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Tamar Friedmann
University of Rochester, tfriedmann@smith.edu

C. R. Hagen
University of Rochester

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Tamar Friedmann and C. R. Hagen*
Department of Physics and Astronomy
University of Rochester
Rochester, N.Y. 14627-0171

The spectrum of the square of the angular momentum in arbitrary dimensions is derived using only group theoretical techniques. This is accomplished by application of the Lie algebra of the noncompact group $O(2,1)$.

The famously soluble problems of quantum mechanics include the simple harmonic oscillator, the hydrogen atom, and angular momentum. Each of these is known to have solutions obtainable from careful analysis of the underlying wave equations as well as from purely algebraic methods. In the case of the oscillator, the algebraic method is arguably preferable while for the hydrogen atom the choice between a wave equation approach and an algebraic one based on the Lenz vector may be less compelling. Nonetheless, it is generally recognized that the two approaches constitute a valuable complement to each other. In the case of angular momentum, Louck *et al.* [1] have shown that the eigenvalues of angular momentum squared in q spatial dimensions (namely, $\ell(\ell+q-2)$) can be obtained equally well (if tediously!) from the relevant differential equations or from the underlying algebra. It is the object of the present note to demonstrate that these eigenvalues follow in a remarkably simple way from a third method based on group theoretical considerations.

The technique employed is based on the Lie algebra of $O(2,1)$. This particular algebra was invoked some decades ago by Bacry and Richard [2] to obtain the spectra of the q-dimensional oscillator and the relativistic hydrogen atom [3]. Their techniques were in a sense a hybrid using the tools of both differential equations as well as group theory. The techniques presented here, however, are based solely on group theory, and they allow a totally group theoretical derivation when applied to the cases considered in [2]. The absence of such a result in the published literature constitutes a gap which this note proposes to address.

One begins by noting that the coordinate operators x_i and p_i, where $i = 1, 2, \ldots q$ with $[x_i, p_j] = i\delta_{ij}$ and $\hbar = 1$, may be expressed in terms of the operators a_i and $a_i^\dagger$ by

$$x_i = (a_i + a_i^\dagger)/\sqrt{2}$$

and

$$p_i = (a_i - a_i^\dagger)/\sqrt{2}i$$

where $[a_i, a_j^\dagger] = \delta_{ij}$. One readily infers that

$$x_i p_j - x_j p_i = -i[a_i^\dagger a_j - a_j^\dagger a_i]$$

so that the square of the angular momentum operator L_q^2, as defined by

$$L_q^2 = \frac{1}{2}\sum_{i,j=1}^{q} L_{ij}^2 \ ,$$

where $L_{ij} \equiv x_i p_j - x_j p_i$, becomes

$$L_q^2 = N_q(N_q + q - 2) - A_q^\dagger A_q,$$

where $N_q \equiv a_i^\dagger a_i, A_q \equiv a_i a_i$. The operators N_q and A_q satisfy the commutation relations

$$[N_q, A_q] = -2A_q$$

*mail to: tamarf@pas.rochester.edu, hagen@pas.rochester.edu

and

$$[A_q, A_q^\dagger] = 4N_q + 2q.$$

Upon defining $K_{q-} = \frac{1}{2}A_q$, $K_{q+} = \frac{1}{2}A_q^\dagger$, $K_{q1} = \frac{1}{2}(K_{q+} + K_{q-})$, $K_{q2} = \frac{1}{2i}(K_{q+} - K_{q-})$, and $J_{q3} = \frac{1}{2}N_q + \frac{q}{4}$, one obtains the $o(2,1) \sim su(1,1)$ Lie algebra [4]

$$[J_{q3}, K_{q1}] = iK_{q2},$$

$$[J_{q3}, K_{q2}] = -iK_{q1},$$

$$[K_{q1}, K_{q2}] = -iJ_{q3}.$$

The Casimir operator

$$Q_q \equiv J_{q3}^2 - K_{q1}^2 - K_{q2}^2$$

is readily seen to commute with each of the individual members of the algebra J_{q3}, K_{q1}, and K_{q2}. Comparison with the result for L_q^2 yields the relation

$$L_q^2 = 4Q_q - \frac{1}{4}q^2 + q, \tag{1}$$

so that the spectrum of L_q^2 is determined by values of the Casimir operator of representations of $o(2,1)$ [5]. Generalizing the earlier work of Bargmann [6] (who considered the group $O(2,1)$, and not its covering group $\overline{SU(1,1)}$), these values have been obtained via algebraic techniques by Barut and Fronsdal [7] who showed that there are four series of representations of $o(2,1)$ characterized by a pair of complex numbers (Φ, E_0). The series are

1. $D(\Phi, E_0)$, where $-\frac{1}{2} < Re(E_0) \leq \frac{1}{2}$, $\Phi \pm E_0 \neq \pm n$ (n is a non-negative integer) and the J_{q3} spectrum is $E_0 \pm n$;

2. $D^+(\Phi)$, where $E_0 = -\Phi$, $2\Phi \neq n$, and the J_{q3} spectrum is $E_0 + n$;

3. $D^-(\Phi)$, where $E_0 = \Phi$, $2\Phi \neq n$, and the J_{q3} spectrum is $E_0 - n$; and

4. $D(\Phi)$, where $E_0 = 0$, $2\Phi = n$, and the J_{q3} spectrum is $-\Phi, -\Phi+1, \ldots, \Phi-1, \Phi$.

Additional conditions on Φ and E_0 are needed for these representations to be unitary: $D(\Phi, E_0)$ is unitary if either $Im(E_0) = 0$ and $\Phi = -\frac{1}{2} + i\lambda$ ("principal series") or $Im(E_0) = Im(\Phi) = 0$ and $|\Phi + \frac{1}{2}| < \frac{1}{2} - |E_0|$ ("supplementary series"); $D^+(\Phi)$ and $D^-(\Phi)$ are unitary if $Im(E_0) = 0$ and $\Phi < 0$; and $D(\Phi)$ is unitary only for the trivial representation $\Phi = 0$. Also note that there are no degeneracies in the J_{q3} spectra.

For all the above representations, the Casimir operator is given by

$$Q_q = \Phi(\Phi + 1). \tag{2}$$

Since $J_{q3} = \frac{1}{2}N_q + \frac{q}{4}$ and the number operator N_q is a non-negative integer [8], one knows that J_{q3} is bounded below and positive. Only $D^+(\Phi)$ with $E_0 > 0$ satisfies this condition. Since

$$E_0 = -\Phi,$$

one has $\Phi < 0$ so the representation is unitary, and the spectrum of $J_{q3} = \frac{1}{2}N_q + \frac{q}{4}$ is $E_0, E_0 + 1, E_0 + 2, \ldots$.

Thus the spectrum of L_q^2 consists at most of the values obtained from equation (1) by inserting $Q_q = \Phi(\Phi + 1) = E_0(E_0 - 1)$ for $E_0 = \frac{1}{2}\ell_q + \frac{q}{4}$ for some integer $\ell_q \geq 0$. It remains to be shown that all integers $\ell_q \geq 0$ are obtained.

Rewriting the Casimir operator as

$$Q_q = J_{q3}(J_{q3} - 1) - K_{q+}K_{q-}$$

shows that if one finds a state $|\psi\rangle$ in a representation of $o(2,1)$ that is annihilated by K_{q_-}, that state will satisfy

$$Q_q|\psi\rangle = J_{q3}(J_{q3} - 1)|\psi\rangle, \tag{3}$$

so that the state's J_{q3} eigenvalue will equal the E_0 of that representation. It may also be noted that the operator K_{q-} plays the role of a lowering operator in the sense that it lowers the eigenvalues of J_{q3} by one unit. One now proceeds to obtain such a state $|\psi_{\ell_q}\rangle$ for each $\ell_q \geq 0$.

To this end one considers the Hilbert space of number eigenstates

$$|\{n_i\}\rangle = |n_q, n_{q-1}, \ldots, n_1\rangle$$

on which the operators $N_{qj} \equiv a_j^\dagger a_j$ (no sum), a_i, and $a_i^\dagger$ act via

$$N_{qj}|\{n_i\}\rangle = n_j|\{n_i\}\rangle;$$

$$a_j|\{n_i\}\rangle = \sqrt{n_j}|\{n_i - \delta_{ij}\}\rangle;$$

$$a_j^\dagger|\{n_i\}\rangle = \sqrt{n_j + 1}|\{n_i + \delta_{ij}\}\rangle.$$

One then defines operators

$$a_\pm = \frac{1}{\sqrt{2}}(a_2 \pm i a_1)$$

so that $[a_\pm, a_\pm^\dagger] = 1$ is the only nonvanishing commutator in the set of $a_\pm$ and $a_\pm^\dagger$. Writing

$$K_{q-} = a_+ a_- + \frac{1}{2}\sum_{i=3}^{q} a_i a_i$$

it follows that

$$[K_{q-}, (a_\pm^\dagger)^{\ell_q}] = \ell_q (a_\pm^\dagger)^{\ell_{q-1}} a_\mp.$$

Thus an appropriate state $|\psi_{\ell_q}\rangle$ which vanishes when acted on by K_{q-} is easily seen to be

$$|\psi_{\ell_q}\rangle = (a_\pm^\dagger)^{\ell_q}|\psi_0\rangle \qquad (4)$$

where $N_q|\psi_0\rangle = 0$. This establishes that *all* eigenvalues of J_{q3} are possible E_0's and upon insertion into the relations (1) and (3) one sees that the eigenvalues of L_q^2 are given by

$$L_q^2 = \ell_q(\ell_q + q - 2),$$

where ℓ_q is any non-negative integer. Thus the eigenvalue spectrum of L_q^2 follows from strictly group theoretical considerations.

The extension to the additional members of the complete set of commuting operators (CSCO) $L_q^2, L_{q-1}^2, \ldots, L_3^2, L_{12}$ is now immediate. One notes first that the state (4) is is one of [9]

$$(q + 2\ell_q - 2)\frac{(q + \ell_q - 3)!}{\ell_q!(q-2)!} \qquad (5)$$

states satisfying $N_q|\psi\rangle = \ell_q|\psi\rangle$ and $K_{q-}|\psi\rangle = 0$ and forming an irreducible representation of $SO(q)$. For each value of $q' < q$ one defines operators $K_{q'\pm}$, $J_{q'3}$, and $N_{q'}$. One readily infers the eigenvalues of $L_{q'}^2$, which are Casimir operators of $SO(q') \subset SO(q)$, to be of the form $\ell_{q'}(\ell_{q'} + q' - 2)$ where the $\ell_{q'}$ are integers satisfying the relations

$$\ell_q \geq \ell_{q-1} \geq \ell_{q-2} \geq \cdots \geq \ell_2 \geq 0. \qquad (6)$$

One deduces from the branching rules [10] of the above representation that all possible combinations of $\ell_{q'}$ are attained.

Since this procedure yields only the eigenvalues of the square of L_{12} despite the fact that the *sign* of L_{12} is to be included in the commuting set of operators, some additional comment is warranted in that case. To this end one notes that

$$L_{12} = a_+^\dagger a_+ - a_-^\dagger a_-$$

which anticommutes with the parity operator P. The latter has the property that

$$P a_1 P^{-1} = -a_1$$

while leaving all a_i unchanged for $i > 1$. Since P commutes with the operators $L_q^2, L_{q-1}^2, \ldots L_2^2$, a CSCO necessarily requires that P be added to this set. Thus all states with nonzero ℓ_2 are twofold degenerate. In this two-dimensional space one can readily diagonalize the operator L_{12} to obtain eigenvalues $\pm\ell_2$. This completes the determination of the full set of eigenvalues of the operators $L_q^2, L_{q-1}^2, \ldots, L_3^2, L_{12}$ [11].

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[1] J. D. Louck, *Theory of Angular Momentum in N-Dimensional Space*, Los Alamos Scientific Laboratory monograph LA-2451 (LASL, Los Alamos, (1960); J. D. Louck, J. Mol. Spec **4**, 298 (1960); J. D. Louck and H. W. Galbraith, Rev. Mod. Phys **44**, 540 (1972).

[2] H. Bacry and J. L. Richard, J. Math. Phys. **8**, 2230 (1967).

[3] The results of [2] are described as *partial* by these authors inasmuch as they use the theory of spherical harmonics in q dimensions in their otherwise group theoretical derivation.

[4] Following standard usage one denotes the group and its associated Lie algebra by upper and lower cases respectively.

[5] For a different relation between Casimir operators of $o(2,1)$ and angular momentum, in the context of the magnetic vortex and magnetic monopole, see R. Jackiw, Ann. Phys. **129** 183 (1980); **201**, 83 (1990).

[6] V. Bargmann, Ann. Math. **48**, 568 (1947).

[7] A. O. Barut and C. Fronsdal, Proc. Roy. Soc. (London) **A 287**, 532 (1965).

[8] It is worth noting that the nonnegative integer spectrum of N_q can be obtained directly from the J_3 spectrum of suitable representations of $o(2,1)$ by applying the group theoretical approach of [2] to a sum of q one-dimensional oscillators. No recourse to analytic or algebraic methods is required.

[9] The result (5) is obtained by subtracting the dimension of the space with $N_q = \ell_q - 2$ from the dimension of the space with $N_q = \ell_q$.

[10] H. Boerner, *Representations of Groups*, North-Holland Publishing Company, 1970.

[11] As noted in the introductory paragraph of this work these results allow the *partial* group theoretical treatment of [2] to be extended to a *totally* group theoretical one. This has the additional satisfying consequence that all the quantum mechanical systems known to have both analytic and algebraic solutions are also amenable to group theoretical methods.