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Comparing Two Thickened Cycles: A Generalization of Spectral Inequalities

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COMPARING TWO THICKENED CYCLES: A GENERALIZATION OF SPECTRAL INEQUALITIES

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Abstract. Motivated by an effort to simplify the Watts-Strogatz model for small-world networks, we generalize a theorem concerning interlacing inequalities for the eigenvalues of the normalized Laplacians of two graphs differing by a single edge. Our generalization allows weighted edges and certain instances of self loops. These inequalities were first proved by Chen et. al in [2] but our argument generalizes the simplified argument given by Li in [8].

Contents

1. Introduction

Complex networks play an important role in a variety of disciplines, ranging from computer science, physics, sociology, and biology. One of the most significant classes of graphs are those that demonstrate "small-world" phenomenom; meaning that they are highly connected and display local clustering, but have a relatively small diameter. Several random graph models that exhibit small-world phenomena have been studied. In this paper, we will briefly examine the Watts-Strogatz model, and explore a slight variation in its construction.

1.1. **Watts-Strogatz Small World Network.**

The creation of the first small-world network model was motivated by an interest in introducing more complex structure into random graphs to capture natural phenomena in biological, technological and social networks [11]. This small-world network model, called the Watts-Stogatz random graph exhibits high clustering among local neighborhoods of vertices while maintining a short average path length.

To construct a graph G from the Watts-Strogatz Model, we begin with a base circulant graph on n vertices.

Definition 1.1. A *base circulant graph*, C_n^k, with jumps $1, ..., k$, is a graph on n vertices labelled $0, 1, ..., n-1$, where each node i is adjacent to the $2k$ nodes with labels $i \pm 1, ..., i \pm k \mod n$.

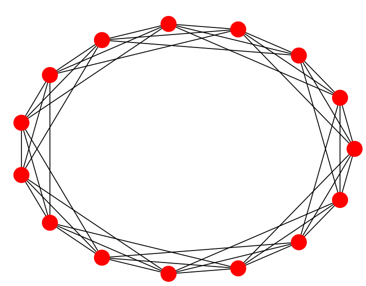


FIGURE 1. C_{15}^3; A circulant graph on 15 vertices with jumps $1, 2, 3$

Then for each edge of C_n^k, we leave one end fixed and with a given probability, p, move the other end to a different vertex chosen at random from the other $n-2$ vertices in G. We can consider these adjustments in edge endpoints as shortcuts, causing the average shortest path length of G to be shorter than it was before the random process. While the Watts-Strogatz model captures small-world phenomena, the process can be difficult to analyze since the additions of random edges and the deletions of base graph edges are not independent.

1.2. **The Base: An Alternative to the Circulant Graph.**

The Watts-Strogatz random graph is often used as a model in the study of complex networks [5]. The study of complex networks uses standard tools from random graphs that often rely on the independence of adding edges [10], [1]. Since the process with which random edges are added in the Watts-Strogatz model is not independent, computational difficulties arise. We will propose using a different type of thickened cycle as a base graph and simply add random shortcut edges; greater independence may result in a random graph exhibiting small world properties that is easier to analyze.

Definition 1.2. An (m, n) *cluster graph*, $T(m, n)$ is a collection of mn vertices labelled from 0 to $mn-1$, grouped into n clusters of m vertices such that every vertex in a cluster is in the same equivalence class $\mod n$. Two vertices of this graph, i and j are adjacent if and only if $(i-j) \mod n = 1$.

We use the notation $T(m, m)$ because this is essentially a thick cycle on m vertices.

In this paper, we will be comparing the cluster graph with the base circulant graph and analyzing their spectral properties as we begin to add random edges. In order to have the same number of vertices, edges, and vertex degree, we will be looking at the cluster graph with m clusters of m vertices, $T(m, m)$, and the circulant graph, $C^m_{m^2}$, on m^2 vertices with jumps $1, 2, \ldots, m$.

2. Geometric Properties of Base Circulant and Cluster Graphs

The graph theoretic vocabulary and definitions that we will use are standard and can be found in [4]. By construction, $C^m_{m^2}$ and $T(m, m)$ both have m^3 edges, m^2 vertices and every vertex has degree $2m$.

2.1. **Diameter.**

Definition 2.1. The *diameter* of a graph G is the greatest distance between any two vertices in G.

Theorem 2.2. *The diameter of the circulant graph, $C^m_{m^2}$ is $\lceil \frac{m}{2} \rceil$.*

Proof. Since $C^m_{m^2}$ is symmetric, choose any vertex, j. If the vertices are arranged clockwise by label, it is clear that the vertex furthest from j is the vertex labelled $j + m^2/2 \mod m^2$. To take the shortest path, we will want to take as many edges that connect vertices i and $i + m$ as possible. There are $\lfloor \frac{m}{2} \rfloor$ of them on the path between vertices j and $j + m^2/2 \mod m^2$. However, if m is odd, then $\frac{m^2}{2} \not\equiv 0 \mod m$. In this case, we must take one more edge to get to vertex $j + m^2/2 \mod m^2$. Therefore, the diameter is $\lceil \frac{m}{2} \rceil$. $\square$

Theorem 2.3. *The diameter of the cluster graph, $T(m,m)$ is given by $\lfloor \frac{m}{2} \rfloor$ when $m \geq 4$ and 2 when $m < 4$.*

Proof. If $m < 4$, this is clear. So let $m \geq 4$. Since the labelled vertices of $T(m,m)$ are grouped into clusters determined by their equivalence classes mod n and each of those clusters is completely connected to the two adjacent clusters, we can view the distances between vertices in two different clusters as the distance between the clusters themselves. Once we consider the vertices in each cluster to be lumped together, we reduce this problem to finding the diameter of a cycle on m vertices, which is just $\lfloor \frac{m}{2} \rfloor$.

$\square$

As $m \to \infty$, the diameters of the two graph families are both asymptotic to $\frac{m}{2}$ as $m \to \infty$.

2.2. **Average Shortest Path.**

Definition 2.4. The *average shortest path* of a graph G is the average number of edges on the shortest path between all pairs of vertices in the graph.

Theorem 2.5. *The average shortest path length of $C^m_{m^2}$ is given by*
(1)
$$\frac{m^2\left(m(\lceil \frac{m}{2} \rceil - 1)(\lceil \frac{m}{2} \rceil - 2) + \lceil \frac{m}{2} \rceil (m^2 - 1 \mod 2m)\right)}{2\binom{m^2}{2}} \sim \frac{m}{4}, \quad \textit{as } m \to \infty.$$

Proof. Consider an arbitrary vertex i. Via the symmetry in the circulant graph, vertex i has $2m$ neighbors, $2m$ vertices at distance 2, and so on, up to and including distance $\lceil \frac{m}{2} \rceil - 1$. These shortest path use edges that connect vertices i and $i + m$. There are an additional $(m^2 - 1 \mod 2m)$ vertices at distance $\lceil \frac{m}{2} \rceil$. Therefore, the average distance between any two vertices in the graph is given by,

$$\frac{m^2\left(2m\sum_{k=1}^{\lceil \frac{m}{2} \rceil - 1} k + \lceil \frac{m}{2} \rceil (m^2 - 1 \mod 2m)\right)}{2\binom{m^2}{2}}.$$

Using the formula for a sum of consecutive integers and simplifying, we get that the average shortest path is

$$\frac{m^2\left(m(\lceil \frac{m}{2} \rceil - 1)(\lceil \frac{m}{2} \rceil - 2) + \lceil \frac{m}{2} \rceil (m^2 - 1 \mod 2m)\right)}{2\binom{m^2}{2}}.$$

$\square$

Theorem 2.6. *The average shortest path length of $T(m,m)$ is given by*

$$\frac{m^2\left(2(m-1)+m(\frac{m}{2}-1)(\frac{m}{2}-2)+\frac{m^2}{2}\right)}{2\binom{m^2}{2}} \sim \frac{m}{4} \quad \textit{when } m \textit{ is even,} \tag{2}$$

and

$$\frac{m^2\left(2(m-1)+\lfloor\frac{m^2}{2}\rfloor(\lfloor\frac{m}{2}\rfloor-1)\right)}{2\binom{m^2}{2}} \sim \frac{m}{4} \quad \textit{when } m \textit{ is odd.} \tag{3}$$

Proof. Begin by choosing an arbitrary vertex j. Via the way the cluster graph is connected, it is clear that the distance between vertex j and vertex i, where $i \equiv j \mod m$ will be 2. Additionally, there are $2m$ vertices are every distance up to and including $\frac{m}{2}-1$. If m is even, there is only one cluster of m vertices at distance $\frac{m}{2}$. Therefore, we get that the average shortest path will be,

$$\frac{m^2\left(2(m-1)+2m\sum_{k=1}^{\frac{m}{2}-1}k+\frac{m^2}{2}\right)}{2\binom{m^2}{2}}.$$

When m is odd, there will be two clusters of vertices at distance $\lfloor\frac{m}{2}\rfloor$ and so we see that the average shortest path length in this case is,

$$\frac{m^2\left(2(m-1)+2m\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor}k\right)}{2\binom{m^2}{2}}.$$

From here, we can use the formula for a sum of consecutive integers to simplify the equation. ☐

Since all of the expressions are asymptotic to $\frac{m}{4}$, the difference between the average shortest path of the base circulant and cluster graphs will only differ by a constant as $m \to \infty$.

3. Spectral Properties of Base Circulant and Cluster Graphs

3.1. Graph Laplacians.

Now that we have seen some of the geometric properties of the base circulant and cluster graphs are similar, we can turn our attention to their spectral properties. Given a graph G on n vertices, there are several different ways of representing G in matrix form. Among these are adjacency matrices, Laplacians, and normalized Laplacians. While we will discuss all three, this paper will primarily be concerned with the eigenvalues of the normalized Laplacian.

Given the graph G, let d_i denote the degree of vertex i. We define the Laplacian, L, of G as follows:

$$L(i,j) = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

We can alternatively define L as $L = D - A$ where A is the adjacency matrix associated with G and $D = \mathrm{diag}(d_1, \ldots, d_n)$.

We define the normalized Laplacian of G as $\mathcal{L} = D^{-1/2} L D^{-1/2}$. Note that if $d_i = 0$, by convention, we define $D^{-1/2}(i,i) = 0$. Therefore, the entries of $\mathcal{L}$ are

$$\mathcal{L}(i,j) = \begin{cases} 1 & \text{if } i = j \text{ and } d_i \neq 0, \\ -\frac{1}{\sqrt{d_i d_j}} & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The normalized Laplacians of graphs are well studied and it can be shown that all eigenvalues of $\mathcal{L}(G)$ lie in $[0,2]$. For a more complete discussion on the eigenvalues of the normalized Laplacian, see [3].

3.2. **The Spectra of Base Circulant Graphs.**

Recall the circulant graph introduced Definition 1.1. The Laplacian for $C^m_{m^2}$ is a particular kind of matrix called a *circulant matrix*.

$$L\left(C^3_{15}\right) = \begin{bmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 0 & \ldots & 0 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & 1 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \ldots & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} \\ & & & & \vdots & & & & & \\ & & & & \vdots & & & & & \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 0 & 0 & \ldots & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 1 \end{bmatrix}$$

FIGURE 2. The normalized Laplacian for C^3_{15}, see Figure 3.

Definition 3.1. A *circulant matrix* is a matrix where each successive row is rotated one element to the right of the previous row.

The jth eigenvalue of an $n \times n$ circulant matrix, C, is given by

$$p_j = p\left(e^{\frac{2\pi i j}{m^2}}\right) = c_1 + c_2\left(e^{\frac{2\pi i j}{m^2}}\right) + ... + c_n\left(e^{\frac{2\pi i j}{m^2}}\right)^{n-1},$$

and $(c_1, ..., c_n)$ are the entries of the first row of C [9].

Using this, we can then show that the jth eigenvalue of $\mathcal{L}(C^m_{m^2})$ is given by

$$\lambda_j = 1 - \frac{1}{2m}\sum_{k=1}^{m}\left(e^{\frac{j2\pi i}{m^2}}\right)^k - \frac{1}{2m}\sum_{k=m^2-m}^{m^2-1}\left(e^{\frac{j2\pi i}{m^2}}\right)$$

$$= \frac{2m+1}{2m} - \frac{1}{2m}\frac{\sin\left((m+1/2)\frac{2\pi j}{m^2}\right)}{\sin(\frac{\pi j}{m^2})}. \tag{4}$$

The trigonometric part of the jth eigenvalue of $C^m_{m^2}$ actually quite well known [7].

Definition 3.2. The *Dirichlet kernel*, $D_m(x)$ is of the form, $D_m(x) = \frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}$. This function is bounded by $f(x) = \sin^{-1}\frac{x}{2}$, for any m.

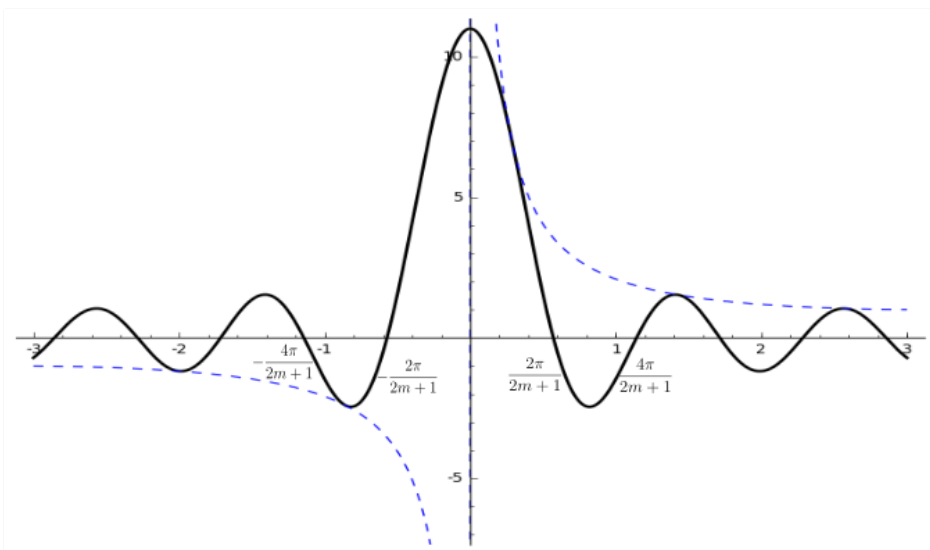


Figure 3. A plot of the Dirichlet kernel.

So now we see that the eigenvalues of $C^m_{m^2}$ are determined by evaluating an affine transformation of the Dirichlet kernel, $D_m(x)$, at the points where $x = \frac{2\pi j}{m^2}, \quad j = 0, \ldots, m-1$. When we plot the eigenvalues with respect to j, this is quite obvious.

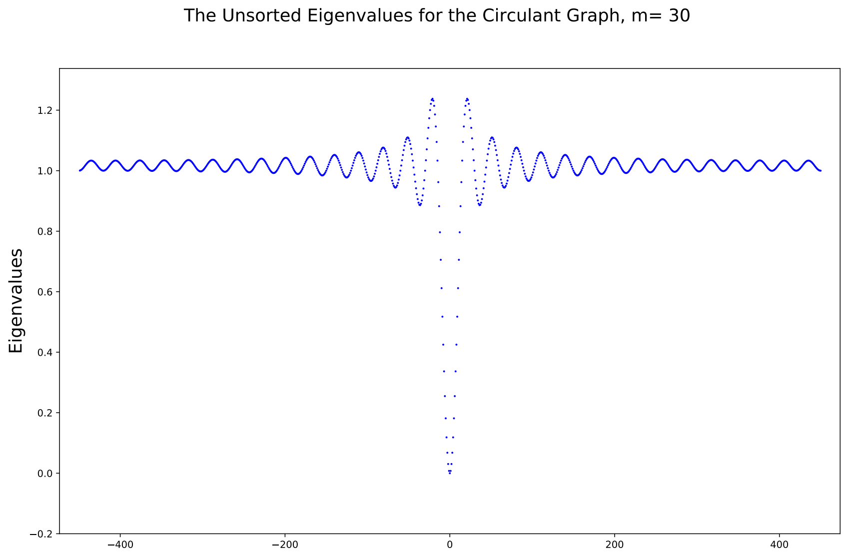


FIGURE 4. The eigenvalues of $C^{30}_{30^2}$.

3.3. **The Spectra of Cluster Graphs.**

Recall the cluster graph introduced in Definition 1.2.

Theorem 3.3. *Let G be an (m, m) cluster graph and H be the cycle on m vertices. Additionally, let $\lambda_0, ..., \lambda_{m-1}$ be the eigenvalues of $\mathcal{L}(H)$ where $\lambda_i = \frac{2m+1}{2m} - \frac{1}{2}\frac{\sin\left((3/2)\frac{2\pi j}{m}\right)}{\sin(\frac{\pi j}{m})}$. Then the eigenvalues of $\mathcal{L}(G)$ are $\lambda_0, \ldots, \lambda_{m-1}, \underbrace{1, \ldots, 1}_{m(m-1)}$.*

Proof. Consider the normalized Laplacian for G in the following way,

$$\mathcal{L}(G) = \begin{pmatrix} A' & A & A & & A \\ A & A' & A & & A \\ A & A & A' & & A \\ & \vdots & & \ddots & \\ & \vdots & & & \ddots \\ A & A & A & & A' \end{pmatrix}$$

where

$$A' = \begin{pmatrix} 1 & -\frac{1}{2m} & 0 & \cdots & 0 & -\frac{1}{2m} \\ -\frac{1}{2m} & 1 & -\frac{1}{2m} & & 0 & 0 \\ \vdots & & \ddots & & & \\ \vdots & & & \ddots & & \\ -\frac{1}{2m} & & & & -\frac{1}{2m} & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & -\frac{1}{2m} & 0 & \cdots & 0 & -\frac{1}{2m} \\ -\frac{1}{2m} & 0 & -\frac{1}{2m} & & 0 & 0 \\ \vdots & & \ddots & & & \\ \vdots & & & \ddots & & \\ -\frac{1}{2m} & & & & -\frac{1}{2m} & 0 \end{pmatrix}.$$

Note that A and A' are both $m \times m$ matrices and that $A' - A = I_n$.

Now having the normalized Laplacian of G defined in this way, we will use block operations to put $\mathcal{L}(G) - \lambda I_{m^2}$ in block diagonal form. First, we define B' as the following:

$$B' = \begin{pmatrix} 1-\lambda & -\frac{1}{2m} & \cdots & -\frac{1}{2m} \\ -\frac{1}{2m} & 1-\lambda & & -\frac{1}{2m} \\ \vdots & & \ddots & \\ -\frac{1}{2m} & & & 1-\lambda \end{pmatrix}.$$

With this definition, we now complete some elementary row and column operations to obtain,

$$\mathcal{L}(G) - \lambda I_{m^2} = \begin{pmatrix} I_m - \text{diag}(\lambda) & 0 & & A \\ 0 & I_m - \text{diag}(\lambda) & & A \\ & \vdots & \ddots & \\ 0 & 0 & & (m-1)A + B' \end{pmatrix}$$

It can be shown that,

$$(m-1)A + B' = \begin{pmatrix} 1-\lambda & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda & & -\frac{1}{2} \\ \vdots & & \ddots & \\ -\frac{1}{2} & & & 1-\lambda \end{pmatrix},$$

which is just the normalized Laplacian for a cycle on m vertices. Since a cycle on m vertices is a base circulant graph with a jump of 1, we can express the eigenvalues of this submatrix using Equation (4), meaning that λ_i is given by,

$$\lambda_i = \frac{2m+1}{2m} - \frac{1}{2}\frac{\sin\left((3/2)\frac{2\pi j}{m}\right)}{\sin(\frac{\pi j}{m})}, \quad i = 0, \ldots, m-1.$$

Since the eigenvalues of a block diagonal matrix are the eigenvalues of each block; we can conclude that $\mathcal{L}(G)$ has the eigenvalues of the cycle on m vertices. The eigenvalues for the rest of $\mathcal{L}(G)$ are given by the eigenvalues of $I_m - \text{diag}(\lambda)$ which will all have value exactly 1.

□

3.4. **A Comparison.**

Definition 3.4. Let G be a graph with n vertices and normalized Laplacian eigenvalues $\lambda_1, \ldots, \lambda_n$. Define the *spectral cumulative distribution function (spectral CDF)* $F_G : [0,2] \to [0,1]$ of G by

$$F_G(\lambda) = \frac{|\{i \,:\, \lambda_i < \lambda\}|}{n}.$$

When we plot the spectral CDF for the circulant and cluster graphs with $m = 30$, in Figure 5 it is apparent that they share a similar shape.

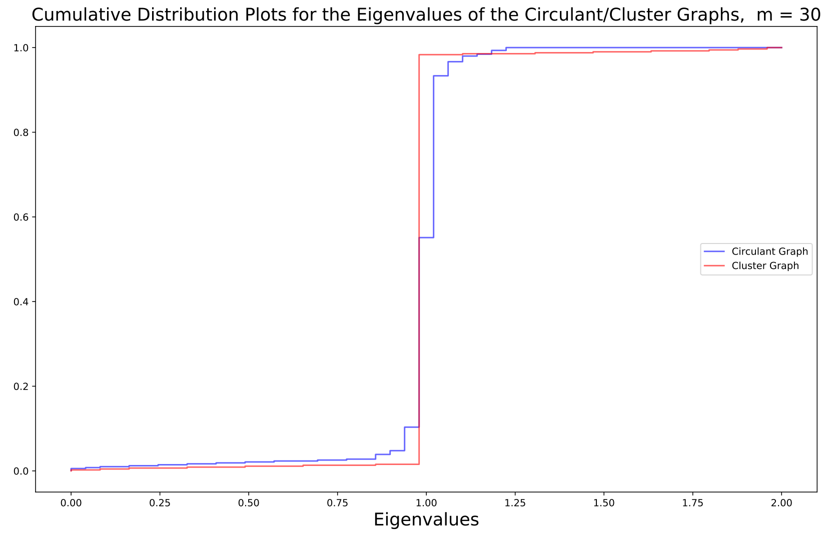


FIGURE 5. Spectral CDF, $m = 30$.

We would like to quantify how similar these distributions are to one another and determine if and how the distributions converge as m gets large. In order to do so, we will divide the eigenvalues into three regions based on the eigenvalues of $T(m,m)$. Region 1 contains the set of eigenvalues less than 1, forming the left tail in Figure 5. Region 2 contains the eigenvalues larger than 1, forming the right tail in Figure 5. The third region will be the eigenvalues that have value 1; these form the vertical region in the middle of the figure.

We will produce bounds on where these regions begin and end for the base circulant graph. In order to do so, we must first understand how the shape of the eigenvalue plots of $C^m_{m^2}$ arises.

Figure 6 shows the eigenvalues of the base circulant graph on 100^2 vertices in order to emphasize the bumps at the boundary of region 1. These bumps are coming from the introduction of the eigenvalues in successive peaks in Figure 4.

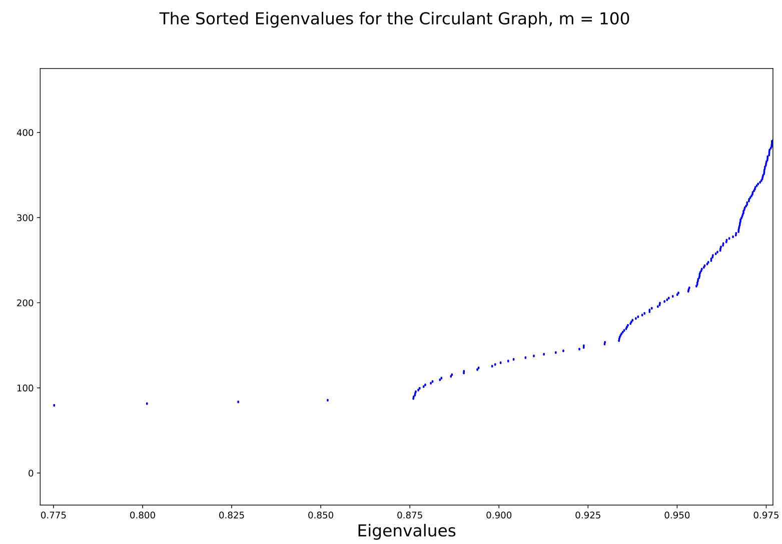


FIGURE 6. A close up of the eigenvalues of $C^{100}_{100^2}$.

However, we are only interested in the bumps arising from the highest peak and the second deepest valley, meaning we are only interested in approximating where the first bump appears in this figure.

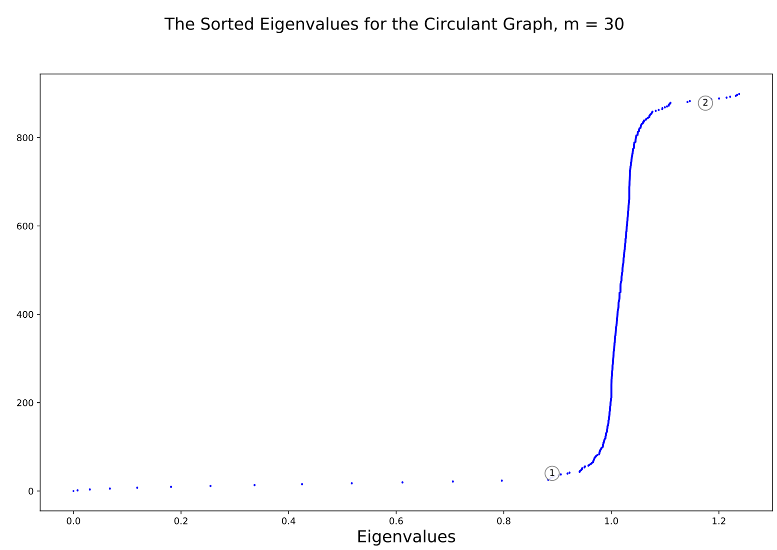
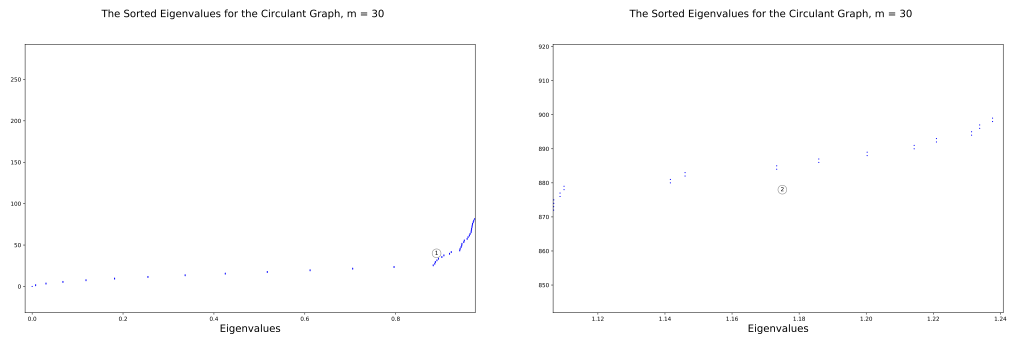


FIGURE 7. The Eigenvalues of $C^{30}_{30^2}$; regions $1, 2$ marked.



(A) Region 1. (B) Region 2.

FIGURE 8. A Close Up of the Eigenvalues of $C^{30}_{30^2}$.

Looking at the eigenvalues in these plots, it is clear that the jumps that mark the boundaries of regions 1 and 2 are coming from the introduction of eigenvalues from the second valley, (1), and the highest peak, (2). In order to see this more clearly, consider the following figure.

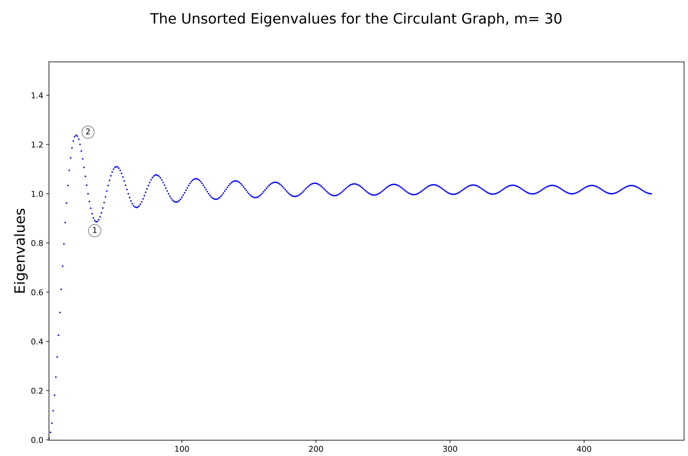


FIGURE 9. The Unsorted Eigenvalues of $C^{30}_{30^2}$; regions $1, 2$ marked.

Thus, in order to determine bounds on the locations of regions 1 and 2, we need only to estimate the height and depth of the highest peak and the second valley respectively.

3.5. **Eigenvalue Convergence.**

In order to approximate the location of the highest peak and second valley, we will differentiate the function for the eigenvalues and consider the zeroes.

Since our eigenvalues are symmetric about the y axis, we will only need to find the positive critical points.

It can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\lambda_x) = \frac{(m+1)\sin(mx) - m\sin((m+1)x)}{sin^2(x/2)} \tag{5}$$

where λ_x is found in Equation (4).

It is clear that the critical points corresponding to the highest peak and the second valley occur when $(m+1)\sin(mx) - m\sin((m+1)x) = 0$, since $\sin^2(x/2) = 0$ at $x = 4k\pi$, which occurs at values too large to be the locations of the highest peak and the second valley.

3.5.1. *Bounding Regions of Convergence.*

First, we will bound the location of the second valley from above. Since we know that the second valley occurs at some x such that $\frac{4\pi}{2m+1} < x < \frac{6\pi}{2m+1}$, we can create an upper bound for the location of the second valley by fashioning a lower bound for the derivative in this region,

$$\frac{m+1}{m}\sin(mx) - \sin((m+1)x) \geq \sin(mx) - \sin((m+1)x) = 2\cos\left(\frac{x(2m+1)}{2}\right)\sin\left(\frac{x}{2}\right).$$

Since $\sin\left(\frac{x}{2}\right) \neq 0$ for $x \in \left[\frac{4\pi}{2m+1}, \frac{6\pi}{2m+1}\right]$, the zeroes are given by the cosine, at $x = \frac{(2k+1)\pi}{2m+1}$. It then follows that the critical point that bounds the location of the second valley from above is $\frac{5\pi}{2m+1}$.

Therefore, the location of the second valley lies at some $x \in \left[\frac{4\pi}{2m+1}, \frac{5\pi}{2m+1}\right]$. We can use a similar strategy with slightly more complicated computations to produce a lower bound for the location as well and see that the location of the second valley lies in,

$$\left[\frac{5\pi - 2\arcsin(\frac{2m+1}{4\pi m})}{2m+2}, \frac{5\pi}{2m+1}\right].$$

For details of these computations, see Appendix A.

Then, we can use this information to construct the following curve that bounds the depth of the second valley in this region from below:

$$y = \frac{2m+1}{2m} - \frac{2\,(2m+2)}{m\left(5\pi \ - \ 2\arcsin\left(\frac{2m+1}{(4\pi m)}\right)\right)}.$$

Now we will turn our attention to region 2. This is the upper interval in which only eigenvalues from the highest peak occur, which will be the largest eigenvalues in the spectrum.

Again we will use Equation (5), but this time, we will be looking at the highest peak, and so we are concerned about the derivative within the region $\left[\frac{6\pi}{2m+1}, \frac{8\pi}{2m+1}\right]$. Using virtually identical methods as we did when estimating the location of the second valley, we see that the location of the highest peak will be between $\left[\frac{7\pi+2\arcsin\left(-\frac{2m+1}{6\pi m}\right)}{2m+2}, \frac{7\pi}{2m+1}\right]$. We then can show that the height of the second peak is less than

$$y \;= \frac{2m+1}{2m} - \;\frac{2m+2}{m\left(7\pi \;-\; 2\arcsin\left(-\frac{2m+1}{(6\pi m)}\right)\right)}.$$

We note that Theorem 3.3 and the bounds proved in this section suffice to show:

Theorem 3.5. *Fix $\epsilon > 0$. As $m \to \infty$, the spectral CDFs of both families of base graphs (circulant and cluster) converge to*

$$F(\lambda) = \begin{cases} 0 & \lambda < 0, \\ 1 & \lambda \geq 1, \end{cases}$$

with pointwise error $O\left(\frac{1}{m}\right)$ for $|\lambda - 1| > \epsilon$.

4. Eigenvalue Interlacing: Adding Edges

Now that we have an understanding of the eigenvalues of the base circulant and cluster graphs, we will begin to add random edges and explore the resulting effect on the spectra.

4.1. Previous Results.

Definition 4.1. A *Hermitian matrix* is a complex square matrix that is equal to its own conjugate transpose.

As graph Laplacians are real-valued symmetric, they are Hermitian. For an arbitrary matrix, the only certain characterization of the eigenvalues we can make is that they are the roots of the characteristic polynomial. However, the eigenvalues of Hermitian matrices can be expressed as the optimal solutions to a series of optimization problems, known as the variational characterizations of eigenvalues [6]. We will rely on the following theorem in later sections.

Theorem 4.2. *(Rayleigh-Ritz) Let $A \in M_n$ be Hermitian. Then,*

$$\lambda_1 x^* x \leq x^* A x \leq \lambda_n x^* x \quad \text{for all } x \in \mathbb{C}^n$$

$$\lambda_{max} = \lambda_n = \max_{x \neq 0} \frac{x^* A x}{x^* x} = \max_{xx^* = 1} x^* A x$$

$$\lambda_{min} = \lambda_1 = \min_{x \neq 0} \frac{x^* A x}{x^* x} = \min_{x^* x = 1} x^* A x$$

The next two theorems are concerned with how the eigenvalues of particular matrices interlace. These statements can be used to infer things about the spectra of graphs after adding edges or deleting vertices. Previous work can be found in [13] and [12].

Theorem 4.3 (Thm 4.3.1, [6]). *Let $A, B \in M_n$ be Hermitian and let the eigenvalues $\lambda_i(A)$, $\lambda_i(B)$, and $\lambda_i(A+B)$ be arranged in increasing order. Then, for each $k = 1, 2, \dots, n$ we have*

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_n(B).$$

Theorem 4.4 (Thm 4.3.15, [6]). *Let $A \in M_n$ be a Hermitian matrix, let r be an integer with $1 \leq r \leq n$, and let A_r denote any $r \times r$ principal submatrix of A. Then, for each integer k such that $1 \leq k \leq r$ we have*

$$\lambda_k(A) \leq \lambda_k(A_r) \leq \lambda_{k+n-r}(A)$$

The following result that we will generalize is due to Chen et. al [2].

Theorem 4.5. *Suppose H is a connected graph obtained from the graph G by removing an edge. Let $\mathcal{L}(G)$ and $\mathcal{L}(H)$ have eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$, respectively. Set $\lambda_0 = 2$ and $\lambda_{n+1} = 2$. Then,*

$$\lambda_{j-1} \geq \mu_j \geq \lambda_{j+1} \quad for \quad j = 1, \dots, n.$$

The original proof uses variational characterizations of eigenvalues, (see Theorem 4.2) and some complicated computations. However, a simpler proof relying on block decomposition and inertia is presented by Chi-Kwong Li [8]. We will present a generalization of Theorem 4.5; our proof directly generalizes the proof presented in [8]. Before we do so, we introduce the concepts we rely on.

4.2. **Inertia.**

Definition 4.6. Let $A \in M_n$ be a Hermitian matrix. The *inertia* of A is the triple, (p, n, z) where p number of positive eigenvalues of A, n is the number of negative positive eigenvalues of A and z is the number of zero eigenvalues of A, all counting multiplicity.

To see a more in depth explanation of inertia, see [6]. The proof of Theorem 4.8 in the next section relies heavily on simplifications made possible by the following theorem.

Theorem 4.7 (Thm 4.5.8, [6]). *Let $A, B \in M_n$ be Hermitian matrices. There is a nonsingular matrix $S \in M_n$ such that $A = SBS^* \iff A$ and B have the same inertia.*

4.3. **A Generalized Interlacing Inequality.**

In this section, we generalize Theorem 4.5 to weighted graphs with self loops that are not necessarily connected. In order to do so, we need to adjust some of the concepts and definitions we've been using throughout this paper for graphs of this generalized form.

Let G be a weighted undirected graph with self loops. By definition, this means that G is associated with a weight function, $w : V \times V \to \mathbb{R}$ such that $w(u,v) = w(v,u)$ and $w(u,v) \geq 0$. Note that if $\{u,v\} \notin E(G)$, then $w(u,v) = 0$. The degree, d_v of vertex v is defined as

$$d_v = \sum_u w(u,v).$$

We can now define the Laplacian and the normalized Laplacian the same way as we did in Section 3.1, using the definition of vertex degree from above. However, since we now allow self loops, the (u,v)th entry of the normalized Laplacian associated with G will be as follows:

$$\mathcal{L}(u,v) = \begin{cases} 1 - \frac{w(v,v)}{d_v} & \text{if } u = v \text{ and } d_v \neq 0, \\ -\frac{w(u,v)}{\sqrt{d_u d_v}} & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

We will use this definition of the normalized Laplacian in the following result and proof. In the previous section, Theorem 4.5 orders the eigenvalues so that those with the smallest indices are the largest in magnitude. In our theorem and proof, we will relabel the eigenvalues so that $\lambda_1 \leq \cdots \leq \lambda_n$, in order to be consistent with the standard literature [3].

Theorem 4.8. *Let G be a graph with weighted edges and self loops and suppose that H is a graph obtained from G by removing an edge. Additionally, let $\mathcal{L}(G)$ and $\mathcal{L}(H)$ have eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$ respectively, and let $\lambda_{n+1} = 2$ and $\lambda_0 = 0$. Then, provided that:*

(1) the edge was removed from vertices v_i and v_j, where v_i, v_j don't have self loops, or
(2) the edge removed was a self loop,

$$\lambda_{j+1} \geq \mu_j \geq \lambda_{j-1}, \quad j = 1, \ldots, n.$$

We can reformulate the conclusion of this theorem as follows,

Lemma 4.9. *Assume that the hyptheses of Theorem 4.8. Then, the conclusion of Theorem 4.8 results from the following. Define D_2 and Z to be the degree matrix and Laplacian for $G/\{v_1, v_2\}$ respectively. Then for any $\mu \in (\mu_1, \mu_n)$ such that $D_2 Z D_2 - \mu I_{n-2}$ is invertible,*

(a) If $\mathcal{L}(H) - \mu I_n$ has p positive eigenvalues, then $\mathcal{L}(G) - \mu I_n$ has at least $p-1$ positive eigenvalues.

(b) If $\mathcal{L}(H) - \mu I_n$ has q negative eigenvalues, then $\mathcal{L}(G) - \mu I_n$ has at least $q-1$ negative eigenvalues.

It will then follow that $\lambda_{j-1} - \mu_j \geq 0$ and $\mu_j - \lambda_{j+1} \geq 0$ for any $j = 1, \ldots n$.

Proof. Let $\mathcal{L}(H)$ have eigenvalues $\mu_1, \ldots, \mu_n$ and $\mathcal{L}(G)$ have eigenvalues $\lambda_1, \ldots, \lambda_n$. Now assume that H has exactly C components. Then, that means $\mu_1 = \cdots = \mu_C = 0$ and that $\mathcal{L}(H) - \mu I_n$ has $n - C$ eigenvalues that are greater than $\epsilon - \mu$, for all $\epsilon > 0$. Then (a) tells us this means that $\mathcal{L}(G) - \mu I_n$ must have at least $n - C - 1$ eigenvalues bigger than $\epsilon - \mu$. But then this implies that

$$-\mu < \epsilon - \mu \leq \lambda_C \leq \cdots \leq \lambda_{n-1} \leq \lambda_n.$$

But since this is tue for any $\epsilon < \mu_{C+1}$, we have that

$$\mu_{C+1} \geq \lambda_C.$$

So now assume that $\mathcal{L}(H)$ has C eigenvalues less than ϵ for all $\epsilon > 0$. Then $\mathcal{L}(G)$ has at least $C - 1$ eigenvalues less than ϵ, for $0 < \epsilon < \mu_{C+1}$ but then we see that

$$\lambda_C \geq \mu_{C-1}.$$

So now consider those eigenvalues not at the end of the spectrum. For all but finitely many values of ϵ, G has $j+1$ eigenvalues strictly less than $\lambda_{j+1} + \epsilon$. This means that $\mu_j \leq \lambda_{j+1} + \epsilon$. Hence H has at least $(j+1) - 1 = j$ eigenvalues strictly less than $\lambda_{j+1} + \epsilon$ and since $\mu_j \leq \lambda_{j+1} + \epsilon$ for all but at most finitely many values of epsilon, we can conclude that $\mu_j \leq \lambda_{j+1}$. The other half of the inequality follows from virtually identical computations. □

We will use this formulation of the conclusion to Theorem 4.8 in the following proof.

Proof. First, relabel the vertices and assume that H is obtained from G by removing either the edge joining vertices 1 and 2, or the self loop attached to vertex 1.

Case 1: The edge removed connected vertices 1 and 2 and neither vertex has a self loop.
Then, we can define the normalized Laplacians of G and H in the following way,

$$\mathcal{L}(G) = \begin{pmatrix} D_1 X D_1 & D_1 Y D_2 \\ D_2 Y^T D_1 & D_2 Z D_2 \end{pmatrix}, \qquad \mathcal{L}(H) = \begin{pmatrix} I_2 & \widetilde{D_1} Y D_2 \\ D_2 Y^T \widetilde{D_1} & D_2 Z D_2 \end{pmatrix}$$

where,

(6)
$$D_1 = \operatorname{diag}\left(1/\sqrt{d_1}, 1/\sqrt{d_2}\right), \qquad \widetilde{D_1} = \operatorname{diag}\left(1/\sqrt{d_1 - w(1,2)}, 1/\sqrt{d_2 - w(2,1)}\right)$$
$$D_2 = \operatorname{diag}\left(1/\sqrt{d_3}, \dots, 1/\sqrt{d_n}\right), \quad X = 2 \times 2 \text{ Laplacian matrix for } v_1, v_2.$$
$$Y = 2 \times (n-2) \text{ Laplacian matrix for } v_1, v_2 \text{ and the rest of the graph.}$$
$$Z = (n-2) \times (n-2) \text{ Laplacian matrix for } G/\{v_1, v_2\}.$$
$$\widetilde{Z} = D_2 Z D_2 - \mu I_{n-2}, \ (\mu \text{ not an eigenvalue of } D_2 Z D_2).$$

Now we define the following matrices:
$$S = \begin{pmatrix} I_2 & -D_1 Y D_2 \widetilde{Z}^{-1} \\ 0 & I_{n-2} \end{pmatrix} \quad \text{and} \quad \widetilde{S} = \begin{pmatrix} I_2 & -\widetilde{D_1} Y D_2 \widetilde{Z}^{-1} \\ 0 & I_{n-2} \end{pmatrix}.$$

Notice that these are both upper triangular and are therefore invertible. Additionally, $\mathcal{L}(G) - \mu I_n$ and $\mathcal{L}(H) - \mu I_n$ are both symmetric, as are the products $S(\mathcal{L}(G) - \mu I_n)S^T$ and $\widetilde{S}(\mathcal{L}(H) - \mu I_n)\widetilde{S}^T$. Therefore, we can use Theorem 4.7 to conclude that the inertias of $S(\mathcal{L}(G) - \mu I_n)S^T$ and $\mathcal{L}(G) - \mu I_n$ are the same; similarly for $\widetilde{S}(\mathcal{L}(H) - \mu I_n)\widetilde{S}^T$ and $\mathcal{L}(H) - \mu I_n$.

We will express these computations for the matrices associated with G, but note that those for H are similar.
$$\begin{aligned} S(\mathcal{L}(G) - \mu I_n)S^T &= \begin{pmatrix} I_2 & -D_1 Y D_2 \widetilde{Z}^{-1} \\ 0 & I_{n-2} \end{pmatrix} \begin{pmatrix} D_1 X D_1 - \mu I_2 & D_1 Y D_2 \\ D_2 Y^T D_1 & \widetilde{Z} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ -D_1 Y D_2 \widetilde{Z}^{-1} & I_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} D_1 X D_1 - \mu I_2 - D_1 Y D_2 \widetilde{Z}^{-1} D_2 Y^T D_1 & 0 \\ D_2 Y^T D_1 & \widetilde{Z} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ -D_1 Y D_2 \widetilde{Z}^{-1} & I_{n-2} \end{pmatrix} \end{aligned}$$

(7)
$$= \begin{pmatrix} D_1 X D_1 - \mu I_2 - D_1 Y D_2 \widetilde{Z}^{-1} D_2 Y^T D_1 & 0 \\ 0 & \widetilde{Z} \end{pmatrix}$$

Completing the computations for $\mathcal{L}(H)$, we obtain
$$\widetilde{S}(\mathcal{L}(H) - \mu I_n)\widetilde{S}^T = \begin{pmatrix} I_2 - \mu I_2 - \widetilde{D_1} Y D_2 \widetilde{Z}^{-1} D_2 Y^T \widetilde{D_1} & 0 \\ 0 & \widetilde{Z} \end{pmatrix}.$$

Since the upper left hand corner of $\widetilde{S}(\mathcal{L}(H) - \mu I_n)\widetilde{S}^T$ and $S(\mathcal{L}(G) - \mu I_n)S^T$ both have the matrix $Y D_2 \widetilde{Z}^{-1} D_2 Y^T$ in common, let $C = Y D_2 \widetilde{Z}^{-1} D_2 Y^T$. With these block decompositions, we see that $S(\mathcal{L}(G) - \mu I_n)S^T$ and $\widetilde{S}(\mathcal{L}(H) - \mu I_n)\widetilde{S}^T$ have the same eigenvalues resulting from $\widetilde{Z}$, and that only two of their eigenvalues differ; specifically those resulting from the 2×2 matrices in the upper left hand corner. Therefore, we can check conditions a) and b) from Lemma 4.9 on

$$B = D_1 X D_1 - \mu I_2 - D_1 C D_1 \quad \text{and} \quad \widetilde{B} = I_2 - \mu I_2 - \widetilde{D_1} C \widetilde{D_1}. \tag{8}$$

Thus, all that remains to verify is if $\widetilde{B}$ has two positive or two negative eigenvalues, then B must have at least one positive or one negative eigenvalue respectively. To see this, we will use Theorem 4.7 again to simplify B and $\widetilde{B}$.

Notice that

$$\widetilde{D}_1^{-1} = \begin{pmatrix} \sqrt{d_1 - w(1,2)} & 0 \\ 0 & \sqrt{d_2 - w(2,1)} \end{pmatrix},$$

is diagonal and nonsingular, so we can compute the products $\widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1}$ and $D_1^{-1}BD_1^{-1}$, and conclude that they have the same inertia as $\widetilde{B}$ and B respectively.

One can show that,

$$\widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1} = (1-\mu)\begin{pmatrix} d_1 - w(1,2) & 0 \\ 0 & d_2 - w(2,1) \end{pmatrix} - C$$

$$\text{and} \quad D_1^{-1}BD_1^{-1} = \widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1} + \begin{pmatrix} w(1,2)(1-\mu) & -1 \\ -1 & w(2,1)(1-\mu) \end{pmatrix}.$$

For details of these computations, see Appendix B. Now consider $A = \begin{pmatrix} w(1,2)(1-\mu) & -1 \\ -1 & w(2,1)(1-\mu) \end{pmatrix}$. The characteristic polynomial can be computed using that $w(1,2) = w(2,1)$, as follows,

$$\begin{aligned} \det(A - I\lambda) &= (w(1,2) - \mu w(1,2) - \lambda)^2 - 1 \\ &= \lambda^2 + 2w(1,2)(\mu - 1)\lambda + w(1,2)^2(\mu^2 - 2\mu - 1 + 1/w(1,2)) \end{aligned}$$

The constant term of the characteristic polynomial is the product of the roots; meaning that it is the product of the eigenvalues. Since $\mu \in [0,2]$, we know that $\mu^2 - 2\mu \leq 0$ and as long as $w(1,2) \geq 1$, then $\frac{1}{w(1,2)} - 1 \leq 0$. Therefore the constant term of the characteristic polynomial is negative; meaning that matrix A has one negative and one positive eigenvalue. Let these eigenvalues be $\lambda_1 > 0 > \lambda_2$ with unit eigenvectors v_1 and v_2 respectively.

Observe the following:

$$\begin{aligned} v_1^t D_1^{-1} B D_1^{-1} v_1 &= v_1^t \widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1} v_1 + v_1^t \begin{pmatrix} w(1,2)(1-\mu) & -1 \\ -1 & w(2,1)(1-\mu) \end{pmatrix} v_1 \\ &= v_1^t \widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1} v_1 + \lambda_1 \\ &> 0. \end{aligned}$$

Since $\widetilde{D_1}^{-1}\widetilde{B}\widetilde{D_1}^{-1}$ has two positive eigenvalues by assumption, and Theorem 4.2 tells us that $v_1^t\widetilde{D_1}^{-1}\widetilde{B}\widetilde{D_1}^{-1}v_1$ must lie between those two eigenvalues, we see that $v_1^t\widetilde{D_1}^{-1}\widetilde{B}\widetilde{D_1}^{-1}v_1 > 0$. Similarly, since $v_1^tD_1^{-1}BD_1^{-1}v_1$ must lie between the two eigenvalues of $D_1^{-1}BD_1^{-1}$, and $v_1^tD_1^{-1}BD_1^{-1}v_1 > 0$, we can conclude that $D_1^{-1}BD_1^{-1}$ and therefore B must have at least one positive eigenvalue.

We can complete virtually identical computations with v_2 and λ_2 to conclude that if $\widetilde{B}$ has two negative eigenvalues, then $D_1^{-1}BD_1^{-1}$ and therefore B must have at least one negative eigenvalue.

Case 2: the edge we remove is a self loop.
So now we have,

$$\mathcal{L}(G) = \begin{pmatrix} D_1XD_1 & D_1YD_2 \\ D_2Y^TD_1 & D_2ZD_2 \end{pmatrix} \qquad \mathcal{L}(H) = \begin{pmatrix} \widetilde{D_1}\widetilde{X}\widetilde{D_1} & \widetilde{D_1}YD_2 \\ D_2Y^T\widetilde{D_1} & D_2ZD_2 \end{pmatrix}$$

where we define the block matrices similarly to (6):

(9)

$$D_1 = \left(1/\sqrt{d_1}\right), \quad \widetilde{D_1} = \left(1/\sqrt{d_1 - w(1,1)}\right), \quad D_2 = \operatorname{diag}\left(1/\sqrt{d_2}, \ldots, 1/\sqrt{d_n}\right)$$

$$X = \left(d_1 - w(1,1)\right), \quad \widetilde{X} = \left(d_1 - 2w(1,1)\right)$$

$Y = 1 \times n-1$ Laplacian matrix for v_1 and the rest of the graph.

$Z = n-1 \times n-1$ Laplacian matrix for $G/\{v_1\}$.

$\widetilde{Z} = D_2ZD_2 - \mu I_{n-1}$, ($\mu$ not an eigenvalue of D_2ZD_2).

We perform the block decomposition computations as we did in the previous case and obtain:

$$S(\mathcal{L}(G) - \mu I_n)S^T = \begin{pmatrix} D_1XD_1 - \mu I_2 - D_1YD_2\widetilde{Z}^{-1}D_2Y^TD_1 & 0 \\ 0 & \widetilde{Z} \end{pmatrix}$$

$$\widetilde{S}(\mathcal{L}(H) - \mu I_n)\widetilde{S}^T = \begin{pmatrix} \widetilde{D_1}\widetilde{X}\widetilde{D_1} - \mu I_2 - \widetilde{D_1}YD_2\widetilde{Z}^{-1}D_2Y^T\widetilde{D_1} & 0 \\ 0 & \widetilde{Z} \end{pmatrix}$$

But now note that these two matrices share the lower left $(n-1)\times(n-1)$ matrix, $\widetilde{Z}$, and must have all of the eigenvalues of $\widetilde{Z}$ in common. Therefore, the spectra of these two matrices differ by the eigenvalue coming from the 1×1 matrix in the upper left corner. But, regardless of what those eigenvalues actually are, conditions a), and b) from Lemma 4.9 will be satisfied; so we are done. $\square$

4.3.1. *The Remaining Case.*

Note that the only case not covered in the previous proof is when the edge removed is adjacent to vertices 1 and 2, and one or both of those vertices have a self loop. If we try to complete the proof for this case, all of the block matrices are the same as Equations (6) presented in Case 1 of the proof, except we also have,

$$\widetilde{X} = \operatorname{diag}(d_1 - w(1,2) - w(1,1), d_2 - w(1,2) - w(2,2).$$

Note that one of $w(1,1)$ or $w(2,2)$ could be zero.

We can complete block decomposition computations as we did previously to obtain $S(\mathcal{L}(G) - \mu I_n)S^T$ as defined in Equation (7) and

$$\widetilde{S}(\mathcal{L}(H) - \mu I_n)\widetilde{S}^T = \begin{pmatrix} \widetilde{D_1}\widetilde{X}\widetilde{D_1} - \mu I_2 - \widetilde{D_1} Y D_2 \widetilde{Z}^{-1} D_2 Y^T \widetilde{D_1} & 0 \\ 0 & \widetilde{Z} \end{pmatrix}.$$

Again, set $C = Y D_2 \widetilde{Z}^{-1} D_2 Y^T$ and define B and $\widetilde{B}$ as in Equation (8). Then, using inertia, we further simplify our matrices to see that,

$$\widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1} = \begin{pmatrix} (1-\mu)(d_1 - w(1,2) - w(1,1) & 0 \\ 0 & (1-\mu)(d_2 - w(2,1) - w(2,2)) \end{pmatrix} - C$$

and if $\quad A' = \begin{pmatrix} w(1,2) - \mu w(1,2) + w(1,1) & -1 \\ -1 & w(2,1) - \mu w(2,1) + w(2,2) \end{pmatrix}$ then,

$$D_1^{-1} B D_1^{-1} = \widetilde{D_1}^{-1} \widetilde{B} \widetilde{D_1}^{-1} + A'.$$

For computational details, see Appendix B.

If we compute the characteristic polynomial for A', we will see that the constant term is,

$$(1-\mu)w(1,2)(w(1,1) + w(2,2)) + w(1,2)^2(1-\mu)^2 - 1.$$

We can simplify the constant term by renormalizing the weights for G with respect to $w(1,2)$, and let the new loop weights on vertices 1 and 2 be v and w respectively. Then, the constant term becomes

$$v + w + vw - (2 + v + w)\mu + \mu^2.$$

This is quadratic in μ and will be negative between the two roots,

$$\left(1 + \frac{v+w}{2} - \sqrt{1 + \frac{(v-w)^2}{4}}, \quad 1 + \frac{v+w}{2} + \sqrt{1 + \frac{(v-w)^2}{4}}\right).$$

However, when $v, w > 0$, this interval is no longer $[0, 2]$. As v, w get large, this interval increases and becomes disjoint from $[0, 2]$ rather quickly. Therefore, since we cannot conclude that A' has a positive and negative eigenvalue, using Theorem 4.2 as we did before to finish the argument will not work.

So our previous proof strategy cannot conclude anything about this case. Experimentally, Theorem 4.8 appears to still hold.

4.3.2. *Relation to the Circulant and Cluster Graphs.*

We can apply this result to the circulant and cluster graphs. Since the eigenvalues of these two graphs come in pairs, we have the following corollary.

Corollary 4.10. *Let G be a circulant or a cluster graph on n vertices. Then,*

$$0 = \lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 \dots \lambda_{n-1} \leq \lambda_n$$

with equality holding when n is odd. Let G' be G with a random edge and let the eigenvalues of G' be $\mu_1 \leq \cdots \leq \mu_n$. Then we have that

$$\lambda_{2i-1} \leq \mu_{2i} \leq \lambda_{2i} \quad \textit{and} \quad \lambda_{2(i+1)} \geq \mu_{2i+1} \geq \lambda_{2i+1}.$$

This corollary can be visualized with the following two plots.

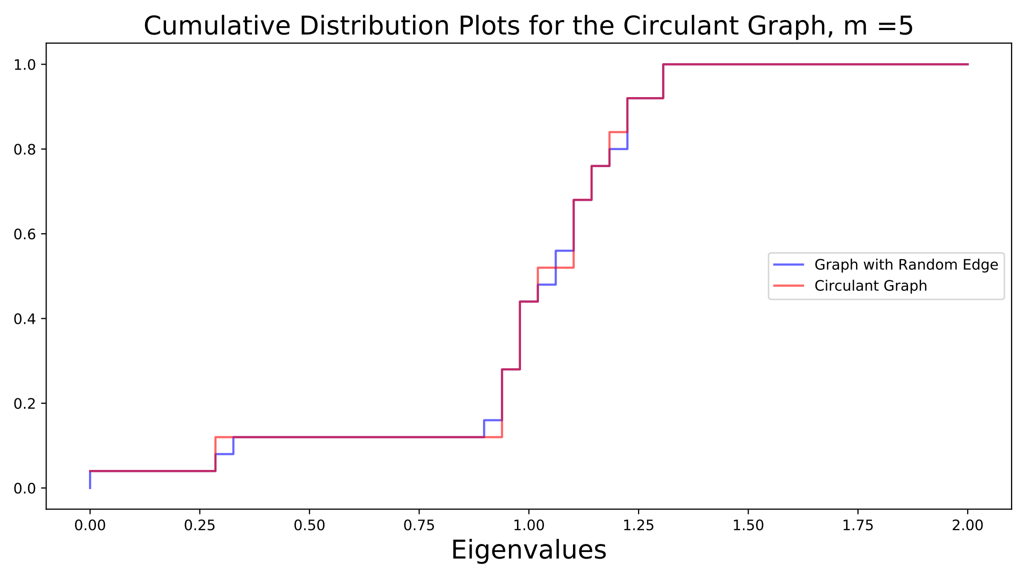


FIGURE 10. Cumulative Distibution for the Eigenvalues of $C^5_{5^2}$; added edge between vertices 15 and 23.

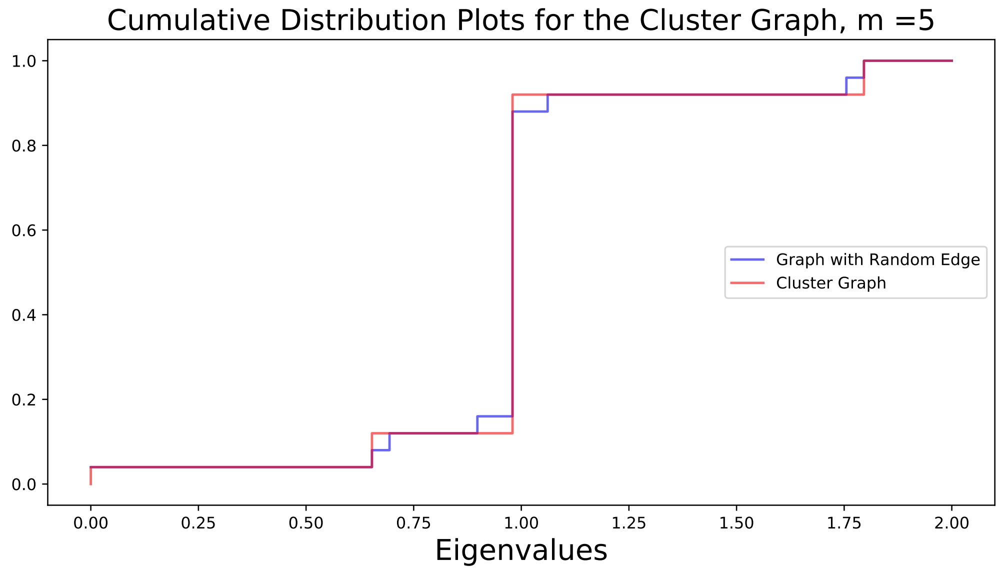


FIGURE 11. Cumulative Distribution for the Eigenvalues of $C(5,5)$; added edge between vertices 2 and 15.

5. Conclusion and Further Work

We gave a weak convergence statement on the spectral CDFs of the cluster and circulant base graphs in Theorem 3.5. The interlacing inequalities from Section 4 allow us to conclude that the spectra of the two graphs still converge as in Theorem 3.5 with the addition of a single edge. We hope that these interlacing inequalities will allow us to conclude that the broad spectrum is displaying some sort of convergence when adding more than 1 edge as well.

Experimentally, Theorem 4.8 seems to still apply to graphs when removing an edge connecting vertices 1 and 2 and one or both of those vertices has a self loop. While our proof in this paper failed to prove this case, we intend to investigate whether the more complicated proof employing variational characterizations of eigenvalues and complicated computations used by Chen et. al can be generalized to this case.

Appendix Appendix A Computations from Section 3.5.1

Claim A.1. *The lower bound for the location of the second valley is*

$$x = \frac{5\pi - 2\arcsin(\frac{2m+1}{4\pi m})}{2m+2}.$$

Proof. We can use inequalities $a)$ $1-\cos(x) \leq x\sin(x)$ and $b)$ $x \geq \sin(x)$ for $x \in \left[\frac{4\pi}{2m+1}, \frac{5\pi}{2m+1}\right]$ to bound the location of the second valley from below by constructing the following upper bound on the derivative.

$$\begin{aligned}\frac{m+1}{m}\sin(mx) - \sin((m+1)x) &\leq \frac{1}{m} + \sin(mx) - sin((m+1)x)\\ &\leq \frac{1}{m} - x\cos(mx) + \sin(mx)(1-cos(x))\\ &\leq \frac{1}{m} - x\cos(mx+x)\end{aligned}$$

From here it is a question of where $x\cos(mx+x) = \frac{1}{m}$. For $x \in \left[\frac{4\pi}{2m+1}, \frac{5\pi}{2m+1}\right]$,

$$\frac{1}{m} - x\cos((m+1)x) \leq \frac{1}{m} - \frac{4\pi}{2m+1}\cos((m+1)x)$$

$$\implies \frac{4\pi}{2m+1}\cos((m+1)x) = \frac{1}{m} \iff x = \frac{2\pi k + \arccos\left(\frac{2m+1}{4\pi m}\right)}{m+1}$$

Note that for this to occur in the region we are interested in, $k = 1$. Because $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$, we we can rewrite this bound as,

$$x = \frac{2\pi + \arccos\left(\frac{2m+1}{4\pi m}\right)}{m+1} = \frac{5\pi - 2\arcsin\left(\frac{2m+1}{5\pi m}\right)}{2m+2}.$$

□

Claim A.2. *The depth of the second valley can be approximated with*

$$y = \frac{2m+1}{2m} - \frac{2\,(2m+2)}{m\left(5\pi\ -\ 2\arcsin\left(\frac{2m+1}{(4\pi m)}\right)\right)}.$$

Proof. We will construct a curve that will bound the depth of the second valley from below. Using small angle approximation in the region, $\left[\frac{5\pi - 2\arcsin(\frac{2m+1}{4\pi m})}{2m+2}, \frac{5\pi}{2m+1}\right]$, we see,

$$\begin{aligned}\frac{2m+1}{2m} - \frac{1}{2m}\frac{\sin\left((m+1/2)x\right)}{\sin\left(x/2\right)} &\geq \frac{2m+1}{2m} - \frac{1}{xm}\sin\left((m+1/2)x\right)\\ &\geq \frac{2m+1}{2m} - \frac{2\,(2m+2)}{m\left(5\pi\ -\ 2\arcsin\left(\frac{2m+1}{(4\pi m)}\right)\right)}\sin\left(\left(m+\frac{1}{2}\right)x\right),\end{aligned}$$

$$\text{evaluated at } x = \frac{5\pi}{2m+1}.$$

This means the depth of the second valley can be approximated with

$$y = \frac{2m+1}{2m} - \frac{2\,(2m+2)}{m\left(5\pi\ -\ 2\arcsin\left(\frac{2m+1}{(4\pi m)}\right)\right)}.$$

□

The computations for bounding the location and height of the highest peak use similar strategies to those above.

Appendix Appendix B Computations from Section 4

Claim B.1.

$$\textit{and}\quad D_1^{-1}BD_1^{-1} = \widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1} + \begin{pmatrix} w(1,2)(1-\mu) & -1 \\ -1 & w(2,1)(1-\mu) \end{pmatrix}.$$

Proof. Recall that

$$\widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1} = (1-\mu)\begin{pmatrix} d_1 - w(1,2) & 0 \\ 0 & d_2 - w(2,1) \end{pmatrix} - C.$$

Then,

$$\begin{aligned} D_1^{-1}BD_1^{-1} &= \begin{pmatrix} d_1 - \mu d_1 & -1 \\ -1 & d_2 - \mu d_2 \end{pmatrix} - C \\ &= (1-\mu)\begin{pmatrix} d_1 - w(1,2) & 0 \\ 0 & d_2 - w(2,1) \end{pmatrix} + \begin{pmatrix} w(1,2)(1-\mu) & -1 \\ -1 & w(2,1)(1-\mu) \end{pmatrix} - C \\ &= \widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1} + \begin{pmatrix} w(1,2)(1-\mu) & -1 \\ -1 & w(2,1)(1-\mu) \end{pmatrix}. \end{aligned}$$

□

Claim B.2.

$$D_1^{-1}BD_1^{-1} = \widetilde{D_1}^{-1}\widetilde{B}\widetilde{D_1}^{-1} + A'.$$

Where

$$A' = \begin{pmatrix} w(1,2) - \mu w(1,2) + w(1,1) & -1 \\ -1 & w(2,1) - \mu w(2,1) + w(2,2) \end{pmatrix}.$$

Proof. Recall that

$$\begin{aligned} \widetilde{D}_1^{-1}\widetilde{B}\widetilde{D}_1^{-1} &= \widetilde{D}_1^{-1}\left(\widetilde{D_1}\widetilde{X}\widetilde{D_1} - \mu I_2 - \widetilde{D_1}C\widetilde{D_1}\right)\widetilde{D}_1^{-1} \\ &= \begin{pmatrix} (1-\mu)(d_1 - w(1,2) - w(1,1) & 0 \\ 0 & (1-\mu)(d_2 - w(2,1) - w(2,2)) \end{pmatrix} - C \end{aligned}$$

Now, computing $D_1^{-1}BD_1^{-1} - \widetilde{D_1}^{-1}\widetilde{B}\widetilde{D_1}^{-1}$ we see that,

$$D_1^{-1}BD_1{}^{-1} - \widetilde{D_1}^{-1}\widetilde{B}\widetilde{D_1}^{-1} = \begin{pmatrix} d_1 - \mu d_1 & -1 \\ -1 & d_2 - \mu d_2 \end{pmatrix} - C$$

$$
\begin{aligned}
&- \begin{pmatrix} d_1 - w(1,1) - w(1,2) & 0 \\ 0 & d_2 - w(2,2) - w(2,1) \end{pmatrix} + \begin{pmatrix} \mu(d_1 - w(1,2)) & 0 \\ 0 & \mu(d_2 - w(2,1)) \end{pmatrix} \\
&+ C \\
&= \begin{pmatrix} w(1,2) - \mu w(1,2) + w(1,1) & -1 \\ -1 & w(2,1) - \mu w(2,1) + w(2,2) \end{pmatrix}.
\end{aligned}
$$

Meaning that

$$
D_1^{-1} B D_1^{-1} = \widetilde{D_1}^{-1} \widetilde{B} \widetilde{D_1}^{-1} + \begin{pmatrix} w(1,2) - \mu w(1,2) + w(1,1) & -1 \\ -1 & w(2,1) - \mu w(2,1) + w(2,2) \end{pmatrix}.
$$

□

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