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ABSTRACT. Three formulations of the rank of a matrix that are equivalent in classical linear algebra give rise to distinct notions of rank over the tropical semiring. This paper explores these three concepts of tropical rank and their relationships with one another, working up to a proof of the inequality that relates the three.

1. INTRODUCTION

The tropical semiring is the set $\mathbb{R} \cup \{\infty\}$ with the two new operations $\oplus, \odot$. These operations are defined such that, for all $x, y \in \mathbb{R} \cup \{\infty\}$,

$$x \oplus y := \min\{x, y\}, \text{ while } x \odot y := x + y,$$

where " $+$ " signifies ordinary addition of real numbers. The tropical semiring has nice algebraic features, and in fact satisfies all field axioms except for existence of additive inverses. For instance, 0 is the multiplicative identity as

$$0 \odot x = 0 + x = x$$

for all $x \in \mathbb{R} \cup \{\infty\}$. The symbol ∞ is taken to be greater than all $x \in \mathbb{R}$, so ∞ acts as the additive identity in the tropical semiring; that is, for all $x \in \mathbb{R} \cup \{\infty\}$,

$$x \oplus \infty = \min\{x, \infty\} = x.$$

Tropical division is regular subtraction (on $\mathbb{R}$; ∞ has no multiplicative inverse), but tropical subtraction is not defined — indeed,

$$5 \oplus x = 22 \text{ if and only if } \min\{5, x\} = 22,$$

which has no solutions $x \in \mathbb{R} \cup \{\infty\}$. Thus $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ is a field without additive inverses, and is therefore a semiring.

Odd as tropical arithmetic may seem at first glance, it actually arises in a natural way. In particular, consider the field $K = \mathbb{C}\{\{t\}\}$, that is, the field of Puiseux series over $\mathbb{C}$. An arbitrary $\alpha \in \mathbb{C}\{\{t\}\}$ is of the form $\alpha = c_1 t^{a_1} + c_2 t^{a_2} + \ldots$, where the coefficients c_i are complex numbers and $a_1 < a_2 < \ldots$ are rational numbers with a common denominator. The natural valuation to define on K is to send α to the lowest exponent that appears in the expansion of α. For example, if $\alpha = a_1 t + a_2 t^2 + a_3 t^3 + \ldots$, then $\mathrm{val}(\alpha) = 1$, and if $\beta = b_1 t^{-1/2} + b_2 + b_3 t^{1/2} + \ldots$, then $\mathrm{val}(\beta) = -1/2$.

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For arbitrary $\alpha, \beta \in \mathbb{C}\{\{t\}\}$, we have that $\mathrm{val}(\alpha+\beta) \leq \min\{\mathrm{val}(\alpha), \mathrm{val}(\beta)\} = \mathrm{val}(\alpha) \oplus \mathrm{val}(\beta)$ (with equality unless there is cancellation of terms of lowest degree) and $\mathrm{val}(\alpha\beta) = \mathrm{val}(\alpha) + \mathrm{val}(\beta) = \mathrm{val}(\alpha) \odot \mathrm{val}(\beta)$; this is the power series version of Lemma 8 from Section 2.2 of [1]. As in Section 3 of [2], we can extend K to $\tilde{K}$, the field of all formal power series over $\mathbb{C}$ with real (not just rational) exponents, and the valuation map behaves in the same manner. In this way, tropical arithmetic is, in a sense, the arithmetic of valuations of power series.

The goal now is to reimagine linear algebra in the context of the tropical semiring. First, define tropical matrix algebra in the obvious way by replacing ordinary addition and multiplication by $\oplus$ and $\odot$. For instance,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \odot 0 \oplus 0 \odot 1 & 1 \odot 1 \oplus 0 \odot 0 \\ 0 \odot 0 \oplus 1 \odot 1 & 0 \odot 1 \oplus 1 \odot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In classical — that is, not tropical — linear algebra, one of the key properties of a matrix M is its rank. The rank of M is normally defined to be the dimension r of the linear space spanned by the columns (or, equivalently, the rows) of M; said differently, the rank of M is the smallest r such that the columns (rows) of M are contained in a linear space of dimension r. Again considering the classical case, the rank of M can be shown to be equivalent to the following:

(1) The largest positive integer r for which M has a non-singular $r \times r$ minor;
(2) The smallest positive integer r such that M is the sum of r rank one matrices,

where a matrix has rank one if is the outer product of two vectors, that is, the product of a column vector on the left with a row vector on the right.

Although these three formulations of rank are equivalent in classical linear algebra, the analogous definitions over the tropical semiring do not coincide in general. Instead, they give rise to different notions of rank, which are referred to as the tropical rank, the Kapranov rank (named for Mikhail Kapranov), and the Barvinok rank (named for Alexander Barvinok). For an $m \times n$ matrix M with real entries, we consider these three ranks separately.

First, before we can define the tropical rank of M, we need the notion of a *tropical determinant*.

Definition 1.1. The *tropical determinant* of an $n \times n$ matrix $A = (a_{ij})$ is taken to be

$$\bigoplus_{\sigma \in S_n} a_{1\sigma(1)} \odot a_{2\sigma(2)} \ldots \odot a_{n\sigma(n)} = \min_{\sigma \in S_n} \{a_{1\sigma(1)} + a_{2\sigma(2)} \ldots + a_{n\sigma(n)}\},$$

with S_n the symmetric group on n elements.

With this notation, A is said to be *tropically non-singular* provided that the minimum in the tropical determinant of A is attained exactly once. In analogy with the classical case, then, the tropical rank of M is defined as follows:

Definition 1.2. The *tropical rank* of M, denoted $\text{Trank}(M)$, is the largest integer r for which M has a tropically non-singular $r \times r$ minor.

The definition of Kapranov rank depends on the notion of *tropical linear spaces*, the formal definition of which will come in Section 2.

Definition 1.3. The *Kapranov rank* of M, written $\text{Krank}(M)$, is the least dimension of any tropical linear space that contains the columns of M.

The Kapranov rank is then the tropical analogue of the usual definition of the rank of a matrix as the dimension of the subspace spanned by the columns. Lastly, we have the Barvinok rank of M.

Definition 1.4. The *Barvinok rank* of M, denoted by $\text{Brank}(M)$, is the smallest positive integer r such that M is the tropical sum of r matrices of Barvinok rank one. As in the classical case, a matrix has Barvinok rank one if it is the tropical outer product of two vectors.

The general relationship between the tropical, Kapranov, and Barvinok ranks is described in the following theorem, which will be the main result of this paper.

Theorem 1.5 ([2], Theorem 1.4; [3], Theorem 5.3.4)**.** *For a matrix M with entries in the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, we have the inequalities*

$$\text{Trank}(M) \leq \text{Krank}(M) \leq \text{Brank}(M).$$

Both inequalities may be strict.

This paper will work up to a proof of this result by examining the structure of each of these three notions of rank. We will need ideas from both tropical geometry and matroid theory, and these concepts are introduced in Sections 2 and 4, respectively. Section 3 analyzes the Barvinok and Kapranov ranks of a matrix and includes most of the proof of Theorem 1.5, lacking only the strictness in the first inequality. Lastly, Section 5 looks at the tropical rank of zero-one matrices and provides an example of a matrix M where the tropical rank of M is strictly less than the Kapranov rank of M, thus completing the proof of Theorem 1.5

This paper is largely based on [2] and Section 5.3 of [3], and most theorems and proofs come from those sources.

Finally, I would like to thank the Mathematics department at Oberlin College for teaching me these past four years, and I would especially like to thank my honors adviser Professor Susan Colley for suggesting this topic and helping me write this paper.

2. Tropical Preliminaries

In order to better understand the various ranks of an $m \times n$ matrix M over the tropical semiring, we first need to develop some tools from tropical geometry. To begin with, we define *tropical projective space*, referred to as the *tropical projective torus* in [3]. To introduce this, consider the following.

Example 2.1. Let " $\sim$ " be the equivalence relation on $\mathbb{R}^n$ defined by

$$(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n) \text{ if and only if } (a_1, \ldots, a_n) = (\lambda \odot b_1, \ldots, \lambda \odot b_n)$$

for some $\lambda \in \mathbb{R}$. Then, for instance,

$$(2, 3, 4) \sim (-2, -1, 0), \text{ and, more generally, } (a_1, a_2, \ldots, a_n) \sim (a_1 - a_n, a_2 - a_n, \ldots, 0).$$

Under this equivalence relation a point in $\mathbb{R}^n$ may be identified with a point in $\mathbb{R}^n$ with last coordinate equal to zero.

With Example 2.1 as motivation, define $\mathbf{1}$ to be the vector $(1, 1, \ldots, 1)$ (so that $\mathbb{R}\mathbf{1} = \{(a, \ldots, a) : a \in \mathbb{R}\}$), and define *tropical projective space* as follows.

Definition 2.2. *Tropical projective space*, denoted $\mathbb{TP}^{n-1}$, is the $(n-1)$-dimensional quotient space $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ under the equivalence relation " $\sim$ " above. As in Example 2.1, identify points in $\mathbb{TP}^{n-1}$ with the corresponding point in $\mathbb{R}^n$ with last coordinate zero.

One way that $\mathbb{TP}^{n-1}$ emerges in tropical geometry is through *Gröbner complexes.* A Gröbner complex G is a polyhedral complex in $\mathbb{R}^n$ that always contains $\mathbb{R}\mathbf{1}$ in its lineality space. It therefore makes sense to identify G with its image in $\mathbb{R}^n/\mathbb{R}\mathbf{1} = \mathbb{TP}^{n-1}$ (see Section 2.5 of [3]). We mainly work over $\mathbb{TP}^{n-1}$ in this paper.

Next, in order to properly define the Kapranov rank of a matrix, we need the notion of a *tropical linear space.* In what follows, $\tilde{K}$ is the field of formal power series over $\mathbb{C}$ as before, although, more generally, $\tilde{K}$ can be any algebraically closed field of characteristic zero with nontrivial valuation.

Definition 2.3. Let I be an ideal in $\tilde{K}[x_1, \ldots, x_n]$ that is generated by affine-linear forms $a_1x_1 + \ldots + a_nx_n + b$. Then I gives rise to a *tropical linear space*, written $\mathcal{T}(I)$, given by

$$\mathcal{T}(I) = \{(\text{val}(u_1), \ldots, \text{val}(u_n)) : (u_1, \ldots, u_n) \in V(I)\} \subset \mathbb{R}^n,$$

where $V(I)$ is the ordinary variety of I in the algebraic torus $(\tilde{K}^*)^n$. The dimension of $\mathcal{T}(I)$ is its topological dimension which, in this case, is $n - r$, where r is the number of minimal generators of I.

In fact, a tropical linear space is a specific type of *tropical variety*, which is defined as the component-wise valuation of a classical variety. That is, if I is an arbitrary ideal in $\tilde{K}[x_1, \ldots, x_d]$ and $V(I)$ is the variety of I, then the tropical variety of I is

$$\mathcal{T}(I) = \{(\text{val}(u_1), \ldots, \text{val}(u_n)) : (u_1, \ldots, u_n) \in V(I)\} \subset \mathbb{R}^n,$$

exactly as above. An important theorem, known as the "fundamental theorem of tropical algebraic geometry" (Theorem 3.2.5 of [3]), proves the equivalence of three different definitions of a tropical variety. For our purposes, a tropical variety will just be the image of a classical variety under the map

$$\text{val} : (\tilde{K}^*)^n \to \mathbb{R}^n, (\alpha_1, \ldots, \alpha_n) \mapsto (\text{val}(\alpha_1), \ldots, \text{val}(\alpha_n)).$$

By Theorem 3.1 of [2], the dimension of $\mathcal{T}(I)$ is equal to the dimension of $V(I)$.

The obvious way of defining a tropical linear space — as all tropical linear combinations of some given set of vectors — actually forms a different object in tropical geometry. The following definition is important in understanding the geometry involved in the Barvinok rank of a matrix.

Definition 2.4. The *tropical convex hull* of $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, where each $\mathbf{v}_i$ is in $\mathbb{R}^n$, is

$$\bigoplus_{i=1}^{n} a_i \odot \mathbf{v}_i = a_1 \odot \mathbf{v}_1 \oplus \ldots \oplus a_n \odot \mathbf{v}_n, \text{ for } a_1, \ldots, a_n \in \mathbb{R}^n.$$

That is, the tropical convex hull of V is all tropical linear combinations of vectors in V.

Example 2.5. Consider the three points $(4, 0, 0)$, $(-2, -2, 5)$, and $(1, 2, 1)$ in $\mathbb{R}^3$. In $\mathbb{TP}^2$ we may equate these points with $(4, 0, 0)$, $(-7, -7, 0)$, and $(0, 1, 0)$, respectively. Then these three points lie on a tropical line through the origin in $\mathbb{TP}^2$, which is their convex hull; see Figure 1.

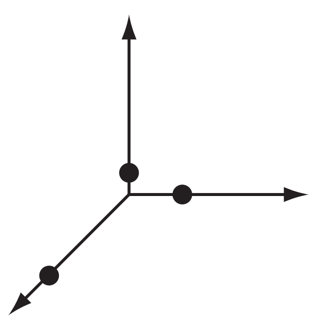

FIGURE 1. The points $(4, 0)$, $(-7, -7)$, and $(0, 1)$ lie on a tropical line through the origin.

These definitions provide the main tools we need from tropical geometry. We can now begin to prove the second inequality, that the Barvinok rank of a matrix is greater than or equal to its Kapranov rank.

3. Barvinok Rank and Kapranov Rank

To begin studying Barvinok rank, consider the following example.

Example 3.1. Let M be the following matrix:

$$M = \begin{pmatrix} 4 & 1 & -2 \\ 0 & 2 & -2 \\ 0 & 1 & 5 \end{pmatrix}.$$

Note that M has the points from Example 2.5 as columns. As the following equation shows, $\operatorname{Brank}(M) \leq 3$:

$$\begin{pmatrix} 4 & 1 & -2 \\ 0 & 2 & -2 \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \\ 30 \end{pmatrix} \odot \begin{pmatrix} 4 & 1 & -2 \end{pmatrix} \oplus \begin{pmatrix} 30 \\ 0 \\ 30 \end{pmatrix} \odot \begin{pmatrix} 0 & 2 & -2 \end{pmatrix} \oplus \begin{pmatrix} 30 \\ 30 \\ 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 5 \end{pmatrix}.$$

As the next theorem will show, the Barvinok rank of M is exactly 3 and it is no accident that the columns of M lie in the tropical convex hull of $\operatorname{Brank}(M) = 3$ points. Also, note that M is tropically singular — the minimum in the expression in Definition 1.1 for the tropical determinant, -1, is achieved twice — and so $\operatorname{Trank}(M) \leq 2$.

The following theorem relates the Barvinok rank of a matrix to tropical convex hulls and tropical matrix multiplication.

Theorem 3.2 ([2], Proposition 2.1; [3], Proposition 5.3.5)**.** *For a matrix $M \in \mathbb{R}^{m\times n}$, the following are equivalent:*

1. *The Barvinok rank of M is at most r;*
2. *The columns of M lie in the tropical convex hull of r points in $\mathbb{TP}^{n-1}$;*
3. *There exist matrices $X \in \mathbb{R}^{m\times r}$ and $Y \in \mathbb{R}^{n\times r}$ such that $M = X \odot Y^T$.*

Proof. Write $M_1, \dots, M_n \in \mathbb{R}^m$ for the columns of M and let $X_1, \dots, X_r \in \mathbb{R}^m$ and $Y_1, \dots, Y_r \in \mathbb{R}^n$ be the columns of generic matrices $X \in \mathbb{R}^{m\times r}$ and $Y \in \mathbb{R}^{n\times r}$. Consider three equations relating M to X and Y:

1. $M = X_1 \odot Y_1^T \oplus \dots \oplus X_r \odot Y_r^T$;
2. $M_j = Y_{j1} \odot X_1 \oplus \dots \oplus Y_{jr} \odot X_r$ for all $1 \leq j \leq n$;
3. $M = X \odot Y^T$.

Here Y_{ij} represents the element in row i, column j of Y, as usual. Proving the theorem reduces to proving the equivalence of these three expressions, for (1) exhibits M as the sum of r Barvinok-rank-one matrices, while (2) writes each column of M as a tropical linear combination of r vectors in $\mathbb{R}^n$, where the elements Y_{ij} are the scalars.

To see the equivalence of these three equations, look at each one term by term. In (1), by using the fact that $[X_k \odot Y_l^T]_{ij} = X_{ik} + Y_{jl}$ for all k, l, we have

$$M_{ij} = [X_1 \odot Y_1^T \oplus \dots \oplus X_r \odot Y_r^T]_{ij} = \min\{X_{i1} + Y_{j1}, X_{i2} + Y_{j2}, \dots, X_{ir} + Y_{jr}\},$$

while from (2), we see that

$$M_{ij} = [Y_{j1} \odot X_1 \oplus \ldots \oplus Y_{jr} \odot X_r]_i = \min\{Y_{j1} + X_{i1}, Y_{j2} + X_{i2}, \ldots, Y_{jr} + X_{ir}\}.$$

Finally, (3) yields

$$M_{ij} = \text{row}_i(X) \odot \text{column}_j(Y^T) = \min\{X_{i1} + Y_{j1}, X_{i2} + Y_{j2}, \ldots, X_{ir} + Y_{jr}\},$$

because, of course, $\text{column}_j(Y^T) = \text{row}_j(Y)$. Therefore (1), (2), and (3) are equivalent, proving the theorem.

□

In light of Theorem 3.2, we can see that the matrix M from Example 3.1 has Barvinok rank exactly 3 since its columns cannot fall in the tropical convex hull of just 2 points.

In order to prove results about Kapranov rank, we need to define a special ideal.

Definition 3.3. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates. Define the ideal J_r in $\tilde{K}[x_{11}, x_{12}, \ldots, x_{mn}]$ to be that generated by all $(r+1) \times (r+1)$ minors of X, which are just formal polynomials.

The significance of J_r is that an $m \times n$ matrix M is in $V(J_r)$ if and only if all of its $(r+1) \times (r+1)$ minors vanish; that is, $M \in V(J_r)$ if and only $\text{rank}(M) \le r$. This ideal appears in the following powerful result about Kapranov rank.

Theorem 3.4 ([2], Theorem 3.3; [3], Theorem 5.3.11)**.** *Let $M = (m_{ij})$ be an $m \times n$ matrix with real entries. The following are equivalent:*

1. *The Kapranov rank of M is at most r;*
2. *M is in $\mathcal{T}(J_r) = \{(\text{val}(u_{11}), \ldots, \text{val}(u_{mn})) : (u_{11}, \ldots, u_{mn}) \in V(J_r)\}$;*
3. *There exists an $m \times n$ matrix $F = (f_{ij})$ with entries in $\tilde{K}^*$ of rank at most r such that, for all i and j, $\text{val}(f_{ij}) = m_{ij}$.*

The matrix F in (3) of Theorem 3.4 is called a *lift* of M, and we write $\text{val}(F) = M$.

Proof. That (2) and (3) are equivalent is a result of the way we have defined tropical varieties, for a matrix in $(\tilde{K}^*)^{m \times n}$ is in $V(J_r)$ if and only if its rank is at most r, as noted above.

Now we show that (1) implies (3), and vice versa. First suppose that $\text{Krank}(M) \le r$, so that the columns of M are contained in some tropical linear space L of dimension r. This means that there is some linear ideal $I \subset \tilde{K}[x_1, \ldots, x_m]$ such that $L = \mathcal{T}(I)$. That the columns of M are in $\mathcal{T}(I)$ implies that each column M_j of M lifts to a $F_j \in (\tilde{K}^*)^m$ such that $F_j \in V(I)$. Let the matrix F be the $m \times n$ matrix over $\tilde{K}$ with $\text{column}_j(F) = F_j$ for all j. Then F is a lift of M and its column space is contained in $V(I)$. But the dimension of $V(I)$ is equal to the dimension of $\mathcal{T}(I)$ by Theorem 3.1 of [2], and therefore $\text{rank}(F) \le r$, so (1) implies (3).

Now assume that there exists $F \in (\tilde{K}^*)^{m\times n}$ of rank at most r such that $\text{val}(F) = M$. Let L' be the r-dimensional linear subspace of $(\tilde{K}^*)^m$ spanned by the columns of F. There is then a linear ideal I' in $\tilde{K}[x_1, \dots, x_m]$ such that $V(I') = L'$. Then each column of M is contained in $\mathcal{T}(I')$. As $\dim(\mathcal{T}(I')) = \dim(V(I')) = r$, the columns of M therefore lie in a tropical linear space of dimension r, so $\text{Krank}(M) \leq r$, as desired. This completes the proof. □

As an immediate corollary of Theorem 3.4, we get that the Kapranov rank of M is the least possible rank of any lift of M.

Example 3.5. Consider the 3×3 matrix over $\mathbb{C}\{\{t\}\}$

$$F = \begin{pmatrix} t^4 & t & t^{-2} \\ 0 & t^2 & t^{-2} \\ 0 & t & t^5 \end{pmatrix}.$$

We then have that $\text{val}(F)$ is the matrix M from Example 3.1, and $\text{rank}(F) = 3$, so $\text{Krank}(M) \leq 3$ by Theorem 3.4. Note that this does not show that the Kapranov rank of M is equal to three, as there might exist a lift of M with lower rank.

Besides providing a means for finding the Kapranov rank of a matrix, Theorem 3.4 allows us to see that the notions of Kapranov rank one and Barvinok rank one are actually the same.

Theorem 3.6. *A matrix $M = (m_{ij})$ in $\mathbb{R}^{m\times n}$ has Kapranov rank one if and only if it has Barvinok rank one.*

Proof. Suppose M has Kapranov rank one. By Theorem 3.4, this means that M has a lift $F = (f_{ij}) \in V(J_1)$, which in turn means that $f_{ij}f_{kl} = f_{il}f_{kj}$ for all i, j, k, and l, using the definition of J_1. Applying valuations to this yields that $m_{ij} + m_{kl} = m_{il} + m_{kj}$ for all i, j, k, and l.

Now we claim that M has this property if and only if there are real row vectors $X = (x_1, \dots, x_m)$ and $Y = (y_1, \dots, y_n)$ such that $m_{ij} = x_i + y_j = x_i \odot y_j$ for all i and j. One direction is easy to see, for if such X and Y exist then certainly, for all i, j, k, and l,

$$m_{ij} + m_{kl} = x_i + y_j + x_k + y_l = x_i + y_l + x_k + y_j = m_{il} + m_{kj}.$$

For the other direction, suppose that indeed $m_{ij} + m_{kl} = m_{il} + m_{kj}$ for all i, j, k, and l. This would then imply that $m_{ij} - m_{il} = m_{kj} - m_{kl}$. From this, we can build appropriate X and Y by letting $X = (m_{11}, m_{21}, \dots, m_{m1})$ and setting $y_j = m_{ij} - m_{i1}$ for all $j = 1, \dots, n$. As $m_{ij} - m_{i1} = m_{kj} - m_{k1}$ for any i and k, each y_j is uniquely defined. Furthermore, $x_i + y_j = m_{i1} + m_{ij} - m_{i1} = m_{ij}$ for all i and j, proving the claim.

Therefore, if M has Kapranov rank one, then $m_{ij} = x_i \odot y_j$ for all i and j and some X and Y as above, which happens if and only if $M = X^T \odot Y$. Put differently, M has Barvinok rank one.

Note that if M has Barvinok rank one, then $M = X^T \odot Y$ lifts to a rank one matrix in $\tilde{K}$ by sending m_{ij} to $t^{m_{ij}}$, so a Barvinok rank one matrix has Kapranov rank one. □

We can now prove the second inequality in Theorem 1.5.

Theorem 3.7 ([2], Proposition 3.6; [3], Proposition 5.3.15)**.** *For any matrix* $M \in \mathbb{R}^{m\times n}$, $\mathrm{Krank}(M) \leq \mathrm{Brank}(M)$.

Proof. Suppose that M has Barvinok rank r, so that we can write

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_r,$$

where each M_i has Barvinok rank one. Then, by Theorem 3.6, each M_i has Kapranov rank one, so Theorem 3.4 implies that M_i has a lift F_i in $(\tilde{K}^*)^{m\times n}$ of rank one for all i. By multiplying each F_i by scalars with zero valuation, we can ensure that there is no cancellation in the sum $F = F_1 + \ldots + F_r$. This then means that $\mathrm{val}(F) = M$. Furthermore, over $\tilde{K}$, we can use classical linear algebra to say that since F can be written as the sum of r rank one matrices, we have that $\mathrm{rank}(F) \leq r$. Therefore Theorem 3.4 gives that $\mathrm{Krank}(M) \leq r$. □

Theorem 3.7 shows that the Kapranov rank of a matrix is always at most its Barvinok rank. That this inequality may be strict is shown by an example.

Example 3.8 ([2], Example 3.5; [3], Example 5.3.14)**.** For $n \geq 3$, let C_n be the classical identity matrix, that is,

$$C_n = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

By Proposition 2.2 of [2] and Theorem 5.3.9 of [3], the Barvinok rank of C_n increases logarithmically with n and so, for large enough n, the Barvinok rank of C_n is not one and hence, by Theorem 3.6, the Kapranov rank of C_n is not one. For such n, the Kapranov rank must then be at least two. Fix distinct, nonzero scalars $a_3, a_4, \ldots, a_n$ with zero valuation, and define F_n over $\mathbb{C}\{\{t\}\}$ as follows:

$$F_n = \begin{pmatrix} t & 1 & t+a_3 & t+a_4 & \ldots & t+a_n \\ 1 & t & 1+a_3t & 1+a_4t & \ldots & 1+a_nt \\ t-a_3 & 1 & t & t-a_3+a_4 & \ldots & t-a_3+a_n \\ t-a_4 & 1 & t-a_4+a_3 & t & \ldots & t-a_4+a_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t-a_n & 1 & t-a_n+a_3 & t-a_n+a_4 & \ldots & t \end{pmatrix}.$$

Note that the only entries of F_n with nontrivial valuation are the terms on the diagonal (which have valuation one), so $\mathrm{val}(F_n) = C_n$. Furthermore, the first two columns of F_n

are linearly independent, but for every $j \geq 3$, we have $\text{column}_j(F_n) = \text{column}_1(F_n) + a_j\text{column}_2(F_n)$. It follows that $C_n = \text{val}(F_n)$ has Kapranov rank two independent of n. The Barvinok rank of C_n increases with n, so for large enough n values, $\text{Krank}(C_n) < \text{Brank}(M)$.

We used Theorem 3.4 to show the second inequality in Theorem 1.5. In fact, Theorem 3.4 is strong enough to show the other inequality in Theorem 1.5 as well.

Theorem 3.9 ([2], Proposition 4.1; [3], Proposition 5.3.17)**.** *If M is any matrix in $\mathbb{R}^{m\times n}$, then* $\text{Trank}(M) \leq \text{Krank}(M)$.

Proof. Any $r \times r$ tropically non-singular minor of M lifts to a non-singular matrix over $\tilde{K}$ because the unique minimum in the tropical determinant of M corresponds to a leading exponent that occurs only once in the determinant of the lift and thus does not cancel. Therefore, if $\text{Trank}(M) = r$, then any lift of M has rank at least r. From this Theorem 3.4 implies that $\text{Trank}(M) = r \leq \text{Krank}(M)$. □

To make the first statement in Theorem 3.9 more concrete, consider the following example.

Example 3.10. Define M as

$$M = \begin{pmatrix} 1 & -2 \\ 1 & 5 \end{pmatrix},$$

so that M is a 2×2 tropically non-singular minor of the matrix in Example 3.1. The unique minimum in the tropical determinant of M is -1. Let F be the following lift of M:

$$F = \begin{pmatrix} (1+i)t & \pi t^{-2} \\ (1-i)t & e^e t^5 \end{pmatrix}.$$

The classical determinant of F is $(1+i)e^e t^6 - (1-i)\pi t^{-1}$. This determinant cannot equal zero as the lowest exponent, -1, appears only once and so cannot cancel. This corresponds to -1 as the unique minimum in the tropical determinant of M.

In order to prove that the inequality in Theorem 3.9 may be strict, we need some elementary ideas from matroid theory. The next section quickly develops the key concepts we need.

4. Basics of Matroids

A matroid is a generalization of the notion of linear independence in linear algebra, and the defining axioms of a matroid can be stated in several different but equivalent ways: in terms of *independent sets*, in terms of *bases*, or in terms of *circuits*. We will use the definition in terms of *independent sets* from Section 1.1 of [4].

Definition 4.1. A *matroid* $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$, where E (called the *ground set*) is a finite set and $\mathcal{I}$ is a collection of subsets of E that satisfy three axioms:

(1) $\emptyset \in \mathcal{I}$;
(2) If $I \in \mathcal{I}$ and $I' \subset I$, then $I' \in \mathcal{I}$;
(3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element e in $I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The sets in $\mathcal{I}$ are called the *independent sets* of $\mathcal{M}$. A set in $\mathcal{I}$ that is maximal with respect to inclusion is called a *basis* for $\mathcal{M}$, and the collection of all bases is denoted $\mathcal{B}$. It turns out that all bases of $\mathcal{M}$ have the same size (Lemma 1.2.1 of [4]), which is called the *rank* of $\mathcal{M}$. A minimal dependent set, called a *circuit* of $\mathcal{M}$, is a set not in $\mathcal{I}$ but all of whose proper subsets land in $\mathcal{I}$. The collection of all circuits of $\mathcal{M}$ is written $\mathcal{C}$.

It is a fact (Proposition 1.1.1 of [4]) that the columns of an $m \times n$ matrix A over a field F give rise to a matroid by taking E as the set of column labels and $\mathcal{I}$ as the subsets of E for which the corresponding columns of A are linearly independent in the vector space F^m. Such a matroid is called the *vector matroid* of A.

Example 4.2. For the 3×3 classical identity matrix over $\mathbb{R}$,

$$C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have $E = \{1, 2, 3\}$ and $\mathcal{I} = \mathcal{P}(\{1, 2, 3\})$, that is, the power set of $\{1, 2, 3\}$, as any subset of $\{e_1, e_2, e_3\}$ (standard basis for $\mathbb{R}^3$) forms a linearly independent set in $\mathbb{R}^3$. In this case, $\mathcal{B} = \{\{1, 2, 3\}\}$ and $\mathcal{C} = \emptyset$.

Informally, two matroids are *isomorphic* if their ground sets have the same size and they have the same independent set structure. To make this more precise, let $E(\mathcal{M})$ denote the ground set of a matroid $\mathcal{M}$. Two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to be *isomorphic* if there exists a bijection ψ from $E(\mathcal{M}_1)$ to $E(\mathcal{M}_2)$ such that, for all $Y \subset E(\mathcal{M}_1)$, Y is a basis for $\mathcal{M}_1$ if and only if $\psi(Y)$ is a basis for $\mathcal{M}_2$.

The notion of isomorphisms between matroids is mainly important for us in the following context.

Definition 4.3. Let $\mathcal{M}$ be a matroid. If $\mathcal{M}$ is isomorphic to the vector matroid of some matrix D over a field F, then $\mathcal{M}$ is said to be *representable* over F, and D is called a *representation* of $\mathcal{M}$ over F.

In particular, matroids that are not representable over a field of characteristic zero will be significant for us as they furnish examples of matrices where the tropical rank is strictly less than the Kapranov rank. We will take as a fact that the so-called *Fano matroid* is representable over a field if and only the field has characteristic two; see

Proposition 6.4.8 of [4]. In particular, this means that the Fano matroid cannot be represented over $\mathbb{C}$.

The last concept we need from matroid theory is that of dual matroids.

Definition 4.4. Let $\mathcal{M}$ be a matroid and let $\mathcal{B}(\mathcal{M})$ be its set of bases. The set

$$\mathcal{B}^*(\mathcal{M}) = \{E(\mathcal{M}) - B : B \in \mathcal{B}(\mathcal{M})\}$$

forms the set of bases for a matroid $\mathcal{M}^*$ called the *dual* of $\mathcal{M}$. The bases of $\mathcal{M}^*$ are called *cobases* of $\mathcal{M}$, the circuits of $\mathcal{M}^*$ are called *cocircuits* of $\mathcal{M}$, and so forth.

These are all the tools we need from matroid theory to prove that the first inequality in Theorem 1.5 may be strict.

5. Zero-One Matrices and Tropical Rank

We begin this section with two definitions.

Definition 5.1. The *support* of a vector in $\mathbb{R}^m$ is the set of the vector's zero coordinates.

Definition 5.2. The *support poset* of a tropical matrix M is the set of all unions of supports of columns of M, partially ordered with respect to inclusion.

Example 5.3. The tropical vectors

$$\mathbf{v}_1 = (1\ 0\ 1\ 0\ 1)^T, \mathbf{v}_2 = (0\ 1\ 0\ 1\ 0)^T, \mathbf{v}_3 = (1\ 1\ 1\ 1\ 1)^T$$

have supports $\{2, 4\}$, $\{1, 3, 5\}$, and $\emptyset$, respectively. If M is the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

then the support poset of M is $\{\emptyset, \{1, 3, 5\}, \{2, 4\}, \{1, 2, 3, 4, 5\}\}$.

These notions come into play in the next theorem about tropical rank. A matrix $M = (m_{ij})$ is a *zero-one* matrix if $m_{ij} = 0$ or $m_{ij} = 1$ for all i and j. This is one of two more involved proofs in the paper.

Theorem 5.4 ([2], Proposition 4.3; [3], Proposition 5.3.18)**.** *The tropical rank of a zero-one matrix M with no column of all ones is equal to the maximum length of a chain in the support poset of M.*

After the proof of Theorem 5.4, we will work through an example to demonstrate some of the key ideas in the proof.

Proof. To begin with, we claim that there is no loss of generality in assuming that every union of supports of columns in M is indeed the support of some column in M. To see this, note that the tropical sum of columns yields a column whose support is the union of the supports of the summands. That is, if $N = M_{i_1} \oplus M_{i_2} \oplus \ldots \oplus M_{i_j}$ where the M_{i_k} are columns of M, then the support of N is the union of the supports of the M_{i_k}, for N has a zero in coordinate l if and only if some M_{i_k} has a zero in coordinate l. Now, appending N to M does not change the tropical convex hull of the columns of M as N is already in this tropical convex hull by construction. By Theorem 4.2 of [2], it therefore follows that appending N to M does not alter the tropical rank of M.

Thus, if the support poset of M contains a chain of length r, we may assume that M has r columns representing this chain. Note that tropical determinants are invariant upon switching two rows or two columns. Number these r columns from 1 to r relative to inclusion of their supports, so that the support of column n contains the support of column $n+1$. Column 1 must have at least r zeroes as M has no column of all ones, which corresponds to $\emptyset$ not appearing in any chain of the poset. Furthermore, there must exist $r-1$ zeroes in column 2, $r-2$ zeroes in column 3, and so forth. Similarly, since the supports of the columns properly contain one another, column 2 must have a one, column 3 must have two ones, and so on.

Again because the supports of the columns properly contain one another, there is an index i_r such that column r has a one in row i_1 and column $r-1$ has a zero in row i_r. Similarly, we can find such a row i_n for all pairs of columns n and $n-1$, down to columns 2 and 1. Extracting these $r-1$ rows and adding a row of zeroes at the bottom (one must exist, as column r has at least one zero and any zero persists from column n to column $n-1$), we create an $r \times r$ minor with zeroes on and below the diagonal and ones above the diagonal, which is tropically non-singular. Therefore, a chain of length r in the support poset of M corresponds to a tropically non-singular $r \times r$ minor of M.

For the other direction, suppose that M has a tropically non-singular $r \times r$ minor N. Assume without loss of generality that the unique minimum in the expression of the tropical determinant of N occurs on the main diagonal. This minimum is at most one, for if n_{ii} and n_{jj} are both one, then switching them for n_{ij} and n_{ji} cannot increase the sum — remember that N is a zero-one matrix — so the minimum cannot be unique in this case.

Suppose that this minimum is zero. We may construct a directed graph on r vertices from the minor N by drawing an oriented edge from vertex i to j if $n_{ij} = 0$. Moreover, this graph is acyclic since a cycle would correspond to a string of zeroes off the diagonal that we could permute with the diagonal elements to get a sum no greater than the sum on the diagonal. Thus this graph admits an ordering, and rearranging the rows of N according to this ordering yields a matrix with zeroes on and below the diagonal and ones above the diagonal — column 1 ordered before all others means column 1 has all zeroes, column 2 ordered before all others except column 1 means column 2 has all zeroes except in the first coordinate, and so on. The supports of the columns form a chain of length r, so M contains a chain of length r in its support poset.

Finally, consider the case when the unique minimum in N is one and let n_{ii} be the element on the diagonal with value one. Then all of row i must be ones, for if n_{ij} is zero, then switching n_{ij} and n_{ji} for n_{ii} and n_{jj} cannot not increase the sum, contradicting the assumption that N is tropically non-singular. Changing row i to a row of all zeroes does not change that N is tropically non-singular and it adds exactly one element to the support of every column of N. Now N is of the form of the last paragraph and so has a chain of supports of length r, as desired.

□

Example 5.5. To make the arguments of Theorem 5.4 more concrete, consider this matrix M:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The support poset of M is $\{\{1\},\{3\},\{1,3\},\{1,2,3\}\}$, so, by Theorem 5.4, we should find $\text{Trank}(M) = 3$. First, to make M fit the form of Theorem 5.4, add a column corresponding to $\{1,3\}$ so that every union of supports in the support poset of M is represented by a column of M; doing so does not change the tropical rank of M.

$$M' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Columns one, two, and four correspond to a chain of supports of length three. Ordering them as in the proof of Theorem 5.4, we get

$$M'' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

From here we pick a row in which column three has a one and column two has a zero — such a row must exist in general since the support of column two contains the support of column three. In this case, we take row 1. Now find a row for which column two has a one but column one has a zero; here it is row 2. Now put these rows in reverse order and choose a row in which column three has a zero (row 3) to add to the bottom to get a minor N in the form of the proof of Thereom 5.4.

$$N = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we found a 3×3 minor of M with zeroes on and below the main diagonal and ones above it, as Theorem 5.4 guarantees.

As an application of Theorem 5.4, we see that the classical identity matrix C_n has tropical rank two for all n. The support poset of C_n consists of $\{1, \ldots, n\}$ and every $(n-1)$-subset of $\{1, \ldots, n\}$, so all chains in the support poset have length two.

Zero-one matrices and matroids meet in the next construction. Recall that a cocircuit of a matroid $\mathcal{M}$ is a circuit (that is, minimally dependent set) of the dual matroid of $\mathcal{M}$.

Definition 5.6. For a matroid $\mathcal{M}$, the *cocircuit matrix* of $\mathcal{M}$, written $\mathcal{C}(\mathcal{M})$, is a zero-one matrix with rows indexed by the ground set of $\mathcal{M}$ and columns indexed by the cocircuits of $\mathcal{M}$. The entry $\mathcal{C}(\mathcal{M})_{ij}$ is 0 if and only if the i-th element of the ground set is in the j-th cocircuit.

The next two theorems demonstrate why this is an important concept: Cocircuit matrices provide examples where the tropical rank is strictly less than the Kapranov rank.

Theorem 5.7 ([2], Proposition 7.2; [3], Proposition 5.3.20)**.** *The tropical rank of the cocircuit matrix $\mathcal{C}(\mathcal{M})$ is equal to the rank of $\mathcal{M}$ as a matroid.*

Proof. By Corollary 1.2.6 of [4], if e is an element in any cocircuit of $\mathcal{M}$, then e is contained in some basis of $\mathcal{M}$. Then the length of any chain of unions of supports of cocircuits is at most equal to the rank of $\mathcal{M}$ and a chain of length rank($\mathcal{M}$) must exist, so applying Theorem 5.4 yields the result. □

Theorem 5.8. *The Kapranov rank of $\mathcal{C}(\mathcal{M})$ over $\mathbb{C}$ is equal to the rank of $\mathcal{M}$ if and only if $\mathcal{M}$ is representable over $\mathbb{C}$.*

This theorem is proven in more generality in Theorem 7.3 of [2] and Theorem 5.3.21 of [3].

Proof. First suppose that the Kapranov rank of $\mathcal{C}(\mathcal{M})$ is equal to the rank of $\mathcal{M}$ for $\mathcal{M}$ a matroid of rank r on the set $\{1, \ldots, m\}$, and suppose that $F \in \tilde{K}^{m \times n}$ is a rank r lift of $\mathcal{C}(\mathcal{M})$. (As before, $\tilde{K}$ is the field of formal power series over $\mathbb{C}$.) For each row $\mathbf{f}_i$ in F, let $\mathbf{v}_i \in \mathbb{C}^m$ be the vector of constant terms from $\mathbf{f}_i$. We will show that the matrix V formed by adjoining the row vectors $\mathbf{v}_i$ is a representation of $\mathcal{M}$ over $\mathbb{C}$. First of all, observe that the rank of V is at most r as any $\tilde{K}$-linear relation among the $\mathbf{f}_i$ becomes a $\mathbb{C}$-linear relation among the $\mathbf{v}_i$.

Now, suppose that $\{i_1, \ldots, i_r\}$ is a basis for $\mathcal{M}$; to prove that $\mathcal{M}$ and the vector matroid of V are isomorphic, we need to show that $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r}\}$ is then a basis for V (and conversely, as well). By Theorem 5.7, $\mathcal{C}(\mathcal{M})$ has tropical rank r and so, as in the proof of Theorem 5.4, we can find a tropically non-singular minor of $\mathcal{C}(\mathcal{M})$ using rows $i_1, \ldots, i_r$ with zeroes on and below the main diagonal and ones above it. The constant term lift of this minor is then lower-triangular (valuation one means no constant term) with nonzero values on the diagonal. This shows that $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r}$ are linearly independent. Given that $\text{rank}(V) \leq r$, it then follows that $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r}\}$ are a basis and that $\text{rank}(V) = r$.

Now suppose that $\{i_1, \ldots, i_r\}$ is not a basis for $\mathcal{M}$. There is then a cocircuit in $\mathcal{M}$ containing none of $i_1, \ldots, i_r$, in which case the column of that cocircuit has ones in each of those r rows. Therefore, each of $\mathbf{f}_{i_1}, \ldots, \mathbf{f}_{i_r}$ have zero constant term in that coordinate (as above, valuation one implies no constant term), which in turn means that $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r}$ are zero in that coordinate. This cocircuit is not empty (no cocircuit can be), so not all $\mathbf{f}_j$ have zero constant term in that coordinate and therefore not all $\mathbf{v}_j$ are zero in that coordinate. In particular, then, $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r}\}$ cannot be a basis for V, thus proving that V does in fact represent $\mathcal{M}$ over $\mathbb{C}$.

For the converse, suppose that $\mathcal{M}$ is representable, and assume that $\mathcal{M}$ has no single-element circuits ("loops"). This is no loss of generality, as such a circuit corresponds to a row of all ones in $\mathcal{C}(\mathcal{M})$ because the element in the circuit would not be contained in any cocircuit. Furthermore, a row of all ones does not raise $\text{Krank}(\mathcal{C}(\mathcal{M}))$ because every column of $\mathcal{C}(\mathcal{M})$ has at least one zero. Let $A \in \mathbb{C}^{m\times n}$ be such that the rows of A represent $\mathcal{M}$ and the sets of non-zero coordinates on the columns of A are the cocircuits of $\mathcal{M}$. Note that A has rank r as a representation of $\mathcal{M}$. Without loss of generality suppose that $\{1, \ldots, r\}$ form a basis for $\mathcal{M}$ and let A' be the first r rows of A. We can then write (classically)

$$A = \begin{pmatrix} C_r \\ D \end{pmatrix} A',$$

with C_r the classical identity matrix over $\mathbb{C}$ and D a $(m-r) \times r$ matrix. As $\mathcal{M}$ has no loops, A has no row of all zeroes, and therefore neither does D. Then, because no rows in either C_r or D are all zero, and because $\mathbb{C}$ is an infinite field, we can find some matrix $B' \in \mathbb{C}^{r\times n}$ such that

$$B = \begin{pmatrix} C_r \\ D \end{pmatrix} B'$$

has no zero entries. Finally, define $F \in \tilde{K}^{m\times n}$ to be

$$F = \begin{pmatrix} C_r \\ D \end{pmatrix} (A' + tB') = A + tB.$$

We see that $\operatorname{rank}(F) = r$, as there can be no cancellation between the terms appearing in the entries of $A+tB$. Consider $\operatorname{val}(F)$, which is a zero-one matrix because every entry in F is either a constant plus a value of t or just a value of t. Note that $[\operatorname{val}(F)]_{ij} = 0$ if and only if A_{ij} is nonzero. By construction of A, this happens exactly when element i of $\mathcal{M}$ is in the j-th cocircuit of $\mathcal{M}$; that is, $\operatorname{val}(F) = \mathcal{C}(\mathcal{M})$. □

The conclusion to be drawn from Theorems 5.7 and 5.8 is that if $\mathcal{M}$ is a matroid that cannot be represented over $\mathbb{C}$, then $\operatorname{Trank}(\mathcal{C}(\mathcal{M})) < \operatorname{Krank}(\mathcal{C}(\mathcal{M}))$. As mentioned in Section 4, the Fano matroid $\mathcal{M}$ cannot be represented over $\mathbb{C}$, so the following matrix has tropical rank strictly less than its Kapranov rank:

$$\mathcal{C}(\mathcal{M}) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

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