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Two-Coloring Cycles In Complete Graphs

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TWO-COLORING CYCLES IN COMPLETE GRAPHS

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Abstract. Inspired by an investigation of Ramsey theory, this paper aims to clarify in further detail a number of results regarding the existence of monochromatic cycles in complete graphs whose edges are colored red or blue. The second half focuses on a proof given by Gyula Károlyi and Vera Rosta for the solution of all $R(C_n, C_k)$.

1. Introduction

Let K_n denote the complete graph on n vertices. The *Ramsey number* $R(n, m)$ is the smallest integer r with the property that any edge coloring of the complete graph with r vertices using red and blue must contain a red K_n or a blue K_m. For example, no matter how we assign the color blue or red to each edge of K_6, there will always be a blue triangle or a red triangle as a subgraph. Additionally, K_6 is the smallest complete graph with this property, since we are in fact able to bi-color the edges of K_5 such that there is no monochromatic triangle. Since a triangle in graph theory is just another word for K_3, we can say that $R(3, 3) = 6$ (see the detailed proof in the next section).

The proof of this fact is one of the most well-known in Ramsey theory; one part of it was even asked as a question in the 1953 William Lowell Putnam Mathematical Competition. The proof can be used to justify the fact that within any group of 6 people, either there is a group of three mutual friends, or there is a group of three mutual strangers[1]. But what about bigger groups of people? Given any positive integers n and m, can we always find a large enough group of people such that there will always be n people in the group who mutually know each other, or m people who mutually do not? In other words, does $R(n, m)$ exist for all positive integer pairs (n, m)? It turns out that while it is incredibly difficult to find the exact value of certain Ramsey numbers, we can prove that $R(n, m)$ exists for all positive integers n and m, and furthermore can prove that $R(n, m)$ is bounded from above.

Theorem 1.1 (Ramsey's Theorem). *For any two positive integers n and m, the number $R(n, m)$ exists and satisfies the inequality $R(n, m) \leq R(n - 1, m) + R(n, m - 1)$.*

One proof of this, as explained in [1], considers an arbitrary red-blue coloring of a complete graph on $R(n-1, m)+R(n, m-1)$ vertices, and proves the existence of either

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[1] Let each vertex represent a person, and let two vertices share a red edge if the two people know each other, or a blue edge if they do not. This has been called the "theorem on friends and strangers."

a red K_n or a blue K_m. The base cases for this inductive proof are established with $R(n, 2) = n$ (likewise $R(2, m) = m$), since a bi-colored complete graph on n vertices is either a completely red K_n, or it has at least one blue edge, which may also be written as K_2. Observe also that for all integers n and m, $R(n, m) = R(m, n)$, since the definition of Ramsey number is inherently symmetric (we can merely flip the colors red and blue).

Only nine Ramsey numbers of this form are known [2] for integer pairs (n, m), with $n \geq m$. Specifically, these include the seven numbers $R(3, 3), R(3, 4), \ldots R(3, 9)$, whose values are $6, 9, 14, 18, 23, 28,$ and 36 respectively. It has also been discovered that $R(4, 4) = 18$ and $R(4, 5) = 25$. These numbers may seem small, but note that a graph on n vertices has a total of $\binom{n}{2}$ edges and therefore $2^{\binom{n}{2}}$ possible red-blue colorings. For a 25-vertex graph, the number is 2^{300}, which is indeed enormous. In larger graphs, the problem is not easily studied on a case-by-case basis. For higher Ramsey numbers, only loose bounds have been established.

It is useful to extend the definition of Ramsey number so that it may be applied to a larger set of combinatorial problems that involve different graph structures. Given two graphs G_1 and G_2, we let the *generalized Ramsey number* $R(G_1, G_2)$ denote the smallest integer r with the property that any edge coloring of the complete graph on r vertices using red and blue must contain either a red G_1 or a blue G_2 subgraph. This extension of our original definition now treats what we have defined as $R(n, m)$ as $R(K_n, K_m)$, naturally. We also maintain the symmetric property, $R(G_1, G_2) = R(G_2, G_1)$, again by just flipping the roles of the two colors. The advantage to this new definition is that we may also let each of the graphs be paths, stars, trees, or cycles, allowing for many more possible areas of study in which concrete results are often easier to obtain.

As a final comment on notation, one other equivalent definition for the Ramsey number $R(G_1, G_2)$ is the smallest number r with the property that any graph G on r vertices will either contain a G_1 subgraph, or its complement $\overline{G}$ will contain a G_2 subgraph. To see the equivalence at play here, let the edges of G be colored entirely red, and color all edges of $\overline{G}$ blue. Taken together, G and $\overline{G}$ form a complete graph on r vertices. It is sometimes useful to speak of a graph G and its complement $\overline{G}$, rather than a complete graph whose edges are colored red and blue. For this paper, however, I prefer the red-blue edge-coloring definition, particularly for its visual appeal and clarity.

A *cycle* is a graph that consists of some vertices connected by edges in a closed chain. Let C_n denote a cycle on n vertices, which we may also write as $x_1 x_2 \ldots x_n x_1$ with each x_i representing a vertex of the cycle. The purpose of this paper is to study and to illustrate results that involve the Ramsey number $R(C_n, C_k)$, for which a specific formula has been discovered for all $n \geq k \geq 3$. In Section 2, I present the proofs for small cases of $R(C_n, C_k)$, aiming to illustrate some of the basic proof techniques that are used in this area of research. Next, in Section 3, I highlight some important results involving the existence of monochromatic cycles in complete graphs under certain preconditions. In Section 4, I state the main theorem presented in [3] that establishes values for all generalized Ramsey numbers where G_1 and G_2 are both cycles. I also present some lemmas used in [3], without full proof. In Sections 5-7, I delve into separate cases of the theorem, with final supporting arguments given in Section 8.

2. Small Cycles

The following proofs help illustrate some important logical concepts and techniques involved in the theory. To start, in order to prove that a Ramsey number equals a specific value N, one must provide an example of a graph on $N-1$ vertices that does not contain either the red or blue specified subgraph. The more difficult part of the proof involves showing that any complete graph on N vertices with edges colored red and blue will contain one of the two subgraphs. Often, it is useful to assume one of the colored subgraphs does not exist, and then show that this implies the existence of the other. Proof by contradiction is also commonly used.

When finding $R(G_1, G_2)$ with $G_1 = G_2$, the task of showing the existence of either the red or blue subgraph is reduced to showing the existence of a monochromatic subgraph. The roles of the two colors are even more easily exchanged to prove the result.

Theorem 2.1. $R(C_3, C_3) = 6$.

Proof. Note that because $C_3 = K_3$, the Ramsey number $R(C_3, C_3) = R(K_3, K_3) = R(3, 3)$, as mentioned in the introduction. Consider any 5-cycle of K_5. Color its edges red. The edges not in this cycle form another C_5; color these edges blue. There exists no monochromatic C_3 in this coloring of K_5 (see Figure 1a), therefore $R(C_3, C_3) > 5$.

Let G be a K_6 graph whose edges are colored red and blue. Consider a single vertex v in G, and note that, since all vertices in G including v have degree 5, at least three of the edges incident to v must be red, or else at least three must be blue. Without loss of generality, assume at least three of these edges are red, and let's say v_1, v_2, and v_3 are vertices adjacent to v by red edges (Figure 1b). Now consider the possible edge colorings of the K_3 graph formed on v_1, v_2, and v_3. If one of its three edges (say v_1v_2) is red, then it forms a red K_3 with two of the already established red edges (vv_1 and vv_2, Figure 1c). Otherwise, all of these three edges are blue, but then we have a blue K_3 (Figure 1d). We have just shown that $R(C_3, C_3) \geq 6$, and this combined with the result from the first paragraph gives $R(C_3, C_3) = 6$. □

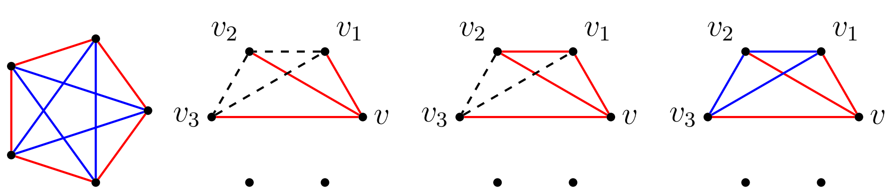

FIGURE 1. Demonstrating that $R(C_3, C_3) = 6$.

Theorem 2.2 (Djang[2])**.** $R(C_4, C_4) = 6$.

Proof. The edge-coloring of K_5 in Figure 1a also shows $R(C_4, C_4) > 5$, since it is free of any monochromatic C_4. Now let G be a complete graph on six vertices with edges colored red and blue. Consider a vertex v in G.

Case 1: Four or more vertices are adjacent to v by red edges. Consider the K_4 subgraph on these other vertices, shown with dashed edges in Figure 2a. Among the six edges of the K_4, if any two adjacent edges are both red, then together with the two red edges joined to v, they form a red C_4 (Figure 2b). Otherwise, there are at most two red edges in this K_4, which are not incident to any of the same vertices. In this case, there are four remaining blue edges that will form a blue C_4, as in Figure 2c.

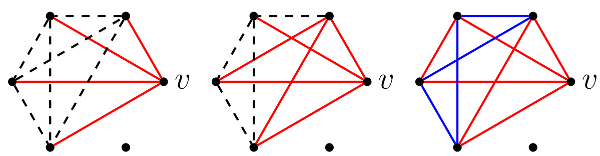

FIGURE 2. Diagrams for Case 1 of Theorem 2.2.

Case 2: The vertex v is joined by red edges to exactly three vertices, and by blue edges to exactly two vertices. Group these five vertices into the sets $V_r = \{r_1, r_2, r_3\}$ and $V_b = \{b_1, b_2\}$, based on the color of the edge connecting each vertex to v. Consider the complete bipartite graph $K_{3,2}$ on said vertex sets, shown by dashed lines in Figure 3a. Note if there exists a vertex $b_i \in V_b$ with two red edges incident to it in this $K_{3,2}$, a red C_4 is formed. Likewise, if there exists a vertex $r_j \in V_r$ with two blue edges incident to it, a blue C_4 exists.

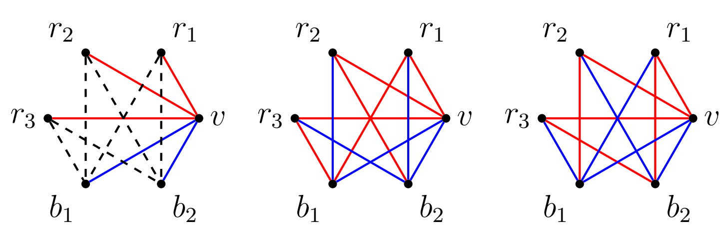

FIGURE 3. Diagrams for Case 2 of Theorem 2.2.

Assume there is no monochromatic C_4, and try to color the edges of this $K_{3,2}$ while avoiding the above two situations. Start by coloring, say, the edge r_1b_1 red. Since b_1 may not have two red edges incident to it, we are forced to color b_1r_2 blue. Then r_2b_2 must be red, etc. This process of coloring edges continues until we reach the last edge

[2]The first known proof of this theorem can be found in [4], but I choose to present my own here.

of the $K_{3,2}$, which will inevitably form part of a red C_4 (namely $vr_1b_1r_3v$, in Figure 3b). If instead, we color r_1b_1 blue, we are still forced to color the rest of the edges (Figure 3c), and again cannot avoid the formation of a red C_4.

By swapping the roles of the colors red and blue in the above two cases, we may conclude that if four or more vertices are adjacent to v by blue edges, or if v is incident to exactly three blue edges and two red edges, inevitably a monochromatic C_4 exists somewhere in G. This covers all possible cases. We have shown that in all red-blue colorings of K_6, there must exist a monochromatic C_4, completing the proof. □

For the following theorem, I reiterate the proof given first by [5], using the language of edge-coloring rather than complementary graphs.

Theorem 2.3 (Chartrand and Schuster)**.** $R(C_4, C_3) = 7$.

Proof. For the lower bound case, consider a red-blue edge-coloring of K_6, consisting of two disjoint red C_3 graphs, with all remaining edges colored blue (these blue edges form a complete bipartite subgraph $K_{3,3}$), as shown in Figure 4. Since this coloring of K_6 contains no red C_4 and no blue C_3, we have $R(C_4, C_3) \geq 7$.

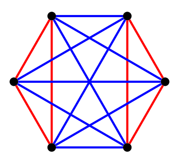

FIGURE 4. Showing a lower bound for $R(C_4, C_3)$.

What remains is to show that a complete graph G on 7 vertices, with edges colored red and blue, will always contain either a red C_4 or a blue C_3. Assume G contains no blue C_3. Since $R(C_3, C_3) = 6$, there must exist a monochromatic C_3 in G, which by our assumption must be red; call it C. Let the remaining four vertices be v_1, v_2, v_3, and v_4.

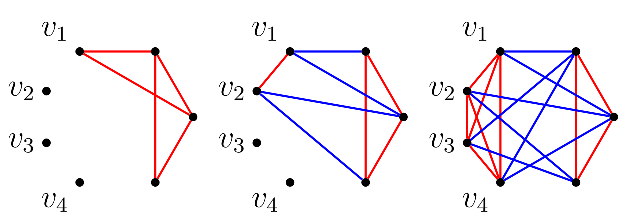

FIGURE 5. Demonstrating $R(C_4, C_3) \leq 7$.

If any v_i is adjacent to two or more vertices of C by red edges, then a red C_4 is formed, as in Figure 5a. Thus, assume all v_i are adjacent to at most one vertex of C by a red edge; therefore each v_i is adjacent to at least two vertices of C by blue edges. Since C consists of only 3 vertices, any two distinct v_i must then be joined by blue edges to at least one common vertex of C. In order to avoid a blue C_3 in G, any pair of distinct v_i must be adjacent by a red edge; see vertices v_1 and v_2 in Figure 5b. But then a red K_4 is formed on the vertices $\{v_i\}$, as shown in Figure 5c, and a red C_4 subgraph exists within that. □

Building on this last result, we are able to prove by induction the values for many more Ramsey numbers for cycles, specifically when one of the subgraphs is a C_3. The proof of this more general result shares similar techniques and constructions as those just demonstrated in the proof of Theorem 2.3.

Theorem 2.4 (Chartrand and Schuster)**.** *For* $n \geq 4, R(C_n, C_3) = 2n - 1$.

Proof. First, consider a complete graph G on $2n-2$ vertices, and color completely red the edges of two separate K_{n-1} subgraphs. Color the rest of the edges in G blue, so that the blue subgraph consists of a bipartite $K_{n-1,n-1}$. See an example of this construction in Figure 6, and note that this is a generalization of the coloring given for the small case in Figure 4. Since bipartite graphs contain no 3-cycles, and since the biggest red cycles that can be found here have length $n-1$, we have constructed a graph that has neither a red C_n nor a blue C_3. Thus, $R(C_n, C_3) > 2n - 2$.

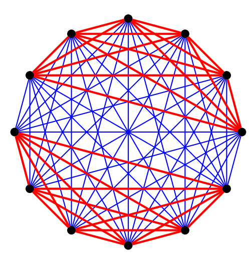

FIGURE 6. Demonstrating $R(C_n, C_3) > 2n - 2$, with the example of $n = 7$.

We now use induction on n, with the result of Theorem 2.3 as the base case. Assume that $R(C_n, C_3) = 2n - 1$ for some $n \geq 4$. From the above lower bound analysis, we have $R(C_{n+1}, C_3) > 2(n+1) - 2 = 2n$. We aim to show that $R(C_{n+1}, C_3) = 2n + 1$. Let G be a graph on $2n+1$ vertices, and assume that G contains no blue C_3. We have $R(C_n, C_3) = 2n - 1$ by our induction hypothesis, and since the number of vertices in G is greater than $2n-1$, there must exist a red C_n (or a blue C_3, but that's impossible by our assumption) in G. Let C be this red cycle of length n, with vertices $u_1, u_2, \ldots, u_n$. Denote the remaining vertices $v_1, v_2, \ldots, v_{n+1}$.

Suppose some vertex u of C is adjacent to all vertices v_i by blue edges. Since G contains no blue C_3, every two distinct vertices v_i are adjacent by red edges. But this creates a red K_{n+1} and therefore a red C_{n+1} subgraph. This contradiction allows us to conclude that every vertex of C is adjacent to at least one v_i by a red edge. There are two cases to consider. We define a pair of *alternate* vertices in C to be two vertices that are not adjacent to one another but are both adjacent to the same vertex in C that lies between them in the cycle.

Case 1: Suppose there exists a pair of alternate vertices of C which are respectively joined to two distinct v_i by red edges. Suppose u is the vertex in C between these alternate vertices, with v_1 and v_2 the mentioned vertices outside C. Note that if any two adjacent vertices in C are connected by red edges to the same vertex outside C, then there exists a red C_{n+1}. To avoid this situation, it must be that both edges uv_1 and uv_2 are blue. Now consider the edge v_1v_2. If it is red, then a red C_{n+1} is formed by replacing u in C by the edge v_1v_2. But if this edge is blue, then the C_3 denoted by uv_1v_2u is entirely blue. Either way, we have a red C_{n+1} or a blue C_3.

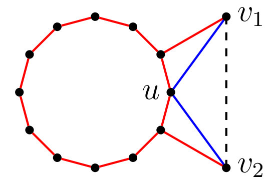

FIGURE 7. Case 1 of Theorem 2.4: A pair of alternate vertices of C are respectively joined by red edges to distinct vertices not in C.

Case 2: Suppose no two alternate vertices of C are joined by red edges to distinct vertices v_i. Since at least one red edge must be incident to each vertex of C, it must be that exactly one red edge is incident to each, otherwise we revert back to Case 1. Now u_1 and u_3 are joined by red edges to the same v_i, say v_1. In fact all u_i with i odd are joined to v_1 by a red edge. If n is odd, then both u_1 and u_n are joined by red edges to v_1, and since u_1 is adjacent to u_n in C, a red C_{n+1} is formed. Assume n is even. Then it follows that each u_i with i even is adjacent by red edges to the same $v_i \neq v_1$, say v_2. No more red edges connect vertices of C to vertices not in C, so each v_j, $3 \leq j \leq n+1$, is joined by only blue edges to every vertex of C. In order to avoid the existence of a blue C_3, each v_j, excluding v_1 and v_2, must be adjacent to one another by red edges. In particular, the edges $v_4v_5, \ldots, v_nv_{n+1}, v_{n+1}v_1$ are red. Also, since the edges v_1u_2 and v_2u_1, v_3u_2 and v_3u_1 are blue, it must be that all edges v_1v_i and v_2v_i, $3 \leq i \leq n+1$ are red (avoiding a blue C_3). Finally, we may now create a red C_{n+1}: $v_1v_3v_2v_4v_5 \ldots v_{n+1}v_1$.

Either hypothetical case yields $R(C_{n+1}, C_3) \leq 2n+1$, and we have equality after putting to use the lower bound mentioned earlier. By induction, we have proven the theorem. □

Chartrand and Schuster [5] also go on to prove similarly that $R(C_n, C_4) = n+1$ for $n \geq 6$, and also that $R(C_n, C_5) = 2n-1$ for $n \geq 5$. These results are encompassed in the complete solution for $R(C_n, C_k)$, which was obtained shortly afterwards by other mathematicians. Nearly 30 years later, in 2001, a simpler proof was published by Gyula Károlyi and Vera Rosta [3]. I choose to elaborate on results from this newer proof in the remaining sections.

3. Observations Regarding Monochromatic Cycles

Before introducing the main theorem as presented in [3], I discuss some interesting preliminary observations made by [3], with the aim of clarifying visually[3] and verbally the arguments behind them. Let G be a complete graph whose edges are colored red and blue. Given a cycle $C = x_1x_2 \dots x_tx_1$, we let an edge of the form x_ix_{i+j} be called a *chord* of length j, or a j-chord. Note also that indices are meant modulo the length of the cycle that is considered.

Lemma 3.1 (Károlyi and Rosta)**.** *Let $C = x_1x_2\dots x_tx_1$ be a monochromatic blue (red) cycle in G. Then either G contains a blue (red) C_{t-1} or every 2-chord of C is red (blue).*

Proof. We can see that if some 2-chord of C were the same color as C, then we can create a same-colored C_{t-1} by using the 2-chord to replace the two adjacent edges of C whose outer endpoints are the vertices of the 2-chord. Figure 8a and 8c show this. Otherwise, every 2-chord of C must be the opposite color, as in Figure 8b, 8d. □

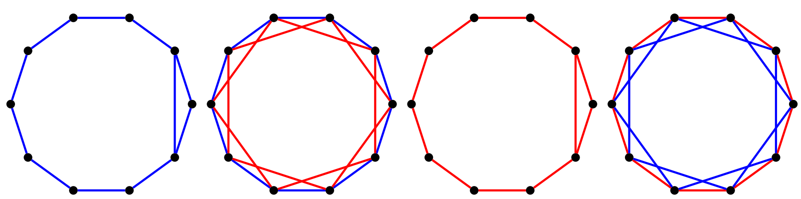

Figure 8. Lemma 3.1, illustrated here using $t = 10$.

Lemma 3.2 (Károlyi and Rosta)**.** *Suppose that G contains a monochromatic C_{2l+1} for some $l \geq 3$. Then G also must contain a monochromatic C_{2l}.*

Proof. Suppose $C = x_1x_2\dots x_{2l+1}x_1$ is a monochromatic (blue) cycle, and there is no monochromatic C_{2l}. By Lemma 3.1 we know that every 2-chord of C is red. Since C is an odd-length cycle, these red 2-chords (there are exactly $2l+1$ of them) form a new C_{2l+1} that is entirely red. By Lemma 3.1, every 2-chord of this new C_{2l+1} must be blue, and these chords are precisely all 4-chords of C. Figure 9a illustrates this situation.

Were some 3-chord, say x_1x_4, blue, then consider $x_1x_4x_3x_2x_6x_7\dots x_{2l+1}x_1$ (shown in Figure 9b). This cycle uses the blue 3-chord x_1x_4 and the blue 4-chord x_2x_6, along with blue edges of C. It includes all vertices of C except x_5, and thus we are left with a blue C_{2l}, a contradiction. We can repeat this argument for any blue 3-chord of C.

[3]In the following diagrams, when appropriate, we let x_1 be the rightmost vertex, with subsequent vertices labeled counterclockwise.

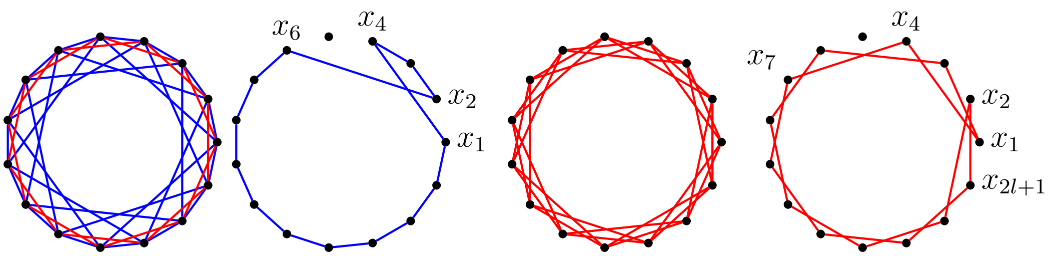

FIGURE 9. Diagrams for Lemma 3.2, with $l = 7$.

Therefore, we can conclude that every 3-chord of C is red. Known red edges are now shown in Figure 9c. Consider the cycle $x_1x_4x_7x_9 \ldots x_{2l+1}x_2x_{2l}x_{2l-2} \ldots x_6x_3x_1$ (Figure 9d), which uses red 2- and 3-chords, and includes all odd and all even vertices of C, except for x_5. This is a red C_{2l}, and again, we have a contradiction. It must be that some monochromatic C_{2l} exists. □

Lemma 3.3 (Károlyi and Rosta)**.** *Suppose that G contains a monochromatic C_{2l} for some $l \geq 3$. Then G also contains a monochromatic C_{2l-2}.*

Proof. First, note that if G contains a monochromatic C_6, then G has at least six vertices. Since $R(C_4, C_4) = 6$ (see proof in Section 2), G definitely contains a monochromatic C_4, located specifically in the K_6 subgraph formed on the vertices of the C_6. Thus we have proven the truth of the lemma for $l = 3$.

Now, assume $l \geq 4$. Suppose $C = x_1x_2 \ldots x_{2l}$ is a monochromatic (blue) cycle. If G contains a monochromatic C_{2l-1}, then by Lemma 3.2, G also contains a monochromatic C_{2l-2} and we are done. Assume then that G contains no monochromatic C_{2l-1}. Then by Lemma 3.1, all 2-chords of C are red. If there is no blue C_{2l-2}, then all 3-chords of C are also red. Figure 10 illustrates this. But now we can find a red cycle of length $2l - 1$, a contradiction. Consider $x_1x_{2l-1} \ldots x_3x_{2l}x_{2l-2} \ldots x_4x_1$, as in Figure 10b, which uses all odd and even vertices of C except x_2.

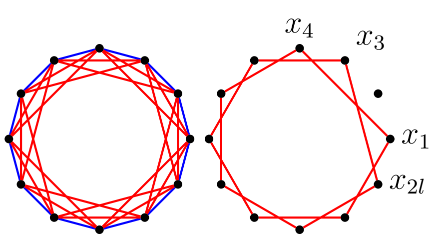

FIGURE 10. Diagrams for Lemma 3.3.

To summarize, if G contains a monochromatic C_{2l-1}, then by Lemma 3.2, G also contains a monochromatic C_{2l-2}. If we then assume that G has no a monochromatic

C_{2l-1} or C_{2l-2}, we reach a contradiction. It must be that G contains a monochromatic C_{2l-2}. □

Lemma 3.4 (Károlyi and Rosta). *Suppose that G contains a (monochromatic) blue cycle $C = x_1x_2 \ldots x_{2l}x_1$, but does not contain any monochromatic C_{2l-1}. Then each of the complete subgraphs G_1 and G_2, induced on the vertex sets $\{x_1, x_3, \ldots, x_{2l-1}\}$ and $\{x_2, x_4, \ldots, x_{2l}\}$ respectively is a red K_l.*

Proof. The two supposed red K_l's are made up precisely of all even chords of C. We must prove that all $2j$-chords of C are red for $1 \leq j \leq l/2$. If $j = 1$, we immediately see from Lemma 3.1 that all 2-chords of C are red. For any $2j$-chord, say x_1x_{2j+1}, consider the following two cycles in G: $x_1x_{2j+1}x_{2j}x_{2j-1} \ldots x_2x_{2j+3}x_{2j+4} \ldots x_{2l}x_1$ and $x_1x_2 \ldots x_{2j-1}x_{2l}x_{2l-1} \ldots x_{2j+1}x_1$, shown in Figure 11.

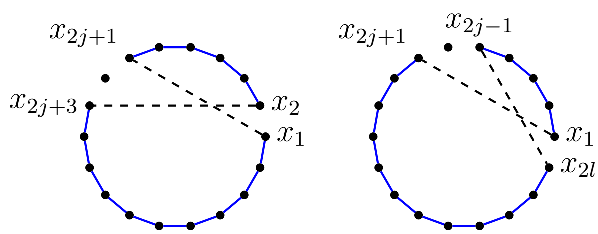

FIGURE 11. Two cycles in G using the $2j$-chord x_1x_{2j+1}. ($j = 3$)

Both of these cycles use $2l-3$ blue edges of C, plus two additional chord edges, whose colors we have yet to determine. The chords x_1x_{2j+1} and x_2x_{2j+3} are used in the first cycle, and $x_{2j-1}x_{2l}$ and x_1x_{2j+1} are used in the second. Since each cycle has total length $2l-1$, each may not be monochromatic (blue). We can conclude that either the edge x_1x_{2j+1} is red, or, if it is blue, then both edges x_2x_{2j+3} and $x_{2j-1}x_{2l}$ must be red, as in Figure 12a and 12b respectively. If this second case occurs, consider the $(2l-1)$-cycle $x_2x_4 \ldots x_{2l}x_{2j-1}x_{2j-3} \ldots x_{2j+3}x_2$, in Figure 12c.

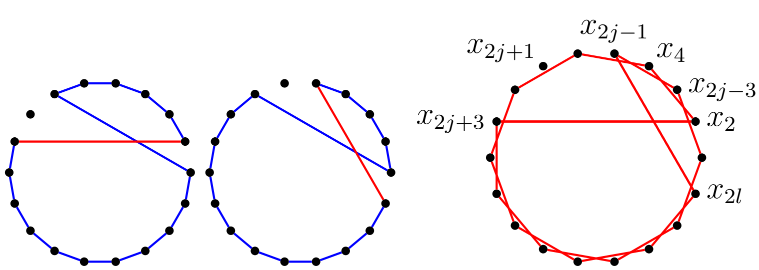

FIGURE 12. If both chords x_2x_{2j+3} and $x_{2j-1}x_{2l}$ are red, we find a red C_{2l-1}.

This final cycle uses all even and odd vertices of C, excluding x_{2j+1}. It's edges are red 2-chords of C, plus the two chords, x_2x_{2j+3} and $x_{2j-1}x_{2l}$, that we assume are both red. Here we have a red C_{2l-1}, a contradiction. Since this case is impossible, it must be that the $2j$-chord x_1x_{2j+1} is red. This argument works for all $1 < j \leq l/2$, and we can rotate the graph to make any vertex the starting vertex x_1. We have shown that all even chords of C are red, which gives the desired result of two red K_l's, respectively spanning the even and odd vertices of C, as shown in Figure 13, using $l = 6$. □

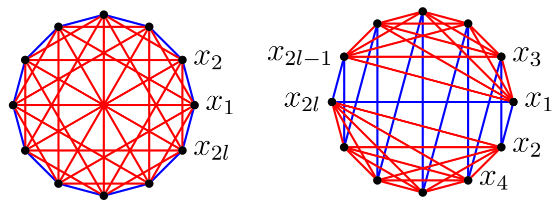

FIGURE 13. Two arrangements of the vertices of C (the blue cycle) with known colored edges. All even chords of C are red (left); equivalently, two red K_l subgraphs span the even and odd vertices of C, respectively (right).

The following important lemma is stated without proof, as it would be too long to include here.

Lemma 3.5 (Károlyi and Rosta). *Suppose that G contains a blue cycle $C = x_1x_2 \ldots x_{2l}x_1$, $l \geq 3$, such that $G_1 = \{x_1, x_3, \ldots, x_{2l-1}\}$ and $G_2 = \{x_2, x_4, \ldots, x_{2l}\}$ are red complete subgraphs with l vertices. Then one of the following 3 possibilities occur.*

(i) G contains a red C_m for each $3 \leq m \leq 2l$.

(ii) G contains a blue C_k for each $3 < k \leq 2l + 1$.

(iii) G contains a blue C_k for each even number $4 \leq k \leq 2l$ and a red C_m for each $3 \leq m \leq min\{\lceil |V(G)| \rceil /2, 2l\}$.

In short, the argument behind this result first considers possible red edges connecting G_1 to G_2. If two independent red edges exist between the two red subgraphs, we have case (i). Otherwise, in both case (ii) and (iii), we can obtain blue cycles of even length that alternate between G_1 and G_2. Next, the vertices in G that lie outside of C are considered. If some vertex d, not in C, is connected by blue edges to some vertices in both G_1 and G_2, we may modify the blue cycles of even length to also obtain cycles of odd length that include d. This gives case (ii). If no such vertices exist, then we are guaranteed to have many vertices outside C that connect to one of the red subgraphs (G_1 or G_2) with only red edges. This grants the existence of red cycles of many lengths. The largest guaranteed red cycle is limited by the number of vertices in G and in C; this is case (iii).

4. The Main Theorem

We have handled several small cases for $R(C_n, C_k)$, and we have investigated how monochromatic cycles of varying sizes may imply the existence of other monochromatic cycles in complete graphs. This next section moves on to examine certain areas of the proof given by Károlyi and Rosta [3] for the solution for all $R(C_n, C_k)$. We state the theorem here.

Theorem 4.1. *Let $3 \leq k \leq n$ be integers. Then,*

$$R(C_n, C_k) = \begin{cases} 6 & \textit{if k=n=3 or 4,} \\ n + k/2 - 1 & \textit{if n,k are even,} \\ max\{n + k/2 - 1, 2k - 1\} & \textit{if n is odd, k is even,} \\ 2n - 1 & \textit{otherwise (i.e. if k is odd).} \end{cases}$$

We have already shown in Section 2 that $R(C_3, C_3) = R(C_4, C_4) = 6$, so we may focus on the remaining three cases. Also note that our previous result of $R(C_n, C_3) = 2n - 1$ is consistent with this theorem, as it falls into the case where k is odd. We also have already established $R(C_4, C_3) = 7$, and since $n \geq k$, we have covered all situations where $n = 4$. Instead of handling these small cases again, we may assume $n > 4$, $k > 3$ for all statements that follow.

Károlyi and Rosta's proof is the newest and shortest for this particular result; it is organized into three main overarching lemmas that each cover all possible combinations of parities of n and k. I briefly summarize these three main points. Let G be a complete graph with edges colored red and blue. The first lemma states that if G has a required number of vertices, we may ensure the existence of either a monochromatic cycle of length $l \geq n$, or we may find a blue C_k in G. Using this lemma, it is argued next that G contains either a monochromatic cycle of length exactly n, or G contains a blue C_k. Finally, the third lemma states that if a blue C_n exists in G, this implies the existence of a red C_n or a blue C_k. This justifies the theorem.

The structure of their proof has the advantage of conciseness and unity of overall strategy, however, the proof of each individual lemma is quite involved, as it must often provide separate explanations for different cases of even or odd n and k. As a reader, it may sometimes be difficult to jump between these cases while remaining confident in the continuity of the proof. Additionally, the chain of arguments may be difficult to follow on the first read, as the ideas put forward are not easily absorbed quickly. It is necessary to pause and visualize the ways in which vertices and colored edges of a graph interact, and no diagrams are provided to aid the reader in this task. The proof leaves out many details that may be trivial for experts, but that are worthwhile for anyone else wishing to gain a more immediate understanding of the concepts.

I state the first main lemma from [3] here, without full proof. While elaborating on the latter two of the main lemmas, the remaining sections seek to fill in some of these gaps for one who is less familiar with the topic at hand. Following this pattern, some lemmas taken from [3] will be stated without proof, while others will be explained with a greater level of detail than was supplied in their paper. The proof ideas are reiterated in three cases, treated separately in Sections 5-7, based on the parity of n and k. Although much overlap does occur in each case, examining each one individually may allow for an easier following of the reasoning involved.

Lemma 4.2. *Let G be a complete graph with $n + k/2 - 1$ vertices, where k is even. Then either there exists a monochromatic C_l with $l \geq n$, or there is a blue C_k in G.*

This is indeed a strong result, and it makes up a good portion of Károlyi and Rosta's proof. To prove it, in short, we first assume that C is the largest monochromatic cycle in G, with length $L \leq n - 1$. If C is red, we may utilize the maximality of L to prove the existence of a blue C_k that alternates along vertices of C and vertices outside C. If C is blue, we may find a red C_k in the same way, and use some other results to prove that this implies the existence of a blue C_k. The following Corollary will also be useful.

Corollary 4.3. *Let G be a graph on at least $(3/2)m - 1$ vertices, with m even. There must exist a monochromatic cycle in G of length $l \geq m$.*

Proof. Since $(3/2)m - 1 = m + m/2 - 1$, and m is even, we apply Lemma 4.2 using the values $n = k = m$. Then either there exists a monochromatic cycle C_l with $l \geq m$, or there is a blue C_m in G. The second possibility falls into the first. □

5. Let $k \leq n$ be even integers with $k \geq 4$, $n \geq 6$. Then $R(C_n, C_k) = n + k/2 - 1$.

For the lower bound case, consider a complete graph G on $n+k/2-2$ vertices. Separate the vertices of G into two sets of sizes $n - 1$ and $k/2 - 1$, respectively, and consider the two disjoint complete subgraphs on each set, K_{n-1} and $K_{k/2-1}$. Color the edges of these graphs completely red. The remaining edges of G form a complete bipartite subgraph $K_{n-1,k/2-1}$; color these edges blue. There cannot be a red C_n in G, since we have only $n - 1$ vertices in the red K_{n-1}, and only $k/2 - 1 \leq n/2 - 1 < n$ vertices in the red $K_{k/2-1}$. Additionally, there is no blue C_k in G, since blue cycles in $K_{n-1,k/2-1}$ can be at most twice the length of the smaller of the vertex sets, and $2(k/2 - 1) = k - 2 < k$. Therefore, $R(C_n, C_k) > n + k/2 - 2$. Now let G be a graph on $n + k/2 - 1$ vertices. We prove that either a red C_n or blue C_k must exist, beginning with the following lemma.

Lemma 5.1. *Let $k \leq n$ be even integers with $k \geq 4$, $n \geq 6$. If $|V(G)| \geq n + k/2 - 1$, then G contains either a monochromatic C_n or a blue C_k.*

Proof. Assume there is no blue C_k in G. Then by Lemma 4.2, there exists a monochromatic cycle of length at least n. Assume by contradiction that G does not contain any monochromatic C_n. Then, since n is even, by Lemma 3.3 in Section 3, it would be impossible for G to contain any monochromatic C_{n+2}, C_{n+4}, C_{n+6} ... in fact, G could not contain any monochromatic C_l for all even $l \geq n$. Additionally, now by using

Lemma 3.2, G could not contain a monochromatic C_{l+1} for the same even $l \geq n$, so that monochromatic cycles of length $n+1$, $n+3$, $n+5$, etc. are not allowed. But this means that G contains no monochromatic cycles of length greater than or equal to n, a contradiction. G must contain a monochromatic C_n. □

If this monochromatic C_n is red, we are done. If it is blue, we continue on to prove that either a red C_n or blue C_k must still exist. In the following lemma, note that we will make an assumption in the proof (and in the proofs of corresponding lemmas in Sections 6 and 7) that will be handled later.

Lemma 5.2. *Let $k \leq n$ be even integers with $k \geq 4$, $n \geq 6$. If G contains a blue C_n, then G also contains either a red C_n or a blue C_k.*

Proof. Let the blue C_n in G be called C, and assume that G contains no blue C_k. If a $(k-1)$-chord of C is blue, it can be used to create a blue C_k. Thus, all $(k-1)$-chords of C must be red. We will also now assume[4] here that all 2-chords of C are red. Now consider the cycle $x_3x_5 \ldots x_1x_kx_{k-2}x_{k-4} \ldots x_{k+2}x_3$, which first uses red 2-chords to traverse all odd vertices of C, and then uses the red $(k-1)$-chord x_1x_k to jump to all even vertices (k is even). Finally, the red $(k-1)$-chord $x_{k+2}x_3$ completes the cycle. Thus, we have found a red C_n in G. Figure 14 shows an example of this red cycle with $n = 10, k = 6$.

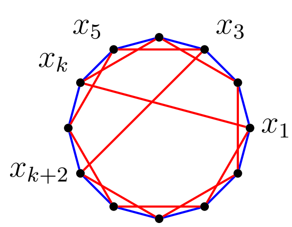

FIGURE 14. When n and k are even, we may find a red C_n on the vertices of a blue C_n, using red 2-chords and $(k-1)$-chords.

□

6. LET $k \leq n$ BE INTEGERS WITH $k \geq 4$ EVEN, AND $n \geq 5$ ODD. THEN $R(C_n, C_k) = max\{n + k/2 - 1, 2k - 1\}$.

Again, we start with the lower bound case. The same construction used in the beginning of Section 5, since it does not depend on the parity of n, establishes that $R(C_n, C_k) \geq n + k/2 - 1$. Still, we consider one additional construction to make our lower bound more specific. Let a graph G on $2k-2$ vertices be colored so that there exists a red complete bipartite graph $K_{k-1,k-1}$. Then color the remaining edges blue, and note that these edges make up two complete K_{k-1} subgraphs. Any red cycle in G can

[4]We postpone the case in which some 2-chord of C is blue until Section 8.

use only edges of the bipartite graph, and therefore can only be of even length. Since n is odd, there cannot be a red C_n. There is also no blue C_k, since the two K_{k-1} subgraphs are disjoint, and each only have $k-1$ vertices. This shows that $R(C_n, C_k) > 2k-2$. We finally may state that $R(C_n, C_k) \geq max\{n + k/2 - 1, 2k - 1\}$. Now, let G be a graph on $max\{n + k/2 - 1, 2k - 1\}$ vertices.

Lemma 6.1. *Let $k \leq n$ be integers with $k \geq 4$ even, and $n \geq 5$ odd. If $|V(G)| \geq max\{n + k/2 - 1, 2k - 1\}$, then G contains either a monochromatic C_n or a blue C_k.*

Proof. Assume there is no blue C_k in G. Since k is even, by Lemma 4.2, there exists a monochromatic cycle of length at least n. Assume that G does not contain any monochromatic C_n. Furthermore, let's assume G contains no monochromatic C_{n+1}. But since $n+1$ is even, using the argument in Lemma 5.1, we have that no C_l exists with $l \geq n+1$. This gives a contradiction, so now assume that G contains a monochromatic C_{n+1}. We can generate yet another contradiction as follows.

Suppose this C_{n+1} is blue. We have a monochromatic C_{n+1}, where $n+1$ is even, but no monochromatic C_n. We can apply Lemma 3.4, and then 3.5 with $l = (n+1)/2$. Combining lemmas 3.4 and 3.5 in this fashion always results in case (iii) of 3.5, since cases (i) and (ii) result in a monochromatic cycle of length $2l-1$ (specifically $2l-1 = n$ here). Thus, G contains a blue C_j for each even number $4 \leq j \leq 2l = n+1$. Since $k \leq n$ and k is even, $k = j$ for some j just described, thus we have shown the existence of a blue C_k, a contradiction.

Now reverse the roles of the colors, letting the C_{n+1} be red. Case (iii) of Lemma 3.5 gives us a blue C_m in G for each $3 \leq m \leq min\{\lceil |V(G)|/2 \rceil, n+1\}$. If $n+1 \leq \lceil |V(G)|/2 \rceil$, then we have found a blue C_n, a contradiction. If $\lceil |V(G)|/2 \rceil < n+1$, we must prove that $k < \lceil |V(G)|/2 \rceil$ in order to ensure a blue C_k. But $k = 2k/2 = \lceil (2k-1)/2 \rceil \leq \lceil |V(G)|/2 \rceil$, since $|V(G)| \geq 2k-1$. Thus a blue C_k is guaranteed. □

If G contains a red C_n, we are done. If G contains a blue C_n, since n is odd, the 2-chords of this blue cycle form another C_n. Assume all 2-chords of the blue cycle are red to form a red C_n. Again, see Section 8 for the explanation of the case when some 2-chord of the blue cycle is blue.

7. Let $5 \leq k \leq n$ be integers with k odd. Then $R(C_n, C_k) = 2n - 1$.

We may color the edges of a graph on $2n-2$ vertices so that there is no red C_n or blue C_k. See the beginning of the proof of Theorem 2.4, that verifies $R(C_n, C_3) \geq 2n-1$. As in the earlier theorem, there are no red C_n subgraphs. The blue edges in this construction form a bipartite graph, and since k is odd, there are no blue cycles of length k. This gives us $R(C_n, C_k) > 2n-2$. Now, let G be a graph on $2n-1$ vertices.

Lemma 7.1. *Let $5 \leq k \leq n$ be integers with k odd. If $|V(G)| \geq 2n-1$, then G contains either a monochromatic C_n or a blue C_k.*

Proof. Since $n > 3$, we can see that $|V(G)| \geq 2n - 1 = (3/2)n + n/2 - 1 = (3/2)(n + 1) - 1 + n/2 - (3/2) \geq (3/2)(n+1) - 1 > (3/2)n + 1$. If n is even, apply Corrollary 4.3 with $m = n$, so that we find a monochromatic cycle C_l, $l \geq n$. The argument used in the proof for Lemma 5.1 (which references Lemmas 3.3 and 3.2) shows that there must be a monochromatic C_n in G.

Now assume n is odd, and apply the special case of Lemma 4.2 to $m = n + 1$, so that we find a monochromatic cycle of length C_l, $l \geq n + 1$. Assume that there is no monochromatic C_n in G. Since n is odd, by the argument used in Lemma 6.1, there must exist a C_{n+1} in G, which we'll assume is blue. Again, now apply Lemmas 3.4 and 3.5 with $l = (n + 1)/2$ to guarantee that we will be in case (iii) of 3.5. Now we see that G has a red C_m for each $3 \leq m \leq min\{\lceil |V(G)|/2 \rceil, 2l\} = min\{\lceil |V(G)|/2 \rceil, n + 1\}$.

To generate a contradiction by showing a red C_n exists, we must prove that $n \leq min\{\lceil |V(G)|/2 \rceil, n+1\}$. Since $|V(G)| \geq 2n-1$, $|V(G)|/2 \geq (2n-1)/2$, and $\lceil |V(G)|/2 \rceil \geq (2n)/2 = n$. Clearly $n = min\{n, n + 1\} \leq min\{\lceil |V(G)|/2 \rceil, n + 1\}$. □

Lemma 7.2. *If k is odd and $|V(G)| \geq 2n - 1$, then if G contains a blue C_n, it also contains either a red C_n or a blue C_k.*

Proof. Assume all 2-chords of the blue C_n are red (again, see Section 8 for explanation if not). If n is odd, the red 2-chords of C form a red C_n. Otherwise, assume n is even.

Note that if $G_1 = \{x_1, x_3, \ldots, x_{n-1}\}$ and $G_2 = \{x_2, x_4, \ldots x_n\}$ are complete red subgraphs, then, on account of Lemma 3.5 (with $n = 2l$), we have the situation in which one of cases (i), (ii), or (iii) must be true. Since we have already assumed that G has no blue C_k, we eliminate case (ii). In case (i), we definitely have a red C_n. In case (iii), we have a red C_n if $n \leq \lceil |V(G)|/2 \rceil$. But this is clear, since by hypothesis $|V(G)|/2 \geq (2n-1)/2$, so $\lceil |V(G)|/2 \rceil \geq \lceil (2n - 1)/2 \rceil = 2n/2 = n$.

We may illustrate that this particular condition on G_1 and G_2 is definitely true if $n = 4$ or $n = 6$. Observe in Figure 15 that the red 2-chords alone give us two red $K_{n/2}$ subgraphs, induced on the even and odd vertex sets of C.

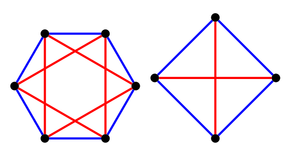

FIGURE 15. The 2-chords of C_4 and C_6 give two complete graphs on 2 and 3 vertices, respectively.

If G_1 and G_2 are not complete red subgraphs, then not all even chords of C are red. In other words, there exists a blue $2j$-chord, say $x_1 x_{2j+1}$, for some $2 \leq 2j \leq n/2$, $2j+1 \neq k$ (since all $(k-1)$-chords of C are red). We may assume, since we have handled lower cases, that $n \geq 8$ and $k > 3$.

If there are indices γ, δ of different parity such that both edges $x_\gamma x_\delta$ and $x_{\gamma-2}x_{\delta+2}$ are red, then they form, together with red 2-chords of C, a red C_n, which we can denote as $x_\gamma x_{\gamma+2} \dots x_{\gamma-2}x_{\delta+2}x_{\delta+4} \dots x_\delta x_\gamma$. Below, see Figure 16a, which illustrates this C_n when $n = 18$, $\gamma = 5$, and $\delta = 12$. We must assume that such indices do not exist, so that for any indices γ, δ of different parity, we must have at least one of $x_\gamma x_\delta$ and $x_{\gamma-2}x_{\delta+2}$ blue.

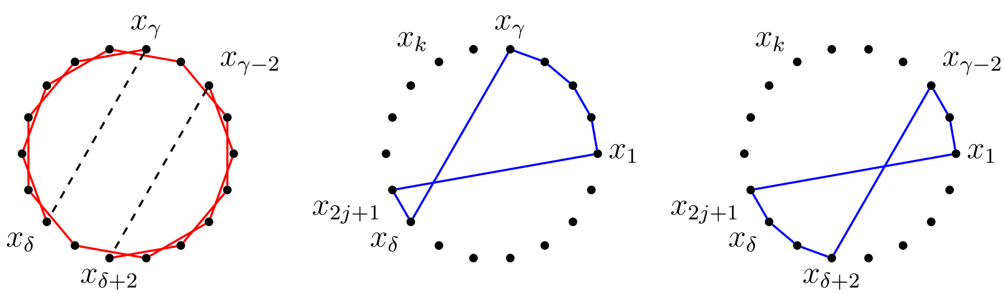

FIGURE 16. Forming a red C_n, or a blue C_k. The blue $2j$-chord is shown ($j = 5$).

Next, let's first assume $k < 2j + 1$. Choose $\gamma = k - 2$ (odd), $\delta = 2j + 2$ (even). This choice is consistent with the indices shown in Figures 16b and 16c. If the edge $x_\gamma x_\delta$ $= x_{k-2}x_{2j+2}$ is blue, then we may find a blue $C_k = x_1x_2 \dots x_\gamma x_\delta x_{2j+1}x_1$ (Figure 16b). If $x_\gamma x_\delta$ is not blue, then $x_{\gamma-2}x_{\delta+2} = x_{k-4}x_{2j+4}$ is blue, but then we may find a blue $C_k = x_1x_2 \dots x_{\gamma-2}x_{\delta+2}x_{\delta+1}x_\delta x_{2j+1}x_1$ (Figure 16c).

Assume next that $2j + 1 < k < n - 1$. Choose $\gamma = 2j$ (even), $\delta = k$ (odd). Again, we have that either $x_\gamma x_\delta = x_{2j}x_k$ is blue or $x_{\gamma-2}x_{\delta+2} = x_{2j-2}x_{k+2}$ is blue, in order to avoid a red C_n (Figure 17a). If $x_{2j}x_k$ is blue, we may find a blue C_k $=$ $x_1x_2 \dots x_{2j}x_kx_{k-1} \dots x_{2j+1}x_1$, as in Figure 17b. But if $x_{2j-2}x_{k+2}$ is blue, then C_k $=$ $x_1x_2 \dots x_{2j-2}x_{k+2}x_{k+1} \dots x_{2j+1}x_1$ is also completely blue, shown in Figure 17c.

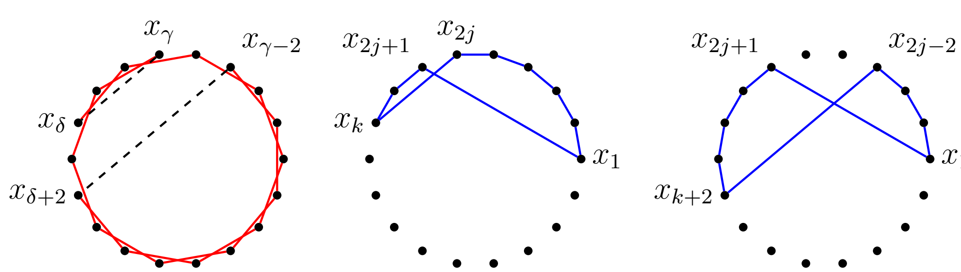

FIGURE 17. Forming a red C_n, or a blue C_k. Here, $n = 18$, $j = 3$, and $k = 9$.

Finally, now suppose $k = n - 1$. Either there is a blue C_k (of length $n - 1$), or, as in the proof of Lemma 3.4 (using $2l = n$), the assumption that the $2j$-chord x_1x_{2j+1} is blue gives us knowledge of a few red edges that must exist in order to avoid the formation of a blue C_{n-1}. These edges are x_2x_{2j+3} and $x_{2j-1}x_n$ (see Figure 12). Very similar constructions also give us the red edges $x_{n-1}x_{2j}$ and x_3x_{2j+2}. I choose not to include the full proof of this case here, but in [3], p.93, it is described how to use these red edges to conclude that either a red C_n or a blue C_k exists. □

8. Additional Lemmas

One large case has not been handled in the second half of the proofs for Sections 5, 6, and 7. We showed that if G contains a blue C_n, G also contains either a red C_n or a blue C_k, *under the assumption that all 2-chords of the blue C_n were red*. This section aims to clarify some of the case in which a 2-chord of the blue C_n is blue. We don't give a full proof, but do explain in detail a necessary proposition that used in it.

Lemma 8.1. *Let G be a complete graph such that $|V(G)| \geq 2n-1$ when n is even and k is odd. Consider any 2-coloring of its edges with red and blue. Suppose that G contains a blue cycle of length n, denoted by $C = x_1x_2\ldots x_nx_1$, and that some 2-chord of C is blue. Then G contains either a red C_n or a blue C_k.*

Assume G does not have a blue C_k and that some 2-chord of C is blue. These assumptions allows us to find many red edges using the following proposition, and we are able to connect them to form a red C_n.

Proposition 8.2. *If a 2-chord x_ix_{i+2} of C is blue, then either G contains a blue C_k or the edges $x_{i+1}x_{i+k-1}$, $x_{i+1}x_{i-k+3}$ and x_jx_{j+k} are red for $i-k+2 \leq j \leq i$.*

Proof. Each edge mentioned above, if colored blue, can form a blue C_k when combined with the blue x_ix_{i+2} edge and the blue edges of C, as we will see. Assume the above edges are blue. Note that both $x_{i+1}x_{i+k-1}$ and $x_{i+1}x_{i-k+3}$ are $(k-2)$-chords of C. By hypothesis, x_ix_{i+2} is a blue edge. Using it, we may form a blue $C_k = x_{i+1}x_ix_{i+2}\ldots x_{i+k-1}x_{i+1}$ (Figure 18a). Similarly, we may use the chord $x_{i-k+3}x_{i+1}$ to form a blue $C_k = x_{i+1}x_{i+2}x_i\ldots x_{i-k+3}x_{i+1}$ (Figure 18b).

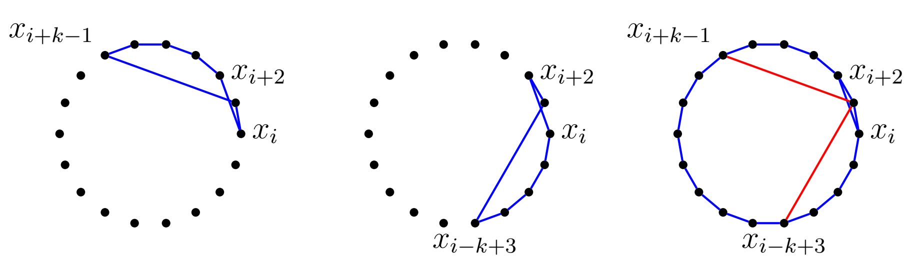

Figure 18. A 2-chord x_ix_{i+2} is blue. In order to prevent a blue C_k, the edges $x_{i+1}x_{i+k-1}$ and $x_{i+1}x_{i-k+3}$ must be red.

To show that x_jx_{j+k} are red for $i-k+2 \leq j \leq i$, note that x_jx_{j+k} is a k-chord of C. We may use such a k-chord along with the edges of C to form a cycle of length $(k+1)$. If this cycle includes the segment $x_ix_{i+1}x_{i+2}$ of C, then we may replace $x_ix_{i+1}x_{i+2}$ with the blue edge x_ix_{i+2}, to form a blue C_k. The k-chords with this particular property are $x_{i-k+2}x_{i+2}$, $x_{i-k+3}x_{i+3}$, $\ldots x_{i-1}x_{i+k-1}$, x_ix_{i+k}, giving us the required bounds for j.

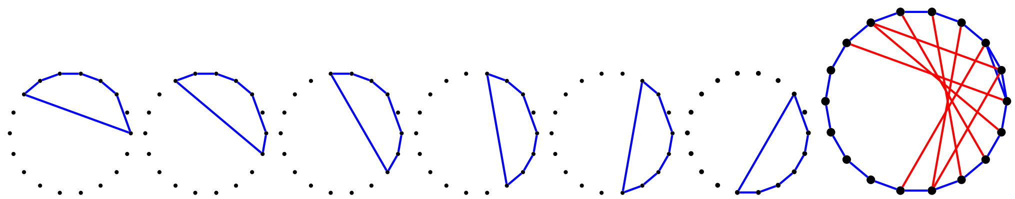

FIGURE 19. A sketch of blue C_k's formed using the k-chords mentioned. We conclude that each must be red. Known red edges, including those in Figure 18 are shown (right).

Because each of these edges, when colored blue, form a blue C_k, each must be red, since we assumed before that no blue C_k exists in G. Observe the above illustration, with $n = 18$ and $k = 7$. □

While we have by no means reconstructed a complete proof for all Ramsey numbers $R(C_n, C_k)$, hopefully this study has illuminated some important strategies used in the proof, and has revealed that it is both complicated and fascinating to prove conjectures about the existence of monochromatic structures within a graph whose edges are randomly colored red and blue.

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