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NORMAL AND Δ-NORMAL CONFIGURATIONS IN TORIC ALGEBRA

LIAM SOLUS

Abstract. Toric algebra is a field of study that lies at the intersection of algebra, geometry, and combinatorics. Thus, the algebraic properties of the toric ideal $I_{\mathcal{A}}$ defined by the vector configuration $\mathcal{A}$ are often characterizable via the geometric and combinatorial properties of its corresponding toric variety and $\mathcal{A}$, respectively. Here, we focus on the property of normality of $\mathcal{A}$. A normal vector configuration defines the toric ideal of a normal toric variety. However, the definition of normality of $\mathcal{A}$ is based entirely on the algebraic structures associated with $\mathcal{A}$ without regard to any of its combinatorial properties. In this paper, we discuss two attempts to provide a combinatorial characterization of normality of $\mathcal{A}$. Particularly, we show that the properties "the convex hull of $\mathcal{A}$ possesses a unimodular covering" and "$\mathcal{A}$ is a Δ-normal configuration" are both sufficient conditions for normality of $\mathcal{A}$, but neither is necessary. This suggests that another combinatorial property is required to provide the desired characterization of normality of $\mathcal{A}$.

1. Introduction

Toric ideals form a special class of ideals in multivariate polynomial rings, the study of which lies at the intersection of algebra, geometry, and combinatorics. For this reason, toric ideals have proven to be quite useful for testing general theories in these areas of mathematics. This quality of toric ideals is inherent in their definition, which begins with a combinatorial object, transforms it into an algebraic one, and then uses a classical correspondence to draw geometric connections.

Let k be an algebraically closed field, and let $k[\mathbf{x}] := k[x_1, \ldots, x_n]$ be a polynomial ring in n variables. To construct a toric ideal, we first pick a finite set of integer vectors $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$. Then let $\mathbf{A} \in \mathbb{Z}^{dxn}$ be the integer matrix with column i equal to $\mathbf{a_i}$ for $i = 1, \ldots, n$. Consider the following semigroup homomorphism

$$\pi : \mathbb{N}^n \longrightarrow \mathbb{Z}^d, \quad \pi : \mathbf{u} = (u_1, \ldots, u_n) \longmapsto u_1\mathbf{a_1} + \cdots + u_n\mathbf{a_n} = \mathbf{Au}.$$

Here $\mathbb{N}$ denotes the set of natural numbers (including 0). The image of π is the semigroup

$$\mathbb{N}\mathcal{A} = \{\lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n : \lambda_1, \ldots, \lambda_n \in \mathbb{N}\} = \{\mathbf{A}u : u \in \mathbb{N}^n\}.$$

We can identify each vector $\mathbf{a}_i$ with the monomial $\mathbf{t}^{\mathbf{a}_i} = t_1^{a_{1i}} t_2^{a_{2i}} \ldots t_d^{a_{dj}}$ in the Laurent polynomial ring $k[\mathbf{t}^{\pm 1}] := k[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}]$. Given this identification, the map π lifts to a homomorphism of the semigroup algebras:

$$\hat{\pi} : k[\mathbf{x}] \longrightarrow k[\mathbf{t}^{\pm 1}], \quad x_i \longmapsto \mathbf{t}^{\mathbf{a}_i} = t_1^{a_{1i}} t_2^{a_{2i}} \ldots t_d^{a_{dj}}.$$

Definition 1.1. The **toric ideal** of $\mathcal{A}$, denoted $I_{\mathcal{A}}$, is the kernel of the map $\hat{\pi}$.

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For example, consider the set of integer vectors corresponding to the columns of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then the map $\hat{\pi} : \mathbb{C}[x, y, z] \longrightarrow \mathbb{C}[t_1^{-1}, t_2^{-1}, t_1, t_2]$ sends $\hat{\pi}(x) = t_1$, $\hat{\pi}(y) = t_2$, and $\hat{\pi}(z) = t_1 t_2$. Thus, $I_{\mathcal{A}} = \ker(\hat{\pi}) = \langle xy - z \rangle$, the ideal generated by $xy - z$ in $\mathbb{C}[x, y, z]$.

The finite set of vectors $\mathcal{A}$ is interesting in its own right as a combinatorial object. For example, we will be particularly interested in examining *triangulations* of $\mathcal{A}$. Moreover, we can use the ideal-variety correspondence of classical algebraic geometry to associate with each toric ideal $I_{\mathcal{A}}$ an *affine toric variety* $\mathcal{V}_{\mathcal{A}}$, consisting of the set of all zeros shared by all polynomials in $I_{\mathcal{A}}$. Thus, we now have a geometric structure associated with the set of integer vectors $\mathcal{A}$, which we can use to study the geometric properties of $\mathcal{A}$ from an algebro-geometric point of view.

In this way, the definition of toric ideals and their associated structures places the study of toric ideals at the intersection algebra, geometry, and combinatorics, and provides useful correspondences between these fields of study. Thus, the natural type of question to ask is, given, for example, an algebraic property P of $I_{\mathcal{A}}$, what can be said about the geometric and combinatorial properties of $\mathcal{V}_{\mathcal{A}}$ and $\mathcal{A}$, respectively? Can we find a geometric property G of $\mathcal{V}_{\mathcal{A}}$ and/or a combinatorial property C of $\mathcal{A}$ such that G of $\mathcal{V}_{\mathcal{A}}$ guarantees P of $I_{\mathcal{A}}$, and similarly for C of $\mathcal{A}$? If so, are either of these conditions necessary and sufficient for P of $I_{\mathcal{A}}$? In other words, can we find a geometric and/or combinatorial characterization of P of $I_{\mathcal{A}}$ via properties of $\mathcal{V}_{\mathcal{A}}$ and $\mathcal{A}$, respectively?

In the following, we explore a question of this type in relation to the property that the semigroup $\mathbb{N}\mathcal{A}$ associated with the toric ideal $I_{\mathcal{A}}$ is *normal*. In other words, we are interested in defining geometric and combinatorial characterizations of normality of $\mathbb{N}\mathcal{A}$ via properties of $\mathcal{V}_{\mathcal{A}}$ and $\mathcal{A}$, respectively. In Section 4, we demonstrate that such a *geometric* characterization has been found. Namely, the semigroup $\mathbb{N}\mathcal{A}$ is normal if and only if the affine toric variety $\mathcal{V}_{\mathcal{A}}$ is a *normal variety*. In this way, the property of normality spans the algebro-geometric correspondence for toric varieties as nicely as we could desire. On the other hand, no such combinatorial characterization of the normality of $\mathbb{N}\mathcal{A}$ is known. In Section 5, we discuss problems related to attempts to provide such a characterization, as well as those related to the best known combinatorial approximation of normality of $\mathbb{N}\mathcal{A}$. In Section 2 we give a brief introduction to the theory of Gröbner bases with a focus on the material necessary for our discussion in Section 4. Then in Section 3, we provide the details of convex polytopal geometry and triangulations that will be critical in the analysis given in Sections 4 and 5.

2. Gröbner Basis Theory

The Division Algorithm for polynomials in one variable is well-known, and its simplicity has consequences for the polynomial ring $k[x]$. In particular, the polynomial ring $k[x]$ is a principal ideal domain, and the polynomial $f \in k[x]$ is in the ideal $I \subset k[x]$ generated by the polynomial g if and only if the remainder of f upon division by g is 0. What if we consider division of polynomials in more than one variable? The polynomial ring $k[x_1, \ldots, x_n]$ is not a principal ideal domain. However, one might hope that to see if $f \in k[x_1, \ldots, x_n]$ is in $I = \langle g_1, \ldots, g_t \rangle$ (the ideal generated by $g_1, \ldots, g_t \in k[x_1, \ldots, x_n]$) we simply need to check that the remainder of f upon division by all of $g_1, \ldots, g_t$ equals 0. For example, say we wanted to see if the polynomial $x^2y^2 + x^2y + y^5$ is in the ideal $I = \langle y^2 + 1, x^2y + x \rangle \subset k[x, y]$. Intuitively, we can try to mimic the process of the univariate division algorithm in two ways. First, we try dividing by $y^2 + 1$ and then $x^2y + x$:

$$
\begin{array}{r|l}
 & x^2 + y^3 - y \\ \hline
y^2 + 1 & x^2y^2 + x^2y + y^5 \\
 & x^2y^2 + x^2 \\ \cline{2-2}
 & \quad y^5 + x^2y - x^2 \\
 & \quad y^5 + y^3 \\ \cline{2-2}
 & \qquad -y^3 + x^2y - x^2 \\
 & \qquad -y^3 - y \\ \cline{2-2}
 & \qquad\quad x^2y - x^2 + y
\end{array} \tag{1}
$$

Since no term in $x^2y - x^2 + y$ is divisible by y^2 we now divide this remainder by $x^2y + x$:

$$
\begin{array}{r|l}
 & 1 \\ \hline
x^2y + x & x^2y - x^2 + y \\
 & x^2y + x \\ \cline{2-2}
 & \qquad -x^2 - x + y
\end{array} \tag{2}
$$

We can check that

$$x^2y^2 + x^2y + y^5 = (y^2 + 1)(x^2 + y^3 - y) + (x^2y + x)(1) - (x^2 + x + y),$$

and so we think of $r = -(x^2 + x + y)$ as our remainder upon division of $x^2y^2 + x^2y + y^5$ by the ordered pair $(y^2 + 1, x^2y + x)$. Now, if we try dividing first by $x^2y + x$ and then by $y^2 + 1$ we see that

$$x^2y^2 + x^2y + y^5 = (x^2y + x)(y + 1) + (y^2 + 1)(y^3 - y) + (y - xy - x),$$

and so our remainder in this case is $y - xy - x$. At this point we should be both confused and frustrated. First, in our intuition-based attempt to perform the division, what prompted us to divide with respect to the terms x^2y and y^2? Why not divide with respect to x and 1? Second, what determined when we stopped dividing by a given divisor? For example, in (2), why didn't we try to divide $-x^2 - x + y$ by x once we had finished dividing by x^2y? Third, why does switching the order of our divisors result in two distinct remainders? Moreover, the above equalities indicate that the difference between the two remainders, $(y - xy - x) + (x^2 + x + y)$, is in the ideal $\langle y^2 + 1, x^2y + x \rangle$, but the division we just performed has no way of determining this, since this difference is not divisible by either of the generators. In this section, we will develop some of the fundamental concepts of the theory of Gröbner bases. In doing so, we will provide a well structured multivariate polynomial division algorithm, along with special sets of divisors, called Gröbner bases, that will give unique remainders regardless of the order of division. This work will play an important role in our discussion of toric ideals and their associated affine varieties. The following material is based mostly on Chapter 1 of [9] with some additional material drawn from [3] and [5].

Let k be a field and let $k[\mathbf{x}] := k[x_1, \dots, x_n]$ be a polynomial ring in n variables over k. We say that an ideal $I \subset k[\mathbf{x}]$ is *finitely generated* if there exists a finite subset $\mathcal{B} = \{f_1, \dots, f_t\} \subset I$ such that $I = \{\sum_{i=1}^{t} h_i f_i : h_i \in k[\mathbf{x}]\}$. We call $\mathcal{B}$ a *basis* of I and write $I = \langle f_1, \dots, f_t \rangle$ to indicate that I is *generated* by $\mathcal{B}$. By the Hilbert Basis Theorem, given in [5, §1.5], every ideal I in $k[\mathbf{x}]$ is finitely generated.

A *monomial* in $k[\mathbf{x}]$ is a product $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in k[\mathbf{x}]$, where $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$. We denote the monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ by $\mathbf{x}^{\mathbf{a}}$. If $c \in k$ and $\mathbf{x}^{\mathbf{a}}$ is a monomial in $k[\mathbf{x}]$, then we call the product $c\mathbf{x}^{\mathbf{a}}$ a *term* in $k[\mathbf{x}]$. A *polynomial* in $k[\mathbf{x}]$ is finite sum of terms in $k[\mathbf{x}]$. The *support* of a polynomial $f = \sum c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}]$ is the set $\mathrm{supp}(f) = \{\mathbf{a} \in \mathbb{N}^n : f = \sum c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, c_{\mathbf{a}} \neq 0\}$.

Example 2.1. Let $k = \mathbb{Q}$, and $f = x^2 y + 3y^5 + \frac{1}{9} y^3 + \frac{3}{4} x^7 y^4 \in \mathbb{Q}[x, y]$. The terms of f are $x^2 y$, $3y^5$, $\frac{1}{9} y^3$, and $\frac{3}{4} x^7 y^4$. Moreover, $\mathrm{supp}(f) = \{(2,1), (0,5), (0,3), (7,4)\}$.

Let k^n denote the Cartesian product of k with itself n times. We refer to k^n as *affine n-space over k*, and we call any $\mathbf{p} = (p_1, \dots, p_n) \in k^n$ a *point* in k^n. If a point $\mathbf{p} = (p_1, \dots, p_n) \in k^n$ *satisfies* a polynomial $f = \sum c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}]$, i.e. $f(p_1, \dots, p_n) = \sum c_{\mathbf{a}} \mathbf{p}^{\mathbf{a}} = 0$, then we say that $\mathbf{p}$ is a *zero* of f. The *affine variety* of a set of polynomials $S \subset k[\mathbf{x}]$, denoted $\mathcal{V}(S)$, is the set of all zeros shared by the polynomials in S. More precisely,

$$\mathcal{V}(S) = \{\mathbf{p} \in k^n : f(\mathbf{p}) = 0 \text{ for all } f \in S\}.$$

It is well known that if $I = \langle f_1, \dots, f_t \rangle \subset k[\mathbf{x}]$ then $\mathcal{V}(I) = \mathcal{V}(f_1, \dots, f_t)$. If $\mathcal{W}$ and $\mathcal{V}$ are varieties in k^n such that $\mathcal{W} \subset \mathcal{V}$ then we call $\mathcal{V} \backslash \mathcal{W} = \{\mathbf{p} \in k^n : \mathbf{p} \in \mathcal{V}, \mathbf{p} \notin \mathcal{W}\}$ a *Zariski open subset* of $\mathcal{V}$. The smallest variety containing $\mathcal{V} \backslash \mathcal{W}$ is called the *Zariski closure* of $\mathcal{V} \backslash \mathcal{W}$.

Our goal in this section is to determine a finite basis $\mathcal{B}$ for a given ideal $I \subset k[\mathbf{x}]$ such that the elements of $\mathcal{B}$ can be used to determine if a given polynomial $f \in k[\mathbf{x}]$ is indeed an element of I. In order to do this, we must define a method for ordering the monomials in $k[\mathbf{x}]$ so that the polynomial $f \in k[\mathbf{x}]$ always has a greatest term.

Definition 2.2. A **term order** $\succ$ on $k[\mathbf{x}]$ is a total order on the monomials of $k[\mathbf{x}]$ such that

(1) $\mathbf{x}^{\mathbf{b}} \succ \mathbf{x}^{\mathbf{a}}$ implies that $\mathbf{x}^{\mathbf{b}} \mathbf{x}^{\mathbf{c}} \succ \mathbf{x}^{\mathbf{a}} \mathbf{x}^{\mathbf{c}}$ for all $\mathbf{c} \in \mathbb{N}^n$, and
(2) $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{0}} = 1$ for all $\mathbf{a} \in \mathbb{N}^n \backslash \{0\}$.

It then follows from the Gordan-Dickson Lemma [3, §4.2] and (2) of Definition 2.2 that every term order on $k[\mathbf{x}]$ is a *well-ordering*. More precisely, every subset of $k[\mathbf{x}]$ has a smallest element in relation to $\succ$.

Example 2.3. Consider the the polynomial ring in n-variables, $k[\mathbf{x}] = k[x_1, \dots, x_n]$, and fix an ordering of the variables such that $x_1 \succ x_2 \succ \cdots \succ x_n$. Two commonly used term orders on $k[\mathbf{x}]$ are called the *lexicographic (lex) order* and *graded reverse lexicographic (grevlex) order*.

In the lex ordering on $k[\mathbf{x}]$, $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{b}}$ if and only if the left-most nonzero term in $\mathbf{a} - \mathbf{b}$ is positive. For example, if $x \succ y$, then

$$x^7 y^4 \succ x^2 y \succ y^5 \succ y^3.$$

In the grevlex ordering on $k[\mathbf{x}]$, $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{b}}$ if and only if either $\deg(\mathbf{x}^{\mathbf{a}}) > \deg(\mathbf{x}^{\mathbf{b}})$ or $\deg(\mathbf{x}^{\mathbf{a}}) = \deg(\mathbf{x}^{\mathbf{b}})$ and the right-most nonzero term in $\mathbf{a} - \mathbf{b}$ is negative. Thus, if $x \succ y$, then

$$x^7y^4 \succ y^5 \succ x^2y \succ y^3.$$

Notice that the lex and grevlex orders have thus given us two different orderings for the terms in $f \in \mathbb{Q}[x, y]$ from Example 2.1.

Fix a term order $\succ$ on $k[\mathbf{x}]$. Given a polynomial $f = \sum c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}]$, the *initial term* or *leading term* of f is the term $c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{a}'}$ for all terms $c_{\mathbf{a}'}\mathbf{x}^{\mathbf{a}'} \neq c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$ of f. We denote the initial term of f with respect to $\succ$ by $in_{\succ}(f)$. We call the monomial $\mathbf{x}^{\mathbf{a}}$ the *initial monomial* of f. Now that we have established a method for ordering the monomials in $k[\mathbf{x}]$, we use this method to generalize the division algorithm for univariate polynomials to a division algorithm for multivariate polynomials.

Algorithm 2.4 ([3] Theorem 3)**.** A Division Algorithm for Multivariate Polynomials.
Input: A dividend $f \in k[\mathbf{x}]$, an ordered set of divisors $(f_1, \ldots, f_t) \subset k[\mathbf{x}]$, and a term order $\succ$.
Output: A set of polynomials $\{a_1, \ldots, a_s, r\} \subset k[\mathbf{x}]$ such that $f = \sum_{i=1}^{s} a_i f_i + r$, where either $r = 0$ or no term in r is divisible by any of $in_{\succ}(f_1), \ldots, in_{\succ}(f_t)$.

Do exactly what we did in the example at the beginning of the section. Starting with $i = 1$ and running through $i = t$, divide $in_{\succ}(f_i)$ into the remaining terms of f until the remainder is equal to zero, or until there is no $i \in \{1, \ldots, t\}$ such that $in_{\succ}(f_i)$ divides any of the remaining terms.

Example 2.5. Consider the polynomial $f = x^2y^2 + x^2 + y^5 \in \mathbb{Q}[x, y]$, and the set of divisors $\{f_1 = y^2 + 1, f_2 = x^2y + x\} \subset \mathbb{Q}[x, y]$ discussed at the beginning of this section. If we fix the ordering $x \succ y$ then the lex order applied to $\mathbb{Q}[x, y]$ gives $in_{\succ}(f) = x^2y^2$, $in_{\succ}(f_1) = y^2$, $in_{\succ}(f_2) = x^2y$. Thus, we see that the division performed in equations (1) and (2) is an example of dividing f by the ordered set of divisors (f_1, f_2) with respect to the lex ordering on $\mathbb{Q}[x, y]$.

However, recall that dividing f by (f_1, f_2) gives $r = -(x^2+x+y)$, and dividing f by (f_2, f_1) gives $r = y - xy - x$. Therefore, our division algorithm alone does not guarantee uniqueness of remainders. Thus, we need to define nice sets of divisors for which the remainder produced is dependent only on the dividend.

A *monomial ideal* is an ideal in $k[\mathbf{x}]$ generated by monomials. For an ideal $I \subset k[\mathbf{x}]$ we call the monomial ideal

$$in_{\succ}(I) := \langle in_{\succ}(f) : f \in I, f \neq 0 \rangle \subset k[\mathbf{x}]$$

the *initial ideal* of I with respect to $\succ$. It follows from the Gordan-Dickson Lemma [3] §4.2 that all monomial ideals have a unique minimal finite basis consisting only of monomials. Thus, there exist $g_1, \ldots, g_s \in I$ such that $in_{\succ}(I) = \langle in_{\succ}(g_1), \ldots, in_{\succ}(g_s) \rangle$.

Definition 2.6.
1. A finite subset of polynomials $\mathcal{G}_{\succ}(I) = \{g_1, \ldots, g_s\} \subset I$ is a **Gröbner basis** of I with respect to $\succ$ if $in_{\succ}(I) = \langle in_{\succ}(g_1), \ldots, in_{\succ}(g_s) \rangle$. We may assume that $in_{\succ}(g_i)$ is a monomial for $i = 1, \ldots, s$.
2. If $\mathcal{G}_{\succ}(I)$ is the unique minimal generating set for $in_{\succ}(I)$, we say that $\mathcal{G}_{\succ}(I)$ is a **minimal Gröbner basis** of I with respect to $\succ$.

(3) We say that a minimal Gröbner basis $\mathcal{G}_{\succ}(I)$ of I is **reduced** if no non-initial term of any $g_i \in \mathcal{G}_{\succ}(I)$ is divisible by any of $in_{\succ}(g_1), \ldots, in_{\succ}(g_s)$.

(4) The monomial of $k[\mathbf{x}]$ that do not belong to $in_{\succ}(I)$ are called the **standard monomials** of $in_{\succ}(I)$.

Theorem 2.7. *For a given term order $\succ$ and ideal $I \subset k[\mathbf{x}]$, the reduced Gröbner basis $\mathcal{G}_{\succ}(I)$ is unique.*

Proof. To see the theorem holds, assume otherwise. Then there exists a term order on $\succ$ such that for some ideal $I \subset k[\mathbf{x}]$, $\mathcal{G}_{\succ}(I) = \{g_1, \ldots, g_t\}$ and $\mathcal{G}'_{\succ}(I) = \{f_1, \ldots, f_s\}$ are both reduced Gröbner bases for I with respect to $\succ$, and there exists some $g_i \in \mathcal{G}_{\succ}(I)$ such that $g_i \neq f_j$, for all $j = 1, \ldots, s$. Since $\mathcal{G}'_{\succ}(I)$ generates $in_{\succ}(I)$, there exists an $f_j \in \mathcal{G}'_{\succ}(I)$ such that $in_{\succ}(g_i) = \mathbf{x}^{\mathbf{a}} in_{\succ}(f_j)$ for some monomial $\mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}]$. However, since $\mathcal{G}_{\succ}(I)$ also generates $in_{\succ}(I)$ there exists some $g_k \in \mathcal{G}_{\succ}(I)$ such that $in_{\succ}(f_j) = \mathbf{x}^{\mathbf{b}} in_{\succ}(g_k)$, and so $in_{\succ}(g_i) = \mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}} in_{\succ}(g_k)$. Thus, the minimality of $\mathcal{G}_{\succ}(I)$ implies that $in_{\succ}(g_i) = in_{\succ}(g_k)$. So, by the uniqueness of the minimal generating set for $in_{\succ}(I)$, it must be that $in_{\succ}(g_i) = in_{\succ}(f_j)$. Then, since $g_i - f_j \neq 0$ and $g_i - f_j \in I$, it must be that $in_{\succ}(g_i - f_j) \in in_{\succ}(I)$. This contradicts the assumption that $\mathcal{G}_{\succ}(I)$ and $\mathcal{G}'_{\succ}(I)$ are reduced, since no term in $g_i - f_j$ is divisible by any of the elements in the unique generating set for $in_{\succ}(I)$. Therefore, it must be that $g_i = f_j$, and so $\mathcal{G}_{\succ}(I) = \mathcal{G}'_{\succ}(I)$. □

Lemma 2.8 ([9] Lemma 1.3.8)**.** *If $\mathcal{G}_{\succ}(I)$ is a reduced Gröbner basis of I with respect to the term order $\succ$ on $k[\mathbf{x}]$, then the remainder of any polynomial after division by $\mathcal{G}_{\succ}(I)$ is unique.*

Proof. Let $f \in k[\mathbf{x}]$ and let $\mathcal{G}_{\succ}(I) = \{g_1, \ldots, g_s\} \subset I$, and assume that we can divide f by $\mathcal{G}_{\succ}(I)$ to obtain two distinct remainders $r_1, r_2 \in k[\mathbf{x}]$. The division algorithm then implies that

$$f = \sum a_i g_i + r_1 = \sum a'_i g_i + r_2.$$

It then follows that

$$r_1 - r_2 = \sum a'_i g_i - \sum a_i g_i \in I.$$

Now, if $r_1 - r_2 \neq 0$, then $in_{\succ}(r_1 - r_2) \in in_{\succ}(I)$ and so there exists some $i \in \{1, \ldots, s\}$ such that $in_{\succ}(g_i)$ divides $in_{\succ}(r_1 - r_2)$. This gives us a contradiction since the Division Algorithm assures that no term in $r_1 - r_2$ is divisible by any $in_{\succ}(g_i)$ for $i = 1, \ldots, s$. Thus, it must be that $r_1 - r_2 = 0$, and so $r_1 = r_2$. □

Corollary 2.9 ([9] Corollary 1.3.9)**.** *If $\mathcal{G}_{\succ}(I) = \{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I with respect to $\succ$, then I is finitely generated by $\mathcal{G}_{\succ}(I)$.*

Corollary 2.10 ([9] Corollary 1.3.10)**.** *If $\mathcal{G}_{\succ}(I) = \{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I with respect to $\succ$, and $f \in k[\mathbf{x}]$, then the remainder of f upon division by $\mathcal{G}_{\succ}(I)$ equals 0 if and only if $f \in I$.*

Now that we have found a basis for I with the nice properties described in Lemma 2.8 and Corollary 2.10, we provide an algorithm developed to compute a reduced Gröbner basis of an ideal $I = \langle f_1, \ldots, f_t \rangle \subset k[\mathbf{x}]$ with respect to any term order $\succ$ on $k[\mathbf{x}]$. The algorithm requires that we first compute a certain type of polynomial for each pair of generators of I.

Given two generators f and g of I, we compute the S-pair of f and g, denoted S-pair(f,g). Let $\text{lcm}(in_\succ(f), in_\succ(g))$ be the least common multiple of $in_\succ(f)$ and $in_\succ(g)$. Then

$$\text{S-pair}(f,g) = \frac{\text{lcm}(in_\succ(f), in_\succ(g))}{in_\succ(f)} f - \frac{\text{lcm}(in_\succ(f), in_\succ(g))}{in_\succ(g)} g.$$

Notice that S-pair(f,g) is the simplest way to cancel the initial terms of f and g. In the following, we let $\overline{f}^{\mathcal{G}}$ denote the remainder of f upon division by $\mathcal{G}$. It is well known that a set of polynomials $\mathcal{G}$ in $k[\mathbf{x}]$ form a Gröbner basis with respect to $\succ$ if and only if $\overline{\text{S-pair}(f,g)}^{\mathcal{G}} = 0$ for every pair $f, g \in \mathcal{G}$ [3]. This fact is critical to the validity of the following algorithm.

Algorithm 2.11 ([3] Theorem 2)**. Buchberger's Algorithm**
Input: $F = \{f_1, \ldots, f_t\}$ a basis of the ideal $I \subset k[\mathbf{x}]$ and a term order $\succ$ on $k[\mathbf{x}]$.
Output: The reduced Gröbner basis $\mathcal{G}_\succ(I)$ of I with respect to $\succ$.

(1) Start by setting $\mathcal{G} := F$ and $\mathcal{G}' := \mathcal{G}$.
(2) For each pair $\{f, g\} \subset \mathcal{G}'$ for which $f \neq g$ compute the remainder of S-pair(f,g) upon division by $\mathcal{G}'$, and label it S. If $S \neq 0$, then set $\mathcal{G} := \mathcal{G} \cup \{S\}$.
(3) Continue this process until $S = 0$ for all such pairs in $\mathcal{G}'$. At this point $\mathcal{G}$ is a Gröbner basis for I.

Producing a minimal Gröbner basis with respect to $\succ$*.*
Let $\mathcal{G}_\succ(I)$ be a Gröbner basis of I with respect to $\succ$. Make all the elements of $\mathcal{G}_\succ(I)$ monic by dividing each element by its leading coefficient. For each $g \in \mathcal{G}_\succ(I)$ remove it from $\mathcal{G}_\succ(I)$ if its leading term is divisible by another element $f \in \mathcal{G}_\succ(I)$.

Producing the reduced Gröbner basis with respect to $\succ$*.*
Set $\mathcal{G}' := \mathcal{G}$ where $\mathcal{G}$ is a minimal Gröbner basis of I with respect to $\succ$. Set $\mathcal{G}_\succ(I) := \emptyset$. Then for each $g \in \mathcal{G}$ do $g' = \overline{g}^{\mathcal{G}\setminus g}$. The set $\mathcal{G}_\succ(I) = \mathcal{G}_\succ(I) \cup \{g'\}$ and $\mathcal{G}' = \mathcal{G}' \setminus \{g\} \cup \{g'\}$.

Example 2.12. Let $I = \langle f_1 = x^2 + xy + y^2, f_2 = x + z \rangle \subset \mathbb{Q}[x,y,z]$ and let $\succ$ be the lex ordering on $\mathbb{Q}[x,y,z]$ with respect to the fixed variable ordering $x \succ y \succ z$. Notice that $in_\succ(f_1) = x^2$ and $in_\succ(f_2) = x$. We want to compute a reduced Gröbner basis, $\mathcal{G}_\succ(I)$ for I with respect to $\succ$. So we first compute

$$\begin{aligned}\text{S-pair}(f_1, f_2) &= \frac{x^2}{x^2} f_1 - \frac{x^2}{x} f_2, \\ &= (x^2 + xy + y^2) - x(x+z), \\ &= xy - xz + y^2.\end{aligned}$$

Applying the division algorithm to S-pair(f_1, f_2) for $G = \{f_1, f_2\}$ gives S-pair$(f_1, f_2) = xy - xz + y^2 = (y-z)(x+z) + (y^2 - yz + z^2)$, and so $\overline{\text{S-pair}(f_1, f_2)}^{\mathcal{G}} = y^2 - yz + z^2 \neq 0$. Thus, we set $G = \{f_1, f_2, f_3 = y^2 - yz + z^2\}$. Now, using the division algorithm, it is easy to verify that

$$\overline{\text{S-pair}(f_1, f_2)}^{\mathcal{G}} = \overline{\text{S-pair}(f_1, f_3)}^{\mathcal{G}} = \overline{\text{S-pair}(f_2, f_3)}^{\mathcal{G}} = 0,$$

and so G is a Gröbner basis for I with respect to $\succ$. However, G is not minimal since $in_\succ(f_2)$ divides $in_\succ(f_1)$. So we set

$$\mathcal{G}_\succ(I) = \{f_2 = x + z, f_3 = y^2 - yz + z^2\}.$$

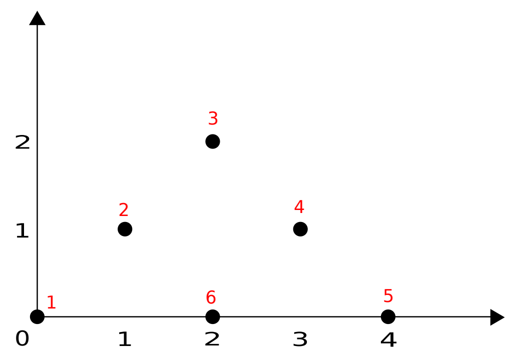
Notice that $\mathcal{G}_\succ(I)$ is in fact the reduced Gröbner basis for I with respect to $\succ$ since $\overline{f_1}^{\mathcal{G}_\succ(I)\setminus f_1} = f_1$, and $\overline{f_2}^{\mathcal{G}_\succ(I)\setminus f_2} = f_2$.

3. Convex Polytopes and Triangulations

Consider the set of points $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_6\} \subset \mathbb{R}^2$ where $\mathbf{a}_i$ is the point given by column i of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 2 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}.$$

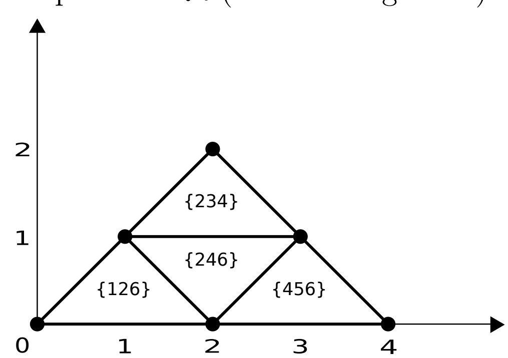
These points form the diagram



If we refer to the point $\mathbf{a}_i$ simply by the index i, then the collection of subsets of $\mathcal{A}$

$$\mathcal{T} = \{\{126\}, \{234\}, \{456\}, \{246\}\},$$

divides $\mathcal{A}$ into a set of triangles, none of which overlap, and together, completely cover all the points in and between the points of $\mathcal{A}$ (and nothing more).



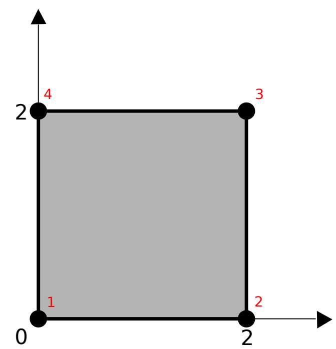
For this reason, we call the set $\mathcal{T}$ a *triangulation* of $\mathcal{P}$ (and of $\mathcal{A}$). The generation of triangulations of such finite sets of points like $\mathcal{A}$ will be the focus of this section, and critical to our construction of toric ideals and their corresponding affine varieties. The following material is drawn from Section 2.3 of [9] and Chapter 2 of [4].

A *convex combination* of a finite set of points $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^m$ is any $\mathbf{x} \in \mathbb{R}^m$ that can be expressed as $\sum_{i=1}^k \lambda_i \mathbf{x}_i$ with $\lambda_i \in \mathbb{R}$ being nonnegative for $i = 1, \ldots, k$, and $\sum_{i=1}^k \lambda_i = 1$. If we do not require nonnegativity, we call $\mathbf{x}$ an *affine combination*. The *convex hull* of a set $\mathbf{X} \subset \mathbb{R}^m$, denoted $\operatorname{conv}(\mathbf{X})$, is the set of all points in $\mathbb{R}^m$ that are convex combinations of a subset of the elements of $\mathbf{X}$. Similarly, the *affine span* of $\mathbf{X}$ is the set of all affine combinations of $\mathbf{X}$. In the following discussion, we think of the points in $\mathbb{R}^n$ as column vectors, but often write them as row vectors to save space.

Example 3.1. The point $\mathbf{x} = (1,1) \in \mathbb{R}^2$ can be written as a convex combination of the finite set of points $\mathcal{A} = \{\mathbf{a}_1 = (0,0), \mathbf{a}_2 = (2,0), \mathbf{a}_3 = (2,2), \mathbf{a}_4 = (0,2)\} \subset \mathbb{R}^2$. For example,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\mathbf{a}_3 + \frac{1}{2}\mathbf{a}_1 = \frac{1}{2}\mathbf{a}_2 + \frac{1}{2}\mathbf{a}_4.$$

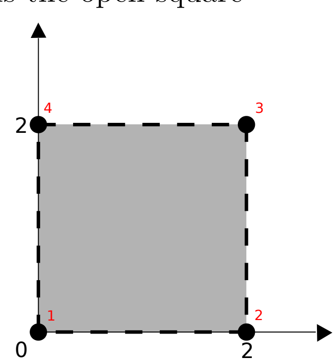
Moreover, conv$(\mathcal{A})$ is the square



Also notice that $(-1,0)$ and $(0,-1)$ are affine combinations of $\mathcal{A}$ since $(-1,0) = -\frac{1}{2}\mathbf{a}_2 + \frac{3}{2}\mathbf{a}_1$, and $(0,-1) = -\frac{1}{2}\mathbf{a}_4 + \frac{3}{2}\mathbf{a}_1$. From this, it is easy to see that the affine span of $\mathcal{A}$ is all of $\mathbb{R}^2$.

The convex hull of a finite set of points $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^m$ is called a *(convex) polytope*. A *face* of a polytope $\mathcal{P}$ is a subset of $\mathbb{R}^m$ of the form $F_c(\mathcal{P}) = \{\mathbf{x} \in \mathcal{P} : \mathbf{c} \cdot \mathbf{x} \geq \mathbf{c} \cdot \mathbf{y}, \text{ for all } \mathbf{y} \in \mathcal{P}\}$. The *dimension* of a polytope, or a face of a polytope, is the dimension of its affine span. If $\mathcal{P}$ is a d-dimensional polytope we call the $(d-1)$-dimensional faces of $\mathcal{P}$ the *facets* of $\mathcal{P}$. The zero dimensional and one dimensional faces of $\mathcal{P}$ are called the *vertices* and *edges* of $\mathcal{P}$, respectively. The *boundary* of $\mathcal{P}$ is the union of all facets of $\mathcal{P}$. The *relative interior* of $\mathcal{P}$, denoted rel$(\mathcal{P})$, is the set all points in $\mathcal{P}$ that do not lie on the boundary of $\mathcal{P}$.

Example 3.2. Consider the finite set of points $\mathcal{A} \subset \mathbb{R}^2$ from Example 3.1. The polytope $\mathcal{P} = \text{conv}(\mathcal{A})$ is a 2-dimensional polytope, since the affine span of $\mathcal{A}$ is all of $\mathbb{R}^2$. The facets of $\mathcal{P}$ are $F_{(0,-1)}(\mathcal{P})$, $F_{(1,0)}(\mathcal{P})$, $F_{(0,1)}(\mathcal{P})$, and $F_{(0,-1)}(\mathcal{P})$, which correspond to the four line segments defining conv$(\mathcal{A})$, depicted in Example 3.1, starting at the origin and traveling counterclockwise about the square. Since $\mathcal{P}$ is 2-dimensional then the facets of $\mathcal{P}$ are also the edges of $\mathcal{P}$, and the vertices are the set of points $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$. The boundary of $\mathcal{P}$ is the empty square given by the set of points $F_{(0,-1)}(\mathcal{P}) \cup F_{(1,0)}(\mathcal{P}) \cup F_{(0,1)}(\mathcal{P}) \cup F_{(0,-1)}(\mathcal{P})$, and so the relative interior of $\mathcal{P}$ is the open square



Given a d-dimensional polytope $\mathcal{P} \subset \mathbb{R}^m$, we may always find a projection $\mathbb{R}^m \longrightarrow \mathbb{R}^d$ that embeds $\mathcal{P}$ in $\mathbb{R}^d$ and preserves many of the properties that will be important to us, including triangulations and volume. We will see that it is often helpful to find such an embedding of $\mathcal{P}$ in $\mathbb{R}^d$ and then think of $\mathcal{P}$ as lying in a hyperplane of $\mathbb{R}^{d+1}$.

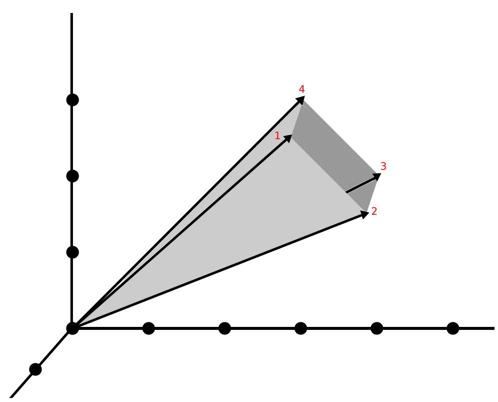
We say that a set of points is *affinely independent* if no point in the set is an affine combination of the rest. Otherwise, the set is called *affinely dependent*. Equivalently, a set of k points is *affinely independent* if the dimension of its convex hull is $k-1$. We call an affinely independent set a *basis* of its affine span. A *k-simplex* is the convex hull of an affinely independent set of $k+1$ points.

Let $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{R}^m$ be a finite set of vectors. Then the *positive hull* of $\mathcal{V}$ is the set

$$\operatorname{cone}(\mathcal{V}) = \left\{ \sum_{\mathbf{v} \in \mathcal{V}} \lambda_{\mathbf{v}} \mathbf{v} : \lambda_{\mathbf{v}} \geq 0, \text{ for all } \mathbf{v} \in \mathcal{V} \right\}.$$

Sets of the form $\operatorname{cone}(\mathcal{V})$ are commonly referred to as *convex polyhedral cones*, or more simply, *cones*. The *dimension* of a cone is the dimension of the linear subspace spanned by the elements of $\mathcal{V}$, and the *lineality space* of $\operatorname{cone}(\mathcal{V})$ is the largest linear subspace contained in the cone. For example, consider $\operatorname{cone}\{1, -1\} \subset \mathbb{R}$. Since $\operatorname{cone}\{1, -1\} = \mathbb{R}$, the lineality space of $\operatorname{cone}\{1, -1\}$ is the entire real line. A cone is called *pointed* if its lineality space is the zero subspace. Just as for polytopes, we define a face of a cone C to be a set of the form $F_c(C) = \{\mathbf{x} \in C : \mathbf{c} \cdot \mathbf{x} \geq \mathbf{c} \cdot \mathbf{y}, \text{ for all } \mathbf{y} \in C\}$. Similarly, if C is a d-dimensional cone, then the $(d-1)$-dimensional faces of C are called the *facets* of C, and the *relative interior* of C is the set of all points in C that do not lie on a facet of C.

Example 3.3. The set of vectors $\mathcal{V} = \{\mathbf{v}_1 = (1,3,3), \mathbf{v}_2 = (1,4,2), \mathbf{v}_3 = (0,4,2), \mathbf{v}_4 = (0,3,3)\} \subset \mathbb{R}^3$ form the convex polyhedral cone



Since $(1,0,0) = \mathbf{v}_1 - \mathbf{v}_4$, $(0,1,0) = \frac{1}{2}\mathbf{v}_2 - \frac{1}{2}\mathbf{v}_1 - \frac{5}{6}\mathbf{v}_4$, and $(0,0,1) = \frac{2}{3}\mathbf{v}_4 - \frac{1}{2}\mathbf{v}_3$, $\operatorname{cone}(\mathcal{V})$ spans a 3-dimensional linear subspace, and so $\operatorname{cone}(\mathcal{V})$ is 3-dimensional. It is easy to see that $\operatorname{cone}(\mathcal{V})$ is contained in the positive orthant of $\mathbb{R}^3$, and therefore the largest linear subspace contained in $\operatorname{cone}(\mathcal{V})$ must be the zero subspace. Thus, $\operatorname{cone}(\mathcal{V})$ is pointed.

We are now ready to define a point configuration in $\mathbb{R}^m$ and triangulations of point configurations. We do so in such a way that the following definitions and properties are easily generalized to a set of vectors in $\mathbb{R}^m$. Both of these perspectives are useful in the study of triangulations of polytopes.

Definition 3.4 ([4] Definition 2.1.10/2.1.11)**.** A **point configuration** in $\mathbb{R}^m$ is a finite set of (perhaps repeated) points with (non-repeated) labels from a selected index set. More precisely, a point configuration $\mathcal{A}$ in $\mathbb{R}^m$ with a finite set of labels $\mathcal{J}$ is a map $\mathcal{A} : \mathcal{J} \longrightarrow \mathbb{R}^m$. The **dimension** of a point configuration is the dimension of its convex hull. The **rank** of a point configuration is equal to the dimension plus one.

Notice that if $\mathcal{A} \subset \mathbb{R}^m$, then we can consider the convex polytope $\mathcal{P} = \mathrm{conv}(\mathcal{A})$, and thus all of the affine geometry of polytopes described previously may be applied directly to the configuration $\mathcal{A}$.

Let $\mathcal{A} \subset \mathbb{R}^m$ be a point configuration with label set $\mathcal{J}$, and $C \subset \mathcal{J}$. We let $\mathrm{conv}(C)$ denote the convex hull of all elements in $\mathcal{A}$ whose labels lie in C. We say that C is *spanning* if $\mathrm{conv}(C)$ has the same dimension as $\mathrm{conv}(\mathcal{J})$. We say that an element $j \in \mathcal{J}$ is *extremal* in $\mathbf{A}$ if the corresponding point $\mathbf{p}_j$ in $\mathcal{A}$ is not repeated and is a vertex of $\mathrm{conv}(\mathcal{J}) = \mathrm{conv}(\mathcal{A})$. We are now ready to define triangulations for point configurations.

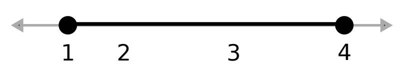
Definition 3.5 ([4] Definition 2.3.1)**.** Let $\mathcal{A}$ be a point configuration in $\mathbb{R}^m$, with label set $\mathcal{J}$. A collection $\mathcal{T}$ of subsets of $\mathcal{J}$ is a **polyhedral subdivision** of $\mathcal{A}$ if it satisfies the following conditions:

(1) (Closure Property) If $C \in \mathcal{T}$ and F is a face of C, then $F \in \mathcal{T}$.
(2) (Union Property) $\mathrm{conv}(\mathcal{J}) \subseteq \bigcup_{C \in \mathcal{T}} \mathrm{conv}(C)$.
(3) (Intersection Property) If $C \neq C'$ are two cells in $\mathcal{T}$, then $\mathrm{rel}(C) \cap \mathrm{rel}(C') = \emptyset$.

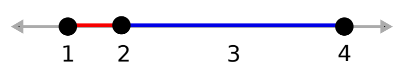
The elements of a polyhedral subdivision are called **cells**. A cell C is called **maximal** if the dimension of C equals the dimension of $\mathcal{A}$. A **triangulation** of $\mathcal{A}$ is a polyhedral subdivision in which all maximal cells are d-simplices, where d is the dimension of $\mathcal{A}$.

Example 3.6. The set $\mathcal{T}$ described at the beginning of the section qualifies as a triangulation of the configuration $\{(0,0),(1,1),(2,2),(3,1),(4,0),(2,0)\} \subset \mathbb{R}^2$. We can also define triangulations for the configuration $\mathcal{A} = \{1,2,4,6\} \subset \mathbb{R}$ given in Example 3.10. We will refer to the points of $\mathcal{A}$ by their corresponding indices in the matrix $\mathbf{A}$ given in Example 3.10.

$\mathcal{T}_1 = \{\{14\}\}$

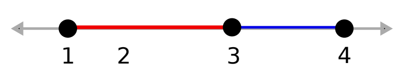


$\mathcal{T}_2 = \{\{12\},\{24\}\}$



$\mathcal{T}_3 = \{\{12\},\{23\},\{34\}\}$

$\mathcal{T}_4 = \{\{13\},\{34\}\}$

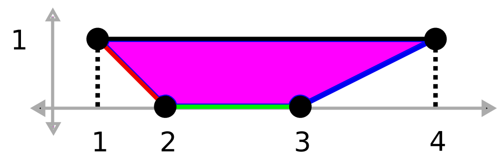


We will be particularly interested in working with *regular triangulations* of point configurations.

Definition 3.7. Let $\mathcal{A} \subset \mathbb{R}^d$ be a d-dimensional point configuration with n elements and let $\omega : \mathcal{J} \longrightarrow \mathbb{R}$ be a "height vector." We let $\omega(j) = \omega_j$, and define the **lifted point configuration** $\mathcal{A}^\omega = \{(\mathbf{p}_j, \omega_j) \in \mathbb{R}^{d+1} : j \in \mathcal{J}\}$. A **lower face** of $\mathcal{A}^\omega$ is any face of $\mathrm{conv}(\mathcal{A}^\omega)$ that lies in the direction of a linear functional with positive last coordinate. A

regular subdivision of $\mathcal{A}$ is a polyhedral subdivision consisting of the set of lower faces of $\mathcal{A}^\omega$ for some height vector ω. A **regular triangulation** of $\mathcal{A}$ is a regular subdivision of $\mathcal{A}$ whose maximal cells consist only of d-simplices.

Example 3.8. First consider the configuration $\mathcal{A} = \{1, 2, 4, 6\} \subset \mathbb{R}^2$ given in Example 3.6. It is a simple task to find height vectors corresponding to all four triangulations $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$. In doing so, we show that all triangulations of $\mathcal{A}$ are regular triangulations. For example, define the height vector $\omega : \mathcal{J} \longrightarrow \mathbb{R}$ such that $\omega(1) = \omega(4) = 1$ and $\omega(2) = \omega(3) = 0$. This gives the lifted configuration $\mathcal{A}^\omega = \{(1, 1), (2, 0), (4, 0), (6, 1)\} \subset \mathbb{R}^2$. Moreover, $\mathcal{P} = \mathrm{conv}(\mathcal{A}^\omega)$ is the polytope



The lower faces of $\mathcal{P}$ are the red, green, and blue edges as well as all the points in $\mathcal{A}^\omega$. Thus, ω gives the regular triangulation $\mathcal{T}_3 = \{\{12\}, \{23\}, \{34\}\}$ of $\mathcal{A}$. Similar liftings can be defined to produce $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_4$.

Also notice that the triangulation $\mathcal{T} = \{\{126\}, \{234\}, \{456\}, \{246\}\}$ of the configuration $\mathcal{A} = \{(0, 0), (1, 1), (2, 2), (3, 1), (4, 0), (2, 0)\} \subset \mathbb{R}^2$ presented at the beginning of this section is a regular triangulation induced by the height vector $\omega : \mathcal{J} \longrightarrow \mathbb{R}$ such that $\omega(1) = \omega(3) = \omega(5) = 1$ and $\omega(2) = \omega(4) = \omega(6) = 0$.

We may now consider the straight-forward generalization of point configurations to vector configurations.

Definition 3.9 (Del, Definition 2.5.1)**.** A **vector configuration** in $\mathbb{R}^m$ is a finite set $\mathcal{A} = (\mathbf{p}_j : j \in \mathcal{J})$ of labeled vectors $\mathbf{p}_j \in \mathbb{R}^m$. Its **rank** is its rank as a matrix of vectors. A **subconfiguration** is any labeled subset of the configuration.

It is often easiest to refer to the elements of a vector configuration $\mathcal{A}$ by their corresponding labels in $\mathcal{J}$. We will usually depict $\mathcal{A}$ as the set of columns of a matrix $\mathbf{A}$. Moreover, it is convenient to represent the point configuration $\mathcal{A} \subset \mathbb{R}^d$ by adding a row of ones to the matrix $\mathbf{A}$. This allows us to consider the affine geometry of $\mathcal{A}$ as a special case of linear algebra without sacrificing many of the geometric properties in question, such as the triangulations of $\mathcal{A}$. We will see that this process, referred to as *homogenization*, of adding an extra coordinate provides a natural way of viewing a point configuration in $\mathbb{R}^m$ as a vector configuration in $\mathbb{R}^{m+1}$.

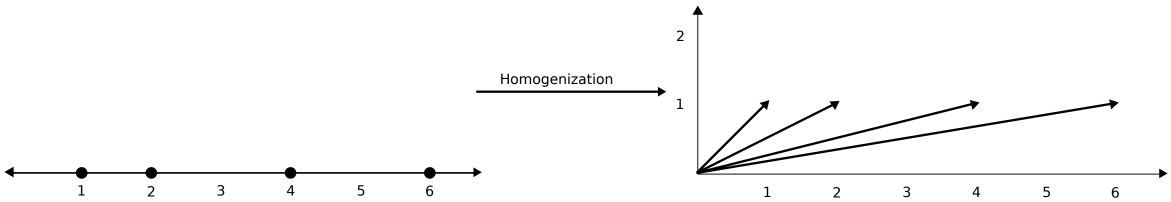
Example 3.10. We represent the point configuration $\mathcal{A} = \{1, 2, 4, 6\} \subset \mathbb{R}$ with the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 & 6 \end{pmatrix}.$$

The convex hull of $\mathcal{A}$ is the line segment defined by the real closed interval $[1, 6] \subset \mathbb{R}$. Embedding $\mathcal{A}$ in the hyperplane $x_2 = 1$ in $\mathbb{R}^2$ via the homogeneous matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

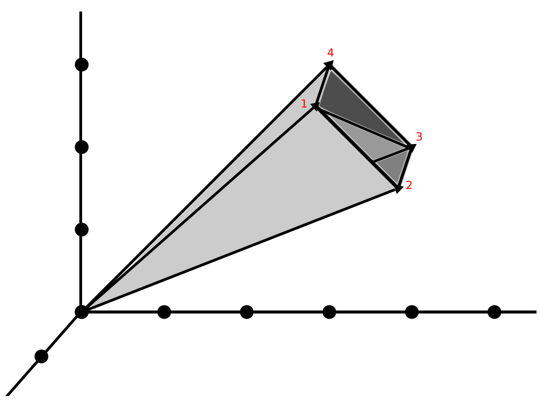
allows us to think about the affine set of points $\mathcal{A}$ as a set of vectors in $\mathbb{R}^2$.



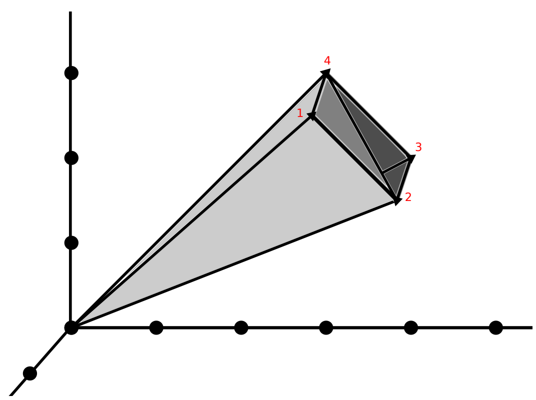
We refer to the homogenization of $\mathcal{A}$ as representing $\mathcal{A}$ in *homogeneous coordinates*. Notice that this process is simply identifying $\mathcal{A} \subset \mathbb{R}^d$ as a configuration lying in the hyperplane $x_{d+1} = 1$ of $\mathbb{R}^{d+1}$. Thus, we more generally refer to a matrix $\mathbf{A}$ (and its corresponding point configuration) as *homogeneous* if all of its columns lie in the same hyperplane.

Notice that this homogenization technique allows us to view any point configuration as a vector configuration. Thus, it is easy to see that all of the properties and constructions we have described for point configurations extend naturally to vector configurations simply by using *cones instead of simplices*.

Example 3.11. The vector configuration $\mathcal{V} = \{\mathbf{v}_1 = (1,3,3), \mathbf{v}_2 = (1,4,2), \mathbf{v}_3 = (0,4,2), \mathbf{v}_4 = (0,3,3)\} \subset \mathbb{R}^3$ has exactly two triangulations.



$\mathcal{T}_1 = \{\{123\}, \{134\}\}$,



$\mathcal{T}_2 = \{\{124\}, \{234\}\}$.

4. The Algebra of Normal Toric Varieties

In this section, we take a closer look at the *toric ideal* as defined in Definition 1.1 generated by the vector configuration $\mathcal{A} \subset \mathbb{Z}^d$ and its corresponding zero set, which we will call the *affine toric variety* associated with $\mathcal{A}$. Here, we let k be an algebraically closed field, and let $k[\mathbf{x}] := k[x_1, \dots, x_n]$. This definition of a toric variety differs from the standard algebraic geometry definition, which defines a toric variety as an irreducible, normal variety $\mathcal{V}$ in affine d-space that contains a d-dimensional algebraic torus $(k^*)^d$ as a Zariski open subset such that the action of $(k^*)^d$ on itself extends to an action of $(k^*)^d$ on $\mathcal{V}$. We will first show that the only property our definition is lacking with regard to the classical definition is normality of $\mathcal{V}$. We then demonstrate exactly when this variety is normal. The following discussion is based on Chapter 3 of [9] and Chapters 4, 7, and 13 of [11]. We begin with a Proposition that guarantees a nice correspondence between the toric ideals, and their associated affine varieties.

Proposition 4.1. *Let $\mathcal{A} \subset \mathbb{Z}^d$ be a finite vector configuration. Then the toric ideal $I_{\mathcal{A}}$ is a prime ideal in $k[\mathbf{x}]$.*

Proof. Notice that the first isomorphism theorem of rings implies that $k[\mathbf{x}]/I_{\mathcal{A}} \cong \hat{\pi}(k[\mathbf{x}]) = k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$. Since $k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$ is an integral domain, then $I_{\mathcal{A}}$ is a prime ideal. $\square$

Notice that Proposition 4.1 together with Hilbert's Nullstellensatz (given in [5]), guarantees that the affine toric variety $\mathcal{V}_{\mathcal{A}}$ is irreducible for all finite vector configurations $\mathcal{A} \subset \mathbb{Z}^d$. We now define a first basis for an arbitrary toric ideal.

Lemma 4.2 ([11] Lemma 4.1)**.** *The toric ideal $I_{\mathcal{A}}$ is spanned as a k-vector space by the set of binomials*

$$\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} | \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ with } \pi(\mathbf{u}) = \pi(\mathbf{v})\}.$$

Proof. A binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ lies in $I_{\mathcal{A}}$ if and only if $\pi(\mathbf{u}) = \pi(\mathbf{v})$. Thus, it suffices to show that if $f \in I_{\mathcal{A}}$, then f is a k-linear combination of these binomials. For the sake of contradiction, assume that $f \in I_{\mathcal{A}}$ cannot be written as a k-linear combination of these binomials. We choose $f \in I_{\mathcal{A}}$ such that $in_{\succ}(f) = \mathbf{x}^{\mathbf{u}}$ is minimal with respect to the term order $\succ$ on $k[\mathbf{x}]$. Since $f \in I_{\mathcal{A}} = \ker(\hat{\pi})$, then when we expand $f(\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n})$ we get zero. This implies that $\hat{\pi}(\mathbf{x}^{\mathbf{u}})$ cancels in this expansion, and so there exists some monomial $\mathbf{x}^{\mathbf{v}} \prec \mathbf{x}^{\mathbf{u}}$ appearing in f such that $\pi(\mathbf{u}) = \pi(\mathbf{v})$. It follows that the polynomial $f' = f - \mathbf{x}^{\mathbf{u}} + \mathbf{x}^{\mathbf{v}}$ also cannot be written as a k-linear combination of the binomials in $I_{\mathcal{A}}$. However, $in_{\succ}(f') \prec in_{\succ}(f)$, which contradicts the minimality of $in_{\succ}(f)$. $\square$

Let $\mathbf{u} \in \mathbb{Z}^n$, and define the vectors $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{Z}^{\mathbf{n}}$ such that $u_i^+ = u_i$ if $u_i > 0$, otherwise $u_i^+ = 0$; and $u_i^- = |u_i|$ if $u_i < 0$, otherwise $u_i^- = 0$. Then every vector $\mathbf{u} \in \mathbb{Z}^n$ can be written uniquely as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$, where both $\mathbf{u}^+$ and $\mathbf{u}^-$ are nonnegative and have disjoint support. If we let $\ker(\pi)$ denote the sublattice of $\mathbb{Z}^n$ containing all vectors $\mathbf{u}$ such that $\pi(\mathbf{u}^+) = \pi(\mathbf{u}^-)$, then we have the following corollary.

Corollary 4.3 ([11] Corollaries 4.3 and 4.4)**.** $I_{\mathcal{A}} = \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} | \mathbf{u} \in \ker(\pi) \rangle$. *Moreover, For every term order $\succ$ on $k[\mathbf{x}]$, there is a finite set of vectors $\mathcal{G}_{\succ} \subset \ker(\pi)$ such that the reduced Gröbner basis of $I_{\mathcal{A}}$ is equal to $\{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} | \mathbf{u} \in \mathcal{G}_{\succ}\}$.*

Proof. First let $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} | \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ with } \pi(\mathbf{u}) = \pi(\mathbf{v})\} \subset I_{\mathcal{A}}$. We will show that there exists a binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in \{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} | \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ with } \pi(\mathbf{u}) = \pi(\mathbf{v})\}$ where $\mathbf{u} \in \ker(\pi)$ and $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \mathbf{x}^{\mathbf{a}}(\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-})$ for some monomial $\mathbf{x}^{\mathbf{a}}$. If $\mathbf{u}$ and $\mathbf{v}$ have disjoint support, then we are done, since $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} | \mathbf{u} \in \ker(\pi)\}$. If $\mathbf{u}$ and $\mathbf{v}$ do not have disjoint support then there exists a monomial $\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{x}^{\mathbf{u}} = \mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{u}'}$ and $\mathbf{x}^{\mathbf{v}} = \mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{v}'}$, where $\mathbf{u}'$ and $\mathbf{v}'$ do have disjoint support. Thus, $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \mathbf{x}^{\mathbf{a}}(\mathbf{x}^{\mathbf{u}'} - \mathbf{x}^{\mathbf{v}'})$, $\mathbf{u} = \mathbf{a} + \mathbf{u}'$, and $\mathbf{v} = \mathbf{a} + \mathbf{v}'$. Therefore,

$$\begin{aligned}
\pi(\mathbf{u}) &= \pi(\mathbf{v}), \\
\mathbf{A}\mathbf{u} &= \mathbf{A}\mathbf{v}, \\
\mathbf{A}(\mathbf{a} + \mathbf{u}') &= \mathbf{A}(\mathbf{a} + \mathbf{v}'), \\
\mathbf{A}\mathbf{a} + \mathbf{A}\mathbf{u}' &= \mathbf{A}\mathbf{a} + \mathbf{A}\mathbf{v}', \\
\pi(\mathbf{u}') &= \pi(\mathbf{v}').
\end{aligned}$$

Thus, since, $\mathbf{u}' - \mathbf{v}' \in \ker(\pi)$, and $\mathbf{u}', \mathbf{v}' \in \mathbb{N}^n$ and have disjoint support, it follows that $\mathbf{u}' - \mathbf{v}' \in \{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} | \mathbf{u} \in \ker(\pi)\}$. Therefore, $I_{\mathcal{A}} = \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} | \mathbf{u} \in \ker(\pi) \rangle$.

To see that the second part of the corollary holds, recall that the Hilbert Basis Theorem allows us to find a finite subset of binomials $\mathcal{G}$ in $\{\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} | \mathbf{u} \in \ker(\pi)\}$ that generate

$I_{\mathcal{A}}$. Since the operations of reduction and forming S-pairs preserve the binomial structure when applied to elements of $\mathcal{G}$, applying the Buchberger Algorithm to $\mathcal{G}$, with respect to some term order $\succ$, will result in the reduced Gröbner basis $\mathcal{G}_{\succ}$ consisting only of binomials. Moreover, any binomial produced in this process must lie in $\{\mathbf{x}^{\mathbf{u}^+}-\mathbf{x}^{\mathbf{u}^-}|\mathbf{u} \in \ker(\pi)\}$ or the resulting Gröbner basis would not be reduced. To see this, assume that $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{G}_{\succ}$ and $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \notin \{\mathbf{x}^{\mathbf{u}^+}-\mathbf{x}^{\mathbf{u}^-}|\mathbf{u} \in \ker(\pi)\}$. Then it must be that $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \mathbf{x}^{\mathbf{a}}(\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-})$ for some $\mathbf{u} \in \ker(\pi)$. Thus, either $\mathbf{x}^{\mathbf{u}}$ or $\mathbf{x}^{\mathbf{v}}$ is divisible by $in_{\succ}(\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-})$, contradicting the reducedness of $\mathcal{G}_{\succ}$. □

The following algorithm is a special case of Theorem 3.3.2 of [3]. Given a toric ideal $I_{\mathcal{A}} \subset k[\mathbf{x}]$ and a term order $\succ$ on $k[\mathbf{x}]$, we can compute the reduced Gröbner basis for $I_{\mathcal{A}}$ with respect to $\succ$, which we will denote $\mathcal{G}_{\succ}$.

Algorithm 4.4 ([11] Algorithm 4.5)**.** Computing a reduced Gröbner basis of a toric ideal.

(1) Introduce $n + d + 1$ indeterminates $t_0, t_1, \ldots, t_d, x_1, x_2, \ldots, x_n$. Let $\succ$ be any term order with $t_i \succ x_j$, for all $1 \leq i \leq d$ and $1 \leq j \leq n$.
(2) Compute the reduced Gröbner basis $\mathcal{G}$ for the ideal
$$\langle t_0 t_1 \cdots t_d - 1, x_1 \mathbf{t}^{\mathbf{a}_1^-} - \mathbf{t}^{\mathbf{a}_1^+}, \ldots, x_n \mathbf{t}^{\mathbf{a}_n^-} - \mathbf{t}^{\mathbf{a}_n^+}\rangle.$$
(3) Output: The set $\mathcal{G} \cap k[\mathbf{x}]$ is the reduced Gröbner basis for $I_{\mathcal{A}}$ with respect to $\succ$.

The correctness of Algorithm 4.4 follows from [3] Theorem 3.3.2. There do exist many faster algorithms, two of which are given in §3.3 of [9].

Example 4.5. Let $\mathcal{A} = \{(1,1),(2,-2),(-3,1),(2,0)\} \subset \mathbb{Z}^2$. Use the indeterminants $t_0, t_1, t_2, x_1, x_2, x_3, x_4$, and choose a term order such that $t_i \succ x_j$ for all $i = 0, 1, 2$ and $j = 1, 2, 3, 4$. We first compute a reduced Gröbner basis $\mathcal{G}_{\succ}(I)$ of
$$I = \langle t_0 t_1 t_2 - 1, x_1 - t_1 t_2, x_2 t_2^2 - t_1^2, x_3 t_1^3 - t_2, x_4 - t_1^2\rangle \subset k[t_0, t_1, t_2, x_1, x_2, x_3, x_4]$$
with respect to the lex ordering on $k[t_0, t_1, t_2, x_1, x_2, x_3, x_4]$. Using the Buchberger Algorithm, we find that
$$\mathcal{G}_{\succ}(I) = \{x_2 x_3^2 x_4^2 - 1, -x_3 x_4^2 + x_1, -x_3^2 x_4^3 + t_2^2, -t_2 x_2 x_3 x_4 + t_1, -x_2 x_3 + t_0\}.$$
Thus, $\mathcal{G} = \mathcal{G}_{\succ}(I) \cap k[x_1, x_2, x_3] = \{x_2 x_3^2 x_4^2 - 1, -x_3 x_4^2 + x_1\}$ is a reduced Gröbner basis for $I_{\mathcal{A}}$ with respect to the lex ordering on $k[x_1, x_2, x_3]$.

Definition 4.6. (a) The *universal Gröbner basis* of $\mathcal{A}$, denoted $\mathcal{U}_{\mathcal{A}}$, is the union of all reduced Gröbner bases $\mathcal{G}_{\succ}$ of the toric ideal $I_{\mathcal{A}}$ as $\succ$ runs through all possible term orders on $k[\mathbf{x}]$.
(b) A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ in $I_{\mathcal{A}}$ is called *primitive* if there exists no other binomial $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-}$ in $I_{\mathcal{A}}$ such that $\mathbf{x}^{\mathbf{v}^+}$ divides $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{v}^-}$ divides $\mathbf{x}^{\mathbf{u}^-}$.
(c) The *Graver Basis* of $\mathcal{A}$, denoted $Gr_{\mathcal{A}}$, is the set of all primitive binomials in $I_{\mathcal{A}}$.

Proposition 4.7 ([11] Lemma 4.6)**.** *Let $\mathcal{A} \subset \mathbb{Z}^d$ be a finite vector configuration. Then $\mathcal{U}_{\mathcal{A}} \subseteq Gr_{\mathcal{A}}$.*

Proof. Fix a term order $\succ$ on $k[\mathbf{x}]$ and let $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ be any binomial in the reduced Gröbner basis $\mathcal{G}_{\succ}$. We assume, without loss of generality, that $\mathbf{u}^+ \succ \mathbf{u}^-$. Since $\mathcal{G}_{\succ}$ is also a minimal Gröbner basis, then $\mathbf{x}^{\mathbf{u}^+}$ is a minimal generator for $in_{\succ}(I_{\mathcal{A}})$, and so $\mathbf{x}^{\mathbf{u}^-}$ must be a standard monomial. Assume for the sake of contradiction that $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ is not primitive. Then there exists $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-}$ with $\mathbf{v} \neq \mathbf{u}$ such that $\mathbf{x}^{\mathbf{v}^+}$ divides $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{v}^-}$ divides $\mathbf{x}^{\mathbf{u}^-}$. If $\mathbf{v}^+ \succ \mathbf{v}^-$ then $\mathbf{x}^{\mathbf{u}^+}$ is not a minimal generator, a contradiction. So it must be that $\mathbf{v}^- \succ \mathbf{v}^+$, which implies that $\mathbf{x}^{\mathbf{u}^-}$ is not standard, another contradiction. $\square$

For a given $\mathcal{A} \subset \mathbb{Z}^d$ the *Lawrence Lifting* of $\mathcal{A}$ is the set of column vectors of the enlarged matrix

$$\Lambda(\mathcal{A}) = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{I_n} & \mathbf{I_n} \end{pmatrix},$$

where $\mathbf{I_n}$ is the $n \times n$ identity matrix and $\mathbf{0}$ is the $d \times n$ zero matrix. Notice that $\mathcal{A}$ and $\Lambda(\mathcal{A})$ have isomorphic kernels since $\ker(\Lambda(\mathcal{A})) = \{(\mathbf{u}, -\mathbf{u}) | \mathbf{u} \in \ker(\pi)\}$. This isomorphism indicates that

$$I_{\Lambda(\mathcal{A})} = \langle \mathbf{x}^{\mathbf{u}^+}\mathbf{y}^{\mathbf{u}^-} - \mathbf{x}^{\mathbf{u}^-}\mathbf{y}^{\mathbf{u}^+} | \mathbf{u} \in \ker(\pi) \rangle \subset k[\mathbf{x}, \mathbf{y}],$$

where $k[\mathbf{x}, \mathbf{y}] := k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, a polynomial ring in $2n$ variables.

Theorem 4.8 ([11] Theorem 7.1)**.** *Let $\mathcal{A} \subset \mathbb{Z}^d$. Then for $\Lambda(\mathcal{A})$ the following sets of binomials coincide:*

(i) the Graver Basis of $\Lambda(\mathcal{A})$,
(ii) the universal Gröbner basis of $\Lambda(\mathcal{A})$,
(iii) any reduced Gröbner basis of $I_{\Lambda(\mathcal{A})}$,
(iv) any minimal generating set of $I_{\Lambda(\mathcal{A})}$ (up to scalar multiples).

As a corollary to Theorem 4.8, we have the following algorithm for computing the Graver basis of a given integer vector configuration.

Algorithm 4.9 ([11] Algorithm 7.2)**.** Computing the Graver basis of an integer vector configuration $\mathcal{A}$.

(1) Fix a term order $\succ$ on $k[\mathbf{x}, \mathbf{y}]$. Compute the reduced Gröbner basis $\mathcal{G}_\succ$ of $I_{\Lambda(\mathcal{A})}$.
(2) Substitute $y_1, \ldots, y_n \mapsto 1$ in $\mathcal{G}_\succ$. The resulting subset of $k[\mathbf{x}]$ is the Graver basis $Gr_{\mathcal{A}}$.

Example 4.10. Recall the configuration $\mathcal{A} = \{(1,1), (2,-2), (-3,1), (2,-1)\} \subset \mathbb{Z}^2$ from Example 4.5. The Lawrence Lifting of $\mathcal{A}$ is the matrix

$$\Lambda(\mathcal{A}) = \begin{pmatrix} \mathbf{A} & \mathbf{0}_{2\times 4} \\ \mathbf{I_4} & \mathbf{I_4} \end{pmatrix} = \begin{pmatrix} 1 & 2 & -3 & 2 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{a} = (a, b, c, d, e, f, g)$, $\mathbf{x} = (x, y, z, w)$ and $\mathbf{y} = (u, v, q, r)$. Then, following the steps of Algorithm 4.4, we pick a term order $\succ$ such that $a \succ b \succ c \succ d \succ e \succ f \succ g \succ x \succ y \succ z \succ w \succ u \succ v \succ q \succ r$, and compute a reduced Gröbner basis $\mathcal{G}_\succ(J)$ for the ideal

$$J = \langle abcdefg - 1, x - bcd, yc^2 - b^2e, zb^3 - cf, wc - b^2g, u - d, v - e, q - f, r - g \rangle \subset k[\mathbf{a}, \mathbf{x}, \mathbf{y}]$$

with respect to the lex ordering on $k[\mathbf{a}, \mathbf{x}, \mathbf{y}]$. Using the Buchberger Algorithm we find that $\mathcal{G}_\succ(J)$ contains 61 elements such that $\mathcal{G} = \mathcal{G}_\succ(J) \cap k[\mathbf{x}, \mathbf{y}] = \{-z^2w^4v + yq^2r^4, xz^3w^4 -$

$uq^3r^4, xyz - uvq, -zw^4uv^2 + xy^2qr^4, -w^4u^2v^3 + x^2y^3r^4\}$ is a reduced Gröbner basis for $\Lambda(\mathcal{A})$ with respect to the lex ordering on $k[\mathbf{x}, \mathbf{y}]$. We then make the substitution $u, v, q, r \mapsto 1$ in $\mathcal{G}$. This gives

$$Gr_{\mathcal{A}} = \{-z^2w^4 + y, xz^3w^4 - 1, xyz - 1, -zw^4 + xy^2, -w^4 + x^2y^3\}.$$

Throughout this discussion we have been regarding a toric variety as the set of zeros of a toric ideal $I_{\mathcal{A}}$ for some configuration $\mathcal{A}$. We now demonstrate that this definition does give a (not necessarily normal) irreducible variety $\mathcal{V}$ in affine d-space that contains a d-dimensional algebraic torus $(k^*)^d$ as a Zariski open subset, such that the action of $(k^*)^d$ on itself extends to an action of $(k^*)^d$ on $\mathcal{V}$. This implies that our definition differs from the standard algebraic geometry definition only on the requirement of normality. We then show exactly when normality arises in toric varieties that satisfy our original definition.

Let σ be a pointed rational polyhedral cone in the vector space $\mathbb{Q}^d$. Then the dual cone to σ,

$$\sigma^\vee = \{\mathbf{u} \in \mathbb{Q}^d | \mathbf{u} \cdot \mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \sigma\},$$

is a d-dimensional cone in $\mathbb{Q}^d$. We then have that the associated semigroup

$$S_\sigma := \sigma^\vee \cap \mathbb{Z}^d = \{\mathbf{u} \in \mathbb{Z}^d | \mathbf{u} \cdot \mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \sigma\}$$

is finitely generated. It follows that the semigroup algebra $k[S_\sigma]$ is also finitely generated.

Lemma 4.11 ([11] Lemma 13.1)**.** *The cone σ is d-dimensional if and only if its dual cone $\sigma^\vee$ is pointed. In this case, the semigroup S_σ has a unique minimal generating set $\mathcal{A} \subset \mathbb{Z}^d$.*

Definition 4.12. The unique minimal generating set $\mathcal{A} \subset \mathbb{Z}^d$ of the semigroup S_σ associated with a d-dimensional, pointed rational polyhedral cone σ is called the *Hilbert basis* of S_σ.

The following algorithm uses Gröbner basis techniques to compute Hilbert bases.

Algorithm 4.13 ([11] Algorithm 13.2)**.** Computing the Hilbert basis for an affine toric variety.

Input: A spanning set for a d-dimensional convex polyhedral cone $\sigma \subset \mathbb{Z}^d$.

Output: The Hilbert Basis $\mathcal{A}$ of the semigroup $S_\sigma \subset \mathbb{Z}^d$.

(1) Replace the given generators of the cone σ by a new generating set $\{\mathbf{v}_1, \ldots, \mathbf{v}_d, \mathbf{v}_{d+1}, \ldots, \mathbf{v}_m\}$ consisting only of lattice points and such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ is a lattice basis of $\mathbb{Z}^d$. Let $\mathbf{V}$ denote the $m \times d$ matrix whose rows are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d, \mathbf{v}_{d+1}, \ldots, \mathbf{v}_m$.

(2) The image of $\mathbf{V}$ in $\mathbb{Z}^m$ is a *saturated* sublattice, i.e. $\mathbb{Z}^m / \operatorname{im}_{\mathbb{Z}}(\mathbf{V})$ is free abelian. Compute an $(m-d) \times m$ integer matrix $\mathbf{B}$ such that $\ker(B) = \operatorname{im}_{\mathbb{Z}}(\mathbf{V})$.

(3) Compute the Graver Basis $Gr_{\mathbf{B}}$ of the vector configuration consisting of the columns of $\mathbf{B}$.

(4) For each *nonnegative* vector $\mathbf{s} = (s_1, \ldots, s_m)$ in $Gr_{\mathbf{B}}$ determine and output the unique vector $\mathbf{u} \in \mathbb{Z}^d$ such that $\mathbf{u} \cdot \mathbf{v}_i = s_i$ for $i = 1, \ldots, d$.

Example 4.14. Let σ be the convex polyhedral cone spanned by $(1,1)$ and $(1,3)$ in rational affine 2-space. We wish to compute the Hilbert basis $\mathcal{A}$ for $S_\sigma = \mathbb{N}\{(1,1),(1,3)\}$. Since $(1,1)$ and $(1,3)$ do not form a lattice basis for $\mathbb{Z}^2$, we include the vector $(1,2) = \frac{1}{2}(1,1) + \frac{1}{2}(1,3)$.

Now, $\{(1,1),(1,2)\}$ is a lattice basis for $\mathbb{Z}^2$ contained in our spanning set for σ. The desired matrices from steps (1) and (2) of Algorithm 4.13 are

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}.$$

We now compute the Graver basis $Gr_{\mathbf{B}}$ of the matrix $\mathbf{B}$ using Algorithm 4.9, and find that

$$Gr_{\mathbf{B}} = \{yz^2 - 1, x - z, xyz - 1, x^2y - 1\}.$$

Thus, $\text{supp}(Gr_{\mathbf{B}}) = \{(0,1,2),(1,0,-1),(1,1,1),(2,1,0)\}$. We then compute the unique preimages under $\mathbf{V}$ of the three nonnegative vectors in $\text{supp}(Gr_{\mathbf{B}})$, and the resulting three vectors are the desired Hilbert basis of the semigroup S_σ:

$$\mathcal{A} = \{(-1,1),(1,0),(3,-1)\} \subset \mathbb{Z}^2.$$

Consider the variety $X_\sigma = \text{Spec}(k[S_\sigma])$. This implies that X_σ is the spectrum of the semigroup algebra $k[S_\sigma] = k[\mathcal{A}] = k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}] = \hat{\pi}(k[\mathbf{x}]) \cong k[\mathbf{x}]/I_{\mathcal{A}}$. Therefore, X_σ is embedded in affine d-space as as the zero set of the toric ideal $I_{\mathcal{A}}$. To see that X_σ contains an algebraic torus $(k^*)^d$, consider the following lemma.

Lemma 4.15 ([11] Lemma 13.4)**.** *Suppose* $\dim(\mathcal{A}) = d$. *Then the set* $X_\sigma \cap (k^*)^n$ *is an algebraic group under coordinate-wise multiplication, and this group is isomorphic to the algebraic torus* $(k^*)^d$.

Proof. For any n-element subset of $\mathcal{A} \subset \mathbb{Z}^d$, the map

$$(k^*)^d \longmapsto X_\sigma \cap (k^*)^n, \quad \mathbf{t} = (t_1, \ldots, t_d) \longmapsto (\mathbf{t}^{\mathbf{a}_1}, \mathbf{t}^{\mathbf{a}_2}, \ldots, \mathbf{t}^{\mathbf{a}_n})$$

is an injective group homomorphism . Since, $\dim(\mathcal{A}) = d$ implies that this map is onto, it follows that this is an isomorphism of groups. ☐

It now remains to consider the issue of normality of X_σ.

Definition 4.16. (a) Let S be an affine semigroup. The *group of differences* of S, denoted $\text{gp}(S)$, is the smallest group (up to isomorphism) which contains S.

(b) An affine semigroup S is called *normal* if every element $x \in \text{gp}(S)$ such that $cx \in S$ (for some $c \in \mathbb{N}$) belongs to S.

(c) A variety X is *normal* if its local rings are integrally closed in its field of fractions.

Definition (b) indicates that for a given semigroup S, the semigroup algebra $k[S]$ is normal if it is integrally closed in its field of fractions. It is well-known that S is a normal semigroup if and only if $k[S]$ is a normal variety (see [1]).

Proposition 4.17. *Let* $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ *be a finite vector configuration. Then the semigroup* $\mathbb{N}\mathcal{A}$ *is normal if and only if* $\mathbb{N}\mathcal{A} = \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A}$.

Proof. First let $\mathbb{N}\mathcal{A}$ be a normal semigroup. We will show that $\mathbb{N}\mathcal{A} = \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A}$ by showing containment in both directions. If $x \in \mathbb{N}\mathcal{A}$, then $x = \lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$. Since $\lambda_1, \ldots, \lambda_n \geq 0$, then $x \in \text{cone}(\mathcal{A})$. Thus, since $x \in \mathbb{N}\mathcal{A} \subset \mathbb{Z}\mathcal{A}$, we have that $x \in \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A}$, and so $\mathbb{N}\mathcal{A} \subseteq \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A}$.

If $x \in \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A}$, then $x \in \text{cone}(\mathcal{A})$, and so $x = \lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}_{\geq 0}$. If we let $\lambda_i = \frac{p_i}{q_i}$ with p_i and q_i in lowest terms for $i = 1, \ldots, n$, then $cx \in \mathbb{N}\mathcal{A}$, where

$c = q_1 q_2 \cdots q_n \in \mathbb{N}$. Since $x \in \mathbb{Z}\mathcal{A}$, and $\mathbb{N}\mathcal{A}$ is normal, it follows that $x \in \mathbb{N}\mathcal{A}$ also. Thus, $\text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A} \subseteq \mathbb{N}\mathcal{A}$, and we conclude that $\mathbb{N}\mathcal{A} = \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A}$.

Conversely, assume that $\mathbb{N}\mathcal{A} = \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A}$. Then if $x \in \mathbb{Z}\mathcal{A}$ such that $cx \in \mathbb{N}\mathcal{A}$ for some $c \in \mathbb{N}$, then $cx \in \text{cone}(\mathcal{A})$, and so $x = \frac{1}{c}cx \in \text{cone}(\mathcal{A})$. Thus, $x \in \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A} = \mathbb{N}\mathcal{A}$, and so $\mathbb{N}\mathcal{A}$ is normal. □

The following proposition of [11] tells us exactly when the toric variety X_σ is normal.

Proposition 4.18. *For a finite vector configuration $\mathcal{A} \subset \mathbb{Z}^d$ the following are equivalent:*

1. *The affine toric variety $X_\mathcal{A}$ is normal.*
2. *The affine toric variety $X_\mathcal{A}$ is isomorphic to X_σ for some rational polyhedral cone σ in $\mathbb{Q}^d$.*
3. *The integral domain $k[\mathcal{A}] \cong k[\mathbf{x}]/I_\mathcal{A}$ is integrally closed in its field of fractions.*
4. *The semigroup $\mathbb{N}\mathcal{A}$ is normal.*

5. Normal and Δ-Normal Vector Configurations

Definition 5.1. The finite vector configuration $\mathcal{A}$ is **normal** if $\mathbb{N}\mathcal{A} = \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A}$. Moreover, $\text{conv}(\mathcal{A})$ is **normal** if $\mathcal{A}$ is normal.

Thus, Proposition 4.18 indicates that the configuration $\mathcal{A}$ is normal if and only if its associated algebraic structures are normal. While this definition assures that $\mathcal{A}$ is normal exactly when its associated affine toric variety is normal, it does not provide any immediate connections between these algebraic and geometric properties of $\mathcal{A}$ and the combinatorial properties of $\mathcal{A}$ as a vector or lattice point configuration in $\mathbb{Z}^d$. Therefore, it is natural to consider what combinatorial properties of $\mathcal{A}$ characterize, approximate, or at the very least, guarantee the normality of $\mathcal{A}$. In particular, we will define a concept called Δ-*normality* for a configuration $\mathcal{A}$ that is combinatorial, and investigate its relationship with normality of $\mathcal{A}$. We will show that Δ-normality of $\mathcal{A}$ implies $\mathcal{A}$ is indeed normal. However, the converse does not hold, indicating that more work is required to identify the correct combinatorial characterization of normality of $\mathcal{A}$.

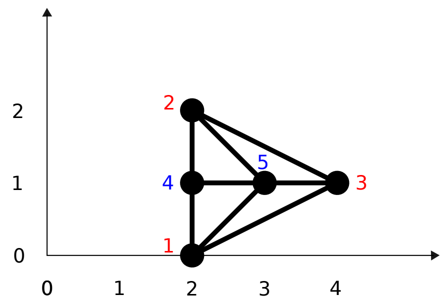
Currently, the best combinatorial approximation for the normality of $\mathcal{A}$ is the existence of a *unimodular covering* of $\text{conv}(\mathcal{A})$. A d-simplex in $\mathbb{R}^d$ is called *unimodular* if its standard Euclidean volume has the smallest possible value, $\frac{1}{d!}$ (or normalized volume 1). A collection of unimodular simplices covering $\text{conv}(\mathcal{A})$ is called a *unimodular covering* of $\text{conv}(\mathcal{A})$. Similarly, a triangulation of $\text{conv}(\mathcal{A})$ is called a *unimodular triangulation* if each of its simplices is unimodular. In an effort to decipher further connections between unimodular coverings of $\mathcal{A}$ and normality, specific types of unimodular coverings, such as regular unimodular triangulations, of $\text{conv}(\mathcal{A})$ have been considered. In particular, if we let

1. $\text{conv}(\mathcal{A})$ possesses a unimodular regular triangulation,
2. $\text{conv}(\mathcal{A})$ possesses a unimodular triangulation,
3. $\text{conv}(\mathcal{A})$ possesses a unimodular covering, and
4. $\text{conv}(\mathcal{A})$ is normal,

then (1) ⇒ (2) ⇒ (3) ⇒ (4). The implications (1) ⇒ (2) ⇒ (3) follow from the definitions of regular triangulation and unimodular covering, respectively. The details of (3) ⇒ (4) are specified in [1].

Unfortunately, this examination has failed to yield a characterization of the normality of $\mathcal{A}$ via unimodularity. In fact, there exist counterexamples to the converses of all three of the above implications. We first present one of many counterexamples to the converse of $(2) \Rightarrow (3)$.

Example 5.2. Consider the homogeneous configuration $\mathcal{A} = \{\mathbf{a}_1 = (2,0,1), \mathbf{a}_2 = (2,2,1), \mathbf{a}_3 = (4,1,1)\} \subset \mathbb{Z}^3$. We can define a unimodular covering of $\text{conv}(\mathcal{A})$ by considering the configuration $\mathcal{A}' = \mathcal{A} \cup \{\mathbf{a}_4 = (2,1,1), \mathbf{a}_5 = (3,1,1)\}$, and the triangulation $\Delta = \{\sigma_1 = \{245\}, \sigma_2 = \{235\}, \sigma_3 = \{135\}, \sigma_4 = \{145\}\}$ of $\mathcal{A}'$. Since all the points in $\mathcal{A}'$ lie in the same hyperplane, we can represent this triangulation in $\mathbb{Z}^2$.



The points in $\mathcal{A}$ are labeled with red indices. It is easy to see that $\text{conv}(\mathcal{A})$ is a 2-simplex. Moreover, we have that

$$\text{Vol}(\sigma_1) = \text{Vol}(\sigma_2) = \text{Vol}(\sigma_3) = \text{Vol}(\sigma_4) = 1,$$

and so Δ defines a unimodular covering of $\text{conv}(\mathcal{A})$. However, since $\mathcal{A}$ is a 2-simplex, it has the unique triangulation $\Delta = \{\{123\}\}$. If we let $\text{Vol}(\mathcal{A})$ be the normalized volume of $\text{conv}(\mathcal{A})$, then

$$\text{Vol}(\mathcal{A}) = \frac{1}{d!}|Det(A)| = \frac{1}{2!}\left| Det \begin{pmatrix} 2 & 2 & 4 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right| = 2 \neq 1.$$

Thus, $\mathcal{A}$ is not a unimodular simplex, and therefore does not possess a unimodular triangulation.

Second, the configuration defined by Ohsugi and Hibi [7] verified that $(2) \not\Rightarrow (1)$ and $(4) \not\Rightarrow (1)$. Third, Firla-Ziegler configurations [10] provide counterexamples for the converse of $(2) \Rightarrow (4)$. Finally, Bruns and Gubeladze [2] discovered multiple *tight cones* to be counterexamples to $(4) \Rightarrow (3)$. Therefore, we also have the hierarchy:

$$(1) \not\Leftarrow (2) \not\Leftarrow (3) \not\Leftarrow (4).$$

The combinatorial properties of these counterexamples will be central to the following discussion, and each will be considered in more detail as the necessity to do so arises. As a result of these counterexamples, even though the existence of a unimodular covering of $\text{conv}(\mathcal{A})$ provides the best known approximation for the normality of $\mathcal{A}$, it does not provide a characterization of normality of $\mathcal{A}$ in terms of its combinatorial properties.

Another approach to this problem is to formally define a concept of combinatorial normality for $\mathcal{A}$ and study the relationship of this property with the normality and unimodularity of $\mathcal{A}$. In doing so, it would be nice if the combinatorial normality of $\mathcal{A}$ guaranteed that $\mathcal{A}$ is in fact normal.

Definition 5.3. Let $\max \Delta$ denote the set of all facets of a triangulation Δ. A configuration $\mathcal{A}$ is **Δ-normal** if it has a regular triangulation Δ such that for each $\sigma \in \max \Delta$, $\mathcal{A} \cap \text{cone}(\mathcal{A}_\sigma)$ is a Hilbert basis of $\text{cone}(\mathcal{A}_\sigma) \cap \mathbb{Z}\mathcal{A}$.

Example 5.4. Recall the triangulation $\Delta = \{\{126\}, \{234\}, \{456\}, \{246\}\}$ of the configuration $\mathcal{A} = \{(0,0),(1,1),(2,2),(3,1),(4,0),(2,0)\} \subset \mathbb{R}^2$ from the beginning of section 3. In Example 3.8 we showed that Δ is in fact a regular triangulation of $\mathcal{A}$. In this case, $\max \Delta = \{\{126\}, \{234\}, \{456\}, \{246\}\}$, all of which are unimodular. Therefore, for a given $\sigma \in \max \Delta$, $\mathcal{A} \cap \text{cone}(\mathcal{A}_\sigma) = \mathcal{A}_\sigma$ is a Hilbert basis of $\text{cone}(\mathcal{A}_\sigma) \cap \mathbb{Z}\mathcal{A}$.

The definition of Δ-normality is rooted in highly studied combinatorial properties of $\mathcal{A}$, as it depends on the existence of a special type of regular triangulation of $\mathcal{A}$. This is in sharp contrast to the definition of normality, which depends solely on properties of algebraic objects associated with $\mathcal{A}$ – namely, the normality of the semigroup generated by $\mathcal{A}$. Now that we have established a concept of combinatorial normality, we must investigate its associations with normality.

Theorem 5.5. *All Δ-normal configurations are normal.*

Proof. Let $\mathcal{A} \subset \mathbb{Z}^d$ be a Δ-normal configuration with respect to the regular triangulation Δ. Then for each $\sigma \in \max \Delta$, $\mathcal{A} \cap \text{cone}(\mathcal{A}_\sigma)$ is a Hilbert basis of $\text{cone}(\mathcal{A}_\sigma) \cap \mathbb{Z}\mathcal{A}$, where $\mathcal{A}_\sigma = \{\mathbf{a}_i \in \mathcal{A} | i \in \sigma\}$. Since Δ is a simplicial complex of $\mathcal{A}$ it follows that

$$\text{conv}(\mathcal{A}) = \bigcup_{\sigma \in \max(\Delta)} \text{conv}(\mathcal{A}_\sigma),$$

and so

$$\text{cone}(\mathcal{A}) = \bigcup_{\sigma \in \max(\Delta)} \text{cone}(\mathcal{A}_\sigma).$$

Since $\mathcal{A} \cap \text{cone}(\mathcal{A}_\sigma)$ is a Hilbert basis of $\text{cone}(\mathcal{A}_\sigma) \cap \mathbb{Z}\mathcal{A}$, then $\mathbb{N}\mathcal{A}_\sigma = \text{cone}(\mathcal{A}_\sigma) \cap \mathbb{Z}\mathcal{A}$. Therefore,

$$\begin{aligned} \text{cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A} &= \left[\bigcup_{\sigma \in \max(\Delta)} \text{cone}(\mathcal{A}_\sigma) \right] \cap \mathbb{Z}\mathcal{A}, \\ &= \bigcup_{\sigma \in \max(\Delta)} \left[\text{cone}(\mathcal{A}_\sigma) \cap \mathbb{Z}\mathcal{A} \right], \\ &= \bigcup_{\sigma \in \max(\Delta)} \mathbb{N}\mathcal{A}_\sigma, \\ &= \mathbb{N}\mathcal{A}. \end{aligned}$$

□

Remark 5.6. All finite vector configurations are Δ-normal with respect to their regular unimodular triangulations. This follows directly from the definition of a unimodular simplex. Therefore, the Δ-normal property for $\mathcal{A}$ is a generalization of the existence of a regular unimodular triangulation of $\mathcal{A}$ [8].

Notice that this does not imply that the Δ-normal property is a generalization of the existence of a unimodular triangulation or unimodular covering of $\text{conv}(\mathcal{A})$. In fact, the set of all configurations possessing a regular unimodular triangulation is a proper subset of the set of all configurations possessing a unimodular triangulation. For example, in [7], Hibi and Ohsugi demonstrated the existence of a normal (0,1)-polytope that possesses no regular unimodular triangulations, but does possess a non-regular unimodular triangulation.

The combined works of [8], [10], and Firla and Ziegler (See [10]) have done well to categorize the different types of Δ-normal configurations. The first of these categories we noted in Remark 5.6. The next category described in [8] consists of all normal configurations $\mathcal{A}$ such that cone($\mathcal{A}$) is *simplicial*. Such a configuration is Δ-normal with respect to its coarsest regular triangulation, $\Delta = \{\{1, \dots, d\}\}$, where we assume that $\text{cone}(\mathcal{A}) = \text{cone}(\{\mathbf{a}_1, \dots, \mathbf{a}_d\})$. In [10], O'Shea and Thomas construct a collection of Δ-normal configurations that are non-simplicial and possess no unimodular triangulations. These configurations are generated as an extension of a collection of normal simplicial configurations known as Firla-Ziegler configurations (defined in [10]).

Definition 5.7. A $\mathbb{N}^4$**-Firla-Ziegler configuration** is a normal, simplicial vector configuration $\mathcal{A} \subset \mathbb{N}^4$ without unimodular triangulations that is the Hilbert basis of the cone generated by the first three standard basis vectors of $\mathbb{R}^4$, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and a vector $\mathbf{v} = (a, b, c, d) \in \mathbb{N}^4$ with $0 < a < b < d$.

The set of $\mathbb{N}^4$-Firla-Ziegler configurations is in fact nonempty. Firla and Ziegler were able to generate hundreds of such configurations via computer search [10].

Example 5.8. The configuration $\mathcal{A}^4$ consisting of the columns of the matrix

$$\mathbf{A^4} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

is the first $\mathbb{N}^4$-Firla-Ziegler configuration. It is the Hilbert basis of the configuration $\mathcal{A}^4_{ext}$ consisting of the columns of the matrix

$$\mathbf{A^4_{ext}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

The configuration $\mathcal{A}^4$ is normal since it is the Hilbert basis for $\text{cone}(\mathcal{A}^4) \cap \mathbb{Z}\mathcal{A}^4 = \text{cone}(\mathcal{A}^4_{ext}) \cap \mathbb{Z}\mathcal{A}^4$. To see that $\mathcal{A}^4$ is simplicial, notice that $\mathbf{v} \notin \text{cone}(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$. Thus, cone($\mathcal{A}^4$) is a 4-dimensional cone in $\mathbb{R}^4$. One can also verify that cone($\mathcal{A}^4$) possesses no unimodular triangulations.

O'Shea and Thomas recursively construct Δ-normal configurations, $\mathcal{A}^d \subset \mathbb{N}^d$ for each $d \geq 5$, that are non-simplicial and possess no unimodular triangulations via the following steps [10]. First, choose a $\mathbb{N}^4$-Firla-Ziegler configuration, $\mathcal{A}^4$, and let $\mathcal{A}^4_{ext} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{v}\}$ be the set of defining vectors for $\mathcal{A}$. Also let $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ be the standard basis for $\mathbb{R}^d$. Then for each $d \geq 5$ define the integer vectors

$$\begin{aligned}\mathbf{p}_d &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4,\\ \mathbf{p}_d^+ &= \mathbf{p}_d + \mathbf{e}_d,\\ \mathbf{p}_d^- &= \mathbf{p}_d - \mathbf{e}_d.\end{aligned}$$

Now, recursively define $\mathcal{A}^{d-1'} := \{(\mathbf{a}, 0) : \mathbf{a} \in \mathcal{A}^{d-1}\}$, $\mathcal{A}_{ext}^{d-1'} := \{(\mathbf{a}, 0) : \mathbf{a} \in \mathcal{A}_{ext}^{d-1}\}$, and $\mathcal{A}^d := \{\mathbf{p}_d^+, \mathbf{p}_d^-\} \cup \mathcal{A}^{d-1'}$. Assuming that $\mathbf{p}_d^+$ and $\mathbf{p}_d^-$ are always the first and second elements of $\mathcal{A}^d$ and that σ is the index set of $\mathcal{A}_{ext}^{d-1'}$ in $\mathcal{A}^d$, let $\sigma_1 = \{1\} \cup \sigma$ and $\sigma_2 = \{2\} \cup \sigma$. Then cone$(\mathcal{A}^d)$ has the triangulation Δ^d consisting of the maximal subcones $K_1 = \text{cone}(\mathcal{A}_{\sigma_1}^d)$ and $K_2 = \text{cone}(\mathcal{A}_{\sigma_2}^d)$.

Example 5.9. Consider the $\mathbb{N}^4$-Firla-Ziegler configuration $\mathcal{A}^4$ from Example 5.8. For $d = 5$, we have

$$\mathbf{p}_5 = (1,1,1,1,0), \quad \mathbf{p}_5^+ = (1,1,1,1,1), \quad \mathbf{p}_5^- = (1,1,1,1,-1).$$

We then define $\mathcal{A}^{4'}$ as the columns of the matrix

$$\mathbf{A^{4'}} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 0 & 2 & 1 & 1 & 2 & 2\\ 0 & 0 & 1 & 3 & 1 & 2 & 2 & 3\\ 0 & 0 & 0 & 5 & 1 & 2 & 3 & 4\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\mathcal{A}_{ext}^{4'}$ as the columns of the matrix

$$\mathbf{A_{ext}^{4'}} = \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 5\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So $\mathcal{A}^5$ consists of the columns of the matrix

$$\mathbf{A^5} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 2\\ 1 & 1 & 0 & 0 & 1 & 3 & 1 & 2 & 2 & 3\\ 1 & 1 & 0 & 0 & 0 & 5 & 1 & 2 & 3 & 4\\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which has the regular triangulation $\Delta^5 = \{\{13456\}\}, \{23456\}\}$, where we are indexing the columns of $\mathbf{A^5}$ with the index set $\{1, \ldots, 9\}$ going from left to right.

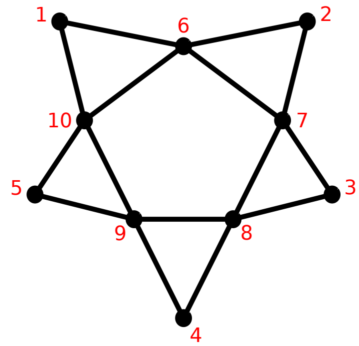
Continuing this recursive process we construct a collection of vector configurations about $\mathcal{A}^4$ for which we have the following theorem.

Theorem 5.10 ([10] Theorem 5.4)**.** *For each $d \geq 5$, the configuration $\mathcal{A}^d$ has the following properties:*

(1) $\mathbb{Z}(\mathcal{A}^d \cap K_1) = \mathbb{Z}(\mathcal{A}^d \cap K_2) = \mathbb{Z}^d$*,*
(2) $\mathcal{A}^d$ *is non-simplicial,*
(3) $\mathcal{A}^d$ *is* Δ^d*-normal, and*
(4) A^d *admits no unimodular triangulations.*

We have now established three distinct categories of vector configurations that are guaranteed to be Δ-normal for some regular triangulation. However, we have yet to consider the existence of a normal configuration $\mathcal{A}$ that is not Δ-normal for any regular triangulation of $\mathcal{A}$. Both [8] and [10] give examples of such configurations, the existence of which indicates that the set of all Δ-normal configurations is a proper subset of the set of normal configurations. The following is a class of such configurations constructed in [10] about the *Hibi-Ohsugi Configuration* [7].

Let HO be the finite, connected graph depicted below, and let $E(HO)$ denote the edge set of HO.



The Hibi-Ohsugi Edge Polytope

The Hibi-Ohsugi configuration is the defined as

$$\mathcal{A}_{HO} = \{\rho(e) = \mathbf{e}_i + \mathbf{e}_j \in \mathbb{R}^{10} : e = \{i, j\} \in E(HO)\}.$$

More generally, if G is a finite connected graph with d vertices and edge set $E(G)$, we call $\mathcal{A}_G = \{\rho(e) = \mathbf{e}_i + \mathbf{e}_j \in \mathbb{R}^d : e = \{i, j\} \in E(G)\}$ the *edge polytope configuration of* G. Hibi and Ohsugi show in [6] that $\mathcal{A}_G$ is normal if and only if, for two arbitrary odd cycles C and C' in G having no common vertex, there exists an edge of G joining a vertex of C with a vertex of C'. It is then clear from the figure above that $\mathcal{A}_{HO}$ is normal. Hibi and Ohsugi then used a Gröbner basis approach based on Corollary 8.9 of [11] to show that $\mathcal{A}_{HO}$ possesses no regular unimodular triangulations [7]. It is also noted in [7] that $\mathcal{A}_{HO}$ does possess a nonregular unimodular triangulation.

O'Shea and Thomas defined the homogenized (and therefore graded) Hibi-Ohsugi configuration

$$\mathcal{A}'_{HO} = \{\rho(e) = \mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_j : e = \{i, j\} \in E(HO), 1 \notin e\} \cup \{\mathbf{e}_1 + \mathbf{e}_i : \{1, i\} \in E(HO)\}.$$

It follows from the main result of [7] that $\mathcal{A}'_{HO}$ is also a normal configuration that possesses no regular unimodular triangulation, although it does possess a nonregular unimodular triangulation. As well, $\text{conv}(\mathcal{A}'_{HO})$ is a (0,1)-polytope in $\mathbb{R}^{10}$, and is consequently contained in the 10-cube. Thus, $\text{conv}(\mathcal{A}'_{HO})$ contains no interior lattice points, and so is referred to as *empty*. Finally, $\text{conv}(\mathcal{A}'_{HO})$ has all 15 vectors in $\mathcal{A}'_{HO}$ as extreme rays, and is therefore nonsimplicial. O'Shea and Thomas then use the following lemma to prove that $\mathcal{A}'_{HO}$ is not Δ-normal for any regular triangulation Δ [10].

Lemma 5.11 ([10] Lemma 5.5)**.** *Let $\mathcal{A} \subset \mathbb{Z}^d$ be a normal graded nonsimplicial configuration in $\{\mathbf{x} \in \mathbb{R}^d : x_1 = 1\}$ such that* $\operatorname{conv}(\mathcal{A})$ *is empty. If $\mathcal{A}$ does not have a regular unimodular triangulation then $\mathcal{A}$ is not Δ-normal for any regular triangulation Δ.*

Let $\mathcal{A}^{10} = \mathcal{A}'_{HO}$, and for each $d \geq 11$ let $\mathbf{p}_d = \mathbf{e}_1 + \mathbf{e}_d \in \mathbb{Z}^d$. O'Shea and Thomas [10] recursively define the configurations

$$\mathcal{A}^{d-1'} := \left\{ \begin{pmatrix} \mathbf{a} \\ 0 \end{pmatrix} : \mathbf{a} \in \mathcal{A}^{d-1} \right\} \text{ and } \quad \mathcal{A}^d := \{\mathbf{p}_d\} \cup \mathcal{A}^{d-1'}.$$

Theorem 5.12 ([10] Theorem 5.7)**.** *For each $d \geq 11$, the configuration $\mathcal{A}^d$ is normal and graded but not Δ-normal for any regular triangulation Δ.*

This analysis has provided many nice examples to help describe the relationship between the Δ-normal property of a given configuration $\mathcal{A}$, and the normality and existence of unimodular triangulations for $\mathcal{A}$. However, these relationships have not been explored to their fullest extent. In particular, these works could be expanded upon by considering the relationship between the Δ-normal property for $\mathcal{A}$, and the existence of a unimodular covering of $\mathcal{A}$. Answers to the following questions could further refine our understanding of this relationship.

Question 5.13. If $\mathcal{A}$ is Δ-normal for some regular triangulation Δ, does $\mathcal{A}$ possess a unimodular covering?

Question 5.14. Does there exist a normal configuration $\mathcal{A}$ that is not Δ-normal for any regular triangulation Δ and possesses no unimodular covering?

Question 5.14 is critical since a negative answer would imply that the union of the set of all Δ-normal configurations with the set of all configurations possessing a unimodular covering equals the set of all normal configurations. In which case, these two properties combined would provide a combinatorial characterization of normality. However, we now present an example of a configuration derived from [2] that answers Question 5.14 positively, and thus suggests further work is needed to develop such a characterization.

In [2], Bruns and Gubeladze show that the configuration C_6 with Hilbert basis consisting of the 10 vectors:

$$\begin{aligned} z_1 &= (0,1,0,0,0,0), & z_6 &= (1,0,2,1,1,2), \\ z_2 &= (0,0,1,0,0,0), & z_7 &= (1,2,0,2,1,1), \\ z_3 &= (0,0,0,1,0,0), & z_8 &= (1,1,2,0,2,1), \\ z_4 &= (0,0,0,0,1,0), & z_9 &= (1,1,1,2,0,2), \\ z_5 &= (0,0,0,0,0,1), & z_{10} &= (1,2,1,1,2,0), \end{aligned}$$

is isomorphic to the configuration C_Q, the set of vertices of a 5-dimensional normal lattice polytope Q. Moreover, they verified that C_6 does not satisfy the Unimodular Hilbert Cover condition, and so Q does not possess a unimodular covering. Among several other properties, [2] points out that

(1) C_6 has 27 facets, of which 5 are not simplicial.
(2) The Hilbert Basis for C_6 is contained in the hyperplane H given by the equation $-5\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 1$. Thus, $z_1, \ldots, z_{10}$ are the vertices of a normal

5-dimensional lattice polytope P_6 (isomorphic to the polytope Q) that possesses no unimodular covering and contains no other lattice points.

It follows immediately from (1) and (2) that C_6 is a normal, nonsimplicial configuration such that $P_6 = \mathrm{conv}(C_6)$ is empty and possesses no unimodular covering. We now prove a slightly more general version of Lemma 5.11.

Lemma 5.15 ([10] Lemma 5.5). *Let $\mathcal{A} \subset \mathbb{Z}^{d+1}$ be a normal nonsimplicial configuration such that* $\mathrm{conv}(\mathcal{A})$ *is empty. If $\mathcal{A}$ does not have a regular unimodular triangulation then $\mathcal{A}$ is not Δ-normal for any regular triangulation Δ.*

Proof. Let Δ be a regular triangulation of $\mathcal{A}$. Since $\mathcal{A}$ has no regular unimodular triangulations, then there exists some $\sigma \in \max(\Delta)$ such that $\mathrm{Vol}(A_\sigma) \geq 2$. It follows that the Hilbert basis of A_σ contains at least one vector $\mathbf{x} \in \mathbb{Z}^{d+1}$ that is not in $\mathcal{A}_\sigma$. Since $\mathrm{conv}(\mathcal{A})$ is empty, then all the vectors in $\mathcal{A}$ are extreme rays of $\mathrm{cone}(\mathcal{A})$, and so none of them lie in $\mathrm{cone}(\mathcal{A}_\sigma)$ unless they are in $\mathcal{A}_\sigma$. Since $\mathcal{A}$ is normal, we have that $\mathcal{A}$ is the Hilbert basis for $\mathbb{N}\mathcal{A}$. Moreover, since $\mathcal{A}_\sigma \subset \mathcal{A}$, it follows that the Hilbert basis of $\mathcal{A}_\sigma$ is contained in the Hilbert basis of $\mathcal{A}$. Therefore, $\mathcal{A} \cap \mathrm{cone}(\mathcal{A}_\sigma)$ is not a Hilbert basis for $\mathrm{cone}(\mathcal{A}_\sigma) \cap \mathbb{Z}\mathcal{A}$, and so $\mathcal{A}$ is not Δ-normal. Notice that the requirement of nonsimpliciality is also necessary, since otherwise $\mathcal{A}$ would be Δ-normal with respect to its coarsest triangulation. $\square$

Corollary 5.16. *C_6 is a normal configuration that possesses no unimodular covering and is not Δ-normal for any regular triangulation Δ.*

Thus, we see that there exist normal configurations that possess neither of the two combinatorial properties in question, and so these properties considered together still fail to provide a combinatorial characterization of normality. The following question is prompted by the Hibi-Ohsugi configuration and Remark 5.6, in which it is demonstrated that the Δ-normal property is a generalization of the existence of a regular unimodular triangulation. A negative answer to this question would indicate that the Δ-normal property is in fact a generalization of the existence of a unimodular triangulation.

Question 5.17. Does there exist a normal configuration $\mathcal{A}$ that is not Δ-normal for any regular triangulation, and possesses no non-regular unimodular triangulations?

More generally, Question 5.13 asks the same question with regard to unimodular coverings.

Recall the hierarchy (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) from the beginning of this section. It has been demonstrated that no converse holds for any implication in this hierarchy. We have also seen that (1) $\Rightarrow$(Δ-normal)$\Rightarrow$ (4), and that the converse of (Δ-normal)$\Rightarrow$ (4) does not hold. The validity of the converses of the implications (Δ-normal)$\Rightarrow$ (1), (Δ-normal)$\Rightarrow$ (2), and (Δ-normal)$\Rightarrow$ (3) are all open questions that we have posed in this paper. Answers to these questions would provide us with a much improved understanding of the relationship between Δ-normality of $\mathcal{A}$ and $\mathrm{conv}(\mathcal{A})$ possessing a unimodular covering. An improved understanding of this relationship may give us more insight into what is missing in terms of our attempts to define a combinatorial characterization of normality of $\mathcal{A}$.

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1548 Copperstone Drive, Brentwood, TN 37027

E-mail address: `liam.solus@gmail.com`