 5.2 or 5.3.

It remains to consider the case when r is even and a is odd. First we consider the case when $rt + s \not\equiv 3$ and s is even.

Lemma 5.11 *Let r, s, a and t be integers with r even and positive, t positive, $a \geq 3$ and odd and $s \geq 1$ even. Then*

$$\pi(r,s,a,t) = \begin{cases} N(r, s, a-1, t) & \textit{if } rt + s \not\equiv 2\ (\textit{mod}\ \ a-1), \\ N(r, s, a-1, t) - r & \textit{if } rt + s \equiv 2\ (\textit{mod}\ \ a-1). \end{cases}$$

Proof First suppose that $rt + s \not\equiv 2(mod\ a-1)$. Let $d = N(r, s, a-1, t) - 1$, so that d is odd. Let F be the $(d, d+s)$ - pseudograph with two components, G_1 consisting of an edge uv with $\frac{d-1}{2}$ loops on u and $\frac{d-1}{2}$ loops on v, and G_2 consisting of an edge wx with $\frac{1}{2}(d+s-1)$ loops on w and $\frac{1}{2}(d+s-1)$ loops on x. Recall that a loop contributes two to the degree of the vertex it is on. In any $(r, r+a)$ - factorization of F, all but at most two of the factors consist entirely of loops and so are $(r, r+a-1)$-factors. If there are x factors in an $(r, r+a)$-factorization of F, then all but one of the factors would have maximum degree at most $(r+a-1)$, and one might have degree as high as $r+a$. Therefore $(r+a) + (x-1)(r+(a-1)) \geq d(w) = d+s$, so that

$$x \geq \frac{d+s-1}{r+a-1}.$$

Similarly all but one of the factors would have minimum degree at least r, and one would have minimum degree at least $r+1$. Therefore $r + 1 + (x-1)r \leq d$ so that

$$x \leq \frac{d-1}{r}.$$

Since r and $a-1$ are even, and $d = N(r, s, a-1, t) - 1$ it follows that

$$d = r\left\lceil \frac{rt+s-1}{a-1} \right\rceil + (t-1)r - 1,$$

so that

$$\begin{aligned}\frac{d-1}{r} &= \left\lceil \frac{tr+s-1}{a-1} \right\rceil + (t-1) - \frac{2}{r}\\ &= \frac{tr+s-1+c}{a-1} + (t-1) - \frac{2}{r}\end{aligned}$$

for some odd c, $1 \le c \le a-2$. But if $c = a-2$ then $(a-1) \mid (rt+s-1+a-2) = rt+s+a-3$, so that $rt+s \equiv 2(mod\ a-1)$, which is not allowed in this case. Thus the odd number c satisfies $0 \le c \le a-4$, and $(a-1) \mid (rt+s-1+c)$.

We also have that

$$d+s = r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + r(t-1) + s - 1$$

so that

$$\begin{aligned}d+s-1 &= r\frac{tr+s-1+c}{a-1} + r(t-1) + s - 2\\ &= \frac{r+a-1}{a-1}(tr+s-1+c) - r - c - 1.\end{aligned}$$

Therefore

$$\frac{d+s-1}{r+a-1} = \frac{tr+s-1+c}{a-1} - \frac{r+c+1}{r+a-1}.$$

Therefore if F has an $(r, r+a)$-factorization with x factors, then

$$\frac{tr+s-1+c}{a-1} - \frac{r+c+1}{r+a-1} \le x \le \frac{rt+s-1+c}{a-1} + (t-1) - \frac{2}{r}.$$

This inequality is satisfied by the following integer values of x:

$$\frac{tr+s-1+c}{a-1} + i$$

for $i = 0, 1, ..., t-2$. As there are only $t-1$ such values of x, it follows that

$$\pi(r,s,a,t) \ge N(r,s,a-1,t).$$

But by Corollary 4.2 if $rt+s \not\equiv 2(mod\ a-1)$ it follows that

$$\pi(r,s,a,t) \le N(r,s,a-1,t).$$

Therefore

$$\pi(r,s,a,t) = N(r,s,a-1,t).$$

Now suppose that $rt+s \equiv 2(mod\ a-1)$. We let $d = N(r,s,a-1,t) - r - 1$. With this value of d we proceed as in the case above when $rt+s \not\equiv 2(mod\ a-1)$ and note that if F is the $(d, d+s)$-pseudograph with the two components G_1 and G_2 as above, and if there are x factors in an $(r, r+a)$-factorization of F, then

$$\frac{d+s-1}{r+a-1} \le x \le \frac{d-1}{r}.$$

Since r and $a-1$ are even and $d = N(r,s,a,t) - r - 1$ it follows that

$$d = r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + r(t-1) - r - 1$$

so that

$$\begin{aligned}\frac{d-1}{r} &= \left\lceil \frac{tr+s-1}{a-1} \right\rceil + (t-2) - \frac{2}{r}\\ &= \frac{tr+s+a-3}{a-1} + (t-2) - \frac{2}{r}.\end{aligned}$$

We also have that

$$d + s = \frac{tr + s + a - 3}{a - 1} r + (t - 2)r - 1 + s,$$

so that

$$\begin{aligned} d + s - 1 &= \frac{r + a - 1}{a - 1}(rt + s + a - 3) + (t - 2)r - 2 + s - tr - s - a + 3, \\ &= \frac{r + a - 1}{a - 1} + (rt + s + a - 3) - 2r + 1, \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d + s - 1}{r + a - 1} &= \frac{tr + s + a - 3}{a - 1} - \frac{2r + 1}{r + a - 1} \\ &= \frac{tr + s + a - 3}{a - 1} - 1 - \frac{r - a}{r + a - 1}. \end{aligned}$$

So if F has an $(r, r + a)$-factorization with x factors, then

$$\frac{tr + s + a - 3}{a - 1} - 1 - \frac{r + a}{r + a - 1} \leq x \leq \frac{tr + a - 3}{t - 2} + (t - 2) - \frac{2}{r}.$$

This double inequality is satisfied by the following integer values of x:

$$\frac{tr + s + a - 3}{a - 1} + i$$

for $i = -1, 0, 1, 2, ..., t - 3$. As there are only at most $t - 1$ such integers, it follows that

$$\pi(r, s, a, t) \geq N(r, s, a, t) - r.$$

But by Lemma 5.4(3)

$$\begin{aligned} \pi(r, s, a, t) &\leq N(r, s - 2, a - 1, t) \\ &= N(r, s, a - 1, t) - r. \end{aligned}$$

Therefore

$$\pi(r, s, a, t) = N(r, s, a - 1, t) - r.$$

∎

The result corresponding to Lemma 5.11 but for the case when s is odd is:

Lemma 5.12 *Let r, s, a and t be integers with r even and positive, t positive, $a \geq 3$ and odd, and $s \geq 1$ odd. Then*

$$\pi(r, s, a, t) = N(r, s, a - 1, t).$$

Proof By Theorem 4.8

$$\pi(r, s, a, t) \leq \pi(r, s, a - 1, t)$$

when $rt + s \not\equiv 1, 3(mod\ a - 1)$ (s is odd here in Lemma 5.12). By Lemma 5.3(3), this is also true when $rt + s \equiv 1(mod\ a - 1)$. In the case when $rt + s \equiv 3(mod\ a - 1)$ we showed in Theorem 4.8 that

$$\pi(r, s + 1, a, t) \leq \pi(r, s, a - 1, t).$$

But since

$$\pi(r, s, a, t) \leq \pi(r, s, a - 1, t)$$

it follows that

$$\pi(r, s, a, t) \leq \pi(r, s, a - 1, t).$$

Thus this holds in every case when $r \geq 2$ even, $a \geq 3$ odd, $t \geq 1$ and $s \geq 1$ odd.

We need to show that $\pi(r, s, a-1, t) \leq \pi(r, s, a, t)$. To this end let $d = N(r, s, a-1, t) - 1$, so that d is odd. Let F be the $(d, d+s)$-pseudograph with two components, G_1 consisting of an edge uv with $\frac{d-1}{2}$ loops on u and $\frac{d-1}{2}$ loops on v, and G_2 consisting of single vertex w on which are placed $\frac{1}{2}(d+s)$ loops. Since a loop contributes two to the degree of the vertex it is on, u and v have degree d, and w has degree $d+s$. Thus F is a $(d, d+s)$-pseudograph.

Since a and d are odd and r is even, and since all but one of the edges of F are loops, in any $(r, r+a)$-factorization of F, all but one of the factors would be $(r, r+(a-1))$-factors. If there are x factors in an $(r, r+a)$-factorization of F, then all but one of the factors would have maximum degree at most $(r+a-1)$, and one might have degree as high as $r+a$. Therefore $(r+a)+(x-1)(r+a-1) \geq d(w) = d+s$ so that $x(r+a-1) \geq d+s-1$. But $d+s-1$ is odd and $r+a-1$ is even, so $x(r+a-1) \geq d+s$, and so

$$x \geq \frac{d+s}{r+a-1}.$$

Each $(r, r+a)$-factor has minimum degree at least r, so $xr \leq d$. But since r is even and d is odd, we have $xr \leq d-1$, so that

$$x \leq \frac{d-1}{r}.$$

Since r and $a-1$ are even, and $d = N(r, s, a-1, t) - 1$ it follows that

$$d = r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + r(t-1) - 1$$

so that

$$\begin{aligned}\frac{d-1}{r} &= \left\lceil \frac{tr+s-1}{a-1} \right\rceil + (t-1) - \frac{2}{r} \\ &= \frac{tr+s-1+c}{a-1} + (t-1) - \frac{2}{r}\end{aligned}$$

for some even c, $0 \leq c \leq a-3$,

We also have that

$$d+s = r\left\lceil \frac{tr+s-1}{a-1} \right\rceil + r(t-1) + s - 1$$

so that

$$\begin{aligned}d+s &= r\frac{tr+s-1+c}{a-1} + r(t-1) + s - 1 \\ &= \frac{r+a-1}{a-1}(tr+s-1+c) + r(t-1) + s - (tr+s-1+c) \\ &= \frac{r+a-1}{a-1}(tr+s-1+c) - r - c.\end{aligned}$$

Therefore

$$\frac{d+s-1}{r+a-1} = \frac{tr+s-1+c}{a-1} - \frac{r+c}{r+a-1}.$$

If F has an $(r, r+a)$-factorization with x factors, then

$$\frac{tr+s-1+c}{a-1} - \frac{r+c}{r+a-1} \leq x \leq \frac{tr+s-1+c}{a-1} + (t-1) - \frac{2}{r}.$$

This double inequality is satisfied by the following integer values of x

$$\frac{tr+s-1+c}{a-1} + i$$

for $i = 0, 1, ..., t-2$. As there are only $t-1$ such integers, it follows that

$$\pi(r,s,a,t) \geq \pi(r,s,a-1,t).$$

Therefore in this case

$$\pi(r,s,a,t) = \pi(r,s,a-1,t).$$

∎

We now have the evaluation of $\pi(r,s,a,t)$ when r is even and a is odd.

Theorem 5.13 *Let r,s,a and t be integers with r even and positive, t positive, $a \geq 2$ and odd and s non-negative. Then*

$$\pi(r,s,a,t) = \begin{cases} N(r,s,a-1,t) & \text{if } rt+s \not\equiv 2 \ (mod \ \ a-1), \\ N(r,s,a-1,t)-r & \text{if } rt+s \equiv 2 \ (mod \ \ a-1). \end{cases}$$

Proof This follows from Lemma 5.11 if s is even and from Lemma 5.12 if s is odd.

∎

Proof of Theorem 5.1(1.23)
(1) If r and a are both even, this is Theorem 3.15.
(2) If r and a are both odd, this is Lemma 5.8
(3) If r is odd and a is even, this is Theorem 5.10.
(4) If r is even and a is odd, this is Theorem 5.13. ∎

5.5 Determination of $\pi(r,s,a,t)$ in the cases when $a = 0$ or 1, or when $a = 2$ and r is odd.

Theorem 5.1(1.23) settles the value of the pseudograph threshold number $\pi(r,s,a,t)$ when $a \geq 3$ or $a = 2$ and r is even. Recall that we noted earlier that $\pi(r,s,0,t) = \infty$, meaning that for no integer d_0 is it true that whenever $d \geq d_0$ then any $(d, d+s)$-pseudograph has an $(r, r+1)$ - factorization with x $(r, r+1)$ - factors for t different values of x (if $t > 0$ and $s > 0$). With two exceptions, it is generally true that $\pi(r,s,a,t) = \infty$ if $a = 0$ or 1.

Proof of Theorem 5.2(1.24)

Case 1: Suppose $a = 0$.

Clearly we cannot r-factorize any irregular graph. Therefore $\pi(r,s,0,t) = \infty$ if $s \neq 0$ and $t \geq 1$. For the case when $s = 0$, if $r \geq 2$ we note that we cannot r-factorize any graph of degree $pr+1$ for any positve integer p. Therefore $\pi(r,0,0,t) = \infty$. For the case when $r = 1$, it is well-known that there are regular graphs of degree $d \geq 2$ which cannot be 1-factorized. Therefore, again, $\pi(r,0,0,t) = \infty$; in particular $\pi(1,0,0,1) = \infty$.

Case 2: Suppose $a = 1$.

Suppose that $r \geq 3$. If d is even and $d \equiv 2 \pmod{r+1}$ if r is odd, or $d \equiv 2 \pmod{r}$ if r is even, and if a pseudograph G contains a component with one vertex on which are placed $\frac{d}{2}$ loops, then clearly G has no $(r, r+1)$-factorization. Therefore $\pi(r,s,1,t) = \infty$ when $r \geq 3$; in particular $\pi(r,0,1,1) = \infty$ for $r \geq 3$.

Now suppose that $r = 2$. If G has a component consisting of one vertex and $\frac{d}{2}$ loops when d is even, then G has an $(r, r+1)$–factorization with x factors only if $x = \frac{d}{2}$. Therefore $\pi(2,s,1,t) = \infty$ unless $t = 1$; in particular $\pi(2,0,1,2) = \infty$.

So now suppose that $a = 1$, $r = 2$, s odd and $t = 1$. Consider the case when d is odd and G contains two components, C_1, with two vertices u and v joined by an edge and with $\frac{d-1}{2}$ loops on each vertex, and

C_2 with one vertex and $\frac{d+s}{2}$ loops. Then G has no $(r, r+1)$–factorization (since any $(r, r+1)$–factorization of C_2 has $\frac{d+s}{2} > \frac{d-1}{2}$ factors). Therefore $\pi(2, s, 1, 1) = \infty$ if s is odd. In particular $\pi(2, 1, 1, 1) = \infty$. Since $\pi(r, s+1, a, t) \geq \pi(r, s, a, t)$ it follows now that $\pi(2, s, 1, 1) = \infty$ for each $s \geq 1$. Therefore $\pi(2, s, 1, t) \geq \infty$ whenever $s \geq 1$.

We now show that $\pi(2, 0, 1, 1) = 2$. Let G be a regular pseudograph of degree $d \geq 2$. If d is even, then, by Theorem 1.7 (Petersen's Theorem [33]), G has a 2-factorization, i.e. in this case an $(r, r+1)$–factorization with $r = 2$.

So we need to consider the case when d is odd. We may pair off the vertices of G, and join each such pair by an edge. Call the graph obtained by adding these extra edges G^+. Each component of G^+ is Eulerian, and so contains an Eulerian circuit. Going one way round each such circuit, we may orient the edges of that circuit. If the vertices of G are $u_1, ..., u_n$ construct a bipartite graph B on vertices $v_1, ..., v_n$ and $w_1, ..., w_n$ and join vertex v_i to vertex w_j for each edge u_iu_j of G when the direction goes from u_i to u_j in the Eulerian circuit of G^+.

Notice that in G a vertex u_i has degree d, and in B the corresponding vertices v_i and w_i have degrees $\frac{d+1}{2}$ and $\frac{d-1}{2}$ in some order. By Theorem 1.5 (on page 7), B has an equitable edge-colouring with $\frac{d-1}{2}$ colours, and then each colour occurs on exactly one edge incident with a vertex of degree $\frac{d-1}{2}$, but at a vertex of degree $\frac{d+1}{2}$, some colour will occur on two edges and the rest on one edge each. Colouring the edges of G with the colours of the corresponding edges of B yields a $(2, 3)$–factorization of G.

Thus every regular pseudograph of degree at least two has a $(2, 3)$–factorization. Therefore $\pi(2, 0, 1, 1) = 2$. (I believe this to be a new result).

Now suppose that $r = 1$. If G is a pseudograph consisting of one vertex and $d/2$ loops, then any $(r, r+1)$ - factorization of G in the case $r = 1$ must in fact be an $(r+1)$–factorization. By the same argument as in the case above when $r = 2$, we have $\pi(1, s, 1, t) = \infty$ unless $t = 1$. In particular $\pi(1, 1, 1, 2) = \infty$.

Now suppose that $a = 1$, $r = 1$ and $t = 1$. If G contains C_1 with one vertex u and $\frac{d}{2}$ loops on it, if d is even, and C_2 with two vertices joined by one edge with $\frac{d+s-1}{2}$ loops on each vertex if d is even and s is odd, then G has no $(r, r+1)$-factorization since any $(r, r+1)$-factorization of C_1 has $\frac{d}{2}$ factors, and of C_2 has $\frac{d+s-1}{2} + 1 = \frac{d+s+1}{2} > \frac{d}{2}$ factors. Therefore $\pi(1, s, 1, 1) = \infty$ unless $s = 0$.

Finally, we show that $\pi(1, 0, 1, 1) = 1$. Let G be a regular pseudograph of degree $d \geq 1$. If $d = 1$ then G is a regular pseudograph of degree 1, so is its own $(r, r+1)$–factorization. If d is even, then by Theorem 1.7 (Petersen [33]), G can be 2 -factorized and thus G has an $(r, r+1)$–factorization. If d is odd, $d \geq 3$, then we may pair off the vertices, and join each pair of vertices by an edge. The graph G^+ obtained this way is regular of degree $d+1$, which is even. Now 2-factorize G^+ , and then remove the extra edges. What remains is an $(r, r+1)$–factorization of G. Thus $\pi(1, 0, 1, 1) = 1$.

It remains to consider the case when $r = 0$. However it is clear that no pseudograph containing at least one loop can have a $(0, 1)$–factorization. Thus $\pi(0, 0, 1, 1) = \infty$.

Case 3: Suppose $a = 2$.

Recall that the case when r is even is covered by Theorem 3.15. We need to consider the case when r is odd. Let G be a graph with one vertex, v, and $\frac{1}{2}x(r+1) + 1$ loops incident with v. Any $(r, r+2)$-factor consists of $\frac{1}{2}(r+1)$ loops on v, and so x $(r, r+2)$-factors utilize $\frac{1}{2}x(r+1)$ loops, leaving one loop over not in any factor. Therefore, provided $r+1 \geq 1$, i.e. $r \geq 3$ (as r is odd), G has no $(r, r+2)$-factorization. Therefore $\pi(r, s, 2, t) = \infty$ if r is odd and $r \geq 3$. So we may suppose that $r = 1$, so we are investigating $(1, 3)$-factorizations.

Let G be a pseudograph with two components, G_1 of degree d and G_2 of degree $d+s$. Suppose s is even. If d is even, let G_1 consist of one vertex with $\frac{d}{2}$ loops on it, and let G_2 consist of one vertex with $\frac{d+s}{2}$ loops on it. Any $(1, 3)$-factor of G contains exactly one loop from G_1 and one loop from G_2. Therefore G cannot have a $(1, 3)$-factorization unless $s = 0$ or $t = 1$.

Now suppose that $r = 1$, $s = 0$ and $t = 1$. We showed above that $\pi(2, 0, 1, 1) = 2$. Therefore every regular pseudograph of degree at least 2 has a $(2, 3)$-factorization, and so has a $(1, 3)$-factorization. But a regular graph of degree 1 has a 1-factorization, and so has a $(1, 3)$-factorization. Therefore $\pi(1, 0, 2, 1) = 1$.

Next suppose that $r = 1$ and s is odd. Let G now be a 2-component graph, one component being G_1 with one vertex and $\frac{d}{2}$ loops (d being even), and G_2 containing an edge v_3v_4 with $\frac{d+s-1}{2}$ loops on each of v_3

and v_4 . Any $(1,3)$-factorization of G_1 has exactly $\frac{d}{2}$ factors (each being a loop), so any $(1,3)$-factorization of G has exactly $\frac{d}{2}$ factors (so $t=1$). In any $(1,3)$-factorization of G, each factor has a loop from G_1, and one factor contains the edge v_3v_4 and a loop on each of v_3 and v_4, the other factors containing a loop on G_1 and two loops from G_2, one on v_3 the other on v_4. Therefore $s=1$ and $t=1$.

Now let G be an arbitrary $(d,d+s)$-pseudograph. Pair off the vertices odd degree in G, and join each such pair with an edge. Call the graph so formed G^+. Then form a bipartite graph B^+, and subsequently a bipartite graph B as described in the proof above that $\pi(2,0,1,1)=2$. Suppose that d is even. Then each extra edge u_iv_j joined two vertices which had degree $d+2$ in G^+. In B one of v_i, w_i has degree $\frac{d}{2}$, the other has degree $\frac{d}{2}+1$. Give the edges of B an equitable edge colouring with $\frac{d}{2}$ colours, and let each edge of G be coloured with the colour of the corresponding edge in B. In G, each vertex will be incident with two or three edges of the same colour. Then G will have a $(2,3)$-factorization, and so will have a $(1,3)$-factorization.

Next suppose that d is odd. Then in G^+ each extra edge u_iu_j joined two vertices of degree $d+1$. In B, one of v_i, w_i has degree $\frac{d-1}{2}$, the other $\frac{d+1}{2}$. Vertices u_i of G^+ not incident with an extra edge have degree $d+1$, and in B both v_i and w_i have degree $\frac{d+1}{2}$. Give the edges of B an equitable edge-colouring with $\frac{d+1}{2}$ colours. Then the corresponding edge-colouring of G is a $(1,2)$-factorization, and so is a $(1,3)$-factorization.

Bearing in mind that a regular graph of degree 1 has a 1-factorization, and hence a $(1,3)$-factorization, it follows that $\pi(1,1,2,1)=1$.

∎

Chapter 6

Simple graphs: an upper bound for $\sigma(r, s, a, t)$

We first remark that the main result in the remainder of this thesis is the following theorem. In each of the ensuing chapters, one of the four cases below is proved.

Theorem 6.1 *(**1.25**) Let $r \geq 1, s \geq 0, a \geq 2$ and $t \geq 1$ be integers. Then*
(i) If r is odd and a is even, then

$$\sigma(r,s,a,t) = \begin{cases} r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r+1 & \text{if } t \geq 2, \text{ or } t=1 \text{ and } a < r+s+1, \\ r & \text{if } t=1 \text{ and } a \geq r+s+1; \end{cases}$$

(ii) If r is even and a is even, then

$$\sigma(r,s,a,t) = r\left\lceil \frac{tr+s-1}{a} \right\rceil + (t-1)r;$$

(iii) If r is even and a is odd, then

$$\sigma(r,s,a,t) = r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r;$$

(iv) If r and a are both odd, and if $t \geq 2$, or $t = 1$ and $a < tr+s$, then

$$\sigma(r,s,a,t) = r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r+1.$$

However if $t = 1$ and $a \geq tr+s$ then

$$\sigma(r,s,a,t) = r.$$

For the case when $a = 1$ see Theorem 1.19.

In this chapter we first prove the following upper bound for $\sigma(r,s,a,t)$, valid for integers $r \geq 1$, $a \geq 2$, $t \geq 1$ and $s \geq 0$.

Theorem 6.2 *Let $r \geq 1$, $a \geq 2$, $t \geq 1$ and $s \geq 0$ be integers. Then*

$$\sigma(r,s,a,t) \leq r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r+1.$$

This upper bound is the best possible, and is achieved in the case when r is odd and a is even, as we show later.

To prove Theorem 6.2 we first recall the following theorem of Hilton and de Werra [23].

Theorem 6.3 ***(1.6) Hilton and de Werra [23].*** *Let x be a positive integer and let G be a simple graph. Suppose that for no vertex v is it true that $x \mid d(v)$. Then G has an equitable edge-colouring with x colours.*

We use this theorem to prove the following very useful fact.

Theorem 6.4 *Let r and a be integers with $r \geq 2$ and $a \geq 2$. Then every simple $(d, d+s)$-graph G has an $(r, r+a)$-factorization with x factors if*

$$\frac{d+s}{r+a} < x < \frac{d}{r}.$$

Proof Let G be a $(d, d+s)$-simple graph satisfying

$$\frac{d+s}{r+a} < x < \frac{d}{r}.$$

Then

$$r < \frac{d}{x} \leq \frac{d+s}{x} < r+a.$$

At each vertex v where $x \mid d(v)$, it follows that

$$r+1 \leq \frac{d(v)}{x} \leq r+a-1.$$

We form a simple graph G^+ from G by adjoining a pendant edge to each vertex of G satisfying $x \mid d(v)$. For each vertex v of the simple graph G^+ we have $x \nmid d_{G^+}(v)$, and so G^+ has an equitable edge-colouring with x colours, by Theorem 6.3(1.6). Restricting this edge-colouring to G gives an edge-colouring of G which is equitable at the vertices v where $x \nmid d(v)$, and is nearly equitable at the vertices v where $x \mid d(v)$. Thus for each pair of colours α and β,

$$\| \alpha(v) \mid - \mid \beta(v) \| \leq 1 \text{ if } x \nmid d(v),$$
$$\| \alpha(v) \mid - \mid \beta(v) \| \leq 2 \text{ if } x \mid d(v).$$

The average number of edges of each colour at v is exactly $\frac{d(v)}{x}$. If $x \nmid d(v)$ then $r < \frac{d(v)}{x} < r+a$, so $r \leq \alpha(v) \leq r+a$ for each colour α. If $x|d(v)$ then $r+1 \leq \frac{d(v)}{x} \leq r+a-1$ so again $r \leq \alpha(v) \leq r+a$ for each colour α. Therefore each colour class is an $(r, r+a)$-factor, and so G has an $(r, r+a)$-factorization with x $(r, r+a)$-factors.

∎

Proof of Theorem 6.2:

Let us first point out that a number p satisfies

$$p = r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r$$

if and only if

$$p = \frac{r}{a}(tr+s+c) + (t-1)r$$

for some integer c such that

$$a \mid tr+s+c$$

and

$$1 \leq c \leq a.$$

We show that

$$\sigma(r,s,a,t) \leq r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r + 1.$$

So we show that

$$\sigma(r,s,a,t) \leq \frac{r}{a}(tr+s+c) + (t-1)r + 1,$$

where $a \mid tr+s+c$ when $1 \leq c \leq a$.

Let

$$d = \frac{r}{a}(tr+s+c) + (t-1)r + k$$

where $k \geq 1$. We show that in this case there do exist at least t integer values of x satisfying $\frac{d+s}{r+a} < x < \frac{d}{r}$. Then it follows by Theorem 6.4 for $r \geq 2$ that every $(d, d+s)$-simple graph is $(r, r+a)$-factorable into x factors for at least t values of x.

Note that

$$\frac{d}{r} = \frac{1}{a}(tr+s+c) + (t-1) + \frac{k}{r}$$

and that

$$\begin{aligned}
d+s &= \frac{r}{a}(tr+s+c) + (t-1)r + k + s \\
&= \frac{r+a}{a}(tr+s+c) + (t-1)r + k + s - (tr+s+c) \\
&= \frac{r+a}{a}(tr+s+c) - (r+c) + k,
\end{aligned}$$

so that
$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) - \frac{r+c}{r+a} + \frac{k}{r+a}.$$

If $1 \le k \le r+c-1$ then the integer values of x satisfying $\frac{d+s}{r+a} < x < \frac{d}{r}$ include
$$\frac{1}{a}(tr+s+c)+i$$
for $i = 0,1,....,t-1$. Thus there are at least t such integer values of x in this case.

For $r+c=k$ then the integer values of x satisfying $\frac{d+s}{r+a} < x < \frac{d}{r}$ include
$$\frac{1}{a}(tr+s+c)+i$$
for $i = 1,2,....,t$. Thus there are at least t such integer values of x in this case.

For $r+c<k$ then the integer values of x satisfying $\frac{d+s}{r+a} < x < \frac{d}{r}$ include
$$\frac{1}{a}(tr+s+c)+i$$
for $i = \left\lfloor \frac{k-r-c}{r+a} \right\rfloor + 1, ..., \left\lfloor \frac{k-r-c}{r+a} \right\rfloor + (t-1)+1$, since
$$\frac{1}{a}(tr+s+c) + \left\lfloor \frac{k-r-c}{r+a} \right\rfloor + t < \frac{d}{r}$$
i.e, $\left\lfloor \frac{k-r-c}{r+a} \right\rfloor + 1 < \frac{k}{r}$, which is true since
$$\left\lfloor \frac{k-r-c}{r+a} \right\rfloor \le \frac{k-r-c}{r+a} + 1 < \frac{k-r}{r},$$
so there are at least t such integer values of i.
Therefore,
$$\sigma(r,s,a,t) \le r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r+1.$$

∎

We showed just above that for $r \ge 2$ and $a \ge 2$, if $\frac{d+s}{r+a} < x < \frac{d}{r}$ then every simple $(d,d+s)$-graph G has an $(r,r+a)$-factorization with x factors. We now show a kind of converse to this fact.

Lemma 6.5 *Let r and d be positive integers and let a and s be non-negative integers. Let G be a simple graph with at least one vertex of degree d and at least one of degree $d+s$. Suppose that G has an $(r,r+a)$-factorization into x factors. Then*
$$\frac{d+s}{r+a} \le x \le \frac{d}{r}.$$

Proof This is just a specialization of Lemma 2.4 to simple graphs.

∎

We next show that there are simple $(d,d+s)$-graphs G which do not have $(r,r+a)$-factorizations with x factors if $x = \frac{d}{r}$ or $x = \frac{d+s}{r+a}$. Therefore the requirement that every $(d,d+s)$-simple graph has an $(r,r+a)$-factorization precludes the values $x = \frac{d}{r}$ and $x = \frac{d+s}{r+a}$, and we are left with values of $r,a,d,d+s$ and x satisfying $\frac{d+s}{r+a} < x < \frac{d}{r}$.

Example 6.1: Let d and r be odd with $d > r$. Let $x = \frac{d}{r}$. Let D be a graph obtained from K_{d+2} by removing a P_3 and $\frac{1}{2}(d-1)$ K_2's, so that D has one vertex of degree $d-1$ and the remaining vertices have degree d. Let G be the regular graph obtained from two copies of D by joining the two vertices of degree $d-1$ by an edge e. Then G is a regular graph of degree d. Any d-factor of G must contain the edge e. Either

x is not an integer, or, if it is an integer, then $x \geq 2$, and, since in any $(r, r+a)$-factorization of G with x factors, each factor must be an r-factor, it follows that G does not have an $(r, r+a)$-factorization with x factors. Thus G is an example of a d-regular graph with $\frac{d}{r}$ not in $F_{\{r,a\}}(G)$.

Example 6.2: Let d be even, r be odd, and let $d > r$. Let $x = \frac{d}{r}$. Consider the regular graph $G = K_{d+1}$. Since r is odd, any r-factor of G has even order, but since G has odd order, G has no r-factor. As in Example 6.1, either x is not an integer, or, if it is then $x \geq 2$; and it follows as in Example 6.1 that G does not have an $(r, r+a)$-factorization with x factors. Thus G is again an example of a d-regular graph with no $(r, r+a)$-factorization with x factors. So again $\frac{d}{r} \notin F_{\{r,a\}}(G)$.

Example 6.3: Let $(d+s)$ and $(r+a)$ both be odd, and let $d+s > r+a$. Let $x = \frac{d+s}{r+a}$. Let D be a graph formed from K_{d+s+1} by removing a P_3 and $\frac{1}{2}(d+s-1)$ K_2's, so that D has one vertex of degree $d+s-1$ and the remaining vertices have degree $d+s$. Take two copies of D and join the two vertices of degree $d+s-1$ by an edge e. Let G be the graph formed in this way. Then G is regular of degree $d+s$ and has even order. Any $(r+a)$-factor of G must contain the edge e. Either x is not an integer, or, if it is an integer, then $x \geq 2$. Since in any $(r, r+a)$-factorization of G with x factors, each factor must be an $(r+a)$-factor, it follows that G does not have an $(r, r+a)$-factorization with x factors. Thus G is an example of a simple $(d+s)$-regular graph with $\frac{d+s}{r+a}$ not in $F_{\{r,a\}}(G)$.

Example 6.4: Let $(d+s)$ be even, $(r+a)$ be odd, and let $d > r+a-s$. Let $x = \frac{d+s}{r+a}$. Since $(r+a)$ is odd, any $(r+a)$-factor of G must have even order. However, since G has odd order, G has no $(r+a)$-factor. Either x is not an integer, or if it is then $x \geq 2$, and it follows as in Example 6.3 that G does not have an $(r, r+a)$-factorization with x factors. Thus G is also an example of a $(d+s)$-regular simple graph with $\frac{d+s}{r+a}$ not in $F_{\{r,a\}}(G)$.

Examples 6.1 and 6.2 together demonstrate that if r is odd and $d > r$, then there is a simple $(d, d+s)$-regular graph G with $\frac{d}{r}$ not in $F_{\{r,a\}}(G)$.

Examples 6.3 and 6.4 together show that if $r+a$ is odd and $d+s > r+a$, i.e. $d > r+a-s$, then there is a simple $(d, d+s)$-regular graph G with $\frac{d+s}{r+a}$ not in $F_{\{r,a\}}(G)$.

We note that if G is the disjoint union of two pseudographs, A and B, then $F_{\{r,a\}}(G) = F_{\{r,a\}}(A) \cap F_{\{r,a\}}(B)$. Thus using Examples 6.1 to 6.4, it follows that if d is odd and a is even and $d > max\{r, r+a-s\}$, then there is a simple d-regular graph G such that neither $\frac{d}{r}$ nor $\frac{d+s}{r+a}$ are in $F_{\{r,a\}}(G)$.

From Theorem 6.4 it follows that if $a \geq 2$ and $r \geq 2$, then every simple $(d, d+s)$-graph G has an $(r, r+a)$-factorization if $\frac{d+s}{r+a} < x < \frac{d}{r}$. Thus

$$Z \cap (\frac{d}{r}, \frac{d+s}{r+a}) \subseteq F_{\{r,a\}}(G).$$

Using this, plus the previous paragraph, it now follows that if $d > max\{r, r+a-s\}$ then there is a simple $(d, d+s)$-graph G such that

$$F_{\{r,a\}}(G) = Z \cap (\frac{d}{r}, \frac{d+s}{r+a}).$$

Taking these examples into account as well as Lemma 6.5 we may say:

Theorem 6.6 *Let $a \geq 2$ and $r \geq 2$.*

(i) If r is odd and a is even, and if every $(d, d+s)$-simple graph with $d \geq max(r, r+a-s)$ has an $(r, r+a)$-factorization with x factors, then

$$\frac{d+s}{r+a} < x < \frac{d}{r}.$$

(ii) If r is even and a is odd, and if every $(d,d+s)$-simple graph with $d > r+a-s$ has an $(r,r+a)$-factorization with x factors, then

$$\frac{d+s}{r+a} < x \leq \frac{d}{r}.$$

(iii) If r and a are both odd, and if every $(d,d+s)$-simple graph with $d \geq r$ has an $(r,r+a)$-factorization with x factors, then

$$\frac{d+s}{r+a} \leq x < \frac{d}{r}.$$

Proof
(i) Follows from Theorem 6.5 and Examples 6.1, 6.2, 6.3 and 6.4.
(ii) Follows from Theorem 6.5 and Examples 6.3 and 6.4.
(iii) Follows from Theorem 6.5 and Examples 6.1 and 6.2.

$\blacksquare$

In the case when r is odd and a is even, so that

$$\frac{d+s}{r+a} < x < \frac{d}{r},$$

the upper bound in Theorem 6.2 is achieved as we now show.

Theorem 6.7 *(**1.25**) Let $r \geq 1$ be odd, $a \geq 2$ be even. Let s and t be positive integers. Then*

$$\sigma(r,s,a,t) = \begin{cases} r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r + 1 & \text{if } t \geq 2, \text{ or } t = 1 \text{ and } a < r+s+1, \\ r & \text{if } t = 1 \text{ and } a \geq r+s+1. \end{cases}$$

Proof From Theorem 6.2 we already know that

$$\sigma(r,s,a,t) \leq r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r + 1.$$

First assume that $t \geq 2$, or $t = 1$ and $a < r+s+1$. The equation

$$p = r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r + 1$$

is true if and only if

$$p = r(\frac{tr+s+c}{a}) + (t-1)r + 1$$

for some integer c such that $a \mid tr+s+c$ and $1 \leq c \leq a$.

We first show that if

$$d = r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r$$

and $d \geq max\{r, r+a-s\}$ then there is an example of a $(d,d+s)$-simple graph G which does not have an $(r,r+a)$-factorization with x factors for t different values of x. We recall that just before Theorem 6.6 we noted that if r is odd and a is even, and if $d \geq max\{r, r+a-s\}$ then there is a $(d,d+1)$-graph G such that $F_{\{r,a\}}(G) = Z \cap (\frac{d}{r}, \frac{d+s}{r+a})$, so it suffices to show that there do not exist t integer values of x satisfying

$$\frac{d+s}{r+a} < x < \frac{d}{r}.$$

So suppose that

$$d = \frac{r}{a}(tr+s+c) + (t-1)r$$

where $a \mid tr + s + c$ and $1 \leq c \leq a$. Then

$$\frac{d}{r} = \frac{1}{a}(tr + s + c) + (t - 1)$$

and

$$\begin{aligned} d + s &= \frac{r}{a}(tr + s + c) + (t - 1)r + s \\ &= \frac{r + a}{a}(tr + s + c) + (t - 1)r + s - (tr + s + c) \\ &= \frac{r + a}{a}(tr + s + c) - (r + c) \end{aligned}$$

so that

$$\frac{d + s}{r + a} = \frac{1}{a}(tr + s + c) - \frac{r + c}{r + a}.$$

The integer values of x which satisfy $\frac{d+s}{r+a} < x < \frac{d}{r}$ are

$$\frac{1}{a}(tr + s + c) + i$$

for $i = 0, 1, ..., t - 2$, giving only $t - 1$ values altogether. Therefore

$$\sigma(r, s, a, t) \geq \frac{r}{a}(tr + s + c) + (t - 1)r + 1$$

where $a \mid tr + s + c$ and $1 \leq c \leq a$. In other words

$$\sigma(r, s, a, t) \geq r \left\lceil \frac{tr + s + 1}{a} \right\rceil + (t - 1)r + 1.$$

In view of Theorem 6.2, it now follows that

$$\sigma(r, s, a, t) = r \left\lceil \frac{tr + s + 1}{a} \right\rceil + (t - 1)r + 1.$$

Secondly suppose that $t = 1$ and that $a \geq r + s + 1$. Then Theorem 6.2 tells us that $\rho(r, s, a, t) \leq r + 1$ in this case. However, if $d = r$ then a $(d, d + s)$-graph is already an $(r, r + a)$ factor, so that $\sigma(r, s, a, t) = r$.

∎

Finally we observe:

Theorem 6.8 *Let r be an odd positive integer and a be an even positive integer. Let d and x be positive integers and let s be a non-negative integer. For $d \geq max\{r, r + a - s\}$ the statement "Every $(d, d + s)$-simple graph has an $(r, r + a)$-factorization with x factors" is true if and only if*

$$\frac{d + s}{r + a} < x < \frac{d}{r}.$$

Proof This follows from Theorem 6.4 and the Theorem 6.6(i).

∎

Chapter 7

Simple graphs: a lower bound for $\sigma(r, s, a, t)$

One main concern in this chapter is to give in Theorem 7.1 a lower bound for $\sigma(r,s,a,t)$, valid for general values of r, a, t and s. This is a new result, although quickly deducible from results in Chapter 2. In Theorem 7.2 we prove the same result in a completely different way. The argument is very close to the arguments used to prove the simple graph results in Chapters 6, 8 and 9. The argument is also independent of the argument used to prove Theorems 3.14 and 3.15 which depend ultimately on the bipartite graph results in Chapter 2. It happens to be the case that the argument that we use in Theorem 7.2 works for pseudographs, unlike the similar argument used in Chapters 6, 8 and 9. In Theorem 7.5 we give a completely different proof of Theorem 3.14 designed to be valid for simple graphs, but fortuitously true for pseudographs in general.

Theorem 7.1 *Let $r \geq 2$, $a \geq 2$, $t \geq 1$, and $s \geq 0$ be integers. Then*

$$\pi(r,s,a,t) \geq r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r.$$

Proof We know from Theorem 2.8 that

$$\beta_s(r,s,a,t) = r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r.$$

By Lemma 1.14,

$$\beta_s(r,s,a,t) \leq \sigma(r,s,a,t) \leq \pi(r,s,a,t).$$

Therefore

$$\pi(r,s,a,t) \geq r\left\lceil \frac{tr+s-1}{a} \right\rceil + (t-1)r.$$

∎

Theorem 7.2 is the same result for simple graphs. We give a proof along the same lines as the simple graph results in Chapters 6, 8 and 9. In this case, exceptionally, the proof works for pseudographs as well.

Theorem 7.2 *Let $r \geq 2$, $a \geq 2$, $t \geq 1$, and $s \geq 0$ be integers. Then*

$$\sigma(r,s,a,t) \geq r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r.$$

Before proving Theorem 7.2, let us first state the following lemma which we proved as Lemma 2.4.

Lemma 7.3 *Let r and d be positive integers and s and a be non-negative integers. Let G be a $(d, d+s)$-pseudograph with at least one vertex of degree d and at least one vertex of degree $d+s$. Suppose that G is $(r, r+a)$-factorable with exactly $x \geq 1$ factors. Then*

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$$

Proof of Theorem 7.2: First let us remark that a number p satisfies

$$p = r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r$$

if and only if

$$p = r(\frac{rt+s+c}{a}) + (t-1)r$$

for some integer c such that $a \mid tr+s+c$ and $-1 \leq c \leq a-2$.

By Lemma 2.4, if G is a $(d, d+s)$-graph which is $(r, r+a)$-factorable with x factors for at least t different values of x, then, as stated above,

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$$

for t distinct integers $x_1, x_2, ..., x_t$.

Now suppose that an integer d satisfies

$$d = \frac{r}{a}(tr+s+c) + (t-1)r - 1$$

where $a \mid tr+s+c$ and $-1 \le c \le a-2$. Then

$$\frac{d}{r} = \frac{1}{a}(tr+s+c) + (t-1) - \frac{1}{r}$$

and

$$\begin{aligned} d+s &= \frac{r}{a}(tr+s+c) + (t-1)r - 1 + c \\ &= \frac{r+a}{a}(tr+s+c) + (t-1)r - 1 + c - (tr+s+c) \\ &= (r+a)\frac{1}{a}(tr+s+c) - c - r - 1 \end{aligned}$$

so that

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) - \frac{r+c+1}{r+a}.$$

Since $c+1 < a$, it follows that the values of x which satisfy

$$\frac{d+s}{r+a} \le x \le \frac{d}{r}$$

are

$$\frac{tr+s+c}{a} + j$$

for $0 \le j \le t-2$, but not $j = -1$ or $j = t-1$. So there are indeed fewer than t such integer values of x. So it follows that

$$\sigma(r,s,a,t) \ge r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1).$$

∎

We prove next the following lemma.

Lemma 7.4 *Let a and r both be even. Let*

$$\frac{d+s}{r+a} \le x \le \frac{d}{r}.$$

Then any $(d, d+s)$-simple graph has an $(r, r+a)$-factorization with x factors.

Remark: This follows from Theorem 3.14. However it is of interest to provide an alternative derivation, which we now do:

Proof If

$$\frac{d+s}{r+a} < x < \frac{d}{r}$$

then, by Theorem 6.4, any $(d, d+s)$-simple graph has an $(r, r+a)$-factorization with x factors.

Now suppose that

$$\frac{d+s}{r+a} = x < \frac{d}{r}.$$

We use the fact that $(a+r)$ is even and r is even. Since $r+a$ is even, it follows that $d+s$ is even.

Let G be the given $(d, d+s)$-simple graph. Pair off the vertices of odd degree and insert an extra edge between each such pair. Let G^* be the graph obtained. Each component of G^* is Eulerian. Orient the edges in each component around an Eulerian circuit. Construct a bipartite graph B^* as follows. Let the vertices of G^* be $\{v_1, v_2, ..., v_n\}$. Let $\{u_1, u_2, ..., u_n\}$ and $\{w_1, w_2, ..., w_n\}$ be the two sets of vertices in B^*. If there is an edge $\longrightarrow v_iv_j$ in G^* then place the edge u_iw_j in B^*. In G^* the degree of each vertex v_i is even, so there are same number of "in-edges" as "out-edges", so $d_{B^*}(u_i) = d_{G^*}(w_i) = \frac{1}{2}d_{G^*}(v_i)$ for each vertex v_i.

Now remove the extra edges from G^* and the corresponding edges from B^*, getting G and B. Then $x = \frac{(d+s)/2}{(r+a)/2}$, where $\frac{d+s}{2}$ and $\frac{r+a}{2}$ are integers. Also B has maximum degree at most $\frac{d+s}{2}$ and minimum degree at least $\frac{r+a}{2}$. By the theorem of McDiarmid and de Werra(Theorem 1.5), B has an equitable edge-colouring with x colours. Each colour-class has highest degree at most $\frac{r+a}{2}$, and since $x < \frac{d/2}{r/2}$ and r is even, each colour-class has lowest degree at least $\frac{r}{2}$. Therefore B has x $(\frac{r}{2}, \frac{r+a}{2})$-factors, say $F_1, F_2, ..., F_x$. From $F_1, F_2, ..., F_x$ we obtain factors $G_1, G_2, ..., G_x$ of G as follows. If (u_j, w_k) is an edge of F_i, then (v_i, v_k) is an edge of G_i. Then $d_{G_i}(v) = d_{F_i}(u_j) + d_{F_i}(w_j)$, so that G_i is an $(r, r+a)$-factorization with x factors.

Finally suppose that $x = \frac{d}{r}$. Since r is even, d is also even. As just above, consider the graphs G^* and B^*, and then obtain G and B. Then $x = \frac{d/2}{r/2}$, where $\frac{d}{2}$ and $\frac{r}{2}$ are integers. Also B has maximum degree at most $\frac{d+s}{2}$ and minimum degree at least $\frac{r+a}{2}$. Therefore B has x $(\frac{r}{2}, \frac{r+a}{2})$-factors, say $F_1, F_2, ..., F_x$. From these we obtain x $(r, r+a)$-factors, say $G_1, G_2, ..., G_x$. Then G has an $(r, r+a)$-factorization with x factors. ∎

Next we give a second proof of the special case of Theorem 3.14 for simple graphs.

Theorem 7.5 *Let r, s, a, t be integers with r, a even and positive, t positive and s non-negative. Then*

$$\sigma(r,s,a,t) = r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r.$$

Proof By Theorem 7.2 we know that

$$\sigma(r,s,a,t) \geq r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r.$$

So we need to show that

$$\sigma(r,s,a,t) \leq r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r.$$

Recall from the proof of Theorem 7.2 that a number p satisfies

$$p = r\left\lceil \frac{rt+s-1}{a} \right\rceil + (t-1)r$$

if and only if

$$p = r(\frac{rt+s+c}{a}) + (t-1)r$$

for some integer c such that $a \mid tr+s+c$ and $-1 \leq c \leq a-2$.

Let

$$d = \frac{r}{a}(tr+s+c) + (t-1)r + k$$

where $k \geq 0$. We show that, in this case, there do exist t values of x between $\frac{d+s}{r+a}$ and $\frac{d}{r}$. Then it follows from Lemma 7.3 that every $(d, d+s)$-simple graph is $(r, r+a)$-factorable into x factors for at least t values of x.

First we note that

$$\frac{d}{r} = \frac{1}{a}(tr+s+c) + (t-1) + \frac{k}{r}$$

and that

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) - \frac{r+c}{r+a} + \frac{k}{r+a}.$$

Therefore if $r+c \geq k \geq 0$ then, since $r+a > r+a-2 \geq r+c$, the values of x lying between $\frac{d+s}{r+a}$ and $\frac{d}{r}$ include

$$\frac{1}{a}(tr+s+c), ..., \frac{1}{a}(tr+s+c) + (t-1),$$

so there are at least t such values of x.

Next suppose that $k = r + c + y$ where $(p-1)(r+a) < y \leq p(r+a)$ and $p \geq 1$. Then

$$\begin{aligned}\frac{d+s}{r+a} &= \frac{1}{a}(tr+s+c) + \frac{r+c-k}{r+a}\\ &= \frac{1}{a}(tr+s+c) + \frac{y}{r+a}\\ &\leq \frac{1}{a}(tr+s+c) + p.\end{aligned}$$

and

$$\begin{aligned}\frac{d}{r} &= \frac{1}{a}(tr+s+c) + (t-1) + \frac{k}{r}\\ &= \frac{1}{a}(tr+s+c) + (t-1) + \frac{r+c+y}{r}\\ &= \frac{1}{a}(tr+s+c) + t + \frac{c+y}{r}\\ &\geq \frac{1}{a}(tr+s+c) + t + \frac{y-1}{r}\\ &\geq \frac{1}{a}(tr+s+c) + t + \frac{(p-1)(r+a)}{r}\\ &\geq \frac{1}{a}(tr+s+c) + t + (p-1).\end{aligned}$$

The integer values of x between $\frac{d+s}{r+a}$ and $\frac{d}{r}$ are

$$\frac{1}{a}(tr+s+c) + (p-1) + i$$

for $i = 1, ..., t$. Thus there are at least t such integer values.
So indeed

$$\sigma(r,s,a,t) \geq r\left\lceil \frac{tr+s-1}{a} \right\rceil + (t-1)r$$

as asserted. Therefore

$$\sigma(r,s,a,t) = r\left\lceil \frac{tr+s-1}{a} \right\rceil + (t-1)r.$$

when r and a are both even.

∎

Theorem 7.6 *Let r and a be even positive integers. Let d and x be positive integers, and let s be a non-negative integer. The statement "Every $(d, d+s)$-simple graph has an $(r, r+a)$-factorization with x factors" is true if and only if*

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}.$$

Proof This follows from Lemma 7.3 and Lemma 7.4. It is also a special case of Theorem 3.14.

∎

Chapter 8

$\sigma(r, s, a, t)$ when r is even and a is odd

Even though the upper bound for $\sigma(r,s,a,t)$ given in Theorem 6.1 (which is achieved when r is odd and a is even) and the lower bound given in Theorem 7.1 (which is achieved when r is even and a is even) are very close (often only 1 apart), it is worthwhile to try to evaluate $\sigma(r,s,a,t)$ in the two remaining cases, namely r odd and a odd, and r even and a odd. We do this in the case when r is even and a is odd in this chapter, and in the case when r is odd and a is odd in Chapter 9.

First let us recall the following:

Theorem 8.1 ***(6.6(ii))*** *Let r be even and a be odd, $r \geq 2$, $a \geq 1$. Let s be non-negative. If every $(d,d+s)$-simple graph with $d > r+s-a$ has an $(r,r+a)$-factorization into x factors then*

$$\frac{d+s}{r+a} < x \leq \frac{d}{r}.$$

We prove two theorems about this case.

Theorem 8.2 *Let $r \geq 2$ be even, $a \geq 1$ be odd, and let s be a non-negative integer. Let G be a $(d,d+s)$-simple graph and let*

$$\frac{d+s}{r+a} < x \leq \frac{d}{r}.$$

Then G has an $(r,r+a)$-factorization with x factors.

Theorem 8.3 *Let r be even, $r \geq 2$ and $a \geq 1$ be odd. Let t be a positive integer and s a non-negative integer. Then*

$$\sigma(r,s,a,t) = r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r.$$

We show in Theorem 8.8 and Corollary 8.9 that Theorem 8.2 implies Theorem 8.3.

By Theorems 6.2 and 7.1 we know that

$$r\left\lceil \frac{tr+s-1}{a} \right\rceil + (t-1)r \leq \sigma(r,s,a,t) \leq r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r + 1.$$

In the case when r is even and a is odd, we can start by making a very slight improvement to the lower bound here.

Theorem 8.4 *Let r be even, $r \geq 2$, and $a \geq 1$ be odd. Let t be a positive integer and s a non-negative integer. Then*

$$\sigma(r,s,a,t) \geq r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r.$$

Proof First let us remark that a number p satisfies

$$p = r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r$$

if and only if

$$p = \frac{r}{a}(tr+s+c) + (t-1)r$$

for some integer c such that

$$a \mid tr+s+c$$

and

$$0 \leq c \leq a-1.$$

Suppose that an integer d satisfies

$$d = \frac{r}{a}(tr+s+c) + (t-1)r - 1$$

where
$$a \mid tr + s + c$$
and
$$0 \leq c \leq a - 1.$$

Then
$$\frac{d}{r} = \frac{1}{a}(tr + s + c) + (t - 1) - \frac{1}{r}$$
and
$$\frac{d + s}{r + a} = \frac{1}{a}(tr + s + c) - \frac{r + c + 1}{r + a},$$
since
$$\begin{aligned} d + s &= \frac{r + a}{a}(tr + s + c) + (t - 1)r - 1 - (tr + s + c) + s \\ &= \frac{r + a}{a}(tr + s + c) - r - c - 1. \end{aligned}$$

Since $c + 1 \leq a$ it follows that the integer values of x which satisfy $\frac{d+s}{r+a} < x \leq \frac{d}{r}$ are
$$\frac{1}{a}(tr + s + c) + j$$
for $j = 0, 1, ..., t - 2$, so there are fewer than t such values of x. So it follows that if there are at least t such values of x then
$$d \geq \frac{r}{a}(tr + s + c) + (t - 1)r,$$
so that
$$d \geq r\left\lceil \frac{tr + s}{a} \right\rceil + (t - 1)r.$$
Consequently
$$\sigma(r, s, a, t) \geq r\left\lceil \frac{tr + s}{a} \right\rceil + (t - 1)r$$
when r is even and a is odd.

∎

Corollary 8.5 *When r is even and a is odd,*
$$r\left\lceil \frac{tr + s}{a} \right\rceil + (t - 1)r \leq \sigma(r, s, a, t) \leq r\left\lceil \frac{tr + s + 1}{a} \right\rceil + (t - 1)r + 1.$$

Next we improve Corollary 8.5 by lowering the upper bound; we also show that there are t values of x satisfying $\frac{d+s}{r+a} < x \leq \frac{d}{r}$.

Theorem 8.6 *Let r be even, $r \geq 2$, and $a \geq 1$ be odd. Let t be a positive integer and s a non-negative integer. Then*
$$r\left\lceil \frac{tr + s}{a} \right\rceil + (t - 1)r \leq \sigma(r, s, a, t) \leq r\left\lceil \frac{tr + s + 1}{a} \right\rceil + (t - 1)r;$$

Furthermore if $a \geq \sigma(r, s, a, t)$ then there are t values of x satisfying $\frac{d+s}{r+a} < x \leq \frac{d}{r}$.

Proof The upper bound was established in Theorem 6.2. We make progress by examining the proof of Theorem 6.2 in more detail.

We assumed that $d = \frac{r}{a}(tr+s+c)+(t-1)r+k$, where $a|tr+s+c$ and $1 \le c \le a$, and $k \ge 1$. Then

$$\frac{d}{r} = \frac{1}{a}(tr+s+c)+(t-1)+\frac{k}{r}$$

and

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c)-\frac{r+c}{r+a}+\frac{k}{r+a}.$$

Then, for $k \ge 1$, the number of values of x satisfying

$$\frac{d+s}{r+a} < x < \frac{d}{r}$$

is at least t. If $k=0$ there are only $t-1$ such values of x, but in this case

$$\frac{d}{r} = \frac{1}{a}(tr+s+c)+(t-1)$$

and

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c)+(t-1)-\frac{r+c}{r+a},$$

and the values of x satisfying

$$\frac{d+s}{r+a} < x \le \frac{d}{r}$$

(with $\frac{d}{r}=x$ now being allowed) are

$$\frac{1}{a}(tr+s+c)+i \text{ for } i=0,1,...,t-1,$$

so there are t values of x in this case. Thus in every case, there are at least t values of x satisfying

$$\frac{d+s}{r+a} < x \le \frac{a}{r}.$$

It follows that

$$\sigma(r,s,a,t) \le r\left\lceil \frac{tr+s+1}{a} \right\rceil + (t-1)r.$$

$\blacksquare$

In Theorem 8.6 we showed that if G is a $(d,d+s)$-simple graph with $d \ge \sigma(r,s,a,t)$ then at least t values of x satisfy $\frac{d+s}{r+a} < x \le \frac{d}{r}$. In particular, if $d \ge \sigma(r,s,a,1)$ then every $(d,d+s)$-simple graph G has an $(r,r+a)$-factorization with x factors if $\frac{d+s}{r+a} < x \le \frac{d}{r}$, in the case where $r \ge 2$ is even and $a \ge 1$ is odd. Taken together with Theorem 6.6(ii) this proves:

Theorem 8.7 *(1.26(iii)) Let $r \ge 2$ be even, $a \ge 1$ be odd, and let $s \ge 0$. Then every $(d,d+s)$-simple graph G has an $(r,r+a)$-factorization with x factors, where x is an integer, if and only if*

$$\frac{d+s}{r+a} < x \le \frac{d}{r}.$$

We finally turn to the proof of the equality $\sigma(r,s,a,t) = r\left\lceil\frac{tr+s}{a}\right\rceil+(t-1)r$ where $r \ge 2$ is even and $a \ge 1$ is odd. There is more than one of proving this point, but we want to reinforce the fact that Theorem 8.2 implies Theorem 8.3, and we also want to make the structure of the argument crystal clear. Let $Y(r,s,a,t)$ be the union over all $d \ge r$ of the set of all simple $(d,d+s)$ graphs G which satisfy the inequality

$$\frac{d+s}{r+a} < x \le \frac{d}{r}$$

for each x in $F_{\{r,a\}}(G)$; i.e. for all simple $(d,d{+}s)$-graphs G such that $F_{\{r,a\}}(G) \subseteq (\frac{d+s}{r+a}, \frac{d}{r}]$. Let $\sigma(Y(r,s,a,t))$ be the least value of d, say d_0, for which it is true that if $d \geq d_0$, then all members of Y of degree d have an $(r, r+a)$-factorization with x factors for at least t values of x. Of course, Theorem 8.6 shows that $\sigma(Y(r,s,a,t)) = \sigma(r,s,a,t))$ so the notation $Y(r,s,a,t)$ is not strictly necessary.

Theorem 8.8 *Let r be even, $r \geq 2$, and let $a \geq 1$ be odd. Let $t \geq 1$ and $s \geq 0$ be integers. Then*

$$\sigma(Y(r,s,a,t)) = r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r.$$

Proof The proof of Theorem 8.4 works just as well for $Y(r,s,a,t)$ to show that

$$\sigma(Y(r,s,a,t)) \geq r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r.$$

So we need to show that

$$\sigma(Y(r,s,a,t)) \leq \frac{r}{a}(tr+s+c) + (t-1)r$$

where

$$a \mid tr+s+c$$

and

$$0 \leq c \leq a-1.$$

Let

$$d = \frac{r}{a}(tr+s+c) + (t-1)r + k$$

where $k \geq 0$. We show that there exist t integer values of x satisfying

$$\frac{d+s}{r+a} < x \leq \frac{d}{r}.$$

Then it follows by the definition of $\sigma(Y(r,s,a,t))$ that every $(d,d{+}s)$-simple graph in $Y(r,s,a,t)$ is $(r,r{+}a)$-factorable into x factors for at least t integer values of x.

First we note that

$$\frac{d}{r} = \frac{1}{a}(tr+s+c) + (t-1) + \frac{k}{r}$$

and

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) + \frac{k-r-c}{r+a}.$$

For p a non-negative integer, if $pr \leq k < (p+1)r$ then $\frac{k}{r} \geq p$ and

$$\frac{k-r-c}{r+a} < \frac{(p+1)r-r-c}{r+a} = \frac{pr-c}{r+a} \leq p\frac{r}{r+a} < p$$

so

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) + \frac{k-r-c}{r+a} < \frac{1}{a}(tr+s+c) + p$$

and

$$\begin{aligned}\frac{d}{r} &= \frac{1}{a}(tr+s+c) + (t-1) + \frac{k}{r}\\ &\geq \frac{1}{a}(tr+s+c) + (t-1) + p.\end{aligned}$$

Therefore if $pr \leq k < (p+1)r$ for some non-negative integer p, then the integer values of x satisfying

$$\frac{d+s}{r+a} < x \leq \frac{d}{r}$$

include

$$\frac{1}{a}(tr+s+c)+i$$

for $i = p, p+1, ..., p+(t-1)$ so there are at least t such values of x. Therefore

$$\sigma(Y(r,s,a,t)) \leq r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r$$

as asserted. Theorem 8.8 now follows.

∎

Corollary 8.9 *Theorem 8.2 implies Theorem 8.3.*

What happens if r is even and a is odd and $\frac{d+s}{r+a} = x = \frac{d}{r}$, where x is an integer, $x \geq 1$? Then $d = \frac{sr}{a}$, and so $d > r+s-a$. We know from Theorem 6.6(ii) that if simple graphs with $\frac{d+s}{r+a} = x = \frac{d}{r}$ exist, then there is such a graph G which does not have an $(r, r+a)$-factorization with x factors. In Theorem 8.10, when $d > \frac{a}{r}$ we give examples of graphs which satisfy $\frac{d+s}{r+a} = x = \frac{d}{r}$. but do not have an $(r, r+a)$-factorization with x factors. By a result of Kano and Saito [26] in 1983, such graphs do have at least one $(r, r+a)$-factor (see also the book "Factors and Factorizations of Graphs" by Akiyama and Kano, 2007 [4], Theorem 3.3.7). We do not know whether such graphs must have $x-1$ edge-disjoint $(r, r+a)$-factors.

Theorem 8.10 *Let r be an even and a be an odd positive integer. Let d and x be positive integers such that $\frac{d+s}{r+a} = x = \frac{d}{r} \geq 2$ and $d > \frac{a}{r}$. Let s be a positive integer. Then there is a $(d, d+s)$-simple graph which does not have an $(r, r+a)$-factorization with x factors.*

Proof We separate the cases x even and x odd. Although these are similar, it is easier for the reader if they are treated separately.

Case 1: Let x be even.

Let G be a bi-degreed simple graph with vertex sets M and N, where $|M| = xr+1$ and $|N| = x(r+a)$. Since $d > \frac{r}{a}$ it follows that $\binom{xr+1}{2} > \frac{x(r+a)}{2}$, so there is a simple graph H with $V(H) \subseteq M$ and $|E(H)| = \frac{x(r+a)}{2}$. Label the vertices of H with labels $a_1, a_2, ..., a_{x(r+a)}$ in such a way that if $v \in V(H)$ then v receives $d_H(v)$ labels. Also assign the labels $a_1, a_2, ..., a_{x(r+a)}$ to the vertices of N, assigning one label to each vertex.

We have H placed on the vertices of M. Then to form G from this, join each vertex v of N to each vertex of M except the vertex with the same label as v. Then, for $v \in N$, $d_G(v) = xr$ and, for $v \in M$, $d_G(v) = x(r+a)$.

Notice that

$$|E(G)| = xr(x(r+a)) + \frac{x(r+a)}{2} = x^2r^2 + x^2ra + \frac{x(r+a)}{2}.$$

If G has an $(r, r+a)$-factorization with x factors, let $\{F_1, ..., F_x\}$ be such a set of factors. Each F_i will have $r+a$ edges incident with each vertex of M, so for $1 \leq i \leq x$,

$$\sum_{v \in V(G)} d_{F_i}(v) \geq (xr+1)(r+a) + x(r+a)r = 2xr^2 + 2xra + r + a.$$

Therefore

$$|E(F_i)| \geq xr^2 + xra + \left\lceil \frac{r+a}{2} \right\rceil.$$

Consequently we have

$$\begin{aligned}
|E(G)| &= \sum_{i=1}^{x} |E(F_i)| \\
&\geq x^2r^2 + x^2ra + x\left\lceil \frac{r+a}{2} \right\rceil \\
&> x^2r^2 + x^2ra + \frac{x(r+a)}{2}, \quad \text{since } r+a \text{ is odd,} \\
&= |E(G)|,
\end{aligned}$$

a contradiction.

Therefore G does not have an $(r, r+a)$-factorization into x factors when x is even.

Here we give an example which illustrates the construction used in Theorem 8.7, Case 1. Here $x = r = 2$ and $a = 1$, and the $(4, 6)$-simple graph has no $(2, 3)$-factorization, and $\frac{d+s}{r+a} = \frac{4+2}{2+1} = 2 = x = \frac{4}{2} = \frac{d}{r}$.

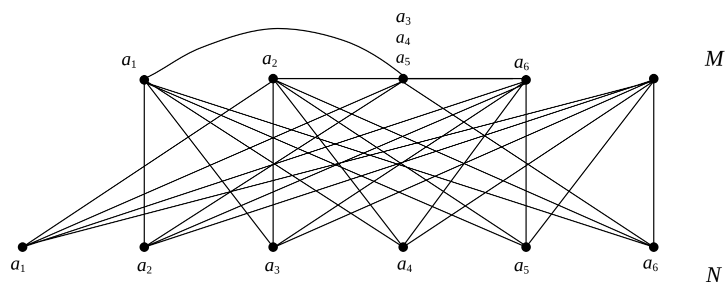


Figure 8.1: A bi-degreed (4, 6)-simple graph with no (2, 3)-factorization.

Case 2: Let x be odd

Let G be a simple graph with vertex sets $M \cup N$ where $|M| = xr + 1$ and $|N| = x(r+a)$. The vertices of M have degree $x(r+a)$ and all except one vertex of N will have degree xr and one vertex of N, say $v_{x(r+a)}$, will have degree $xr + 1$. Let H be a simple graph with $V(H) \subset M$ and $|E(H)| = \frac{x(r+a)-1}{2}$. Label the vertices of H with labels $a_1, a_2, ..., , a_{x(r+a)-1}$ in such a way that if $v \in V(H)$ then v receives $d_H(v)$ labels. Also assign the labels $a_1, ..., a_{x(r+a)-1}$ to the vertices of N, assigning one label to each vertex and leaving one vertex, say $v_{x(r+a)}$, unlabelled.

We have H already placed on the vertices of M. To form G from this, first join each vertex v of N to each vertex of M except the vertex with the same label as v (the vertex $v_{x(r+a)} \in V(N)$ is joined to all the vertices of M). Then, for $v \in V(M)$, $d_G(v) = x(r+a)$, and, for $v \in V(N) \setminus \{v_{x(r+a)}\}$, $d_G(v) = xr$ and $d_G(v_{x(r+a)}) = xr + 1$. Then

$$\begin{aligned}
|E(G)| &= xr(x(r+a)) + 1 + \frac{x(r+a)-1}{2} \\
&= x^2r^2 + x^2ra + \frac{x(r+a)+1}{2}.
\end{aligned}$$

If G has an $(r, r+a)$-factorization with x factors, let $\{F_1, F_2, ..., F_x\}$ be such a set of factors. Then each vertex of M will have $r + a$ edges incident with each of $F_1, F_2, ..., F_x$, and, for all but one i, F_i will have r edges incident with each vertex of N, and the exceptional factor, say F_x, will have r edges incident with each vertex of $V(N) \setminus \{v_{x(r+a)}\}$, and will have $r+1$ edges incident with $v_{x(r+a)}$. Therefore

$$\begin{aligned}\sum_{v \in V(G)} d_{F_i}(v) &\geq \begin{cases}(xr+1)(r+a)+x(r+a)r & \text{if } \ i \neq x, \\ (xr+1)(r+a)+x(r+a)r+1 & \text{if } \ i = x,\end{cases} \\ &= \begin{cases}2xr^2+2xra+(r+a) & \text{if } \ i \neq x, \\ 2x^2r^2+2xra+(r+a)+1 & \text{if } \ i = x.\end{cases}\end{aligned}$$

Therefore

$$\begin{aligned}|E(F_i)| &\geq \begin{cases}xr^2+xra+\left\lceil\frac{r+a}{2}\right\rceil & \text{if } \ i \neq x, \\ xr^2+xra+\frac{r+a+1}{2} & \text{if } \ i = x\end{cases} \\ &= xr^2+xra+\frac{r+a+1}{2},\end{aligned}$$

Therefore

$$\begin{aligned}|E(G)| &\geq x^2r^2+x^2ra+\frac{x(r+a+1)}{2} \\ &= x^2r^2+x^2ra+\frac{x(r+a)+1}{2}+\frac{(x-1)}{2} \\ &> x^2r^2+x^2ra+\frac{x(r+a)+1}{2}, \quad \text{since } \ x > 1, \\ &= E(G),\end{aligned}$$

a contradiction.

Therefore G has no $(r, r+a)$-factorization when $x \geq 3$, x odd.

$\blacksquare$

In Figure 8.2, we give an example which illustrates the construction used in Theorem 8.10, Case 2. Here $x = 3, r = 2, a = 1, d = 6, s = 3$, so $\frac{d+s}{r+a} = \frac{d}{r} = x = 3$, and the $(6, 9)$-simple graph has no $(2, 3)$-factorization.

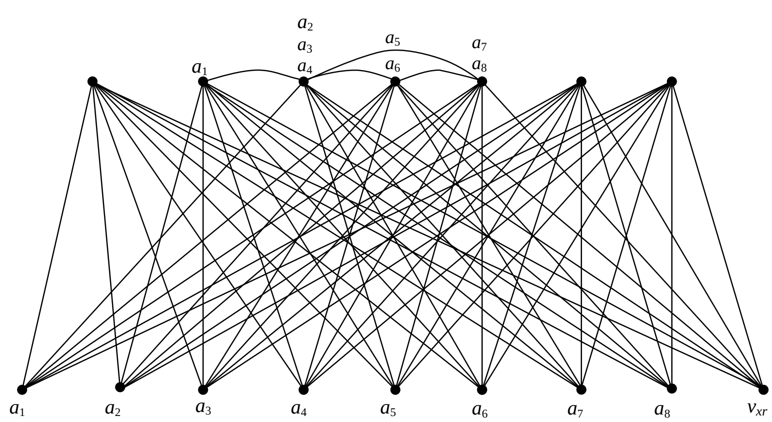


Figure 8.2: A $(6, 9)$-simple graph with no $(2, 3)$-factorization.

Chapter 9

$\sigma(r, s, a, t)$ when r is odd and a is odd

In the case when r is odd and a is odd, we know from Theorem 6.6(iii) that if d is a positive integer and s is a non-negative integer, and if $\frac{d+s}{r+a} \neq \frac{d}{r}$ and $d > r$, then if every simple $(d, d+s)$-graph has an $(r, r+a)$-factorization with x factors, it follows that

$$\frac{d+s}{r+a} \leq x < \frac{d}{r}\ .$$

In the case when r and a are both odd, $d > r$, we show the converse, namely if

$$\frac{d+s}{r+a} \leq x < \frac{d}{r}$$

then all $(d, d+s)$-simple graphs have $(r, r+a)$-factorizations with x factors.

We remark that in all other cases the corresponding statements are known to be true (see Theorems 3.14, 7.6 and 8.7).

We also show that if r and a are both odd, then

$$\sigma(r,s,a,t) = \begin{cases} r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r + 1 & \text{if } t \geq 2, \text{ or } t = 1 \text{ and } a < tr+s, \\ r & \text{if } t = 1 \text{ and } a \geq tr+s. \end{cases}$$

We show in this chapter that when r and a are both odd and $\frac{d+s}{r+a} = x = \frac{d}{r}$ and $d > r$, then there is a $(d, d+s)$-simple graph. This of course does not have an $(r, r+a)$-factorization with x factors.

(Note that this cannot be deduced from the existence of the graph G in Example 6.1, or from any of the graphs in the other examples, for although it might be true that there is a simple $(d, d+s)$-graph G with $d > r$ and $\frac{d}{r}$ not in $F_{\{r,a\}}(G)$, G might not satisfy the condition $\frac{d}{r} = x = \frac{d+s}{r+a}$.)

On our main theme, that of evaluating $\sigma(r,s,a,t)$, we first prove the following theorem.

Theorem 9.1 *Let $r \geq 1$ be odd, $a \geq 3$ be odd, $t \geq 1$, and $s \geq 0$ be integers. Then*

$$\sigma(r,s,a,t) = \begin{cases} r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r + 1 & \text{if } t \geq 2, \text{ or } t = 1 \text{ and } a < tr+s, \\ r & \text{if } t = 1 \text{ and } a \geq tr+s. \end{cases}$$

Proof First suppose that $t \geq 2$ or $t = 1$ and $a < tr + s$. Let us remark that an integer p satisfies

$$p = r\left\lceil \frac{rt+s}{a} \right\rceil + (t-1)r + 1$$

if and only if

$$p = r\left(\frac{rt+s+c}{a}\right) + (t-1)r + 1$$

where $a \mid tr + s + c$ and $0 \leq c \leq a - 1$.

Let an integer d satisfy

$$d = \frac{r}{a}(tr+s+c) + (t-1)r$$

for some c such that $a \mid tr + s + c$ and $0 \leq c \leq a - 1$. Then

$$\frac{d}{r} = \frac{1}{a}(tr+s+c) + (t-1)$$

and

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+c) - \frac{r+c}{r+a},$$

The integer values of x satisfying

$$\frac{d+s}{r+a} \leq x \leq \frac{d}{r}$$

include

$$\frac{tr+s+c}{a} + i$$

for $i = 0, 1, ..., t-2$, since $0 \leq c \leq a-1$. They do not include $i = -1$ or $t-1$ or any other integer values, so there are only $t-1$ such integer values of x. Therefore

$$\sigma(r, s, a, t) > r(\frac{tr+s+c}{a}) + (t-1)r,$$

and so

$$\sigma(r, s, a, t) \geq r\left\lceil \frac{rt+s}{a} \right\rceil + (t-1)r + 1.$$

Now suppose that $t = 1$ and $a \geq tr + s$. If applied in this case, the inequality derived in the other case would (erroneously) say that $\sigma(r, s, a, t) \geq r + 1$. However if $d = r$ and G is an $(r, r+a)$-graph, the G would be a $(d, d+s)$-graph with an $(r, r+a)$-factorization with one factor. Therefore, in this case, $\sigma(r, s, a, t) \geq r$.

Next we provide quite good bounds for $\sigma(r, s, a, t)$ when r and a are both odd, and also show that if $d \geq \sigma(r, s, a, t)$ in this case then there are t integer values of x satisfying $\frac{d+s}{r+a} \leq x < \frac{d}{r}$.

Theorem 9.2 *Let $r \geq 1$ be odd, $a \geq 3$ be odd, $t \geq 2$ or $t = 1$ and $a < rt + s$. Then*

$$r\left\lceil \frac{rt+s}{a} \right\rceil + (t-1)r + 1 \leq \sigma(r, s, a, t) \leq r\left\lceil \frac{rt+s+1}{a} \right\rceil + (t-1)r + 1.$$

Moreover if $d > \sigma(r, s, a, t)$ then there are t values of x satisfying

$$\frac{d+s}{r+a} \leq x < \frac{d}{r}.$$

Proof From Theorem 6.2 and Theorem 9.1, if $r \geq 3$ and $a \geq 3$, then

$$r\left\lceil \frac{rt+s}{a} \right\rceil + (t-1)r + 1 \leq \sigma(r, s, a, t) \leq r\left\lceil \frac{rt+s+1}{a} \right\rceil + (t-1)r + 1.$$

We know from the proof of Theorem 6.2 that if

$$d = r\left\lceil \frac{rt+s+1}{a} \right\rceil + (t-1)r + k$$

where $k \geq 1$, then there are at least t values of x satisfying

$$\frac{d+s}{r+a} < x < \frac{d}{r}.$$

With this value of d, we know that

$$d = \frac{r(rt+s+c)}{a} + (t-1)r + k$$

for some c, $0 < c \leq k$, and $a | rt + s + c$.

However, this is no longer true if $c = 0$, i.e. $a | rt + s$, i.e. $\lceil \frac{rt+s}{a} \rceil \neq \lceil \frac{rt+s+1}{a} \rceil$.

So we have that $\lceil \frac{rt+s}{a} \rceil = \lceil \frac{rt+s+1}{a} \rceil$ if $rt+s \not\equiv 0(mod\ a)$.

If $k=0$ and $c=a$ then there are no longer at least t values of x satisfying $\frac{d+s}{r+a} < x < \frac{d}{r}$. However there are t values of x satisfying $\frac{d+s}{r+a} \le x < \frac{d}{r}$. For then (following the discussion in the proof of Theorem 6.2),

$$\frac{d}{r} = \frac{1}{a}(tr+s+a) + (t-1) = \frac{1}{a}(tr+1) + t$$

and

$$\frac{d+s}{r+a} = \frac{1}{a}(tr+s+a) - \frac{r+a}{r+a} = \frac{1}{a}(tr+s).$$

So the values of x satisfying $\frac{d+s}{r+a} \le x < \frac{d}{r}$ are

$$\frac{1}{a}(tr+s) + i \text{ for } i = 0, 1, ..., t-1$$

so there are t integer values of x as asserted. Note that if

$$d = \frac{r}{a}(\frac{rt+s+a}{a}) + (t-1)r + 0,$$

then

$$d = \frac{r}{a}(\frac{rt+s+0}{a}) + (t-1)r + 1$$
$$= \left\lceil \frac{rt+s}{a} \right\rceil + (t-1)r + 1.$$

Therefore if $d \ge \sigma(r,s,a,t)$ we have

$$\frac{d+s}{r+a} \le x < \frac{d}{r},$$

as required.

In Theorem 9.2 we showed that if $r \ge 3$ is odd and $a \ge 3$ is odd then for every $(d, d+s)$-simple graph with $d \ge \sigma(r,s,a,t)$ there are at least t values of x satisfying $\frac{d+s}{r+a} \le x < \frac{d}{r}$. In particular, if $d \ge \sigma(r,s,a,t)$ then every $(d, d+s)$-simple graph G has an $(r, r+a)$-factorization with x factors for each value of x satisfying $\frac{d+s}{r+a} \le x < \frac{d}{r}$, provided that $r \ge 3$ is odd and $a \ge 3$ is odd. Taken together with Theorem 6.6(iii) this proves:

Theorem 9.3 *(1.26(iv)) Let $r \ge 3$ be odd and $a \ge 3$ be odd, and let $s \ge 0$. Then every $(d, d+s)$-simple graph G has an $(r, r+a)$-factorization with x factors, where x is an integer, if and only if $\frac{d+s}{r+a} \le x < \frac{d}{r}$.*

We finally turn to the proof of our main result in this chapter.

Theorem 9.4 *Let $r \ge 3$ be odd and $a \ge 3$ be odd. Let $s \ge 0$ and $t \ge 1$. Then*

$$\sigma(r,s,a,t) = r\left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r + 1$$

if $t \ge 2$, or if $t = 1$ and $a < rt+s$. If $t = 1$ and $a \ge rt+s$ then $\sigma(r,s,a,t) = r$.

We wish to show that Theorem 9.3 implies Theorem 9.4. For this reason we let $X(r,s,a,t)$ be the union over all $d \ge r$ of the set of all simple $(d, d+s)$-graphs G which satisfy the inequality

$$\frac{d+s}{r+a} \le x < \frac{d}{r}$$

for each x in $F_{\{r,a\}}(G)$; i.e. for all simple $(d, d+s)$-graphs G such that $F_{\{r,a\}}(G) \subseteq \left[\frac{d+s}{r+a}, \frac{d}{r}\right)$. Let $\sigma(X(r,s,a,t))$ be the least value of d, say $d = d_0$, for which it is true that, if $d > d_0$, then all members of X of degree d have an $(r, r+a)$-factorization wih x factors for at least t values of x. Of course Theorem 9.2 shows that $\sigma(X(r,s,a,t)) = \sigma(X(r,s,a,t)$, so the notation $X(r,s,a,t)$ is not strictly necessary.

Proof (Proof of Theorem 9.4) Suppose that $t \geq 2$ or $t = 1$ and $a < tr + s$. The proof of Theorem 9.1 works just as well for $X(r, s, a, t)$ to show that

$$\sigma(X(r, s, a, t)) \geq r \left\lceil \frac{tr + s}{a} \right\rceil + (t - 1)r + 1.$$

So we need to show that

$$\sigma(X(r, s, a, t)) \leq r \left\lceil \frac{tr + s}{a} \right\rceil + (t - 1)r + 1,$$

and then the theorem follows immediately. In other words, we need to show that

$$\sigma(X(r, s, a, t)) \leq \frac{r}{a}(tr + s + c) + (t - 1)r + 1$$

where $a \mid tr + s + c$ and $0 \leq c \leq a - 1$.

Let

$$d = \frac{r}{a}(tr + s + c) + (t - 1)r + k$$

where $k \geq 1$. We show that there do exist t integer values of x satisfying $\frac{d+s}{r+a} \leq x < \frac{d}{r}$. Then it follows by the definition of $\sigma(X(r, s, a, t))$ that every $(d, d + s)$-simple graph in $X(r, s, a, t)$ is $(r, r + a)$-factorable into x factors for at least t values of x.

First we note that

$$\frac{d}{r} = \frac{1}{a}(tr + s + c) + (t - 1) + \frac{k}{r}$$

and

$$\frac{d + s}{r + a} = \frac{1}{a}(tr + s + c) + \frac{k - r - c}{r + a}.$$

For p a non-negative integer, if $pr < k \leq (p + 1)r$ then $\frac{k}{r} > p$ and

$$\frac{k - r - c}{r + a} \leq \frac{(p + 1)r - r - c}{r + a} = \frac{pr - c}{r + a} \leq p\frac{r}{r + a} < p,$$

so

$$\frac{d + s}{r + a} = \frac{1}{a}(tr + s + c) + \frac{k - r - c}{r + a} \leq \frac{1}{a}(tr + s + c) + p$$

and

$$\begin{aligned} \frac{d}{r} &= \frac{1}{a}(tr + s + c) + (t - 1) + \frac{k}{r} \\ &> \frac{1}{a}(tr + s + c) + (t - 1) + p. \end{aligned}$$

Therefore if $pr < k \leq (p + 1)r$ for some non-negative integer p, then the integer values of x satisfying $\frac{d+s}{r+a} \leq x < \frac{d}{r}$ include

$$\frac{1}{a}(tr + s + c) + i \quad \text{for} \quad i = p, p + 1, ..., p + (t - 1),$$

so there are at least t such values of x.

Therefore

$$\sigma(X(r, s, a, t)) \leq r \left\lceil \frac{tr + s}{a} \right\rceil + (t - 1)r + 1,$$

as asserted.

If $t = 1$ and $a \geq tr+1$ then the formula $r \left\lceil \frac{tr+s}{a} \right\rceil + (t-1)r+1$ yields the value $r \neq 1$. However, when $d = r$ then, since $s < r + s \leq a$, a $(d, d+s)$-graph is an $(r, r+a)$-factor. Therefore, in this case, $\sigma(X(r, s, a, t)) = r$.

Theorem 9.4 now follows.

We show next that, in the case when r and a are both odd, then there do exist simple graphs with $\frac{d+s}{r+a} = x = \frac{d}{r} \geq 2$. These of course do not have $(r, r+a)$-factorizations with x factors (similar to what we showed in the case r even and a odd in Theorem 8.7). Again, it follows from a theorem of Kano and Saito [26] that such graphs do have at least one $(r, r+a)$-factor. We do not know if such graphs must have at least $x-1$ edge-disjoint $(r, r+a)$-factors, although this seems very likely.

Theorem 9.5 *Let r and a be odd positive integers, let d and s be positive integers such that $(r+a) \mid (d+s)$, and let $x = \frac{d+s}{r+a} = \frac{d}{r}$, where $x \geq 2$. Then there is a $(d, d+s)$-simple graph which does not have an $(r, r+a)$-factorization with x factors.*

Proof We separate out the cases x even and x odd. Although these are similar, it is easier for the reader if they are treated separately.

Case 1: Let G be a bi-degreed simple graph with vertex sets M and N where $|M| = xr$ and $|N| = x(r+a)+1$. Let H be a simple graph with $V(H) \subset N$ and $|E(H)| = \frac{xr}{2}$. Label the vertices of H with labels $a_1, a_2, ..., a_{xr}$ in such a way that if $v \in V(H)$ then v receives $d_H(v)$ labels. Also assign the labels $a_1, a_2, ..., a_{xr}$ to the vertices of M, assigning one label to each vertex.

We have H placed on the vertices of N. Then to form G from this, join each vertex v of M to each vertex of N except the vertex with the same label as v. Then, for $v \in M$, $d_G(v) = x(r+a)$ and, for $v \in N$, $d_G(v) = xr$.

Notice that
$$|E(G)| = xr(x(r+a)) + \frac{xr}{2} = x^2r^2 + x^2ra + \frac{xr}{2}.$$

If G has an $(r, r+a)$-factorization with x factors, let $\{F_1, ..., F_x\}$ be such a set of factors. Each F_i will have $r+a$ edges incident with each vertex of M, so for $1 \leq i \leq x$,

$$\sum_{v \in V(G)} d_{F_i}(v) \geq xr(r+a) + (x(r+a)+1)r = 2xr^2 + 2xra + r.$$

Therefore for each i, $1 \leq i \leq x$,
$$|E(F_i)| \geq xr^2 + xra + \left\lceil \frac{r}{2} \right\rceil.$$

Therefore

$$\begin{aligned}
|E(G)| &\geq x(xr^2 + xra + \lceil \frac{r}{2} \rceil) \\
&= x^2r^2 + x^2ra + x\lceil \frac{r}{2} \rceil \\
&> x^2r^2 + x^2ra + x\frac{r}{2}, \textit{since } r \textit{ is odd}, \\
&= |E(G)|,
\end{aligned}$$

a contradiction.

Thus G has no $(r, r+a)$-factorization with x factors when x is even.

An aside.

In Figure 9.1 we give an example with $r = a = 1$ and $x = 2$.

Case 2: Let x be odd

Let G be a simple graph with vertex sets $M \cup N$ where $|M| = xr$ and $|N| = x(r+a)+1$. The vertices of N will have degree xr and all except one vertex of M will have degree $x(r+a)$, with one vertex having degree $x(r+a)-1$. Let H be a simple graph with $V(H) \subset N$ and $|E(H)| = \frac{xr+1}{2}$. Label the vertices of H with labels $a_1, a_2, ..., a_{xr}, a_{xr+1}$ in such a way that labels a_{xr} and a_{xr+1} are assigned to different vertices of H and, if $v \in V(H)$, then v receives $d_H(v)$ labels. Also assign the labels $a_1, a_2, ..., a_{xr+1}$ to the vertices of M, with one vertex, say v_{xr} receiving two labels, say a_{xr} and a_{xr+1}, and the remaining $xr-1$ vertices

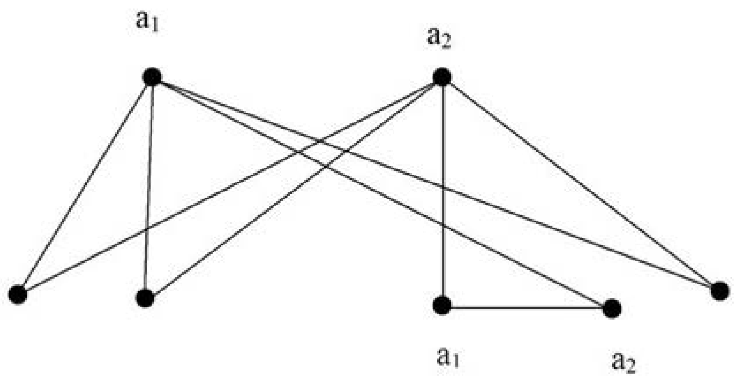


Figure 9.1: A bi-degreed (2 ,4)-simple graph with no (1, 2)-factorization.

receiving one label from $a_1, a_2, ..., a_{xr-1}$ each.

We have H already placed on the vertices of N. To form G from this, first join each vertex v of $M \setminus \{v_{xr}\}$ to each vertex of N except the vertex with the same label as v. Join v_{xr} to all vertices of N except the vertices with labels a_{xr} and a_{xr+1}. Then, for $v \in N, d_G(v) = xr$, $v \in M \setminus \{v_{xr}\}$, $d_{G(v)} = x(r+a)$ and $d_{G(v_{xr})} = x(r+a) - 1$. Then

$$\begin{aligned}|E(G)| &= xr(x(r+a)) - 1 + \frac{xr+1}{2} \\ &= x^2r^2 + x^2ra + \frac{xr}{2} - \frac{1}{2}.\end{aligned}$$

If G has an $(r, r+a)$-factorization with x factors, let $\{F_1, F_2, ..., F_x\}$ be such a set of factors. Then for all except one i, F_i will have $r+a$ edges incident with each vertex of M, but for one i, say $i = x$, F_i will have $r+a-1$ edges incident with v_{xr}, but will have $r+a$ edges incident with each other vertex of M. Therefore

$$\begin{aligned}\sum_{v \in V(G)} d_{F_i}(v) &= \begin{cases} xr(r+a) + (x(r+a)+1)r, & \text{if} \quad i \neq x, \\ xr(r+a) - 1 + (x(r+a)+1)r, & \text{if} \quad i = x, \end{cases} \\ &= \begin{cases} 2xr^2 + 2xra + r, & \text{if} \quad i \neq x, \\ 2xr^2 + 2xra + r - 1, & \text{if} \quad i = x. \end{cases}\end{aligned}$$

Therefore

$$|E(G)| \geq \begin{cases} xr^2 + xra + \left\lceil \frac{r}{2} \right\rceil, & \text{if} \quad i \neq x, \\ xr^2 + xra + \frac{r-1}{2}, & \text{if} \quad i = x. \end{cases}$$

Therefore

$$\begin{aligned}|E(G)| &\geq (x-1)(xr^2 + xra + \left\lceil \frac{r}{2} \right\rceil) + xr^2 + xra + \left\lceil \frac{r-1}{2} \right\rceil \\ &= x^2r^2 + x^2ra + x\left\lceil \frac{r}{2} \right\rceil - 1 \\ &> x^2r^2 + x^2ra + \frac{xr}{2} - \frac{1}{2} \qquad \text{since} \ \ x \geq 3, \\ &= |E(G)|,\end{aligned}$$

a contradiction (noting that $x\left\lceil \frac{r}{2} \right\rceil - 1 > \frac{xr}{2} - \frac{1}{2}$ when $x \geq 3$).

Therefore G has no $(r, r+a)$-factorization when $x \geq 3$, x odd.

In Figure 9.2 we illustrate the construction used in Case 2 in Theorem 9.5.

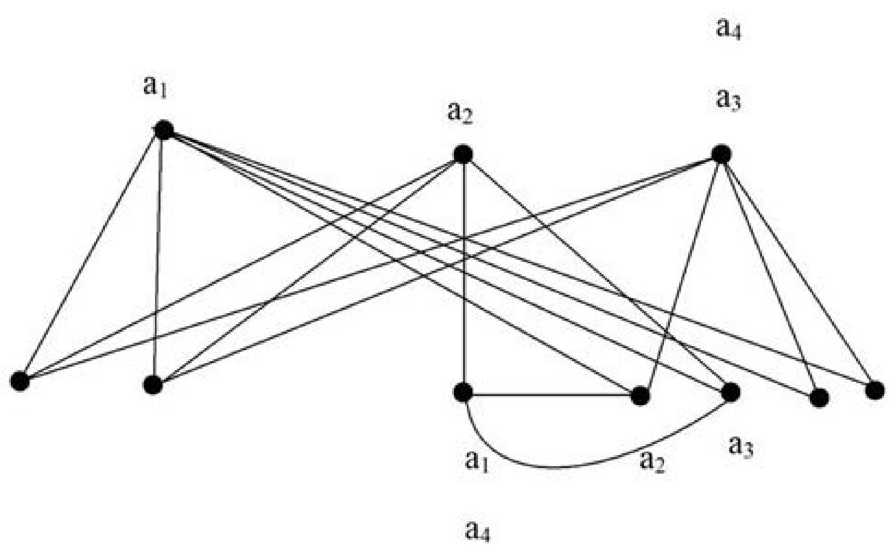


Figure 9.2: A $(d, d+s)$-simple graph which is not $(r, r+a)$-factorizable into x factors. Here $x = 3, s = 3, r = 1, a = 1, a = 3$.

Bibliography

[1] J. Akiyama, D. Avis and H. Era, *On a* $1, 2$*-factor of a graph* TRU Math, **16** (1980), 97–102.

[2] J. Akiyama and M. Kano, *Factors and factorizations of graphs–a survey*, J. Graph Theory **9** (1985), 1–42.

[3] J. Akiyama and M. Kano, *Almost regular factorizations of graphs*, J. Graph Theory **9** (1985), 123–128.

[4] J. Akiyama and M. Kano, *Factors and factorizations of graphs*, Version 1.0, book published online, 2007.

[5] K. Appel, W. Haken and J. Koch, *Every planar graph is four-colorable*, Illinois J. Math **21** (1977), 429–567.

[6] Cai Mao-cheng, $[a, b]$*-factorizations of graphs*, J. Graph Theory **15** (1991), 283–301.

[7] A.G. Chetwynd and A.J.W. Hilton, *Partial edge-colourings of complete graphs or of graphs which are nearly complete, Graph Theory and Combinatorics (Academic Press, Vol. in honour of P. Erdös' 70th birthday*, (1984), 81–98.

[8] A.G. Chetwynd and A.J.W. Hilton, *1-factorizing regular graphs of high degree–an improved bound, Discrete Math.,* **75** (1989), 103–112.

[9] B. Czaba, D. Kühn, A. Lo, D. Osthus, A. Treglown, *Proof of the 1-factorization and Hamilton decomposition conjectures, Memoires of the American Mathematical Society*, **224** (2016), monograph 1154, 170 pages.

[10] Y. Egawa, *Era's conjecture on* $[k, k+1]$*-factorizations of regular graphs*, Ars Combin., **21** (1986), 217–220.

[11] H. Era, *Semiregular factorizations of regular graphs*, in: F. Harary, J. Maybee (Eds), Graphs and Applications, Proceedings of the First Colorado Symposium of Graph Theory, Wiley, New York, 1984, pp. 101–116.

[12] M. Ferencak and A.J.W. Hilton, *Semiregular factorization of regular multigraphs*, Mathematika, **56** (2010), 357–362.

[13] M. Ferencak and A.J.W. Hilton, *Regular multigraphs and their semiregular factorizations*, Congressus Numerantium, **209** (2011), 149–159.

[14] Yan Guiying, *Some new results on (g,f)-factorizations of graphs*, J. Combin. Math. Combin. Comput.**18** (1995), 177–185.

[15] R.P. Gupta, *A theorem on the cover index of an* s*-graph*, Notices Amer. Math. Soc., **13** (1966), 714.

[16] A.J.W. Hilton, *The cover index, the chromatic index and the minimum degree of a graph*, in: Proceedings of the 5th British Combinatorial Conference, Congressus Numerantium, vol. XV, 1975, pp. 307–318.

[17] A.J.W. Hilton, *Factorizations of regular graphs of high degree*, J. Graph Theory, **9** (1985), 193–196.

[18] A.J.W. Hilton, *$(r, r+1)$-factorizations of $(d, d+1)$-graphs*, Discrete Math., **308** (2008), 645–669.

[19] A.J.W. Hilton, *On the number of $(r, r+1)$-factors in an $(r, r+1)$-factorization of a simple graph*, J. Graph Theory, **60** (2009), 257–268.

[20] A.J.W. Hilton, *Degree bounded factorizations of bipartite multigraphs and of pseudographs*, Discrete Math., **310** (2010), 288–303.

[21] A.J.W. Hilton and A. Rajkumar, *The pseudograph (r, s, a, t)-threshold number*, Discrete Applied Math., **209**(2016), 155-163.

[22] A.J.W. Hilton and A. Rajkumar, *The simple graph threshold number $\sigma(r, s, a, t)$ when r is odd and $a \geq 2$ is even*, Congressus Numerantium, **223** (2015), 33-44.

[23] A.J.W. Hilton and D. de Werra, *A sufficient condition for equitable edge-colourings of simple graphs*, Discrete Math., **128** (1994), 179–201.

[24] A.J.W. Hilton and J. Wojciechowski, *Semiregular factorization of simple graphs*, AKCE International Jour. of Graphs and Combinatorics, **2** (2005), 57–62.

[25] M. Kano, *$[a, b]$-factorizations of a graph*,J. Graph Theory **9** (1985), 129–146.

[26] M. Kano and A. Saito, *$[a, b]$-factors of graphs*, Discrete Math., **47** (1983), 113–116.

[27] D. König, *Die Theorie der Endlichen and Unendlichen Graphen*, Chelsea Pub Co., New York,(1950).

[28] L. Lovaśz, *Combinatorial Problems and Exercises*, North Holland, Amsterdam (1979), p. 50.

[29] Guizhen Liu, *(g,f)-factors and (g,f)-factorizations of graphs*, Acta Mathematica Sinica (Chinese series) **37** (1994), 230–237.

[30] C.J.H. McDiarmid, *The solution of a time-tabling problem*, J. Inst. Math. Appl., **9** (1972), 23–34.

[31] C.J.H. McDiarmid, *Integral decomposition in polyhedra*, Mathematical Programming, **25** (1983), 183–198.

[32] T. Niessen and L.Volkmann, *Class 1 conditions depending on the minimum degree and the number of vertices of maximum degree*, J. Graph Theory, **14** (1990), 225–246.

[33] J. Petersen, *Die Theorie der Regulären Graphen*, Acta Math., **15** (1891), 193–220.

[34] M. D. Plummer, *Factors and factorization*, Section 5.4 of Handbook of Graph Theory, eds. Jonathan L. Gross and Jay Yellen, Discrete Mathematics and its Applications, CRC Press, 2004, pp. 403–430.

[35] M. D. Plummer, *Graph factors and factorization: 1987-2003*, Discrete Math., **307** (2007), 791-821.

[36] W. T. Tutte, *The factorization of linear graphs*, J. London Math. Soc. **22** (1947), 107–111.

[37] V.G. Vizing, *Critical graphs with a given chromatic class*, Diskret. Analiz., **5** (1965), 9–17.

[38] V.G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz., **3** (1964), 25–30 (in Russian).

[39] D. de Werra, *Equitable colorations of graphs*, Rev. française d'Inform. Recherche Operationelle, **R-3** (1971), 3–8.

[40] D. de Werra, *A few remarks about chromatic scheduling*, in Combinatorial Programming: Methods and Applications (B. Roy, ed.), Reidel, Dordrecht, 1975, pp. 337–342.

[41] D. de Werra, *On a particular conference scheduling problem*, INFOR **13** (1975), 308–315.

[42] Xia Zhang and Guizhen Lin, *Equitable edge-colorings of simple graphs*, J. Graph Theory, **66** (2011), 175–197.