STOCHASTIC PROPERTIES OF WATER STORAGE

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ABSTRACT

This paper is basically a review or survey of range and deficit analyses in hydrology. The bulk of the paper consists of a catalogue of results, both exact and approximate, that are known about the various properties of range and deficit, with particular emphasis upon the mean, variance and asymptotic distributions; these results are categorized according to whether the sample is small or large, whether the input is dependent or independent, and whether or not seasonality is included in the model. A discussion of the problems associated with statistical inference in range and deficit analyses is included, as well as a discussion of the role and importance of range and deficit analyses in practical hydrologic applications. Several areas where further research would be of value are suggested.

PREFACE

Substantive bridges between the water storage theory, of various schools of approaches and contributions to storage problems, and the ongoing applications in planning and operation of reservoirs, have not been made yet. The storage theory has not become part of practice yet. The reason is simple. Both the water supply and the water demand, as inputs and outputs of reservoirs, are complex time processes. Not only the periodicity in parameters and the complex stochasticity superimposed but also the man-made interventions are responsible for this lack of bridges.

Whenever input and output processes are complex, it becomes difficult to develop the exact mathematical, analytical and numerical, methods that would be theoretically justified and implementable in practice. Therefore, simplifications for inputs and outputs in theoretical treatments are unavoidable in first approximations. The more simplified these processes, the farther is the theory from the practice. It is quite likely that the future of the basic water storage theory will be able to tackle many complex water problems. Only then will it be accepted by practitioners, and considered as useful tools. The numerical methods of solving storage problems, helped by theory, will constitute some of the future bridges between the theory and the practice.

The application of the range and deficit analyses, as statistical techniques for solving storage problems, has been initiated by H. E. Hurst in 1951 by his now classical paper "Long-Term Storage Capacity of Reservoirs," published in Transaction of American Society of Civil Engineers, Volume 116, pages 770-779. Basically, the study of the over-the-year regulation problems of the Nile River basin led Hurst to analytical and experimental studies that triggered the work on the water storage theory in general, and by the range and deficit analysis in particular.

The analyses of range and deficit are often considered only as theoretical approaches for either infinite or semi-infinite reservoirs, but not for finite reservoirs with two boundaries, empty and full reservoir. However, this distinction is controversial, because the range and deficit analyses can be applied to any type of reservoir: infinite, positive semi-infinite, negative semi-infinite or bounded by both sides, with either absorbing or reflecting boundaries, or with a combination of these two types of boundaries in any relationship or even with a stochastic ratio between the absorbing and reflecting aspects at the boundaries. It should be expected that the range analysis, and the deficit analysis as one of the subtopics of the range analysis, will be the major probabilistic-stochastic approach in making the future bridges between the water storage theory and the practice of planning and operation of storage reservoirs.

Analysis made by Hurst using long records of geophysical time series appear to show that the log-log plot of the range (rescaled range) versus the sample size n is a straight line with the slope greater than 1/2 or that the range is proportional to n^h, with $h > 1/2$. On the other hand, Hurst and also W. Feller in 1951 showed that for normal independent variables, the range is proportional to n^h, with $h = 1/2$. This apparent discrepancy has been called the "Hurst phenomenon." Some authors have called this the "Hurst law." However, it seems more appropriate to use the first term because studies after Hurst have shown that only in the asymptotic domain, for very large sample sizes, the straight-line fit is acceptable.

The study of the range, deficit and the other associated properties of input and output of reservoirs, has been a subject of much interest to many researchers at Colorado State University in the last two decades. The professors, research associates, and graduate studies in the Department of Civil Engineering, Department of Statistics, as well as in some other departments, have been involved. Many scientists from abroad visited Colorado State University. Their visits have been very beneficial to both, scientists at Colorado State University, as well as to visiting scientists. One of these visits was by Dr. G. G. S. Pegram from South Africa. The idea of writing this paper was discussed and agreed upon by the four writers of this paper. The first draft of the paper was written by Dr. Pegram, Dr. D. C. Boes and Dr. J. D. Salas. Dr. V. Yevjevich contributed in several sections of the draft. In the continuing work by the four authors and by an exchange of subsequent drafts, the present hydrology paper was shaped. Because Dr. Pegram has been most instrumental in assembling data and in writing parts of the paper, he is assigned the role of the senior author.

This paper presents a kind of state-of-the-art on the Hurst phenomenon, and the range and deficit analyses. However, several positions in the paper, hypotheses advanced and views on the potential of future contributions, are also given at many places in the paper. Therefore, the paper is a combination of the state-of-the-art and positions of the four writers. In writing a paper by four authors, compromise on positions, terms, interpretations of who did what, when and how, has to be made. Therefore, all the writers may not share all the statements in detail as presented in the paper.

The writers have tried their best to find all the relevant references and to show their contributions as much as feasible. However, as always the case may be, no state-of-the-art or no position paper, or their combination can exhaust all references, in all the parts of the world, and do it completely and objectively, by finding and presenting contributions by all the authors. The writers apologize if they have either missed, omitted by the lack of information, any reference, paper or contribution that bears on the Hurst phenomenon as related to the range and deficit analyses.

The writers of this paper hope that its contents will be beneficial to all those who would like to find the state-of-the-art on this subject sometime by the end of 1978 and the middle of 1979. They are welcome to challenge concepts or results presented, to advance new approaches, and to contribute new knowledge on this evolving subject.

Some contributors to the knowledge on the range and deficit take the position that everything on range and deficit analyses is already known. Others think that the problem of storage theory is really in its infancy and the the present state-of-the-art is nothing more than the first phase on a long path of mastering and developing the realistic water storage theory that will be of significant practical impact. At the water storage capacities, both surface and subsurface, will increase in the world with time by the proper planning, development and management, a logical expectation is that the water storage theory for planning and operation of storage capacities will be more and more needed, and in the proper focus, than the case is at present.

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CHAPTER 1
INTRODUCTION

1-1 Motivation of the Study of Range, Deficit and Storage

Whenever we wish to store anything, the immediate question to be asked is: "How much storage is needed?" The questions which immediately follow are: "What is the character of the inflow to storage?" and, likewise, "What is the character of the outflow from storage?"

These questions and their answers follow from the mathematical storage equation (which is the equation of conservation of mass), written as

$$\frac{dV_t}{dt} = Z_t - Y_t \qquad (1\text{-}1)$$

with V_t = the volume of storage, Z_t = the rate of inflow and Y_t = the rate of outflow all at time t. Integrating Eq. (1-1) over a finite interval of time (t_1, t_2) then

$$V_{t_2} = V_{t_1} + \int_{t_1}^{t_2} Z_t\, dt - \int_{t_1}^{t_2} Y_t\, dt =$$

$$= V_{t_1} + \int_{t_1}^{t_2} [Z_t - Y_t]dt \quad . \qquad (1\text{-}2)$$

Note that the change in storage can be written as the difference between the accumulated inflow and the accumulated outflow or, equivalently, as the cumulative sum of differences between inflows and outflows. Note further that Y_t is the total outflow from the storage, including the planned withdrawals and the uncontrolled spills. It can thus be stated unequivocally that the change in storage in an interval of time Δt is equal to the "net input," which is the integral of net inflow.

Any storage facility to be constructed must of necessity be finite. Furthermore, the amount of usable storage may fluctuate with time or in fact diminish with time. For example, if one is thinking primarily of conservation storage in a reservoir, the amount available may be affected by the need to provide flood storage, varying seasonally over a year, while the amount of storage lost to sediment accumulation will increase with time unless steps are taken to divert or remove the sediment.

Effectively then one can define the lower and upper limits of storage on some scale as V_o and $V_o + C$, where C = the capacity of the beneficial storage with $-\infty \leq V_o < V_o + C \leq +\infty$. In general, V_o and C are functions of time: $V_o(t)$ and $C(t)$.

The net inflow $X_t = Z_t - Y_t$ is a function of V_t, in addition to t. This may be so because Y_t, following some policy of withdrawals, varies with V_t, particularly when V_t reaches either of the two boundaries, $V_{o,t}$ or $V_{o,t} + C_t$, with X_t constrained to be zero.

Now the first question can be rephrased as: "With this characterization, what does one have to do to find out how much storage is needed?" Given the lower bound $V_{o,t}$ and the storage capacity C_t and the outflow Y_t, one can ascertain the net inflow X_t for a given gross inflow Z_t. In general, Z_t will be the combination of a deterministic and a stochastic component. Hence, X_t will also result as a combination of deterministic and stochastic components. Frequently X_t can be thought of as being the sum of the two components, but in general this combination will not be linear. Nevertheless, X_t will always result as a stochastic process. It remains to ascertain (or specify) the stochastic model and estimate its parameters to characterize the storage problem which can now be variously phrased as: "What is the probability distribution of $\{V_t\}$ or $\{V_t; X_t\}$ or $\{Y_t\}$?" There are other associated questions that can be asked, such as: "What is the minimum storage that for a given probability of assurance is needed to ensure that no spills nor shortages are experienced in some interval (t_o,t_1)?"; or "Given that the reservoir starts full, what size of reservoir will be needed, on the average, to prevent shortages with a given risk, to occur in some interval (t_o,t_1)?"; and similar questions.

This study is addressed to these questions. They can be answered by analyzing the statistical characteristics of V_t, and this is called general storage analysis. When $V_o \to -\infty$, but $V_o + C = 0$ and in addition $V_{t_o} = 0$, then we examine $D_{\Delta t} = -\min V_t$ in $\Delta t = (t_o,t_1)$. This is called the deficit analysis. When $V_o > -\infty$ and $C < +\infty$ we have a finite reservoir and we look for the distribution of V_t, etc. We call this the finite storage analysis. When $V_o \to -\infty$ and $V_o + C \to +\infty$ and in addition $V_{t_o} = 0$, we have an infinite reservoir and we look for the distribution of $R_{\Delta t} = \max V_t - \min V_t$ in $\Delta t = (t_o,t_1)$. We call this the infinite storage analysis or range analysis.

1-2 Objectives of this Study

The various properties of the range, deficit and storage are related to the stochastic model and the parameters of the net inflow. The primary objective of this study is then to catalogue and discuss the relationships that can be found in the literature and to fill in the gaps whenever feasible. In cases where these relations have not yet been found, we will suggest useful lines of inquiry.

The objective of this study is of a tutorial nature, pulling together what is known about range, deficit, stochastic modeling and storage theories into a unified treatment, in an attempt to show the correspondence between what appear to be separate subjects. As a result, we deliberately break with the traditional report form in which one lists previous contributions in chronological order. By contrast, we will acknowledge previous contribution in the appropriate sections. The intent is to make the whole more readable and hence more informative.

1-3 Definitions of Ranges and Deficits

1-3-1 Net Inputs

Throughout this study, except where expressly stated otherwise, input is taken to mean net input to an infinite, a semi-infinite or a finite reservoir. The net input X_t in the discrete interval [t, t+1] is taken as the difference between the gross input Z_t and gross output or withdrawal Y_t in the interval. The level of development (or the degree of regulation) is defined as the ratio $\alpha = E[Y_t]/E[Z_t]$, where $E[Y_t]$ is the expected value of beneficial withdrawals. The mean net input $E[X_t]$ (invariantly called μ) is related to α by $\mu = E[Z_t] - E[Y_t] = (1-\alpha)\, E[Z_t]$ or $\alpha = 1-\mu/E[Z_t]$; (α is sometimes expressed as a percentage). Most of the results in this paper refer to $\alpha = 1$, i.e. $\mu = 0$, and this should be understood unless stated otherwise. The variance $\mathrm{Var}[X_t]$ will be denoted by σ^2. Note also that in the above definitions the expected values can be estimated by their sample means. For instance $\hat{\alpha}$ may be defined as $\hat{\alpha} = \overline{Y}/\overline{Z}$.

The high value of α (approximately $\alpha = 1$) does not require a constant output close to its mean. A large installed capacity of a hydroelectric power plant, say n times the mean (n = 3-5), would enable nearly all the water to be released beneficially, provided there is a market for the power all the time. Therefore, α depends on many factors related to the supply and demand.

Attention is confined herein to discrete time inputs. However, the distributions and time structure of inputs dealt with will be both continuous (gamma, ARMA, etc.) and discrete (binomial, Markov chains, etc.). Furthermore, the net input X_t may be either a stationary or nonstationary process. For instance, if the net inputs are annual values of hydrologic series, they are assumed to be stationary, i.e. a process with constant mean, constant variance and constant autocovariance structure. On the other hand, if the net inputs are periodic-stochastic as in seasonal hydrologic series, they are nonstationary, i.e. a process that has periodic mean, periodic variance, periodic skewness, and often periodic autocovariance structure.

1-3-2 Accumulated Net Input

Define $S_t = S_{t-1} + X_t$, $t = 1,2,\ldots,n$; where $S_o = 0$. The process $\{S_t\}$ is the accumulated sum of all previous net inputs to an infinite reservoir. (There will be cases where S_t applies to the storage level in a semi-infinite or finite reservoir, but the meaning will be clear from the context.)

The sum of the adjusted net inputs is distinguished by an asterisk:

$$S_t^* = S_{t-1}^* + X_t - \alpha\omega \qquad (1\text{-}3)$$

where the two adjustments $\omega = \mu$ or $\omega = \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = S_n/n$ are considered. The term $S_i = \sum_{t=1}^{i} X_t$ is also called the partial sum and $S_i^* = S_i - \alpha\omega$ is the adjusted partial sum.

Define also

$$\begin{aligned} M_n &= \max\,(0, S_1, S_2, \ldots, S_n) \\ m_n &= \min\,(0, S_1, S_2, \ldots, S_n) \\ M_n^* &= \max\,(0, S_1^*, S_2^*, \ldots, S_n^*) \\ m_n^* &= \min\,(0, S_1^*, S_2^*, \ldots, S_n^*) \end{aligned} \qquad (1\text{-}4)$$

where M_n and m_n are called the maximum and minimum partial sums, respectively, and M_n^* and m_n^* are the maximum and minimum adjusted partial sums, respectively. Some definitions do not start with $S_o = 0$ or $S_o^* = 0$ but with S_1 or S_1^*. Such definitions are not used in this study.

1-3-3 The Range

Define $R_n = M_n - m_n$ as the range. This has also been called the crude range (Anis and Lloyd, 1975) as well as the unadjusted range, but these terms will not be used herein. The range is thus the difference between the maximum and the minimum of the partial sums in n time-steps. Figure 1-1 is a graphical representation of the partial sum S_i, the maximum partial sum M_n, the minimum partial sum m_n and the range R_n.

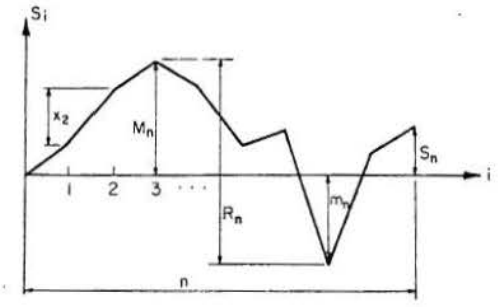

Fig. 1-1 Definition of the maximum partial sum M_n, the minimum partial sum m_n, and the range R_n.

1-3-4 The Adjusted Range

Conventionally when $\alpha = 1$, and $\omega = \overline{X}_n$, the adjusted range is defined as

$$R_n^* = M_n^* - m_n^* \quad . \qquad (1\text{-}5)$$

In this case $S_n^* = 0$, so that the adjusted range is the difference between the maximum and minimum of the adjusted sums in the first n steps. We will use this notation and definition herein, although in some cases R_n^* has been denoted simply as the range. Figure 1-2 is a graphical representation of the adjusted partial sum S_i^*, the maximum adjusted partial sum M_n^*, the minimum adjusted partial sum m_n^* and the adjusted range R_n^*.

1-3-5 The Rescaled Range

Define $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \overline{X}_n)^2$, the (biased) sample variance of the inputs. Then $R_n^{**} = R_n^*/\sigma_n$ is called

the rescaled range. Sometimes the term rescaled adjusted range is used, but it has not been the practice to rescale the unadjusted range, so the term rescaled range will be used to describe R_n^{**}. Instead of $\hat{\sigma}_n^2$, often the unbiased sample variance is used.

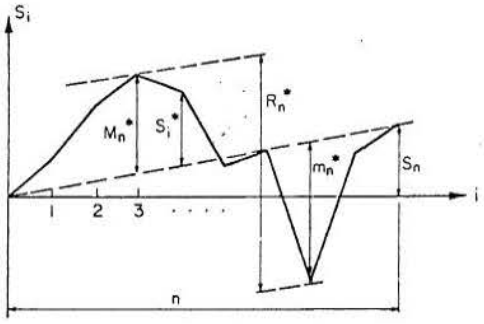

Fig. 1-2 Definition of the adjusted partial sum S_i^*, the maximum adjusted partial sum M_n^*, the minimum adjusted partial sum m_n^*, and the adjusted range R_n^*.

Other notations have been used, perhaps in an attempt to make the definitions more appealing visually. For example, ${}_aR_n$ has been used for R_n^* and ${}_rR_n$ for R_n^{**}. However, it is felt that a proliferation of notations will only complicate what is already a highly complex subject, and the simpler notation will be used herein.

1-3-6 The Conditional Range

A recent arrival, appropriate to discrete inputs, is ${}_cR_n = [R_n | S_n = 0]$ called the conditional range. It is useful as a close approximation to the adjusted range R_n^*.

1-3-7 The Maximum Accumulated Deficit

We call D_n the maximum accumulated deficit to distinguish it from m_n and m_n^* which have been called "deficit" in some publications.

Let $S_o \equiv 0$ as previously. If $S_{t-1} + X_t \geq 0$, set $S_t = 0$, otherwise $S_t = S_{t-1} + X_t \leq 0$. $\{S_t\}$ is now interpreted as the storage level in a semi-infinite reservoir which spills at $S_t = 0$, when $X_t \geq 0$, but has no bottom.

Define $D_n = -\min_i (0, S_i)$, $i = 1,2,\ldots,n$; this quantity is the maximum accumulated deficit (to a semi-infinite reservoir). Maximum accumulated deficit analysis which is to be abbreviated to the term "deficit analysis" is akin to the "sequent peak" or mass-curve analysis applied to a semi-infinite reservoir.

An equivalent definition of D_n is

$$D_n = \max \{0, M_1 - S_1, M_2 - S_2, \ldots, M_n - S_n\} \qquad (1\text{-}6)$$

with S_i and M_i as defined above. The definition of D_n is shown in Fig. 1-3.

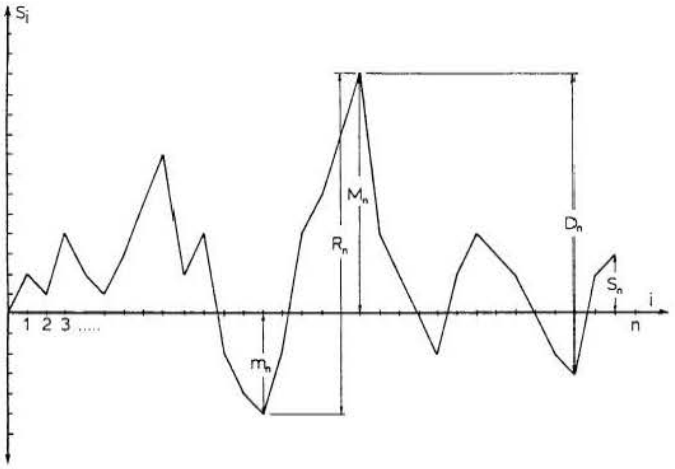

Fig. 1-3 Definition of the maximum accumulated deficit D_n as compared with the deficit of the range R_n.

1-3-8 A Comment on Sampling

All of X_t, S_t, R_n, R_n^*, R_n^{**} and D_n are random variables. It is unfortunate that in the literature little distinction has been made between *random variables* and *samples*. In what follows, most of the results apply to distributions or expectations of the random variables. Where simulation experiments are involved, reference will be made to estimates of these; for example, $\bar{r}_n$ will be an estimate of $E[R_n]$ found by performing a simulation or from sampling data from a geophysical process. Actual realizations of R_n, which we would prefer to call $r_{n,j}$ (for the jth sample of R_n) will not be referred to specifically, but it is understood that if, for example, m simulations have to be performed to estimate $E[R_n]$, $\bar{r}_n$ will be understood to mean $\frac{1}{m} \sum_{j=1}^{m} r_{n,j}$.

1-4 Organization of the Paper

Section 1-5 of this chapter reviews Hurst's personal work and the interpretation of his work by others.

Chapter 2 lists many of the known and no-so-well-known results such as expectations, variances and distribution functions of the four ranges and the deficit; (we may only give references to the more lengthy results). These are laid out as to whether n is small or large, and whether the input is dependent or independent with periodic or non-periodic parameters.

Chapter 3 touches on inference in range and deficit analysis which have not received much attention in the past, and is intended to initiate new avenues of inquiry.

Chapter 4 discusses the importance of range and deficit analysis in hydrology and is naturally a collectively personal view of where we see the studies we describe having an impact in our research and operational efforts.

Chapter 5 attempts to pull the whole of this rather loose subject together.

1-5 A Review of Hurst's Work

1-5-1 Hurst Studies

In studying the planning of the storage capacity of the Aswan Dam on the Nile River and the regulation schemes of the Great Lakes of the same river basin, H. E. Hurst (1951) realized that by using only the historical records misleading results may be obtained as to the size of the reservoir needed and as to the benefit to be expected from it. Hurst then used probability theory and experimental statistics and derived equations for the expected conditional range of independent variables and for the expected rescaled range of geophysical variables which led to much controversy and various positions in statistical hydrology during the past 30 years.

Hurst (1951) derived the expected conditional range $E({}_cR_n)$ of independent inputs which could assume only the values +1 and -1 (tossing coins for heads and tails) each with probability 1/2 as

$$E({}_cR_n) = \frac{2^n}{{}_nC_{n/2}} - 1 \quad , \tag{1-7}$$

where ${}_nC_{n/2}$ denotes the number of combinations of n objects in groups of n/2; n is, of course, even. Furthermore, using the Stirling approximation to factorials when n is large, Eq. (1-7) can be written as

$$E({}_cR_n) \doteq \left(\frac{\pi n}{2}\right)^{0.5} \doteq 1.2533\, n^{0.5} \quad . \tag{1-8}$$

Based on the fact that the expansion of the binomial distribution leads to the normal distribution Hurst showed that Eq. (1-8) is asymptocially valid for independent normal variables and that in general it can be written as

$$E({}_cR_n) \doteq 1.2533\, \sigma\, n^{0.5} \tag{1-9}$$

where σ is the standard deviation of net inputs. It is worth mentioning that the above equations derived by Hurst have always been thought to be the expected adjusted ranges until Boes (personal communication) and Gomide (1975) realized that what Hurst actually studied analytically was the conditional range. In order to verify his analytical results, Hurst conducted experiments with random events by tossing ten six-pences 1,000 times, cutting probability cards 1,000 times and with serial numbers of bonds. In each case the rescaled range was computed and a close agreement was found between the experimental results and those found analytically. It should be mentioned that Hurst actually compared two different equations, Eq. (1-9) for the expected conditional range written as $E({}_cR_n)/\sigma\sqrt{n}$ and the expected conditional range, written as $\bar{r}_n^{**}/\sqrt{n} = (\Sigma\, r_n^*/\hat{\sigma}_n\sqrt{n}/m)$ where m is the number of samples. What Hurst should preferably have used was σ instead of $\hat{\sigma}_n$, so that the estimated adjusted range $\bar{r}_n^*/\sigma\sqrt{n}$ would have compared better with $E({}_cR_n)/\sigma\sqrt{n}$. Anyway, his comparisons were still valid since $E({}_cR_n)/\sigma \approx E(R_n^*)/\sigma \approx E(R_n^{**})$ for large n.

Having derived analytically the asymptotic expected conditional range of normal independent variables, and having demonstrated, by simulation experiments with random events, that a similar expression exists for the rescaled range, Hurst wanted to find the mean rescaled ranges of series of natural phenomena. Thus Hurst (1951) conducted studies with a large number of geophysical time series such as river discharges, lake and river levels, rainfall, temperature, pressure, tree rings, layers of mud (varves) and sunspot numbers, which all together constituted 75 time series of varying length which were divided into 690 cases of lengths ranging from 30 to 2,000. Later Hurst et al. (1965, 1966) extended his experiments to 120 time series with a total of 872 cases of lengths varying from 10 to 2,000. Since Hurst's analytical and experimental studies showed that for large n the expected conditional range was proportional to σ and $n^{1/2}$ (Eq.1-9) and that the estimated mean rescaled range was proportional to $n^{1/2}$, he was expecting to find similar $n^{1/2}$ behavior for the ranges of natural time series. Instead, he found that the estimated mean rescaled range $\bar{r}_n^{**}$ increased more rapidly than $n^{1/2}$. In general, he found that

$$\bar{r}_n^{**} \doteq c\, n^h \tag{1-10}$$

with c a constant and h an exponent greater than 1/2. However, Hurst's original equation for the rescaled range was

$$r_n^{**} = \left(\frac{\bar{r}_n^*}{\hat{\sigma}_n}\right) = (n/2)^K \quad , \tag{1-11a}$$

where (although he did not imply it) K is a random variable, so

$$\log\,(\bar{r}_n^{**}) = \log\left(\frac{\bar{r}_n^*}{\hat{\sigma}_n}\right) = K \log\,(\tfrac{n}{2}) \tag{1-11b}$$

a straight line in the log domain. It should be mentioned that the reason why Hurst chose to use Eq. (1-11b) instead of Eq. (1-10) was because he wanted the line to go through $\bar{r}_n^{**} = 1$ for n = 2 since that is known for all cases (we will discuss this more fully later.)

Equation (1-11b) was used in various ways to estimate the value of K. He first used (his Fig. 4 in Hurst, 1951 and in Hurst et al., 1965) an estimate by fitting straight lines to graphical plots of log $(\bar{r}_n^{**})$ vs. log (n) so that the line goes through the point [log (1) and log (2)] and through the center of gravity of all the observations. In this way, he found values of $\hat{K} \approx 0.75$ for river statistics (river levels and discharges), $\hat{K} \approx 0.70$ for rainfall, temperature and pressure, $\hat{K} \approx 0.80$ for tree rings and similar values for other data. He secondly used a point-by-point estimate of the slope

$$\hat{K} = \log\,(r_n^{**})/\log\,(n/2) \tag{1-12}$$

for each of the 690 (Hurst, 1951) individual values of r_n^{**} (837 cases in Hurst et al., 1965) obtaining an arithmetic mean value of $\hat{K} = 0.729$, a standard deviation of 0.092 and a range of individual values of 0.46 to 0.96 (see Fig. 5 in Hurst, 1957, and Fig. 6 and Appendices 4 through 11 in Hurst et al., 1965). The third slope estimate used by Hurst and generally used by investigators is

$$\hat{K} = \log\,(\bar{r}_n^{**})/\log\,(n/2) \quad , \tag{1-13}$$

as shown in Table 7 (Hurst, 1951) where the values of $\hat{K}$ varied between 0.60 and 0.91 for the 690 cases analyzed.

As a result of Hurst's discovery that (a) his estimators of K were > 1/2, (b) his graphical approach to support fitting a curve and (c) the fact that he used some very long time series (he assumed n = 2000 was long enough to demonstrate asymptotic behavior), Hurst concluded that for geophysical time series K > 1/2 for very large n. It was his conclusion that the combined K value (0.73) derived for finite n, being different from the asymptotic result for independent normals (Eq. 1-9) was evidence that there was something special about geophysical time series. Hurst (1951) was not sure about the theoretical significance that $\hat{K}$ is approximately 3/4 for geophysical series, but comparing Eqs. (1-9) and (1-11a) he observed that while in Eq. (1-9) for random events, i.e. normal independent events ${}_cR_n$ (used as an approximation to R_n^*) is a function of σ and n, in Eq. (1-11a) for natural events, $\bar{r}_n^*$ is a function of $\hat{\sigma}_n$, n and K where "K depends on the order in which variates occur." Actually, Eq. (1-11a) would not really imply exactly that since the average is over r_n^{**} and not over r_n^*, but Hurst used Eq. (1-11a) in a rather loose sense. So Hurst himself realized that the difference he found may be due to the dependence of the time series, but he suggested that that dependence would be due to groupings of high or low values, which occur in random events, but "their tendency to occur in natural events is greater" (Hurst, 1951). In fact, he seemed to neglect the year-to-year correlation (Hurst, 1957). Referring to annual values of river flows, rainfall and temperature, he said "their principal characteristic, however, is the occurrence of periods when on the whole, values tend to be high and others when they tend to be low, though low values may occur in a high period and vice versa. So far as is shown, there is no regularity in the occurrence or the length of these periods, and usually there is not sigificant correlation over one of them between a year and its successor."

In an attempt to devise a model that would explain what he (Hurst, 1951) had found in natural time series, Hurst (1957) devised an ingenious experiment with playing cards involving shifts in the mean. He started with a pack of fifty-two cards that he called "probability cards." Each of the cards was labelled with one of the following numbers: -7, -5, -3, -1, +1, +3, +5, or +7; there were thirteen of each of the ones, eights of the threes, four of the fives, and one of the sevens. The distribution of the cards in the pack provides a crude approximation to a normal distribution with mean zero and standard deviation three.

Hurst performed the following experiment. First he cut a card from the deck, noted its number, and replaced it. He then randomly divided the fifty-two cards into two hands of twenty-six cards, and selected one of the hands. If the initially cut card was +j, he removed the j lowest cards from his hand and replaced them with the j highest cards from the other hand; and, if the cut card was -j, he removed the j highest cards from his hand and replaced them with the j lowest cards from the other hand. Thus, if he had cut a positive card, he had a tendency to give his hand a positive bias. He added a joker to his hand of twenty-six and drew cards at random, with replacement from the twenty-seven cards. The number on the first card drawn is the first number in the series, the numbert on the second card drawn is the second number in the series, etc. He continued to draw cards until he drew the joker, at which time he started over from the beginning. Four different experiments of 1000 generated numbers each were performed and analyzed, resulting in the usual diagram of log rescaled-range versus log n. His mean computed $\hat{K}$ using Eq. (1-5) for all experiments was 0.73. These experiments were further extended as described in Hurst et al. (1965).

1-5-2 Interpretation of Hurst's Findings: The Hurst Phenomenon

Hurst (1951) showed that the mean conditional range of normal independent variables was asymptotically proportional to $n^{1/2}$ (Eq. 1-9). This square root behavior was further confirmed by Feller's (1951) analytical equations when he found that the mean range $E(R_n)$ and the mean adjusted range $E(R_n^*)$ were also asymptotically proportional to $n^{1/2}$ for normal independent variables. But Hurst also showed that the mean rescaled range was on the average proportional to $n^{0.73}$ for natural series. This discrepency between theoretical results stating that the exponent of n is asymptotically 1/2 and Hurst's empirical findings that the exponent of n is greater than 1/2 for natural series, has become known as the "Hurst Phenomenon." Some statisticians and hydrologists also use the term "Hurst law" to denote the fact that when straight lines are fitted to the rescaled ranges of natural phenomena using log scales, they have Hurst slopes greater than 1/2. (The assumption is made that the available n's of natural series are long enough to guarantee asymptotic behavior.)

Interpretation of the Hurst phenomenon has been varied and controversial in statistical hydrology ever since Hurst published his first results. One interpretation has been that the expected rescaled range of hydrologic samples is asymptotically proportional to n^h with h > 1/2. Another interpretation is that series exhibiting the Hurst phenomenon have Hurst slope h greater than 1/2 for small or moderate values of n, but that it tends to 1/2 asymptotically. Hurst's findings and consequent interpretations initiated a volume of investigations by statisticians, hydrologists and geophysicists in general, intent on seeking a mathematical and/or physical explanation of the phenomenon and to ascertain its operational consequences in water resources systems.

Since the Hurst phenomenon has centered around the exponent of n (slope) being either 1/2 or greater than 1/2, comments and criticisms of Hurst work began with the way he estimated the slope and some alternative estimates have been proposed. Chow (1951) first questioned Hurst's graphical slope estimate (line going through the point log r** = 0 and log 2 = 0.30) and suggested instead a least squares estimate. Using Hurst's (1951) data, Chow found a slope of 0.87 and graphically showed that his equation $\bar{r}_n^{**} = 0.31\ n^{0.87}$ would give a better fit than Hurst's $\bar{r}_n^{**} = 0.65\ n^{0.72}$. In response to Chow's comments, Hurst indicated that the main reason why he used the slope estimates of Eqs. (1-12) or (1-13) was for the practical convenience of using just one parameter instead of two as in Chow's least squares procedure. Besides, he said, the fit obtained with just one parameter appeared to be as good as Chow's equation with two parameters.

Mandelbrot and Wallis (1969, p. 321-340) suggested that the Hurst slope should be discarded because "actual pox diagrams have a straight trend of slope H

that fails to pass through the point of abscissa log 2 and ordinate 0. Hurst's average K is thus a very poor estimate of the slope H. It tends to be too low when $H > 0.72$ and too high when $H < 0.72$." [Note that in this passage H is used instead of an (implied) constant h, neglecting the fact that the exponent h in $E[R_n^{**}] = C\, n^h$ actually varies with n. Already notational confusion abounds, but we will continue to use h as an asymptotic exponent, K as a random variable (with $\hat{K}$ its estimate) and k as a sample value, unless otherwise stated.] They referred to Figs. 2 through 7 in Mandelbrot and Wallis (1969, p. 242-259) to verify such statements. However, their statements appear to be exaggerated because while it is true that the best fit lines would not pass through the point of abscissa log 2 and ordinate 0 (obviously because the relation of $E[\overline{R}_n^{**}]$ vs. n is actually a curve, even in log-log plot, and not a line) Mandelbrot and Wallis' computations of $\overline{r}_n^{**}$ seemed to be based on unbiased variances. The equations for the variances shown in their papers are actually biased standard deviation, so they must have changed the estimate while doing the programming, which for small n actually distorts the estimate $\overline{r}_n^{**}$, causing, for instance for n = 2, values of $\overline{r}^{**} < 1$ when it should be the opposite since $\overline{r}^{**} = 1$ exactly. This can be easily verified by observing for instance Fig. 2 in Mandelbrot and Wallis (1969, p. 323), Fig. 3 in Mandelbrot and Wallis (1969, p. 245) and similar other figures in other pages of the same authors. This method of computation was later modified as it appears in a later paper by Wallis and Matalas (1970, p. 1583-1594); for instance compare Fig. 1 of this paper with Fig. 5 in Mandelbrot and Wallis (1969, p. 247).

Gomide (1975) discusses the Hurst K estimator of Eq. (1-13), acknowledging that the estimator is sound in the sense that one would like to have $R_n^* = \hat{\sigma}_n$ for n = 2 independently of the value of K. This is so when the biased estimator of $\hat{\sigma}_n$ is used. He noted that perhaps that was the reason why Hurst did not use the Eq. (1-10) with c = 1.2533 (to match the coefficient of Eq. (1-9)) which would lead to the estimator

$$\hat{K} = \frac{\log\,(\overline{r}_n^{**}) - \log\,(1.2533)}{\log\,(n)} \quad . \qquad (1\text{-}14)$$

Gomide also noted that if Eq. (1-14) is used for Hurst's 30 sequences of generated random events (Hurst, 1951, Table 6) the mean value of $\hat{K}$ is 0.50. On the other hand, if Eq. (1-13) is applied to the same data, the mean value of $\hat{K}$ is 0.64. These results indicate that Eq. (1-13) actually forces $\hat{K}$ to be > 1/2 for independent normal random events (of course this occurs in varying degree, not only with $\hat{K}$ estimated from (1-13), but with any other since $\hat{K}$ actually depends on the sample size n) and this is also to be expected when dealing with natural time series which are usually dependent. Gomide also found that when Eq. (1-14) is used, the mean value of $\hat{K}$ is 0.57 for the 690 natural series used by Hurst. More about the estimation of the slopes of ranges is given in Chapter 3.

Several attempts to explain the Hurst phenomenon have been made since Hurst's (1951) paper. Practically all those attempts center around the so-called transience, because the expected ranges (as functions of n) even of independent normal summands do display transient regions. Several models or model changes, that produce transient regions compatible with Hurst's empirical findings and consequently provide an asymptotic explanation of the Hurst phenomenon, have been previously proposed and are reviewed herein. These models and resultant explanations of the Hurst phenomenon are categorized as: (i) non-normal skewed marginal distributions, (ii) dependence structure, and (iii) "nonstationarity" in the mean or shifting levels.

Non-normal skewed marginal distributions. Commenting on Hurst's (1951) paper, Milleret (1951) questioned the assumption of normality when Hurst analyzed the 690 natural series. He said there are objections to using the normal distribution since natural phenomena always have a certain skew. Hurst replied in the sense that "the difference between R/σ "by which he meant $R_n^*/\hat{\sigma}_n$" for natural and for pure chance events is not due to skewness, but to the fact that the distribution of natural events is not haphazard. It happens that irregular successions of high or low years tend to occur in any long series of years." Hurst had at least good intuitive feeling (although without mathematical proof) that the skewness would not be the cause of the difference found for normal random events and natural events (the Hurst phenomenon) but instead the cause would be the "groupings of high and low values." It seems that this is why Langbein (1955) in commenting Hurst's (1955) second paper was interested in the reason for the grouping tendency observed by Hurst.

Based on statistical tests using the long records of some American rivers, Langbein concluded that the skewness would not be the cause of the grouping tendency, thus confirming Hurst's previous statement. In another comment on the Hurst (1956) paper, Lloyd and Anis (1955) mentioned that a possible explanation of the Hurst phenomenon might lie in the nature of the distribution of the individual increments and in particular their non-normality. However, they also mentioned that such a line of reasoning would not work, making reference to Feller's (1951) paper where he argued that the asymptotic distribution of the range (meaning R_n and R_n^* only) were independent of the distribution of the increments and so the asymptotic $n^{1/2}$ law followed even when they were non-normal.

Assuming that the net inputs are independent stable distributions having characteristics function $\exp\{-|t|^\gamma\}$, $1 < \gamma < 2$, Moran (1964) showed that the expected range is proportional to $n^{1/\gamma}$ and Boes and Salas (1973) showed that the expected adjusted range is also proportional to $n^{1/\gamma}$. Therefore, although stable distributions are not quite appropriate for geophysical series, it was thought that, at least mathematically, distributions with heavy tails (such as the stable distributions) would provide expected ranges proportional to n^h with $h > 1/2$ as geophysical series were thought to have. However, even though both $E(R_n)$ and $E(R_n^*)$ were shown to be proportional to $n^{1/\gamma}$ there was no assurance that $E(R_n^{**})$ would behave in the same way. Simulation studies with independent stably distributed summands with $\gamma = 1.3$ carried out by McLeod and Hipel (1978) gave r_n^{**} values nearly the same as those of independent normal variables, suggesting that $E(R_n^{**}) \sim n^{1/2}$ even though $E(R_n) \sim n^{0.77}$ and $E(R_n^*) \sim n^{0.77}$ in this case.

Yevjevich (1965) studied the effect of skewness by generating a very large number of samples of independent gamma processes, with various values of the

skewness coefficient. The range and adjusted range behavior showed relatively modest deviations from their corresponding values for normal independent processes. Deviations from the normal case increased both with a decrease of n and an increase of the skewness coefficient, but the overall effect was not very great. Matalas and Huzzen (1967) used a simulation of normal and lognormal lag-one Markov variables. They concluded that the skewness had virtually no effect on the Hurst slope. Moran (1968) suggested using gamma variables with large skewness. Mandelbrot and Wallis (1969c), using computer simulation with the skewed marginal distributions of truncated normal, lognormal, and hyperbolic form showed that the asymptotic $n^{1/2}$-law is valid for both the adjusted and resclaed ranges. Furthermore, they showed heuristically that for gamma variables with large skewness, the adjusted range is asymptotically proportional to $n^{1/2}$.

Although previous experiments by Mandelbrot and Wallis (1969c) and by Matalas and Huzzen (1967) appeared to show that skewed distributed independent variables would yield $\overline{r}_n^{**} \sim n^{1/2}$, there was still the question whether the transience induced by skewness would be such that it would be an explanation of the Hurst phenomenon. In this direction, Anis and Lloyd (1975) derived expressions for the expected range and the expected adjusted range of independent gamma variables and showed that for particular parameters $E[R_n]$ and $E[R_n^*]$ behave as n^h with $h > 1/2$ for $100 < n < 1000$. They suggested a possibility of similar behavior for $E[R_n^{**}]$. Using relatively large skewness values, they were able to produce local slopes of the order observed by Hurst for $100 < n < 1000$ for both the range and adjusted range. Thus, they showed that severe skewness was capable of producing a transient region consistent with Hurst's finding, at least for the range and adjusted range.

However, Salas et al. (1979) generated 10,000 values of independent exponential and gamma variables with skewness equal to 2 and 10, respectively and compared the $\overline{r}_n^*$ and $\overline{r}_n^{**}$ obtained for these variables with the corresponding ranges of independent normal variables (known exactly in this case). In the case of the adjusted range $\overline{r}_n^*$ decreases as the skewness increases therefore showing different transience. However, the transience depends on what slope estimator is used. For instance, if the Hurst slope $\hat{K}$ of Eq. (1-13) is used, the transience decreases as the skewness increases, but if other estimators such as the Mandelbrot-Wallis H or the local slope

$$H_n = \log (\overline{r}_{n+j}^{**}/\overline{r}_{n-j}^{**})/\log ((n+j)/(n-j)) \qquad (1\text{-}15)$$

for some j close to 1, are used, then the transience increases with the skewness (the term transience used herein denotes how fast (with n) the slope tends to the limiting slope). On the other hand, the computed $\overline{r}_n^{**}$ for both exponential and gamma variables are indistinguishable from the $E(R_n^{**})$ of normal variables. Since geophysical series have skewness much smaller than those tested here, it is safe to conjecture that for all practical purposes the skewness affects $\overline{r}_n^*$ in the transient region, but does not affect $\overline{r}_n^{**}$ appreciably.

Dependence structure. Since many geophysical series are time dependent, and since theoretical results for $E(R_n^{**}) \sim n^{1/2}$ have been confined to independent variables, it is reasonable to hypothesize that the dependence structure may contribute to prolonging the transient region and hence provide an explanation of the Hurst phenomenon. It is fair to mention that Hurst (1951, p. 783) himself hypothesized that the groupings of high and low flows (long-term persistence) observed in natural series may be a cause of the Hurst phenomenon. But he puts this concept in a different context to the typical year-to-year serial correlation and, in fact, he later (Hurst, 1956, p. 575; Hurst 1957, p. 494; Hurst et al. 1965, p. ; Hurst et al. 1966, p. 67) emphasized that the observed groupings were due to a persistence over an indefinite period (long-term persistence) rather than due to the year-to-year correlation. Evidently Hurst considered two types of persistence or dependence structure, one long-term and another short-term and thus he attributed the cause of the observed groupings (and therefore the cause of the Hurst phenomenon) to the long-term persistence. However, since long-term persistence was little understood in the 1950's (and in fact is still not well understood today), and on the contrary since short-term persistence and corresponding models (say autoregressive models) have been pretty well understood, it was natural to first look at these vis à vis the Hurst phenomenon.

Feller (1951) suggested the autocorrelation structure as a possible explanation of the Hurst phenomenon, and in fact he specifically referred to the Markov process for the modeling of the increments but he also warned about the difficulty of finding an analytical solution to the problem since it would demand the solution of the general Fokker-Planck equation whose solution is not explicitly known. Barnard (1956), however, claimed that no single set of autocorrelation functions could account for the Hurst phenomenon. Thomas (1956) discussing the causes of the "groupings" observed by Hurst referred to Laushey's (1956) discussion on Hurst's (1956) paper and said he had no doubts that the value of the Hurst slope K was closely associated with the degree of autocorrelation. Also Langbein (1956) in analyzing long records of some American rivers, as well as some of the Hurst data, concluded that the persistence was more complex than that of an AR(1) model. Moran also suggested that the serial correlation of the underlying process would have to be of a very peculiar kind for the slope h to remain greater than 0.50 as $n \to \infty$. He went on saying that "a more plausiable theory is that the experimental series used by Hurst are, as a result of serial correlation, not long enough for the asymptotic formula to become valid, and thus for small n, $E(R_n^{**})$ may vary like n^h with $h \neq 1/2$."

Since the use of autoregressive models (in fact the use of stochastic models in general) for generation of hydrologic data began at the beginning of the 1960's with the work of those like Thomas and Fiering (1962) and Yevjevich (1963), it was natural to study the range properties of these models despite previous suggestions (Hurst, 1956; Barnard, 1956; Langbein, 1956 and Moran, 1959) that the Hurst phenomenon may not be explained with a simple dependence structure, but with a very special type of persistence. Thus Yevjevich (1965, 1967, 1972) did some analytical work to find the distribution of the range complemented with extensive computer experiments. He argued and showed by plots that the Hurst phenomenon is the result of transience. Fiering (1967) was especially concerned in finding out whether for common design lines of hydraulic structures (say for time periods of 10 to 100 years) the use of autoregressive models

would be adequate. He made extensive computer experiments with these models and compared the storages obtained from simulation (using the sequent-peak algorithm) with those obtained by the Hurst equation. Fiering concluded that for the periods of 10 to 100 years the AR models were adequate for most design purposes. However, he admitted that for larger periods such as those studied by Hurst, more elaborate models may be needed. Therefore, although both Yevjevich's and Fiering's studies were useful for explaining the applicability of AR models for storage design, they were limited in fully explaining the Hurst phenomenon because they dealt with relatively small values of n compared with those used by Hurst.

In referring to Hurst's analysis of geophysical records, Mandelbrot and Wallis (1969, p. 321) stated: "were the records in question generated by a random process such that observations far removed in time can be considered independent R(t,n)/S(t,n) should become asymptotically proportional to $n^{1/2}$, which means that Hurst's Law would have to 'break' for large enough lags. But no such break has been observed. Thus, for practical purposes, geophysical records must be considered to have an 'infinite' span of statistical interdependence." Based on this interpretation of the Hurst phenomenon, Mandelbrot and Wallis (1968, p. 910) suggested that models in the Brownian domain cannot account for the Noah and Joseph effects. (They defined the Noah effect as the observation that extreme precipitation can be very extreme and the Joseph effect as the observation that a long period of unusual high or low precipitation can be extremely long.) And they stated "therefore, our sweeping assertion can only be controversial in its blanket condemnation of all models in the Brownian domain."

The Brownian domain contains those models for which the ultimate behavior for $\overline{r}_n^{**}$ follows the $n^{1/2}$ law, and the interpretation by Mandelbrot and Wallis of the Hurst phenomenon is that $\overline{r}_n^{**}$ asymptotically behaves like n^h with $h > 0.50$. Following this interpretation, all models within the Brownian domain of attraction, e.g., autoregressive models AR(p), autoregressive-moving average models ARMA(p,q) and even Hurst-like shifting level models, fall under this "blanket condemnation." Using the above interpretation as justification, Mandelbrot and Wallis proposed a fractional Gaussian noise model for hydrologic modeling and simulation. This model uses the concept of self-similarity, implying that the autocorrelation structure is such that the infinite past exerts small but non-negligible effects on the present. This model produces an asymptotic behavior of $\overline{r}_n^{**}$ as n^h with $h > 0.50$. However, some hydrologists (Yevjevich, 1968 at the Tucson Symposium, personal communication; Scheidegger, 1970, p. 754; Klemes, 1974, p. 676-678) have questioned the physical basis of fractional Gaussian noise models as well as the above interpretation of the Hurst phenomenon put forward by Mandelbrot and Wallis.

We note that the frequently stated justification for the use of FGn takes a form similar to the opinion expressed by Wallis (1977): "... most geophysical records and some laboratory experimental noises (Brophy, 1970) have [sic] $h > 0.5$, while all stationary independent and short memory stochastic processes have $h = 0.5$." We do not argue with the possibility that some "real" processes may have asymptotic $h > 0.5$, but content that $n < 2000$ is too short to enable a statistician to reject the null hypothesis that asymptotic $h = 0.5$ for these processes. We have only to examine the slope of the log $[E[R_n^{**}]]$ vs. log [n] curve for independent normal inputs as given by Anis and Lloyd [1976] to see that at $n = 10^3$ we get $K = 0.587$ and $H_n = 0.514$, while at $n = 10^6$ we get $K = 0.544$ and $H_n = 0.501$. Dogmatic statements like Wallis' above are unfortunate because they cloud the issue, in the sense that they cannot stand there without a rebuttal. Regrettably, we cannot prove Wallis wrong because we do not have long enough records as yet, but we do feel that he has rather over-stated the case for FGN.

Markovian models, characterized by the property of short memory, have been criticized as not being capable of simulating the Hurst phenomenon. This is based on the interpretation of the Hurst phenomenon as being the asymptotic behavior of $\overline{r}_n^{**}$ versus n for geophysical time series, neglecting the transience or the pre-asymptotic effects. Yevjevich (1972, p. 167) using relatively short samples of generated AR(1) series, computed the Hurst $\hat{K}$ from Eq. (1-13), using $\overline{r}_n$ instead of $\overline{r}_n^{**}$, and gave the mean of $\hat{K}$ as a function of n.

Gomide (1975) using samples as long as those used by Hurst, and assuming as Hurst (1951) that $E(_cR_n)$ is an approximation of $E(R_n^*/\sigma_n)$, computed the Hurst slopes for first-order autoregressive models with $0 \leq \rho \leq 0.90$, and found that the computed slopes for n up to 1000 were similar to those found by Hurst for his data. The mean of Gomide's computed $\hat{K}$ was around 0.75 as compared to Hurst's 0.73. The above results do not imply that all the geophysical time series follow the first-order autoregressive model. However, this simple dependence model shows that transience (as a result of serial correlation) by itself can explain the Hurst phenomenon. Klemes (1974) also demonstrated that infinite dependence was not a prerequisite for a process to exhibit the Hurst phenomenon, in fact, processes of short-term or no persistence, as a result of a storage, can produce the Hurst phenomenon.

ARMA models were suggested by Carlson et al. (1970) for modeling streamflow series and O'Connell (1971, 1974) suggested the ARMA(1,1) model for reproducing the Hurst phenomenon. Extensive simulation experiments were conducted to investigate the long-term characteristics of this model by computing the rescaled range r_n^{**} and observing the slopes of the "pox-diagrams" r_n^{**} versus n. It was shown that the ARMA(1,1) model could yield slopes of the order of those found by Hurst. Further experiments carried out by Boes and Salas (1978) and Salas et al. (1979) indicated a strong transient behavior of $\overline{r}_n^{**}$ versus n for the ARMA(1,1) models, in that $\overline{r}_n^{**} \sim n^{H_n}$ with $H_n >> 1/2$ for n's that could be of the order of 100-2000 and ultimately $H_n \to 1/2$ as $n \to \infty$ (we here use H_n of Eq. (1-15)). In fact, they showed that $\overline{r}_n^{**}$ versus n results in an S-curve (Fig. 1-4), which may be divided into three regions: (1) an initial region with slopes somewhat greater than 1/2; (2) a pre-asymptotic region with slopes much greater than 1/2; and (3) the asymptotic region, or the region of ultimate behavior, with slopes converging to 1/2. Also, McLeod and Hipel (1978) and Hipel and McLeod (1978) demonstrate that ARMA models may reproduce the Hurst phenomenon.

Shifting levels. As mentioned above, Hurst (1957) himself tried to build a model that could reproduce the Hurst phenomenon in geophysical series. His

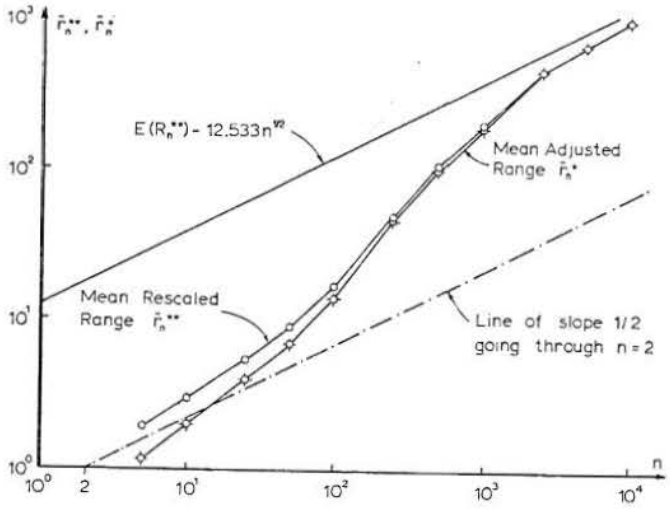

Fig. 1-4 Mean adjusted range and mean rescaled range of equivalent models of ARMA(1,1) and non-stationary means GNN in logarithmic scales. The two curves for $\bar{r}_n^{**}$ and $\bar{r}_n^*$ of each of these two models are mutually indistinguishable, and appear as unique curves.) Taken from Salas et al., 1979.

ingenious experiment with playing cards involved shifts in the mean. Four different experiments each of 1000 generated numbers were performed and then analyzed, resulting in a plot of log $\bar{r}_n^{**}$ versus log n that produced a Hurst slope of about 0.70, with a break toward the asymptotic $n^{1/2}$-law somewhere around n = 1000. Hurst concluded that his experiment led to time series very similar to the geophysical series he had earlier analyzed. He was first to demonstrate that shifting levels provide a possible explanation of the Hurst phenomenon.

Klemes (1974) made a thorough review of the Hurst phenomenon, including comments on possible physical factors that could cause it. Among various causes he followed Hurst and investigated the effect of non-stationarity in the process "central tendency" by simulation of random series with periodic changes in the mean, and with means alternating randomly. His results showed that the plot of $\bar{r}_n^{**}$ versus n yielded slopes much greater than 1/2 for moderated n with the slope tending to 1/2 as $n \to \infty$. In discussing Klemes' paper, Potter (1975) also reported experiments using time series with shifting means obtaining similar results to Klemes. Potter (1976a) extended the argument of shifting means as a physical explanation of the Hurst phenomenon by analyzing six precipitation records from the East Coast of the United States and argued that nonstationarity of the mean is a viable explanation (although later Potter (1978) seemed to conclude that the apparent shift in the mean may be attributed to nonhomogeneity). Boes and Salas (1978) generalized the concepts of Hurst, Klemes and Potter and proposed a mixture model for shifting levels. In particular, they showed that special shifting level models can have correlation structures identical to ARMA models and consequently, since it is the correlation structure that dictates the preasymptotic behavior of the rescaled range, the shifting level models and ARMA models have equal capacity for mimicking the Hurst phenomenon.

1-5-3 Concluding Remarks

Extensive analyses reported by Hurst (1951, 1956) and Hurst et al. (1965, 1966) of geophysical records of lengths varying from 10 to 2000 years showed that $(\overline{r_n^*/\sigma_n}) = n^K$ with a mean value $\hat{K} = 0.73$ and a standard deviation of 0.092. On the other hand, Hurst's analytical result with independent normal variables yielded $E[_cR_n] \sim n^{1/2}$ for $n \to \infty$ (this analytical result was later shown to be valid for $E(R_n)$, $E(R_n^*)$ and $E(R_n^{**})$). This apparent discrepancy of a slope greater than 1/2 for geophysical series and an asymptotic slope equal to 1/2 for independent normal series has become known as the Hurst phenomenon. Two main interpretations of the Hurst phenomenon have been postulated in the literature: (1) that the Hurst phenomenon is caused by the supposition that geophysical time series have ultimate or asymptotic slopes greater than 1/2, and (2) that the Hurst phenomenon is caused by the supposition that geophysical time series have slopes greater than 1/2 in the preasymptotic region, becoming 1/2 asymptotically. Based on these two main interpretations, hydrologists and statisticians have analyzed and suggested models to exhibit the Hurst phenomenon, and sought for explanations. Research results appear to indicate that transience may be the major reason for the Hurst phenomenon, that transience is inherent, even in independent normal processes but can be accentuated by dependence (short memory or long memory), shifts in the mean and skewness. Models within the Brownian domain of attraction (autoregressive, ARMA, shifting levels) yield Hurst slopes similar to those found by Hurst for geophysical series, but asymptotically the slopes are known to converge to 1/2. On the other hand, models outside the Brownian domain of attraction such as Fractional Gaussian Noise models, can yield asymptotic Hurst slopes between 1/2 and 1.

CHAPTER 2
EXACT AND APPROXIMATE RESULTS

In this chapter, where we concentrate by far the greatest effort, will be found some exact and approximate results for the expectations, variances and distributions, for finite and infinite n, periodic and non-periodic, dependent and independent inputs in each of the cases that have been distinguished in the introduction: The Range, The Conditioned Range, The Adjusted Range, The Rescaled Range and The (maximum accumulated) Deficit. The results also depend on the marginal distribution of the inputs, so where known, results will be given under the following hierarchy of headings: Preasymptotic results; independent inputs with constant parameters; dependent inputs with constant parameters; dependent inputs with periodic parameters; and marginal distribution.

Much is well known, so will only be listed where appropriate. Less well-known results will be enlarged upon a little, and although the intention is that the paper should be of a review nature, we have occasionally succumbed to temptation and reported some new work when we felt this would help the development; this is clearly stated and acknowledgements are given where appropriate.

2-1 Preasymptotic Results of the Range R_n

2-1-1 Independent Net Inputs with Constant Parameters

Bernoulli inputs. Let $P[X_t = +1] = P[X_t = -1] = 1/2$. $S_n = \sum_{t=1}^{n} X_t$ is then the classic one-dimensional random walk. Gomide (1975, Eqs. (4.22) and (4.23)) gave the explicit recursive expressions for $E[R_n]$ and $E[R_n^2]$ as:

$$E[R_{n+1}] - E[R_n] = v_n(1,1) + v_n(1,2)$$

$$E[R_{n+1}^2] - E[R_n^2] = v_n(1,1) + v_n(1,2) + 2\sum_{k=1}^{k=K-1}$$

$$\{v_n(1,1) + v_n(1,2) - \sum_{j=-\infty}^{\infty} [v_n(2j(k+1) + 1,1)$$

$$+ v_n(2j(k+1) + 1,2) - v_n(2j(k+1) - 1, k-1)$$

$$- v_n(2j(k+1-1),k)]\},$$

where $v_n(r,t) = {}_nC_{(n+r-t)/2}(1/2)^n = {}_nC_{(n+t-r)/2}(1/2)^n$, and K is a "very large number." Gomide gives a table (his Table 4.2) for $E[R_n]$ and $\mathrm{var}[R_n]$ for $n = 1,2,\ldots,100$. He also gives an algorithm for the explicit evaluation of the p.d.f. of R_n for a discrete input of which the Bernoulli inputs are a particular case.

Binomial inputs. Let $P[X_t = i] = {}_mC_{(\frac{m}{2}+i)}(1/2)^m$ for $i = -\frac{m}{2}, -\frac{m}{2}+1,\ldots,0,\ldots,\frac{m}{2}-1,\frac{m}{2}$. Gomide (1975) used this distribution with $m = 100$ as an approximation to the normal distribution.

Normal inputs. Let X_t be independent and normal (0,1) distributed random variables. For this case, Anis and Lloyd [1953] derived the result

$$E[R_n] = \sqrt{\left(\frac{2}{\pi}\right)} \sum_{i=1}^{n-1} i^{-1/2} \quad , \tag{2-1a}$$

where $R_n = \max(S_1,\ldots,S_n) - \min(S_1,\ldots,S_n)$. Equation (2-1a) should read

$$E(R_n) = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} i^{-1/2} \tag{2-1b}$$

when $R_n = \max(0,S_1,\ldots,S_n) - \min(0,S_1,\ldots,S_n)$.

However, Anis' and Lloyd's formula can be written in a more general form, based on Spitzer's (1957) identity

$$\sum_{j=1}^{\infty} \Phi_j(t)\, Z^j = \exp\left[\sum_{j=1}^{\infty} j^{-1}\psi_j(t)\, Z^j\right] \tag{2-2}$$

where $\Phi_j(t)$ and $\psi_j(t)$ are the characteristic functions of M_j and S_j^+, respectively; that is $\Phi_j(t) = E\{\exp(i \cdot M_j)\}$ and $\psi_j(t) = E\{\exp(i \cdot S_j^+)\}$ and $S_j^+ = \max(0, S_j)$. The above identity is actually valid even for exchangeable random variables of any distribution function. Based on the above identity, Salas (1972) demonstrated that Anis and Lloyd formula can be written as

$$E(R_n) = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} i^{-1} (\mathrm{Var}\, S_i)^{1/2} \quad . \tag{2-3}$$

Gomide (1975) derived the p.d.f. for R_n, $n = 1,2,3$, explicity. As $f(R_2)$ and $f(R_3)$ require evaluation of $\Phi(\cdot)$ (the c.d.f. of the standardized normal) no closed form solutions exist for the evaluation of $f(R_n)$, $n \geq 2$. As an alternative, Gomide provided an algorithm for the numerical evaluation of the p.d.f. of R_n by binomial approximation.

Laplace inputs. Let X have a p.d.f.: $f_X(x) = (\frac{1}{\sqrt{2}})\exp[-\sqrt{2}|x|]; -\infty<x<\infty$. Gomide (1975) derived explicit expressions for the evaluation of the p.d.f. of the range for $n = 1,2,3,4$, and gave a recursive relation for $f(R_n)$. Further, he derived the expressions for the mean range

$$E[R_n] = \sqrt{2} \sum_{i=1}^{n} {}_{2i}C_i (1/2)^{2i} \quad , \tag{2-4}$$

and for the second moment of the range

$$E[R_n] = 2 \sum_{i=1}^{n} \sum_{j=0}^{n-1} j\, b_{ij}/i\sqrt{2})^{j+1} \tag{2-5}$$

where b_{ij} are found from complicated recursive relations. He tabulated $E[R_n]$ and $\mathrm{var}[R_n]$ for $n = 1,2,\ldots,30$.

Exponential inputs. Gomide (1975) derived $f(R_n)$ analytically for n = 2 and numerically for n = 8, 50.

Stable inputs. Let $X, X_1, X_2, \ldots, X_n, X, X_1, X_2, \ldots, X_n$ be independent random variables with common characteristic function $\psi(t) = \exp(-|t|^{\gamma})$ where $1 < \gamma \leq 2$. X then has a stable distribution with finite first moment (for examples see Feller (1966)).

Moran (1964) showed that in this case

$$E[R_n] = E[|X|] \sum_{i=1}^{n} i^{(1/\gamma)-1} \quad . \tag{2-6}$$

In particular, when $\gamma = 2$, X is normally distributed.

2-1-2 Dependent Net Inputs with Constant Parameters

Bernoulli inputs. Let $P[X_t=+1] = P[X_t=-1] = 1/2$ and further, let $P[X_{t+1}=\pm 1 | X_t=\pm 1] = (1+\rho)/2 = p$, and $P[X_t+1=\pm 1 | X_t=\mp 1] = (1-\rho)/2 = q$. These correlated Bernoulli inputs X_t form a Markov chain $\{X_t\}$ with $\text{corr}[X_{t+s}, X_t] = \rho^s$, s=0,1,2,...

Gomide (1975) found $f(R_n)$ for this process for n up to 1000 and for ρ = 0,0.3,0.6,0.8, and 0.9. He demonstrated that the standardized range $(R_n - E[R_n])/\sqrt{\text{var}[R_n]}$ of this two-state input process converged in distribution to the asymptotic standardized range with n as low as 100.

Binomial input. In this section new developments are presented. Readers interested in the final numerical results should proceed to the end of the section. Let $P[X_t=i] = {}_mC_{(\frac{m}{2}+i)} (\frac{1}{2})^m$, $i = -\frac{m}{2}, -\frac{m}{2}+1, \ldots, \frac{m}{2}-1, \frac{m}{2}$, and let $L = \rho I + (1-\rho)p1'$ be the transition matrix of the homogeneous Markov chain $\{X_t\}$. Hereafter in this section, $p = \{P[X_t=i]\}$ is the equilibrium vector marginal distribution of X_t, I is an (m+1)-square identity matrix and the 1' is a row vector of m+1 ones, where the prime denotes the matrix transpose. It can be shown that $\text{corr}[X_{t+s}, X_t] = \rho^s$, s = 0,1,2,... If we define v' = [-m,-m+1,-m+2,...,m-1,m] and V = diag(v), then $E[X_t] = v'p = 0$; $\sigma_x^2 = E[X_t^2] = v'Vp$ and $E[X_{t+1}X_t] = v'LVp = v'[\rho I + (1-\rho)\, p1']Vp = \rho v'Vp + (1-\rho)v'p\ 1'Vp = \rho\, \sigma_x^2$.

Further as

$$\begin{aligned} L^2 &= [\rho I + (1-\rho)p1']^2 \\ &= \rho^2 I + 2\rho(1-\rho)p1' + (1-\rho)^2 p1'p1' \\ &= \rho^2 I + (2\rho - 2\rho^2 + 1 - 2\rho + \rho^2)p1' \quad (\text{as } 1'p = (1)) \\ &= \rho^2 I + (1-\rho^2)p1' \quad , \end{aligned}$$

then $\text{corr}[X_{t+s}, X_t] = \rho^s$ as asserted. This process is then the discrete analog of the lag-one autoregressive normal process, and the approximation improves with an increase of m.

It has been demonstrated by Gomide (1975) that for independent binomial inputs the standardized range $(R_n - E[R_n])/\sqrt{\text{var}[R_n]}$ converges (in distribution) to the asymptotic distribution for n as low as 6 when m = 100. For moderate n then, it is sufficient to know the standardized asymptotic p.d.f. of R_n together with $E[R_n]$ and $\text{var}[R_n]$ for an adequate description of the distribution of the range for arbitrary ρ, and the results for the dependent binomial input converge to those of the dependent normal (autoregressive) process when m or n is large.

In this section, an algorithm for computing $E[R_n]$ and $\text{var}[R_n]$ for dependent binomial net inputs is given. The method is an adaption and extension of the Gomide (1975) algorithm and leans heavily on his theoretical treatment.

The key equation is Gomide's Eq. (4.9), which applies to independent and dependent discrete inputs equally well, namely

$$P[R_n=k] = \lambda_{k+1}^{(n)} - 2\lambda_k^{(n)} + \lambda_{k-1}^{(n)} \tag{2-7}$$

where

$$\lambda_k^{(n)} = \sum_{u=1}^{k} \sum_{s=1}^{k} q_k^{(n)}(s,u) \tag{2-8}$$

and "$\sum_{s=1}^{k} q_k^{(n)}(s,u)$ denotes the probability that the system $[S_n]$ does not reach the boundaries (states 0 and k+1) in the first n steps given the initial state u."

In the independent input case $q_k(s,u)$ is an element of the k-square "restricted" transition probability matrix Q_k of a Markov chain $\{S_n : S_n = \sum_{t=1}^{n} X_t\}$ where the states 0 and k+1 are taboo states. Thus $q_k^{(n)}(s,u)$ is the (s,u) element of the matrix $\{q_k(s,u)\}^n = Q_k^n$.

In the case of dependent inputs we interpret $q_k^{(n)}(s,u)$ as $P[S_{t+n} = u | S_t = s,\ 0 < S_{t+i} < k+1: i=0,1,2,\ldots,n]$ and it remains to define this conditional probability. To accomplish this, we treat the problem as one where $\{S_t, X_t\}$ is a bivariate Markov chain.

Let $q_{si|uj} = P[S_{t+1} = s,\ X_{t+1} = i | S_t = u,\ X_t = j]$ (s,u = 0,1,2,...,k+1; i,j = 0,1,...,m; t = 0,1,...) be the conditional probabilities of the bivariate homogeneous Markov chain $\{S_t, X_t\}$. Inadmissable transitions will result in the values of $q_{si|uj}$ being zero.

The joint probability $P[S_{t+1} = s,\ X_{t+1} = i,\ S_t = u,\ X_t = j]$ can be written as $q_{si|uj}\ P[S_t = j,\ X_t = j] = q_{si|uj}\ P[S_t = u]\ P[X_t = j]$ because X_t is independent of S_t.

Then $P[S_{t+1} = s|S_t = u] = \{ \sum_{i,j} P[S_{t+1} = s, X_{t+1} = i, S_t = u, X_t = j]\} / P[S_t = u] = \{ \sum_{i,j} q_{si|uj} P[S_t = u] P[X_t = j]\}/P[S_t = u] = \sum_{i,j} \{q_{si|uj} P[X_t = j]\}$.

By similar reasoning, it is found (by employing the Chapman-Kolmogorov theorem) that

$$P[S_{t+n}=s|S_t=u,\ 0 < S_{t+i} < k+1,\ i=0,.,\ldots,n]$$

(i.e. the probability that $S_{t+n} = s$ given that $S_t = u$ without any intermediate S-values reaching either of the boundaries 0 and k+1)

$$= \sum_{i,j} q^{(n)}_{si|uj} P[X_t = j] \quad .$$

Here, the conditional probability $q^{(n)}_{si\ uj}$ is an element of the "restricted" (to state s = 1,2,...,k) n-step transition probability matrix (t.p.m.)

$$Q_k^n = Q_k^{n-1} Q_k, \ (n = 1,2,\ldots) \quad .$$

The matrix Q_k is defined as follows. The net input process $\{X_t\}$ is an (m+1)-state Markov chain described by an (m+1)-square t.p.m. L such that Lp = p; 1'p = 1. L can be written as $L = L_{-m/2} + L_{-m/2+1} + \ldots + L_{m/2-1} + L_{m/2}$ (for m even), where L_i is an (m+1)-square matrix, void except for the column corresponding to the state $i \in \{-m/2,\ldots,0,\ldots,m/2\}$.

Now since attention is restricted to the states S = 1,2,...,k, the nature of states 0 and k+1 is immaterial, hence the restricted transition matrix Q_k becomes:

$$Q_k = \begin{bmatrix} L_o & L_{-1} & L_{-2} & L_{-k+1} \\ L_1 & L_0 & L_{-1} & L_{-k+2} \\ L_2 & L_1 & L_0 & L_{-k+3} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ L_{k-1} & L_{k-2} & L_{k-3} & L_0 \end{bmatrix}$$

where $L_j \equiv 0$ for $j > m/2$ (if $m/2 < k-1$). One further definition is required and then the algorithm for the evaluation of $\lambda_k^{(n)}$ follows immediately.

Let $\pi = [p'\ p'\ \ldots\ p']'$ be a k(m+1) element vector consisting of k identical vectors p, then the scalar

$$\lambda_k^{(n)} = 1' Q_k^n \pi$$

where, in this context, 1' is a row vector of k(m+1) ones performing the necessary summation.

For any particular choice of n, we assemble $\lambda_k^{(n)}$ for k = 1,2,..,K, where K is chosen so that $\lambda_K^{(n)} - \lambda_{K-1}^{(n)} = 1+e$ where $|e| < 10^{-12}$. Then (Gomide 1975, Eq. 4-12)

$$E[R_n] = K - \lambda_k^{(n)} + e \qquad (2\text{-}9)$$

and

$$var[R_n] = 2 \sum_{i=1}^{K-1} \lambda_i^{(n)} + \lambda_K^{(n)}(1-\lambda_K^{(n)}) \qquad (2\text{-}10)$$

which is more convenient than Gomide's (1975, Eq. 4.12 combined with his Eq. 4.13).

To get reasonable approximation to the normal distribution, m must be large, especially for small n. Exploratory computations yielded the following table for $var[R_2]$ as a function of m. The value corresponding to $m = \infty$ is Gomide's (1975) result (p. 24).

m	4	16	100	∞
$Var[R_2]$	0.6619	0.6157	0.6023	0.5997

Thus for small n, if a binomial approximation to the normal is desired, m must be set 100. A straight-forward application of the algorithm requires a Q_k which can be up to 5000-square! Exploiting the sparse and patterned nature of Q_k, the computation can be compressed to one in which the largest matrices are of the order of 5000 elements. Even so, one is daunted by the amount of computation required for the evaluation of $E[R_n]$ and $var[R_n]$ for large m up to large n, which explains the censored appearance of the following Table 2-1 of results.

Table 2-1 $E[R_n]$ and $Var[R_n]$ for binomial inputs.

ρ	m	n=2	n=4	n=8	n=16	n=32	
0	∞	1.3621	2.2217	3.4879	5.3171	7.9322	$E[R_n]$
0	100	1.3594					
	64	1.3579	2.2165				
	36	1.3546	2.2124	3.4772			$E[R_n]$
	16	1.3453	2.2010	3.4640	5.2908		
	4	1.2969	2.1408	3.3943	5.2139	7.8220	
	100	0.7761					
	64	0.7770	1.0256				
	36	0.7790	1.0271	1.3976			$\sqrt{var[R_n]}$
	16	0.7847	1.0312	1.4008	1.9408		
	4	0.8136	1.0518	1.4178	1.9540	2.7266	
0.1	100	1.3826					
	64	1.3810	2.3006				
	36	1.3776	2.2967	3.6722			$E[R_n]$
	16	1.3679	2.2854	3.6598	5.6630		
	4	1.3172	2.2262	3.5950	5.5942	8.4686	
	100	0.8528					
	64	0.8340	1.1446				
	36	0.8565	1.1462	1.5764			$\sqrt{var[R_n]}$
	16	0.8437	1.1504	1.5784	2.1818		
	4	0.8800	1.1731	1.5889	2.1843	3.0362	
0.3	100	1.4291					
	64	1.4274	2.4761				
	36	1.4236	2.4719	4.0993			$E[R_n]$
	16	1.4130	2.4599	4.0876	6.6061		
	4	1.3578	2.3969	4.0243	6.4461	9.9586	
	100	0.9342					
	64	0.9359	1.3940				
	36	0.9394	1.3962	1.9843			$\sqrt{var[R_n]}$
	16	0.9493	1.4026	1.9853	2.7575		
	4	0.9954	1.4399	1.9945	2.7431	3.7949	
0.5	100	1.4756					
	64	1.4737	2.6619				
	36	1.4697	2.6569	4.5933			$E[R_n]$
	16	1.4582	2.6428	4.5798	7.5504		
	4	1.3984	2.5679	4.5016	7.4821	11.8771	
	100	1.0236					
	64	1.0236	1.6628				
	36	1.0300	1.6668	2.5069			$\sqrt{var[R_n]}$
	16	1.0423	1.6779	2.5105	3.5614		
	4	1.1028	1.7417	2.5452	3.5460	4.8993	
0.7	100	1.5221					
	64	1.5200	2.8592				
	36	1.5157	2.8528	5.1827			$E[R_n]$
	16	1.5033	2.8349	5.1624	0.9493		
	4	1.4391	2.7396	5.0404	8.8276	-14.6463	
	100	1.1038					
	64	1.1062	1.9530				
	36	1.1114	1.9597	3.2213			$\sqrt{var[R_n]}$
	16	1.1259	1.9784	3.2367	4.8713		
	4	1.1967	2.0785	3.3424	4.9178	6.8719	

Table 2-2 gives the results for $m = 4$ and $\rho = 0$ extended to $n = 1024$. The first row ($m = \infty$) is computed from Anis and Lloyd's (1953) equation for independent normal net inputs.

Table 2-2 $E[R_n]$ and Var $[R_n]$ for the dependent binomial from Eqs. (2-9) and (2-10).

n	64	128	256	512	1024	m
$E[R_n]$	11.6508	16.9241	24.3920	34.9606	49.9119	∞
$E[R_n]$	11.5355	16.8052	24.2706	34.8372	49.7813	4
$\sqrt{var[R_n]}$	3.8399	5.3979	7.6210	10.7686	15.2227	4

Gaussian inputs. Because of the labor and complexity of the mathematics involved, there are very few exact results for $E[R_n]$, $var[R_n]$ and $f[R_n]$ for finite n in the case of dependent Gaussian processes, and these are confined to small n. On the other hand, a large amount of simulation of varying degrees of approximation to the exact results has been performed and reported on.

Arbitrary correlation structure of normal inputs. Salas (1972) derived exact expressions for $E[R_n]$ for $n = 1,2,3$ when X_1, X_2, X_3 have a multivariate normal distribution with general mean and covariance. He considered as special cases exchangeable and first-order autoregressive random variables. The expressions are rather unwieldy and will not be reproduced here. For comparison purposes we single out his expressions in the autoregressive case.

Autoregressive lag-one model of normal inputs. Salas (1972) gave the following expressions for $E[R_n]$, $n = 1,2,3$ when $corr[X_{t+s},X_t] = \rho^s$, $\sigma_X = 1$, $\mu_X = 0$.

$$E[R_1] = \sqrt{\frac{2}{\pi}}$$

$$E[R_2] = \sqrt{\frac{2}{\pi}}\left[1 + \frac{1}{\sqrt{2}}\sqrt{1+\rho}\right]$$

$$E[R_3] = \sqrt{\frac{2}{\pi}}\left\{\left[\frac{3}{4} + \frac{2}{2\pi}\cdot\arctan(1+\rho)\right] + \sqrt{2(1+\rho)}\cdot\left[\frac{1}{4} + \frac{1}{2\pi}\cdot\arctan\frac{2+2\rho-\rho^2}{2\rho\sqrt{2(1+\rho)}}\right] + \sqrt{(3+4\rho+2\rho^2)}\cdot\left[\frac{1}{4} + \frac{1}{2\pi}\cdot\arctan\left(\frac{(1+\rho)^2}{\sqrt{3+4\rho+2\rho^2}}\right)\right]\right\} \qquad (2\text{-}11)$$

Extending Salas' work, Troutman (1974) derived the expected range for $n = 4$ as

$$E[R_4] = \sqrt{\frac{2}{\pi}}\cdot\sum_{i=1}^{4} c_i\cdot i^{-1}\sqrt{var(S_i)} \qquad (2\text{-}12)$$

where

$$c_1 = \frac{1}{2} + \frac{1}{2\pi}\arctan\frac{(1+\rho)(2+\rho+\rho^2)}{\sqrt{2(1+\rho+\rho^2)}} + \frac{1}{2\pi}\cdot\arctan\frac{(1+\rho+\rho^2)}{\sqrt{2+2\rho+\rho^2}} + \frac{1}{\pi}\arctan(1+\rho)$$

$$c_2 = \frac{3}{2} - \frac{1}{2\pi}\cdot\arctan\frac{2\sqrt{(1+\rho)(1+\rho+\rho^2)}}{\rho^2} - \frac{1}{\pi}\arctan\frac{\sqrt{2+2\rho+\rho^2}}{\sqrt{2(1+\rho)}} - \frac{1}{\pi}\arctan\frac{\sqrt{2\rho(2+2\rho)}}{(2+2\rho-\rho^2)}$$

$$c_3 = \frac{3}{2} - \frac{3}{2\pi}\arctan\frac{\rho(1+\rho)^2}{\sqrt{2(1+\rho+\rho^2)(3+4\rho+2\rho^2)}} - \frac{3}{2\pi}\arctan\left[\rho\sqrt{\frac{2+2\rho+\rho^2}{3+4\rho+2\rho^2}}\right] - \frac{3}{2\pi}\arctan\frac{\sqrt{3+4\rho+2\rho^2}}{(1+\rho)^2}$$

$$c_4 = \frac{1}{2} + \frac{1}{\pi}\arctan\left[\frac{1+2\rho+4\rho^2+2\rho^3+\rho^4}{2\sqrt{(1+\rho)(2+\rho+\rho^2)(1+\rho+\rho^2)}}\right] + \frac{2}{\pi}\arctan\sqrt{\frac{(1+\rho)^3(2+\rho+\rho^2)}{2(2+2\rho+\rho^2)}}$$

with $var[S_1] = 1$, $var[S_2] = 2(1+\rho)$, $var[S_3] = 3 + 4\rho + \rho^2$, and $var[S_4] = 2(1+\rho)(2+\rho+\rho^2)$.

Salas (1974) and Troutman (1974) used the above formulae to check the following very close approximation (hypothesized by Yevjevich, 1967) to the exact values of $E[R_n]$

$$E[R_n] \approx \sqrt{\frac{2}{\pi}}\cdot\sum_{i=1}^{n} i^{-1}\sqrt{var[S_i]} \quad , \qquad (2\text{-}13)$$

which is exact for the case of independent or exchangeable (see Eq. 2-3) random variables $X_1, X_2, \ldots, X_n$, as shown by Boes and Salas (1973). Salas (1974 and Troutman (1974) found that the approximation was to within 0.1% for $E[R_3]$ and 0.2% for $E[R_4]$, respectively. Troutman (1976) showed, inter alia, that Eq. (2-13) for $E[R_n]$ holds exactly as $n \to \infty$ in any case where the correlogram of X_t is integrable and where $\mu_X = 0$ and $0 < \sigma_X^2 < \infty$.

Sutabutra (1967) had suggested (based on his simulations) that Eq. (2.11) may be a satisfactory approximation to the expected range of a first-order autoregressive process. Yevjevich (1967) showed with extensive simulation that this was indeed the case and showed further that it applied equally well in the case of a second-order autoregressive process and a first-order moving-average process.

Results for $var[R_n]$ are harder to come by. Salas (1972) derived $E(R_2^2)$ and $Var(R_2)$ for the AR(1) model as

$$E(R_2^2) = 2(1+\rho) + \frac{3(1-\rho^2)^{1/2}}{\pi} - \frac{(1+2\rho)}{\pi} \arctan\left[\frac{(1-\rho^2)^{1/2}}{\rho}\right], \quad (2\text{-}14)$$

and

$$\mathrm{Var}(R_2) = 2(1+\rho) - \frac{2}{\pi} + \frac{(1+\rho)^{1/2}}{\pi}[3(1-\rho)^{1/2} - (1+\rho)^{1/2} - 2\sqrt{2}] - \frac{(1+2\rho)}{\pi} \arctan\left[\frac{(1-\rho^2)^{1/2}}{\rho}\right] \quad (2\text{-}15)$$

The results in Table 2-1 appear to be the first attempt to get analytical approximations (via binomial approximation) to the variance of the range of the lag-one autoregressive (AR(1)) input process for $n > 2$. Salas (1972) also demonstrated by simulation that $\mathrm{var}[R_n]$ was apparently linearly proportional to n.

Gomide (1975) showed that a linear relationship between $\mathrm{var}[R_n]$ and n holds to a high degree of approximation for the case of independent normal inputs.

Table 2-3 compares the expression

$$\mathrm{var}[R_n] \doteq 0.2261 \frac{1+\rho}{1-\rho} \cdot n \quad (2\text{-}16)$$

shown by Troutman (1976) to hold asymptotically for AR(1) processes with the binomial approximations of the previous section (Table 2-1) given in the upper and lower line respectively for each ρ.

Table 2-3 Comparison of $\mathrm{var}[R_n]$ from Eq. (2-16) (upper line) with those obtained from the binomial approximation (Table 2-1) (lower line).

n	2	4	8	16	32
ρ= 0	.4522 .6023	.9044 1.0519	1.8088 1.9533	3.6176 3.7667	7.2352 7.4343
ρ=0.1	.5527 .6936	1.1054 1.3101	2.2108 2.4850	4.4215 4.7603	8.8430 9.2185
ρ=0.3	.8398 .8727	1.6796 1.9432	3.3592 3.9374	6.7184 7.6038	13.4368 15.7998
ρ=0.5	1.3566 1.0478	2.7132 2.7649	5.4264 6.2845	10.8528 12.6836	21.7056 24.0031
ρ=0.7	2.5625 1.2184	5.1249 3.8142	10.2499 10.3768	20.4997 23.7296	40.9995 47.2230

From an examination of the results for the dependent binomial process of the previous section, it will be seen that these results are upper bounds to the values of the variance, whereas for higher n and ρ values we can be sure that the asymptotic formula provides a lower bound. Nevertheless, as n becomes large both approximations will improve. (They both converge to the true variance as $n \to \infty$.)

For practical purposes then, the distribution of the range of the important class of linearly dependent Gaussian processes can be computed with fair accuracy from reasonably simple formulae, (or the tables herein) together with Feller's (1951) expression for the asymptotic p.d.f. of the range:

$$f_R(\lambda) = 8 \sum_{j=1}^{\infty} (-1)^{j+1} j^2 \phi(j\lambda), \quad \lambda > 0, \quad (2\text{-}17)$$

where $R = R_n/\sigma_X\sqrt{n}$ and $\phi(\cdot)$ is the standardized normal p.d.f.; see Section 2-2-1.

Exchangeable random variables. When $\mathrm{corr}(X_i, X_j) = \rho$ for $i,j = 1,2,\ldots,n$; $i \neq j$, the X_i are exchangeable or symmetrically correlated. Based on Spitzer's (1956) lemma, Salas (1972) demonstrated that the following result holds for exchangeable normal summands for n - 1,2,3:

$$E[R_n] = \sqrt{\left(\frac{2}{\pi}\right)} \sum_{i=1}^{n} i^{-1} \sqrt{\mathrm{var}(S_i)} \quad . \quad (2\text{-}18)$$

Boes and Salas (1973) showed that this expression holds for any n and for an arbitrary degree of development, α. The case of full development ($\alpha = 1$) coincides with the problem of the range considered here, and in that case

$$\mathrm{var}(S_i) = \sigma_X^2 \cdot i[1 + \rho(i-1)] \text{ , so that}$$

$$E[R_n] = \sqrt{\frac{2}{\pi}}\,\sigma_X \cdot \sum_{i=1}^{n} \{[1 + \rho(i-1)]/i\}^{1/2} \} \quad . \quad (2\text{-}19)$$

2-1-3 Independent Net Inputs with Periodic Parameters

Little work has been done to determine the range properties of independent net inputs with varying or periodic parameters. The analytical results available are those of Salas (1972). He derived the expected ranges for n = 2 and n = 3 for independent normal variables with variances σ_1^2, σ_2^2 and σ_3^2 as

$$E(R_2) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2 + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)^{1/2}\right] \quad , \quad (2\text{-}20)$$

and

$$E\{R_3\} = \sqrt{\frac{2}{\pi}}\{\frac{1}{4}(\sigma_1+\sigma_2+\sigma_3) + \frac{1}{4}[(\sigma_1^2+\sigma_2^2)^{1/2} + (\sigma_2^2+\sigma_3^2)^{1/2} + (\sigma_1^2+\sigma_2^2+\sigma_3^2)^{1/2}] + \sigma_1 \frac{1}{2\pi} \arctan\left(\frac{\sigma_2}{\sigma_3}\right) + \sigma_3 \frac{1}{2\pi} \arctan\left(\frac{\sigma_2}{\sigma_1}\right) + (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)^{1/2} \frac{1}{2\pi} \arctan\left(\frac{\sigma_1\,\sigma_3}{\sigma_2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)^{1/2}}\right)\}. \quad (2\text{-}21)$$

As may be observed from the above equations, $E[R_n]$ becomes a complex function of σ_1^2, σ_2^2 and σ_3^2, as n increases. Salas proposed a simple approximate equation containing the variances of all possible combinations of partial sums as

$$E(R_n) \doteq \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \frac{i^{-1}}{\binom{n}{i}} \sum_{j=1}^{\binom{n}{i}} [\mathrm{Var}(S_i)_j]^{1/2} \qquad (2\text{-}22)$$

where $(S_i)_j$ denotes the j-th sum of size i out of $\binom{n}{i}$ possible sums. In other words, for given values of n and i, there are $\binom{n}{i}$ possible ways in which S_i may be formed. For example, for n = 3, Eq. (2-22) becomes

$$E(R_3) = \sqrt{\frac{2}{\pi}} \{(\mathrm{Var}\, S_i)^{1/2} + \frac{1}{6} [\mathrm{Var}\, S_2)_1^{1/2} + (\mathrm{Var}\, S_2)_2^{1/2} + (\mathrm{Var}\, S_2)_3^{1/2}] + \frac{1}{3} (\mathrm{Var}\, S_3)^{1/2}\}$$

which in terms of the variances of the summands becomes

$$E(R_3) = \sqrt{\frac{2}{\pi}} \{\sigma_1 + \frac{1}{6} [(\sigma_1^2 + \sigma_2^2)^{1/2} + (\sigma_1^2 + \sigma_3^2)^{1/2} + (\sigma_2^2 + \sigma_3^2)^{1/2}] + \frac{1}{3} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)^{1/2}\} . \qquad (2\text{-}23)$$

It may be shown that Eq. (2-22) leads to the Anis and Lloyd Eq. (2.1) if $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$. The approximation of Eq. (2-23) with respect to the exact Eq. (2-21) was checked for various combinations of values of σ_1, σ_2 and σ_3. For all cases tested, the error was within $\pm$ 1.0%.

The approximation of Eq. (2-22) was also tested for increasing, decreasing and periodic functions of the variance of the summands and for various values of n (Salas, 1972). In these cases, the comparison was made between the $E[R_n]$ computed for Eq. (2-22) and the corresponding values obtained from computer simulation. The results obtained were good as may be observed in Fig. 2-1 for the case of increasing variance, Fig. 2-2 for the case of decreasing variance and Fig. 2-3 for the case of periodic variance.

Although the approximation by Eq. (2-22) is very good, its evaluation takes much computer time for large n. Salas (1972) found further approximations. For instance, since Eq. (2-22) requires that, for given n and i the average of the standard deviation of all possible sums of size i must be computed, instead one can take a random sample of a given size, say 100, out of all the possible sums of size i and then take the average over the sample size. Using this idea, Salas gave the equation

$$E(R_n) = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \frac{i^{-1}}{m} \sum_{j=1}^{m} [\mathrm{Var}\, (S_i)_j]^{1/2} , \qquad (2\text{-}24)$$

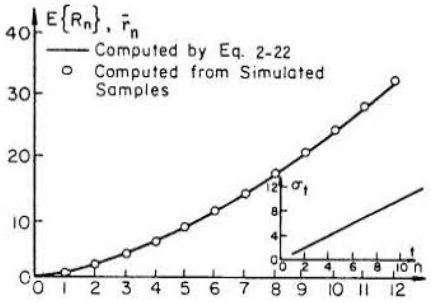

Fig. 2-1 Comparison of mean ranges obtained from simulated samples and the expected values of range computed by Eq. (2-22) for independent random variables with standard deviation increasing with t (Salas, 1972).

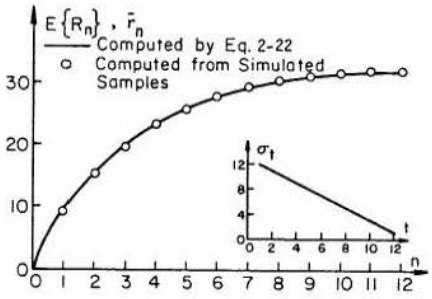

Fig. 2-2 Comparison of mean ranges obtained from simulated samples and the expected values of range computed by Eq. (2-22), for independent random variables with standard deviation decreasing with t (Salas, 1972).

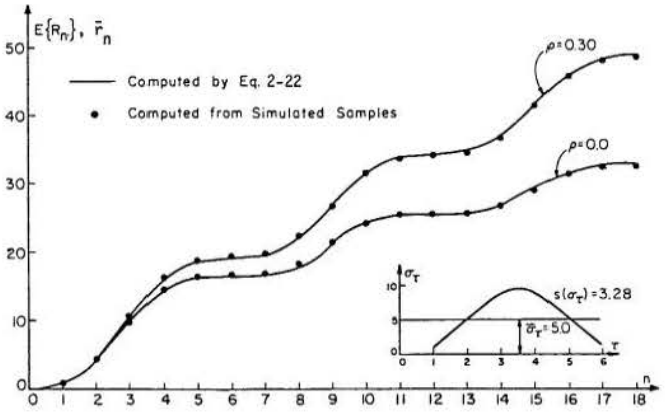

Fig. 2-3 Comparison of mean ranges obtained from simulated samples and the expected values of range computed by Eq. (2-22), for non-stationary exchangeable random variables (Salas,1977).

where m denotes the sample size of the sums computed and the subscript j denotes a particular realization of the sum of size i taken at random. He also tested the approximation of this equation concluding that for practical use of this procedure a compromise should be made between the accuracy of results and the amount of computer time required, both of which depend on the size of the sample considered.

Another equation suggested by Salas (1972) for estimating the expected range of independent variables with varying variance is

$$E(R_n) = \sqrt{\frac{2}{\pi}}\,\hat{\sigma}_n \sum_{i=1}^{n} i^{-1/2} \qquad (2\text{-}25)$$

with $\hat{\sigma}_n$ defined by

$$\hat{\sigma}_n = \sqrt{\frac{1}{n}\sum_{\tau=1}^{n}\sigma_\tau^2} \qquad (2\text{-}26)$$

where σ_τ^2 denotes the variance at time τ. For the particular case of periodic variance σ_τ^2 with $\tau = 1,\ldots,\omega$ and ω the main period (say $\omega = 12$ for monthly values), Eqs. (2-25) and (2-26) give

$$E[R_n] = \sqrt{\frac{2}{\pi}}\sqrt{\frac{1}{\omega}\sum_{\tau=1}^{\omega}\sigma_\tau^2}\sum_{i=1}^{n} i^{-1/2} \qquad (2\text{-}27)$$

Equation (2-27) is the same as Eq. (2-25) only for values of $n = p\omega$, say for $n = 12,24,\ldots,12p$ with p an integer and $\omega = 12$. However, as n becomes large, Eqs. (2-25) and (2-27) yield similar results. The goodness of the approximation of Eq. (2-25) is tested by comparing results obtained by this equation and by computer simulation. The results are shown in Fig. 2-4 for the cases of a linear increase and a linear decrease of variance and in Fig. 2-5 for the case of periodic variance.

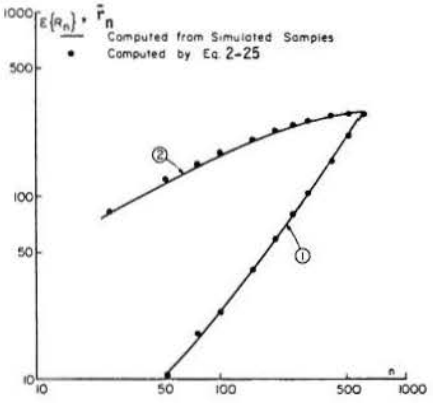

Fig. 2-4 Comparison of mean range obtained from simulated samples and the expected range computed by Eq. (2-25) for independent variables with standard deviation: (1) increasing with t, and (2) decreasing with t (Salas, 1972).

2-1-4 Dependent Net Inputs with Periodic Parameters

The results of a systematic study of basic characteristics of the expected range of dependent net inputs with periodic parameters, carried out by Salas (1972), are reported herein.

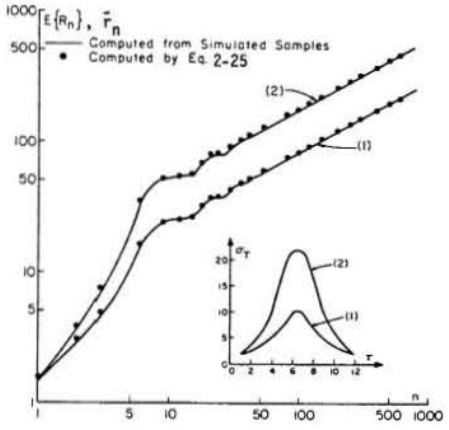

Fig. 2-5 Comparison of mean range obtained from simulated samples and the expected range computed by Eq. (2-25), for two cases of independent variables with periodic standard deviation: (1) $\bar{\sigma}_\tau = 5$ and $s(\sigma_\tau) = 2.79$, and (2) $\bar{\sigma}_\tau = 10$ and $s(\sigma_\tau) = 6.87$ (Salas, 1972).

Model with periodic variance and exchangeable dependence. A model with periodic variance σ_τ^2, $\tau = 1,\ldots,\omega$ (ω is the period), and equal autocorrelation coefficients $\rho_k = \rho$ for any lag k (exchangeable or symmetrically correlated random variables) may be written as (Owen and Steck, 1962)

$$x_t = \sigma_\tau\,(\sqrt{\rho}\;\varepsilon_o + \sqrt{1-\rho}\;\;\varepsilon_t) \qquad (2\text{-}28)$$

where ε_o and ε_t are independent normal variables with mean zero and variance one, both uncorrelated. Salas showed that Eqs. (2-22), (2-24), (2-25) and (2-27) are good approximations for the expected range of summands derived from Eq. (2-28). For instance, a good approximation is obtained between the expected range computed from Eq. (2-22) and those obtained from computed simulation for $\rho = 0.30$ and σ_τ with $\bar{\sigma}_\tau = 5$, and $s(\sigma_\tau) = 3.28$ (Fig. 2-3).

Autoregressive model with periodic coefficients. AR models with periodic (seasonal) autoregressive coefficients may be written as (Salas, 1972)

$$x_{v,\tau} = \sum_{j=1}^{p}\phi_{j,\tau-j}\,x_{v,\tau-j} + b_\tau\varepsilon_{v,\tau} \qquad (2\text{-}29)$$

with $b_\tau = \left(1 - \sum_{i=1}^{p}\sum_{j=1}^{p}\phi_{i,\tau-i}\,\phi_{j,\tau-j}\,\rho_{|i-j|,\tau-\ell}\right)^{1/2}$, $\ell = \max(i,j)$, where $\phi_{j,\tau}$ is the periodic autoregressive coefficient of order j and time τ, $\rho_{j,\tau}$ is the periodic autocorrelation coefficient, p is the order of the model and $\varepsilon_{v,\tau}$ is the independent normal variable with mean zero and variance one. Salas used Eq. (2-3), (valid for exchangeable random variables and shown to be an excellent approximation for AR models with constant autoregressive coefficients) for

approximating the expected range when the inputs are AR models with periodic parameters as in Eq. (2-29). Therefore, using Eq. (2-3), the expected range may be written as

$$E[R_n] = \sqrt{\frac{2}{\pi}}\,\sigma \sum_{i=1}^{n} i^{-1} \left[i + 2 \sum_{t=1}^{i-1} \sum_{u=1}^{i-t} \sum_{j=1}^{p} \phi_{j,t+u-k}\, \rho(u-j,t) \right]^{1/2} \quad (2\text{-}30)$$

where $t = (p-1)\omega + \tau$ and $\rho(u-j,t)$ is the periodic autocorrelation function which may be determined from an extension of the Yule-Walker equation given by Salas (1972, p. 13).

The validity of Eq. (2-30) was tested by computer simulation for AR(1), AR(2) and AR(3) models and various cases of periodic parameters. Figure 2-6 gives a good approximation between $E[R_n]$ computed by Eq. (2-30) and those obtained by computer simulation. It also shows $E[R_n]$ to be greater for AR(1) models with periodic parameters than for AR(1) models with constant parameters.

Autoregressive model with periodic variance and constant coefficients. AR models with periodic variance and constant autoregressive parameters are expressed by

$$x_{v,\tau} = \sigma_\tau \left[\sum_{j=1}^{p} \phi_j\, z_{v,\tau-j} + b\, \varepsilon_{v,\tau} \right], \quad (2\text{-}31)$$

with

$$b = \left[1 - \sum_{i=1}^{p} \sum_{j=1}^{p} \phi_i\, \phi_j\, \rho_{i-j} \right]^{1/2}$$

where σ_τ^2 is the periodic variance of $x_{v,\tau}$, ϕ_j is the j-th constant autoregressive coefficient, ρ_j is the constant autocorrelation coefficient of order j, $z_{v,\tau}$ is the standardized dependent variable $x_{v,\tau}/\sigma_\tau$ and $\varepsilon_{v,\tau}$ is an independent variable with mean zero and variance one. Computer simulation by Eq. (2-31), carried out for p = 1, periodic σ_τ with $\bar{\sigma}_\tau = 5.0$ and $S(\sigma_\tau) = 2.79$, and autocorrelation coefficients $\rho = 0$, $\rho = 0.3$, $\rho = 0.6$ and $\rho = 0.9$, gave the expected ranges presented in Fig. 2-7. The mean range is an increasing periodic function, with the same period as for σ_τ and maximum amplitudes that are out of phase with respect to σ_τ. Salas (1972) gives an approximate equation for the expected range as

$$E[R_n] = \sqrt{\frac{2}{\pi}} \left\{ \hat{\sigma}_n \sum_{i=1}^{n} i^{-1/2} + \bar{\sigma}_\tau \left[\sum_{i=1}^{n} i^{-1} (\text{var}\, S_i)^{1/2} - \sum_{i=1}^{n} i^{-1/2} \right] \right\} \quad (2\text{-}32)$$

where $\hat{\sigma}_n$ is defined by Eq. (2-26), $\bar{\sigma}_\tau$ is the mean of the periodic standard deviation and $\text{var}(S_i)$ is the variance of the partial sums of AR summands with constant autoregressive coefficients. The approximate mean ranges of AR models obtained by Eq. (2-32) with $\bar{\sigma}_\tau = 5.0$, $S(\sigma_\tau) = 2.79$ and $\rho = 0.6$, and with $\bar{\sigma}_\tau = 10.0$, $S(\sigma_\tau) = 6.87$ and $\rho = 0.6$ were compared with those directly obtained by simulation, with a good agreement (Salas, 1972).

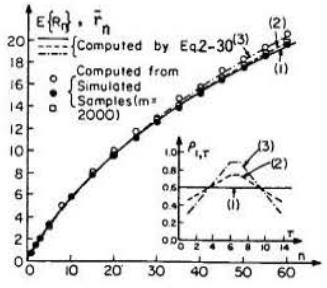

Fig. 2-6 Comparison of mean range obtained from simulated samples and the expected range computed by Eq. (2-30) for the first-order Markov model with $\bar{\rho}_{1,\tau} = 0.60$, and: (1) $s(\rho_{1,\tau}) = 0.0$; (2) $s(\rho_{1,\tau}) = 0.102$; and (3) $s(\rho_{1,\tau}) = 0.207$ (Salas, 1972).

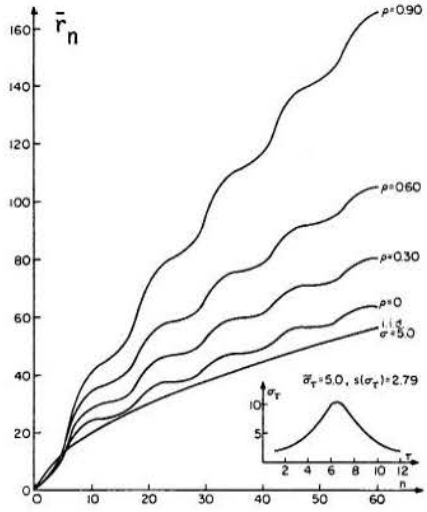

Fig. 2-7 Mean range obtained from simulated samples for the model $x_{p,\tau} = \sigma_\tau(x_{p,\tau-1} + \sqrt{1-\rho^2}\,\varepsilon_{p,\tau})$, with periodic standard deviation σ_τ and constant first autocorrelation coefficient ρ (Salas, 1972).

Autoregressive model with periodic mean, periodic variance and constant autoregressive coefficients. The model is

$$x_{v,\tau} = \mu_\tau + \sigma_\tau \left[\sum_{j=1}^{p} \phi_j\, z_{v,\tau-j} + b\, \varepsilon_{v,\tau} \right], \quad (2\text{-}33)$$

where μ_τ is the periodic mean, $z_{v,\tau}$ is the standardized dependent variable $(x_{v,\tau}-\mu_\tau)/\sigma_\tau$ and the other terms as defined for Eq. (2-31). Salas (1972) computed the expected range of net inputs equal to $x_{v,\tau}-\bar{\mu}_\tau$ by simu-

lation, showing that differences between $\bar{r}_n$ of net inputs of Eq. (2-33) and $\bar{r}_n$ of net inputs of Eq. (2-31) are constant, or independent of n after a short transience and also independent of ρ, but are dependent on the standard deviation $s(\mu_\tau)$, on the mean $\bar{\sigma}_\tau$ and on the standard deviation $s(\sigma_\tau)$. Figure 2-8 is an example of results obtained by simulation. They show that the $\bar{r}_n$ of AR net inputs with parameters μ_τ, σ_τ and ρ can be made up of two parts, the deterministic part, as a function of $s(\mu_\tau)$, $\bar{\sigma}_\tau$ and $\rho(\bar{\sigma}_\tau)$, and the stochastic part, as a function of $\bar{\sigma}_\tau$, $\rho(\sigma_\tau)$, ρ and n.

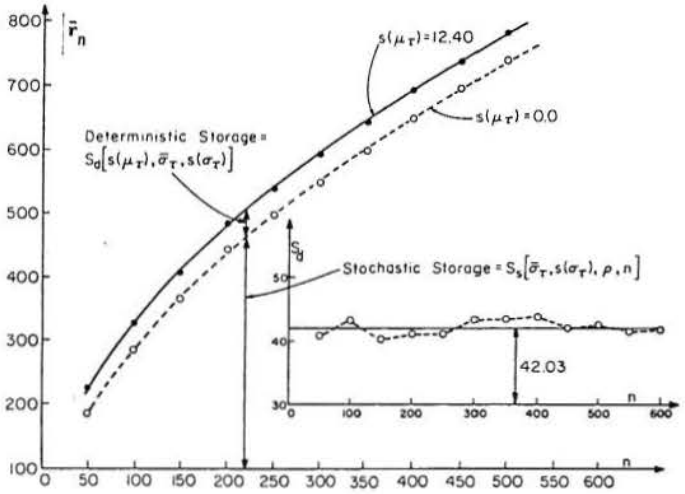

Fig. 2-8 Deterministic and stochastic storage capacities in case of inputs with periodic mean μ_τ, periodic standard deviation σ_τ and constant serial correlation coefficient ρ with $\bar{\sigma}_\tau$ = = 10, $s(\sigma_\tau)$ = 6.87 and ρ = 0.60 (Salas, 1972).

2-2 Asymptotic Results of the Range R_n

2-2-1 Independent Net Inputs with Constant Parameters

Input distribution belonging to the Brownian domain of attraction and no drift. Results of this section appeared originally in the initial pages of Hurst (1951) and Feller (1951) on this subject. Recent derivations utilizing so-called weak convergence theory appear in Troutman (1976), (1978) and Siddiqui (1976); these works will be followed here.

Results are complete in the sense that the asymptotic distribution from which asymptotic moments can be obtained, is known. Let

$$F_R(r) = [\int_0^r f_R(z)\,dz]\, I_{(0,\infty)}(r) \quad , \tag{2-34}$$

where (as in Eq. 2-17)

$$f_R(z) = [8 \sum_{j=1}^{\infty} (-1)^{j+1} j^2 \phi(jz)]\, I_{(0,\infty)}(z) \tag{2-35}$$

and $\phi(u) = (2\pi)^{-1/2} e^{-u^2/2}$.

$F_R(\cdot)$ is the c.d.f. of the range in the continuous time standard Wiener process or Brownian motion. For independent net inputs $\{X_t\}$ with $\mu = E[X_t] = 0$ and belonging to the Brownian domain of attraction, the following holds:

$$\lim_{n\to\infty} P[R_n/\sigma n^{1/2} \le r] = F_R(r) \tag{2-36}$$

where, as before, R_n is the range and $\sigma^2 = \text{var}[X_t]$.

As $E[R] = 2(2/\pi)^{1/2}$, then

$$E[R_n] \sim 2(2/\pi)^{1/2} \sigma n^{1/2} \quad , \tag{2-37}$$

and as $E[R^2] = 4 \ln 2$, then

$$\text{var}[R_n] \sim [(\ln 2) - (2/\pi)]\, 4\sigma^2 n \quad . \tag{2-38}$$

The first two moments of R are not obtained by termwise integration of Eq. (2-35); rather Eq. (2-34) is first expressed in the following alternate form and then integrated.

$$f_R(r) = 4\phi(r) + 4 \sum_{j=1}^{\infty} \{(2j-1)^2 \phi[(2j-1)r] + (2j+1)^2 \phi[(2j+1)r] - 8j^2 \phi(2jr)\}\, I_{(0,\infty)}(r) \quad .$$

Input distribution belonging to the Brownian domain of attraction with drift. Assume now that $\{X_t\}$ is independent and identically distributed with $E[X_t] = \mu > 0$ and $\text{var}[X_t] = \sigma^2 < \infty$. It is intuitively clear that $E[R_n]$ should grow with n rather than $n^{1/2}$. In fact, $E[R_n] \ge E[|S_n|] \ge |E[S_n]| = n|E[X_1]|$ and $E[R_n] \le 2E[\sum_1^n |X_i|] = 2n\,E[|X_1|]$ (see Boes and Salas (1973)).

The main result is given in Troutman (1976) as

$$\lim_{n\to\infty} P\left[\frac{R_n - n\mu}{\nu(n/E[J])^{1/2}} \le r\right] = \Phi(r) \tag{2-39}$$

where $\Phi(\cdot)$ is the standard normal c.d.f., J is the first index $j \ge 1$, for which $S_j \ge 0$ and $\nu^2 = \text{var}[S_J - \mu J]$. So we see that R_n is asymptotically normally distributed with both mean and variance increasing with n.

Stable inputs. As previously stated, Moran (1964) showed that

$$E[R_n] = E[|X|] \sum_{i=1}^{n} i^{(1/\gamma)-1} \tag{2-40}$$

for $X, X_1, \ldots, X_n, \ldots$ independent random variables with common characteristic function $\exp[-|t|^\gamma]$ where $1 < \gamma \le 2$. Such X has a symmetric stable distribution. From Eq. (2-40) it follows that

$$E[R_n] \sim \gamma\, E[|X|]\, n^{1/\gamma} \quad . \tag{2-41}$$

For $1 < \gamma \leq 2$, $1/2 \leq 1/\gamma < 1$ so $E[R_n]$ "can be made to grow asymptotically like n^h for any $1/2 \leq h < 1$ even for independent net inputs."

2-2-2 Net Inputs with Constant and Periodic Parameters

Dependent, periodic inputs belonging to the Brownian domain of attraction. Dependent stationary inputs can be treated as a special case of the more general dependent periodic inputs by following Troutman (1976) and (1978).

Suppose now that the input process $\{X_t\}$ is periodic of period length ω in the following sense: $(X_{t_1}, X_{t_2}, \ldots, X_{t_j})$ has the same joint distribution as $(X_{t_1+h\omega}, X_{t_2+h\omega}, \ldots, X_{t_j+h\omega})$ for any time indices $t_1, \ldots, t_j$ and any integer h. In particular, if we are dealing with monthly values and the period is one year, then the period length $\omega = 12$; also, the definition of periodicity would say, for instance, that the joint distribution of May and July of one year would be the same as the joint distribution of May and July of any other year.

Define $E[X_t] = \mu_t$, $\text{var}[X_t] = \sigma_t^2$, and $\text{corr}[X_t, X_{t+\ell}] = \rho_{\ell,t}$, $\ell = 0,1,2,\ldots$, with μ_t, σ_t^2, and $\rho_{\ell,}$ all periodic in t of period length ω. Set

$$\bar{\mu} = \omega^{-1} \sum_{\tau=1}^{\omega} \mu_\tau \quad \text{and} \quad \bar{\sigma}^2 = \omega^{-1} \sum_{\tau=1}^{\omega} \sigma_\tau^2 \quad , \tag{2-42}$$

the average values, respectively, of μ_τ and σ_τ^2 over one cycle. Now define

$$X'_p = \sum_{t = (p-1)\omega+1}^{p\omega} X_t \quad , \tag{2-43}$$

which is the total net input over the pth cycle. Note that the sequence $\{X'_p\}$ will be stationary and in general dependent. $\omega = 1$ reduces the periodic dependent input to stationary, dependent input.

The type of dependence allowed for in the dependent periodic input $\{X_t\}$ is constrained by the type of dependence allowed for in the $\{X'_p\}$ process, and the type of dependence allowed for in the $\{X'_p\}$ process is quite general, yet complicated enough not to take the time and space to delineate it here. Suffice it to say that it encompasses the traditional ARMA-processes, Markov chain processes, log-linear processes, etc. For details, refer to Troutman (1978) who in turn references Billingsley (1968), a text on weak convergence theory. Essentially, the type of dependence permitted is restricted by the types of dependence sufficient for application of the weak convergency theory. Let $F_R(\cdot)$ be as in Eq. (2-34). The main result is the following.

Let X_t be periodic and dependent as specified above with $\bar{\mu} = 0$ (no drift) then

$$\lim_{n\to\infty} P[R_n/\gamma n^{1/2} \leq r] = F_r(r) \tag{2-44}$$

$$\text{where } \gamma^2 = \bar{\sigma}^2 + 2\,\omega^{-1} \sum_{\tau=1}^{\omega} \sum_{\ell=1}^{\infty} \sigma_\tau\, \sigma_{\tau+\ell}\, \rho_{\ell,\tau} \tag{2-45}$$

and it is assumed $0 < \gamma^2 < \infty$.

One notes that the specification of the periodicity and dependence enters only through $\bar{\mu} = 0$ and the scale parameter γ. If $\omega = 1$ (non-periodic), γ^2 is the sum of the covariance function. Thus, for example, if $\{X_t\}$ is assumed to be stationary AR(1) process, then $\gamma^2 = \frac{1+\rho_1}{1-\rho_1}\sigma^2$, and one can readily see the influence of the dependence. It is important to observe that dependence or periodicity enters the asymptotic distribution only through the first and second order moments! Results for drift ($\bar{\mu} \neq 0$) are available, but not presented here. See Troutman (1976).

Exchangeable random variables. Assume that $\{X_t\}$ are exchangeable or symmetrically correlated; that is, $X_1, \ldots, X_n$ are exchangeable random variables if any permutation, say $X_{i_1}, \ldots, X_{i_n}$ of $X_1, \ldots, X_n$ has the same joint distribution as $X_1, \ldots, X_n$. Let $\rho = \text{corr}[X_i, X_j]$ for $i \neq j$. If a sequence $\{X_t\}$ is to be exchangeable, necessarily $\rho \geq 0$. For $\rho > 0$, exchangeability forces a strong (long memory) type of dependence and as such it manifests itself in rapid growth of $E[R_n]$. Boes and Salas (1973) showed that for exchangeable normal random variables with $\mu = 0$ (no drift)

$$E[R_n] = \frac{2\sigma}{\sqrt{2\pi}} \sum_{i=1}^{n} [\rho + \frac{1-\rho}{i}] n^{1/2} \tag{2-46}$$

which implies $E[R_n]$-constant times n. That is, one gets the same rapid growth of $E[R_n]$ with no drift and strong dependence that one gets with drift in the weak dependence case.

Fractional Gaussian noise. A long-memory stationary stochastic process, known as fractional Gaussian noise, (FGN), was introduced into hydrology by Mandelbrot (1965), and has been recently studied by several writers, including Lawrence and Kottegoda (1977) and McLeod and Hipel (1978). Mandelbrot and Van Ness (1968) and Mandelbrot and Wallis (1969 a,b,c) present the initial analytical treatment, and in particular showed that the correlation function at lag ℓ of FGN is given by

$$\rho_\ell = (1/2)[(\ell+1)^{2h} - 2\ell^{2h} + (\ell-1)^{2h}] \tag{2-47}$$

for $0 < h < 1$ and $\ell = 1,2,\ldots$. Such correlation function damps with increasing lag but at such a rate that it is not summable for $1/2 < h < 1$. Here h is a parameter of the model. Furthermore,

$$E[R_n] \sim a\, n^h \quad , \quad 0 < h < 1 \tag{2-48}$$

where a is a constant not depending on n. Equation (2-48) is the principal result from the theory of fractional Gaussian noise relating to this paper. It shows

that asymptotically $E[R_n]$ can be made to grow like n^h for $1/2 < h < 1$, which certainly includes the exponents observed by Hurst. Indeed, FGN is sometimes referred to as the model of Hurst's geophysical series. Here we see that n^h growth of $E[R_n]$, with $1/2 < h < 1$, is possible with a strong type of dependence (long memory) and not heavy tails; in the previous subsection similar growth of $E[R_n]$ was possible with no dependence but heavy-tailed stable distributions.

2-3 Preasymptotic Results of the Conditional Range $_cR_n$

Only two studies have been made of the conditional range as far as is known to the writers. The first was by Hurst (1951) for the case of independent inputs and the second for dependent inputs was by Gomide (1978) (where the conditional range is first defined). (The latter publication grew out of Gomide's (1975) earlier work where the conditional range was incorrectly called the adjusted range.)

Basically, the conditional range is $[R_n|S_n = 0]$, i.e. those cases of the range where $S_n = \sum_{i=1}^{n} X_i = 0$.

Clearly, this conditioning limits the study to discrete random variables (although one could of course define $_cR_n$ for Gaussian processes by binomial approximation in the limit).

Thus let $_cS_t=[(X_1+X_2+\ldots+X_t)|S_n=0]$; $t=1,2,\ldots,n$,

$$_cM_n=\max(0,{}_cS_1,{}_cS_2,\ldots,{}_cS_{n-1})$$

$$_cm_n=\min(0,{}_cS_1,{}_cS_2,\ldots,{}_cS_{n-1}).$$

Then $_cR_n \equiv {}_cM_n$ is the conditional range of partial sums of X_t.

Hurst (1951) implicity compared the behavior of $E[_cR_n]$ with that of $\bar{r}_n^{**}$. The validity of this comparison is discussed by Gomide (1978), who demonstrates that the approximation is fair for simple cases of dependence when n is large.

2-3-1. Independent Net Inputs with Constant Parameters

Consider Bernoulli inputs such that $P[X_t=\pm 1] = \frac{1}{2}$. Hurst studied this process and derived the following result (Eq. 1-7) which obviously only holds for even n:

$$E[_cR_n] = \left(\frac{2^n}{C_{n/2}^n}\right) - 1 \quad . \tag{2-49}$$

2-3-2 Dependent Net Inputs with Constant Parameters

Consider as before Bernoulli inputs such that $P[X_t = \pm 1] = \frac{1}{2}$ and further let

$$P[X_{t+1} = \pm 1|X_t = \pm 1] = (1 + \rho)/2 = p \quad \text{and}$$

$$P[X_{t+1} = \pm 1|X_t = \mp 1] = (1 - \rho)/2 = q = 1-p.$$

So let $\{X_t\}$ be a two-state Markov chain with $\text{corr}[X_{t+s},X_t] = \rho^s$; $s = 0,1,2,\ldots$, $E[X_t] = 0$ and $\text{var}[X_t] = 1$.

Gomide (1978) studied the conditional range of this input process and showed that

$$E[_cR_n] = K - v_K^{(n)}/P[S_n = 0] \tag{2-50}$$

where K is large enough to ensure $P[_cR_n \le K] = 1$, and

$$P[S_n=0] = p^{n-1} \sum_{j=1}^{n-1} \left(\frac{n-2}{2}\right)^C \left[\frac{j}{2}\right] \cdot \left(\frac{n-2}{2}\right)^C \left[\frac{j-1}{2}\right] \cdot \left(\frac{q}{p}\right)^j$$

(where $[\cdot]$ denotes the integer part of the argument), and where $v_k^{(n)} = \sum_{u=1}^{k} q_k^{(n)}(u,u)$ in which $q_k^{(n)}(u,u)$ is the probability of a transition from "state" u back to "state" u in n steps without reaching 0 or k+1. For a full treatment of the algorithm, the reader is referred to Gomide (1975).

2-4 Asymptotic Results of the Conditional Range $_cR_n$

2-4-1 Independent Net Inputs

Hurst (1951) showed by taking Stirling's approximation to the factorials in (2-49) that

$$E[_cR_n] \to \sqrt{\pi n/2} - 1 \approx \sqrt{\pi n/2} \approx 1.2533\sqrt{n} \ . \tag{2-51}$$

2-4-2 Dependent Inputs

Following Hurst's approach, Gomide (1978) showed that

$$E[_cR_n] \to \sqrt{\left[\frac{n(1+\rho)}{2(1-\rho)}\right]} = 1.2533\sqrt{\left(n\,\frac{1+\rho}{1-\rho}\right)} \tag{2-52}$$

for the case of the two state Markov chains considered. Moreover, he demonstrated numerically that the distribution of the standardized conditional range:

$$(_cR_n - E[_cR_n])/\sqrt{\text{var}[_cR_n]} \text{ for } \rho = 0.5$$

converges to that of the asymptotic standardized adjusted range:

$$(R_n^* - E[R_n^*])/\sqrt{\text{var}[R_n^*]} \text{ for } \rho = 0.0 \text{ with n as}$$

low as 50.

For the sake of comparison, Table 2-4 shows the exact values of $E[R_n]$, $E[_cR_n]$, its asymptotic approximation $\sqrt{(\pi n/2)} - 1$, the result by Solari and Anis (1957) for the adjusted range and Anis and Lloyd's (1975) result for the rescaled range for $n = 2^i$, $i = 1, 2,\ldots,10$. Except for $E[_cR_n]$ and its approximation, the other results are for independent normal inputs. The conclusion is that $E[R_n^*] < E[R_n^{**}] \le E[_cR_n]$ for all n, while it is clear that the approximation $(\sqrt{n\pi/2}-1)$ is very good to all three of the ranges other than R_n.

Table 2-4 Comparison of results for mean ranges.

n	$E[R_n]$	$E[{}_cR_n]$	$\sqrt{n\pi/2}-1$	$E[R_n^*]$	$E[R_n^{**}]$
2	1.3621	1.0000	0.7724	0.5642	1.0000
4	2.2217	1.6667	1.5066	1.3203	1.6547
8	3.4879	2.6571	2.5449	2.3692	2.6246
16	5.3171	4.0922	4.0133	3.8428	4.0355
32	7.9322	6.1454	6.0898	5.9220	6.0655
64	11.6508	9.0658	9.0265	8.8600	8.9656
128	16.9241	13.2074	13.1796	13.0138	13.0906
256	24.3920	19.0726	19.0530	18.8875	18.9431
512	34.9606	27.3731	27.3592	27.1939	27.2338
1024	49.9119	39.1167	39.1060	38.9408	38.9693

Troutman (1976) comments that ${}_cR_n$ will converge (asymptotically) in distribution to the asymptotic distribution $f_{R_n^*}$ of the adjusted/rescaled range, originally given by Feller (1951); (see Eq. 2.70 of Section 2-6-1).

We let $R^* = {}_cR_n/\sqrt{n}\sigma = R_n^*/\sigma\sqrt{n} = R_n^{**}/\sigma\sqrt{n}$ for large n, as the conditional, adjusted and rescaled range all converge to the same distribution asymptotically as shown by Troutman (1976).

Troutman (1976) proves that the mean and variance of ${}_cR_n$, R_n^* and R_n^{**} are given asymptotically by:

$$E[R^*] = \gamma\sqrt{(\pi/2)},\ \text{var}[R^*] = (\pi^2/6 - \pi/2)\gamma^2 \qquad (2\text{-}53)$$

where γ^2 is defined in Eq. (2-45).

Thus for a lag-one autoregressive process X_t, $\gamma^2 = \sigma_X^2\,(1+\rho_1)/(1-\rho_1)$, corroborating Gomide's (1978) result for $E[{}_cR_n]$ and his conjecture that

$$\text{var}[R^*] = 0.0741 \cdot \frac{1+\rho}{1-\rho} \quad . \qquad (2\text{-}54)$$

2-5 Preasymptotic Results of the Adjusted Range R_n^*

2-5-1 Independent Net Inputs with Constant Parameters

Normal inputs. Solari and Anis (1957) derived the first two moments of the maximum adjusted partial sums of independent standard normal variables as:

$$E(M_n^*) = \frac{1}{2}\sqrt{\frac{n}{2\pi}}\sum_{i=1}^{n} i^{-1/2}\,(n-i)^{-1/2}\ , \qquad (2\text{-}55)$$

and

$$E(M_n^{*2}) = \frac{1}{6}\left[\frac{n^2-1}{n} + \frac{\sqrt{n}}{2\pi}\sum_{i=2}^{n-1}\sum_{j=1}^{i-1}\frac{i(2i-n)}{\sqrt{j^3(n-1)(i-j)^3}}\right]. \qquad (2\text{-}56)$$

Then from Eq. (2-55) the Solari and Anis equation for the expected adjusted range is

$$E(R_n^*) = \sqrt{\frac{n}{2\pi}}\sum_{i=1}^{n} i^{-1/2}\,(n-i)^{-1/2}\ . \qquad (2\text{-}57)$$

However, Boes and Salas (1973) showed that since the net inputs $(x_i-\bar{x}_n)$ are exchangeable, Spitzer's identity of Eq. (2-2) for exchangeable random variables applies. Thus Boes and Salas showed that $E(R_n^*)$ for independent normal random variables may be written as

$$E(R_n^*) = \sqrt{\frac{2}{\pi}}\sum_{i=1}^{n} i^{-1}\,(\text{Var}\,S_i^*)^{1/2} \qquad (2\text{-}58)$$

which is of the same form as Eq. (2-3) for $E(R_n)$. They also showed that Eq. (2-58) applies as well when the net inputs are $(x_i-\alpha\bar{x}_n)$ where $0 \le \alpha \le 1$. Thus in this case

$$E(R_n^*) = \sqrt{\frac{2}{2\pi}}\ \sigma\sum_{i=1}^{n} i^{-1/2}\,[(n-1) + i(1-\alpha)^2]^{1/2}\ . \qquad (2\text{-}59)$$

Gamma inputs. Following Boes and Salas' (1973) results, Anis and Lloyd (1975) give $E(R_n^*)$ for gamma net inputs with pdf $f(x) = x^{m-1}\,e^{-x}/\Gamma(m)$ for $x > 0$ as

$$E(R_n^*) = \frac{2\Gamma(nm)}{\sqrt{m}\,n^{nm}}\sum_{i=1}^{n-1}\frac{i^{mi-1}(n-i)^{(n-i)m}}{(mi)(n-i)m}\ ,\ n > 1\ . \qquad (2\text{-}60)$$

Stable inputs. For $X_1,\ldots,X_n$ variables having characteristic function $\exp\{-|t|^\gamma\}$, $1 < \gamma \le 2$ and net inputs equal to $(X_i-\alpha\bar{X}_n)$, Boes and Salas (1973) derived the expected adjusted range as

$$E(R_n^*) = E[|X|]\sum_{i=1}^{n}\{i[(1/i)-(\alpha/n)]^\gamma + (n-i)(\alpha/n)^\gamma\}^{1/\gamma} \qquad (2\text{-}61)$$

which leads to $E(R_n^*) \sim n^{1/\gamma}$.

Inputs of any distribution. For $X_1,\ldots,X_n$ independent variables of any distribution function, the net inputs $(X_i-\alpha\bar{X}_n)$ are exchangeable variables. Thus Boes and Salas (1973) used Spitzer's identity of Eq. (2.2) to derive the expected adjusted range as

$$E(R_n^*) = \sum_{i=1}^{n} i^{-1}\,E[|S_i^*|] = \sum_{i=1}^{n} E[|\bar{X}_i-\alpha\bar{X}_n|]. \qquad (2\text{-}62)$$

The above Eqs. (2-58) and (2-59) for normal variables, Eq. (2-60) for gamma variables and Eq. (2-61) for stable variables can be derived from Eq. (2-62) which, as said before, is applicable for any distribution.

2-5-2 Dependent Net Inputs with Constant Parameters

Exchangeable random variables. Boes and Salas (1973) showed that if the variables $X_1,\ldots,X_n$ are exchangeable, the net inputs $(X_i-\alpha\bar{X}_n)$ are also exchangeable. Therefore, based on Spitzer's identity of Eq. (2-2), they showed that Eq. (2-62) also holds for the case of exchangeable random variables of any distribution. In particular, if the X_i variables are normal with mean $E[X_i] = \mu$ and variance $\text{Var}[X_i] = \sigma^2$, the expected adjusted range becomes

$$E(R_n^*) = \sum_{i=1}^{n} i^{-1}\{E(S_i^*)[2\Phi(K_i)-1] + 2(\text{Var}\,S_i^*)^{1/2}\,\phi(K_i)\}, \qquad (2\text{-}63)$$

where $K_i = \dfrac{E(S_i^*)}{(\text{Var } S_i^*)^{1/2}}$. In addition, if $\mu = 0$ or $\alpha = 1$, then Eq. (2-63) takes the form of Eq. (2-58).

AR dependent inputs. Few analytical results are available for the expected adjusted range of AR net inputs. Based on Salas (1972, p. 24-32) results of the expected range of multivariate normal variables Salas et al. (1979b) showed that the expected adjusted ranges $E(R_2^*)$ and $E(R_3^*)$ are

$$E(R_2^*) = \sqrt{\frac{2}{\pi}}\ (\text{Var } S_1^*)^{1/2} \tag{2-64}$$

and

$$E(R_3^*) = \sqrt{\frac{2}{\pi}}\ \{\tfrac{1}{2}\ (\text{Var } S_1^*)^{1/2} + \tfrac{1}{2}\ (\text{Var } S_2^*)^{1/2} + \tfrac{1}{2}\ [\text{Var } (S_2^*-S_1^*)]^{1/2}\}. \tag{2-65}$$

In particular, if the inputs are first-order autoregressive, Eq. (2-65) gives

$$E(R_3^*) = \frac{2\sigma}{3\sqrt{\pi}}\ [(3-\rho-2\rho^2)^{1/2} + (3-4\rho+\rho^2)^{1/2}]. \tag{2-66}$$

This result serves to demonstrate (Salas et al., 1979b) that Eq. (2-58) is not applicable for AR variables. In fact, the relative errors obtained by using Eq. (2-58) vary from +1.62 percent for $\rho = 0.1$ to +12.72 percent for $\rho = 0.9$.

Results for larger n are available from simulations carried out by Yevjevich (1965). He obtained the empirical distributions of R_n^* for AR(1) inputs with $\rho = 0.0$, 0.1, 0.2, 0.4, 0.6, and 0.8 and n ranging from 2 to 50. Yevjevich also obtained by simulation the corresponding means, variances and skewness coefficients of R_n^*.

2-5-3 Net Inputs with Periodic Parameters

No results are available for the expected adjusted range of inputs (either independent or dependent) with periodic parameters or in general, inputs with parameters varying with time. However, assuming that the variables X_1, X_2, and X_3 are independent with variances σ_1^2, σ_2^2, and σ_3^2, Eq. (2-65) yields

$$E(R_3^*) = \frac{1}{6}\ \frac{2}{\pi}\ \{(4\sigma_1^2+\sigma_2^2+\sigma_3^2)^{1/2} + (\sigma_1^2+4\sigma_2^2+\sigma_3^2)^{1/2} + (\sigma_1^2+\sigma_2^2+4\sigma_3^2)^{1/2}\}. \tag{2-67}$$

2-6 Asymptotic Results of the Adjusted Range R_n^*

2-6-1 Independent Net Inputs with Constant Parameters

Input distribution belonging to the Brownian domain of attraction. This section is introduced by quoting the opening paragraphs of Feller's (1951) paper.

"Let X_k be a sequence of mutually independent random variables with a common distribution $V(x)$, and suppose that $E[X_k] = 0$, $\text{var}(X_k) = 1$. Put $S_n = X_1 + \ldots + X_n$ and let

$$\begin{aligned} M_n &= \max[0,S_1,S_2,\ldots,S_n]. \\ m_n &= \min[0,S_1,S_2,\ldots,S_n]. \end{aligned} \tag{1.1}$$

The random variable

$$R_n = M_n - m_n \tag{1.2}$$

will be called the range of the cumulative sums S_n.

In applications '[c.f Hurst (1951)]' it is advantageous to modify this definition. One considers instead of the values of the sums S_k their deviations from the straight line joining the origin to the point (n,S_n). Thus, we replace the random variables S_k by

$$S_k^* = S_k - kS_n/n \qquad (k = 1,\ldots,n) \tag{1.3}$$

and define the corresponding variables M_n^*, m_n^*, R_n^* in analogy with (1.1) and (1.2). The variable R_n^* will be called the adjusted range of the cumulative sums S_n.

The adjusted range has a greater sampling stability, but its main advantage is probably due to the fact that it eliminates the trend when $E[X_k] \neq 0$, so that it can be used even when the means do not vanish."

Feller goes on to derive the asymptotic distribution of the adjusted range for this input and gives:

$$E(R_n^*) \to \sqrt{(n\pi/2)} \tag{2-68}$$

$$\text{var}(R_n^*) \to (\pi^2/6 - \pi/2)n. \tag{2-69}$$

$$f_{R_n^*/\sqrt{n}}(r) = 4\sqrt{2\pi} \sum_{k=1}^{\infty} \{k(k-1)[\phi'(2(k-1)r) - \phi'(2kr)] + (k-1)^2 r\phi''(2(k-1)r) + k^2 r\phi''(2kr)\} \tag{2-70}$$

where $\phi''(u) = \frac{\partial}{\partial u}\phi'(u) = \frac{\partial^2}{\partial u^2}\phi(u)$ and $\phi(u)$ is the standard normal density.

Gomide (1978) derives a version of the density of the adjusted range which is more amenable to computation.

$$f_{R^*}(r) = \sum_{k=1}^{\infty} 8k^2 r(4k^2r^2 - 3)\exp(-2k^2r^2), \tag{2-71}$$

where R^* is the limit in distribution of

$$R^*_n/\sigma\sqrt{n} \tag{2-72}$$

The infinite series in Eq. (2-71) converges very fast.

Input with positive drift. As a generalization of Feller's definition of R_n^*, Salas and Boes (1974) introduced

$$S_i^* = S_{i-1}^* + (Z_i - \alpha w), \quad (i = 1,2,\ldots,n),$$

where w can be chosen to be the sample mean S_n/n or population mean μ_Z of the input process Z_i. (Recall

that X_i is the mean net input $Z_i - Y_i$ where Y_i is the withdrawal.)

Thus in this context, μ_Z need not be zero, and $\alpha(0 \leq \alpha \leq 1)$ is a constant which, in the language of storage theory, can be considered to be the degree of development or draft.

Asymptotically, $w \to \mu$, so $S_i^* \to S_{i-1}^* + (Z_i - \alpha\mu Z)$. When either $\alpha = 1$ or $\mu = 0$ the net input $X_i = (Z_i - \alpha\mu)$ has zero mean, so S_i^* becomes S_i and the problem reverts to that of studying the range.

On the other hand, if $0 \leq \alpha < 1$ and $\mu > 0$ then based on Boes and Salas (1973), Salas and Boes (1974) show that "$E(R_n^*)$ increases asymptotically as fast as n," and this result applies regardless of the underlying distribution of the process.

To date, there are no other aymptotic results for R_n^* when $\mu > 0$, $0 \leq \alpha < 1$, although Troutman (1976) outlines an approach for R_n. Thus, the results reported in the rest of this section confine themselves to the case where the mean net input $\mu_X = 0$, although α will be variable. Evidently in the limit we could include the special case $\mu_Z = 0$, $\alpha > 0$.

Troutman (1976, p. 89) gives the p.d.f. $f_{R^*}(r;\alpha)$ of the asymptotic adjusted range R^* defined by Eq. (2-72) for $0 \leq \alpha < 1$, and demonstrates that it converges (as $\alpha \to 1$) to Feller's (1951) result. It is a very lengthy formula and will not be reproduced here. However, we have performed some computations to see whether, by standardizing the random variables, it was possible to find a common distribution, independent of α.

Troutman (1976) gave the first two moments of R^*, $(0 \leq \alpha < 1)$ for large n as

$$E(R^*) \to \sqrt{(2/\pi)} \cdot [1 - \alpha + (\arcsin \sqrt{\theta})/\sqrt{\theta}] \qquad (2\text{-}73)$$

$$E(R^{*2}) \to (1/2\theta/2\theta)\{ \sum_{i=1}^{\infty} (\sqrt{[1 + 4i(i+1)(1-\theta)]}/ /\sqrt{[i(i+1)]}) - 2\theta^2 + 6\theta - 1\} \qquad (2\text{-}74)$$

where $\theta = \alpha(2 - \alpha)$. Letting $\mu_\alpha^* = E(R^*)$ and $(\sigma_\alpha^*)^2 = E(R^*)^2 - (\mu_\alpha^*)^2$, Table 2-5 gives μ_α^* and σ_α^* as a function of α.

Table 2-5 Mean and standard deviation of asymptotic adjusted range as functions of α.

α	μ_α^*	σ_α^*
0.5	1.3637	.3146
0.6	1.3284	.2976
0.7	1.2983	.2857
0.8	1.2748	.2779
0.9	1.2591	.2737
1.0	1.2533	.2723

Define $q_\alpha = (r - \mu_\alpha^*)/\sigma_\alpha^*$ as a standardized variable, then we put $r = q_\alpha \sigma_\alpha^* + \mu_\alpha^*$ in $f_{R^*}(r;\alpha)$ in order to compare the distributions of the standardized variate q as a function of α. Further, define

$$R^{\#} = \lim_{n\to\infty} (R_n^* - E(R_n^*))/\sqrt{\text{var}(R^*_n)} \qquad (2\text{-}75)$$

then by direct computation it was found that the p.d.f. of $R^{\#}$ $(0 \leq \alpha < 1)$ equals that of $R^{\#}$ $(\alpha = 1)$ to within three significant figures for $\alpha = 0.6$. The approximation improves markedly as $\alpha \to 1$. Thus a large amount of unnecessary computation can be obviated if this fact is taken into account.

Input distributions belonging to the stable domain of attraction. Only one result exists for R_n^* and that is from Salas and Boes (1974) where they demonstrate that

$$E(R_n^*) \to \text{constant} \times n^{1/\gamma}. \qquad (2\text{-}76)$$

Here γ is a parameter which characterizes a stable which has a characteristic function: $\exp(-|t|^\gamma)$, $1 < \gamma \leq 2$.

2-6-2 Dependent Inputs with Constant Parameters

The c.d.f. of $R^{\#}$ is given in Table 2-7 in Section 2-7-1.

Distributions in the Brownian domain of attraction. Troutman (1976) was the first to find asymptotic results for R_n^* where the input process is a Markov chain, autoregressive, moving average or mixed autoregressive moving average. He even gives results for log ARMA which apply equally well. The basis of his development is the definition of γ^2 which is the sum of the covariance function of X as given in Eq. (2-45). He shows that $R_n^*/\gamma\sqrt{n}$ is distributed as R^* asymptotically, and gives the formula for arbitrary α.

Thus for example if X_t is AR(1) with $E[X] = \mu = 0$, $\text{var}[X] = \sigma^2 = 1$, $\text{corr}[X_{t+s}, X_t] = \rho^s$, $(S = 1,2,\ldots,)$, then

$$E(R_n^*) \to \left[\frac{n\pi(1+\rho)}{2(1-\rho)}\right]^{1/2} = \sqrt{n} \cdot \mu^* \cdot \gamma \,. \qquad (2\text{-}77)$$

It has already been shown above that $R^{\#}$, the standardized asymptotic distribution of R_n^* for $0.5<\alpha<1$ closley resembles that of $R^{\#}$ when $\alpha = 1$. Thus for all practical purposes, a knowledge of μ_α^*, σ_α^* as given in Table 2-5 and $f_{R^{\#}}$ given by Eq. (2-84) will enable anyone studying R_n^* to obtain the information he wants (for large n), provided that he can define γ^2.

Symmetrically correlated inputs. Given $\text{corr}[X_{t+s}, X_t] = \rho$ for every $s \neq 0$, Boes and Salas (1973) give $E(R_n^*)$ for X_t normally distributed $(0,\sigma^2)$ as:

$$E(R_n^*) = \frac{\sigma}{\sqrt{2\pi}} (1-\alpha) \sqrt{n(1+(n-1)\rho)} \to \text{constant} \times n \qquad (2\text{-}78)$$

for $\rho > 0$.

2-6-3 Net Inputs with Periodic Parameters

Troutman (1976) examines the case where the distribution of the net input X has periodic properties with period ω, i.e. $E[X_\tau] = \mu_\tau$, $\text{var}[X_\tau] = \sigma_\tau^2 < \infty$ and $\text{corr}[X_\tau, X_{\tau+\ell}] = \rho_{\ell,\tau}$ $(\rho = 0,1,2,\ldots,)$ $(\tau = 1,2,\ldots,\omega)$.

If further the net inputs are to be adjusted by α_τ, also periodic in τ, then define

$$\bar{\mu} = \omega^{-1} \sum_{\tau=1}^{\omega} \mu_\tau \ , \quad \bar{\sigma}^2 = \omega^{-1} \sum_{\tau=1}^{\omega} \sigma_\tau^2 \ , \quad \bar{\alpha} = \omega^{-1} \sum_{\tau=1}^{\omega} \alpha_\tau \ ,$$

and $\gamma^2 = \bar{\sigma}^2 + 2\omega^{-1} \sum_{\tau=1}^{\omega} \sum_{\ell=1}^{\infty} \sigma_\tau \sigma_{\tau+\ell} \cdot \rho_{\ell,\tau}$, as in Eq. (2-45).

He shows that

$$\lim_{n\to\infty} P(R_n^*/\gamma\sqrt{n} \leq r) = F_{R^*}(r;\bar{\alpha}) \qquad (2\text{-}79)$$

where $F_{R^*}(r;\alpha) = \int_0^r f_{R^*}(Z;\bar{\alpha})\, dZ \ , \ (r > 0)$

$= 0 \quad (r \leq 0)$.

$f_{R^*}(Z;\alpha)$ is given by Troutman, but as we have shown above, this distribution can be recovered with reasonable accuracy for $\alpha \geq 0.6$ from that of $f_{R^*}(Z;1)$ given by Eq. (2-71) if suitable standardization is employed.

The remarkable fact that emerges from Troutman's work is that if $\mu = 0$, the asymptotic distributions of the range and the adjusted range depend only on mean values of the adjustment factors and the covariance sum, i.e. $\bar{\alpha}$ and γ^2.

2-7 Preasymptotic Results of the Rescaled Range R_n^{**}

2-7-1 Independent Net Inputs with Constant Parameters

Normal inputs. Anis and Lloyd (1975) made use of Boes and Salas' (1973) version of Spitzer's (1956) lemma on exchangeable random variables to produce an exact formula for the expected rescaled range, when $\mu_X = 0$:

$$E(R_n^{**}) = \frac{\Gamma[(n-1)/2]}{\Gamma[n/2]\sqrt{\pi}} \sum_{i=1}^{n-1} [(n-i)/i]^{1/2} \qquad (2\text{-}80)$$

and computed this expression for selected n values up to 10^6. (See Table 2-4 of this paper for a comparison of the various expected ranges, and a close approximation).

Wallis and O'Connell (1973) computed the e.c.d.f. (empirical cumulative distribution function) of R_n^{**} for n = 20,30,40,50,75,100 for independent normal as well as autoregressive (lag-one) inputs. Thirty thousand samples were used for each set, and they produce curves of the e.c.d.f.'s plotted on a probability scale. This was apparently the first attempt to find the distribution of R_n^{**}. Hipel and McLeod (1977) produce e.c.d.f.'s for R_n^{**} in tabular form which are easier to use for the purpose of reading percentiles directly or extracting information about $E(R_n^{**})$ and $var(R_n^{**})$. Comparing their results with those of Wallis and O'Connell, it will be seen that there is no difference in the percentiles as far as a visual comparison on the scale of the plots will permit. This comparison adds credence to both sets of e.c.d.f.'s as they were produced independently.

Hipel and McLeod (1978) computed the distribution of $E(R_n^{**})$ for X_t i.i.d. normal (0,1) empirically by simulation, for selected $n \leq 200$. The percentiles were published in terms of Hurst's K defined as $K = \log(\bar{r}_n^{**})/\log(n/2)$, where $\bar{r}_n^{**}$ were sample values of R_n^{**}, 10,000 samples for each n. McLeod and Hipel (1978), in a companion paper, compare $\bar{r}_n^{**}$ for their simulation with Eq. (2-80) and show a good measure of agreement. To date there have been no reports of values of $var(R_n^{**})$, and it was decided that this position could be rectified by using Hipel and McLeod's tables of the e.c.d.f. (The treatment extends itself to the case of AR(1) inputs, which will be summarized in the next sub-section.)

The justification for using this approach rests on the following considerations.

(i) Hipel and McLeod's e.c.d.f.'s required a horrendous amount of computing which it was thought would be folly to repeat;

(ii) Gomide (1975) shows that the standardized range and conditional range converge to their respective asymptotic distributions extremely fast, the implication being that the same should hold for the rescaled range;

(iii) Should values of $var(R_n^{**})$ become available, then utilizing the exact expression for $E(R_n^{**})$ in Eq. (2-80), both the p.d.f. and c.d.f. for R_n^{**} can be computed with reasonable accuracy, say for $n > n_0$ (with n_0 a value of n at which the distribution of the standardized rescaled range, $R_n^{\#} = (R_n^{**} - E[R_n^{**}])/\sqrt{var[R_n^{**}]}$, has converged to its asymptotic equivalent $R^{\#}$ to within a desired degree of accuracy).

(iv) Salas et al. (1977) demonstrated by simulation that $\bar{r}_n^{**}$ for normal, gamma, and McLeod and Hipel (1978) show that, in addition, stable and Cauchy variables, have virtually indistinguishable behavior for n up to 200, which justifies concentrating attention on the normal.

Hipel and McLeod (1978) give percentiles of K: $\hat{P}[K<k]$ for the case of X i.i.d. normal with $\mu = 0$. For example for n = 20 they give the values of Table 2-6.

Table 2-6 Distribution of R_n^{**} for n = 20.

Percentile	0.5	1.0	2.5	5.0	10	20	30	40	50
k	.406	.426	.459	.486	.520	.565	.600	.628	.655

Percentile	60	70	80	90	95	97.5	99	99.5
k	.681	.708	.738	.775	.804	.827	.882	.870

To find $\hat{var}(R_n^{**})$ from the given e.c.d.c.f., the following procedure was used. First transform the K

values to $r = (n/2)^K$. Next fit two cubic polynomials to the five percentiles at each end of the e.c.d.f. by least squares, and compute the values of r_0 and r_{100} where, effectively, $P(R_n^{**} \le r_o) = 0$ and $P(R_n^{**} \le r_{100}) = 1.0$, respectively. Then using cubic splines, fit eight cubic polynomials (with continuous first and second derivatives matching at the end-points of the intervals) to the values $r_{10}, r_{20}, \ldots, r_{90}$. There are then 10 cubic polynomials $f_i(r)$, $i = 1,2,\ldots,10$ such that $f_i(r) = a_{i0} + a_{i1}r + a_{i2}r^2 + a_{i3}r^3$. It follows that the first and second moments are:

$$\hat{E}(R_n^{**}) = \sum_{i=0}^{9} [a_{i1}(r_{i+1}^2 - r_i^2)/2 + a_{i2}(r_{i+1}^3 - r_i^3)/3 + a_{i3}(r_{i+1}^4 - r_i^4)/4] \qquad (2\text{-}81)$$

$$\hat{E}(R_n^{**2}) = \sum_{i=0}^{9} [a_{i1}(r_{i+1}^3 - r_i^3)/3 + a_{i2}(r_{i+1}^4 - r_i^4)/4 + a_{i3}(r_{i+1}^5 - r_i^5)/5] \qquad (2\text{-}82)$$

whence $\hat{var}(R_n^{**})$ and $\hat{C}_{v_n}^{-1} = \hat{E}(R_n^{**})/\sqrt{\hat{var}(R_n^{**})}$ are obtainable.

To corroborate the results of this treatment, a second method of finding the moments of R_n^{**} from Hipel and McLeod's tables was devised. It was to compare Hipel and McLeod's percentiles for various n values with those of the asymptotic distribution given by Feller (1951).

Now, Gomide's (1975) version of Feller's formula is Eq. (2-71):

$$f_{R^*}(x) = \sum_{k=1}^{\infty} 8k^2x(4k^2x^2 - 3)\exp(-2k^2x^2) \qquad (2\text{-}83)$$

where $R_n^*/\sigma_X\sqrt{n}$ and $R_n^{**}/\sigma_X\sqrt{n}$ both tend to R^* as $n \to \infty$.

$E(R^*) = \sqrt{\pi/2}$, $var(R^*) = (\pi^2/6 - \pi/2)$ which we call μ^* and σ^{*2}, respectively. By writing $R^* = \mu^* + \sigma^* R^{\#}$, we define a standardized variate $R^{\#}$ whose p.d.f. is

$$f_{R^{\#}}(r) = \sigma^* . f_{R^*}(\mu^* + \sigma^* r). \qquad (2\text{-}84)$$

The c.d.f. of $R^{\#}$ can be computed by numerical integration of this exact function to any desired accuracy, and Table 2.7 gives the percentiles of $R^{\#}$ to within five decimal places.

Let $\hat{r}_i$ stand for an experimentally derived percentile for a particular n as found by Hipel and McLeod. What is sought is $\hat{E}(R_n^{**})$ and $\sqrt{\hat{var}(R_n^{**})}$, such that the standardized values, $y_i = \hat{E}(R_n^{**}) + \hat{r}_i\sqrt{\hat{var}(R_n^{**})}$ matches (as closely as possible) the corresponding values of r given in the above table. $\hat{E}(R_n^{**})$ and $\sqrt{\hat{var}(R_n^{**})}$ were found by least squares, and differed very little, except for a bit more scatter, from the values found by integrating the fitted cubic polynomials, thus lending credence to the results.

Using Hipel and McLeod's data for $\rho = 0$, the results of the integrated cubic spline computations are reported in Table 2.8 in columns 4, 5 and 6 together with Anis and Lloyd's expression for $E(R_n^{**})$ in column 2 and Hipel and McLeod's results for $\hat{E}(R_n^{**})$ in column 3, as functions of n.

Table 2.7 Percentile of the standardized asymptotic adjusted range $R^{\#}$ computed from Eq. (2-84).

100 x $F_{R^{\#}}(r)$	r	100.$F_{R^{\#}}(r)$	r
0.1	-2.18388	90	1.34459
0.5	-1.95488	95	1.81343
1	-1.83071	97.5	2.23640
2.5	-1.63079	99	2.74503
5	-1.44038	99.5	3.10060
10	-1.19741	99.9	3.85591
20	- .86632		
30	- .60071		
40	- .35523		
50	- .11022		
60	.14968		
70	.44365		
80	.80753		

Table 2-8 Results of integrated cubic spline computations.

n	$E(R_n^{**})$ (A+L)	$\hat{E}(R_n^{**})$ (H+M)	$\hat{E}(R_n^{**})$	$\hat{var}(R_n^{**})$	$\hat{C}_v^{-1}$
5	1.9274	1.9273	1.933	.080	6.825
10	3.0233	3.0302	3.034	.357	5.079
20	3.8812	3.8826	3.889	.684	4.703
25	4.6111	4.6047	4.594	1.040	4.504
30	5.2576	5.2540	5.259	1.401	4.442
35	5.8443	5.8770	5.843	1.751	4.415
40	6.3851	6.4214	6.363	2.120	4.370
45	6.8895	6.8920	6.925	2.602	4.293
50	7.3640	7.3595	7.389	2.901	4.338
60	8.6502	8.6246	8.650	3.908	4.376
70	9.4210	9.4453	9.401	4.651	4.359
80	10.1392	10.1349	10.159	5.496	4.333
90	10.8143	10.8208	10.853	6.321	4.317
100	11.4533	11.4775	11.438	7.005	4.321
125	12.9243	12.9617	12.922	8.680	4.386
150	14.2556	14.1956	14.285	10.966	4.314
175	15.4806	15.4198	15.457	12.580	4.358
200	16.6214	16.5938	16.658	13.964	4.357

Scrutiny of Table 2-8 will show that there is good correspondence between $E(R_n^{**})$ and the two values of its estimates, in fact better than 0.6% difference in all cases. This encourages one to feel that the values for $\hat{var}(R_n^{**})$ are reasonably accurate. Examination of the standardized percentiles of the e.c.d.f. for n = 35 shows that they are close to what appear to be the asymptotic values. Furthermore, on comparing the percentiles (see Tables 2-12 and 2-13 at the end of this section), it seems that the standardized distribution of R_n^{**} has effectively converged to its asymptotic equivalent by n = 35. Note further that the values of C_v^{-1} appear to follow a reasonably smooth curve, asymptotic to a value of about 4.34. But the

asymptotic value of C_v^{-1} for R_n^{**} is $\mu^*/\sigma^* = \sqrt{3/(\pi - 3)} = 4.6030...$ which is six percent larger than 4.34. Does this imply that there is an error in analysis?

Checking the cubic spline algorithm with the exact values of the standardized asymptotic c.d.f. for R_n^{**} (given as $F_{R^\#}(r)$ in Table 2-7 above), it was found that the error in computing the standard deviation was less than 0.5 percent, which exonerates the cubic spline algorithm. This fact together with the observed accuracy of $\hat{E}(R_n^{**})$ in Table 2-8 seems to indicate that the values of the variance given here may be too large by about twelve percent. (This phenomenon was also observed for the cases where Hipel and McLeod (1978) computed the e.c.d.f.'s of R_n^{**} for an input following a Gaussian AR(1) process with $\rho > 0$.)

A simulation check appeared to be in order, so $\hat{E}(R_n^{**})$, $\hat{var}(R_n^{**})$ and $\hat{C}_v$ were computed for ten lots of 500 samples of length n = 35. $\hat{Var}(R_n^{**})$ lay between 1.887 and 2.274, and for the whole set of 5,000 samples, for n = 35, the result was $\hat{E}(R_n^{**}) = 6.439$ (compared to the exact value of 6.3851), and $\hat{var}(R_n^{**}) = 2.109$ (compared to 2.120 computed from Hipel and McLeod's tables) with $\hat{C}_v^{-1} = 4.4336$. Results of simulations cannot prove anything conclusively, but the indicationis that the results given in Table 2-8 computed from Hipel and McLeod's values appear to be reasonable, especially when it is recalled that their values correspond well with those of Wallis and O'Connell.

Two explanations are offered for this apparent paradox. First, the simulation check reported here used the Box-Muller transformation to get normally distributed pseudo-random numbers from uniformly distributed ones, and this may give a value of X which is occasionally too high. Did Wallis and O'Connell and Hipel and McLeod also use this transformation? The second explanation is that n = 200 is too small for C_v^{-1} to have reached its asymptotic value of 4.6030, although the good agreement between the standardized percentiles of (i) the e.c.d.f. and (ii) the asymptotic c.d.f. seem to indicate otherwise.

At any rate, the results reported here seem to be consistent, but it still remains to prove that they are fair approximations to the true values. Some zealot with a large research grant at his disposal will no doubt remedy the situation some day! In the meantime, the conclusion is that Hipel and McLeod's tables (and Wallis and O'Connell's diagrams) of the e.c.d.f. of R_n^{**} may possibly err on the conservative side as far as the variance is concerned.

Skewed input distributions. McLeod and Hipel (1978) compute $\hat{E}(R_n^{**})$ using 10,000 samples for $n \leq 200$, when the input is variously distributed as gamma, stable and Cauchy. Incredibly, there is very little difference in these values from the corresponding exact result for the normal given by Anis and Lloyd (1975).

This is supported by Matalas and Huzzen (1967) and Salas et al. (1979) who reported simulation experiments with correlated and independent skewed inputs. They found that skewness has virtually no effect on $\hat{E}(R_n^{**})$ for any value of n. This property of R_n^{**} is in marked contrast to the behavior of R_n and R_n^*.

2-7-2 Dependent Net Inputs with Constant Parameters

All the results that have appeared to date are from simulation experiments. Without exception, they are confined to net inputs with normal marginal distributions. That this is not a deficiency follows from the conclusion in the last paragraph that skewness has negligible effect on the behavior of $E(R_n^{**})$, coupled with the proof by Troutman (1976) that the asymptotic distribution of R_n^{**} is unaffected by the form of the marginal distribution of those inputs belonging to the Brownian domain of attraction. The variation of the behavior of R_n^{**} thus rests primarily upon the nature of the serial correlation structure of the net inputs, hence the headings which follow describe the type of dependence examined.

Autoregressive, lag-one or AR(1) model. Matalas and Huzzen (1967) were the first to report experiments on R_n^{**} for the AR(1) input process. They give a table of $\bar{K}^n$ for $\rho = 0.1,...,0.9$ and selected values of n from 5 to 1,000. Ten thousand values of $K = \log(r_n^{**})/\log(n/2)$ were computed for each ρ and n and $\bar{K}$ was given as the average of these $K(\rho,n)$ values. Their objective was to explain the Hurst phenomenon, so they were not particularly interested in $E(R_n^{**})$. However, their values of $\bar{K}$ (second row) and K computed from the exact Anis and Lloyd (1975) result (third row) appear in Table 2-9 for comparison purposes. It is evident that because of the difference in the skewness of the distributions of R_n^{**} and K, there is no easy way to find $E(R_n^{**})$ accurately from these data.

Table 2-9 Values of $\bar{K}$ and K.

n	5	10	25	50	100
$\bar{K}$	.59	.66	.64	.63	.61
K	.716	.687	.657	.639	.623

Wallis and O'Connell (1973) gave curves of e.c.d.f.'s (computed from 30,000 sequences for each ρ and n) for n = 20, 30, 40, 50, 75 and 100 and ρ = = 0, 0.1,..., 0.5, 0.75, 0.9. As mentioned already, these curves agree well with Hipel and McLeod's (1978) tables of percentiles of the e.c.d.f. of R_n^{**} (given for = 0.1, 0.2, ..., 0.9 and n = 5(5)50(10)100(25)200 using 10,000 sequences for each ρ and n).

As in the case of the independent inputs, Hipel and McLeod's tables will not be reproduced here, but by using the same (integrated fitted cubic spline) technique, the entries in Tables 2-10 and 2-11 were calculated.

Inspection of the standardized percentiles of these e.c.d.f.'s indicates that asymptotic convergence in distribution appears to have been effectively achieved for $\rho = 0.2$ by $n_0 = 70$ and for $\rho = 0.4$ by $n_0 = 150$. (Recall that for $\rho = 0.0$, $n_0 \simeq 35$.) For higher values of ρ, convergence in distribution has not been achieved by n = 200. This invalidates neither the computations of $\hat{E}(R_n^{**})$ and $\hat{var}(R_n^{**})$ nor Hipel and McLeod's tables, nor Wallis and O'Connell's curves of the e.c.d.f.'s, but is a comment on the

rate of convergence in distribution (transience) as a function of ρ.

For illustration purposes, Table 2-12 gives some percentiles of the standardized e.c.d.f.'s of R_n^{**} for the case $\rho = 0$ and variable n. Table 2-13 gives the e.c.d.f.'s of R_n^{**} for n = 200 and variable ρ, which should be compared with the last two lines of Table 2-12.

Table 2-10 $\hat{E}(R_n^{**})$ values computed from Hipel and McLeod's tables.

n	ρ=0.2	ρ=0.4	ρ=0.6	ρ=0.8
10	3.24	3.44	3.64	3.80
20	5.11	5.70	6.35	7.06
30	6.63	7.55	8.68	10.08
40	7.92	9.14	10.69	12.80
50	9.03	10.55	12.46	15.28
60	10.04	11.78	14.09	17.60
70	10.98	12.93	15.58	19.70
80	11.90	14.07	17.05	21.81
90	12.74	15.11	18.41	23.82
100	13.46	16.01	19.60	25.53
125	15.29	18.30	22.55	29.86
150	16.95	20.37	25.24	33.79
175	18.38	22.15	27.58	37.29
200	19.85	23.99	29.98	40.81

Table 2-11 $\sqrt{\text{var}(R_n^{**})}$ values computed from Hipel and McLeod's tables.

n	ρ=0.2	ρ=0.4	ρ=0.6	ρ=0.8
10	.609	.611	.597	.573
20	1.099	1.179	1.226	1.217
30	1.477	1.630	1.774	1.844
40	1.779	2.010	2.233	2.423
50	2.060	2.339	2.682	2.996
60	2.282	2.641	3.073	3.545
70	2.518	2.945	3.474	4.071
80	2.753	3.232	3.841	4.631
90	2.942	3.465	4.140	5.039
100	3.118	3.687	4.438	5.467
125	3.496	4.179	5.083	6.481
150	3.932	4.738	5.833	7.506
175	4.252	5.129	6.339	8.409
200	4.477	5.463	6.847	9.092

Table 2-12 Standardized percentiles for $\rho = 0$.

n	10	20	30	40	50	60	70	80	90
5	-1.318	-.844	-.512	-.258	-.032	.183	.532	.898	1.260
10	-1.305	-.916	-.603	-.328	-.037	.255	.565	.901	1.350
20	-1.258	-.903	-.601	-.341	-.074	.199	.501	.859	1.336
50	-1.220	-.868	-.599	-.349	-.096	.158	.444	.813	1.338
100	-1.185	-.863	-.596	-.356	-.116	.156	.445	.793	1.318
200	-1.211	-.865	-.608	-.351	-.118	.171†	.457	.808	1.343
∞	-1.1974	-.8663	-.6007	-.3552	-.1102	.1497	.4436	.8075	1.3446

†This value was computed from the corresponding value given by Hipel and McLeod as 0.619. To show how sensitive the calculations are to errors, if a value K = 0.618 is used instead of 0.619, 0.171 changes to 0.150.

Table 2-13 Standardized percentiles for n = 200.

ρ	10	20	30	40	50	60	70	80	90
.2	-1.207	-.874	-.606	-.352	-.099	.158	.461	.814	1.341
.4	-1.209	-.876	-.602	-.347	-.106	.164	.466	.817	1.342
.6	-1.214	-.879	-.596	-.340	-.096	.171	.470	.823	1.343
.8	-1.242	-.873	-.595	-.331	-.074	.201	.506	.842	1.337

Autoregressive - moving average: ARMA(1,1) models. O'Connell (1971) used an ARMA(1,1) model for X with ϕ varying from 0.8 to 0.99 and θ varying from 0.5 to 0.9 to find $\bar{H}$ from ten sequences of X of length 9,000 for each pair of (ϕ,θ) values. $\bar{H}$ is the mean of the ten values of H that he defined as the slope of the line fitted by least squares to $\log(r_n^{**}) \sim \log n$ for $n < 2{,}000$, where r_n^{**} are the sampled values of the rescaled range. Except for five pox diagrams for selected (ϕ,θ) pairs, no estimates were given for $E(R_n^{**})$.

Wallis and O'Connell (1973) give diagrams for the e.c.d.f. of a long ARMA(1,1) model with $\phi = 0.9$ and $\theta = 0.64$ for n = 20, 30, 40, 50, 75, 100. They used the same technique for the computation of these as they did for the AR(1) input, so they should be accurate.

Boes and Salas (1978) computed $\hat{E}(R_n^{**})$ for (ϕ,θ) = = (0.99, 0.8676) and (0.9, 0.6268) for comparison with conditionally nonstationary models of identical correlation structure.

Nonstationary models. Hurst (1957), Klemes (1974, 1975) and Potter (1975, 1976a and 1976b) and Boes and Salas (1978) give values of $\hat{E}(R_n^{**})$ for some restricted sets of parameters. The rescaled range behavior of these and the matching ARMA(1,1) models is indistinguishable from the results of the simulations, reinforcing the observation made earlier that for finite n, the resclaed range depends little, if at all, on the marginal distribution of the inputs, and depends only on the covariance function.

Exchangeable variables. For X_t symmetrically correlated ($\text{corr}(X_{t+s}, X_t) = \rho$, for $s \neq 0$) Anis and Lloyd (1975) showed that $E(R_n^{**})$ is independent of ρ. Hence symmetrically correlated and independent inputs have the same mean rescaled range for a given n.

Fractional Gaussian noise: FGN: The FGN model, proposed by Mandelbrot (1965) and a mathematical derivation given by Mandelbrot and Van Ness (1968) and Mandelbrot and Wallis (1969), and literature concerning the FGN model summarized by Wallis and O'Connell (1973) and Lawrence and Kottegoda (1977), will not be repeated here.

There have been several attempts at obtaining approximations to FGN, but McLeod and Hipel (1978) were the first to give an algorithm for the exact generation of FGN samples. It appears to be tractable for sequences up to length n = 200, as they compute $\hat{E}(R_n^{**})$ for n = 5(5)50(10)100(25)200 and H = 0.7, 0.9. For each value of n and H, 10,000 simulated sequences were generated. (McLeod and Hipel give an excellent exposition of the FGN model, its fitting to data and a comparison with ARMA models, and the interested reader is referred to their paper.)

What distinguishes FGN from models in the Brownian domain of attraction is that when $0.5 < H < 1$ the correlogram of the former is not summable, unlike those of the latter. Nevertheless, it can be shown that $\hat{E}(R_n^{**}) \sim n^H$ for FGN, whereas models in the Brownian domain of attraction yield $E(R^{**}) \sim n^{1/2}$ when $n \to \infty$. It is this difference that has prompted a long controversy over the Hurst phenomenon.

2-7-3 Net Inputs with Periodic Parameters

There are no known results for R_n^{**} when X is distributed with periodically varying parameters. This state of affairs includes the asymptotic case.

2-8 Asymptotic Results of the Rescaled Range R_n^{**}

Numerous references to the asymptotic case were made in the previous section and will not be repeated here. Hurst (1951) was the first to observe that sample means of R_n^{**} values seemed to grow at a rate proportional to n^h with $h > 1/2$ for a wide variety of geophysical time series. The principal theoretical result here (Troutman, 1976) concerns the asymptotic distribution of R_n^{**} for stationary, dependent input $\{X_t\}$ belonging to the Brownian domain of attraction. Let

$$F_{R^*}(r) = \left(\int_0^r f_{R^*}(u)\, du\right) I_{(0,\infty)}(r) \qquad (2\text{-}85)$$

where $f_{R^*}(u)$ is given by Eq. (2-71), then

$$\lim_{n} P[R_n^{**}/\beta n^{1/2} \leq r] = F_{R^*}(r) \quad , \qquad (2\text{-}86)$$

where $\beta^2 = 1 + 2 \sum_{\ell=1}^{\infty} \rho_\ell$, and it is assumed $\beta^2 < \infty$.

The limiting distribution here is the same as the limiting distribution in the case of the adjusted range R_n^* (see Section 2.6); only the scale parameter has changed from γ there to β here, where $\gamma^2 = \sigma^2\beta^2$, to account for the fact that the rescaled range has the sample standard deviation as a divisor. One notes again that dependence enters the asymptotic distribution only through the parameter β. Also, one can readily derive

$$E(R_n^{**}) \sim (\pi/2)^{1/2}\, \beta n^{1/2} \qquad (2\text{-}87)$$

and

$$\mathrm{var}(R_n^{**}) \sim [(\pi^2/6) - \tfrac{\pi}{2}]\, \beta^2\, n \, . \qquad (2\text{-}88)$$

For the case of independent stationary inputs $\beta = 1$.

Comments for the cases of exchangeable random variables and fractional Gaussian noise were made in the previous section.

2-9 Preasymptotic Results of the Maximum Deficit, D_n

Using the mass-curve concept of Rippl (1883), Hurst (1951) studied the maximum adjusted deficit D_n^* particularly in the case where the level of development (regulation) is less than 100 percent, his reasoning being that the range (rescaled) is not relevant where withdrawals are less than the arithmetic mean gross input. Why the choice of the adjusted deficit D_n^* (and indeed the adjusted and rescaled ranges R_n^* and R_n^{**}) should be more relevant for sizing reservoirs than its (their) unadjusted counterpart(s) is not clear. Thomas and Fiering (1962) used simulation and deficit analysis (which they dubbed the "sequent peak algorithm") to size reservoirs. The first analytical treatment of the (maximum accumulated) deficit was provided by Gomide (1975) in his milestone paper. Since then Troutman (1976) has treated the asymptotic case. Following Gomide's lead, Mutreja (1976) and a follow-up paper Mutreja and Yevjevich (1977) have examined the asymptotic behavior of the deficit of ARMA processes, evidently unaware that their results had been preempted by Troutman's work.

2-9-1 Independent Net Inputs with Constant Parameters

Gomide (1975) devised an algorithm for the evaluation of the exact distribution of D_n, defined in Section 1-3, for discrete inputs based on Moran's (1955) concept of a discrete reservoir. This algorithm permits the computation of $P[D_n \leq k]$ directly by summing the elements of the (k+1)-square taboo transition matrix of the Markov chain describing the storage, and is applicable to an arbitrary distribution of net inputs. In other words, $P[D_n \leq k]$ is the probability that the semi-infinite reservoir level does not reach state zero in the first n steps given the initial state is k+1 (full reservoir).

Gomide then proceeds to derive the asymptotic distribution of D_n for the case of full regulation and gives,

$$\text{for } D = \lim_{n \to \infty} D_n/\sqrt{n} \, , \qquad (2\text{-}89)$$

$$f_D(x) = 4 \sum_{(j+1)2=1}^{\infty} (-1)^{(j-1)/2} . j\phi(jx) \qquad (2\text{-}90)$$

$$F_D(x) = 4 \sum_{(j+1)/2=1}^{\infty} \{(-1)^{(j-1)/2} . \Phi(jx)\} - 1 \qquad (2\text{-}91)$$

By using the binomial approximation he demonstrates that the standardized deficit for normal inputs: $D_n^{\#} = (D_n - E[D_n])/\sqrt{\mathrm{var}[D_n]}$ converges in distribution to its asymptotic equivalent: i.e., $D_n^{\#} \to D^{\#} = (D - E[D]/\sqrt{\mathrm{var}[D]})$ by n = 15. As he says, "the convergence of the standardized exact density to the standardized asymptotic density is slower" than in the case of R_n "because of the influence of the probability mass at $D_n = 0$." For the case of full regulation, Gomide shows analytically that $P[D_n = 0] = [\phi(\mu)]^n$ for normal inputs when $E[X_t] = \mu$.

In addition to this general result, he gives curves for $E[D_n]$ and $\mathrm{var}[D_n]$ for $\mu = 0, 0.25, 0.3, 1$ for $0 \leq n \leq 50$, when X_t is normally distributed.

Because of its usefulness in computation, the p.d.f. and c.d.f. of $D^{\#}$, the standardized asymptotic deficit are tabulated in Table 2-14, for some selected values of the standardized variate z. (Note that

Gomide gives $E[D] = \sqrt{\pi/2} = 1.25331..$, and an infinite series for $E[D^2]$ requiring the evaluation of about 1,000 terms for six figure accuracy; then $var[D] \simeq 0.511014$).

Table 2-14 p.d.f. and c.d.f. of $D^{\#}$, the standardized asymptotic deficit

z	$f_D^{\#}(z)$	$F_D^{\#}(z)$
-2	10^{-8}	0
-1.5	.07629	.00698
-1	.41828	.13568
-0.5	.46799	.36877
0	.36972	.58016
0.5	.26116	.73732
1	.17196	.84465
1.5	.10605	.91320
2	.06126	.95423
2.5	.03315	.97724
3	.01680	.98934
3.5	.00798	.99530
4	.00355	.99805
4.5	.00148	.99923

2-9-2 Dependent Net Inputs with Constant Parameters

Let us consider Bernoulli inputs. Using the two state input Gomide used in the study of the range:

$$P[X_{t+1} = \pm 1 | X_t = \pm 1] = p, \quad P[X_{t+1} = \pm 1 | X_t = \mp 1] = q, \; P[X_t = 1] = P[X_t = -1] = 1/2,$$

Gomide (1975) extended his algorithm for the independent input case to that for correlated inputs. This two-state Markov chain has the same correlation structure as an AR(1) process, so that when n is large the results approximate those which correspond to an AR(1) Gaussian input. He demonstrates that for $\mu = 0$ and $\rho = 0.5$ the standardized deficit has effectively converged (in distribution) to its asymptotic counterpart, $D^{\#}$. He gives diagrams of $E[D_n]$ (but not, alas, $var[D_n]$) for n up to 100 and $\rho = 0$, 0.20, 0.50. It remains for $var[D_n]$ to be computed for correlated inputs, and it will be easy to extend the algorithm for R_n in Section 2-1-1 and 2-1-2 to the case of D_n. In the meantime, Troutman's (1976) asymptotic expression for D_n could be used, giving a conservative estimate of the variance, for use with $f_D^{\#}$ and $F_D^{\#}$; his expression is

$$var[D_n] \to 0.511015\gamma^2 n \tag{2-92}$$

where γ^2 is given by Eq. (2-45).

2-10 Asymptotic Results of the Maximum Deficit, D_n

Results for no drift will be presented here for the general case of dependent, periodic net inputs and the special cases of dependent but non-periodic inputs and finally independent inputs will be noted as special cases. The main contributors are Gomide (1975), (1979) and Troutman (1976), (1978). Gomide (1979) presents a succinct derivation of the asymptotic distribution of the maximum deficit for independent inputs using the theory of Markov chains. Troutman (1978) will be followed here.

The maximum deficit was defined in Section 1-3; it can also be defined by either Eqs. (2-93) or (2-94), namely

$$D_n = \max [0, M_1 - S_1, M_2 - S_2, \ldots, M_n - S_n] \tag{2-93}$$

where the S_j are the partial sums of the X's and M_n is the maximum of the first n of these partial sums. Or,

$$D_n = (-1) \min_{1 \le i \le j \le n} [X_i + X_{i+1} + \ldots + X_j]. \tag{2-94}$$

Let X_t denote the dependent periodic input random variables where the nature of the dependence and periodicity is just as in the latter part of Section 2-2. Set

$$F_D(d) = (4/\pi) \left\{ \sum_{j=0}^{\infty} [(-1)^j/(2j+1)] \cdot \exp[-\pi^2 (2j+1)^2 / 8d^2] \right\} I_{(0,\infty)}(d) \tag{2-95}$$

then

$$\lim_{n \to \infty} P[D_n / \gamma n^{1/2} \le d] = F_D(d) \quad , \tag{2-96}$$

where, as before, the constant γ is given by

$$\gamma^2 = \bar{\sigma}^2 + 2 K^{-1} \sum_{k=1}^{K} \sum_{\ell=1}^{\infty} \sigma_k \sigma_{k+\ell} \rho_{\ell,k} \; . \tag{2-97}$$

Once again we see that D_n is scaled by $n^{1/2}$ regardless of the distribution of the X_t's, the type of dependence, or type of periodicity present, subject, of course, to the mild constraints that permit utilization of the weak convergence theory. The contribution of the dependence and periodicity to the asymptotic distribution is completely encompassed within the parameter γ and it in turn fully describes the effects of the dependence and periodicity in terms of second order moments. For instance, if there are no periodic components, then $\omega = 1$, and γ is defined by

$$\gamma^2 = \sigma^2 + 2 \sigma^2 \sum_{\ell=1}^{\infty} \rho_\ell \; , \tag{2-98}$$

and if further there is no dependence γ^2 reduces to σ^2. The asymptotic mean and variance of D_n are given by

$$E[D_n] \sim (\pi/2)^{1/2} \gamma n^{1/2} \quad , \tag{2-99}$$

and

$$var[D_n] \sim \left[2 \sum_{j=0}^{\infty} (-1)^j (2j+1)^{-2} - \pi/2 \right] \gamma^2 n. \tag{2-100}$$

The asymptotic mean is the same as for R_n^*.

CHAPTER 3
INFERENCE ON RANGE AND DEFICIT

3-1 Introduction

For the most part, Chapters 1 and 2 are reviews of those results concerning the range and deficit that were able to be deduced from assumed models. The problem of drawing inferences from data regarding such results was relegated to this chapter. The general problem of inference is often categorized into the following types of inference: (a) model building or goodness of fit; (b) point estimation; (c) interval estimation; and (d) hypothesis testing. Each of these can in turn be studied from the parametric or non-parameteric viewpoint and both will be employed. Also, as has been the case throughout this paper, results obtained by either simulation or analytical methods are presented.

Admittedly, model building and goodness of fit of various models is important, but since it is somewhat divorced from the subject at hand, namely range and deficit analysis, it will not be addressed in this paper. Suffice it to say that certain models, specifically fractional Gaussian noise, were proposed because they were thought to be able to mimic behavior of the range. The real problem of deciding which of several alternative models best fit a given geophysical phenomenon will not be discussed. Among the other three modes of inference, namely point estimation, interval estimation, and hypothesis testing, only the first two will be considered.

The chapter is divided into three further subsections. The Section 3-2 considers estimation of γ^2, the sum of the covariance function. It was noted in Chapter 2 that γ^2 is an important parameter as far as range and deficit analyses are concerned; in fact, γ^2 characterized much of the asymptotic theory given there. Section 3-3 looks at the relationship between the range and/or deficit statistics obtained from overlapping time intervals of observations. Often times, for a given say, 100 year span of data, possibly simulated, the range statistic is computed for n = 100 and also for n = 50, say by dividing the 100 years of data into two sets of 50 each. How then is R_{50} related to R_{100}? The final Subsection 3-4, considers estimation of the Hurst slope. It will be noted that the Hurst slope for the rescaled range is nearly independent of the underlying model marginal distribution, and consequently is a poor statistic for discriminating between models. The problem of just how stable the Hurst slope is for moderate length records is broached.

3-2 The Integral of the Covariance Function γ^2

In Chapter 2 it was noted that γ^2 played a key role in the asymptotic theory of the range and deficit; in fact, γ or β, where $\gamma^2 = \sigma^2\beta^2$, enters as a scale parameter in most limiting distributions given there. Consequently, estimation of γ, as well as an indication of the precision of such estimates, is fundamental in range and deficit study. Two approaches will be mentioned, the first is parametric in which a parametric model is assumed and then γ is a function of the model parameters and hence amenable to the usual parametric estimation techniques such as maximum likelihood or the method of moments; the second is nonparametric and uses techniques not associated with a particular model.

Recall that $\gamma^2 = \sigma^2\beta^2$ where $\beta^2 = 1 + 2\sum_{j=1}^{\infty} \rho_j$ for the nonperiodic dependent Brownian domain of attraction case. To give some idea of the estimation of γ^2, consider a simple AR(1) model given by

$$X_t = \rho X_{t-1} + \varepsilon_t \qquad (3\text{-}1)$$

where $\{\varepsilon_t\}$ is white noise with variance parameter σ_ε^2 and $-1 < \rho < 1$ is a parameter. It is known that $\rho_j = \rho^j$ and $\sigma^2 = \sigma_\varepsilon^2/(1-\rho^2)$ for an AR(1) model, hence

$$\beta^2 = 1 + 2\sum_{1}^{\infty} \rho^j = (1 + \rho)/(1 - \rho) \qquad (3\text{-}2)$$

and

$$\gamma^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2} \cdot \frac{1+\rho}{1-\rho} = \frac{\sigma_\varepsilon^2}{(1-\rho)^2} \qquad (3\text{-}3)$$

a function of the model parameters ρ and σ_ε^2.

The general theory of maximum likelihood estimation states that the maximum likelihood estimator of γ^2, say $\hat{\gamma}^2$, is given by

$$\hat{\gamma}^2 = \hat{\sigma}_\varepsilon^2/(1 - \hat{\rho})^2 \quad , \qquad (3\text{-}4)$$

where $\hat{\rho}$ and $\hat{\sigma}_\varepsilon^2$ are the maximum likelihood estimators of ρ and σ_ε^2, respectively. Furthermore, such theory states that $\hat{\gamma}^2$ is asymptotically distributed as a normal with mean = γ^2 and variance

$$= \frac{1}{n}\left\{\frac{4\sigma_\varepsilon^4\,\mathrm{var}[\hat{\rho}]}{(1-\rho)^6} + \frac{\mathrm{var}[\hat{\sigma}^2]}{(1-\rho)^4}\right\} =$$

$$= \frac{1}{n}\left\{\frac{4\sigma_\varepsilon^4(1-\rho^2)}{(1-\rho)^6} + \frac{\sigma_\varepsilon^4}{2(1-\rho)^4}\right\} \qquad (3\text{-}5)$$

From Eq. (3-5) one can gain some idea as to the precision of the maximum likelihood estimator. For instance, if $\rho = 0.5$ and $\sigma_\varepsilon^2 = 1$ then $\hat{\gamma}^2$ is asymptotically distributed as N(4, 200/n) and so one would expect about 95 percent of the values of the estimate $\hat{\gamma}^2$ to fall within $4 \pm 20\sqrt{2/n}$, and about 67 percent to fall within $4 \pm 10\sqrt{2/n}$.

Now if one were interested in the estimation of β^2, rather than γ^2, then the maximum likelihood estimator of β^2 is $\hat{\beta}^2 = (1+\hat{\rho})/(1-\hat{\rho})$ where again $\hat{\rho}$ is the maximum likelihood estimator of ρ. Here $\hat{\beta}^2$ is asymptotically distributed as a normal with mean = β^2 and variance

$$= \frac{4}{(1-\rho)^4}\,\mathrm{var}[\hat{\rho}] = \frac{4(1-\rho^2)}{n(1-\rho)^4} = \frac{4(1+\rho)}{n(1-\rho)^3} \quad (3\text{-}6)$$

so if $\rho = 0.5$ then $\hat{\beta}^2$ is asymptotically distributed as a $N(3, \frac{48}{n})$. Note that in either case the precision worsens as ρ approaches 1. Similar comments can be made for models other than an AR(1) with the understanding that as the complexity of the model increases so does the corresponding asymptotic theory. In any case, the asymptotic distribution of the maximum likelihood estimator of γ^2 is specific to the model.

To gain some insight into the nonparametric estimation of γ^2 or β^2, emphasis will be placed on β^2 and use will be made of the relation

$$\beta^2 = 1 + 2\sum_{k=1}^{\infty} \rho_k = 2\pi f(0) \quad (3\text{-}7)$$

where $f(0)$ is the spectral density of the process evaluated at zero frequency. Estimation of β^2 is now tantamount to estimation of the spectral density evaluated at zero. A variety of spectral density estimators are available, but only the usual window or smoothed estimators will be considered here. The asymptotic theory associated with such estimators states that $\hat{f}(0)$ is asymptotically distributed with mean $f(0)$ and variance $2f^2(0)\sum_{j=-d_n}^{d_n} W_n^2(j)$ where $W_n(j)$ are the so-called window weights (Fuller, 1976). In comparison with the AR(1) model given above, that is, $\sigma_z^2 = 1$ and $\rho = 0.5$, the asymptotic variance of $2\pi\hat{f}(0)$ for a rectangular window is given by

$$4\pi^2\,(2\,f^2(0))\,\frac{1}{2d_n+1} = \frac{2\,\beta^4}{2d_n+1} \quad . \quad (3\text{-}8)$$

Since this latter technique is nonparametric, it would serve better in those cases when the model is more complicated but with an attendant loss in precision.

A variety of studies regarding inferences of γ^2 can be envisioned but will not be pursued here. Since our purpose is primarily one of review, and not development, and the literature on inferences concerning γ^2 is void, little can be said; the above just sketches an approach to the problem.

3-3 The Correlation Between Successive Values of Range and Deficit

Notwithstanding the intent expressed in the last sentence of the previous section, in this section we present some new results obtained by simulation, supported by limited analytical exploration. They concern the correlation between successive values of the range R_n and also of the (accumulated maximum) deficit D_n, where the input $\{X_t\}$ is a two-state Markov chain.

To motivate this part of the study, the following question might well be posed: Why is the correlation between R_n and R_{n+k} or D_n and D_{n+k} $(n \geq 1,\ k \geq 1)$ of interest? This immediately raises the question: Why study R_n and D_n at all?

In partial answer to the second question:

(a) Because Hurst proposed R_n^{**} and D_n as quantities or statistics useful for defining the size of a reservoir;

(b) Because it has been demonstrated that different input processes have different range and deficit behavior, and thus it has been inferred that by examining the range or deficit it may be possible to detect important differences in characterization of inputs.

Leading on from (b), in order to quantify the range behavior, some statistics need to be derived; and before anything meaningful can be inferred from these, it would be helpful if something was known about their distributions.

One statistic of interest is the Hurst slope K defined (inter alia) as a function of the rescaled range by Hurst(1951) (see Eq. 1-12) as $\log(R_n^{**})/\log[n/2]$; another is the local slope H_n defined by Boes and Salas (1973); see Eq. (1-15). The behavior of these statistics (and their information content) must depend on the correlation between successive values of R_n^{**}, for if R_n^{**} and R_{n+k}^{**} are correlated, then $\hat{E}(R_n^{**})$ (and hence any statistics based on a finite sample) will have greater variance than if there were no correlation between successive values of the range. A similar argument applies to the deficit, and hence an answer has been provided to the first of the questions posed above.

As a first step to understanding, the simplest cases are treated initially: the range R_n and the (maximum accumulated) deficit D_n, where the input process is a two-state Markov chain. It is conjectured that the results for R_n will translate to R_n^* and R_n^{**} with perhaps an intensification of the degree of intercorrelation.

Two-state Markov chain input. Let $P[X_t = -1] = P[X_t = +1] = 1/2$. Further, let $p = P[X_{t+1} = \pm 1 \mid X_t = \mp 1] = (1+\rho)/2$ and $q = P[X_{t+1} = \pm 1 \mid X_t = \mp 1] = (1-\rho)/2$. Then $\{X_t\}$ is a two-state Markov chain with $\mathrm{corr}[X_{t+s}, X_t] = \rho^s$, $s = 0,1,2,\ldots$.

The cumulative sums of this chain $(S_n = \sum_{i=1}^{n} X_i)$ converge in distribution to a Wiener process as shown by Troutman (1976), and for moderate n these S_n mimic the cumulative sums of an autoregressive process with the same correlation structure. This fact was exploited by Gomide (1975). Hence the results reported in this section will apply with a fair degree of approximation to cases with Gaussian inputs.

The range, R_n. For small n it is easy to find the exact joint distribution of $\{R_1, R_2, \ldots, R_m\}$ by direct evaluation. For large n the procedure is tedious and recourse is simulation; however, the exact results in Table 3-1 give us a bench-mark for comparison purposes, where it is understood that $P[R_0=0] = P[R_1=1] = 1$.

Table 3-1 p.m.f. for $[R_2,R_3,R_4]$.

R_2	R_3	R_4	$P[R_2,R_3,R_4]$
1	1	1	q^3
1	1	2	q^2p
1	2	2	q^2p
1	2	3	qp^2
2	2	2	q^2p+qp^2
2	3	3	qp^2
2	3	4	p^3

where $p = (1+\rho)/2$ and $q = (1-\rho)/2$ from above. From Table 3-1:

$$E[R_2] = 1 + p$$

$$var[R_2] = pq \quad (= \sigma^2[R_2] \text{ say})$$

$$E[R_4] = q^3 + 6q^2p + 8qp^2 + 4p^3$$

$$var[R_4] = q^3 + 12q^2p + 22qp^2 + 16p^3 - -E[R_4]^2 \quad (=\sigma^2[R_4] \text{ say})$$

$$E[R_2R_4] = q^3 + 8q^2p + 13\,qp^2 + 8p^3, \text{ and}$$

$$cor[R_2,R_4] = (\sqrt{1+\rho}/\sqrt{1-\rho})(1-\rho)(3+\rho^2)/(47+22\rho-23\rho^2-20\rho^3-23\rho^4-2\rho^5-\rho^6)1/2. \qquad (3\text{-}9)$$

Hence, for comparison with the numerical simulation results we give:

Table 3-2 Expected values, standard deviation and autocorrelation coefficients of successive ranges.

ρ	$E[R_2]$	$\sigma[R_2]$	$E[R_4]$	$\sigma[R_4]$	$corr[R_2,R_4]$
0	1.5	0.5000	2.3750	0.8570	0.4376
0.3	1.65	0.4770	2.8021	0.8909	0.4136
0.6	1.8	0.4000	3.2720	0.8331	0.4033
0.9	1.95	0.2179	3.8049	0.5122	0.4053

The values of $E[R_2]$, $E[R_4]$, $\sigma[R_2]$ and $\sigma[R_4]$ agree with Gomide (1975). Note the behavior of $corr[R_2,R_4]$ as ρ increases, sketched in Fig. 3-1. This behavior seems to extend to $corr[R_n,R_{2n}]$ for $n > 2$ as demonstrated by the simulation results which follow in Tables 3-3 and 3-4.

The results are computed from different sets of 1000 independent samples. Table 3-3 gives $\hat{E}[R_n]$, where Gomide's (1975) results appear in the first line marked * for comparison.

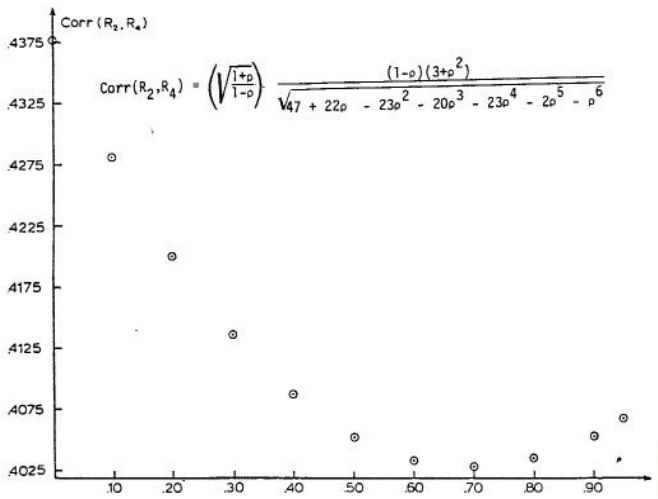

Fig. 3-1 $Corr[R_2,R_4]$ for correlated Bernoulli inputs as a function of ρ.

Table 3-3 $\hat{E}[R_n]$ for various values of ρ and n.

ρ \ n	2	4	8	16	32	64	128	256
0*	1.5	2.37	3.65	5.48	8.10	11.82	[16.92]	[24.39]
0	1.485	2.423	3.686	5.602	8.183	11.733	17.048	24.102
0.3	1.639	2.796	4.495	7.134	10.758	15.951	23.218	32.760
0.6	1.815	3.259	5.574	9.209	14.507	21.960	32.580	47.766
0.9	1.951	3.845	7.385	13.717	24.158	39.860	63.226	98.051

In the first row of Table 3-3, the last two entries [bracketed] are from Anis and Lloyd's (1953) result. Table 3-4 is for $\hat{\sigma}[R_n]$, where the first row * is from Gomide (1975).

Table 3-4 $\hat{\sigma}[R_n]$ for various values of ρ and n.

ρ \ n	2	4	8	16	32	64	128	256
0*	0.50	0.86	1.28	1.85	2.66	3.78	-	-
0	0.499	0.844	1.296	1.936	2.689	3.712	5.486	7.700
0.3	0.480	0.876	1.487	2.319	3.504	5.192	7.276	9.772
0.6	0.388	0.840	1.621	2.957	4.839	7.190	10.472	15.005
0.9	0.216	0.461	1.178	2.932	6.371	12.063	19.704	31.296

The corresponding sample correlation matrices $\{corr(R_n,R_m)\}$, $n,m = 2,4,\ldots,256$ are given in Table 3-5. It is seen that for $n \geq 4$, $\hat{corr}(R_n,R_{2n})$ generally exceeds 0.5 regardless of ρ while $\hat{corr}(R_n,R_{4n})$ is generally in excess of 0.25. This indicates a surprisingly consistent (with varying ρ) and by no means negligible correlation for large values of n.

The deficit, D_n. The particular case of full regulation development (i.e. where mean net input is zero) is treated here. As with the range, an explicit

Table 3-5 Corr $[R_n, R_m]$, n,m = 2,4,...,256 for various values of ρ.

n \ m	2	4	8	16	32	64	128	256
					ρ = 0.0			
2	1	0.3956	0.1904	0.0786	0.0582	0.0439	0.00209	0.00559
4		1	0.5412	0.2499	0.1259	0.0731	0.0215	0.0189
8			1	0.5824	0.3152	0.1753	0.1021	0.0670
16				1	0.6017	0.3363	0.1625	0.0948
32					1	0.5947	0.2961	0.1638
64						1	0.5831	0.3527
128							1	0.6082
256								1
					ρ = 0.30			
2	1	0.4141	0.1830	0.0650	0.0402	0.0190	0.00735	-0.00951
4		1	0.5070	0.2073	0.0813	0.0213	0.0118	-0.0129
8			1	0.5164	0.2676	0.1221	0.0722	0.0514
16				1	0.5937	0.3217	0.1790	0.0897
32					1	0.5821	0.3095	0.1736
64						1	0.6058	0.3507
128							1	0.6388
256								1
					ρ = 0.60			
2	1	0.3676	0.1115	0.0363	0.0185	0.00165	-0.00607	0.0442
4		1	0.4387	0.1654	0.0887	0.0808	0.0419	0.0341
8			1	0.5138	0.2358	0.1548	0.0587	0.0578
16				1	0.5668	0.3040	0.1443	0.0837
32					1	0.5848	0.3097	0.1703
64						1	0.5540	0.2767
128							1	0.5822
256								1
					ρ = 0.90			
2	1	0.4357	0.1843	0.0429	-0.0300	-0.0118	-0.0503	-0.0470
4		1	0.4465	0.1597	0.0553	0.0618	-0.00022	-0.0197
8			1	0.5285	0.1705	0.0772	0.0395	0.0127
16				1	0.5017	0.1581	0.0685	0.00606
32					1	0.5271	0.2019	0.0795
64						1	0.4745	0.1964
128							1	0.5358
256								1

evaluation of the joint probabilities of successive values of the random variable D_n yields the following exact results when the input is the two-state Markov chain described above. Now $P[D_0=0] = 1$, and in Table 3-6 $p = (1+\rho)/2$ and $q = (1-\rho)/2$ as before.

Table 3-6 p.m.f. of $[D_1, D_2, D_3, D_4]$.

D_1	D_2	D_3	D_4	$P[D_1, D_2, D_3, D_4]$
0	0	0	0	$p^3/2$
0	0	0	1	$p^2q/2$
0	0	1	1	$pq^2/2$
0	0	1	2	$p^2q/2$
0	1	1	1	$(pq^2 + q^3)/2$
0	1	2	2	$pq^2/2$
0	1	2	3	$p^2q/2$
1	1	1	1	$(p^2q + pq^2 + q^3)/2$
1	2	2	2	$(p^2q + pq^2)/2$
1	2	3	3	$p^2q/2$
1	2	3	4	$p^3/2$

From Table 3-6,

$$E[D_2] = 1$$

$$\text{var}[D_2] = p = (1+\rho)/2 = \sigma^2[D_2]$$

$$E[D_4] = (4p^3 + 12p^2q + 9pq^2 + 2q^3)/2$$

$$\text{var}[D_4] = (16p^3 + 28p^2q + 15pq^2 + 2q^3)/2 - E[D_4]^2 \text{ and}$$

$$E[D_2D_4] = 4p^3 + p^2q + 5pq^2 + q^3,$$

from which we compile Table 3-7, giving some exact results for values of p which correspond to the later simulations.

Table 3-7 Expected values, standard deviations and autocorrelation coefficients of successive deficits.

ρ	$E[D_2]$	$\sigma[D_2]$	$E[D_4]$	$\sigma[D_4]$	$\text{corr}[D_2, D_4]$
0	1	.7071	1.6875	.9823	.6299
0.3	1	.8062	1.8377	1.2370	.7389
0.6	1	.8944	1.9440	1.5352	.8506
0.9	1	.9747	1.9963	1.8771	.9626

The results of four different sets of 1000 independent simulations are shown in Table 3-8, 3-9 and 3-10.

Table 3-8 $\hat{E}[D_n]$ for various values of ρ and n.

ρ \ n	2	4	8	16	32
0	0.98	1.66	2.63	4.01	6.11
0.3	0.98	1.78	3.08	5.07	8.01
0.6	0.99	1.90	3.53	6.36	10.49
0.9	1.03	2.03	4.03	7.88	14.94

Table 3-9 $[D_n]$ for various values of ρ and n.

ρ \ n	2	4	8	16	32
0	0.72	0.99	1.41	2.02	2.92
0.3	0.78	1.23	1.83	2.75	3.99
0.6	0.88	1.55	2.44	3.71	5.39
0.9	0.98	1.89	3.52	6.31	10.54

The bracketed figures in Table 3-10 are the exact values for $\text{corr}[D_2, D_4]$ from Table 3-7. These results are much as anticipated; however, it is interesting to note the high correlation between D_n and D_{2n}. Further, it would appear from the simulation (although it may be fortuitous) that the sampling variation for D_n is smaller than that for R_n.

The short conclusion that can be drawn from this study is that R_n and R_{n+k} or D_n and D_{n+k} are highly correlated for $k > 1$. Thus great care should be taken in making inferences on statistics such as K, but more on this in Section 3-4.

Table 3-10 Sample correlation matrices $\hat{corr}[D_n, D_m]$, n,m = 2,4,8,16,32 and various ρ.

m \ n	2	4	8	16	32
			ρ = 0.0		
2	1	0.645 (0.6299)	0.349	0.190	0.114
4		1	0.623	0.334	0.145
8			1	0.660	0.330
16				1	0.641
32					1
			ρ = 0.30		
2	1	0.740 (0.7389)	0.405	0.173	0.082
4		1	0.691	0.343	0.147
8			1	0.654	0.348
16				1	0.670
32					1
			ρ = 0.60		
2	1	0.856 (0.8506)	0.574	0.312	0.154
4		1	0.782	0.440	0.218
8			1	0.703	0.373
16				1	0.665
32					1
			ρ = 0.90		
2	1	0.965 (0.9626)	0.857	0.675	0.451
4		1	0.934	0.754	0.515
8			1	0.886	0.632
16				1	0.822
32					1

3-4 The Hurst Slope

In this section some results concerning inferences on the Hurst slope are reviewed. The primary references are the recent studies of McLeod and Hipel (1978) and Hipel and McLeod (1978).

Before commencing with the review, a clear understanding of what we are about is required. The Hurst slope can be considered from the asymptotic or the preasymptotic viewpoint. The asymptotic viewpoint is the following: Assume that the model is such that

$$E[R_n^{**}] \sim a\, n^h \tag{3-10}$$

where "~" is defined as $\lim_{n\to\infty} E[R_n^{**}]/(a\, n^h) = 1$. Here h is a resultant characteristic of the model. As noted in Chapter 2, for any model belonging to the Brownian domain of attraction h is 1/2 and for fractional Gaussian noise models indexed by h, h can vary from 0 to 1. Now, while it is of interest to make inferences regarding h from data for fixed n in the sense that, say the distribution of some estimator of h be studied, such will not be pursued here. Rather, the preasymptotic notion of the Hurst slope will be considered. Again, such slope can be viewed in several ways, but the easiest is to follow Hipel and McLeod (1978) who defined the Hurst slope as the random variable K_n in

$$R_n^{**} = \left(\frac{n}{2}\right)^{K_n} . \tag{3-11}$$

Such definition of the Hurst slope is certainly motivated by Hurst's original work in which he computed a slope, say k_n, from a record of length n by estimating a value of R_n^{**}, say r_n^{**}, from the data and then setting

$$k_n = \log r_n^{**} / \log (n/2) . \tag{3-12}$$

If the definition of Eq. (3-11) is adopted, as it is here, inferences concerning the Hurst slope are embodied in the distribution of K_n and such distribution can be theoretically obtained from the distribution of R_n^{**} via the transformation

$$K_n = \log R_n^{**} / \log (n/2) . \tag{3-13}$$

For the case of R_n^{**} a continuous random variable with

$$f_{K_n}(k) = \left(\frac{n}{2}\right)^k \log\left(\frac{n}{2}\right) f_{R_n^{**}}\left(\left(\frac{n}{2}\right)^k\right) , \tag{3-14}$$

where $f_{R_n^{**}}$ the density of R_n^{**}, the only problem is that the distribution of R_n^{**} is not exactly known except in very special cases. If n is large enough, then the asymptotic distribution of R_n^{**} is given in Section 2.8, but the form of this asymptotic distribution is such that the corresponding asymptotic distribution of K_n coming out of Eq. (3-14) is not nice. It could, however, be written down and probabilities could be obtained numerically.

Employing the simple propagation of errors formulae and still assuming n large enough that the asymptotic results for R_n^{**} can be invoked, one gets

$$E[K_n] \approx \frac{\log E[R_n^{**}]}{\log (n/2)} \approx \frac{1}{2} + \frac{\log \beta\sqrt{\pi}}{\log (n/2)} - \frac{(\frac{\pi}{6} - \frac{1}{2})}{\log (n/2)} \tag{3-15}$$

and

$$\text{var}[K_n] \approx \frac{\text{var}[R_n^{**}]}{\{E[R_n^{**}] \log (n/2)\}^2} \approx$$

$$\approx \frac{(\pi^2/6) - \pi/2}{(\pi/2)(\log (n/2))^2} = (\pi/3 - 1)\frac{1}{(\log (n/2))^2} \approx$$

$$\frac{.0472}{(\log (n/2))^2} . \tag{3-16}$$

Admittedly, propagation of errors formulae are crude, but for an AR(1) model with ρ = 0.4 and n = 100, one obtains $\beta^2 = (1+\rho)/(1-\rho) = 7/3$, and

$$E[K_{100}] \approx 0.75 \tag{3-17}$$

and standardard deviation

$$K_{100} \approx 0.056 . \tag{3-18}$$

Hipel and McLeod (1978) tabled the empirical cumulative distribution function of K_n for various ARMA models and various n via Monte Carlo simulation. For each model and for each n their empirical cumulative

distribution function of K_n was computed from a sample of 10^4 K_n's; 10^4 is certainly large enough that their empirical c.d.f.'s are very close to the true c.d.f.'s (but see comments in Section 2-7). For example, for an AR(1) with $\rho = 0.4$ and $n = 100$, their empirical c.d.f. gives (0.585, 0.813) as a 95 percent probability interval for K_{100}. Such interval is not that different from the corresponding interval of (0.64, 0.86) obtained using asymptotic theory and propagation of errors (see Eqs. (3-17) and (3-18). Not much more can be said about K_n than to give its distribution and Hipel and McLeod do give such distributions, via empirical distributions that are accurate enough for practical considerations, for the models and parameters that they simulated; other models and parameters could be handled similarly.

Other definitions of the Hurst slope could be defined for the preasymptotic case; including one suggested by Gomide (1975) and another by Siddiqui (1976). These could then be analyzed as K_n was, but there does not seem to be much point in doing so since Hurst actually made his calculations using the equality given in the definition (Eq. 3-13) of K_n. Mandelbrot and Wallis (1969) define a slope estimator, labelled H, as the slope of the least squares regression line: $\log [\overline{r}_n^{**}] = a + H \log n$, $M_o \leq n \leq N$. Such definition seems appropriate for those cases when h in $E[R_n^{**}] \propto n^h$ is constant over n, as is the case for the fractional Gaussian noise models. They give an indication of the variability of such H. Also, Wallis and Matalas (1970) further studied H type estimators and presented simulation results that indicated the distribution of H for various models and various n. Their results are compatible with the empirical distributions of Hipel and McLeod.

In summary, some distribution results concerning the Hurst slope are known and well documented, the primary references being Hipel and McLeod (1978), Mandelbrot and Wallis (1969), and Wallis and Matalas (1970).

CHAPTER 4
IMPORTANCE OF RANGE AND DEFICIT ANALYSES IN HYDROLOGY

4-1 Storage Design

There are several ways of sizing reservoirs to regulate flows with a given level of assurance. The methods vary from rule of thumb through the intuitively appealing (and operationally flexible) simulation approach to the analytic method, with an attempt to yield results in exact form. Of necessity, the last approach requires that certain simplifying assumptions be made about the form of input, output and reservoir conditions, but it has the advantage that exact results are forthcoming, usually with a relatively small amount of computation. However, the great virtue of the analytic method is to allow for objective comparisons between the results of the various assumptions that are made concerning the model specification.

Here, we concentrate on the analytic approach, drawing together some well known results from range and deficit analyses, which are juxtaposed with some from stochastic reservoir theory. Pains are taken to ensure that they are compared in the same framework. To facilitate this comparison, two simple and perhaps unrealistic inputs are used. They are the normal and discrete($\pm$1) input processes. They have been chosen because they are so dissimilar.

We will compare the results of range, deficit and storage behavior for both these types of input when the respective net inputs have zero mean and are serially independent. We then compare the effect of the choice of the two types of input on the behavior of a finite reservoir when the net input has non-zero mean. To facilitate this, we introduce the concept of the standard reservoir which is shown to be a powerfully unifying one. We shall end with a brief excursion into reservoir behavior when the input is serially correlated.

4-1-1 Preliminaries

Range analysis can be conceived among others, as the study of the behavior of the accumulated net input to an infinite reservoir, where, of necessity, the net input must have zero mean. Deficit analysis can also be conceived, among others, as the study of the accumulated net input to a semi-infinite reservoir, i.e. it has a top but no bottom. This form of the semi-infinite reservoir is chosen because usually the reservoirs in practice may be operated with a positive mean net input. However, this is none other than the reflection of the semi-infinite reservoir treated in classical stochastic reservoir theory in which the reservoir has a bottom and no top and the mean net input is negative, ensuring ergodicity of the probability distribution of storage. Thus the semi-infinite reservoir is useful as a first approximation to a finite reservoir. Storage analysis will be understood to mean the study of the behavior of a finite reservoir subjected to a net input whose mean can be positive, negative or zero. Before we proceed, we list some essential definitions, repeating some from Section 1-3 for the purpose of clarity.

Net input. The net input X_t in the interval $(t,t+1)$ is the difference between the gross input Z_t and all abstraction Y_t, which are assumed to occur simultaneously. (For simplicity evaporation and other losses may or may not be included in the model abstractions.) We consider two types of net input: normal and discrete. The discrete input employed here has three states: -1, 0, +1. Much of the study employs the particular choice $P[X_t=-1] = 1-p$, $P[X_t=0] = 0$, $P[X_t=1] = p$. In this case, the three-state binomial input is reduced to 2-state binomial input.

We denote $E[X_t]$, $\mathrm{var}[X_t]$ and $\mathrm{corr}[X_t,X_{t-1}]$ by μ, σ^2 and ρ respectively, as we assume stationarity throughout the sequel. Skewness is not explicity included as a parameter, although its importance is tacitly recognized. Note that this formulation can accomodate a stochastic demand process.

Demand draft. The demand is assumed to be constant (all the variance being embodied in Z_t), and is the amount that we would like the reservoir to supply in any interval. The draft (or abstraction) Y_t on the other hand is a variable, conditional on the reservoir level: $Y_t = \delta$ unless the reservoir level reaches one of the boundaries (if they exist).

Storage. Consider a reservoir of finite capacity V. Define $S_{t+1} = \min[V, \max[0,S_t+X_t]]$ where there is no restriction on the value of μ. Depending on the application, the initial storage S_0 can be any point on the interval $[0,V]$. (We may be interested in the first passage time from 0 to V or from V to 0, or more traditionally, in the equilibrium distribution of storage $\lim_{t\to\infty} P[S_t \le s]$, $(0\le s\le V)$, which is independent of S_0.)

Effective capacity. If, as we have assumed, Z_t and Y_t occur simultaneously, then the effective capacity of a reservoir of size V is equal to V. If, on the other hand, either Z_t or Y_t (or both) occur instantaneously, the former before the latter, then at t+1 the level of the reservoir will be lower by δ than it was the instant before the withdrawal commenced.

The effective capacity of the reservoir is thus reduced by δ in the case of "staggered" net input. All the results of the sequel will be interpreted in terms of V. Thus, they apply to a reservoir of capacity V when the net input is "simultaneous" (either instantaneous or evenly spread in $[t,t+1)$) and to a reservoir capacity $V + \delta$ when the net input is "staggered."

Drift. The drift ε is defined as the standardized mean net input μ/σ. It is thus the inverse of the coefficient of variation of the net input and is related to the level of development α by $\varepsilon = (1-\alpha)\cdot E[Z_t]/\sigma$.

Standardized reservoir. We define a standardized reservoir to be one of standardized capacity $c = V/\sigma$ fed by a net input with drift ε. For the independent inputs considered above, it will be shown that the behavior of the finite reservoir is a function only of c and ε. The scaling of the reservoir capacity by σ is in marked contrast to the common practice of scaling the capacity by $E[Z_t]$, the mean gross input.

Probability of emptiness. In storage analysis, a quantity of much interest is $v_0 = \lim_{t\to\infty} P[S_t=0]$, the probability of emptiness. Also of interest is the probability of failure, $y_0 = \lim_{t\to\infty} P[Y_t < \delta]$, to supply the full demand. The relationship between v_0 and y_0 depends on whether $\{X_t\}$ has a continuous or discrete distribution. If $\{X_t\}$ has a continuous distribution, then

$$y_0 = \lim_{t\to\infty} P[S_t=c].P[X_t<c] + \int_{0+}^{c-} P[S_t=s].$$

$$P[X_t<s]ds + P[S_t=0].P[X_t<0].$$

Because of the continuity of the input, $P[X_t=s] = 0$, for all s, so we can replace the inequalities $<$ with $\leq$ in the above expression, which is then numerically equal to v_0.

Thus, for continuous input distributions, the probability of emptiness, and the probability of failure are equal, if not synonymous.

The practice shows, however, that most operated reservoirs have approximately a β-distribution of contents (levels) with two different probability values of full and empty reservoir. This comes from the fact that the operational rules are such that a full reservoir (and dangers of spillovers) and an empty reservoir (and dangers of shortages) are penalized by various operational criteria, penalties and optimizations.

When $\{X_t\}$ has a discrete distribution, then $P[X_t = i] > 0$ for at least some s, so that following the above argument, $y_0 \leq s$. In fact, for the classic case where $\delta = 1$ and $X_t \in \{-1,0,1,\ldots,n-1\}$ we get $y_0 = v_0.P[X_t = -1] < v_0$.

This suggests that when comparing the results for continuous and discrete distributions, we should look at v_0, not y_0. This is what is done in the sequel.

Mean first passage times. In studying the behavior of a finite Markov chain with discrete state and homogeneous (time invariant) transition probability matrix, we denote by m_{ij} the mean number of time-steps taken to reach state i from state j for the first time. In particular, if 0 is the empty state of the chain, m_{00} is the mean recurrence time of emptiness. Thus, if we have a discrete input to a suitably defined reservoir with a discrete state space, then m_{00} is none other than the reciprocal of v_0, the probability of emptiness. Again, if c is the label of the full state, m_{c0} is the mean first passage time from empty to full or the "mean time to fill," while conversely, m_{0c} is the "mean time to empty." m_{cc} is the mean recurrence time of being full, which may not be usually of much interest.

The mean first passage times give us a key to the comparison of deficit and storage, because as Gomide (1975) pointed out: "the probability that a reservoir of size [k+1], initially full, is empty for the first time at discrete time n, regardless of the occurrence of overflows is simply $P[D_n > k] - P[D_{n-1} > k]$."

Troutman (1976) interpreted this to mean $P[N_{0,c} > n] = P[D_n/\sigma < c]$ where $N_{0,c}$ is the first passage time from full to empty. Because of the sense of the inequalities we intuitively expect m_{0c} to be greater than $\{n : E[D_n/\sigma] = c\}$. This is born out by the comparison.

4-1-2 Algorithm for Evaluation of Mean First Passage Times of a Markov Chain

Kemeny and Snell (1960) detail the development of the algorithm. We quote their results and show how they can be applied in storage analysis.

Let a discrete-state stochastic process $\{S_t\}$ be a lag-one finite Markov chain with n+1 states, defined by its (n+1)-square transition probability matrix Q. $\{S_t\}$ will be a stationary process if Q is constant. The marginal probability distribution vector of S_t, denoted by $p_t = \{P[S_t=i]\}$ $i = 0,1,2,\ldots,n$, is given by $p_t = Qp_{t-1} = Q^t p_0$ where p_0 is the initial distribution vector. If Q has at least one positive diagonal element and no zero rows or columns, then S_t is ergodic, which implies that $p_t \to \pi$, (the equilibrium distribution vector of S) as $t \to \infty$. It follows that $Q\pi = \pi$, which when combined with the condition $1'\pi = 1$ (where 1' is a row of ones), yields π on solution of the set of n+1 independent linear simultaneous equations.

Kemeny and Snell give the following algorithm for evaluation of the mean first passage time matrix $M = \{m_{ij}\}, i,j = 0,1,\ldots,n$, where each element, m_{ij}, is equal to the mean number of steps taken to go from j to i for the first time. (It is usual to call m_{ii} the mean first recurrence time for state i, but we will dispense with that distinction where we are talking collectively of m_{ij}.)

They find M as follows. Define the "fundamental matrix" Z as $Z = [I - Q + \pi 1']^{-1}$ then

$$M = D[I - Z + d1'] \qquad (4\text{-}1)$$

where D is a diagonal matrix with $1/\pi_i$ as the i-th entry and d is a vector formed from the diagonal elements of Z. When $\{S_t\}$ is a random walk with partially reflecting barriers at 0 and n, so that

$$P[S_t = 1|S_{t-1} = 0] = P[S_t = n|S_{t-1} = n] = p,$$

$$P[S_t = 0|S_{t-1} = 0] = P[S_t = n-1|S_{t-1} = n] =$$

$$= q = 1-p,$$

$$P[S_t = i+1|S_t = i] = p \text{ and } P[S_t = i-1|S_{t-1} = i] =$$

$$= q$$

for $i = 1,2,\ldots,n-1$, then $\{S_t\}$ is a finite discrete ergodic Markov chain.

Kemeny and Snell also give the following explicit formulae for m_{ij} for this case.

If $p = 1/2$,

$$m_{ij} = \begin{cases} n + 1 & i=j, \\ (2n-i+1)i - (2n-j+1)j & i>j, \\ j(j+1) - i(i+1) & i<j. \end{cases} \quad (4\text{-}2)$$

If $p \neq 1/2$,

$$m_{ij} = \begin{cases} \frac{r^{n+1}-1}{r-1}\frac{1}{r^i} & i=j \\ \frac{1}{p-q}\left[\frac{r^{n-j+1}-r^{n-i+1}}{r-1} - (i-j)\right] & i>j \\ \frac{1}{p-q}\left[(j-i) - \frac{r^j - r^i}{(r-1)r^{i+j}}\right] & i<j\ , \end{cases} \quad (4\text{-}3)$$

where $r = p/q$. These formulae will prove useful when the discrete finite reservoir fed by a ± 1 input is studied.

4-1-3 Mean First Passage Times for the Finite Reservoir with Normal Input

In the case of the normal input, we have to devise a finite difference algorithm for the evaluation of the m_{ij}. Consider a standardized reservoir of capacity c, with drift ε, discretized into k equal slices of size $\Delta c = c/k$. As $k \to \infty$, the results for this finite difference approximation will tend to those for the continuous reservoir. Fortunately, it has been found that adequate convergence is achieved for $k \geq 2c$, when the following scheme is adopted.

The k+2 states of this approximating discrete reservoir are i = 0,1,2,...,k+1, i.e. zero, the mid-point of each of the sub-intervals and full.

For the case of independent continuously distributed net inputs (in particular the normal) the probability of transition from one state at t to another at t+1 depends on whether S_t is a boundary or interior state.

If $S_t = 0$ then

$$P[S_{t+1} = 0 | S_t = 0] = \int_{-\infty}^{-\varepsilon} \phi(x).dx, = \Phi[-\varepsilon]$$

$$P[S_{t+1} = i | S_t = 0] = \Phi(i\Delta c - \varepsilon) - \Phi[(i-1)\Delta c - \varepsilon]$$

for $i = 1,2,...,$

$$P[S_{t+1} = k+1 | S_t = 0] = 1 - \Phi[c-\varepsilon]$$

where $\phi(x)$ and $\Phi(x)$ are the standardized normal p.d.f. and c.d.f.

For intermediate states:

$$P[S_{t+1} = i | S_t = i] = \Phi[\Delta c/2 - \varepsilon] - \Phi[-\Delta c/2 - \varepsilon]$$

$$P[S_{t+1} = i+1 | S_t = i] = \Phi[3\Delta c/2 - \varepsilon] - \Phi[\Delta c/2 - \varepsilon]\ , \text{ etc.}$$

These transition probabilities can be assembled in a (k+2)-square transition probability matrix Q, where $q_{ij} = P[S_{t+1} = i | S_t = j]$, which will be symmetric if $\varepsilon = 0$. We can then proceed to find M from Eqs. (4-1) and (4-2), increasing k until satisfactory convergence is achieved.

4-1-4 Results and Discussion of Computing Mean First Passage Times

In this section we report the results of the computations made using the above algorithm, which enable us to do the comparisons.

Normal net inputs with zero drift

Storage. Once $\Delta c = c/k$ is small enough, the mean first passage times become independent of k, thus we will denote the mean filling time by m_{c0} which equals the mean time to empty m_{0c} because of symmetry. The mean recurrence time of emptiness is $m_{00} = m_{cc}$. Table 4-1 gives these quantities for the case $\varepsilon = 0$ as a function of c.

Table 4-1 Mean first passage times for normal input with zero drift as a function of standardized capacity c.

c	$m_{00}=m_{cc}$	$m_{0c}=m_{c0}$
1/4	2.219	2.462
1/2	2.476	3.065
1	3.082	4.749
2	4.473	10.002
4	7.307	26.65
8	12.98	83.57
16	24.47	287.8

A comment may be appropriate here. Contrary to expectations based on the behavior of the discrete reservoir, the probability density of the intermediate levels of storage is not uniform, unless c is either small or large. The distortion from uniformity seems to be largest when c is about 4. In this case, the ratio between the densities at $S_t = c/2$ and at $S_t = c -$ (or 0+) is approximately 1.30.

Range. For independent normally distributed net input with $\varepsilon = 0$, $E[R_n/\sigma]$ is given as a function of n by Eq. (2-1). A few tabulated values of this function will be helpful.

n	2	4	8	16	32	64	128	256
$E[R_n/\sigma]$	1.362	2.222	3.488	5.317	7.932	11.651	16.924	24.392

Deficit. For independent normal inputs with $\varepsilon = 0$, Gomide (1975) devised a finite difference scheme for the evaluation of $E[D_n/\sigma]$ as a function of n. He does not tabulate any values, but gives a graph (his Fig. 5.9) from which the following values have been taken; they are therefore approximate.

n	2	4	8	16	32
$E[D_n/\sigma]$	.80	1.44	2.46	3.93	5.96

Figure 4-1 shows $E[R_n/\sigma]$ and $E[D_n/\sigma]$ plotted against n. Plotted coaxially with them is c against m_{c0} for the standardized reservoir with $\varepsilon = 0$. All these for independent normal net inputs, plotted as full lines.

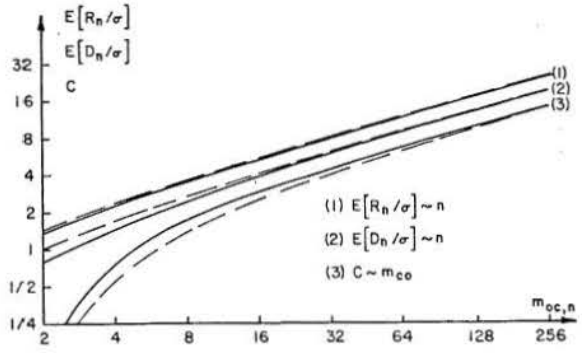

Fig. 4-1 Range and deficit as a function of n, with mean first passage time from full to empty of a standard reservoir of size c with independent normal (full lines) and independent ± 1 discrete (dashed lines) inputs.

Note that for $c \leq 8$, $n \leq 64$, values of $E[R_n/\sigma]$ and $E[D_n/\sigma]$ yield reservoir sizes respectively about 1.7 and 1.3 times as big as those computed from m_{0c}. These factors increase as c and n get smaller.

Discrete net inputs with zero drift

When the net input is discrete and takes the form $P[X_t = \pm 1] = 1/2$, range analysis is associated with an unrestricted random walk, deficit analysis is associated with a random walk where there is partial reflection at the upper boundary and storage analysis is associated with a random walk with two partially reflecting boundaries. Gomide (1975) developed the algorithms for range and deficit analysis for this X_t, while Eqs. (4-2) apply to storage analysis.

Note that in this case, $E[X_t] = 0$, $var[X_t] = 1$.

Range. From Table 4-2 of Gomide (1975) it results

n	2	4	8	16	32	64
$E[R_n/\sigma]$	1.5	2.375	3.6484	5.4806	8.0967	11.8157
$var[R_n/\sigma]$	0.25	.7344	1.6342	3.4393	7.0552	14.2897

Deficit. Gomide (1975) outlines an algorithm for obtaining $E[D_n/\sigma]$, but does not tabulate any values. The algorithm is based on the theory of Markov chains with absorbing barriers. The following is an adaptation of his method.

Let $f_{i,n} = P[D_n \leq i]$, the c.d.f. of D_n; then $E[D_n] = n - \sum_{i=0}^{n-1} f_{i,n}$ and $var[D_n] = \sum_{i=0}^{n-1} (2i+1)(1-f_{i,n}) - E[D_n]^2$. To compute $f_{i,n}$, $i = 1,2,\ldots,n$, we proceed as follows. Let g be an i-element vector, void except for the first element which is 1. For n successvie steps, compute the following, $g_0 = (g_0 + g_1)/2$, $g_i = g_{i-1}/2$ and $g_j = (g_{j-1} + g_{j+1})/2$, $j = 1,2,\ldots,i-1$. At the n-th step compute $f_{i,n} = \sum_{j=0}^{i} g_j$, and put $f_{0,n} = 1/2^n$. Then compute $E[D_n]$ and $var[D_n]$.

The results of this computation are:

n	2	4	8	16	32	64
$E[D_n]$	1	1.6875	2.6712	4.1035	6.1541	9.0721
$var[D_n]$	0.5	.9648	2.0173	4.1115	8.2873	16.6432

Storage. To apply the theory of random walks to storage analysis, we must take care with the specification of the state space of the discrete reservoir. To have the correct properties, the discrete reservoir of "size" k must have k+2 states, 0,1,...,k+1 where, as before 0 and k+1 are the "empty" and "full" states. Transitions from one state to another will only occur if the input consists of amounts which are multiples of the difference between adjacent storage levels. Thus, the "distance" between the adjacent states of a discrete reservoir are all equal. By contrast, in the finite difference version of the continuous reservoir, the distance between the adjacent interior states is Δc, whereas the distance from 0 to 1 and from k to k+1 is $\Delta c/2$. Despite this disparity, the results for the continuous and discrete reservoirs compare favorably, as will be shown in the next section.

When discussing first passage time, we will continue to use the subscript c to denote full; then from Eq. (4-2), with k = c,

$$m_{00} = m_{cc} = c + 2 \quad , \tag{4-4}$$

$$m_{0c} = m_{c0} = (c+1)(c+2) \quad . \tag{4-5}$$

Comparing these results with those of the normal distribution listed in Table 4-1, it will be seen that the m_{00} values match quite well for $c < 4$, while m_{c0} values match well throughout.

The results for $E[R_n/\sigma] \sim n$, $E[D_n/\sigma] \sim n$ and $c \sim m_{c0}$ are plotted in Fig. 4-1 as dashed curves and will be seen to match their corresponding continuous curves, especially in the cases of R_n/σ and D_n/σ.

Note that although a discrete reservoir cannot be defined for non-integer k, if we allow the above expressions to be interpreted for $0 < k < 1$, they closely approximate the normal curves.

The correspondence of results of the continuous and discrete symmetric distributions seems to be fairly good. This suggests that more realistic discrete models should perform even better than the ± 1 model employed here. This observation is relevant, because a thorough study of reservoir behavior for various skewnesses and correlation structures would require a daunting amount of computation if done by simulation or by finite difference approximation to the continuous input.

Before proceeding to examine the effect of drift on reservoir behavior, some concluding remarks on range-deficit-storage analyses seem to be appropriate. Notwithstanding their intellectual appeal because of the interesting (but often difficult) problems associated with range and deficit analyses, and their undoubted influence on time series analysis in hydrology, their practical application in the sizing of reservoirs may leave a lot to be desired, if only because storage analysis is far richer in results than the former, and often yields unequivocal answers to a wide variety of problems with relatively simple

algorithms. Without even invoking the practical aspects of optimization, there are other considerations that may speak against the use of range and deficit analyses in storage problems:

(i) They are each approximations of storage analysis which yield over-conservative reservoir sizes unless adjustments are made.

(ii) The simplicity of the expression for the range may once have been a justification for its use when storage analysis was in its infancy. This condition may no longer hold, since it is seen from Eqs. (4-4) and (4-5) that the equivalent expressions for the first passage times for a discrete reservoir are trivially simple, but accurate.

(iii) Effects of non-zero serial correlation and drift, although not beyond the grasp of range and deficit analysis (Gomide, 1975), are easily computed from simple formulae derived from analysis of the discrete reservoir. The convergence to exact results of continuous reservoirs and net inputs by the use of discrete reservoirs and discrete net inputs are still problems to be taken into account.

4-1-5 Storage Analysis with Non-Zero Drift

Independent, normally distributed inputs

Using the finite difference scheme outlined above, we obtain the following values for m_{00}, m_{0c}, m_{c0} and m_{cc} as functions of c and ε. Note that, because of the symmetry of the normal distribution, $[m_{00}|\varepsilon] = [m_{cc}|-\varepsilon]$ and $(m_{c0}|\varepsilon) = (m_{0c}|-\varepsilon)$.

Table 4-2 Mean recurrence time of emptiness, m_{00}.

c \ ε	0	.2	.4	.6	.8	1.0
1/4	2.219	2.738	3.491	4.604	6.279	8.852
1/2	2.467	3.186	4.270	5.951	8.607	12.89
1	3.082	4.376	6.595	10.48	17.43	30.06
2	4.473	7.998	16.12	35.70	84.57	209.2
4	7.307	21.60	84.73	397.1	2057	11280
8	12.98	114.8	1920	41670	1013000	26×10^6
16	24.47	2627	95×10^4	45×10^7	-	-

Table 4-3 Mean first passage time from full to empty, m_{0c}.

c \ ε	0	.2	.4	.6	.8	1.0
1/4	2.462	3.024	3.825	4.989	6.719	9.352
1/2	3.065	3.891	5.106	6.936	9.761	14.23
1	4.750	6.466	9.220	13.78	21.58	35.29
2	10.00	15.93	27.99	54.19	114.3	258.1
4	26.65	59.89	175.9	653.1	2871	14080
8	83.57	408.6	4250	69350	14×10^5	33×10^6
16	287.8	10300	21×10^5	76×10^7	-	-

Table 4-4 Mean first passage time from empty to full, m_{c0}.

c \ ε	0	.2	.4	.6	.8	1.0
1/4	2.462	2.062	1.772	1.560	1.404	1.289
1/2	3.065	2.492	2.085	1.793	1.579	1.421
1	4.750	3.644	2.908	2.401	2.042	1.780
2	10.00	6.858	5.063	3.960	3.238	2.738
4	26.65	14.91	9.868	7.272	5.742	4.748
8	83.57	33.54	19.81	13.94	10.74	8.752
16	287.8	73.16	39.80	27.27	20.75	16.75

Table 4-5 Mean first recurrence time of fullness, m_{cc}.

c \ ε	0	.2	.4	.6	.8	1.0
1/4	2.219	1.858	1.603	1.422	1.294	1.202
1/2	2.476	2.003	1.684	1.467	1.317	1.214
1	3.082	2.311	1.839	1.544	1.356	1.233
2	4.473	2.852	2.052	1.629	1.390	1.247
4	7.307	3.498	2.200	1.666	1.400	1.250
8	12.98	3.935	2.256	1.680	1.406	1.253
16	24.46	4.066	2.267	1.684	1.408	1.254

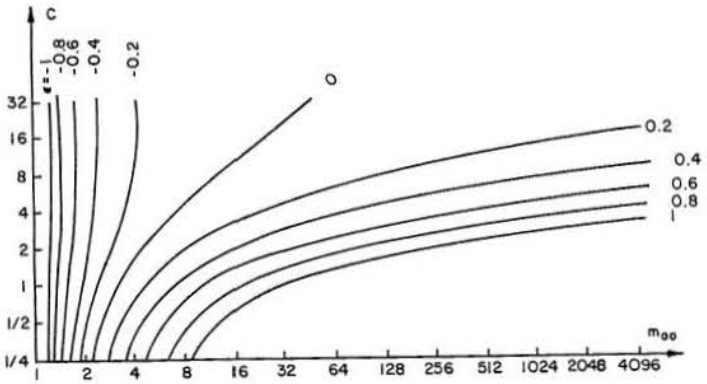

Fig. 4-2 Mean recurrence time of emptiness, m_{00}, of a standard reservoir of capacity c fed by an independent input of drift ε.

These tabled results are plotted in Figs. 4.2 and 4.3.

Independent discrete inputs

The discrete ±1 net input with non-zero drift is given by $P[X_t=1] = p$, $P[X_t=-1] = q = 1-p$ with $p \neq 1/2$. To compare the results of the normal and discrete net inputs, we must define the standardized reservoir for discrete net inputs with $\varepsilon \neq 0$.

Firstly, $\mu = p-q$, $\sigma = 2\sqrt{pq}$, so the drift, $\varepsilon = (p-q)/(2\sqrt{pq})$. Secondly, for a given k, the standard

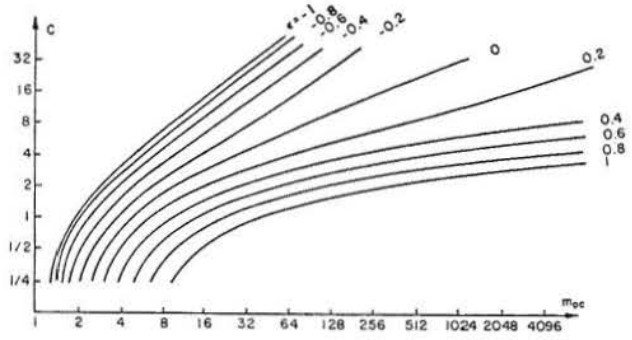

Fig. 4-3 Mean first passage time from full to empty, m_{0c}, of a standard reservoir of size c fed by an independent normal input with drift ε.

capacity $c = k/\sigma$ and vice versa. When $\varepsilon \neq 0$, we have $r = p/q \neq 1$. In terms of ε,

$$p = [\sqrt{(1+\varepsilon^2)} + \varepsilon]/[2\sqrt{(1+\varepsilon^2)}]$$

$$r = [\sqrt{(1+\varepsilon^2)} \quad \varepsilon]/[\sqrt{(1+\varepsilon^2)} - \varepsilon]$$

$$(p-q) = \varepsilon/\sqrt{(1+\varepsilon^2)} ,$$

so that we get, from Eq. (4-3)

$$\left.\begin{aligned} m_{00} &= (r^{k+2} - 1)/r - 1) \\ m_{0c} &= [m_{00} - (k+2)]/(p-q) \\ m_{cc} &= m_{00}/r^{k+1} \\ m_{c0} &= [k + 2 - m_{cc}]/(p-q) \end{aligned}\right\} \quad p \neq q$$

These expressions clearly interpolate for non-interger k, so that we can use them to compute first passage times for given values of c and ε, by which they are completely specified.

It may be more convenient to compute c from a given m_{00} and ε. In that case

$$\sigma = 1/\sqrt{(1+\varepsilon^2)}$$

$$k = \ell n[1 + m_{00}(r-1)]/\ell nr - 2.$$

Whence $c = k/\sigma$ is the standardized capacity. Figure 4-4 shows this interrelationship, where a family of curves in terms of m_{00} (the recurrence interval of failure), has been plotted on ε and c axes. The appearance of this graph is reminiscent of the storage-draft-frequency curves that are sometimes employed in storage problems. The similarity is not coincidental. If we recall that $\varepsilon = (1-\alpha)E[Z_t]/\sigma$, where α is the level of development, we get $\alpha = 1 - \varepsilon\sigma/E[Z_t]$. Thus when $\varepsilon = 0$, $\alpha = 1.00$, and if the coefficient of variation of the gross input is 1/2, say, $\alpha = 0.50$ when $\varepsilon = 1$.

The important point to note about Fig. 4-5 is that the standardized capacity and the drift are respectively the capacity and the mean net input scaled by the standard deviation of the input and not the mean gross input (or mean annual runoff) which is usually the case in practice. It is evident that more meaningful regionalizations are likely if first passage times are expressed in terms of the standardized capacity and drift.

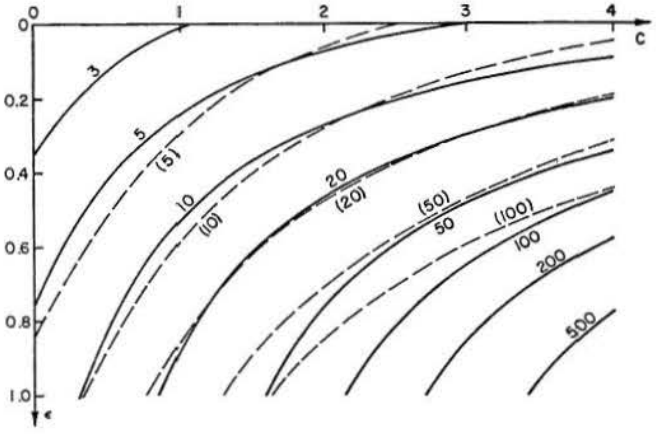

Fig. 4-4 The relationship between size c and drift ε of a standard reservoir fed by discrete ±1 input (full lines) and normal input (dashed lines) for given m_{00}.

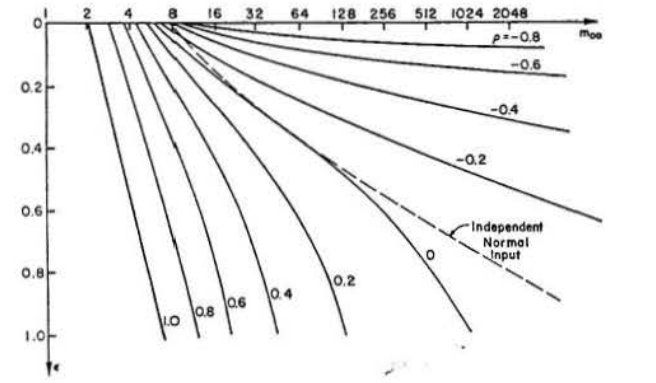

Fig. 4-5 The effect of correlation on the mean recurrence time of emptiness of a standard reservoir of size c = 4 fed by a discrete ±1 input with drift ε, with the corresponding curve for the independent normal input shown dotted for comparison.

Superimposed on Fig. 4-4 are curves from the section of normal net inputs, for various m_{00} values, shown dotted. Note that the discrete model requires a larger c for a given ε when m_{00} is 100. It is thus (compared to the normal) slightly conservative in this region, but the overall behavior is quite satisfactory.

Other discrete 3-state inputs

Of course, one should not be restricted to the ±1 process. If X_t is chosen to have a binomial (2,p) distribution, then the following relationships hold:

$$\left.\begin{aligned} \sigma &= \sqrt{2pq} \\ \varepsilon &= (p-q)/\sigma \end{aligned}\right\} \qquad \begin{aligned} p &= [1 + \varepsilon/\sqrt{(2+\varepsilon^2)}]/2. \\ q &= [1 - \varepsilon/\sqrt{(2+\varepsilon^2)}]/2. \end{aligned}$$

$$r = p^2/q^2 = \left[\frac{\sqrt{(2+\varepsilon^2)} + \varepsilon}{\sqrt{(2+\varepsilon^2)} - \varepsilon}\right]^2 .$$

For example, when $c = 4$ and $\varepsilon = 0.2$, we get $\sigma = 0.70014$, and $r = 1.75737$; yielding $m_{00} = 18.46$.

Comparing this with 19.51 for the ± 1 input, and 21.60 for the normal, there is not a large disparity. For $c = 2$ and $\varepsilon = 1$, the normal yields $m_{00} = 209.2$, the binomial 314.1 and the ± 1 input 84.9, so the expressions appear not to be so close when ε is large (or conversely when α is small).

Serially correlated discrete inputs with arbitrary drift

Since the approximation of the continuous reservoir by the discrete model is so satisfactory in the case of independent inputs, some confidence is felt that the same will occur when the inputs are no longer independent. For the input, the discrete analog of the lag-one autoregressive model is the lag-one Markov chain, specified by the transition probability matrix,

$$L = \rho I + (1-\rho)\,\pi 1! \qquad (4\text{-}6)$$

where ρ is the first serial correlation coefficient, I is the identity matrix and π is the equilibrium vector probability distribution of $\{X_t\}$. This input model was introduced by Pegram (1974) where it was applied to reservoir problems. Explicit expressions for y_0, the probability of failure to meet full demand, were given by Pegram (1978), however, as we have seen above, for meaningful comparison with continuous reservoirs, we need to compute v_0 (or its inverse m_{00}) rather than y_0.

A brief derivation of the expressions will be given, and we then explore the behavior of m_{00} as a function of c, ε, ρ. The joint equilibrium vector probability distribution of $[S_t, X_t]$ is $v = \{v_{ij}\}$ where $v_{ij} = \lim_{t\to\infty} P[S_t = i,\ X_t = j]$, $i = 0,1,2,\ldots,k+1$; $j = -1,0,+1$. Then the equilibrium storage distribution is given by the elements

$$v_i = \lim_{t\to\infty} P[S_t = i] = \sum_{j=-1}^{+1} v_{ij} \ .$$

It was shown by Pegram (1974) that in particular

$$v_0 = a/[a + b(1+\lambda+\lambda^2+\ldots+\lambda^{k-1}) + d\lambda^k] \qquad (4\text{-}7)$$

for a k-state reservoir.

Pegram (1978) derived expressions for the terms in Eq. (4-7) when the input has the particular form given by Eq. (4-6). In that case, $\pi = [q(1-p-q)\ p]'$ is the equilibrium vector distribution of X_t and it was found that, in terms of p, q and ρ,

$$a = 1/[p(1-\rho)]$$

$$b = (1+\rho)/(q+\rho p)$$

$$d = 1/[q(1-\rho)]$$

$$\lambda = (p+\rho q)/(\rho p+q) \ . \qquad (4\text{-}8)$$

There are two cases of interest: zero and non-zero drift. For zero drift, $p = q$ and the substitution of Eq. (4-8) into Eq. (4-7) yields, after simplification

$$m_{00} = m_{cc} = 1/v_0 = 2 + k(1-\rho) \qquad (4\text{-}9)$$

for $\varepsilon = 0$, which in turn yields Eq. (4-4) when $\rho = 0$. Note that this expression is quite independent of p or q and hence the variance of the input.

When $\varepsilon \neq 0$, we find that

$$m_{00} = (p\lambda^{k+1}/q - 1)(\rho p+q)/(p-q) \qquad (4\text{-}10)$$

which becomes $(r^{k+2}-1)/(r-1)$, as expected, when $\rho = 0$, since in that case, $\lambda = p/q = r$. We also have that $m_{cc} = m_{00}/(p\lambda^k/q)$ becomes m_{00}/r^{k+1} when $\rho = 0$.

We note that p and q can in general be chosen independently (subject to the constraint that all elements of π are non-negative). However, if we specifically fix $p+q = 1$, then we only have one parameter, p, describing the equilibrium marginal distribution of the discrete ± 1 input, while ρ fixes the serial correlation.

We then have that, as before $p = [\sqrt{(1+\varepsilon^2)} + \varepsilon]/[2\sqrt{(1+\varepsilon^2)}]$ and $k = 2c\sqrt{pq}$, so that m_{00} is in terms of c, ε and ρ only. For a fixed $c = 4$, we show, in Fig. 4-5, how m_{00} varies as a function of ε and ρ. Note that for $\rho = 1$, $m_{00} = 1/q$ as expected.

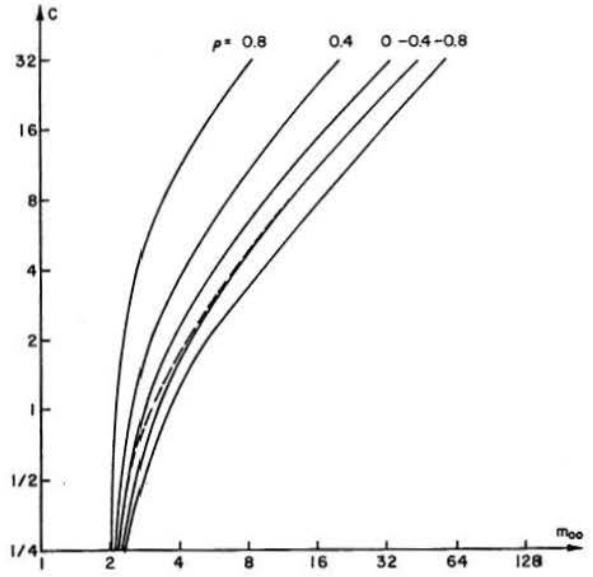

Fig. 4-6 Mean recurrence time of emptiness, m_{00}, of a standard reservoir of capacity c fed by a discrete input (± 1) with zero drift and first serial correlation coefficient ρ. The independent normal input shown dotted for comparison.

In Fig. 4-6, we show how m_{00} varies as a function of c and ρ for $\varepsilon = 0$. Plotted coaxially is the dashed curve for the independent normal input for comparison,

which is seen to hug the discrete input curve for $\rho = 0$ at low c, moving across to the curve for $\rho = -0.4$ at about $c = 8$. The discrete model is thus conservative in its estimate of m_{00} when compared to the normal, but undoubtedly indicates the general response of a reservoir fed by a serially correlated input.

4-1-6 Concluding Remarks

One point of view is that the range and deficit analyses cannot compete with the storage analysis in the realm of reservoir sizing and risk evaluation. The reasoning is that not only are they relatively cumbersome and arcane, but the greater richness of results available from storage analysis makes their application to reservoir problems obsolete. However, there are different viewpoints on the applicability of range and deficit in practice.

Reservoirs fed by two different net input types, distributed as the normal and the ± 1 discrete, coming from opposite ends of the spectrum, as it were, might be expected to behave quite differently. Not so. Provided that the state-description is carefully done in terms of a standardized reservoir, with the correct drift, then the correspondence is remarkably good; so good, in fact, that better correspondence between the results of continuous and discrete reservoirs can be expected for more realistic discrete input distributions.

The final optimistic note is that, because of the fair approximation of the continuous by the discrete reservoir, effects of correlation structure and perhaps skewness can be ascertained with relatively small amounts of exploratory computations. The major differences in viewpoints can result from the interpretation of what a "fair approximation" means.

A word on the application of the range may be useful at this end. If a standardized storage capacity V/σ, is assumed in planning process or is already fixed (decided upon, being constructed, or is built), then the range analysis may provide some useful information. If one looks at what is the probability that this capacity would be sufficient for an overyear regulation and a prescribed draft, for a given number n of years, the distribution of R_n for the given type of net inputs would provide the probability $P(R_n \leq V/\sigma)$. The difference $1 - P(R_n \leq V/\sigma)$, or its reciprocal, would give the risk of V/σ not being able to satisfy the draft in the next n years. In other words, this risk may give the number of n-year periods, out of a given number m of n-year periods, that the storage capacity V would not be able to regulate the inflows to a desired demand outflow. In this analysis, the initial condition of the reservoir storage is crucial. If the regulation starts with V/2 (half storage), the above probability information may be close to reality. However, to apply the above information, not only R_n, but also M_n and m_n, as the partial maximum and minimum sums, are also important, because the conditions $(V/2) \geq M_n$ and $(V/2) \geq |m_n|$ should also be satisfied. For a proper application of the range, one should develop its additional properties, namely the conditional range-storage distributions,

$$P(R_n = r_n \mid [(V_j/\sigma) \geq \hat{M}_n;\ (V-V_j)/\sigma] \geq |\hat{m}_n|) \quad ,$$

where V_j = the state of the reservoir, V = total capacity, $\hat{M}_n$ = the sample value of maximum partial sum, and $\hat{m}_n$ = the sample value of minimum partial sum.

The initial storage state ($0 \leq V_j \leq V$) must enter as a condition for any reasonable application and realistic probability statements related to any range and deficit analysis, and the storage design, with both $\hat{M}_n$ and $\hat{m}_n$ of the range being important statistics.

4-2 Modeling Hydrologic Time Series

The stochastic modeling of hydrologic time series is one of the basic tools for planning and operation of water resources systems in general and of water storage related systems in particular. Modeling is necessary in order to generate new samples that are statistically indistinguishable from the historical record and use them at the planning stage, for instance for designing an appropriate storage capacity of a reservoir, or at the operation level for instance for developing or testing reservoir operational rules. In modeling hydrological time series in general, there are several important aspects to consider such as the selection of the class of models, the model identification, estimation of parameters and testing the goodness of fit of the models selected, and finally the use of the model. It is not the intention to review here in detail all those aspects, but to point out some that are related to the range and deficit analysis and the Hurst phenomenon.

Hydrologists and water resources planners in general have been discussing for several years the best model to be used for planning and operating of water resources systems. Some have advocated the use of autoregressive models (Thomas and Fiering, 1962; Yevjevich, 1963; Beard, 1965, etc.). Others suggested the use of fractional Gaussian noise models (Mandelbrot and Wallis, 1968; Matalas and Wallis, 1971, etc.). Others advised the use of broken-line models (Mejia et al., 1972; Garcia et al., 1972, etc.). Others suggested the use of mixed autoregressive and moving average (ARMA) models (Carlson et al., 1970; O'Connel, 1971; Hipel et al., 1977, etc.), and more recently a new class of mixture models for instance (shifting-level models) has been proposed (Boes and Salas, 1978). Supposedly, all these models have been developed and proposed with the objective of reproducing some or most of the main statistical characteristics which are identifiable in observed hydrologic series but (and most importantly) which have a bearing on the design and/or operation of the water system under study.

Although generally it is not difficult to find and estimate the main statistical characteristics of proposed models by an analytical or a data generation technique (for instance Yevjevich, 1965; Fiering, 1967; Matalas and Wallis, 1971; O'Connel, 1971; Salas, 1972; Hipel and McLeod, 1978) unfortunately it is more difficult to detect or identify those which are the true statistical characteristics of sampled hydrologic series. This problem arises due to the inherent uncertainty present in hydrologic samples because of the prevalence of short records in hydrology. Precisely, difficulties of dealing with observed samples of hydrologic series, i.e., the identification of design or operational statistics from the sample, their understanding and interpretation, and their ultimate use in selecting an appropriate model, has led to controversy and a sort of "cold war" among hydrologists and engineers around the world.

These controversies have been mainly centered around what estimates from historical series should be reproduced in modeling. In most cases hydrologists have agreed on the necessity for reproducing statistics such as the mean, variance and sometimes skewness coefficient (the latter at least in the order of magnitude sense), as well as the first serial correlation coefficient, the main arguments have been with statistics that represent long-term persistence and extreme values. The first serial correlation coefficient, especially when used in connection with estimating the parameter of AR(1) model, has meant to represent the "short memory" or short-term persistence of the time series in question. Similarly, the "long memory" or long-term persistence of a time series has been represented by the Hurst K (or similar slopes), or by the corresponding rescaled range. On the other hand, the frequency and magnitude of high or low values (extreme events) has been respresented by the run-length and run-sum statistics. Since the persistence and run characteristics are related to the Hurst phenomenon, the main argument among hydrologists has been connected to the interpretation of the Hurst phenomenon per se, the statistics to be reproduced and by which models, and their impact on the design and operation of water resources systems. Here specifically, hydrologists and water resources planners and operators usually ask questions such as: Is it necessary to reproduce the Hurst phenomenon when modeling streamflow or other hydrologic series? Is there really a Hurst phenomenon or is it the result of a transient behavior of the rescaled range? If the Hurst phenomenon is the result of a transient behavior of the rescaled range, what class of models or more precisely what model should be selected as most appropriate?

If a planner believes or considers necessary to reproduce the Hurst phenomenon, what model should be chosen? Is the Hurst phenomenon important in designing reservoirs of say 50-100 years of economic design life? Is the Hurst phenomenon important for developing, checking or updating reservoir operational rules for say 50 years of economic horizon, especially when the environmental, technological, social, political, legal and economic conditions are changing very rapidly and many times in an unpredictable manner?

Undoubtedly the answers to all or most of these questions depend very much on the basic philosphy of modeling, statistical and design and operational practical experience, biases of the analyst (for instance some hydrologists do not see the necessity for models other than AR, while others do not believe in the appropriateness of models other than fractional Gaussian noise) and similar factors. Based on practically 30 years of experience (since 1951) of mathematical and experimental analysis, of stochastic modeling, of the Hurst phenomenon and of the design and operation of water resources systems in general, and of water storage reservoirs in particular (experience gained and shared by mathematicians, hydrologists, and water engineers in general), one can safely make conclusions on most of the above raised questions. For instance, it has been demonstrated that the Hurst phenomenon can be the result of transcience of the rescaled range (Salas et al., 1979b). Therefore, simple models such as autoregressive models in some cases and mixed autoregressive moving average models (ARMA) or shifting level models, in most cases, are capable of reproducing transient rescaled ranges of the order of those exhibited by hydrologic records (Salas, et al., 1977; Hipel and McLeod, 1978; Salas et al., 1979b). However, based on the typical lengths of hydrologic records, it is practically infeasible to discriminate statistically between short-term and long-term persistence (Wallis and O'Connell, 1973), which discrimination would be necessary for justifying the use of models outside the Brownian domain of attraction. Several hydrologists have been able to statistically reproduce historical K's (or rescaled ranges) with AR and ARMA models (for instance see Hipel and McLeod, 1978). This also resolves the question about the necessity to reproduce the Hurst phenomenon for design and operation of the water resources systems in general or of reservoirs in particular.

However, it is fair and necessary to say that in some cases the hydrologist or water resources planner or operator may still wish to test his design or operational schemes with other models. He might wish to test the sensitivity of his design or operational rule to the type of model, so in this case he may need fractional Gaussian noise or broken line models. In other cases he might wish to have a model to reproduce both the ρ_1 and the asymptotic h slope. This will not be feasible with AR or ARMA models, nor with fractional Gaussian noise models, then broken line models might be the answer.

Likely, many years will pass until the above controversies will be properly settled. The trend among some practitioners, to generate new samples by the selected models by reproducing identically nearly all the properties of the historic sample(s), runs against the basic theory of sampling statistics. Properties of newly generated samples should preserve the inferred statistics only within the limits and variations as prescribed by the sampling theory. Furthermore, two future types of investigations: (1) physical backgrounds of stochastic models of hydrologic time series; and (2) tests with a large number of historic hydrologic time series on the reliable (homogeneous, consistent) samples all over the world, will likely provide the physical and statistical backings for the simplest, sufficiently accurate, hydrologic stochastic models. Sensitivity analyses of how the various models and methods affect the results of reservoir sizing and operation would be the final criteria in selecting between the simple and the complex models.

In summary, then, from the point of view of water resources planning, the model choice is as always, a pragmatic one. As to the importance of range and deficit analyses for modeling hydrologic time series, suffice it to say that given the (of necessity subjective) choice of stochastic model, there is no substitute for efficiently estimating the parameters of the model; to infer from a derived statistic is to cloud the issue.

CHAPTER 5
POSTSCRIPT

Two titles might be appropriate for this paper: "Reservoirs, Rippl and Range" or "A Guided Tour through the Phantasmagoria of the Hurst Phenomenon."

The bulk of the paper is a catalog of results, approximate and exact, that are known about the various properties of range and deficit, particularly about their first two moments and their asymptotic distributions. Contributions for dependent and/or periodic inputs make the results more realistic from the point of view of hydrologic series and applications, especially when applied to reservoir system studies. Care has been taken to unify definitions, notations and results in this assembled body of knowledge.

Coming out of this stock-taking are some areas that are felt to be profitable for research. The behavior of D_n for correlated inputs can easily be found, based on results outlined in Section 2-9. Further, a whole new field is opened up on inference (Chapter 3). One has yet to discover accurate (as against approximate by simulation) methods of calculating or defining the distributions of ranges and deficits for intermediate n. The need exists for further investigation of the behavior of quasi-stationary models for hydrologic phenomena, for example, weather patterns in combination with storage of moisture and heat in the oceans and atmosphere suggest that conventional hydrologic time series models may be far too restrictive, specifically as their deployment implies stationarity. Time series modeling in hydrology will surely further evolve. This paper has collected some of the tools and ideas that may assist in devising meaningful models and testing them. The asymptotic distributions of range and deficit depend, as would be expected, only on the sum of the covariance function, γ^2, when the inputs belong to the Brownian domain of attraction. This is interesting in itself, but the implication is far-reaching.

There are problems for which currently favored statistical methods applied to time series are not of much use. Consider, for example, an ARMA(1,1) process with $\phi = 0.99$ and $\theta = 0.95258$. For such process, the variance of the white noise is equal to $0.93426\sigma_x^2$ and $\gamma^2 = 21\sigma_x^2$ (compared to $\gamma^2 = \sigma_x^2$ for a white noise process). Estimating the parameters of this ARMA process is fraught with difficulty, at least when the sample size is small, because the theoretical autocorrelation function is $\rho_k = 0.1\ (0.99)^{k-1}$ for $k = 1,2,\ldots$, and, since the confidence limits for the null hypothesis of $\rho_k = 0$ are proportional to $1/\sqrt{n}$, most samples are likely to be estimated to be white noise. Thus, the relatively large γ^2 will be missed. Statistics and the black-box approach only in stochastic hydrology are inadequate; what seems to be needed is a combination of statistics and a reliable description of the underlying physical mechanism, based on the physics of the phenomenon, for adequate model specification.

The evidence that is collected here should be sufficient to persuade those concerned with statistical and probabilistic applications that one must be incredibly careful in phrasing the problems and arguments. Hurst sowed good seed; researchers must now continue to nurture, cultivate, weed and ultimately harvest the crop.

REFERENCES

Anis, A. A., and E. H. Lloyd, 1953. "On the range of partial sums of a finite number of independent normal variates," Biometrika, 40, pp. 35-42.

Anis, A. A., and E. H. Lloyd, 1975. "Skew inputs and the Hurst effect," Journal of Hydrology, 26, pp. 39-53.

Anis, A. A., and E. H. Lloyd, 1976. "The expected value of the adjusted rescaled Hurst range of independent normal summands," Biometrika, Vol. 63, pp. 111-116, 1976.

Barnard, G. A., 1956. "Discussion on methods of using long-term storage in reservoirs," Proc. Inst. Civ. Engrs., 1, pp. 552-553.

Beard, L. R., 1965. "Use of interrelated records to simulate streamflows," Jour. of the Hydr. Div. ASCE, 1, HY5, pp. 13-22.

Billingsley, P., 1968. "Convergence of probability measures," John Wiley, New York.

Boes, D. C., and J. D. Salas-La Cruz, 1973. "On the expected range and expected adjusted range of partial sums of exchangeable random variables," Journal of Applied Probability, 10, pp. 671-677.

Boes, D. C., and J. D. Salas, 1978. "Nonstationarity of the mean and the Hurst phenomenon," Water Resour. Res., 14(1): pp. 135-143.

Brophy, J. A., 1970. "Statistical randomness and 1/f noise," Naval Res. Rev., October, pp. 8-21.

Carlson, R. F., A. J. A. McCormick, and P. G. Watts, 1970. "Application of linear models to four annual streamflow series," Water Resour. Res., 6, pp. 1070-1078.

Chow, V. T., 1951. "Discussion on long-term storage capacity of reservoirs," Transactions Amer. Soc. of Civ. Engrs., Vol. 116, pp. 800-802.

Feller, W., 1951. "The asymptotic distribution of the range of sums of independent random variables," Ann. Math. Stat., 22, pp. 427-432.

Feller, W., 1966. "An Introduction to probability theory and its applications," Vol. 2, John Wiley, New York.

Fiering, M. B., 1967. "Streamflow synthesis," Harvard University Press, Cambridge, Massachusetts.

Garcia, L. E., D. R. Dawdy, and J. M. Mejia, 1972. "Long memory monthly streamflow simulation by a broken line model," Water Resour. Res., 8(4): pp. 1100-1105.

Gomide, F. L. S., 1975. "Range and deficit analysis using Markov chains," Hydrology Paper No. 79, Colorado State University, Fort Collins.

Gomide, F. L. S., 1978. "Markovian inputs and the Hurst phenomenon," J. Hydrol., 37, pp. 23-45.

Gomide, F. L. S., 1979. "The asymptotic distribution of the maximum deficit of partial sums of independent random variables," Water Resour. Res. 15(2): pp. 495-497.

Hipel, K. W., A. I. McLeod, and W. C. Lennox, 1972. "Advances in Box-Jenkins modelling, part one, model construction," Water Resour. Res., 13(2): pp. 567-575.

Hipel, K. W., and A. I. McLeod, 1978. "Preservation of the rescaled adjusted range, 1, a reassessment of the Hurst phenomenon," Water Resour. Res., 14(3): pp. 491-516.

Hurst, H. E., 1951. "Long-term storage capacity of reservoirs," Trans. American Society of Civil Engineers, 116, pp. 770-808.

Hurst, H. E., 1956. "Methods of using long-term storage in reservoirs," Proc. Instn. Civ. Engrs., 1, pp. 519-543.

Hurst, H. E., 1957. "A suggested statistical model of some time series which occur in nature," Nature, Vol. 180, p. 494.

Hurst, H. E., R. P. Black, and Y. M. Simaika, 1965. "Long-term storage," Constable, London.

Hurst, H. E., R. P. Black, and Y. M. Simaika, 1966. "The Nile Basin, Vol. X, The Major Nile Projects," Ministry of Irrigation, Cairo.

Klemes, V., 1974. "The Hurst phenomenon - a puzzle?" Water Resour. Res., 10(4): pp. 675-688.

Klemes, V., 1975. "Reply to K. W. Potter," Water Resour. Res., 11(2): pp. 375-376.

Langbern, W. B., 1956. "Discussion on methods of using long-term storage in reservoirs," Proc. Inst. Civ. Engrs., Vol. 1, pp. 565-568.

Lanshey, L. M., 1951. "Discussion on long-term storage capacity of reservoirs," Transaction Amer. Soc. of Civ. Engrs., 116, pp. 803-804.

Lawrence, A. J., and N. T. Kottegoda, 1977. "Stochastic modelling of riverflow time series," Journal of the Royal Statistical Society, Series A, Vol. 140.

Lloyd, E. H., and A. A. Ariis, 1956. "Discussion on methods of using long-term storage in reservoirs," Proc. Inst. Civ. Engrs., Vol. 1, pp. 568-569.

Lloyd, E. H., 1963. "Reservoirs with serially correlated inflows," Technometrics, Vol. 5, pp. 85-93.

Mandelbrot, B. B., 1965. "Une classe de processus stochastiques homothetiques a soi: application a la loi climatologique de H. E. Hurst," Compt. Rend. Acad. Sci. Paris, 260, pp. 3274-3276.

Mandelbrot, B. B., and J. W. Van Ness, 1968. "Fractional Brownian motions, fractional noises and applications," Soc. Ind. Appl. Math. Rev., 10(4): pp. 422-437.

Mandelbrot, B. B., and J. R. Wallis, 1968. "Noah, Joseph and operational hydrology," Water Resour. Res., 4(5): pp. 909-918.

Mandelbrot, B. B., and J. R. Wallis, 1969a. "Computer experiments with fractional Gaussian noises, parts 1, 2 and 3," Water Resour. Res., 5(1): pp. 228-267.

REFERENCES (continued)

Mandelbrot, B. B., and J. R. Wallis, 1969b. "Some long run properties of geophysical records," Water Resour. Res., 5(2): pp. 321-340.

Mandelbrot, B. B., and J. R. Wallis, 1969c. "Robustness of the rescaled range R/S in the measurement of non-cyclic long-run statistical dependence," Water Resour. Res., 5(5): pp. 967-988.

Matalas, N. C., and C. S. Huzzen, 1967. "A property of the range of partial sums," Proceedings of the International Hydrology Symposium, Colorado State University, Fort Collins, Vol. 1, pp. 252-257.

Matalas, N. C., and J. R. Wallis, 1971. "Statistical properties of multivariate fractional moise processes," Water Resour. Res., 7(6): pp. 1460-1468.

Mejia, J. M., I. Rodriguez-Iturbe, and D. R. Dawdy, 1972. "Streamflow simulation, 2, the broken line process as a potential model for hydrologic simulation," Water Resour. Res., 8(4): pp. 931-941.

Milleret, H., 1951. "Discussion on long-term storage capacity of reservoirs," Transaction Amer. Soc. of Civ. Engrs., Vol. 116, p. 802.

Moran, P. A. P., 1959. "The theory of storage," Methuen and Co. Ltd, London.

Moran, P. A. P., 1964. "On the range of cumulative sums," Ann. Inst. Stat. Math., 16, pp. 109-112.

Moran, P. A. P., 1968. "An introduction to probability theory," Clarendon, Oxford.

Mutreja, K. N., 1976. "Reservoir capacity for periodic-stochastic input and periodic output," Hydrology Paper 86, Colorado State University, Fort Collins.

Mutreja, K. N., and V. M. Yevjevich, 1977. "Asymptotic distribution of maximum deficit for linearly dependent inflows in full flow regulation. Journal of Hydrology, Vol. 35, pp. 299-309.

O'Connell, P. E., 1971. "A simple stochastic modeling of Hurst's law," Int. Symp. on Mathematical models in hydrology, Warsaw, pp. 327-358.

O'Connell, P. E., 1974. "Stochastic modelling of long-term persistence in streamflow sequences," Ph.D. thesis, Imperial College, London, England.

Odoom, S., and E. H. Lloyd, 1965. "A note on the equilibrium distribution of levels in a semi-infinite reservoir subject to Markovian inputs and unit withdrawals," Jnl. App. Prob., Vol. 2, pp. 215-222.

Owen, P. B., and G. P. Steck, 1962. "Moments of order statistics from the equicorrelated multivariate normal distribution," Ann. Math. Stat., 33, pp. 1286-1291.

Pegram, G. G. S., 1974. "Factors affecting draft from a Lloyd reservoir," Water Resour. Res., Vol. 10, pp. 63-66.

Pegram, G. G. S., 1978. "Some simple expressions for the probability of failure of a finite reservoir with Markovian input," Geophysical Research letter, in press.

Potter, K. W., 1975. "Comment on the Hurst phenomenon: a puzzle?" by V. Klemes, Water Resour. Res., 11(2): pp. 373-374.

Potter, K. W., 1976a. "Evidence for nonstationarity as a physical explanation of the Hurst phenomenon," Water Resour. Res., 12(5): pp. 1047-1052.

Potter, K. W., 1976b. "A stochastic model of the Hurst phenomenon: non-stationarity in hydrologic processes, unpublished," Ph.D. dissertation, Johns Hopkins University, Baltimore, MD.

Potter, K. W., 1979. "Annual precipitation in the northeast United States: long memory, short memory, or no memory," Water Resour. Res., 15(2): pp. 340-346.

Ripple, W., 1883. "The capacity of storage reservoirs for water supply," Proc. Inst. Civ. Engrs., Vol. 71, pp. 270-278.

Salas-La Cruz, J. D., 1972. "Range analysis for storage problems of periodic stochastic processes, Hydrology Paper 57, Colorado State University, Fort Collins.

Salas, J. D., and V. Yevjevich, 1972. "Stochastic structure of water use time series," Hydrology Paper 52, Colorado State University, Fort Collins.

Salas, J. D., and D. C. Boes, 1974. "Expected range and adjusted range of hydrologic sequences," Water Resour. Res., 10(3): pp. 457-463.

Salas, J. D., 1974. "Range of cumulative sums, I. Exact and approximate expected values," Jour. of Hydr., 2, 3, pp. 39-66.

Salas, J. D., D. C. Boes, V. Yevjevich, and G. G. S. Pegram, 1977. "On the Hurst phenomenon," paper presented at the Third International Hydrology Symposium, Colorado State University, Fort Collins.

Salas, J. D., B. M. Troutman, and D. C. Boes, 1979. "Comment on the small sample estimation of h, the small sample expectation of population and adjusted ranges and the small sample expectation of rescaled population and rescaled adjusted ranges by 2. Sen. Water Resour. Res., 15(3): pp. 967-970.

Salas, J. D., D. C. Boes, V. Yevjevich, and G. G. S. Pegram, 1979. "Hurst phenomenon as a pre-asymptotic behavior," Jour. of Hydr., 44, pp. 1-15.

Scheidegger, A. E., 1970. "Stochastic models in hydrology," Water Resour. Res., 6(3): pp. 750-755.

Siddiqui, M. M., 1976. "The asymptotic distribution of the range and other functions of partial sums of stationary processes," Water Resour. Res., 12(6): pp. 1271-1275.

Solari, M. E., and A. A. Anis, 1957. "The mean and variance of the maximum of the adjusted partial sums of a finite number of independent normal variates," Ann. Math. Stat., 28, pp. 706-716.

Spitzer, F., 1956. "A combinational lemma and its application to probability theory," Amer. Math. Soc. Trans., Vol. 82, pp. 323-339.

Sutabutra, P., 1967. "Reservoir storage capacity required when water inflow was a periodic and a stochastic component," unpublished Ph.D. dissertation, Colorado State University, Fort Collins.

Thomas, A. R., 1956. "Discussion on methods of using long-term storage in reservoirs," Proc. Inst. Civ. Engrs., 1, pp. 563-565.

Thomas, H. A., and M. B. Fiering, 1962. "Mathematical synthesis of stream flow sequences for the analysis of river basins by simulation," in Design of Water Resource Systems, Harvard University Press, Cambridge, Massachusetts.

Troutman, B. M., 1974. "Expected range of partial sums," unpublished M.S. thesis, Colorado State University, Fort Collins.

Troutman, B. M., 1976. "Limiting distributions in storage theory," Ph.D. dissertation, Colorado State University, Fort Collins.

Troutman, B. M., 1978. "Reservoir storage with dependent periodic net inputs," Water Resour. Res., 14(3): pp. 395-401.

Wallis, J. R., and N. C. Matalas, 1970. "Small sample properties of H- and K-estimators of the Hurst coefficient h," Water Resour. Res., 6(6): pp. 1583-1594.

Wallis, J. R., and P. E. O'Connell, 1973. "Future reservoir yield - How reliable are historical hydrological records?" Hydrol. Sci. Bull., Vol. 18, pp. 347-365.

Wallis, J. R., 1977. "Climate, climatic change and water supply," EOS, Trans. A.G.J., 58(11): pp. 1012-1024.

Yevjevich, V., 1963. "Fluctuations of wet and dry years. Part 1. Research data assembly and mathematical models," Hydrology Paper 1, Colorado State University, Fort Collins.

Yevjevich, V. M., 1965. "The application of surplus, deficit and range in hydrology," Hydrology Paper 10, Colorado State University, Fort Collins.

Yevjevich, V. M., 1967. "Mean range and linearly dependent normal variables with applications to storage problems," Water Resour. Res., Vol. 3, pp. 633-671.

Yevjevich, V. M., 1972. "Stochastic Processes in hydrology," Water Resources Publication, Colorado State University, Fort Collins.

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