DISSERTATION

STOCHASTIC MODELING OF SEASONAL STREAMFLOW

Submitted by

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY ANTONIO SERGIO FERREIRA MENDONÇA ENTITLED STOCHASTIC MODELING OF SEASONAL STREAMFLOW BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

STOCHASTIC MODELING OF SEASONAL STREAMFLOW

This research examines topics on seasonal (monthly, bimonthly, etc.) hydrologic time-series modeling.

A family of periodic models was derived by allowing parameters for a particular Multiplicative Autoregressive Integrated Moving Average model(Multiplicative ARIMA) to vary from season to season. The derived model presents parameters relating data for seasons in the same year and parameters relating data for the same season for consecutive years. PARMA models are particular cases of the proposed model, here called Multiplicative Periodic Autoregressive Moving Average(Multiplicative PARMA). Least-squares estimation based on the Powell algorithm for nonlinear optimization was developed for determining the model parameters.

Properties such as seasonal variances and autocorrelations were derived analitically for particular cases of the general model. Analysis of sensitivity of the annual autocorrelograms to the parameters of the model showed that the yearly autoregressive parameters are the most important for the reproduction of high annual autocorrelations.

Tests of model were made through data generation. The model was applied to four-and six-season series for river discharge presenting distinct characteristics of variabilty and dependence. Tests for goodness-of-fit and selection criteria of models for seasonal series were also discussed.

Results from data generation indicate that the estimation procedure is able to estimate parameters for the Multiplicative PARMA models and can also be used for refinement of estimations made by method-of-moments for other models. Application to discharge data from St. Lawrence, Niger, Elkhorn and Yellowstone rivers showed that the proposed modeling technique is able to preserve long term dependence better than models currently used in practical hydrology. Direct consequence of this improvement is better reproduction of floods and droughts and more accuracy in the design and operation of water resource structures.

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To my wife Vera Lucia, my parents Clóvis and Lourdes
and my parents-in-law Francisco and Lourdes

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

1.1 General Remarks

A fundamental problem both in design and in operation of water resource systems is the appropriate consideration of the variability of the natural inflows. Stochastic models for these inflows are generally utilized for generation of equi-probable future sequences of inflows that preserve basic statistical characteristics, of the historic series available, that could influence the performance of the systems in analysis. Generation of such synthetic streamflow sequences is required because observed series of streamflows commonly do not include the most extreme cases of floods and droughts and cannot provide good estimates of risks involved in the operation of the systems.

Generated hydrological sequences can be used in conjunction with simulation models to test and evaluate various proposed strategies for water resource systems or to optimize the sizing and operation of the systems.

Seasonal models are particularly important in water supply studies, reservoir operation and drought planning. Hydrologic phenomena are periodic and stochastic in nature. Hydrologic time series with intervals that are fractions of the year present periodicities in statistical characteristics caused by the annual cycle of revolution of the earth around the sun. The environment responds to the solar energy, modifying some characteristics of the cyclic input to the hydrologic system, without eliminating periodicity but adding randomness to the input.

Autoregressive Moving Average models (ARMA) are very useful in modeling hydrologic time series that present second-order stationarity (stationarity in the mean and in the covariance).Common procedures for removing periodicity from seasonal series are standardization (by subtracting seasonal means and dividing by seasonal standard deviations) and filtering (by utilization of linear time invariant filters). Multiplicative Autoregressive Integrated Moving Average models (ARIMA) may also be applied in modeling seasonal time series. However, many hydrologic time series cannot be stationarized by standardization and filtering because of their seasonal correlation structure, and the multiplicative ARIMA model does not account for periodicities in variance and in correlation structure. Periodic Autoregressive Moving Average models (PARMA) allow for periodicity in statistical characteristics presented by seasonal time series. PARMA modeling is in an early stages of development and PARMA models are beginning to be recognized as important tools for modeling time series.

Another class of models utilized for seasonal hydrologic series modeling includes disaggregation models. Disaggregation modeling is composed of the temporal aggregation of seasonal data and the modeling of the aggregated series (annual, for example) without taking advantage of the knowledge of the structure of the original periodic time series. Then the generated annual series are distributed into seasonal intervals.

The loss of information seen during the aggregation of seasonal series made researchers develop another procedure called Aggregation Modeling.

The present research deals specifically with the development of a family of models presenting periodic parameters denominated here as Multiplicative PARMA models.

1.2 Research Objectives

1.2.1 General Objectives

Seasonal streamflows result from a very complex interaction of many variables and largely depend on the watershed characteristics. These general characteristics and, consequently, the runoff and its within-the-year distribution are unique. Many models were developed for generating synthetic hydrologic sequences that could preserve particular historic statistical properties. The complexity of seasonal stochastic models is related to their purpose. The modeler must decide which statistics of the observed streamflow series should be preserved, based on the future uses of the model. The uniqueness of basin characteristics and objectives of stochastic modeling suggests that groups of models with different degrees of sophistication should be available for application by hydrologists.

One of the most desired properties for seasonal stochastic models is the ability for dual preservation of seasonal and annual dependence. However, this dual preservation is a very difficult task to perform for many streamflow series. This research aims at the development of a family of models which could preserve seasonal properties and provide better reproduction of annual autocorrelations than those resulting from models currently utilized. This development includes formulation, study of properties, search for techniques for estimation of parameters, development of method for data generation of and application to synthetic and historic data.

1.2.2 Specific Objectives

The specific objectives of this study are:

(a) Formulate a new class of stochastic periodic models, from which particular cases could reproduce seasonal statistical characteristics and long term dependence from historic streamflow series.

(b) Develop estimation and generation techniques for the formulated family of models.

(c) Find the model fitting the series obtained by adding consecutive seasons from a low order PARMA model.

(d) Analyze the most common methods for analysis of goodness of fitting and comparison of seasonal models.

(e) Apply models from the proposed family of models to actual river data and compare with currently used models.

CHAPTER II

LITERATURE REVIEW

2.1 General Remarks

Two basic approaches have been taken by hydrologists in generation of monthly or seasonal streamflow series. The first approach is to generate annual flows with an appropriate model and then divide those flows among the modeled periods within each year, using disaggregation procedures developed for this purpose. The second basic approach for modeling seasonal or monthly flows is to model the seasonal data directly. In dealing with periodicities present in these flows, two sub-approaches are utilized. One is to try to remove the periodicity by using a time invariant filter or a standardization technique. The second approach is the utilization of models presenting periodic parameters for fitting the periodic sequences.

Aggregation Modeling is a technique for modeling aggregated series(annual, for example) by utilizing the characteristics of the seasonal historic data and by deriving relationships between models and parameters for this seasonal data and models and parameters for the aggregated data.

2.2 Models for Monthly and Seasonal Flows

2.2.1 Removal of Periodicities

Hydrologic time series of time scales of less than a year usually present strong seasonal nonstationarity.Time series are called periodic or seasonal when their statistical properties change periodically during the year. Stationary models, like the regular Autoregressive Moving Average models (ARMA) have been widely accepted for modeling time sequences that present second-order stationarity.

One way to remove seasonality is by standardization of the series by subtracting the seasonal mean and dividing by the seasonal standard deviation, transforming the seasonal data into a zero-mean, unit-variance series. This cyclic standardization may be represented by the following expression:

$$y_{v,\tau} = \frac{x_{v,\tau} - \mu_\tau}{\sigma_\tau} \tag{2.1}$$

where μ_τ and σ_τ are the mean and standard deviation for the season τ.

Another technique for removing periodicity is the utilization of a simple autoregressive integrated process. In this procedure seasonal differencing is utilized for transforming the original series. For example, if only first-order differencing is utilized, the process may be described by :

$$U_t = X_t - X_{t-w} \tag{2.2}$$

where X represents the original series and w is the time lag of differentiation. This operation could be repeated several times until a stationary sequence is achieved. If the series becomes deseasonalized after D operations and the tranformed series can be fitted by a stationary ARMA (P,Q) model, the general model for the initial series is called ARIMA $(P,D,Q)_w$.

Multiplicative Autoregressive Integrated Moving Average models, ARIMA$(p,d,q)\mathrm{x}(P,D,Q)_w$, consist of a seasonal ARMA(P,Q) fitted to the D-th seasonal difference of the data, integrated with an ARMA(p,q) model fitted to the d-th difference of the residuals from the former model. The application of the latter model assumes that these residuals achieved stationarity. The multiplicative model may be represented by:

$$\Phi_P(B^w).\phi_p(B)\ \nabla^d\ \nabla_w{}^D\ X_t = \underline{\overline{\Theta}}_Q(B^w).\theta_q(B)\ \epsilon_t \qquad (2.3)$$

where B is a backward shifting operator such that,

$$B^w\ X_t = X_{t-w} \qquad (2.4)$$

and ∇ is a difference operator,

$$\nabla_w{}^D\ X_t = (1-B^w)\ X_t \qquad (2.5)$$

Φ_P, ϕ_p, $\underline{\overline{\Theta}}_Q$ and θ_q are polynomials :

$$\Phi_P(B^w) = 1 - \Phi_1 B^w - \Phi_2 B^{2w} -\ldots- \Phi_P B^{Pw} \qquad (2.6)$$

$$\phi_p(B) = 1 - \phi_1 B - \phi_2 B^2 -\ldots- \phi_p B^p \qquad (2.7)$$

$$\underline{\overline{\Theta}}_Q(B^w) = 1 - \underline{\overline{\Theta}}_1 B^w - \underline{\overline{\Theta}}_2 B^{2w} -\ldots- \underline{\overline{\Theta}}_Q B^{Qw} \qquad (2.8)$$

$$\theta_q(B) = 1 - \theta_1 B - \theta_2 B^2 -\ldots- \theta_q B^q \qquad (2.9)$$

Many hydrologic time series cannot be stationarized because of the periodicity present in their correlation structure. For these series, models with periodic parameters are alternatives.

2.2.2 - Periodic Parameters

Thomas and Fiering (1962) suggested an AR(1) model with periodic coefficients that may be used to fit a time series presenting seasonal lag-one correlations. The model for monthly data is composed of twelve linear regression lines with different parameters for each month :

$$(X_{v,\tau}-\mu_\tau) = r_\tau . \frac{\sigma_\tau}{\sigma_{\tau-1}} . (X_{v,\tau-1}-\mu_{\tau-1}) + \sigma_\tau . (1-r_\tau{}^2)^{1/2} . \epsilon_{v,\tau} \tag{2.10}$$

where $X_{v,\tau}$ is the observed streamflow at year v and season τ, r_τ is the lag 1 correlation coefficient between seasons τ and τ-1. μ_τ is the mean and σ_τ is the standard deviation for season τ.

$\epsilon_{v,\tau}$ is an independent normal random variable with mean 0 and variance 1.

Yevjevich (1970) suggested the use of higher order AR(p) models with periodic parameters for modeling seasonal hydrologic time series. Salas and Yevjevich (1972) derived moment estimates of periodic parameters, and Salas (1972) derived Yule-Walker equations for these models.

Delleur et al. (1976) and Tao and Delleur (1976) developed approximated equations for estimation of parameters for the MA(1), MA(2) and ARMA(1,1) periodic models, patterned on Yule-Walker equations for ARMA(p,q) with constant parameters.

Salas et al. (1982) derived Yule - Walker equations for periodic ARMA(p,q) models from which exact moment estimates of parameters can be obtained. They also show that for the case of ARMA (p,1) models the periodic autoregressive parameters can be calculated by solving a system of linear equations and that the periodic moving average parameters satisfy a system of equations that can be solved iteratively.

Vecchia (1983) developed approximation to the exact likelihood function and algorithm for computing approximate maximum likelihood estimates for PARMA(1,1) models.

Thompstone(1983) formulated a class of models grouping seasons that present similar autoregressive characteristics in order to consider a single AR model for those seasons.

Deustch and Ramos(1986) described a Space-time ARIMA model that is reduced to a Multiplicative ARIMA if only one site is considered.

2.3 Disaggregation and Aggregation Modeling

Disaggregation modeling is a technique by which time series of lower-time scale are derived from time series of higher-time scale already generated. For example, an annual model could be fitted to an annual streamflow series and utilized for generating other annual sequences.

Monthly and seasonal series could be derived from these series by using a disaggregation scheme.

Harms and Campbell (1967) proposed a two-tier model to preserve both seasonal and annual flows. The monthly flows generated by a Thomas-Fiering model are adjusted against the annual flows generated by an annual model. The adjustment procedure preserves annual parameters at the expense of the monthly parameters. The model may be expressed as :

$$X'_{v,\tau} = \frac{Y_v}{\sum_{i=1}^{12} X_{v,\tau}} \cdot X_{v,\tau} \qquad (2.11)$$

where Y_v is the generated annual flow for year v. This model is considered the first disaggregation method.

Valencia and Schaake(1972) developed a model that provided a basis for almost all subsequent disaggregation techniques. The model has the form :

$$Y = A\ X + B\ \underline{\epsilon} \qquad (2.12)$$

For disaggregation of annual flows into monthly flows Y is a column matrix containing the monthly flows and X is a column matrix presenting the annual flow volume. A and B are matrices of parameters, and $\underline{\epsilon}$ is the stochastic term.

Mejia and Roussselle (1976) proposed an extension to the previous model aiming at the preservation of seasonal covariances between seasons of consecutive years.

Lane (1979) developed a condensed model setting to zero parameters of the model not considered important.

Santos and Salas (1983) proposed a step disaggregation model for utilization in operational hydrology that saves computer storage and parameters.

Stedinger et al. (1985) developed a condensed version of the Valencia-Schaake disaggregation model.

Contrary to the disaggregation procedure, during the aggregation modeling the seasonal data is first modeled and the model and parameters are utilized for deriving the model and parameters for the annual series.

Vecchia(1983) concluded that a low-order PARMA process at seasonal level is consistent with an ARMA(1,1) process at annual level and derived the relationship between annual and seasonal parameters.

Vecchia et al.(1983) demonstrated that gain in efficiency of estimation occurs at annual level when seasonal data and their model are utilized instead of annual data and their model.

Rao et al.(1985) investigated the relationship between parameters of original and aggregated data for cases in which the AR model is valid for the aggregated process, and an AR model or a periodic model(terms in sines and cosines) is valid for the seasonal process. For such cases, they concluded that the parameter estimates for a model for the aggregated sequence can be obtained by using the parameter estimates for the original series and that a valid model and its parameter estimates for the original sequence can provide more accurate forecasts for the aggregated series.

Aggregation and disaggregation modeling aim at the preservation of both seasonal (monthly, for example) and aggregated (annual, for example) statistical characteristics. In disaggregation modeling, the aggregated time series is first modeled and then distributed to the different seasons. In practice, however, the aggregated time series are obtained by adding up the seasonal series (see Figure 2.1 on the next page).

Researchers also verified that important loss of efficiency of estimation of parameters for the annual series may be caused by the aggregation (Vecchia et al.(1983), Mendonça(1985) and Rao et al.(1985)). This justifies further development of new models and techniques, dealing primarily with seasonal time series, able to preserve both seasonal and annual statistical dependence.

Preservation of multi-lag correlations among the various seasons and years is very important for generation of periodic series because if significant correlations are not taken into account, serious errors could happen in analysis. Generated series would tend to present droughts and floods less severe than those presented by the historic sample (Bras and Rodriguez-Iturbe,1984).

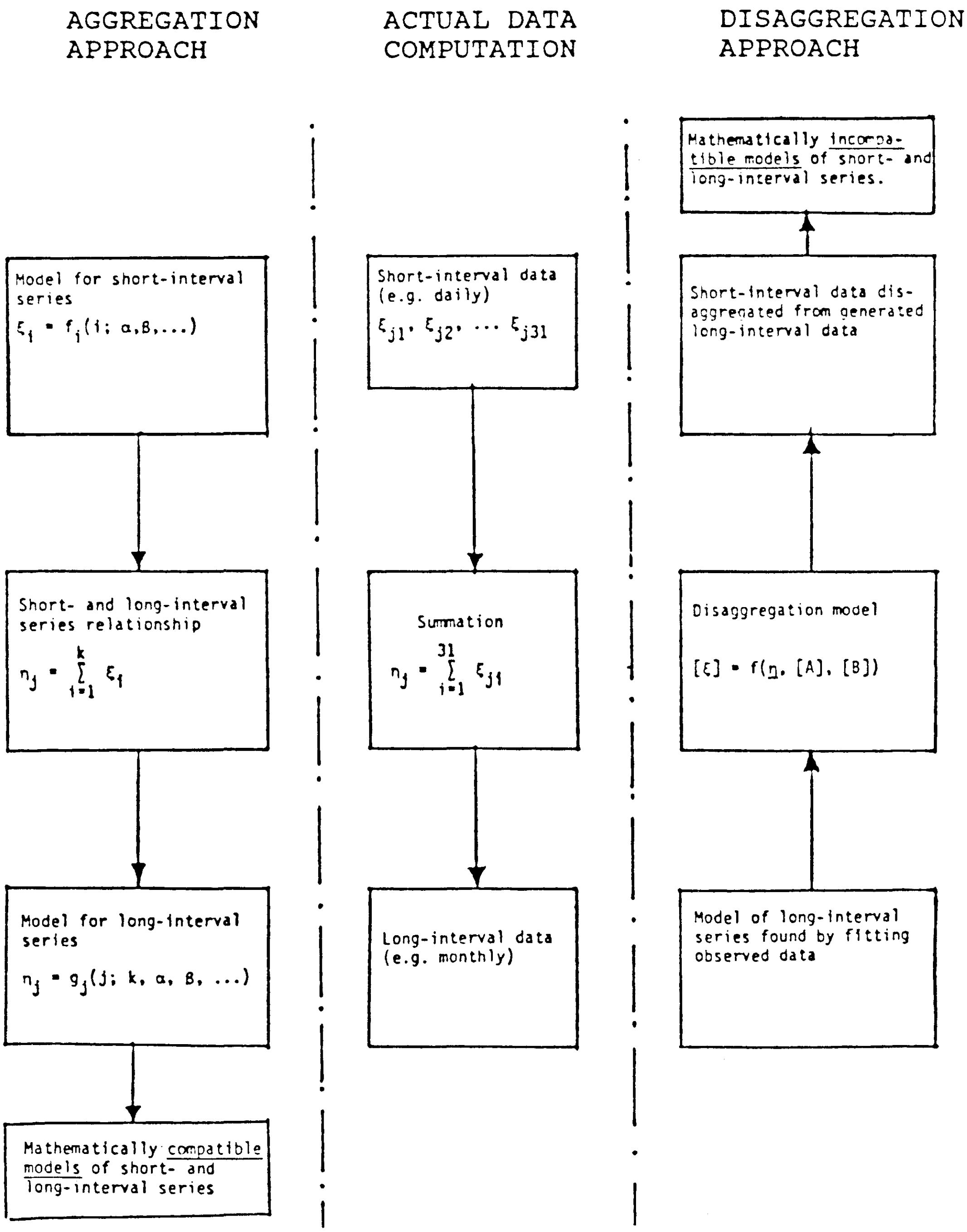

Figure 2.1 - Comparative diagram for aggregation and disaggregation modeling

CHAPTER III

MULTIPLICATIVE PARMA MODEL

3.1 General Remarks

The major concern of researchers who first developed disaggregation models was the preservation of autocorrelation characteristics at both seasonal(monthly, for example) and aggregated(annual, for example) levels, for hydrologic series. These models follow the concept by which annual series should be first synthesized and then disaggregated into seasonal sequences through matricial algebra. However, the inverse way is followed in practical hydrology. The higher-time scale data are derived from those of lower-time scale and consequently aggregated data can not contain more information than that contained in seasonal data. Furthermore, results from Vecchia *et al.*(1983), Mendonça(1985) and Rao *et al.*(1985) show that significant gain in parameter estimation efficiency at aggregated level may occur when seasonal series and their models are utilized rather than aggregated series and their models.In addition, Rao et al. (1985) shows that the aggregated data can be more accurately predicted by using a valid model of the original data than by using a valid model of the annual data.

These conclusions suggest that the ideal model for seasonal series would be a single periodic model that could reproduce automatically both seasonal and annual characteristics.

ARIMA and Multiplicative ARIMA models do not reproduce periodicities in seasonal variance and in seasonal autocorrelation. Low-order PARMA models may preserve seasonal and annual characteristics for series presenting low annual dependence but generally do not reproduce annual autocorrelation for highly dependent series.

This Chapter presents the development and analysis of a new class of models, called here the Multiplicative PARMA model, that has as its objective, besides preservation of seasonal characteristics the improvement of the preservation of annual dependence, over that obtained by the PARMA models for highly correlated series.

The term "Multiplicative PARMA" comes from the fact that members of this new class present periodic parameters relating consecutive seasons in the same year,as the PARMA models do, and periodic parameters relating seasons for consecutive years, as a generalization of the Multiplicative Box-Jenkins models that present constant parameters relating the same seasons.

3.2 Justifification

Box and Jenkins(1976) argue that for monthly data there are two time intervals of greater importance that are the month and the year. Specifically, it is expected that two main relationships occur, one between data for successive months of a year and the other between data in the same month of consecutive years. This idea originated the ARIMA and Multiplicative ARIMA models. However, these models do not allow for different relationships for different months, and most of the hydrologic seasonal data present strong periodicity in statistical characteristics, such as mean, variance and autocorrelation. PARMA models allow for periodicity in parameters but do not emphasize the correlation between data for the same month in consecutive years.

The above analysis indicates that an important step in seasonal hydrologic time-series modeling would be the development of a class of models with periodic parameters that could simultaneously relate data of consecutive seasons for the same year and data of consecutive years for the same season.

3.3 Formulation

The model was derived from the multiplicative ARIMA$(p,d,q)x(P,D,Q)_w$ model defined in Chapter II. Clarke (1973) stated that in practice models with $d=D=0$ would likely be the rule in many hydrological applications, and since it is intended to assume different relationships for each season, the differencing parameters in the expression (2.3) are assumed null, resulting in :

$$\Phi_P(B^W)\ \phi_p(B)\ Y_t = \underline{\overline{\theta}}_Q(B^W)\ \theta_q(B)\ \epsilon_t \tag{3.1}$$

Making this expression more explicit :

$$\begin{aligned}&[1-\phi_1 B-\phi_2 B^2-\ldots-\phi_p B^p].[1-\Phi_1 B^W-\Phi_2 B^{2W}-\ldots-\Phi_P B^{PW}].Y_t = \\ &[1-\theta_1 B-\theta_2 B^2-\ldots-\theta_q B^q].[1-\underline{\overline{\theta}}_1 B^W-\underline{\overline{\theta}}_2 B^{2W}-\ldots-\underline{\overline{\theta}}_q B^{QW}].\epsilon_t\end{aligned} \tag{3.2}$$

This expression considers the same parameters valid for all seasons. However, many hydrological series present strong variability in statistical characteristics from season to season. These series could be better fitted by allowing parameters to vary:

$$\begin{aligned}&[1-\phi_{1,\tau} B-\phi_{2,\tau} B^2-\ldots-\phi_{p,\tau} B^p].[1-\Phi_{1,\tau} B^W-\Phi_{2,\tau} B^{2W}-\ldots- \\ &\Phi_{P,\tau} B^{PW}].\ Y_{v,\tau} = \\ &[1-\theta_{1,\tau} B-\theta_{2,\tau} B^2-\ldots-\theta_{q,\tau} B^q].[1-\underline{\overline{\theta}}_{1,\tau} B^W-\underline{\overline{\theta}}_{2,\tau} B^{2W}-\ldots- \\ &\underline{\overline{\theta}}_{Q,\tau} B^{QW}].\ \epsilon_{v,\tau}\end{aligned} \tag{3.3}$$

where

$$Y_{v,\tau} = Z_{v,\tau} - \mu_\tau .$$

$Z_{v,\tau}$ is the original series and μ_τ represents the seasonal mean of $Z_{v,\tau}$.

$$B(Y_{v,\tau}) = Y_{v,\tau-1} ,$$

$$B(\epsilon_{v,\tau}) = \epsilon_{v,\tau-1} ,$$

$$B^{w}(Y_{v,\tau}) = Y_{v-1,\tau} ,$$

$$B^{w}(\epsilon_{v,\tau}) = \epsilon_{v-1,\tau} ,$$

$$B^{p}(Y_{v,\tau}) = Y_{v,\tau-p} ,$$

$$B^{q}(\epsilon_{v,\tau}) = \epsilon_{v,\tau-q} ,$$

$$B^{Pw}(Y_{v,\tau}) = Y_{v-P,\tau} , \quad \text{and}$$

$$B^{Qw}(\epsilon_{v,\tau}) = \epsilon_{v-Q,\tau}$$

Expression (3.3) represents the general Multiplicative PARMA $(p,q)x(P,Q)_w$ if $\epsilon_{v,\tau}$ are residuals with expected value zero, and w is the number of seasons.

3.4 Properties

3.4.1 Definition

The Multiplicative Periodic Autoregressive Moving Average, (PARMA), model of order $(p,q)x(P,Q)_w$ can be written as follows :

$$\begin{aligned}
Y_{v,\tau} = {} & \phi_{1,\tau} Y_{v,\tau-1} + \phi_{2,\tau} Y_{v,\tau-2} + \cdots + \phi_{p,\tau} Y_{v,\tau-p} \\
& + \Phi_{1,\tau} Y_{v-1,\tau} + \Phi_{2,\tau} Y_{v-2,\tau} + \cdots + \Phi_{P,\tau} Y_{v-P,\tau} \\
& - \phi_{1,\tau} \Phi_{1,\tau} Y_{v-1,\tau-1} - \phi_{1,\tau} \Phi_{2,\tau} Y_{v-2,\tau-1} - \cdots \\
& - \phi_{1,\tau} \Phi_{P,\tau} Y_{v-P,\tau-1} - \cdots - \phi_{p,\tau} \Phi_{1,\tau} Y_{v-1,\tau-p} \\
& - \phi_{p,\tau} \Phi_{2,\tau} Y_{v-2,\tau-p} - \cdots - \phi_{p,\tau} \Phi_{P,\tau} Y_{v-P,\tau-p} \\
& + \epsilon_{v,\tau} \qquad\qquad (3.4) \\
& - \theta_{1,\tau} \epsilon_{v,\tau-1} - \theta_{2,\tau} \epsilon_{v,\tau-2} - \cdots - \theta_{q,\tau} \epsilon_{v,\tau-q} \\
& - \overline{\underline{\theta}}_{1,\tau} \epsilon_{v-1,\tau} - \overline{\underline{\theta}}_{2,\tau} \epsilon_{v-2,\tau} - \cdots - \overline{\underline{\theta}}_{Q,\tau} \epsilon_{v-Q,\tau} \\
& + \theta_{1,\tau} \overline{\underline{\theta}}_{1,\tau} \epsilon_{v-1,\tau-1} + \theta_{1,\tau} \overline{\underline{\theta}}_{2,\tau} \epsilon_{v-2,\tau-1} + \cdots \\
& + \theta_{1,\tau} \overline{\underline{\theta}}_{Q,\tau} \epsilon_{v-Q,\tau-1} + \cdots + \theta_{q,\tau} \overline{\underline{\theta}}_{1,\tau} \epsilon_{v-1,\tau-q} \\
& + \theta_{q,\tau} \overline{\underline{\theta}}_{2,\tau} \epsilon_{v-2,\tau-q} + \cdots + \theta_{q,\tau} \overline{\underline{\theta}}_{Q,\tau} \epsilon_{v-Q,\tau-q}
\end{aligned}$$

where the parameters $\phi_{1,\tau}$'s, $\phi_{2,\tau}$'s and $\phi_{p,\tau}$'s are the periodic lag 1, lag 2 and lag p autoregressive coefficients, respectively. $\Phi_{1,\tau}$'s, $\Phi_{2,\tau}$'s and $\Phi_{P,\tau}$'s are the periodic lag w, lag 2w and lag Pw autoregressive coefficients.

The parameters $\theta_{1,\tau}$'s, $\theta_{2,\tau}$'s and $\theta_{q,\tau}$'s represent, respectively, the lag 1, lag 2 and lag q moving average coefficients; while $\underline{\bar{\theta}}_{1,\tau}$'s, $\underline{\bar{\theta}}_{2,\tau}$'s and $\underline{\bar{\theta}}_{Q,\tau}$'s represent, respectively, the lag w, lag 2w and lag Qw moving average coefficients.

$Y_{v,\tau}$ and $\epsilon_{v,\tau}$ are as defined before and present mean zero and variances respectively $\sigma_\tau{}^2(y)$ and $\sigma_\tau{}^2(\epsilon)$.

A simpler model, called Multiplicative PARMA$(1,1)x(1,1)_w$, can be obtained from expression (3.4) by assuming p=q=P=Q=1 :

$$Y_{v,\tau} = \phi_{1,\tau}\, Y_{v,\tau-1} + \Phi_{1,\tau}\, Y_{v-1,\tau} - \phi_{1,\tau}\Phi_{1,\tau}\, Y_{v-1,\tau-1} \quad (3.5)$$
$$+\epsilon_{v,\tau} - \theta_{1,\tau}\, \epsilon_{v,\tau-1} - \underline{\bar{\theta}}_{1,\tau}\, \epsilon_{v-1,\tau} + \theta_{1,\tau}\underline{\bar{\theta}}_{1,\tau}\, \epsilon_{v-1,\tau-1}$$

The variable $Y_{v,\tau}$ presents seasonal mean zero. Consequently, the Multiplicative PARMA $(1,1)x(1,1)_w$ model presents the parameter set $\{\phi_{1,\tau}, \Phi_{1,\tau}, \theta_{1,\tau}, \underline{\bar{\theta}}_{1,\tau}, \sigma_\tau{}^2(\epsilon),$ $\tau=1,2,\ldots,w\}$ that can be estimated from data.

The expression (3.5) can be written in matricial form. As an example, for the two-season case, i.e., w=2, the model can be represented by :

$$\begin{vmatrix} 1 & 0 \\ -\phi_{1,2} & 1 \end{vmatrix} \begin{vmatrix} Y_{v,1} \\ Y_{v,2} \end{vmatrix} + \begin{vmatrix} -\Phi_{1,1} & -\phi_{1,1} \\ 0 & -\Phi_{1,2} \end{vmatrix} \begin{vmatrix} Y_{v-1,1} \\ Y_{v-1,2} \end{vmatrix}$$
$$+ \begin{vmatrix} \phi_{1,1}\Phi_{1,1} & 0 \\ 0 & \phi_{1,2}\Phi_{1,2} \end{vmatrix} \begin{vmatrix} Y_{v-2,2} \\ Y_{v-1,1} \end{vmatrix} =$$
$$\begin{vmatrix} 1 & 0 \\ -\theta_{1,2} & 1 \end{vmatrix} \begin{vmatrix} \epsilon_{v,1} \\ \epsilon_{v,2} \end{vmatrix} + \begin{vmatrix} -\underline{\bar{\theta}}_{1,1} & -\theta_{1,1} \\ 0 & \underline{\bar{\theta}}_{1,2} \end{vmatrix} \begin{vmatrix} \epsilon_{v-1,1} \\ \epsilon_{v-1,2} \end{vmatrix} \tag{3.6}$$
$$+ \begin{vmatrix} \theta_{1,1}\underline{\bar{\theta}}_{1,1} & 0 \\ 0 & \theta_{1,2}\underline{\bar{\theta}}_{1,2} \end{vmatrix} \begin{vmatrix} \epsilon_{v-2,2} \\ \epsilon_{v-1,1} \end{vmatrix}.$$

In multivariate form :

$$U(B)\ \underline{Y}_v = V(B)\ \underline{\epsilon}_v \tag{3.7}$$

where

$$U(B) = \begin{vmatrix} 1-\Phi_{1,1}B & -\phi_{1,1}B+\phi_{1,1}\Phi_{1,1}B^2 \\ -\phi_{1,2}+\phi_{1,2}\Phi_{1,2}B & 1-\Phi_{1,2}B \end{vmatrix} \tag{3.8}$$

and

$$V(B) = \begin{vmatrix} 1-\underline{\bar{\theta}}_{1,1}B & -\theta_{1,1}B+\theta_{1,1}\underline{\bar{\theta}}_{1,1}B^2 \\ -\theta_{1,2}B+\theta_{1,2}\underline{\bar{\theta}}_{1,2}B & 1-\underline{\bar{\theta}}_{1,2}B \end{vmatrix} \tag{3.9}$$

where $B(Y_{v,\tau}) = Y_{v-1,\tau}$

3.4.2 Seasonal Variance

The expression for the seasonal variance of $Y_{v,\tau}$ for the particular Multiplicative PARMA$(1,1)x(1,1)_w$ model, that is the simplest model presenting all kinds of parameters, ϕs, θs, Φs and $\underline{\bar{\theta}}$s, can be obtained by multiplying all terms of the expression (3.5) by $Y_{v,\tau}$:

$$Y_{v,\tau}Y_{v,\tau} = \phi_{1,\tau}Y_{v,\tau-1}Y_{v,\tau} + \Phi_{1,\tau}Y_{v-1,\tau}Y_{v,\tau}$$
$$- \phi_{1,\tau}\Phi_{1,\tau}Y_{v-1,\tau-1}Y_{v,\tau} \qquad (3.10)$$
$$+ \epsilon_{v,\tau}Y_{v,\tau} - \theta_{1,\tau}\epsilon_{v,\tau-1}Y_{v,\tau} - \underline{\overline{\theta}}_{1,\tau}\epsilon_{v-1,\tau}Y_{v,\tau}$$
$$+ \theta_{1,\tau}\underline{\overline{\theta}}_{1,\tau}\epsilon_{v-1,\tau-1}Y_{v,\tau}$$

Replacing the expression (3.5) into the expression (3.10) and taking expected values results in the following expression for the variance of the Multiplicative PARMA$(1,1)x(1,1)_w$ model:

$$\sigma_\tau^2(y)\ [1- \Phi_{1,\tau}^2 - \underline{\overline{\theta}}_{1,\tau}^2] \qquad (3.11)$$
$$- \sigma_{\tau-1}^2(y)\ [\phi_{1,\tau}^2 + \phi_{1,\tau}^2\Phi_{1,\tau}^2 + \theta_{1,\tau}^2 + \theta_{1,\tau}^2\underline{\overline{\theta}}_{1,\tau}^2] =$$

$$\sigma_\tau^2(\epsilon)\ [1- 2\phi_{1,\tau}\underline{\overline{\theta}}_{1,\tau}]$$
$$- 2\sigma_{\tau-1}^2(\epsilon)\ [\phi_{1,\tau}\theta_{1,\tau} + \phi_{1,\tau}\Phi_{1,\tau}\theta_{1,\tau}\underline{\overline{\theta}}_{1,\tau}]$$
$$+ \sigma_{\tau-1}^2(\epsilon)\ [\phi_{1,\tau}\theta_{1,\tau}\underline{\overline{\theta}}_{1,\tau}]\ [\phi_{1,\tau} + \Phi_{1,\tau}]$$
$$+ 2\phi_{1,\tau}\Phi_{1,\tau}\ r_{w-1,\tau-1} - 2\phi_{1,\tau}^2\Phi_{1,\tau}\ r_{w,\tau-1} - 2\phi_{1,\tau}\Phi_{1,\tau}^2\ r_{1,\tau}$$
$$-2\phi_{1,\tau}\underline{\overline{\theta}}_{1,\tau}E(\epsilon_{v-1,\tau}Y_{v,\tau-1}) + 2\phi_{1,\tau}\theta_{1,\tau}\underline{\overline{\theta}}_{1,\tau}E(\epsilon_{v-1,\tau-1}Y_{v,\tau-1})$$

$r_{v,\tau}$ represents the estimates of the seasonal autocorrelations, and E represents the expectance operator.

For the case of two seasons (i. e., w=2), this expression may be transformed into :

$$
\begin{aligned}
&\sigma_\tau^2(y)\ [1-\ \Phi_{1,\tau}^2-\ \underline{\overline{\theta}}_{1,\tau}^2] \\
&-\ \sigma_{\tau-1}^2(y)\ [\phi_{1,\tau}^2+\ \phi_{1,\tau}^2\Phi_{1,\tau}^2+\ \theta_{1,\tau}^2+\ \theta_{1,\tau}^2\underline{\overline{\theta}}_{1,\tau}^2] = \\
&\sigma_\tau^2(\epsilon)\ [1-\ 2\phi_{1,\tau}\underline{\overline{\theta}}_{1,\tau}-\ 2\phi_{1,\tau}\phi_{1,\tau-1}\theta_{1,\tau}] \\
&-\ \sigma_{\tau-1}^2(\epsilon)\ \ \phi_{1,\tau}\theta_{1,\tau}\ \ [2+\ \ \Phi_{1,\tau}\underline{\overline{\theta}}_{1,\tau}-\ \ \phi_{1,\tau}\underline{\overline{\theta}}_{1,\tau}-\ \ \Phi_{1,\tau}\underline{\overline{\theta}}_{1,\tau}- \\
&2\theta_{1,\tau}+\ +\ 2\theta_{1,\tau}\underline{\overline{\theta}}_{1,\tau-1}] \\
&+\ 2\phi_{1,\tau}\Phi_{1,\tau}\ [r_{w-1,\tau-1}-\ \phi_{1,\tau}r_{w,\tau-1}-\ \Phi_{1,\tau}r_{1,\tau}]
\end{aligned}
\tag{3.12}
$$

For the Multiplicative PARMA(1,0)x(1,0)$_w$, from the expression (3.11), the results are :

$$
\begin{aligned}
&\sigma_\tau^2(y)\ [1-\ \Phi_{1,\tau}^2]\ -\ \sigma_{\tau-1}^2(y)\ [\phi_{1,\tau}^2+\ \phi_{1,\tau}^2\Phi_{1,\tau}^2] = \\
&\sigma_\tau^2(\epsilon)\ +\ 2\phi_{1,\tau}\Phi_{1,\tau}\ [r_{w-1,\tau-1}-\ \phi_{1,\tau}r_{w,\tau-1}-\ \Phi_{1,\tau}\ r_{1,\tau}]
\end{aligned}
$$

Expression (3.13) may be written in matricial form, and replacing the index $(1,\tau)$ by (τ) :

$$
\begin{vmatrix}
(1-\Phi_1^2) & 0 & & & -(\phi_1^2+\phi_1^2\Phi_1^2) \\
-(\phi_2^2+\phi_2^2\Phi_2^2) & (1-\Phi_2^2) & & & \\
0 & -(\phi_3^2+\phi_3^2\Phi_3^2) & \ddots & & \\
 & 0 & & \ddots & \\
 & & & -(\phi_w^2+\phi_w^2\Phi_w^2) & (1-\Phi_w^2)
\end{vmatrix}
\begin{vmatrix}
\sigma_1^2(y) \\ \sigma_2^2(y) \\ \vdots \\ \\ \sigma_w^2(y)
\end{vmatrix}
\tag{3.14}
$$

$$
= \begin{vmatrix}
\sigma_1^2(\epsilon) \\ \sigma_2^2(\epsilon) \\ \vdots \\ \sigma_w^2(\epsilon)
\end{vmatrix}
+ 2
\begin{vmatrix}
\phi_1\Phi_1\ [r_{w-1,w}-\ \ \phi_1 r_{w,w}-\ \Phi_1 r_{1,1}] \\
\phi_2\Phi_2\ [r_{w-1,1}-\ \ \phi_2 r_{w,1}-\ \Phi_2 r_{1,2}] \\
\vdots \\
\phi_w\Phi_w[r_{w-1,w-1}-\ \phi_w r_{w,w-1}-\ \Phi_w r_{1,w}]
\end{vmatrix}
$$

3.4.3 Moment Equations and Seasonal Autocorrelationsa

Here, the objective is to derive equations that could allow estimation of parameters and determination of expressions for the seasonal autocorrelations in function of the parameters of the model. The utilized approach is similar to that used by Salas et al. (1982) for some PARMA models.

The Multiplicative PARMA$(1,1)x(1,1)_W$ model may be expressed as:

$$(1-\phi_{1,\tau}B).(1-\Phi_{1,\tau}B^W)Y_{v,\tau} = (1-\theta_{1,\tau}B).(1-\underline{\overline{\theta}}_{1,\tau}B^W)\epsilon_{v,\tau} \quad (3.15)$$

This equation is equivalent to

$$\begin{aligned}(1- \phi_{1,\tau}B- \Phi_{1,\tau}B^W+ \phi_{1,\tau}.\Phi_{1,\tau}B.B^W)\ Y_{v,\tau} &= \\ (1- \theta_{1,\tau}B- \underline{\overline{\theta}}_{1,\tau}B^W+ \theta_{1,\tau}.\underline{\overline{\theta}}_{1,\tau}B.B^W)\ \epsilon_{v,\tau}\end{aligned} \quad (3.16)$$

This can be made more explicit as :

$$\begin{aligned}Y_{v,\tau}- \phi_{1,\tau}Y_{v,\tau-1}- \Phi_{1,\tau}Y_{v-1,\tau}+ \phi_{1,\tau}.\Phi_{1,\tau}Y_{v-1,\tau-1} &= \\ \epsilon_{v,\tau}- \theta_{1,\tau}\epsilon_{v,\tau-1}- \underline{\overline{\theta}}_{1,\tau}\epsilon_{v-1,\tau}+ \theta_{1,\tau}.\underline{\overline{\theta}}_{1,\tau}\epsilon_{v-1,\tau-1}\end{aligned} \quad (3.17)$$

Five moment equations are necessary for finding moment estimates of the parameters $\phi_{1,\tau}$, $\Phi_{1,\tau}$, $\theta_{1,\tau}$, $\underline{\overline{\theta}}_{1,\tau}$, $\sigma_\tau{}^2(\epsilon)$. Multiplying expression (3.17) by $Y_{v,\tau-1}$, taking expected values and simplifying, the first equation is obtained :

$$\begin{aligned}&r_{1,\tau}- \phi_{1,\tau}\sigma_{\tau-1}{}^2(y)- \Phi_{1,\tau}r_{w-1,\tau-1}+ \phi_{1,\tau}.\Phi_{1,\tau}r_{w,\tau-1} = \\ &- \theta_{1,\tau}\sigma^2{}_{\tau-1}(\epsilon) \\ &+ \underline{\overline{\theta}}_{1,\tau}E(Y_{v,\tau-1}\epsilon_{v-1,\tau})+ \theta_{1,\tau}\underline{\overline{\theta}}_{1,\tau}E(Y_{v,\tau-1}\epsilon_{v-1,\tau-1})\end{aligned} \quad (3.18)$$

where $r_{v,\tau}$ represents the estimates of seasonal autocorrelations and E represents the expectance operator.

Proceeding in the same way, but replacing the multiplicative factor consecutively by $Y_{v-1,\tau}$, $Y_{v-1,\tau-1}$, $Y_{v,\tau-2}$ and $Y_{v,\tau}$, the corresponding moment equations are derived :

$$r_{w,\tau}-\phi_{1,\tau}r_{w-1,\tau-1}-\Phi_{1,\tau}\sigma_\tau^{2}(y)+\phi_{1,\tau}\cdot\Phi_{1,\tau}r_{1,\tau} = \qquad (3.19)$$
$$-\underline{\overline{\theta}}_{1,\tau}\sigma^2{}_\tau(\epsilon)+\theta_{1,\tau}\cdot\underline{\overline{\theta}}_{1,\tau}\cdot\phi_{1,\tau}\sigma^2{}_{\tau-1}(\epsilon)-\theta_{1,\tau}{}^2\underline{\overline{\theta}}_{1,\tau}\sigma^2{}_{\tau-1}(\epsilon)$$

$$r_{w+1,\tau}-\phi_{1,\tau}r_{w-1,\tau-1}-\Phi_{1,\tau}r_{w,\tau-1}+\phi_{1,\tau}.\Phi_{1,\tau}\sigma_{\tau-1}{}^{2}(y) =$$
$$\theta_{1,\tau}\cdot\underline{\overline{\theta}}_{1,\tau}\sigma^2{}_{\tau-1}(\epsilon) \qquad (3.20)$$

$$r_{2,\tau}-\phi_{1,\tau}r_{1,\tau-1}-\Phi_{1,\tau}r_{w+2,\tau-2}+\phi_{1,\tau}\cdot\Phi_{1,\tau}r_{w+1,\tau-2} =$$
$$\theta_{1,\tau}\underline{\overline{\theta}}_{1,\tau}E(Y_{v,\tau-2}\epsilon_{v-1,\tau-1})-\underline{\overline{\theta}}_{1,\tau}E(Y_{v,\tau-2}\epsilon_{v-1,\tau}) \qquad (3.21)$$

$$\sigma_\tau{}^2(y)-\phi_{1,\tau}r_{1,\tau}-\Phi_{1,\tau}r_{w,\tau}+\phi_{1,\tau}\cdot\Phi_{1,\tau}r_{w+1,\tau} =$$
$$\sigma^2{}_\tau(\epsilon)-\phi_{1,\tau}\cdot\theta_{1,\tau}\sigma^2{}_{\tau-1}(\epsilon)+\theta_{1,\tau}{}^2\sigma^2{}_{\tau-1}(\epsilon)$$
$$+\theta_{1,\tau}\underline{\overline{\theta}}_{1,\tau}E(Y_{v,\tau}\epsilon_{v-1,\tau-1})-\underline{\overline{\theta}}_{1,\tau}E(Y_{v,\tau}\epsilon_{v-1,\tau}) \qquad (3.22)$$

For the Multiplicative PARMA(1,1)x$(1,0)_w$ the above moment equations may be reduced to :

$$r_{1,\tau}-\phi_{1,\tau}\sigma_{\tau-1}{}^{2}(y)-\Phi_{1,\tau}r_{w-1,\tau-1}+\phi_{1,\tau}\cdot\Phi_{1,\tau}r_{w,\tau-1} =$$
$$-\theta_{1,\tau}\sigma_{\tau-1}{}^{2}(\epsilon) \qquad (3.23)$$

$$r_{w,\tau}-\phi_{1,\tau}r_{w-1,\tau-1}-\Phi_{1,\tau}\sigma_\tau{}^{2}(y)+\phi_{1,\tau}\cdot\Phi_{1,\tau}r_{1,\tau} =$$
$$0 \qquad (3.24)$$

$$r_{w+1,\tau} - \phi_{1,\tau} r_{w-1,\tau-1} - \Phi_{1,\tau} r_{w,\tau-1} + \phi_{1,\tau} . \Phi_{1,\tau} \sigma_{\tau-1}{}^2(y) = 0 \qquad (3.25)$$

$$r_{2,\tau} - \phi_{1,\tau} r_{1,\tau-1} - \Phi_{1,\tau} r_{w+2,\tau-2} + \phi_{1,\tau} \cdot \Phi_{1,\tau} r_{w+1,\tau-2} = 0 \qquad (3.26)$$

$$\sigma_\tau{}^2(y) - \phi_{1,\tau} r_{1,\tau} - \Phi_{1,\tau} r_{w,\tau} + \phi_{1,\tau} \cdot \Phi_{1,\tau} r_{w+1,\tau} = \sigma^2{}_\tau(\epsilon) \qquad (3.27)$$

The moment equations are extremely nonlinear, very difficult to solve and would require a complex optimization procedure for solution of the system. It was decided that, instead of trying to solve the above system of equations, it would be more practical to estimate the parameters by utilizing a least-squares technique directly applied to the data.

Seasonal autocorrelations can be computed from the parameters of the model by utilizing the above moment equations. It was verified that this could be achieved by matricial algebra. As an example, for the two-season case, w=2, the system of equations for calculating autocorrelations up to lag 4 is :

$$r_{2,2} - \phi_{1,2} r_{1,1} - \Phi_{1,2} r_{4,2} + \phi_{1,2} \Phi_{1,2} r_{3,2} = 0 \qquad (3.28)$$

$$r_{3,2} - \phi_{1,2} r_{1,1} - \Phi_{1,2} r_{2,1} + \phi_{1,2} \Phi_{1,2} \sigma_1{}^2(y) = 0 \qquad (3.29)$$

$$r_{2,2} - \phi_{1,2} r_{1,1} - \Phi_{1,2} \sigma_2^2(y) + \phi_{1,2}\Phi_{1,2} r_{1,2} = 0 \tag{3.30}$$

$$r_{1,2} - \phi_{1,2} \sigma_1^2(y) - \Phi_{1,2} r_{1,1} + \phi_{1,2}\Phi_{1,2} r_{2,1} = -\theta_{1,2}\sigma_1^2(\epsilon) \tag{3.31}$$

$$r_{2,1} - \phi_{1,1} r_{1,2} - \Phi_{1,1} r_{4,1} + \phi_{1,1}\Phi_{1,1} r_{3,1} = 0 \tag{3.32}$$

$$r_{3,1} - \phi_{1,1} r_{1,2} - \Phi_{1,1} r_{2,2} + \phi_{1,1}\Phi_{1,1} \sigma_2^2(y) = 0 \tag{3.33}$$

$$r_{2,1} - \phi_{1,1} r_{1,2} - \Phi_{1,1} \sigma_1^2(y) + \phi_{1,1}\Phi_{1,1} r_{1,1} = 0 \tag{3.34}$$

$$r_{1,1} - \phi_{1,1} \sigma_2^2(y) - \Phi_{1,1} r_{1,2} + \phi_{1,1}\Phi_{1,1} r_{2,2} = -\theta_{1,1}\sigma_2^2(\epsilon) \tag{3.35}$$

The system of equations (3.28) through (3.35) may be represented by:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & -\Phi_1 & \phi_1\Phi_1 & 0 & 0 \\ \phi_1\Phi_1 & 1 & 0 & 0 & -\phi_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\phi_1 & -\Phi_1 & 0 & 0 \\ 0 & 1 & \phi_1\Phi_1 & -\Phi_1 & -\phi_1 & 0 & 0 & 0 \\ -\Phi_2 & \phi_2\Phi_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\phi_2 & \phi_2\Phi_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\phi_2 & -\Phi_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\phi_2 & 0 & 0 & 0 & 0 & 1 & \phi_2\Phi_2 & -\Phi_2 \end{vmatrix} \begin{vmatrix} r_{1,1} \\ r_{2,1} \\ r_{3,1} \\ r_{4,1} \\ r_{1,2} \\ r_{2,2} \\ r_{3,2} \\ r_{4,2} \end{vmatrix} = \tag{3.36}$$

$$\begin{vmatrix} \phi_1\sigma_2{}^2(y) - \theta_1\sigma_2{}^2(\epsilon) \\ \Phi_1\sigma_1{}^2(y) \\ -\phi_1\Phi_1\sigma_2{}^2(y) \\ 0 \\ \phi_2\sigma_1{}^2(y) - \theta_2\sigma_1{}^2(\epsilon) \\ \Phi_2\sigma_2{}^2(y) \\ -\phi_2\Phi_2\sigma_1{}^2(y) \\ 0 \end{vmatrix}$$

This matricial equation can be written as:

$$A \,.\, \underline{r} = \underline{u} \tag{3.37}$$

and this can be solved for r,

$$\underline{r} = A^{-1}.\ \underline{u}, \tag{3.38}$$

where A^{-1} is the inverse matrix of A.

The vector $\underline{r}$ presents the seasonal autocorrelations.

For the Multiplicative PARMA(1,0)x$(1,0)_2$ the same procedure can be followed. The matrix A and the vector $\underline{r}$ remain the same, while the vector $\underline{u}$ is transformed into:

$$\underline{u} = \begin{vmatrix} \phi_1\sigma_2{}^2(y) \\ \Phi_1\sigma_1{}^2(y) \\ -\phi_1\Phi_1\sigma_2{}^2(y) \\ 0 \\ \phi_2\sigma_1{}^2(y) \\ \Phi_2\sigma_2{}^2(y) \\ -\phi_2\Phi_2\sigma_1{}^2(y) \\ 0 \end{vmatrix} \tag{3.39}$$

3.5 Aggregation

3.5.1 Annual Series

The objective here is to investigate the structure of the annual hydrologic series given the structure of the seasonal series. Due to the complexity of the complete Multiplicative PARMA$(p,q)x(P,Q)_w$, the analysis will be made by utilizing the simplest model that presents all kinds of parameters (ϕs, θs, Φs and $\underline{\overline{\Theta}}$s) that is the Multiplicative PARMA$(1,1)x(1,1)_2$.

For this model,

$$Y_{v,1} = \phi_{1,1}Y_{v-1,2} + \Phi_{1,1}Y_{v-1,1} - \phi_{1,1}\Phi_{1,1}Y_{v-2,2} \qquad (3.40)$$
$$+ \epsilon_{v,1} - \theta_{1,1}\epsilon_{v-1,2} - \underline{\overline{\Theta}}_{1,1}\epsilon_{v-1,1} + \theta_{1,1}\underline{\overline{\Theta}}_{1,1}\epsilon_{v-2,2}$$

and

$$Y_{v,2} = \phi_{1,2}Y_{v,1} + \Phi_{1,2}Y_{v-1,2} - \phi_{1,2}\Phi_{1,2}Y_{v-1,1} \qquad (3.41)$$
$$+ \epsilon_{v,2} - \theta_{1,2}\epsilon_{v,1} - \underline{\overline{\Theta}}_{1,2}\epsilon_{v-1,2} + \theta_{1,2}\underline{\overline{\Theta}}_{1,2}\epsilon_{v-1,1}$$

Adding these two expressions for obtaining the annual series and replacing the indices $(1,\tau)$ by (τ) for the parameters :

$$X_v = Y_{v,1} + Y_{v,2} = \phi_2Y_{v,1} + (\phi_1 + \Phi_2)Y_{v-1,2} + (\phi_1 - \phi_2\Phi_2)Y_{v-1,1}$$
$$- \phi_1\Phi_1Y_{v-2} + \epsilon_{v,2} + (1 - \theta_2)\epsilon_{v,1} - (\theta_1 + \underline{\overline{\Theta}}_2)\epsilon_{v-1,2}$$
$$+ (\theta_2\underline{\overline{\Theta}}_2 - \underline{\overline{\Theta}}_1)\epsilon_{v-1,1} + \theta_1\underline{\overline{\Theta}}_1\epsilon_{v-2,2} \qquad (3.42)$$

Replacing $Y_{v,1}$ and $Y_{v-1,2}$ utilizing expressions (3.40) and (3.41) :

$$
\begin{aligned}
X_v = {} & \phi_2(\phi_1 Y_{v-1,2} + \Phi_1 Y_{v-1,1} - \phi_1\Phi_1 Y_{v-2,2} + \epsilon_{v,1} - \theta_1\epsilon_{v-1,2} \\
& - \underline{\overline{\theta}}_1\epsilon_{v-1,1} + \theta_1\underline{\overline{\theta}}_1\epsilon_{v-2,2}) \\
& + \phi_1(\phi_2 Y_{v-1,1} + \Phi_2 Y_{v-2,2} - \phi_2\Phi_2 Y_{v-2,1} + \epsilon_{v-1,2} - \theta_2\epsilon_{v-1,1} \\
& - \underline{\overline{\theta}}_2\epsilon_{v-2,2} + \theta_2\underline{\overline{\theta}}_2\epsilon_{v-2,1}) \\
& + \Phi_2(\phi_2 Y_{v-1,1} + \Phi_2 Y_{v-2,2} - \phi_2\Phi_2 Y_{v-2,1} + \epsilon_{v-1,2} - \theta_2\epsilon_{v-1,1} \\
& - \underline{\overline{\theta}}_2\epsilon_{v-2,2} + \theta_2\underline{\overline{\theta}}_2\epsilon_{v-2,1}) \\
& + (\Phi_1 - \phi_2\Phi_2)Y_{v-1,1} - \phi_1\Phi_1 Y_{v-2,2} \\
& + \epsilon_{v,2} + (1-\theta_2)\epsilon_{v,1} - (\theta_1 + \underline{\overline{\theta}}_2)\epsilon_{v-1,2} + (\theta_2\underline{\overline{\theta}}_2 - \underline{\overline{\theta}}_1)\epsilon_{v-1,1} \\
& + \theta_1\underline{\overline{\theta}}_1\epsilon_{v-2,2} \qquad (3.43)
\end{aligned}
$$

Simplifying this expression, taking into account that the annual term for the previous year may be expressed as

$$X_{v-1} = Y_{v-1,1} + Y_{v-1,2}, \qquad (3.44)$$

and writing only the terms presenting the variable Y, since Rose(1977) proved that the sum of independent moving average processes each of order one is representable as a single moving average process also of order 1 :

$$
\begin{aligned}
X_v = {} & \phi_1\phi_2 X_{v-1} + (\Phi_1\phi_2 + \Phi_1)Y_{v-1,1} \\
& + (\Phi_1{}^2 + \phi_1\Phi_2 - \phi_1\Phi_1 - \phi_2\phi_1\Phi_1)Y_{v-2,2} \qquad (3.45) \\
& - (\Phi_2{}^2\phi_2 + \phi_1\phi_2\Phi_2)Y_{v-2,1} + \ldots
\end{aligned}
$$

Iterating one more time and making the same simplifications :

$$X_v = \phi_1\phi_2 X_{v-1} + (\phi_1\Phi_2 + \Phi_2^2) Y_{v-2,2}$$
$$+ (\Phi_1^2 + \Phi_1^2\phi_2 - \phi_1\phi_2\Phi_2 - \phi_2\Phi_2^2) Y_{v-2,1} \tag{3.46}$$
$$- (\phi_1\Phi_1^2 + \phi_1\phi_2\Phi_1^2) Y_{v-3,2} + \ldots$$

After one more iteration :

$$X_v = \phi_1\phi_2 X_{v-1} + (\Phi_1^2 + \phi_2\Phi_1^2) Y_{v-2,1}$$
$$+ (\Phi_2^3 + \phi_1\Phi_2^2 - \phi_1\Phi_1^2 - \phi_2\Phi_1^2\phi_1) Y_{v-3,2} \tag{3.47}$$
$$- (\phi_2\Phi_2^3 + \phi_1\phi_2\Phi_2^2) Y_{v-3,1} + \ldots$$

Expressions (3.45), (3.46) and (3.47) present an autoregressive term of order 1. However, the other terms could not be simplified into new autorgressive terms. Each new iteration introduces a new term of the variable Y and the order of the product of parameters(coefficients) increases by 1. This means that the annual series represents an ARMA process only if the parameters Φs are null . In this case, the seasonal Multiplicative PARMA$(1,1)x(1,1)_2$ would be simplified to a PARMA(1,1) model. Data generation also showed that the annual series for the Multiplicative PARMA resulted in an ARMA with autoregressive parameter approximately equal to the product of the seasonal autoregressive parameters only when the parameters other than $\phi_{1,\tau}$'s and $\Theta_{1,\tau}$'s were close to zero.

Annual aggregation for the low-order PARMA model was studied by Vecchia(1983) and Vecchia <u>et al</u>(1983). Analysis of partial aggregation for this model (addition of any number of consecutive seasons) is shown in the next sub-chapter.

3.5.2 Aggregation of PARMA(1,1) Model

3.5.2.1 PARMA(1,1) Model

The $PARMA_w(1,1)$ is equivalent to the Multiplicative $PARMA(1,1)x(1,1)_w$ when the parameters Φ_S and Θ_S are null. It presents parameters relating consecutive seasons.

The PARMA (1,1) model may be represented by :

$$Y_{v,\tau} = \phi_\tau \; Y_{v,\tau-1} - \theta_\tau \; \epsilon_{v,\tau-1} + \epsilon_{v,\tau} \qquad (3.48)$$

where $Y_{v,\tau}$ is the seasonal variable during season τ and year v, $\tau = 1,2...w$, and w is the number of seasons in the year. $\epsilon_{v,\tau}$ is an independent random variable with mean zero and variance $\sigma_\tau{}^2(\epsilon)$. ϕ_τ and θ_τ are the periodic autoregressive and moving average coefficients.

In multivariate form :

$$U(B) \; \underline{Y}_v = V(B) \; \underline{\epsilon}_v \qquad (3.49)$$

where

$$U(B) = \begin{vmatrix} 1 & & & & -\phi_1 B \\ -\phi_2 & 1 & & & \\ 0 & -\phi_3 & 1 & & \\ \cdot & & \cdot & \cdot & \\ \cdot & & \cdot & & \cdot \\ 0 & 0\ldots & -\phi_w & & 1 \end{vmatrix}$$

$$\underline{Y}_v' = | \; Y_{v,1};\; Y_{v,2};\ldots;Y_{v,w} \; |$$

$$V(B) = \begin{vmatrix} 1 & & & & -\theta_1 B \\ -\theta_2 & 1 & & & \\ 0 & -\theta_3 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \ldots & -\theta_w & 1 \end{vmatrix}$$

$$\underline{\epsilon}_v' = | \; \epsilon_{v,1};\; \epsilon_{v,2};\ldots;\epsilon_{v,w} \; |$$

and B is a lag operator defined by

$$B \; Y_{v,\tau} = Y_{v-1,\tau}$$

3.5.2.2 Aggregation Modeling of Low Θrder PARMA

Vecchia(1983) and Vecchia et al.(1983) showed that the total aggregation(annual) of a low-order Periodic Autoregressive Moving Average model, summing over the seasons, follows a regular ARMA(1,1) model.

First it will be shown, through an example for easier understanding, that not only the annual series but also the series obtained by aggregating some consecutive seasons can be fitted by this model.

Later this conclusion will be generalized for aggregation of any number of consecutive seasons from any PARMA(1,1) model.

3.5.2.2.1 Aggregation of a $PARMA_4(1,1)$ Series

a) Annual

Let the series be represented by :

$$
\begin{array}{cccc}
Y_{1,1} & Y_{1,2} & Y_{1,3} & Y_{1,4} \\
Y_{2,1} & Y_{2,2} & Y_{2,3} & Y_{2,4} \\
\vdots & \vdots & \vdots & \vdots \\
Y_{v-1,1} & Y_{v-1,2} & Y_{v-1,3} & Y_{v-1,4} \\
Y_{v,1} & Y_{v,2} & Y_{v,3} & Y_{v,4} \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$

From the definition of PARMA models :

$$Y_{v,1} = \phi_1 Y_{v-1,4} - \theta_1 \epsilon_{v-1,4} + \epsilon_{v,1} \tag{3.50}$$

$$Y_{v,2} = \phi_2 Y_{v,1} - \theta_2 \epsilon_{v,1} + \epsilon_{v,2} \tag{3.51}$$

$$Y_{v,3} = \phi_3 Y_{v,2} - \theta_3 \epsilon_{v,2} + \epsilon_{v,3} \tag{3.52}$$

$$Y_{v,4} = \phi_4 Y_{v,3} - \theta_4 \epsilon_{v,3} + \epsilon_{v,4} \tag{3.53}$$

The annual series may be written as :

$$X_v = Y_{v,1} + Y_{v,2} + Y_{v,3} + Y_{v,4} \tag{3.54}$$

Substituting expressions (3.50)...(3.53) into the present equation :

$$
\begin{aligned}
X_v = {} & \phi_1 Y_{v-1,4} - \theta_1 \epsilon_{v-1,4} + \epsilon_{v,1} + \\
& \phi_2 Y_{v,1} - \theta_2 \epsilon_{v,1} + \epsilon_{v,2} + \\
& \phi_3 Y_{v,2} - \theta_3 \epsilon_{v,2} + \epsilon_{v,3} + \\
& \phi_4 Y_{v,3} - \theta_4 \epsilon_{v,3} + \epsilon_{v,4}
\end{aligned} \tag{3.55}
$$

Iterating, utilizing relationships (3.50)...(3.53), results in the expression (3.56) :

$$\begin{aligned} X_v =&\phi_1(\phi_4\ Y_{v-1,3} - \theta_4\ \epsilon_{v-1,3} + \epsilon_{v-1,4}) - \theta_1\ \epsilon_{v-1,4} + \epsilon_{v,1} + \\ &\phi_2(\phi_1\ Y_{v-1,4} - \theta_1\ \epsilon_{v-1,4} + \epsilon_{v,1}) - \theta_2\ \epsilon_{v,1} + \epsilon_{v,2} + \\ &\phi_3(\phi_2\ Y_{v,1} - \theta_2\ \epsilon_{v,1} + \epsilon_{v,2}) - \theta_3\ \epsilon_{v,2} + \epsilon_{v,3} + \\ &\phi_4(\phi_3\ Y_{v,2} - \theta_3\ \epsilon_{v,2} + \epsilon_{v,3}) - \theta_4\ \epsilon_{v,3} + \epsilon_{v,4} \end{aligned}$$

Iterating two more times :

$$\begin{aligned} X_v =&\phi_1\phi_2\phi_3\phi_4(Y_{v-1,1} + Y_{v-1,2} + Y_{v-1,3} + Y_{v-1,4}) + \\ &(-\phi_1\phi_4\phi_3\theta_2)\epsilon_{v-1,1} + \\ &(1 + \phi_2 - \theta_2 + \phi_3\phi_2 - \phi_3\theta_2 + \phi_4\phi_3\phi_2 - \phi_4\phi_3\theta_2)\epsilon_{v,1} + \\ &(\phi_1\phi_4\phi_3 - \phi_1\phi_4\theta_3 - \phi_2\phi_1\phi_4\theta_3)\epsilon_{v-1,2} + \\ &(1 + \phi_3 - \theta_3 + \phi_4\phi_3 - \phi_4\theta_3)\epsilon_{v,2} + \\ &(\phi_1\phi_4 - \phi_1\theta_4 + \phi_2\phi_1\phi_4 - \phi_2\phi_1\theta_4 - \phi_3\phi_2\phi_1\theta_4)\epsilon_{v-1,3} + \\ &(1 + \phi_4 - \theta_4)\epsilon_{v,3} \\ &+(\phi_1 - \theta_1 + \phi_2\phi_1 - \phi_2\theta_1 + \phi_3\phi_2\phi_1 - \phi_3\phi_2\theta_1 - \phi_4\phi_3\phi_2\theta_1)\epsilon_{v-1,4} + \epsilon_{v,4} \end{aligned} \tag{3.57}$$

The last four lines of the above expression show four independent moving average processes. Box and Jenkins(1976) and Rose(1977) proved that the sum of independent moving average processes each of order 1 is representable as a single moving average process also of order 1. Hence, expression (3.57) may be expressed as :

$$X_v - \phi^* X_{v-1} = \eta_v - \theta^* \eta_{v-1} \tag{3.58}$$

This equation represents a regular ARMA(1,1) process. It can be concluded that the annual series follows an ARMA(1,1) model with autoregressive parameter ϕ^* ,moving average parameter θ^*. η's are independent with expected value zero and variance σ_η^2, and $\phi^*=\phi_1\phi_2\phi_3\phi_4$.

b) Partial

Aggregating only the first two seasons from the same $PARMA_4(1,1)$:

$$X'_{v,1} = Y_{v,1} + Y_{v,2} \tag{3.59}$$

Replacing expressions (3.50)...(3.53) :

$$\begin{aligned} X'_{v,1} = {} & \phi_1 Y_{v-1,4} - \theta_1 \epsilon_{v-1,4} + \epsilon_{v,1} + \\ & \phi_2 Y_{v,1} - \theta_2 \epsilon_{v,1} + \epsilon_{v,2} \end{aligned} \tag{3.60}$$

Iterating three more times :

$$\begin{aligned} X'_{v,1} = {} & \phi_1\phi_2\phi_3\phi_4 \ (Y_{v-1,1} + Y_{v-1,2}) + \\ & (-\phi_1\phi_4\phi_3\theta_2) \ \epsilon_{v-1,1} + (1+\phi_2-\theta_2) \ \epsilon_{v,1} + \\ & (\phi_1\phi_4\phi_3-\phi_1\phi_4\theta_3-\phi_2\phi_1\phi_4\theta_3) \ \epsilon_{v-1,2} + \epsilon_{v,2} + \\ & (\phi_1\phi_4-\phi_1\theta_4+\phi_2\phi_1\phi_4-\phi_2\phi_1\theta_4) \ \epsilon_{v-1,3} + \\ & (\phi_1-\theta_1+\phi_2\phi_1-\phi_2\theta_1) \ \epsilon_{v-1,4} \end{aligned} \tag{3.61}$$

Analogously, aggregating the two last seasons :

$$X'_{v,2} = Y_{v,3} + Y_{v,4} \tag{3.62}$$

and

$$X'_{v,2} = \phi_1\phi_2\phi_3\phi_4 \ (Y_{v-1,3} + Y_{v-1,4})+ \qquad (3.63)$$
$$(-\phi_3\phi_2\phi_1\theta_4 \ \epsilon_{v-1,3} + (1+\phi_4-\theta_4) \ \epsilon_{v,3} +$$
$$(\phi_3\phi_2\phi_1-\phi_3\phi_2\theta_1-\phi_4\phi_3\phi_2\theta_1) \ \epsilon_{v-1,4} + \epsilon_{v,4} +$$
$$(\phi_3\phi_2-\phi_3\theta_2+\phi_4\phi_3\phi_2-\phi_4\phi_3\theta_2) \ \epsilon_{v-1,1} +$$
$$(\phi_3-\theta_3+\phi_4\phi_3-\phi_4\theta_3) \ \epsilon_{v-1,2}$$

Expressions (3.61) and (3.62) may be written as:

$$X'_{v1} - \phi^* \ X'_{v1-1} = \cap'_{v1} - \theta_1^* \ \cap'_{v1-1} \qquad (3.62)$$
$$X'_{v2} - \phi^* \ X'_{v2-1} = \cap'_{v2} - \theta_2^* \ \cap'_{v2-1} \qquad (3.63)$$

These equations show that the series obtained by partial aggregation of $PARMA_4(1,1)$ samples follows ARMA (1,1) models with the same autoregressive parameters as those for the annual series but different moving average parameters.

3.5.2.2.2 General Proof

In order to generalize the previous conclusions for any number of consecutive seasons of a $PARMA_w(1,1)$, consider initially the explicit expressions corresponding to a $PARMA_w(1,0)$ Model :

$$\begin{aligned} Y_{v,1} - \phi_1 \ Y_{v-1,w} &= \epsilon_{v,1} \\ Y_{v,2} - \phi_2 \ Y_{v,1} \quad &= \epsilon_{v,2} \\ \vdots \qquad \vdots \qquad &\quad \vdots \\ Yv,w - \phi_w \ Y_{v,w} \quad &= \epsilon_{v,w} \end{aligned} \qquad (3.66)$$

Aggregating m consecutive seasons :

$$
\begin{aligned}
X_v'' &= Y_{v,1} + Y_{v,2} + \ldots + Y_{v,m} \\
&= \phi_1 Y_{v-1,w} + \epsilon_{v,1} + \\
&\quad \phi_2 Y_{v,1} + \epsilon_{v,2} + \\
&\quad \ldots\ldots\ldots\ldots + \\
&\quad \ldots\ldots\ldots\ldots + \\
&\quad \phi_m Y_{v,m-1} + \epsilon_{v,m}
\end{aligned}
\tag{3.67}
$$

Replacing equations (3.66) into the last expression :

$$
\begin{aligned}
X_v'' &= \phi_1(\phi_w Y_{v-1,w-1} + \epsilon_{v-1,w}) + \epsilon_{v,1} + \\
&\quad \phi_2(\phi_1 Y_{v-1,w} + \epsilon_{v,1}) + \epsilon_{v,2} + \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots + \\
&\quad \phi_m(\phi_{m-1} Y_{v,m-2} + \epsilon_{v,m-1}) + \epsilon_{v,m}
\end{aligned}
\tag{3.68}
$$

After w iterations, the expression (3.69) results :

$$
\begin{aligned}
X_v'' = {} & \phi_1\phi_2\phi_3\cdots\phi_{w-1}\phi_w (Y_{v-1,1} + Y_{v-1,2} + \ldots + Y_{v-1,m}) + \\
& \phi_1\phi_w\phi_{w-1}\cdots\phi_4\phi_3\ \epsilon_{v-1,2} + \phi_2\phi_1\phi_w\cdots\phi_5\phi_4\ \epsilon_{v-1,3} + \\
& \ldots\ldots\ldots\ldots + \phi_m\phi_{m-1}\cdots\phi_1\cdots\phi_{m+2}\ \epsilon_{v-1,m+1} + \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
& \phi_1\phi_w\phi_{w-1}\ \epsilon_{v-1,w-2} + \phi_2\phi_1\phi_w\ \epsilon_{v-1,w-1} + \ldots\ldots + \\
& \ldots\ldots\ldots\ldots + \phi_w\phi_{w-1}\phi_{w-2}\ \epsilon_{v,m-3} + \\
& \phi_1\phi_w\ \epsilon_{v-1,w-1} + \phi_2\phi_1\ \epsilon_{v-1,w} + \ldots + \phi_m\phi_{m-1}\ \epsilon_{v,m-2} + \\
& \phi_1\ \epsilon_{v-1,w} + \phi_2\ \epsilon_{v,1} + \ldots + \phi_m\ \epsilon_{v,m-1} + \\
& \epsilon_{v,1} + \epsilon_{v,2} + \ldots + \epsilon_{v,m}
\end{aligned}
$$

This equation presents an autoregressive term and a series of independent moving average processes of order one. Thus, this expression is equivalent to that for an ARMA(1,1) model. This shows that the partial aggregation of $PARMA_w(1,0)$ can be fitted by an ARMA(1,1) model. The same conclusion can be drawn for the partial aggregation of $PARMA_w(1,1)$ because the introduction of seasonal moving average terms to the group of expressions (3.66) would add only moving average processes of order one to the right-hand side of equation (3.69).

3.5.2.2.3 Model for Partial Aggregation

The global seasonal series resulting from partial aggregation cannot be fitted (in the general case) by any of the models described in the literature review. The multiplicative Box and Jenkins model does not allow for periodicity in parameters, and the PARMA (1,1) does not correlate the data of the same season for consecutive years.

The model fitting all the seasons of the partially aggregated series would be one equivalent to different ARMA(1,1) fitting each season and able to be represented by a specific case of expression (3.4) where ϕ and θ are equal to zero and P=Q=1, such as :

$$(1 - \Phi_{1,\tau}\ B^{W})\ Y_{V,\tau} = (1 - \underline{\overline{\Theta}}_{1,\tau}\ B^{W})\ \epsilon_{V,\tau} \quad (3.70)$$

Expliciting this expression results in :

$$Y_{v,\tau} - \Phi_{1,\tau}\, Y_{v-1,\tau} = \epsilon_{v,\tau} - \underline{\bar{\theta}}_{1,\tau}\, \epsilon_{v-1,\tau} \qquad (3.71)$$

3.5.2.3 Parameters Relationship

It was shown previously that series obtained by partially aggregating PARMA (1,1) series follow a regular ARMA (1,1) model. Here, the objective is to derive a relationship between parameters for original and aggregated series.

Vecchia (1983) derived the following expressions for the parameters of the annual series, starting from the multivariate form of PARMA (1,1) model, (3.49):

$$X_v - \phi^* X_{v-1} = \cap_v - \theta^* \cap_{v-1} \qquad (3.72)$$

where $\phi^* = \phi_1 \phi_2 \ldots \phi_w$ (3.73)

and θ^* can be obtained by solving the matricial system :

$$\Gamma_0{}^* = \sigma_w{}^2(1+ \theta^{*2}) = \underline{1}'W_0\; D(0)\; W_0'\underline{1} + \underline{1}'W_1\; D(0)\; W_1'\underline{1}$$

$$\Gamma_1{}^* = -\sigma_w{}^2 = \underline{1}'W_1\; D(0)\; W_0'\underline{1} \qquad (3.74)$$

The solution for θ^* is

$$\theta^* = -R + \sqrt{R^2 - 1} \quad \text{if} \quad \Gamma_1{}^* \geq 0 \qquad (3.75)$$

and $\theta^* = -R - \sqrt{R^2 - 1}$ if $\Gamma_1{}^* < 0$

where $R = \Gamma_0^*/(2\Gamma_1^*)$ and $|R| \geq 1$ (3.76)

W_0 and W_1 are square matrices with elements :

$$W_0 = \begin{bmatrix} 0 & (i<j) \\ 1 & (i=j) \\ \phi(j+1)-\theta(j+1) & (i=j+1) \\ \pi(j+1,i)-\theta(j+1)\pi(j+2,i) & (i>j+1) \end{bmatrix}$$

(3.77)

$$W_1 = \begin{bmatrix} \pi(j+1,w+i) - \theta(j+1)\pi(j+2,w+i) & (i<j) \\ -\theta(j+1)\pi(j+2,w+j) & (i=j) \\ 0 & (i>j) \end{bmatrix}$$

$\underline{1}$ is the unit column vector.

D(0) is a diagonal matrix=diag$(\sigma^2(1),\sigma^2(2),\ldots,\sigma^2(w))$ and $\pi(l,m)= \phi(l)\phi(l+1)\ldots\phi(m)$.

Vecchia(1983) developed the above expressions for annual aggregation by matricial algebra technique, utilizing the PARMA(1,1) covariance structure. For partial aggregation, most of the derivation deals with the same covariance structure, since the model is the same in both cases. Parts of the sequence of derivation that remain the same will not be repeated. The key difference is related with the matricial procedure for aggregation.

Repeating the multivariate representation for the PARMA(1,1):

$$U(B)\ \underline{Y}_v = V(B)\ \underline{\epsilon}_v \qquad (3.78)$$

Applying the same operator $\det(U(B))\ U^{-1}(B)$ to both sides of the expression above:

$$\det(U(B))\ \underline{Y}_v = \det(U(B))\ U^{-1}(B)\ V(B)\ \underline{\epsilon}_v \qquad (3.79)$$

The inverse matrix may be expressed as:

$$U^{-1}(B) = U^a(B)/\det(U(B)) \qquad (3.80)$$

where $U^a(B)$ is the classical adjoint matrix of $U(B)$ and det represents the determinant operator. Substituting into (3.79) :

$$\det(U(B))\ \underline{Y}_v = U^a(B)\ V(B)\ \underline{\epsilon}_v \qquad (3.81)$$

Each component of $\underline{Y}_v$ is subjected to the same autoregressive operator. Hence, the components may be summed. Multiplying each term of (3.81) by the transposed of a vector $\underline{c}$ presenting values $\underline{1}$ corresponding to the seasons to be aggregated and values $\underline{0}$ otherwise, the expression containing the aggregated value X_v may be represented by :

$$\det U(B)\ X_v = \underline{c}'\ U^a\ (B)\ V(B)\ \underline{\epsilon}^v \qquad (3.82)$$

For annual aggregation, the vector $\underline{c}$ is the unit vector represented as $\underline{1}$ by Vecchia(1983). Throughout Vecchia's derivation, the matricial operations were made in such a way that this unit vector was not changed or introduced into other matrices. For partial aggregation, the same procedure could be followed utilizing the vector $\underline{c}$ instead of the unit vector. Thus, it can be concluded that the equations relating seasonal parameters of the PARMA model and the parameters for the ARMA fitting aggregated series can be obtained by replacing the unit vector by the vector $\underline{c}$ in the group of expressions (3.75).

The parameters ϕ^*, θ^* and $\sigma_w{}^2$ for the ARMA(1,1) fitting the series resulting from partial aggregation of PARMA(1,1) series can be computed by employing the following matricial equations :

$$\phi^* = \phi_1\phi_2\ldots\phi_w \qquad (3.83)$$

$$\sigma_w{}^2(1+\theta^{*2}) = \underline{c}'W_0\ D(0)\ W_0'\underline{c} + \underline{c}'W_1\ D(0)\ W_1'\underline{c} \qquad (3.84)$$

$$-\ \sigma_w{}^2 = \underline{c}'W_1\ D(0)\ W_0'\underline{c} \qquad (3.85)$$

where all variables were previously defined. The technique for the estimation of parameters utilizing the above expressions could be called "Matricial Method" .

The validity of these expressions was verified by data generation utilizing a computer code developed by the author.

3.6 Particular Cases

The general Multiplicative PARMA model can present a large number of parameters if the orders of parameters, p, q, P and Q, are all assumed greater than 1.

The total number of parameters may be calculated by :

$$NP = (p+q+P+Q+1)w \tag{3.86}$$

Hence, for practical applications it is necessary to make assumptions that could reduce the number of parameters of the model, keeping the most important relationships incorporated in the model.

Reasonable assumptions for dealing with hydrologic sequences are:

a) The most important parameters are those relating values for two consecutive seasons in the same year (lag 1) and those relating values for the same season in consecutive years (lag w), (Box and Jenkins,1976).

b) If, besides the above parameters, others are to be taken into account, these should be initially the parameters related with autoregression between data, for different lags (in increasing order), of seasons in the same year.

Another factor that influences the choice of a particular case is that the larger the number of parameters the more difficult the estimation procedure.

Most of the models utilized for fitting seasonal hydrologic time series are specific cases of the general model (3.4), including PARMA(p,q), PAR(p), Multiplicative ARIMA(p,0,q)x(P,0,Q)$_w$, and ARIMA(p,0,q) models.

From expression (3.4), fixing all parameters the same throughout the seasons results in the ARIMA(p,0,q)x(P,0,Q)$_w$. If besides that the parameters relating data for the same season in different years are not taken into account (P=Q=0), the ARIMA(p,0,q) model results.

If parameters are allowed to vary from season to season, but no parameter linking data for the same season in different years is considered, results the PARMA(p,q) model. The PARMA(1,1) model, defined previously, is a PARMA(p,q) presenting only one lag-one autoregressive parameter and one lag-one moving average parameter for each season. PAR(p) are PARMA(p,q) presenting only autoregressive parameters.

It is important to note that the seasonal series composed of aggregating partially PARMA(1,1) samples also can be fitted by a particular case of the general Multiplicative PARMA model. As shown before, these series can be expressed as follows :

$$(1-\Phi_{1,\tau}B^{w})\ Y_{v,\tau} = (1-\underline{\overline{\Theta}}_{1,\tau}B^{w})\epsilon_{v,\tau} \qquad (3.87)$$

This expression presents only parameters relating the same season for consecutive years and is equivalent to the equations (3.72) and (3.73) represents a Multiplicative PARMA(0,0)x(1,1) model.

3.7 Estimation of Parameters

Estimation techniques for ARIMA (p,0,q) and Multiplicative ARIMA $(p,0,q)x(P,0,Q)_w$ are described by Box and Jenkins(1976). Salas et al.(1980) described application of the ARMA model with periodic parameters. Salas et al.(1982) derived equations for exact-moment estimation of parameters for some PARMA (p,q) models and Vecchia(1983) developed an algorithm for approximate maximum likelihood estimation for PARMA(1,1) models. As shown previously, the moment equations for the Multiplicative PARMA models are difficult to solve even for simple particular cases. In this part of the dissertation, the search for an adequat estimation procedure for these models is described.

3.7.1 Selection of Estimation Procedure

Delleur and Kavvas(1978), Salas et al.(1980) and Bras and Rodriguez-Iturbe(1985) suggest that, given a set of possible models, parameters could first be estimated roughly and then refined in several iterative procedures.

Most of the techniques for this refinement rely on the evaluation of the sum of squares of residuals. Salas et al.(1980) cites the use of a modified steepest-gradient algorithm for minimizing the sum of squares of residuals in order to find the maximum likelihood estimate of parameters for a classical ARMA(p,q) model fitting an annual time series. Delleur and Kavvas (1978) also applied this technique to find maximum likelihood estimates of parameters for ARMA(1,1), ARIMA(1,1,1) and some Multiplicative ARIMA models for 15 basins in Indiana, Illinois and Kentucky.

As stated by Bras and Rodriguez-Iturbe (1985), a similar procedure could be applied to other seasonal models. For models presenting few parameters, graphics for the sum-of-squares surface or conditional sum-of-squares surfaces could be drawn for visualization of a suitable starting point. The Multiplicative PARMA(1,1)x(1,1) model, for example, presents 5 parameters for each season. These graphics would not help and moment estimates are difficult to derive, as shown before. It was verified that a good starting point could be, instead, the moment estimates for Multiplicative PARMA(1,0)x(0,0) or Multiplicative PARMA (1,1)x(0,0) model. These models are respectively equivalent to the PAR(1) and PARMA(1,1), for which exact moment estimates were derived by Salas el al.(1982). The utilization of one of these two points as initial has the advantage that if, for any reason, the optimal parameters are not achieved, it can at least be guaranteed that the derived Multiplicative PARMA(1,1)x(1,1) model, presenting smaller sum of squares of residuals, would show a better fitting to the data than that obtained by the initial

PAR(1) or PARMA (1,1) model.

3.7.2 Search Technique

A number of search methods for minimization of a function are described in the literature. The objective function for the Multiplicative PARMA(1,1)x(1,1) is :

$$\text{Minimize} \quad \sum_{v=1}^{n} \sum_{\tau=1}^{w} \epsilon^2_{v,\tau} \tag{3.88}$$

where n symbolizes the number of years, w the number of seasons and

$$\epsilon_{v,\tau} = Y_{v,\tau} - \phi_{1,\tau} Y_{v,\tau-1} - \Phi_{1,\tau} Y_{v-1,\tau} + \phi_{1,\tau} \cdot \Phi_{1,\tau} Y_{v-1,\tau-1}$$
$$+ \theta_{1,\tau} \epsilon_{v,\tau-1} + \underline{\overline{\theta}}_{1,\tau} \epsilon_{v-1,\tau} - \theta_{1,\tau} \cdot \underline{\overline{\theta}}_{1,\tau} \epsilon_{v-1,\tau-1}$$

Complexity of the objective function indicates that a non-derivative method should be employed in the search (Gill et al.(1981)).

The Powell algorithm, that is an expanded variation of the univariate gradient search, is the most used method for optimization without derivatives and is described by, among many other authors, Powell (1964), Brent (1973) and Gill et al. (1981). Applications of this procedure to water resources problems are shown by Wurbs(1978) and Fontane(1982). This algorithm is commonly employed for maximum-likelihood estimation of parameters for the ARMA(1,1) model. Powell (1964) showed that his algorithm is more efficient than the Rosenbrock's (1960) that is also frequently applied. The Powell method, that was chosen as the search procedure for this research, can be summarized as follows for the case of n-dimensional search :

(1) Starting with the best previous value position $(\underline{X}_o)^k$ and a series of linearly independent directions of search $(\underline{M}_1)^k$, $(\underline{M}_2)^k, \ldots, (\underline{M}_n)^k$, begin the search by finding the position of the optimum along the line passing through $(\underline{X}_0)^k$ which is parallel to $(\underline{M}_1)^k$. At this optimum point $(\underline{X}_1)^k$, begin a second search from this new point in the $(\underline{M}_2)^k$ direction and continue this procedure until all n search directions have been explored.

(2) Find the particular point $(\underline{X}_m)^k$ for which the greatest improvement of the objective function over its previous value is realized. The point $(\underline{X}_m)^k$, therefore, yields the largest change D of any n moves, where $D = |F(\underline{X}_m)^k - F(\underline{X}_{m-1})^k|$. Also determine the vector $\underline{\mu} = (\underline{X}_n)^k - (\underline{X}_0)^k$

(3) Determine $y[(2\underline{X}_n)^k - (\underline{X}_0)^k]$

(4) If $(F_t)^k \geq (F_0)^k$ and/or

$$[(F_0)^k - 2(F_n)^k + (F_t)^k].[(F_0)^k - (F_n)^k - D]^2 \geq \frac{D[(y_0)^k - (y_t)^k]^2}{2}$$

then μ is not a good direction(no progress and/or function in this region is a valley), and the search is started again at the last point, using the same directions :

$(\underline{X}_0)^{k+1} = (\underline{X}_n)^k$

and $(\underline{M})^{k+1} = (\underline{M})^k$ for i=1,2,...,n. Step (1) is then repeated.

If neither of these inequalities is satisfied, search along direction μ until the minimum is found. This point is defined $(\underline{X}_0)^{k+1}$ and new search directions for the k+1 stage are $(\underline{M}_i)^{k+1} = (\underline{M}_i)^k$ for $i=1,2,...,m-1$; $(\underline{M}_i)^{k+1} = (\underline{M}_{i+1})^k$ for $i=m,...n-1$ and $(\underline{M}_n)^{k+1} = \mu$.

Then, repeat the entire process, starting with the step (1). The stop criterion consists of verifying if the difference between values of the decision variables in the two iterations is less than the specified limits,

$$| (\underline{X}_i)^k - (\underline{X}_i)^{k-1} | < E(i) \text{ for } i = 1,2,...,n.$$

Another parameter utilized by the algorithm is the maximum step size multiplier in the single variable searches. Each variable is not increased by more than the product of this multiplier by the corresponding convergence limit. The diagram for the algorithm is shown in Figure 3.1 (Kuester and Mize,1973).

3.7.3 Application to Synthetic Series

Applicability of the Powell algorithm for search of

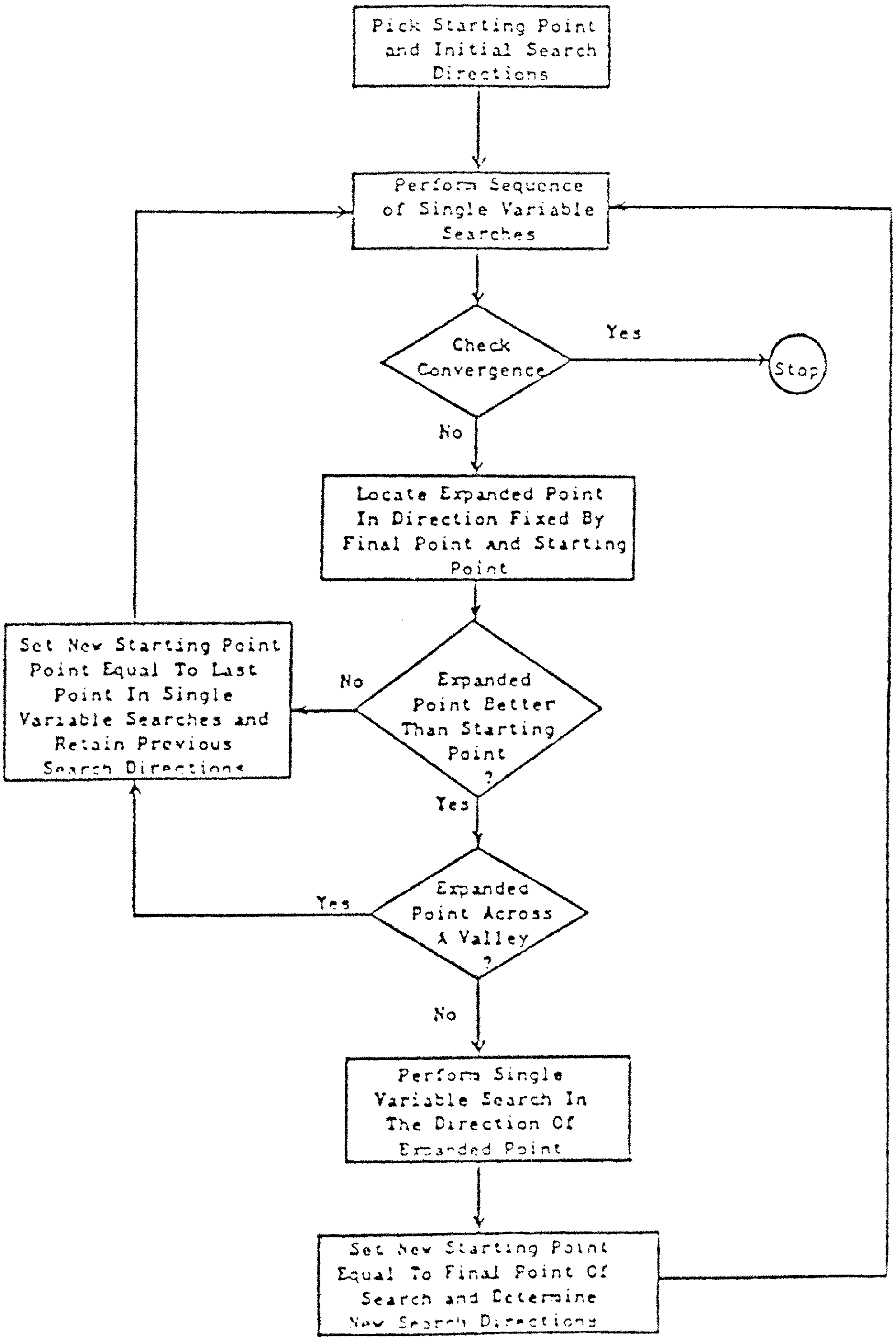

Figure 3.1 - Logical Diagram of the Powell Method

the periodic parameters for seasonal models was verified through data generation. PARMA(1,1) series were simulated and estimated to check if the developed search technique could present results comparable with those obtained by exact method of moments that is the most commonly used. Multiplicative PARMA(1,0)x(1,0) and and Multiplicative PARMA(1,1)x(1,1) were also generated to analyze the applicability of the proposed procedure for the estimation of parameters for members of the class of models previously developed in this chapter. Large samples allow estimations closer to the true value of the parameters because they present smaller variability of estimation. Here, the objective is to find out if the method gives estimates close enough to the true values, and large and moderately large samples were utilized when estimation for individual samples were analyzed. Some statistical properties of the residuals were also computed to check if they are compatible with the assumptions of the model.

3.7.3.1 Individual Large Samples

Sample 1 - $PARMA_4(1,1)$ -

Parameters : ϕ_1= 1.2, ϕ_2= 0.7, ϕ_3= 0.9, ϕ_4= 0.8

θ_1=-0.2, θ_2=-0.4, θ_3= 0.3, θ_4= 0.5

σ_1= 1.0 σ_2= 2.5, σ_3= 1.5, σ_4= 2.0

Initial parameters :

$\phi_1=\phi_2=\phi_3=\phi_4$= 0.5,

$\theta_1=\theta_2=\theta_3=\theta_4$= 0.1

Accuracy for each parameter : E(i)= 0.1, i=1,...8

Sample size : 300 years

Sum of squares of residuals for initial values : 84.33

Evolution of the search process :

Iterations :		Total number of trials :	Sum of squares :
	1	41	44.43
	2	99	43.77
	3	142	43.75
	4	179	43.73
	5	211	43.72
	6	245	43.67
	7	280	43.66
	8	306	43.65
	9	337	43.63

Table 3.1 - Estimated parameters and bias for Sample 1

ϕ			θ			σ		
Value	Estim.	Bias	Value	Estim.	Bias	Value	Estim.	Bias
1.20	1.18	0.01	-0.2	-0.26	0.06	1.0	0.98	0.02
0.70	0.72	0.02	-0.4	-0.35	0.05	2.5	2.53	0.03
0.90	0.92	0.02	0.3	0.33	0.03	1.5	1.48	0.02
0.80	0.79	0.01	0.5	0.45	0.05	2.0	2.00	0.00

Table 3.2 - Statistics of residuals for Sample 1

Season	Mean	Skewness	$r_{1,\tau}$	$r_{2,\tau}$	$r_{3,\tau}$	$r_{4,\tau}$
1	0.08	0.18	0.04	-0.14	0.03	-0.02
2	0.18	0.11	0.02	0.07	-0.01	0.03
3	0.11	-0.12	-0.03	-0.10	0.12	0.09
4	-0.09	0.10	0.01	-0.04	-0.02	0.01

Sample 2 - Same parameters, initial values, sample size and accuracy :

Sum of squares for initial values : 81.14

Evolution of the search process :

Iterations :	1	Total :	43	Sum of squares :	39.26
	2	number	111		38.77
	3	of	153		38.60
	4	trials	201		38.60

Table 3.3 - Estimated parameters and bias for Sample 2

ϕ			θ			σ		
Value	Estim.	Bias	Value	Estim.	Bias	Value	Estim.	Bias
1.20	1.16	0.04	-0.20	-0.31	0.11	1.00	0.99	0.01
0.70	0.71	0.01	-0.40	-0.30	0.05	2.50	2.59	0.09
0.90	0.89	0.02	0.30	0.27	0.03	1.50	1.60	0.10
0.80	0.84	0.04	0.50	0.55	0.05	2.00	2.15	0.15

Table 3.4 - Statistics of residuals for Sample 2:

Season	Mean	Skewness	$r_{1,\tau}$	$r_{2,\tau}$	$r_{3,\tau}$	$r_{4,\tau}$
1	-0.04	0.03	0.00	0.06	-0.02	-0.04
2	-0.16	-0.08	-0.06	-0.02	-0.03	-0.05
3	0.15	-0.12	0.01	-0.03	-0.07	0.06
4	-0.07	0.04	0.07	0.04	0.06	-0.07

Sample 3 - Multiplicative PARMA$(1,1)\times(1,1)_4$

Parameters: $\phi_1= 1.2,\ \phi_2= 0.7,\ \phi_3= 0.9,\ \phi_4= 0.8$

$\theta_1=-0.2,\ \theta_2=-0.4,\ \theta_3= 0.3,\ \theta_4= 0.5$

$\Phi_1= 0.3,\ \Phi_2= 0.5,\ \Phi_3= 0.4,\ \Phi_4= 0.2$

$\underline{\overline{\theta}}1= 0.5,\ \underline{\overline{\theta}}_2=-0.3,\ \underline{\overline{\theta}}_3= 0.1,\ \underline{\overline{\theta}}_4= 0.4$

$\sigma_1= 1.0,\ \sigma_2= 2.5,\ \sigma_3= 1.5,\ \sigma_4= 2.0$

Initial parameters :

$$\phi_1=\phi_2=\phi_3=\phi_4=\Phi_1=\Phi_2=\Phi_3=\Phi_4= 0.5$$

$$\theta_1=\theta_2=\theta_3=\theta_4=\underline{\overline{\theta}}_1=\underline{\overline{\theta}}_2=\underline{\overline{\theta}}_3=\underline{\overline{\theta}}_4= 0.1$$

Accuracy for each parameter : $E(i)= 0.1$, $i= 1,\ldots,16$

Sample size : 300 years

Sum of squares for initial values : 88.58

Evolution of search process :

Iterations :		Total number of trials :	Sum of squares :
	1	114	43.52
	2	162	42.32
	3	235	42.12
	4	320	42.11
	5	390	42.11

Table 3.5 - Estimated parameters and bias for Sample 3

ϕ			θ			Φ		
Value	Estim.	Bias	Value	Estim.	Bias	Value	Estim.	Bias
1.20	1.20	0.00	-0.20	-0.21	0.01	0.30	0.13	0.27
0.70	0.67	0.03	-0.40	-0.40	0.00	0.50	0.52	0.02
0.90	0.92	0.02	0.30	0.30	0.00	0.40	0.40	0.00
0.80	0.79	0.01	0.50	0.51	0.01	0.20	0.03	0.17
$\underline{\overline{\theta}}$			σ					
0.50	0.08	0.42	1.00	1.00	0.00			
-0.30	-0.27	0.03	2.50	2.57	0.07			
0.10	0.08	0.02	1.50	1.52	0.02			
0.40	0.13	0.27	2.00	2.02	0.02			

Table 3.6 - Statistics of residuals for Sample 3

Season	Mean	Skewness	$r_{1,\tau}$	$r_{2,\tau}$	$r_{3,\tau}$	$r_{4,\tau}$
1	0.05	0.05	-0.04	0.01	0.04	0.04
2	0.13	0.06	-0.03	0.01	-0.01	0.00
3	0.03	-0.01	0.03	0.05	0.05	-0.02
4	0.00	0.15	-0.03	0.00	0.03	-0.01

Sample 4 - Same parameters, initial values, sample size and accuracy:

Sum of squares for initial values : 80.16

Evolution of the search process :

Iterations :		Total number of trials :	Sum of squares :
	1	153	54.10
	2	228	47.64
	3	292	44.19
	4	365	44.01
	5	430	43.88
	6	506	43.88

Table 3.7 - Estimated parameters and bias for Sample 4

ϕ			θ			Φ		
Value	Estim.	Bias	Value	Estim.	Bias	Value	Estim.	Bias
1.20	1.24	0.04	-0.20	-0.08	0.12	0.30	0.33	0.03
0.70	0.66	0.04	-0.40	-0.39	0.01	0.50	0.53	0.03
0.90	0.91	0.01	0.30	0.32	0.02	0.40	0.55	0.15
0.80	0.80	0.00	0.50	0.47	0.03	0.20	0.11	0.09
$\underline{\bar{\theta}}$			σ					
0.50	-0.30	0.80	1.00	0.99	0.01			
-0.30	-0.30	0.00	2.50	2.58	0.08			
0.10	0.22	0.12	1.50	1.58	0.08			
0.40	0.28	0.12	2.00	2.13	0.13			

Table 3.8 - Statistics of residuals for Sample 4

Season	Mean	Skewness	$r_{1,\tau}$	$r_{2,\tau}$	$r_{3,\tau}$	$r_{4,\tau}$
1	0.15	0.24	0.06	0.06	-0.08	-0.04
2	0.17	0.07	-0.20	0.08	-0.08	-0.04
3	0.11	-0.12	0.04	-0.02	0.14	-0.03
4	-0.08	0.10	-0.02	0.00	0.09	-0.01

3.7.3.2 PARMA(1,1) Estimation

Simulations of $PARMA_4$ (1,1) samples were conducted and parameters were estimated by the developed least-square technique applying the Powell algorithm, and by the Exact Method of Moments, to verify if the procedures give similar estimates. Parameters estimated from the logarithm of flows of 90 years of discharge data for the St. Lawrence River near Ogdensburg (1861-1950),by the method of moments, were utilized for generation. The objective of the utilization of the historic data estimates for data generation is only to make comparisons based on real data.

The starting point for the least-square method was the origin, that is, all parameters starting equal to zero. The accuracy of each parameter was assumed 0.001, and different sample sizes were used for generation: ninety (historic sample size), fifty and twenty five years. For each sample size, one hundred samples were generated. Tables 3.9., 3.10 and 3.11 show historic parameters, average estimated parameters, bias and standard deviations for estimations by the two procedures.

Bias, standard deviations and Root Mean Square Errors, (RMSE), computed through the two techniques are very close for the 90-year and 50-year samples. For 25-year samples the least-squares method gives smaller bias but higher standard deviations than those from the method of moments, resulting in almost equal Root Mean Square Errors by the two methods.

These results suggest that the proposed method can be an alternative for estimation of parameters for PARMA(1,1) models. However, it must be stated that the computer time for estimation through the least-squares technique is higher than that required by the method of moments. However, if computer time is not considered as an important factor, an alternative for estimation of parameters for PARMA(1,1) models could be the utilization of the method of moments estimates as starting point for the least-squares technique.

This procedure is called " refinement of estimation" by Bras and Rodriguez-Iturbe(1985).

Table 3.9 presents results for estimations from one hundred simulations of 90-year samples. Tables 3.10 and 3.11 present results for estimations from 50-year and 25-year samples, respectively.

Table 3.9 - One hundred 90-year samples - $PARMA_4(1,1)$

Histor.		MOM				MLS			
τ	ϕ_τ	Estim.	Bias	STDV	RMSE	Estim.	Bias	STDV	RMSE
1	0.892	0.880	0.012	0.051	0.052	0.887	0.005	0.059	0.059
2	0.693	0.724	0.031	0.069	0.076	0.715	0.022	0.074	0.077
3	1.023	0.899	0.124	0.139	0.186	0.909	0.114	0.142	0.181
4	0.881	0.950	0.069	0.045	0.083	0.943	0.061	0.057	0.083
τ	θ_τ								
1	-0.337	-0.356	0.019	0.081	0.083	-0.356	0.019	0.106	0.108
2	-0.613	-0.572	0.041	0.172	0.176	-0.679	0.066	0.194	0.205
3	0.688	0.571	0.177	0.195	0.227	0.604	0.083	0.195	0.212
4	-0.169	-0.169	0.000	0.080	0.080	-0.200	0.031	0.101	0.106
τ	σ_τ								
1	0.030	0.030	0.000	0.000	0.000	0.028	0.002	0.000	0.002
2	0.062	0.060	0.002	0.000	0.002	0.059	0.003	0.000	0.003
3	0.056	0.049	0.007	0.000	0.007	0.049	0.007	0.000	0.007
4	0.036	0.034	0.002	0.000	0.002	0.034	0.002	0.000	0.002

Table 3.10 - One hundred 50-year samples - $PARMA_4(1,1)$

Histor.		MOM				MLS			
τ	ϕ_τ	Estim.	Bias	STDV	RMSE	Estim.	Bias	STDV	RMSE
1	0.892	0.866	0.026	0.050	0.056	0.881	0.011	0.056	0.057
2	0.693	0.725	0.032	0.094	0.099	0.714	0.021	0.089	0.091
3	1.023	0.884	0.139	0.123	0.186	0.899	0.124	0.134	0.183
4	0.881	0.957	0.076	0.070	0.103	0.945	0.064	0.062	0.089
τ	θ_τ								
1	-0.337	-0.367	0.030	0.138	0.141	-0.351	0.014	0.134	0.135
2	-0.613	-0.525	0.088	0.334	0.345	-0.652	0.039	0.267	0.269
3	0.688	0.554	0.134	0.189	0.232	0.574	0.114	0.189	0.220
4	-0.169	-0.176	0.007	0.150	0.150	-0.196	0.027	0.124	0.127
τ	σ_τ								
1	0.030	0.033	0.003	0.000	0.003	0.028	0.002	0.000	0.002
2	0.062	0.059	0.003	0.000	0.003	0.058	0.004	0.000	0.004
3	0.056	0.049	0.007	0.000	0.007	0.048	0.008	0.000	0.008
4	0.036	0.035	0.001	0.000	0.001	0.034	0.002	0.000	0.002

Table 3.11 - One hundred 25-year samples - $PARMA_4(1,1)$

Histor.		MOM				MOLS			
τ	ϕ_τ	Estim.	Bias	STDV	RMSE	Estim.	Bias	STDV	RMSE
1	0.892	0.826	0.066	0.086	0.108	0.869	0.023	0.090	0.093
2	0.693	0.731	0.038	0.150	0.154	0.714	0.021	0.147	0.148
3	1.023	0.838	0.185	0.192	0.266	0.856	0.167	0.208	0.266
4	0.881	0.921	0.040	0.099	0.107	0.926	0.045	0.108	0.117
τ	θ_τ								
1	-0.337	-0.439	0.102	0.295	0.312	-0.389	0.052	0.206	0.212
2	-0.613	-0.532	0.081	0.462	0.469	-0.644	0.031	0.467	0.468
3	0.688	0.570	0.118	0.247	0.274	0.569	0.119	0.255	0.281
4	-0.169	-0.198	0.029	0.245	0.247	-0.178	0.009	0.247	0.247
τ	σ_τ								
1	0.030	0.035	0.005	0.000	0.005	0.027	0.003	0.000	0.003
2	0.062	0.058	0.004	0.000	0.004	0.057	0.005	0.000	0.005
3	0.056	0.047	0.009	0.000	0.009	0.046	0.010	0.000	0.010
4	0.036	0.030	0.006	0.000	0.006	0.030	0.006	0.000	0.006

3.7.3.3 Multiplicative PARMA(1,0)x(1,0) Estimation

In order to verify if the least-squares technique can estimate parameters other than ϕs and θs efficientely, presented by the PARMA(1,1) model, samples for a Multiplicative PARMA(1,0)x(1,0) were generated and estimated. It was done by checking if the biases, standard deviations and Root Mean Square errors for estimations of Φ parameters (not presented by the PARMA(1,1) model) present values compatible with those computed for this latter model. The Multiplicative PARMA(1,0)X(1,0) was chosen because it presents the same number of parameters as the PARMA(1,1) does.

Tables 3.13, 3.14 and 3.15 show the computed values for 100 generated samples with, respectively, 100, 50 and 25 years.

For all estimations the accuracy E(i)= 0.0001, for i=1,...,8 was adopted. The origin was always utilized as the starting point for the Powell algorithm .

It can be seen in the tables that the proposed technique is able, in this particular case, to compute estimations presenting low biases, standard deviations and root mean square errors, even lower than those obtained previously by the same technique for some PARMA(1,1) samples (Tables 3.9 through 3.12).

The average user CPU time for each estimation on the Cyber 205 Computer was about 12 seconds.

Table 3.13 - One hundred 100-year samples
Multiplicative PARMA(1,0)x(1,0)

Param.		MOLS			
τ	ϕ_τ	Estim.	Bias	STDV	RMSE
1	0.850	0.858	0.008	0.100	0.100
2	0.650	0.649	0.001	0.042	0.042
3	0.650	0.653	0.003	0.059	0.059
4	0.850	0.853	0.003	0.109	0.109
τ	Φ_τ				
1	0.400	0.368	0.032	0.087	0.093
2	0.200	0.178	0.022	0.100	0.102
3	0.200	0.195	0.005	0.085	0.085
4	0.400	0.375	0.025	0.100	0.011
τ	σ_τ				
1	0.150	0.149	0.001	0.003	0.003
2	0.100	0.097	0.003	0.001	0.003
3	0.100	0.099	0.001	0.001	0.001
4	0.150	0.146	0.004	0.003	0.005

Table 3.14 - One hundred 50-year samples
Multiplicative PARMA(1,0)x(1,0)

Param.		MOLS			
τ	ϕ_τ	Estim.	Bias	STDV	RMSE
1	0.850	0.863	0.013	0.141	0.142
2	0.650	0.641	0.009	0.149	0.150
3	0.650	0.656	0.006	0.010	0.110
4	0.850	0.847	0.003	0.144	0.144
τ	Φ_τ				
1	0.400	0.336	0.064	0.149	0.162
2	0.200	0.154	0.046	0.157	0.160
3	0.200	0.188	0.012	0.115	0.116
4	0.400	0.376	0.024	0.152	0.154
τ	σ_τ				
1	0.150	0.142	0.008	0.005	0.009
2	0.100	0.095	0.005	0.002	0.005
3	0.100	0.095	0.005	0.002	0.005
4	0.150	0.142	0.008	0.004	0.009

Table 3.15 - One hundred 25-year samples
Multiplicative PARMA(1,0)x(1,0)

Param.		MOLS			
τ	ϕ_τ	Estim.	Bias	STDV	RMSE
1	0.850	0.825	0.025	0.209	0.211
2	0.650	0.649	0.001	0.109	0.109
3	0.650	0.653	0.003	0.171	0.171
4	0.850	0.899	0.099	0.218	0.239
τ	Φ_τ				
1	0.400	0.299	0.101	0.209	0.232
2	0.200	0.137	0.063	0.207	0.216
3	0.200	0.166	0.034	0.148	0.152
4	0.400	0.323	0.077	0.241	0.253
τ	σ_τ				
1	0.150	0.149	0.001	0.003	0.003
2	0.100	0.097	0.003	0.001	0.003
3	0.100	0.099	0.001	0.001	0.001
4	0.150	0.146	0.004	0.003	0.005

3.7.4 Initial Values and Accuracy

Two kinds of variables that one needs to choose and that are important for performance of the Powell algorithm are the acceptable accuracies for parameters, E(i), and the starting vector of parameters, $(\underline{X}_0)^k$. To study the effects of different accuracies and starting points, one hundred Multiplicative PARMA(1,1)x(1,1) samples with 100 years each were generated. Parameters for these samples were estimated by using the developed procedure for three different accuracies: 0.1, 0.01 and 0.001. For accuracy 0.01, two different starting points were chosen : the origin, that is, all parameters assumed zero; and the method-of-moments estimates of the PARMA(1,1) model for seasonal parameters ϕs and θs and zero for seasonal parameters Φs and $\underline{\bar{\theta}}$s. Average parameters, standard deviation of estimates for each parameter and average statistical characteristics for residuals were computed. Estimation for one of the samples (called here Sample A) is described with more details for better understanding of the search of parameters.

Parameters :

$\phi_1= 0.9 \quad \phi_2= 0.6 \quad \phi_3= 0.6 \quad \phi_4= 0.9$

$\theta_1= 0.4 \quad \theta_2= 0.2 \quad \theta_3= 0.2 \quad \theta_4= 0.4$

$\Phi_1= 0.4 \quad \theta_2= 0.2 \quad \theta_3= 0.2 \quad \theta_4= 0.4$

$\underline{\bar{\theta}}_1= 0.25 \quad \underline{\bar{\theta}}_2= 0.05 \quad \underline{\bar{\theta}}_3= 0.05 \quad \underline{\bar{\theta}}_4= 0.25$

$\sigma_1= 0.1 \quad \sigma_2= 0.2 \quad \sigma_3= 0.2 \quad \sigma_4= 0.1$

Initial values :

PARMA(1,1) parameters for the sample, obtained through the exact method of moments:

$\phi_1= 0.99, \ \phi_2= 0.67, \ \phi_3= 0.66, \ \phi_4= 0.99$

$\theta_1= 0.55, \ \theta_2= 0.27, \ \theta_3= 0.07, \ \phi_4= 0.51$

Sample size : 100 years

Sum of squares for initial values : 10.528

a) Sample A , E(i)= 0.1, i=1,...,16

Iterations :	1	Total :	50	Sum of squares: 10.177
	3	number	121	10.084
	5	of	189	10.062
	7	trials	268	9.968
	9		344	9.944
	11		413	9.938
	13		477	9.933

Table 3.16-Estimated parameters and bias for Sample A
PARMA(1,1)x(1,1) - E(i)=0.1
Initial point - PARMA(1,1) moments estimates

Value	Estim.	Bias	Value	Estim.	Bias	Value	Estim.	Bias
	ϕ			θ			Φ	
0.90	1.01	0.11	0.40	0.29	0.21	0.40	0.61	0.21
0.60	0.52	0.08	0.20	0.24	0.04	0.20	0.08	0.12
0.60	0.40	0.20	0.20	-0.19	0.39	0.20	0.76	0.56
0.90	0.96	0.06	0.40	0.49	0.09	0.40	0.18	0.22
	$\underline{\bar{\theta}}$			σ				
0.25	0.43	0.18	0.10	0.10	0.00			
0.05	-0.25	0.30	0.20	0.18	0.02			
0.05	0.59	0.54	0.20	0.22	0.02			
0.25	0.22	0.03	0.10	0.10	0.00			

a.1)Average estimated parameters, standard deviations, bias and root mean square errors for 100 samples generated under the same conditions and with the same accuracy of estimation are shown in Table 3.17.

Table 3.17 - One hundred 100-year samples - E(i)= 0.1
MOLS - Multiplicative PARMA(1,1)x(1,1)
Initial Point : PARMA(1,1) moments estimates

ϕ_T	Est.	Bias	S.D.	RMSE	θ_T	Est.	Bias	S.D.	RMSE
0.90	0.89	0.01	0.09	0.10	0.40	0.43	0.03	0.16	0.16
0.60	0.61	0.01	0.19	0.20	0.20	0.28	0.08	0.35	0.36
0.60	0.59	0.01	0.33	0.34	0.20	0.18	0.02	0.36	0.37
0.90	0.87	0.03	0.17	0.19	0.40	0.40	0.00	0.18	0.18
Φ_T	Est	Bias	S.D.	RMSE	$\underline{\bar{\theta}}_T$	Est.	Bias	S.D.	RMSE
0.40	0.32	0.08	0.40	0.41	0.25	0.12	0.13	0.42	0.44
0.20	0.20	0.00	0.36	0.36	0.05	0.01	0.04	0.35	0.36
0.20	0.22	0.02	0.35	0.35	0.05	0.05	0.00	0.38	0.38
0.40	0.27	0.13	0.43	0.45	0.25	0.09	0.16	0.43	0.46
σ_T	Est.	Bias	S.D.	RMSE					
0.10	0.10	0.00	0.00	0.00					
0.20	0.19	0.01	0.01	0.01					
0.20	0.19	0.01	0.01	0.01					
0.10	0.10	0.00	0.00	0.00					

Table 3.18 -Average statistics of residuals for <u>Sample A</u>
One hundred 100-year samples - E(i)=0.1
Starting point: PARMA(1,1) moments estimates

Season	Mean	Skewness	$r_{1,\tau}$	$r_{2,\tau}$	$r_{3,\tau}$	$r_{4,\tau}$
1	0.00	-0.01	0.02	0.00	0.01	-0.01
2	0.00	-0.04	0.01	-0.01	0.00	-0.01
3	0.00	-0.02	-0.00	0.00	0.01	-0.02
4	0.00	-0.02	0.01	-0.02	0.00	0.00

b) <u>Sample A</u>, E(i) = 0.01, i=1,...,16

Iterations :	Trials :	Sum of squares :
1	66	10.177
3	147	10.080
5	236	10.034
7	314	9.999
9	394	9.995
11	465	9.934
13	534	9.931
15	601	9.930
17	667	9.930
19	737	9.929
21	804	9.929
23	872	9.929
24	904	9.929

Table 3.19 - Estimated parameters and bias for <u>Sample A</u>
PARMA(1,1)x(1,1) - E(i)=0.01
Initial point - PARMA(1,1) moments estimates

Value	Estim.	Bias	Value	Estim.	Bias	Value	Estim.	Bias
	ϕ			θ			Φ	
0.90	1.02	0.12	0.40	0.59	0.19	0.40	0.57	0.17
0.60	0.50	0.10	0.20	0.21	0.01	0.20	0.12	0.08
0.60	0.45	0.15	0.20	-0.20	0.40	0.20	0.80	0.60
0.90	0.94	0.04	0.40	0.47	0.07	0.40	0.27	0.13
	$\bar{\Theta}$			σ				
0.25	0.40	0.15	0.10	0.10	0.00			
0.05	-0.23	0.28	0.20	0.18	0.02			
0.05	0.64	0.71	0.20	0.21	0.01			
0.25	0.10	0.15	0.10	0.10	0.00			

b.1) Average estimated parameters, standard deviations, bias and root mean square errors for 100 samples generated under the same conditions and with the same accuracy of estimation, E(i)= 0.01, are shown in Table 3.20.

Table 3.20 - One hundred 100-year samples - E(i)= 0.01
MOLS - Multiplicative PARMA(1,1)x(1,1)
Initial Point : PARMA(1,1) moments estimates

ϕ_τ	Est.	Bias	S.D.	RMSE	θ_τ	Est.	Bias	S.D.	RMSE
0.90	0.89	0.01	0.10	0.10	0.40	0.42	0.02	0.16	0.17
0.60	0.62	0.02	0.21	0.21	0.20	0.29	0.09	0.38	0.39
0.60	0.59	0.01	0.35	0.35	0.20	0.29	0.09	0.36	0.37
0.90	0.87	0.03	0.16	0.18	0.40	0.38	0.02	0.17	0.18
Φ_τ	Est	Bias	S.D.	RMSE	$\underline{\overline{\theta}}_\tau$	Est.	Bias	S.D.	RMSE
0.40	0.34	0.06	0.42	0.42	0.25	0.14	0.11	0.45	0.46
0.20	0.20	0.19	0.01	0.43	0.05	0.00	0.05	0.42	0.42
0.20	0.23	0.03	0.42	0.42	0.05	0.02	0.03	0.43	0.43
0.40	0.29	0.11	0.46	0.47	0.25	0.11	0.14	0.47	0.49
σ_τ	Est.	Bias	S.D.	RMSE					
0.10	0.10	0.00	0.00	0.00					
0.20	0.19	0.01	0.01	0.01					
0.20	0.19	0.01	0.01	0.01					
0.10	0.10	0.00	0.00	0.00					

Table 3.21 - Average statistics of residuals for one hundred 100-year samples - E(i)= 0.01
Starting point: PARMA(1,1) moments estimates

Season	Mean	Skewness	$r_{1,\tau}$	$r_{2,\tau}$	$r_{3,\tau}$	$r_{4,\tau}$
1	0.00	-0.02	0.02	0.00	0.02	0.00
2	0.00	-0.03	0.01	-0.01	-0.01	-0.00
3	0.00	0.02	0.00	0.00	0.01	-0.01
4	0.00	-0.03	0.01	0.02	-0.02	0.01

c) <u>Sample A</u> - Same accuracy as in item <u>b</u> but starting from a different point. All starting parameters equal to zero :

Sum of squares for initial values : 20.079

Iterations :	Trials :	Sum of squares :
1	326	10.403
3	419	10.123
5	500	10.041
7	585	9.981
9	656	9.956
11	729	9.944
13	799	9.939
15	868	9.937
17	940	9.934
19	1008	9.932
21	1076	9.931
23	1143	9.929
25	1212	9.929
27	1273	9.929

Table 3.22 - Estimated parameters and bias for <u>Sample A</u>
Multiplicative PARMA(1,1)x(1,1)
Initial point - origin

Value	Estim.	Bias	Value	Estim.	Bias	Value	Estim.	Bias
	ϕ			θ			Φ	
0.90	1.02	0.12	0.40	0.59	0.19	0.40	0.57	0.17
0.60	0.50	0.10	0.20	0.21	0.01	0.20	0.23	0.03
0.60	0.57	0.03	0.20	-0.16	0.36	0.20	0.80	0.60
0.90	0.94	0.04	0.40	0.47	0.07	0.40	0.27	0.13
	$\underline{\bar{\Theta}}$			σ				
0.25	0.40	0.15	0.10	0.10	0.00			
0.05	-0.23	0.28	0.20	0.18	0.02			
0.05	0.64	0.59	0.20	0.22	0.02			
0.25	0.10	0.15	0.10	0.10	0.00			

These estimated parameters are almost exactly equal to those obtained, for the same accuracy, using as starting point the PARMA(1,1) estimates by the method of moments (case <u>b</u>). The computed averages of parameters and averages of statistics for the residuals for the one hundred samples were also equal to those computed using the method of moments estimates (case b1). However, the utilization of PARMA(1,1) estimates resulted in fewer iterations and less computer time for searching the parameters.

Estimations utilizing accuracy E(i)= 0.01 generally presented smaller bias than the estimations for accuracy 0.1. Estimations utilizing accuracy for parameters equal to 0.001 were also realized, resulting in parameters and statistics very close to those obtained for accuracy 0.01. This indicates that the last is sufficient for these particular samples. In every case the seasonal autoregressive parameters ϕ and seasonal standard deviations σ presented smaller variability, and average estimates for all parameters were very close to the original. Average statistical characteristics of residuals satisfy the assumptions of mean zero and independence for residuals of the proposed model.

The analysis of the examples above and results of various other simulations indicate that the developed procedure is able to estimate parameters for Multiplicative PARMA models well. The number of trials, CPU time for computing on CYBER 205 and optimal sum of squares of residuals are shown in Table 3.23. This table shows that the savings of computer time by using as starting point the exact- moments estimates for the PARMA(1,1) increases if E(i) diminishes.

Table 3.23 - Estimation from <u>Sample A</u> for different accuracies and starting points.

Accuracy	Starting point	Iterations	Trials	CPU time (seconds)	Sum of squares
0.100	PARMA(1,1)	13	477	1.8	9.93366
0.010	PARMA(1,1)	24	904	3.1	9.92874
0.010	origin	27	1273	3.5	9.92874
0.001	PARMA(1,1)	33	3019	7.4	9.92874
0.001	origin	51	4703	12.3	9.92874

The utilization of estimates of PARMA(1,1) parameters for the starting point, besides saving computer time spent on iterations, guarantees that the Mixed Model would give better fit to the data than that from the PARMA(1,1), the most complete model with varying parameters actually utilized in practical hydrology. Furthermore, it may avoid the problem of finding local optimal points corresponding to parameters that are not good for fitting the data in analysis. If estimations of parameters for PARMA models are not available, it would be helpful in the search of parameters to make various estimations for the same sample, utilizing different starting points as suggested by Powell(1964).

Although the results of application were reported for only a few cases, various other simulations were proceeded and results from estimation showed that conclusions drawn for the described cases are valid for most cases.

3.9 Sensitivity of Annual Autocorrelogram

One of the objectives of the development of the Multiplicative PARMA models is the improvement of the preservation of the annual autocorrelations over that obtained by other models currently applied in practice in hydrology. Hence, it is necessary to check if the new parameters affect the annual autocorrelations. Data generation was employed for this purpose. First of all, PAR(1) samples were generated with parameters

$$\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$$

and the annual autocorrelograms for 40 samples with 250 years (10,000 years) were averaged.

New parameters were added to those for the PAR(1) model, one kind each time, to verify if these parameters affect the autocorrelations. Large positive and negative values (close to 1) were chosen for these new parameters. Average autocorrelograms for the PAR(1) and for the Multiplicative PARMA models (same number of samples and sample sizes) are shown in Figures 3.2, 3.3, 3.4 and 3.5 on the following pages.

Figure 3.2 presents annual autocorrelograms for the PAR(1), or Multiplicative PARMA(1,0)x(0,0) model, presenting the above parameters, and for the Multiplicative PARMA (1,0)x(1,0) models with the following parameters :

a) $\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$

$\Phi_{1,1} = \Phi_{1,4} = 1.0 \qquad \Phi_{1,2} = \Phi_{1,3} = 0.8$

b) $\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$

$\Phi_{1,1} = \Phi_{1,4} = -1.0 \qquad \Phi_{1,2} = \Phi_{1,3} = -0.8$

Figure 3.3 presents annual autocorrelograms for the same PAR(1) model and for Multiplicative PARMA(1,0)x(0,1) models with parameters :

a) $\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$

$\underline{\bar{\theta}}_{1,1} = \underline{\bar{\theta}}_{1,4} = 1.0 \qquad \underline{\bar{\theta}}_{1,2} = \underline{\bar{\theta}}_{1,3} = 0.8$

b) $\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$

$\underline{\bar{\theta}}_{1,1} = \underline{\bar{\theta}}_{1,4} = -1.0 \qquad \underline{\bar{\theta}}_{1,2} = \underline{\bar{\theta}}_{1,3} = -0.8$

Figure 3.4 shows the same curves for the PAR(1) and for PAR(2), or Multiplicative PARMA(2,0)x(0.0), models with parameters :

a) $\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$

$\phi_{2,1} = \phi_{2,4} = 1.0 \qquad \phi_{2,2} = \phi_{2,2} = 0.8$

b) $\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$

$\phi_{2,1} = \phi_{2,4} = -1.0 \qquad \phi_{2,2} = \phi_{2,3} = -0.8$

Figure 3.5 shows the same curves for the PAR(1) and for PAR(3), or Multiplicative PARMA(3,0)x(0,0), models presenting parameters :

a) $\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$

$\phi_{2,1} = \phi_{2,2} = \phi_{2,3} = \phi_{2,4} = 0.0$

$\phi_{3,1} = \phi_{3,4} = 1.0 \qquad \phi_{3,2} = \phi_{3,3} = 0.8$

b) $\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$

$\phi_{2,1} = \phi_{2,2} = \phi_{2,3} = \phi_{2,4} = 0.0$

$\phi_{3,1} = \phi_{3,4} = -1.0 \qquad \phi_{3,2} = \phi_{3,3} = -0.8$

All the figures present different graphs for the PAR(1) models and for the other considered models showing that the additional parameters for the latter models can have significant effects on the annual autocorrelations. Figure 3.2 shows the highest increase on the annual correlation, indicating the high importance of the parameters $\Phi_{1,\tau}$s. Figure 3.3 shows some increase for negative $\underline{\overline{\theta}}_{1,\tau}$s. Figures 3.4 and 3.5 showed that the annual dependence diminished for both positive and negative $\phi_{2,\tau}$ and $\phi_{3,\tau}$ parameters. Hence, it seemed important to make a more detailed analysis for the two first cases by generating samples with different $\phi_{1,\tau}$, $\Phi_{1,\tau}$ and $\underline{\overline{\theta}}_{1,\tau}$.

The influence of the parameters $\Theta_{1,\tau}$ was studied by adding this seasonal parameter to the parameters $\phi_{1,\tau}$ and $\underline{\overline{\theta}}_{1,\tau}$, obtaining a Multiplicative PARMA(1,1)x(0,1). The addition of $\Theta_{1,\tau}$ parameters to the $\phi_{1,\tau}$ parameters alone would not change the annual correlogram since the aggregation of both PAR(1) and PARMA(1,1) model results in regular annual ARMA models presenting the same autoregressive parameter, equal to the product of the seasonal autoregressive parameters $\phi_{1,\tau}$.

Figure 3.6 shows the annual autocorrelograms for the same seasonal autoregressive parameters $\phi_{1,1}=\phi_{1,4}= 1.0$ and $\phi_{1,2}=\phi_{1,3}= 0.8$ and for seven different groups of seasonal lag 4 parameters. Figures 3.7, 3.8, 3.9 and 3.10 show, respectively, the seasonal lag 1, lag 2, lag 3 and lag 4 autocorrelations corresponding to the annual

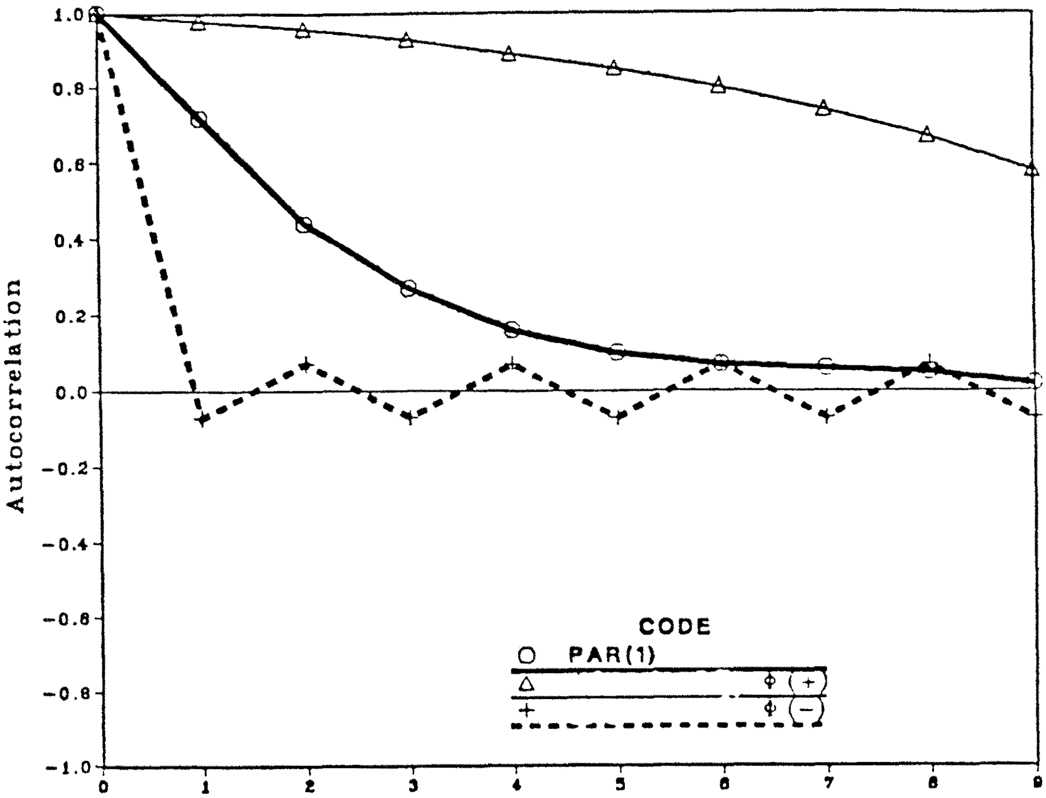

Figure 3.2 - Annual autocorrelograms for PAR(1) and Multiplicative PARMA(1,0)x(1,0) models - Four seasons

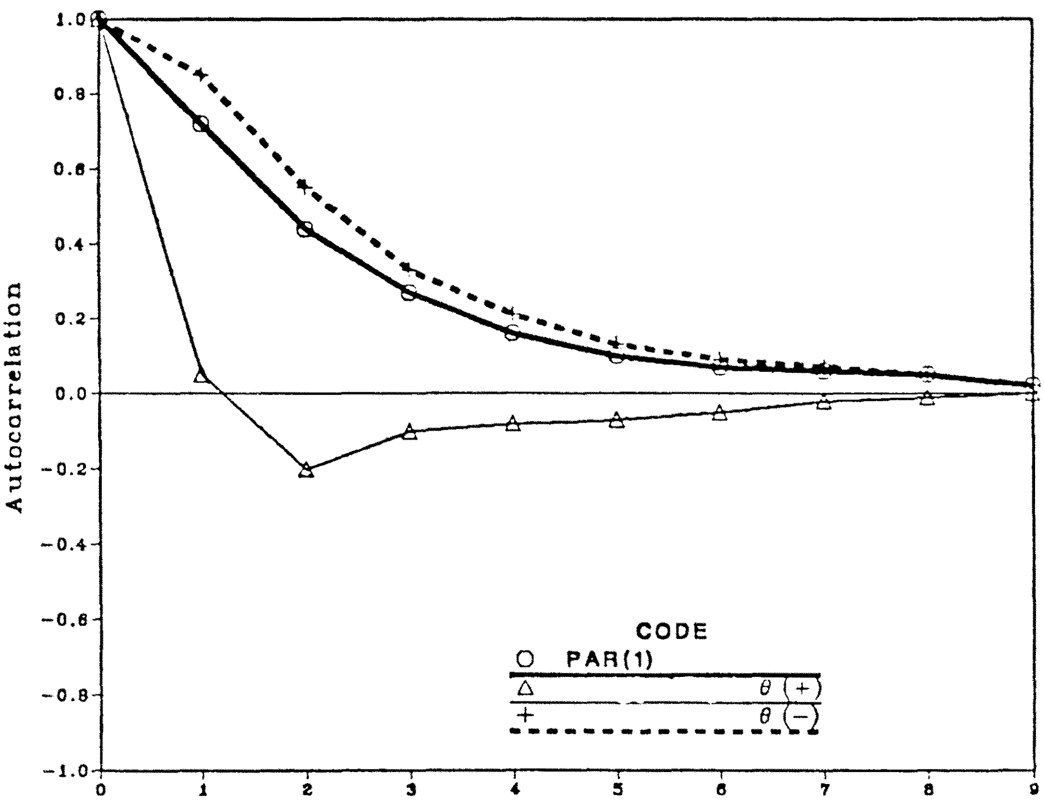

Figure 3.3 - Annual autocorrelograms for PAR(1) and Multiplicative PARMA(1,0)x(0,1) models - Four seasons

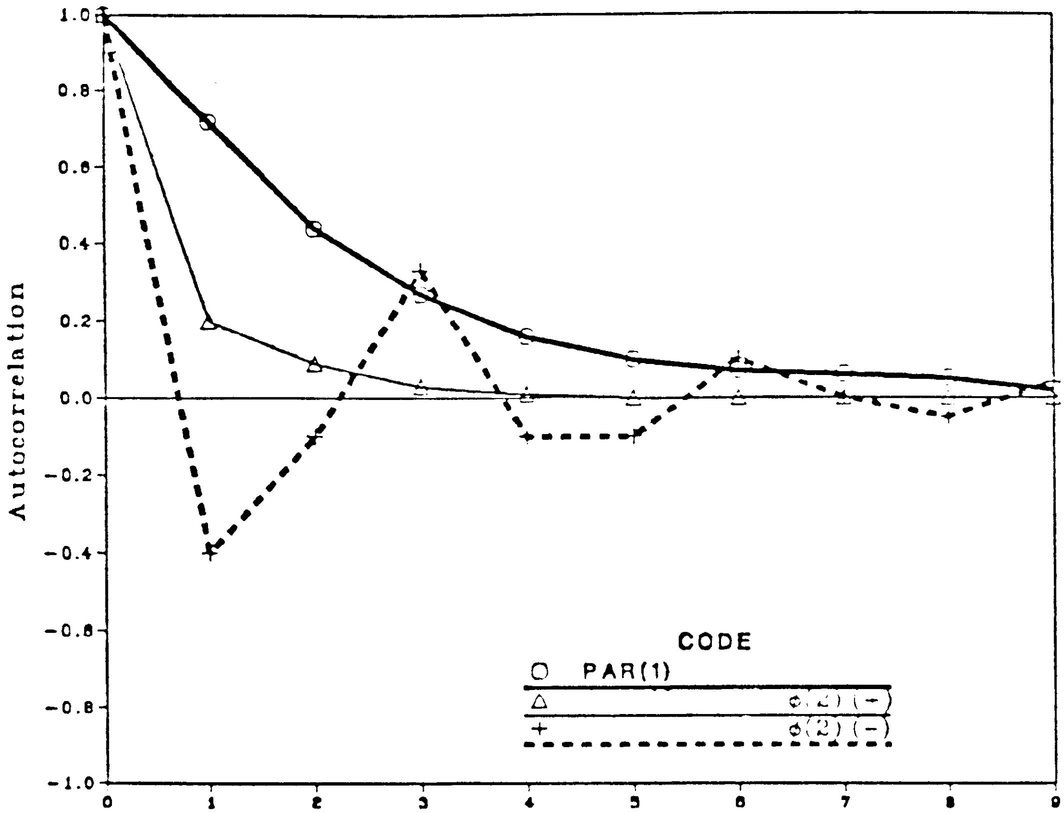

Figure 3.4 - Annual autocorrelograms for PAR(1) and Multiplicative PARMA(2,0)x(0,0) models - Four seasons

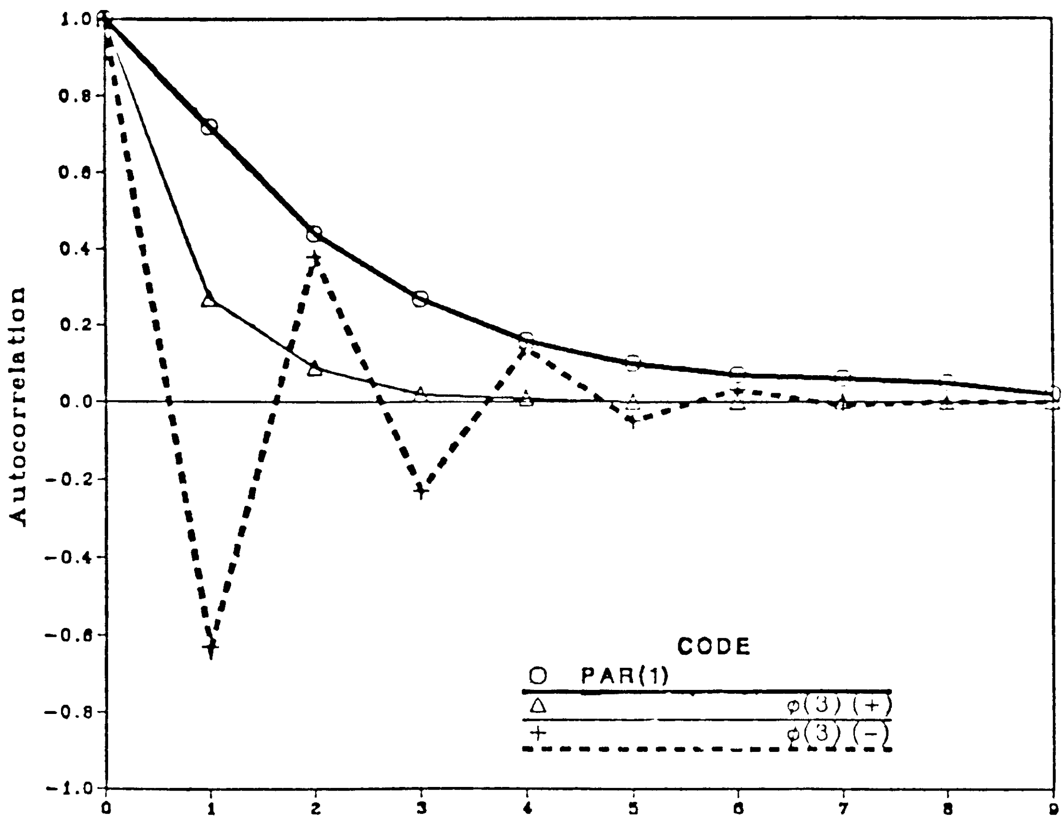

Figure 3.5 - Annual autocorrelograms for PAR(1) and Multiplicative PARMA(3,0)x(0,0) models - Four seasons

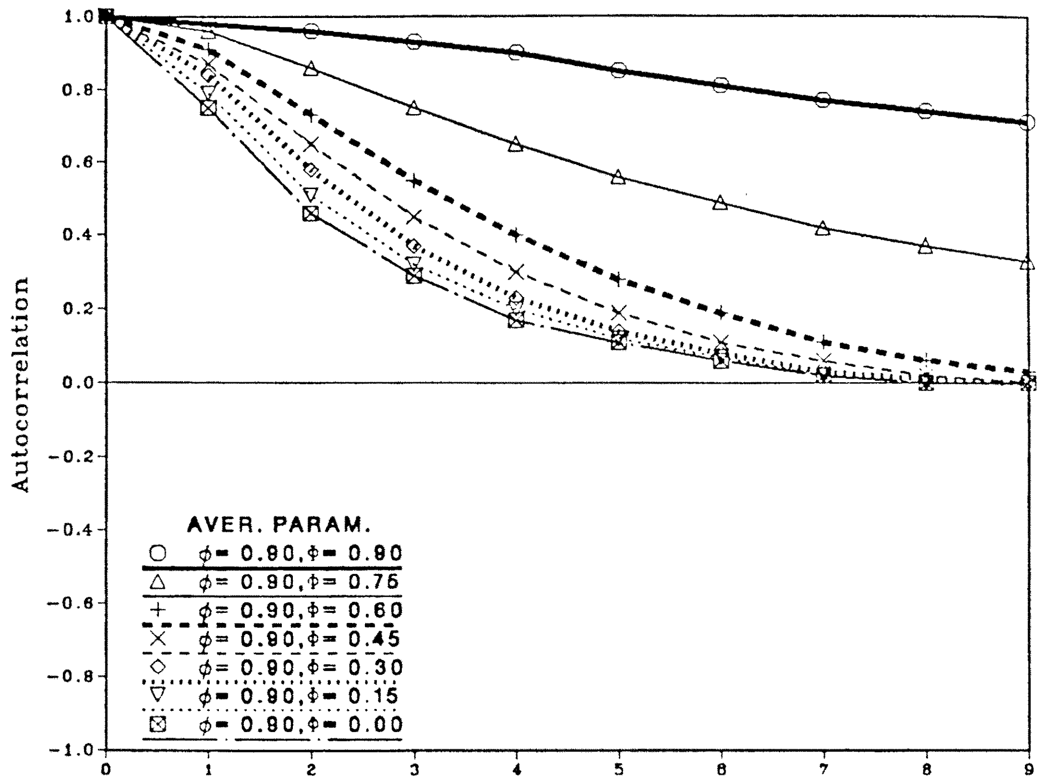

Figure 3.6 - Annual autocorrelograms for Multiplicative PARMA(1,0)x(1,0)-Average parameter $\phi_{1,\tau}$=0.9 - 4 seasons

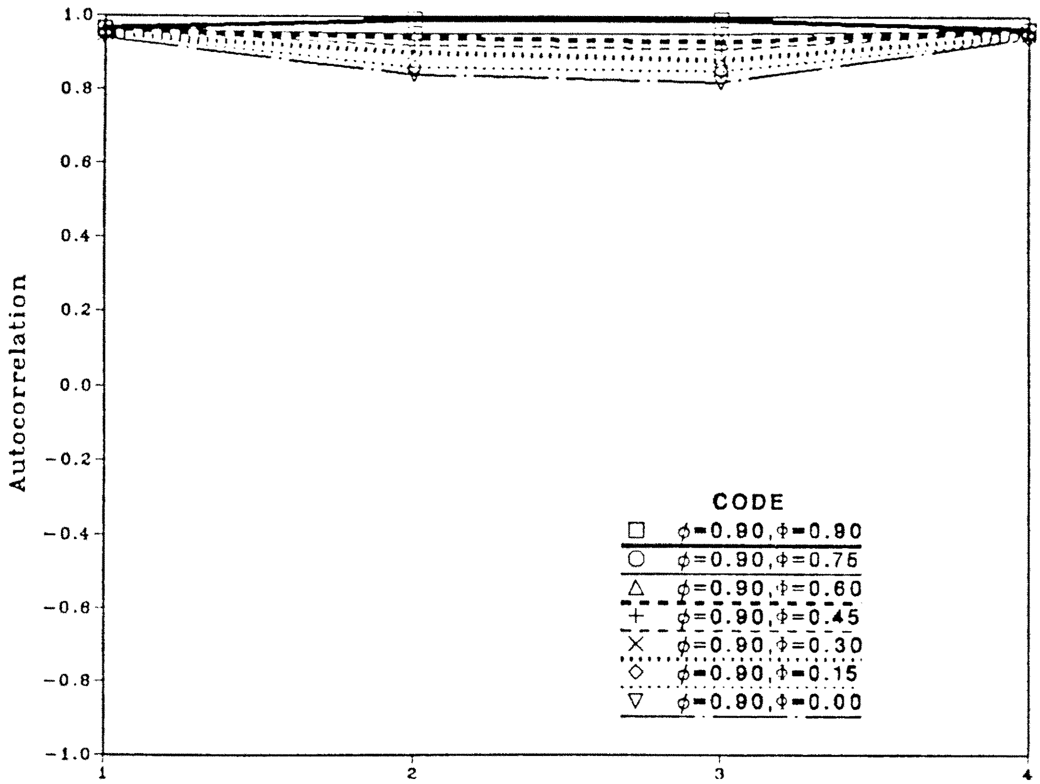

Figure 3.7-Lag 1 seasonal autocorrelograms for Multipicative PARMA(1,0)x(1,0) models - Average $\phi_{1,\tau}$ = 0.9

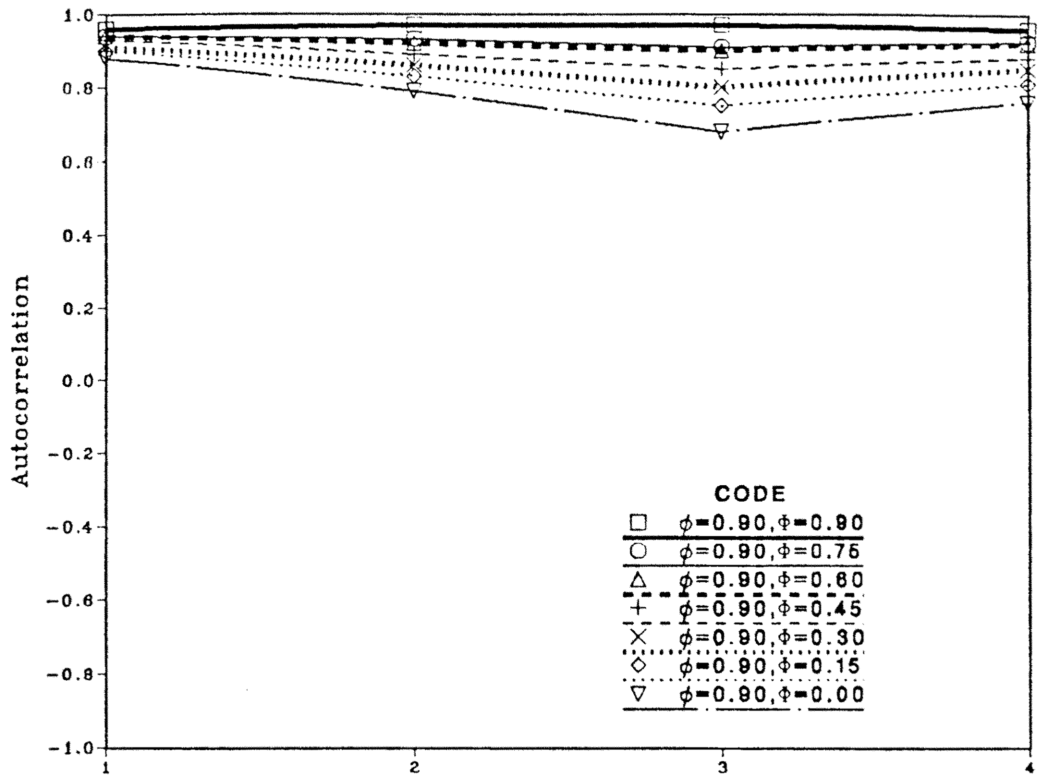

Figure 3.8 - Lag 2 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ =0.9

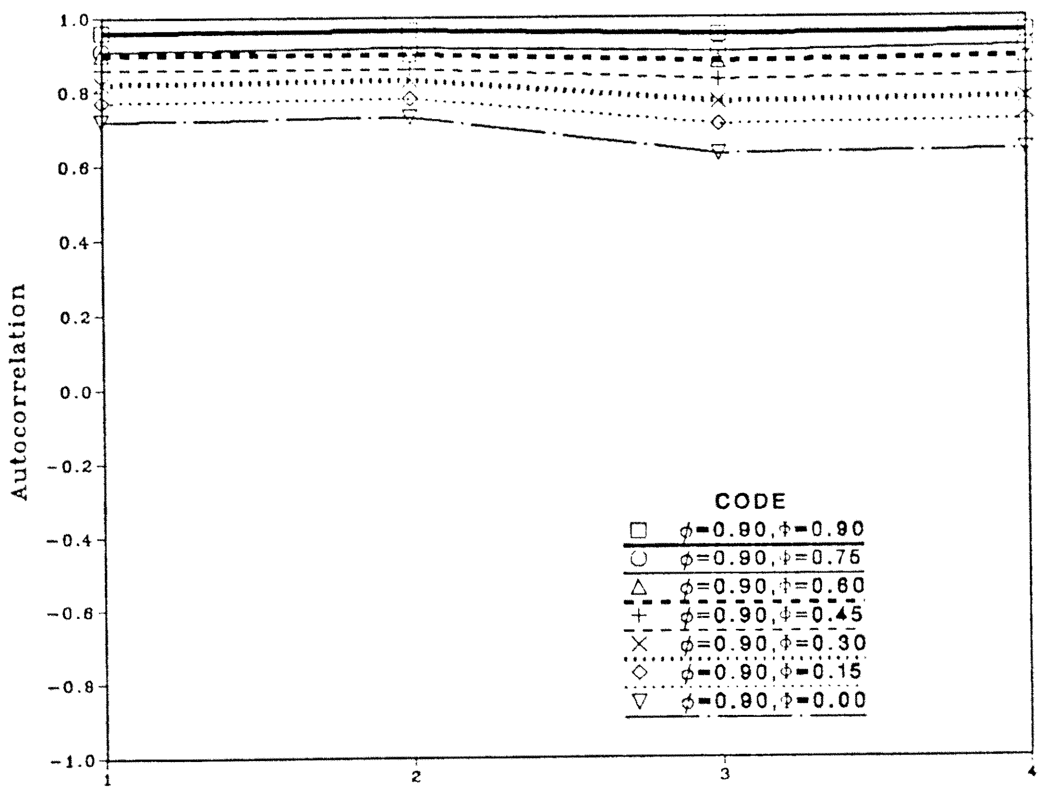

Figure 3.9 - Lag 3 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ =0.9

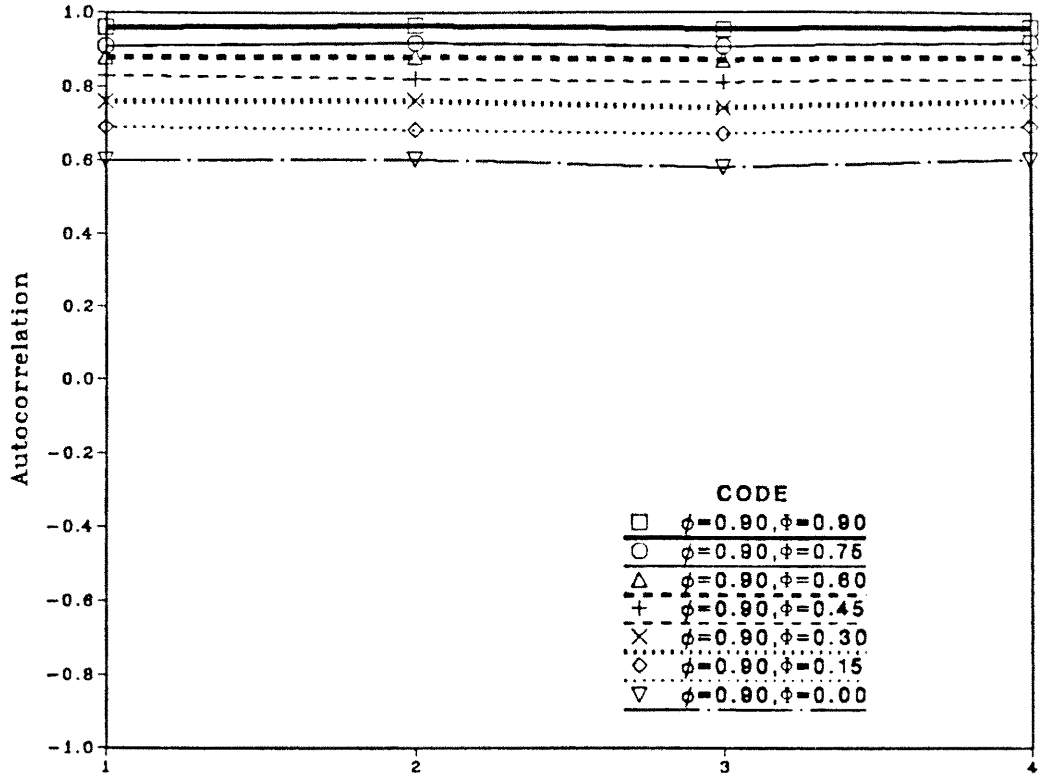

Figure 3.10 -Lag 4 seasonal autocorrelations for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ = 0.9

Figure 3.11, on the next page, presents the annual autocorrelograms for PARMA(1,0)x(1,0) models presenting the same seasonal autoregressive coefficients

$$\phi_{1,1} = \phi_{1,4} = 0.85$$

$$\phi_{1,2} = \phi_{1,3} = 0.65$$

(average 0.75) and the same seven different groups of seasonal parameters $\Phi_{1,\tau}$ utilized for generations corresponding to figures 3.6 through 3.10.

Figures 3.12 through 3.15 show the seasonal lag 1, lag 2, lag 3 and lag 4 autocorrelograms corresponding to the annual autocorrelograms plotted on figure 3.11.

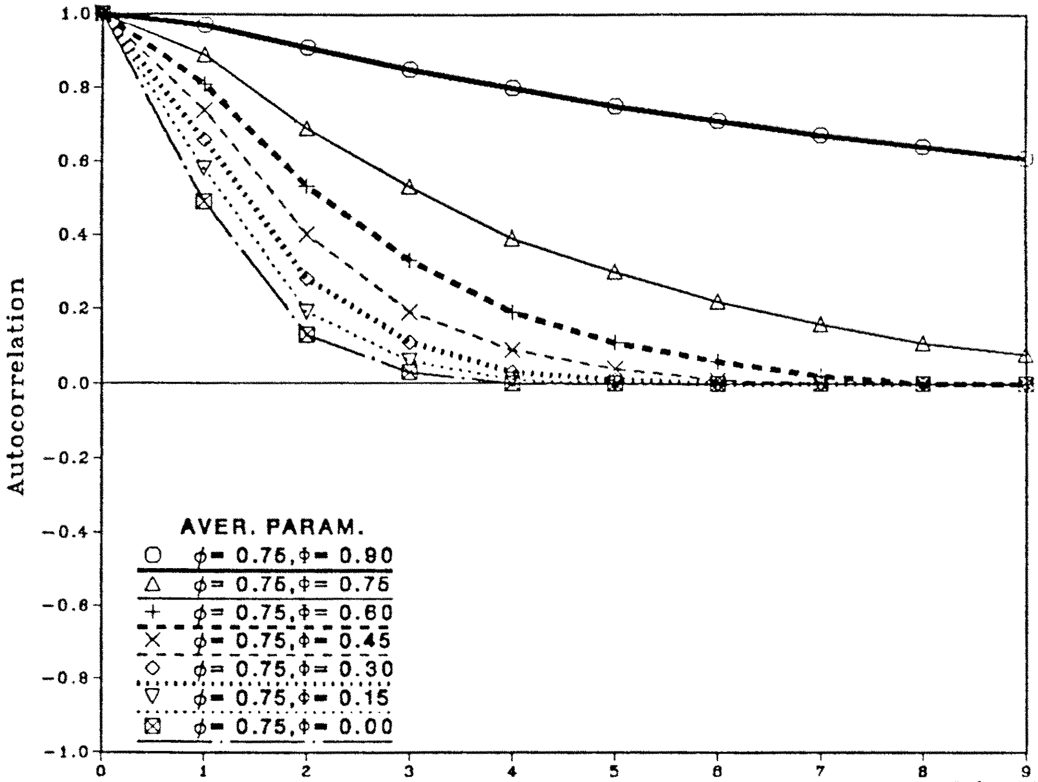

Figure 3.11-Annual autocorrelograms for Multiplicative PARMA(1,0)x(1,0) -Average parameter $\phi_{1,\tau}$=0.75-4 seasons

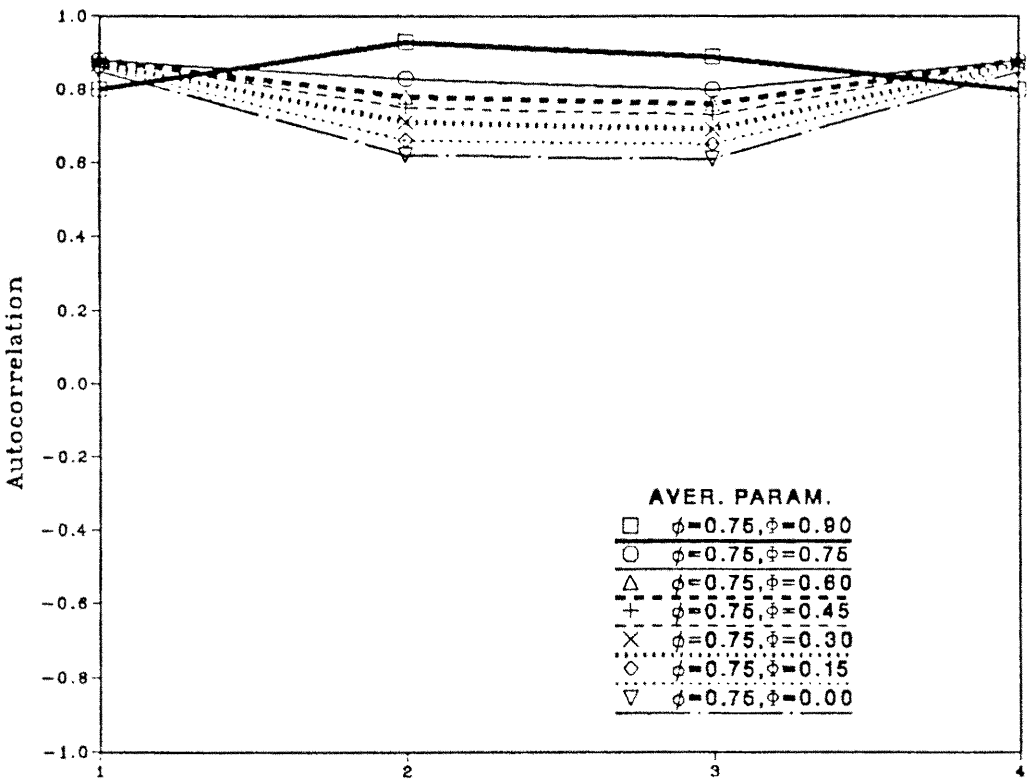

Figure 3.12 -Lag 1 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ =0.75

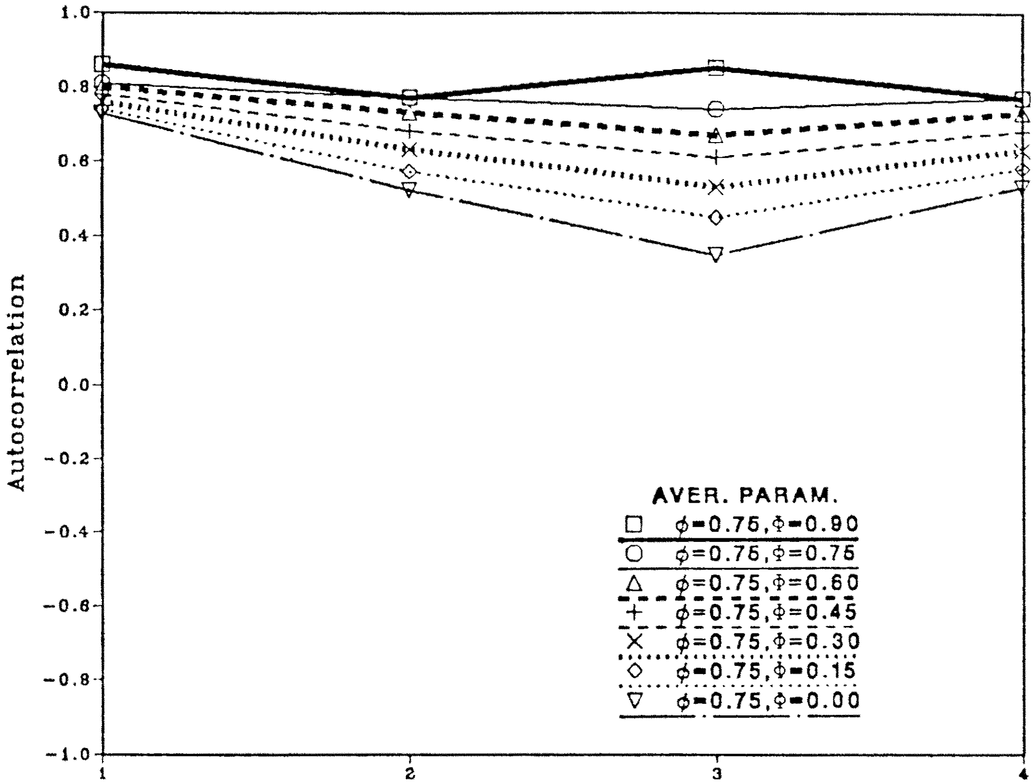

Figure 3.13 -Lag 2 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$= 0.75

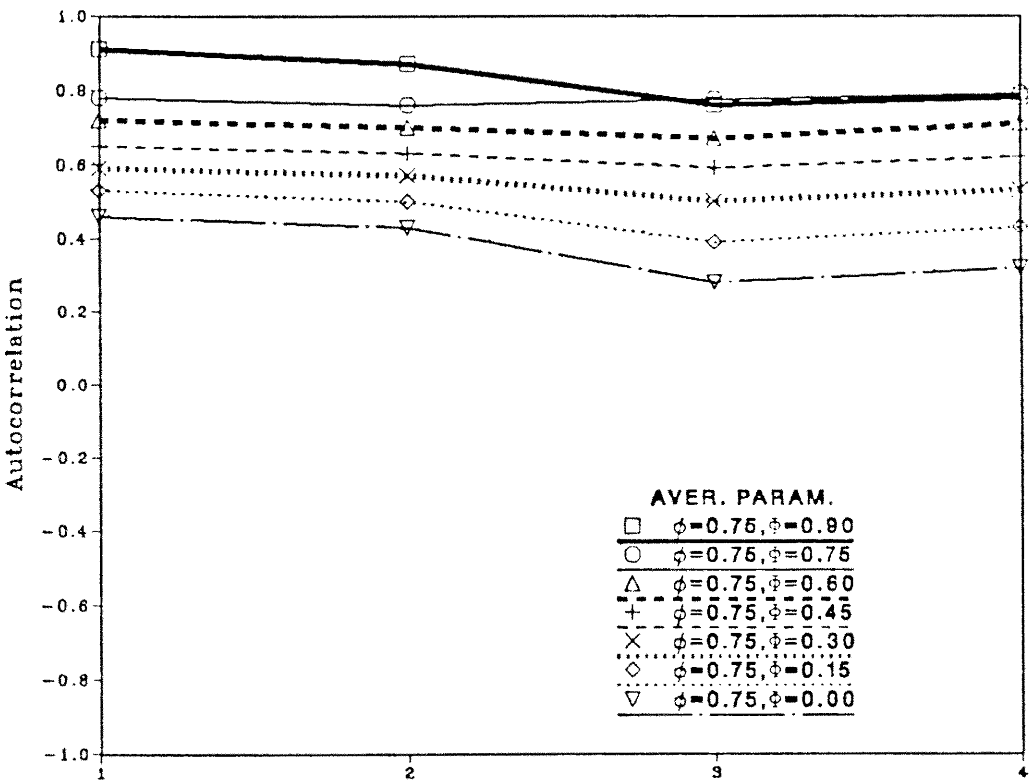

Figure 3.14 -Lag 3 seasonal autocorrelations for Multiplicative PARMA(1,0)x(1,0) models -Average $\phi_{1,\tau}$= 0.75

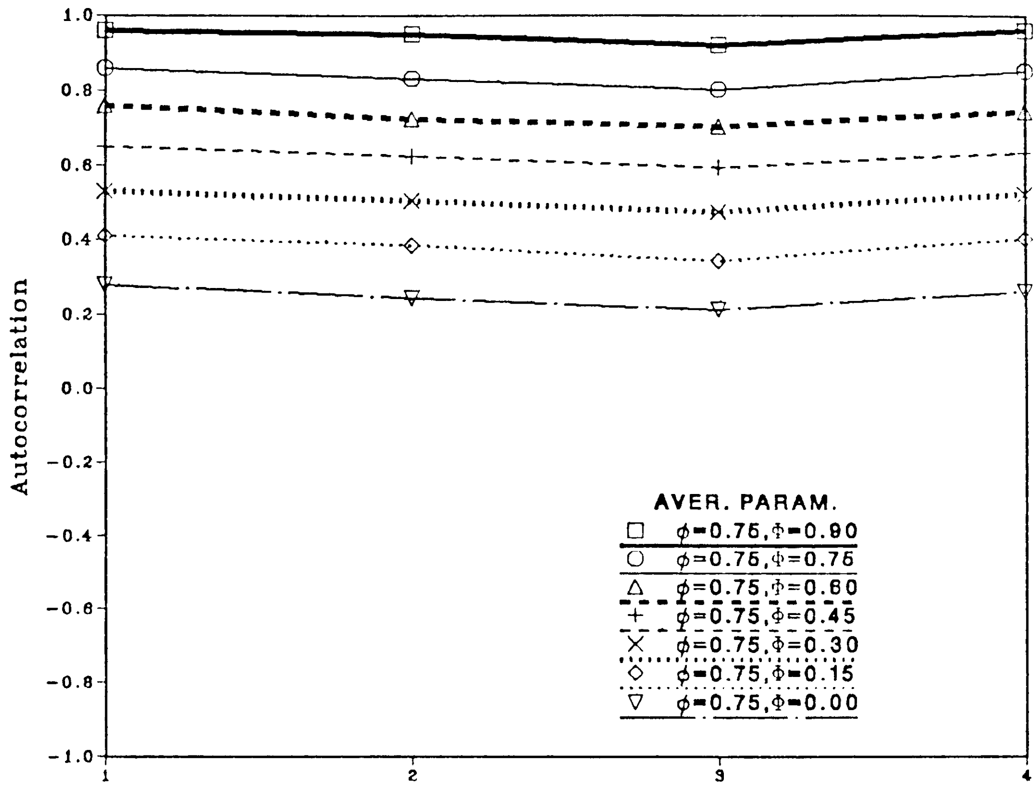

Figure 3.15 -Lag 4 seasonal autocorrelograms for MultiplicativePARMA(1,0)x(1,0) models - Average $\phi_{1,\tau}$ =0.75

The annual autocorrelograms for seven Multiplicative PARMA(1,0)x(1,0) models presenting the same set of seasonal autoregressive parameters

$$\phi_{1,1} = \phi_{1,4} = 0.40$$

$$\phi_{1,2} = \phi_{1,3} = 0.20$$

(average = 0.30) and different sets of parameters $\Phi_{1,\tau}$ are shown in Figure 3.16.

The seasonal lag 1, lag 2, lag 3 and lag 4 autocorrelograms corresponding to the seven annual autocorrelograms from Figure 3.16 are presented in Figures 3.17 through 3.20.

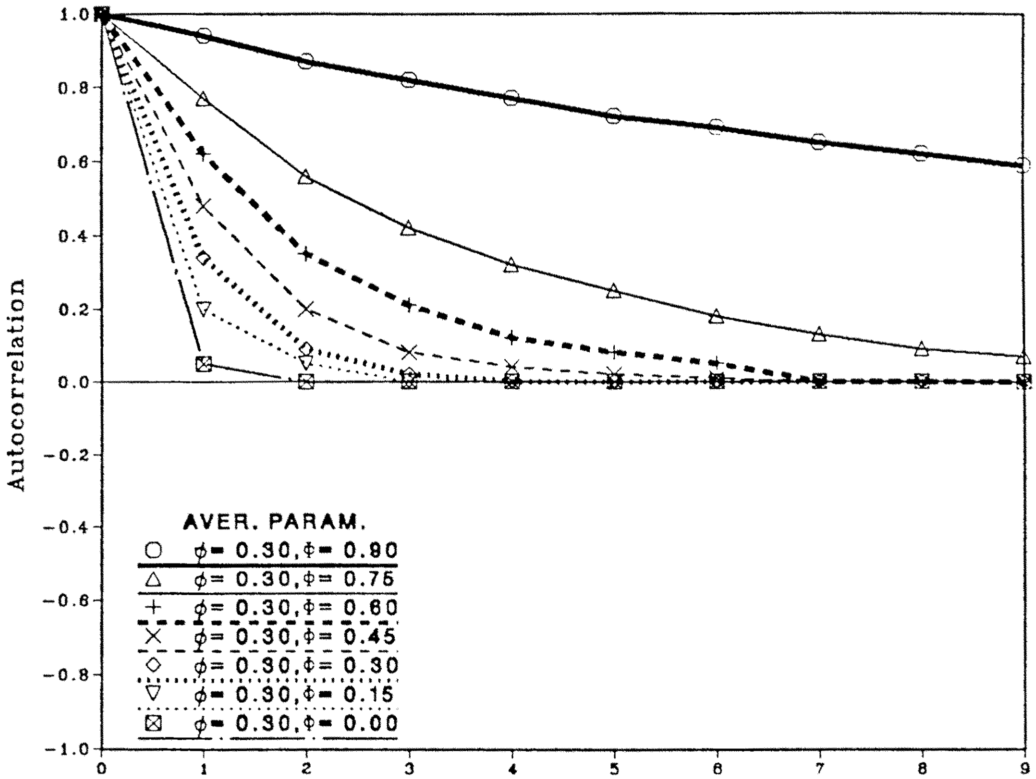

Figure 3.16 -Annual autocorrelograms for Multiplicative PARMA(1,0)x(1,0) - Average parameter $\phi_{1,\tau}$=0.3- 4 seasons

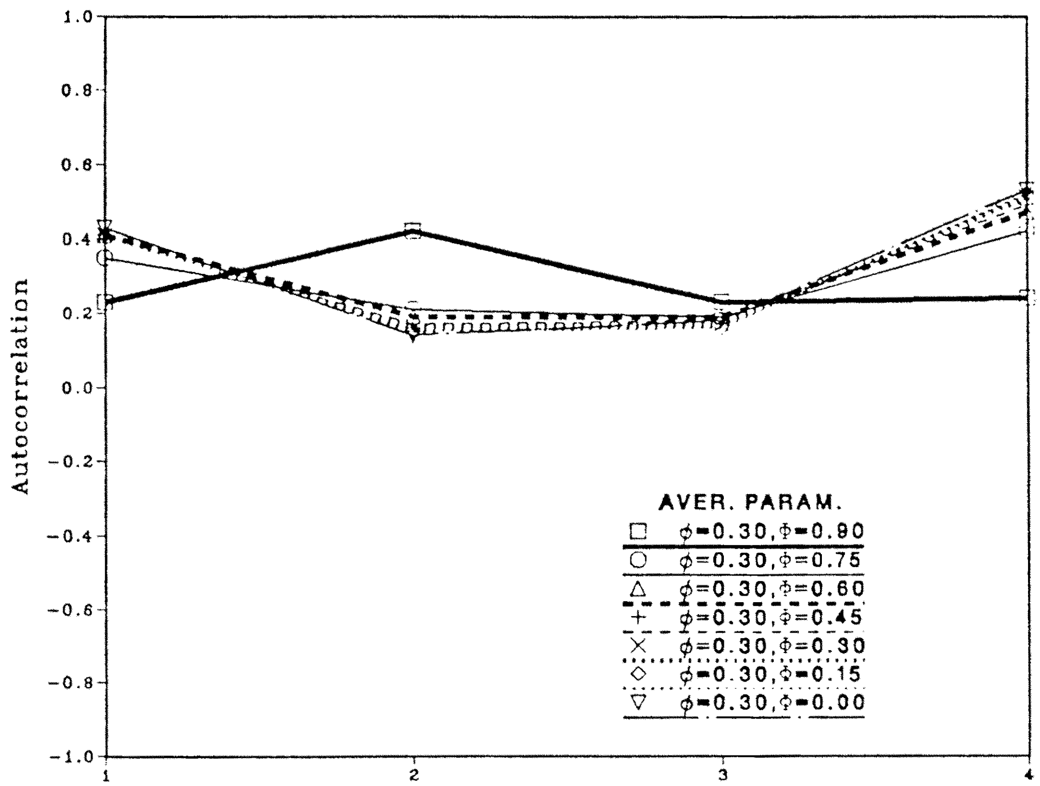

Figure 3.17 -Lag 1 seasonal autocorrelograms for Multplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ =0.3

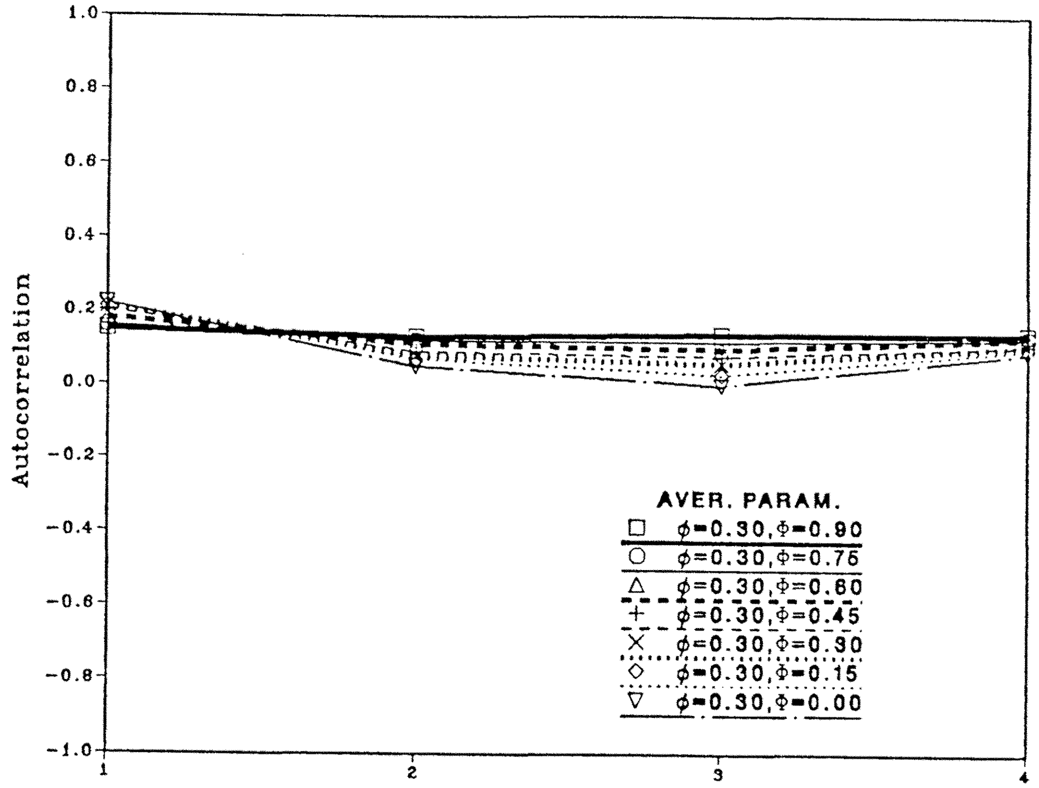

Figure 3.18 -Lag 2 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ = 0.3

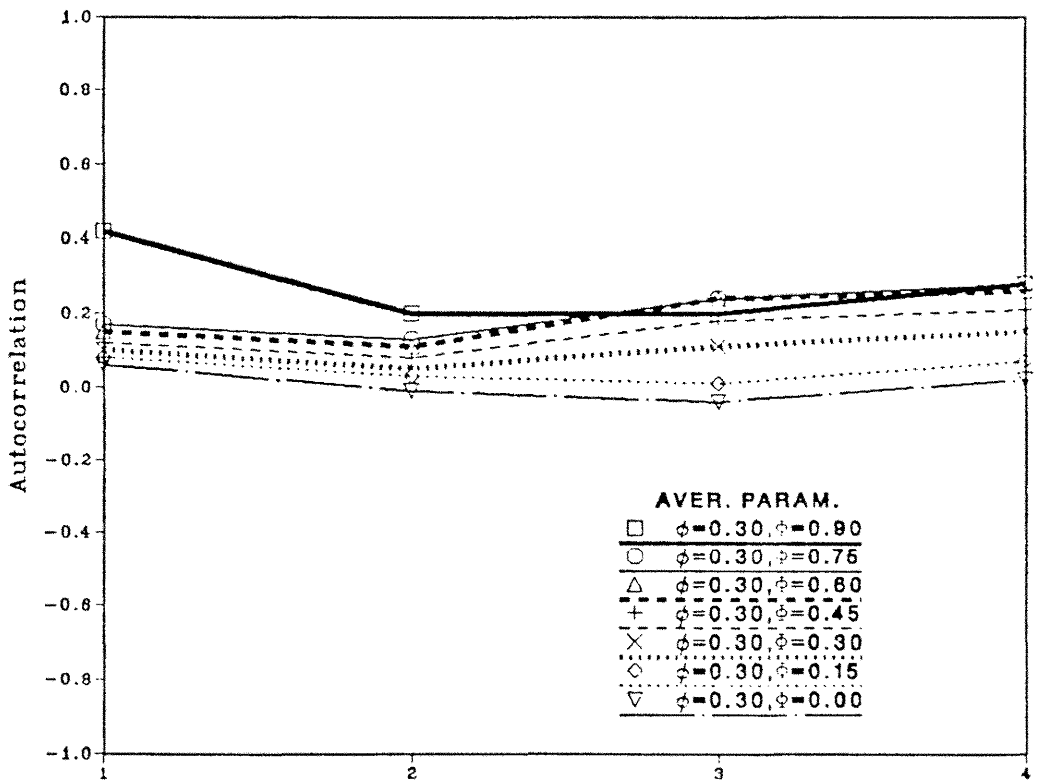

Figure 3.19 -Lag 3 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ = 0.3

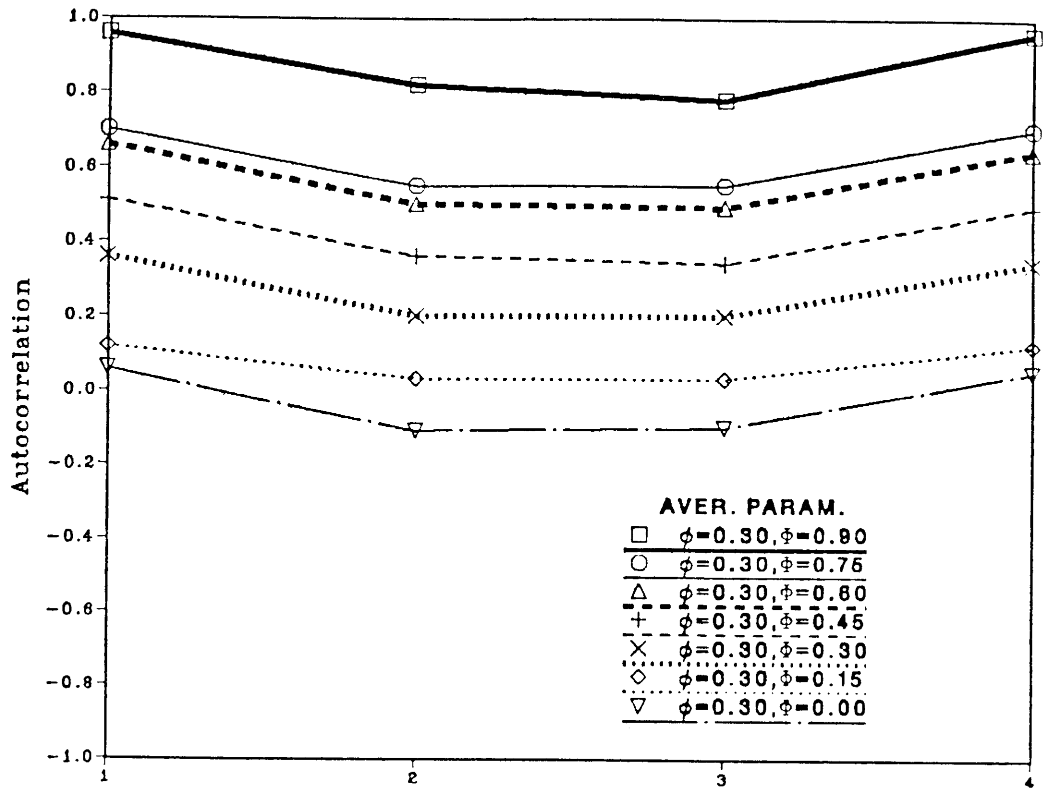

Figure 3.20 -Lag 4 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ = 0.3

Figure 3.21 presents the annual autocorrelograms for a series of Multiplicative PARMA(1,0)x(1,0) models presenting autoregressive parameters

$$\phi_{1,1} = \phi_{1,4} = 0.1$$

$$\phi_{1,2} = \phi_{1,3} = -0.1$$

(average 0.00) and different groups of seasonal parameters $\Phi_{1,\tau}$.

Figures 3.22, 3.23, 3.24 and 3.25 show, respectively, the lag 1, lag 2, lag 3 and lag 4 seasonal autocorrelations corresponding to the annual autocorrelograms presented in Figure 3.21.

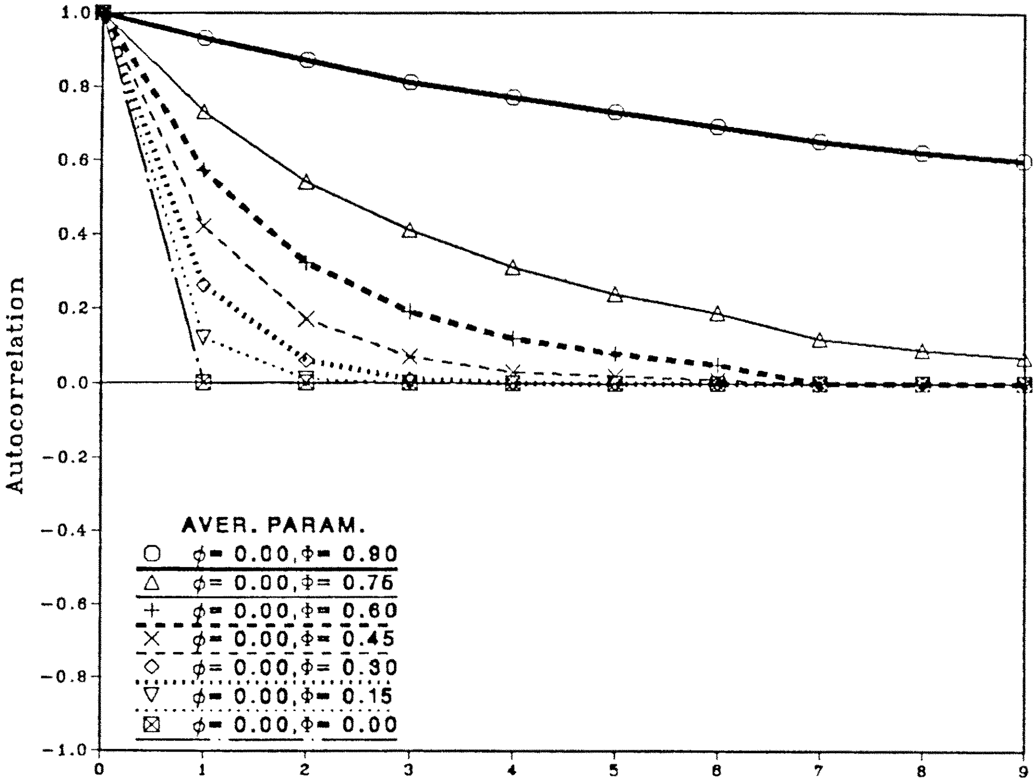

Figure 3.21 -Annual autocorrelograms for Multiplicative PARMA(1,0)x(1,0)- Average parameter $\phi_{1,\tau}$=0.0- 4 seasons

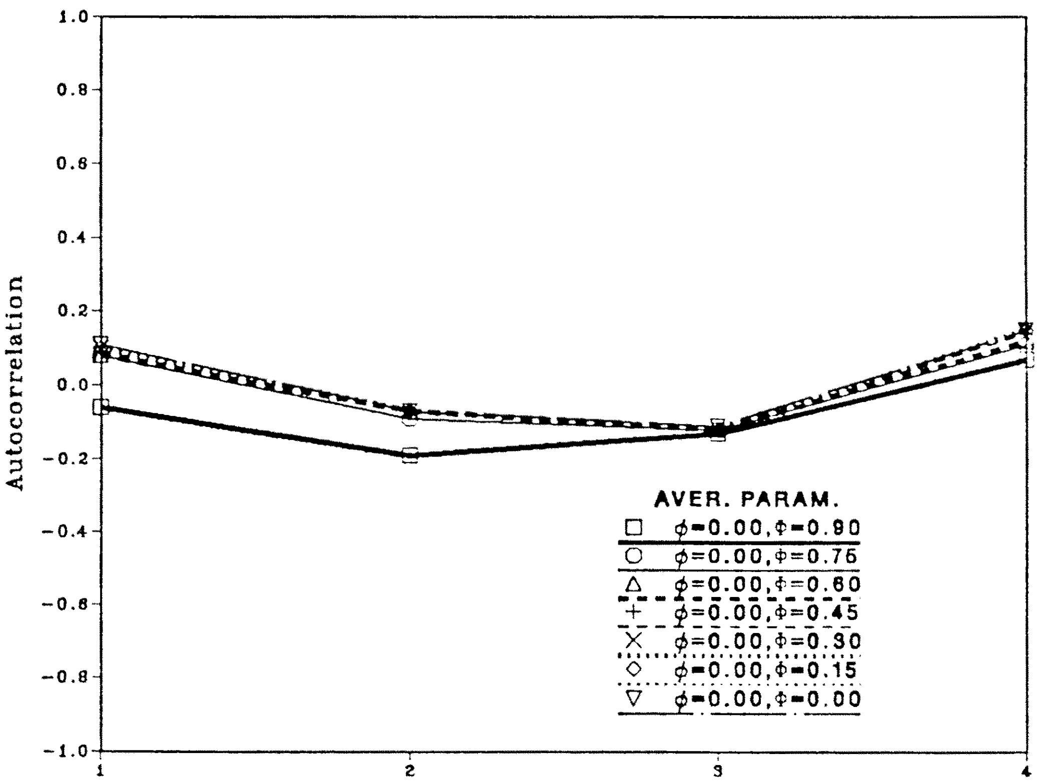

Figure 3.22 - Lag 1 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ = 0.0

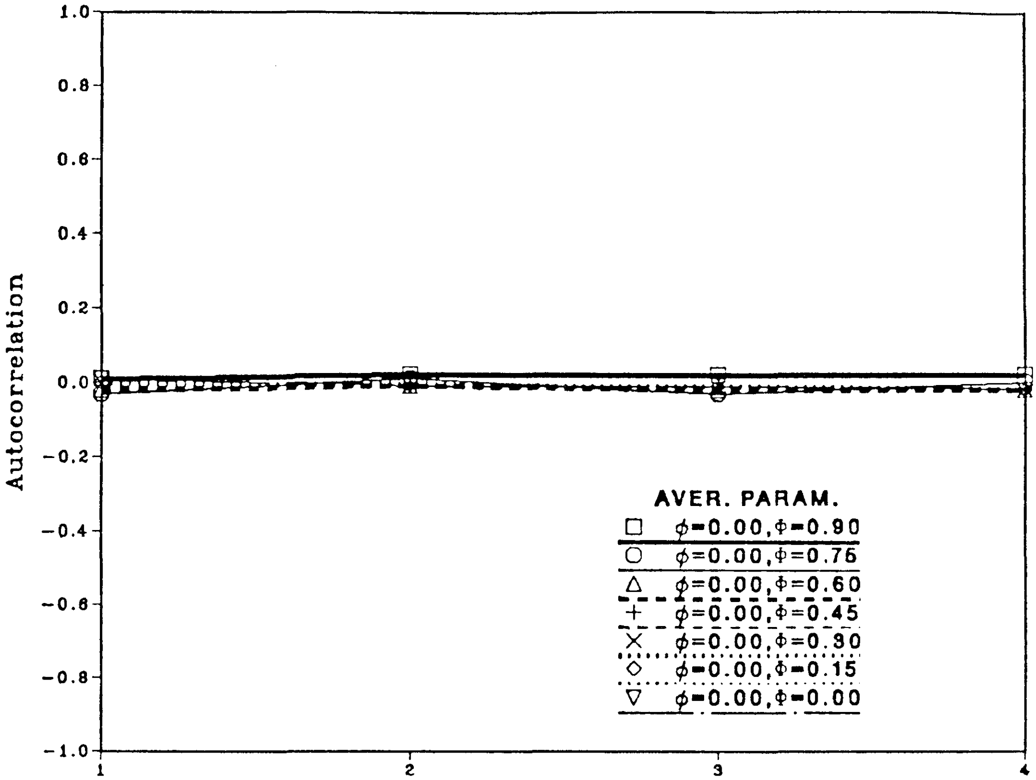

Figure 3.23 - Lag 2 seasonal autocorrelations for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ = 0.0

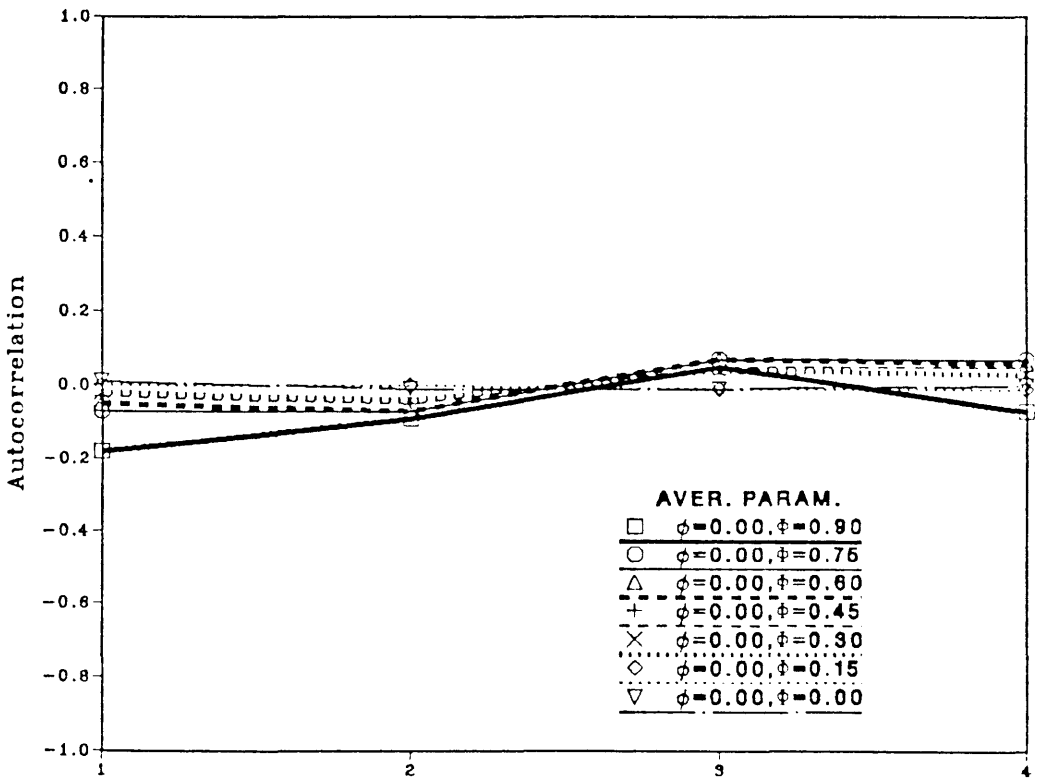

Figure 3.24 - Lag 3 seasonal autocorrelations for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ = 0.0

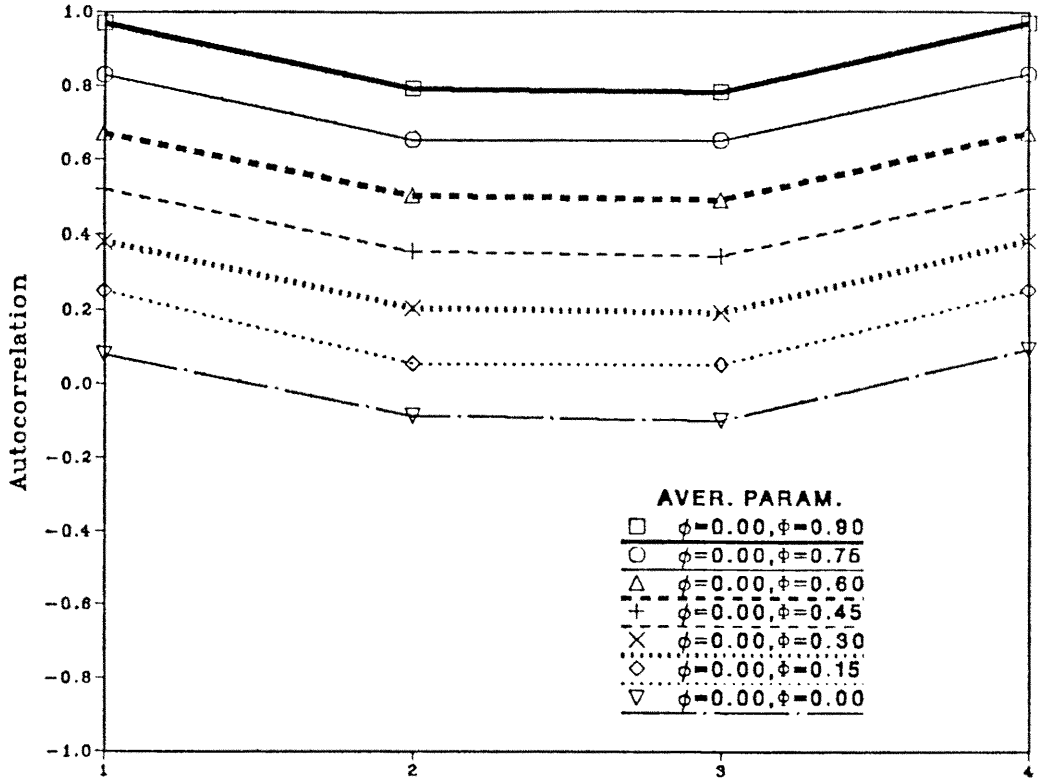

Figure 3.25 - Lag 4 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models- Average $\phi_{1,\tau}$ = 0.0

Figures 3.26 through 3.31 deal with the influence of the lag 4 moving-average parameters, $\underline{\bar{\Theta}}_{1,\tau}$, over a given set of autoregressive parameters $\phi_{1,\tau}$:

$$\phi_{1,1} = \phi_{1,4} = 1.00$$

$$\phi_{1,2} = \phi_{1,3} = 0.80$$

Figure 3.26 shows annual correlograms, while figures 3.27 through 3.31 present the corresponding lag 1, lag 2, lag 3 and lag 4 seasonal autocorrelations for the Multiplicative PARMA(1,0)x(0,1) for seven different sets of parameters $\underline{\bar{\Theta}}_{1,\tau}$.

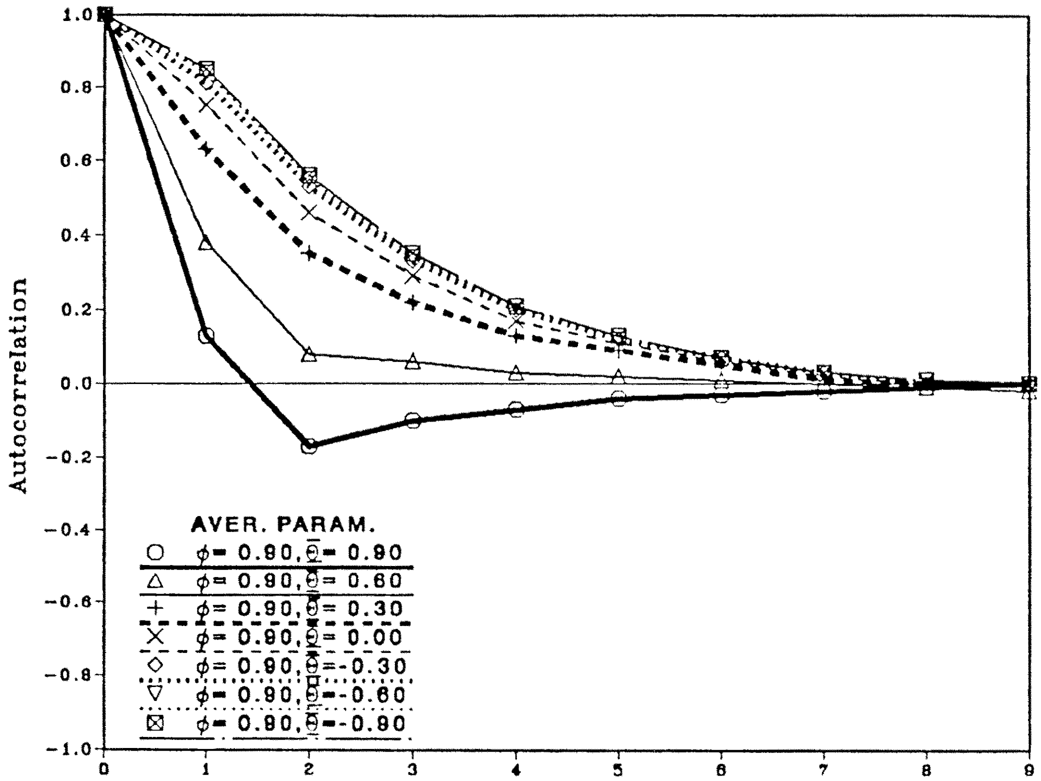

Figure 3.26 -Annual autocorrelograms for Multplicative PARMA(1,0)x(0,1)- Average parameter $\phi_{1,\tau}$=0.9- 4 seasons

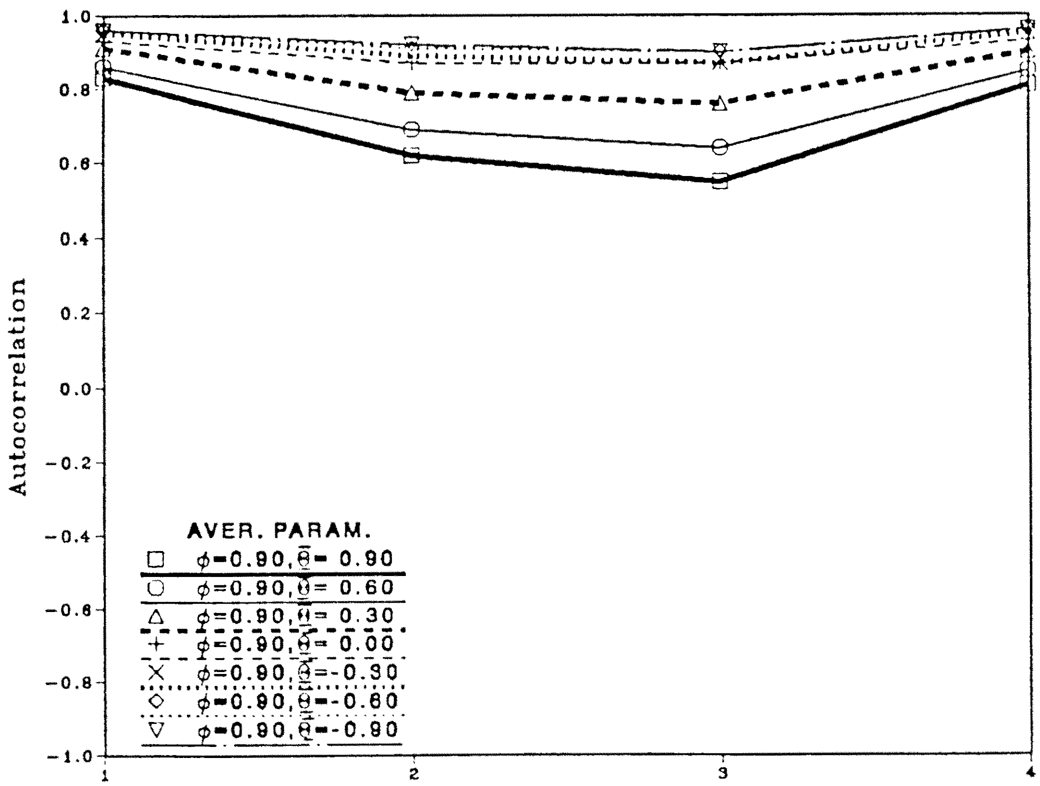

Figure 3.27 - Lag 1 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(0,1) models- Average $\phi_{1,\tau}$= 0.9

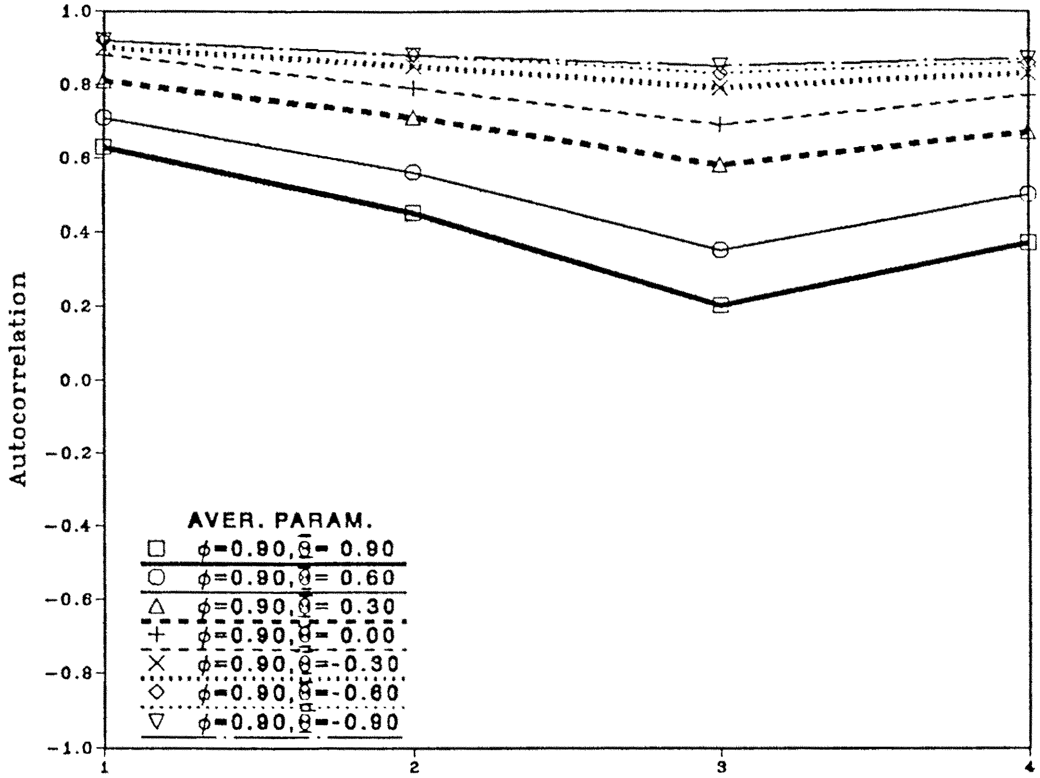

Figure 3.28 - Lag 2 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(0,1) models- Average $\phi_{1,\tau}$ = 0.9

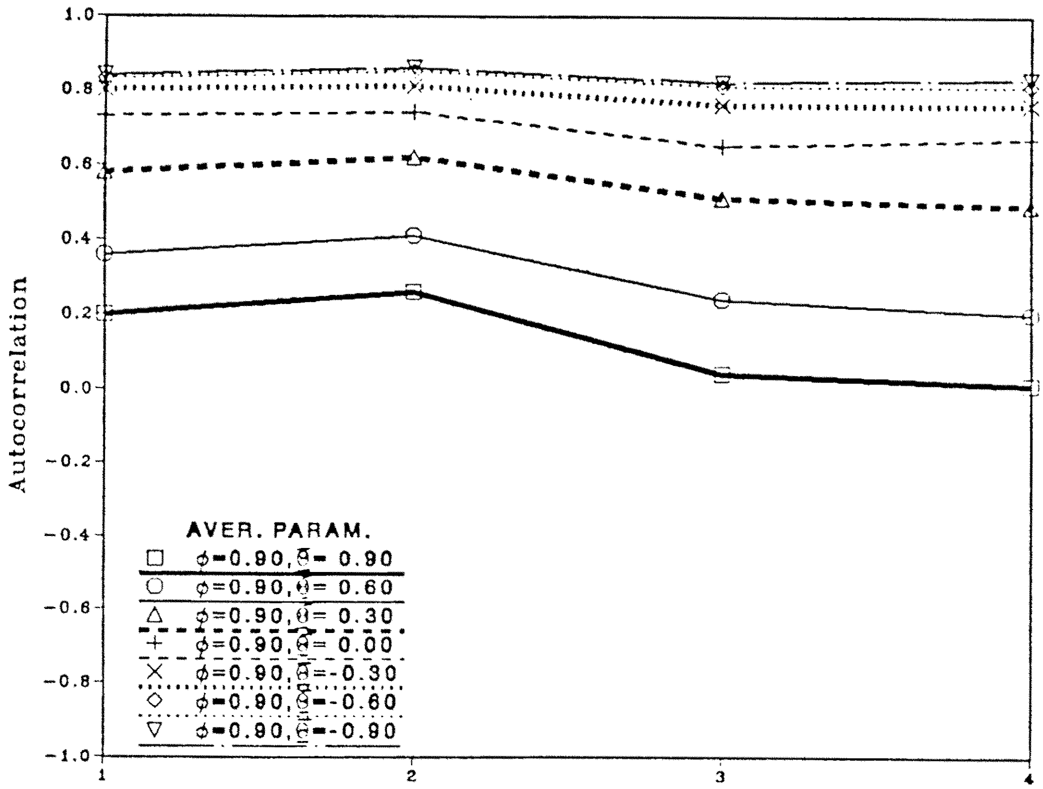

Figure 3.29 - Lag 3 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(0,1) models- Average $\phi_{1,\tau}$ = 0.9

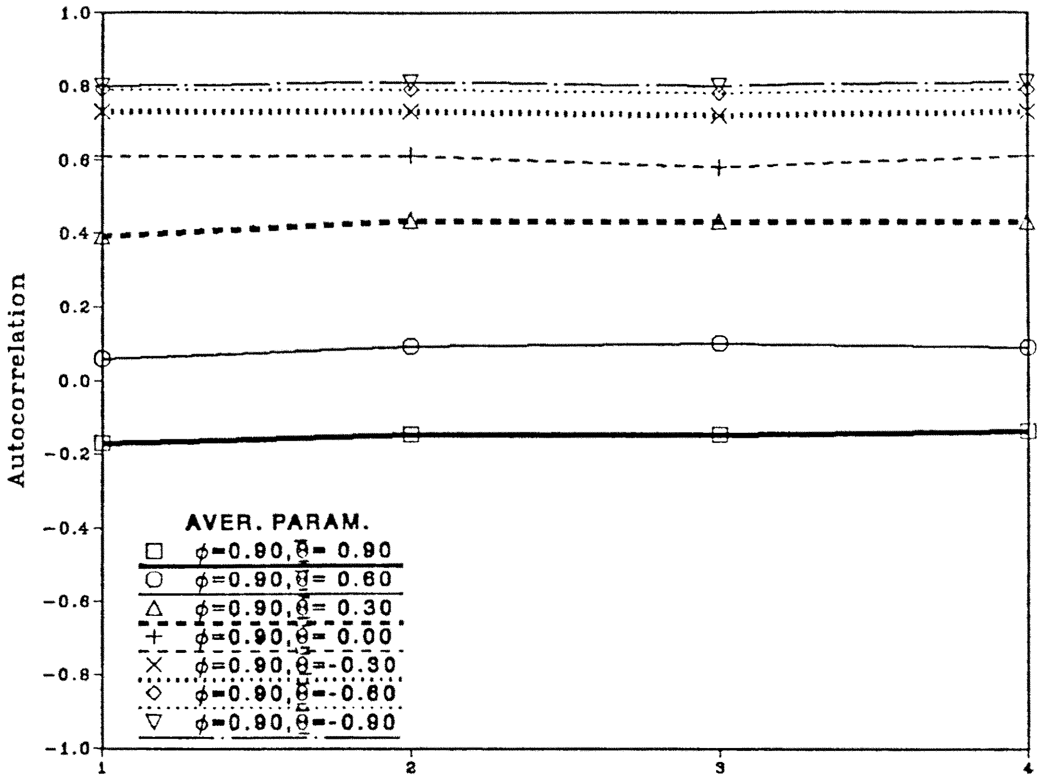

Figure 3.30 - Lag 4 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(0,1) models- Average $\phi_{1,\tau} = 0.9$

Figures 3.31 through 3.35 show the effects of the addition of lag 1 moving-average parameters, $\theta_{1,\tau}$ on the model presenting parameters

$$\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$$
$$\underline{\bar{\theta}}_{1,1} = \underline{\bar{\theta}}_{1,4} = 1.0 \qquad \underline{\bar{\theta}}_{1,2} = \underline{\bar{\theta}}_{1,3} = 0.8$$

Figures 3.36 through 3.40 show the same for the model presenting parameters

$$\phi_{1,1} = \phi_{1,4} = 1.0 \qquad \phi_{1,2} = \phi_{1,3} = 0.8$$
$$\underline{\bar{\theta}}_{1,1} = \underline{\bar{\theta}}_{1,4} =-1.0 \qquad \underline{\bar{\theta}}_{1,2} = \underline{\bar{\theta}}_{1,3} =-1.0$$

The models resulting from inclusion of the $\theta_{1,\tau}$ parameters are Multiplicative PARMA(1,1)x(0,1).

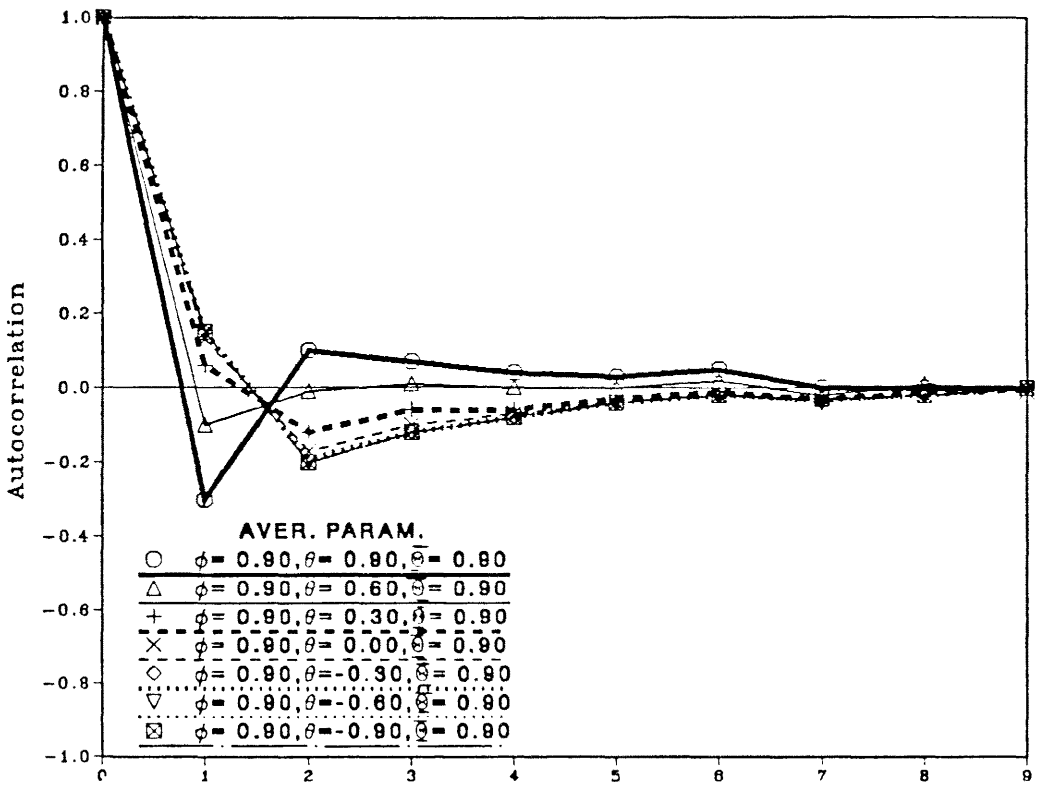

Figure 3.31 - Annual autocorrelograms for PARMA(1,1)x(0,1) Multiplicative -Averages $\phi_{1,\tau}$=0.9 and $\underline{\Theta}_{1,\tau}$=0.9- 4 seasons

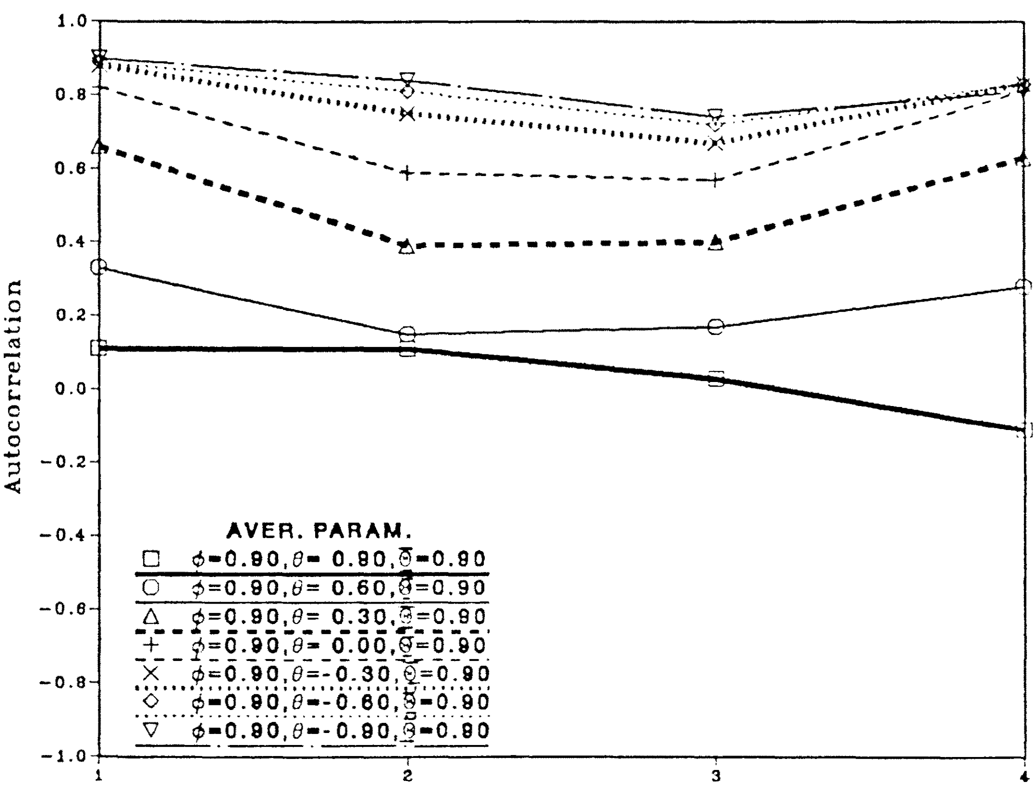

Fig. 3.32-Lag 1 seasonal autocorrelograms for Mult. PARMA(1,1)x(0,1) - Averages $\phi_{1,\tau}$=0.9, $\underline{\Theta}_{1,\tau}$=0.9

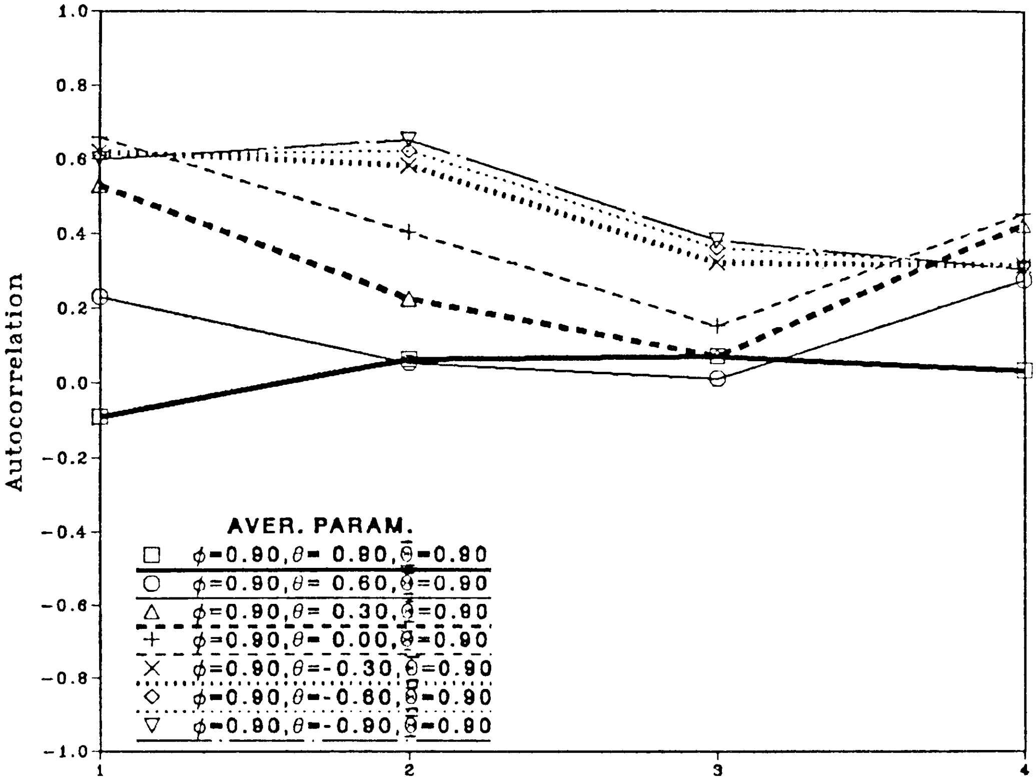

Fig. 3.33-Lag 2 seasonal autocorrelograms for Mult. PARMA(1,1)x(0,1) - Averages $\phi_{1,\tau}$=0.9, $\underline{\Theta}_{1,\tau}$=0.9

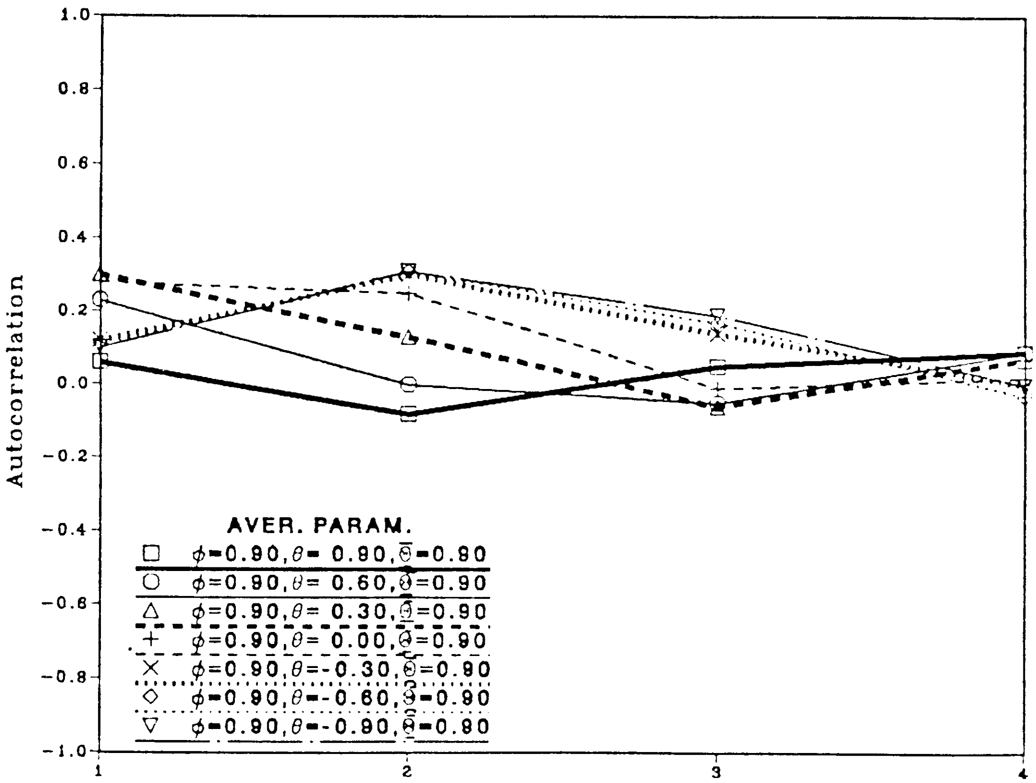

Figure 3.34-Lag 3 seasonal autocorrelograms for Mult. PARMA(1,1)x(0,1) - Averages $\phi_{1,\tau}$=0.9, $\underline{\Theta}_{1,\tau}$=0.9

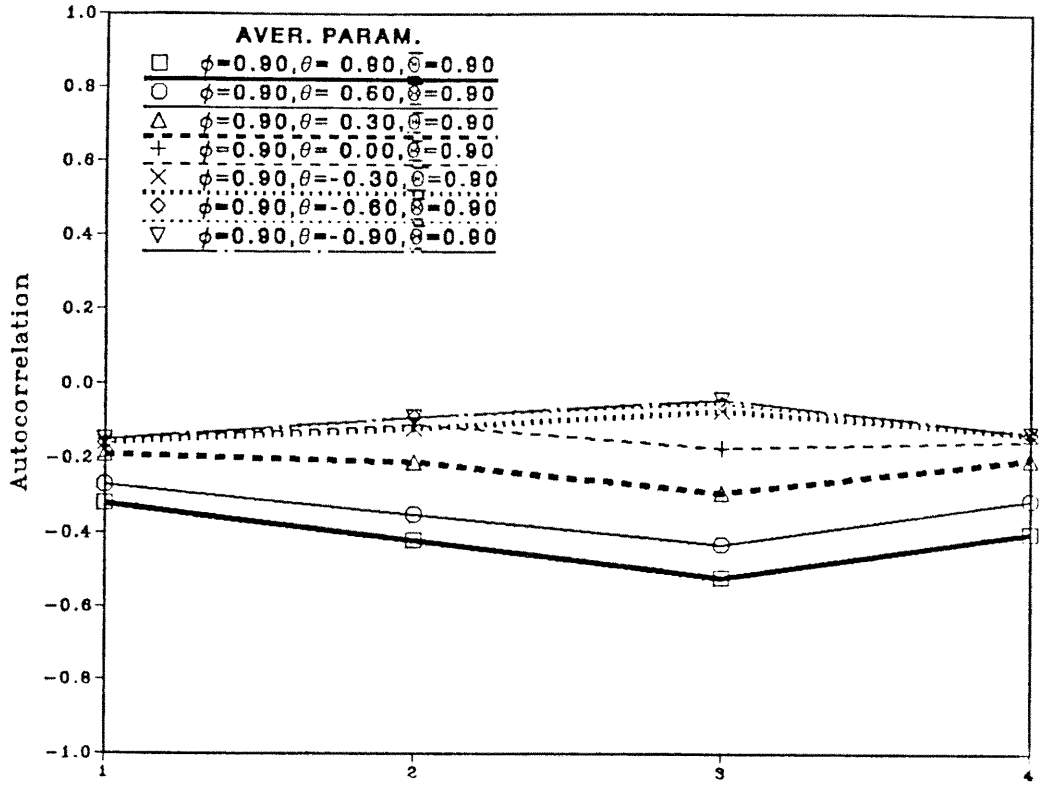

Figure 3.35-Lag 4 seasonal autocorrelograms for Mult. PARMA(1,1)x(0,1) - Averages $\phi_{1,\tau}=0.9$, $\underline{\Theta}_{1,\tau}=0.9$

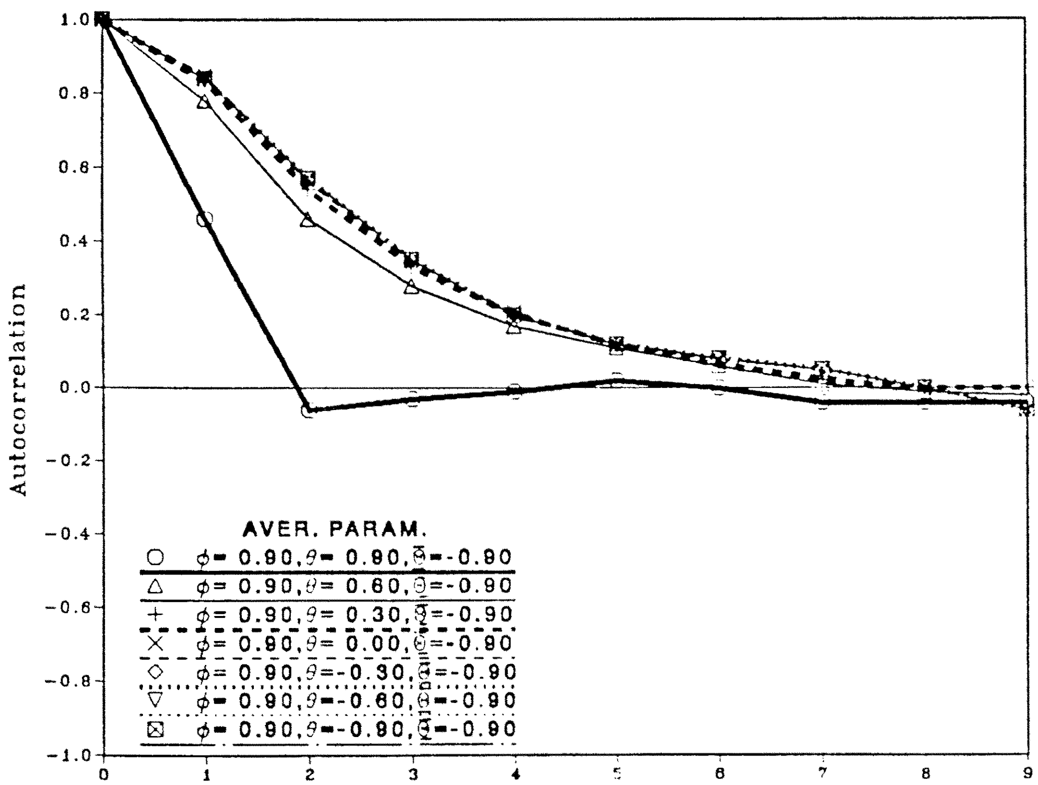

Figure 3.36-Annual autocorrelograms for Multiplicative PARMA(1,1)x(0,1)- Averages $\phi_{1,\tau}=0.9$, $\underline{\Theta}_{1,\tau}=-0.9$ - 4 seasons

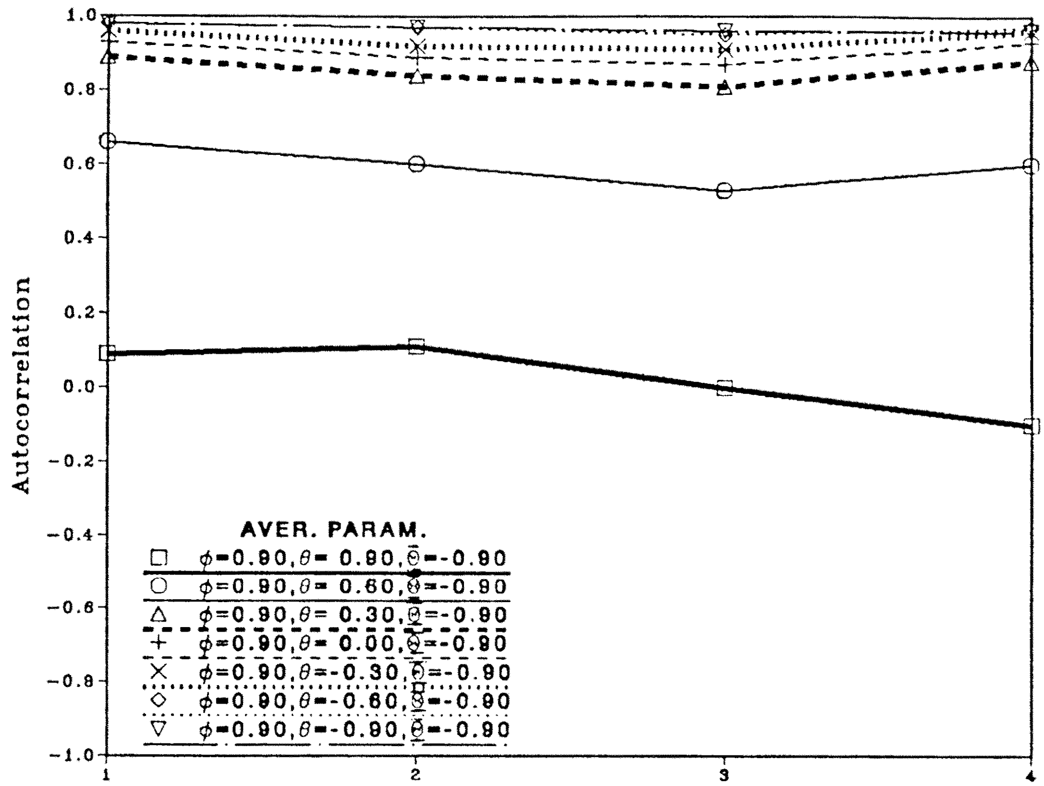

Figure 3.37-Lag 1 seasonal autocorrelograms for Mult. PARMA(1,1)x(0,1) - Averages $\phi_{1,\tau}$=0.9, $\underline{\Theta}_{1,\tau}$=-0.9

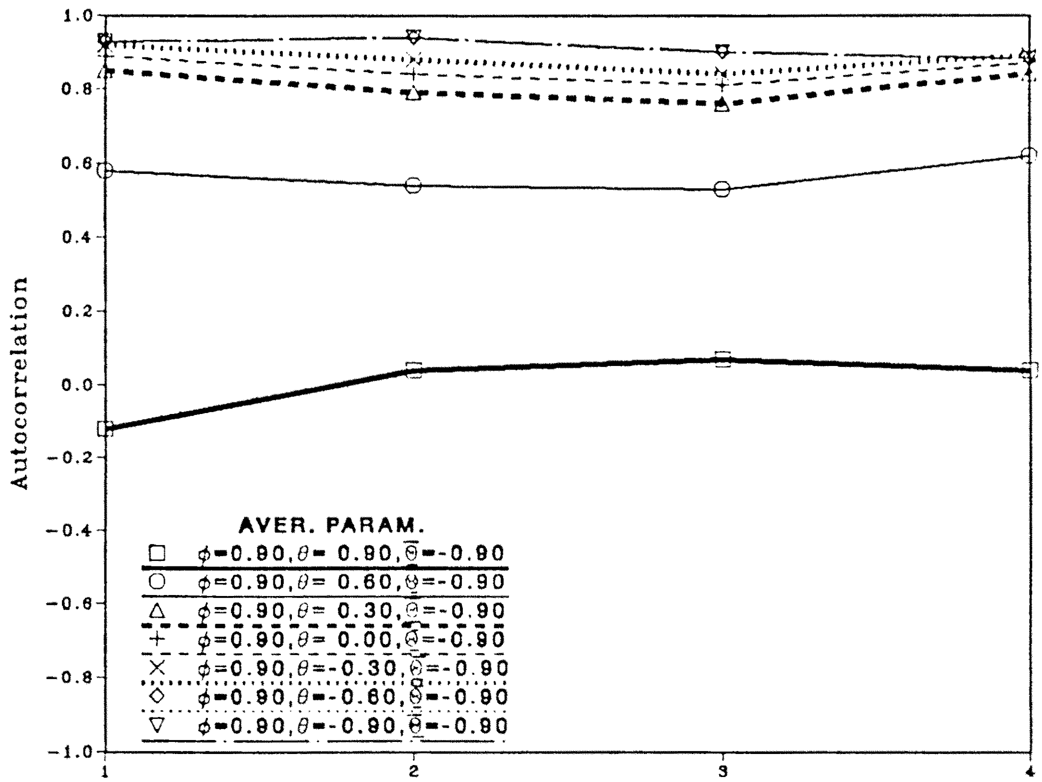

Figure 3.38-Lag 2 seasonal autocorrelograms for Mult. PARMA(1,1)x(0,1)- Averages $\phi_{1,\tau}$=0.9, $\underline{\Theta}_{1,\tau}$=-0.9

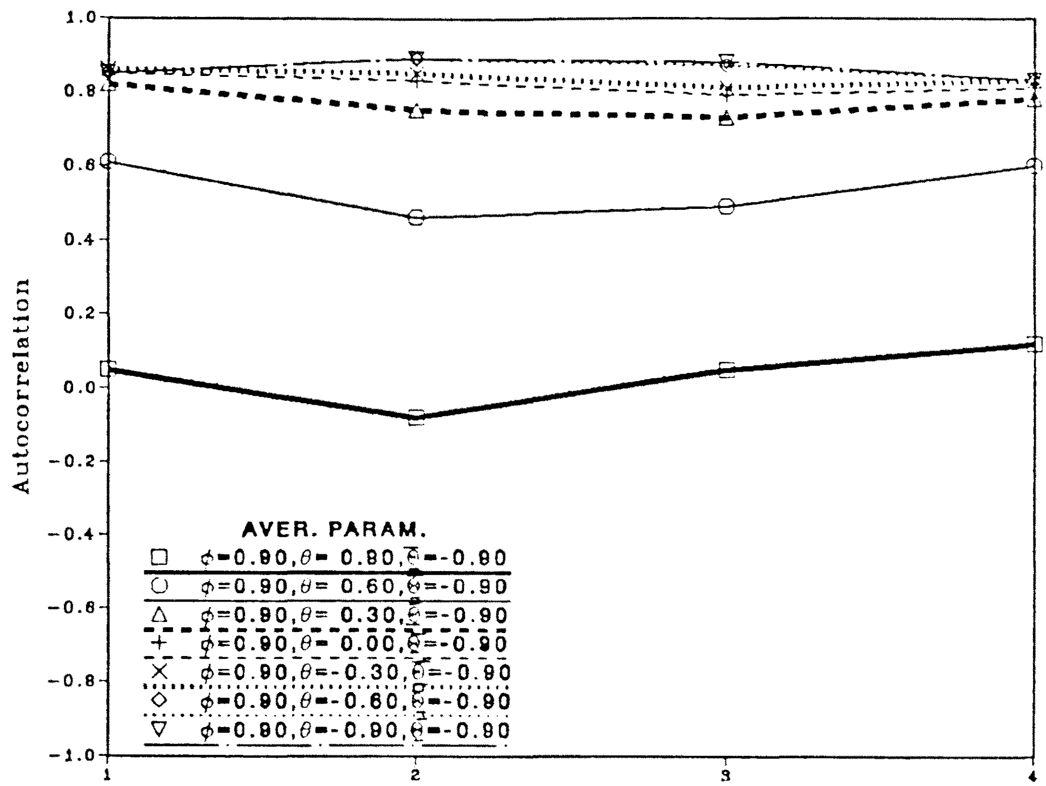

Figure 3.39-Lag 3 seasonal autocorrelograms for Mult. PARMA(1,1)x(0,1) - Averages $\phi_{1,\tau}$=0.9, $\Theta_{1,\tau}$=-0.9

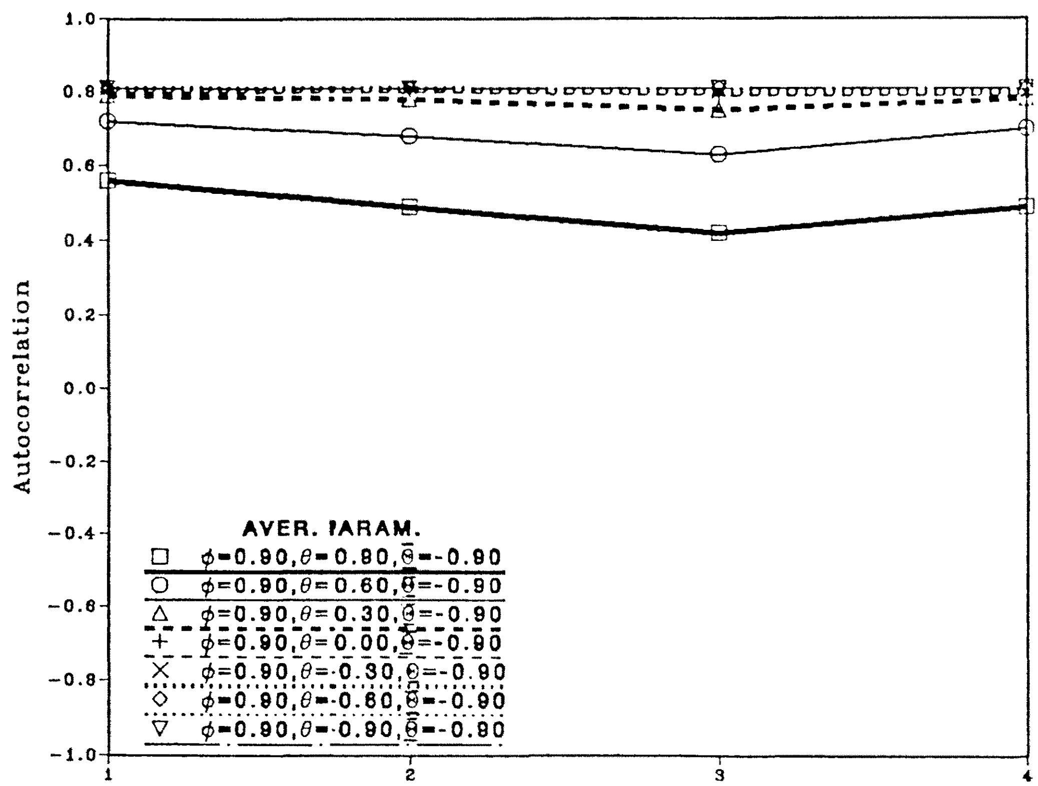

Figure 3.40-Lag 4 seasonal autocorrelograms for Mult. PARMA(1,1)x(0,1)-Averages $\phi_{1,\tau}$=0.9, $\Theta_{1,\tau}$=-0.9

Analysis of figures 3.02 through 3.40 shows that the seasonal parameters $\Phi_{1,\tau}$ are the most important with respect to the capability of increasing the annual and large lag dependence. The PARMA(1,1) models have proven to be very reliable for modeling seasonal dependence for low and moderately autocorrelated seasonal series, which are the most common in practical hydrology. However, Obeysekera and Salas (1986) found out that reproduction of higher annual autocorrelation by this model is worsened when the number of seasons is increased.

It was verified previously in this dissertation that for PAR(1), or Thomas-Fiering, and PARMA(1,1) models, the corresponding annual series fits a regular ARMA(1,1) model with autoregressive parameters equal to the product of all the seasonal autoregressive parameters. Since the parameter space of the seasonal autoregressive parameters for the PARMA(1,1) models is given by the expression

$$\phi_{1,1} \cdot \phi_{1,2} \cdots \phi_{1,w} < 1 \qquad (3.89)$$

and in most practical cases these seasonal autoregressive parameters are less than 1, the number of seasons can be an important factor affecting the annual autocorrelation. For example, assuming that the seasonal parameters are all equal to 0.90(high values), the annual autoregressive parameters for the case of 2 seasons, w=2, would be

$$\phi = 0.9^2 = 0.81$$

For the case of 4 seasons ,w=4 , would result in :

$\phi = 0.9^4 = 0.66$

For monthly data, w=12, the result would be :

$\phi = 0.9^{12} = 0.28$

These values show that, for the above models, the annual lag 1 autocorrelation decreases exponentially when the number of seasons increases. Hence, it would be important to verify if the models presenting the parameters $\Phi_{1},_{\tau}$ are able to reproduce high annual dependence for larger number of seasons. Simulations of twelve-season Multiplicative PARMA(1,0)x(1,0) samples were made. Annual autocorrelograms are shown in figure 3.41. Seasonal autocorrelograms for lags 1, 2 and 12 are shown on figures 3.42, 3.43 and 3.44.

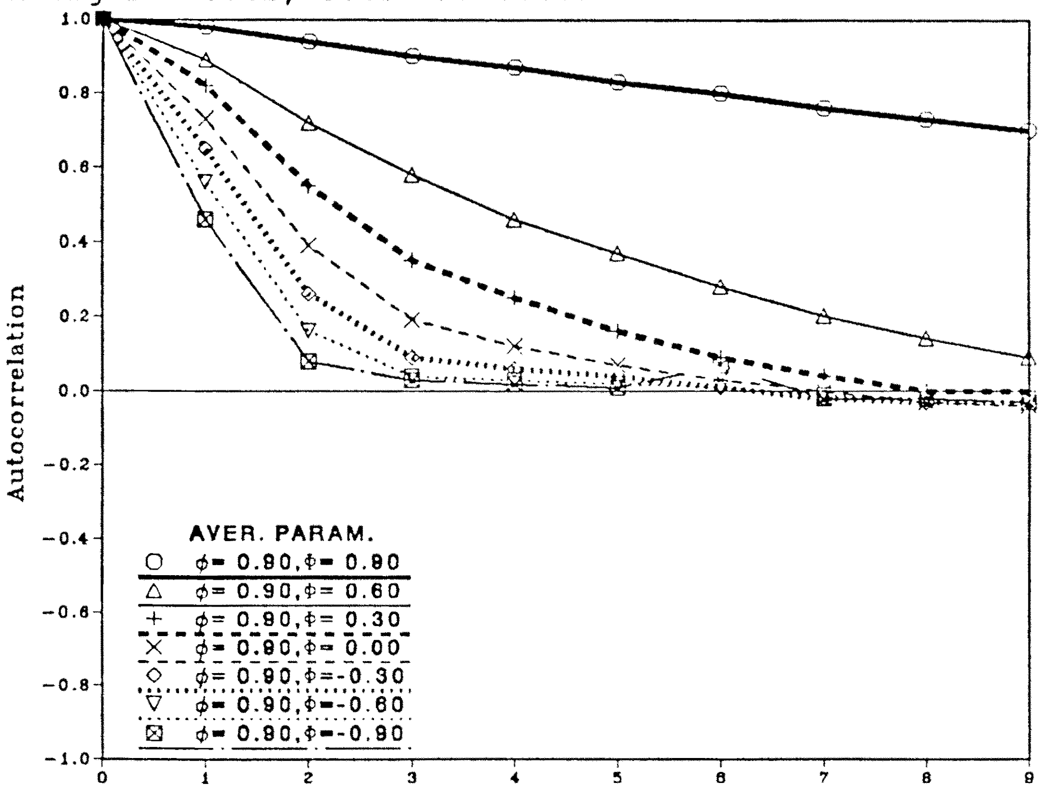

Figure 3.41-Annual autocorrelograms for Multiplicative PARMA(1,0)x(1,0)- Average parameter $\phi_{1,\tau}$=0.9 -12 seasons

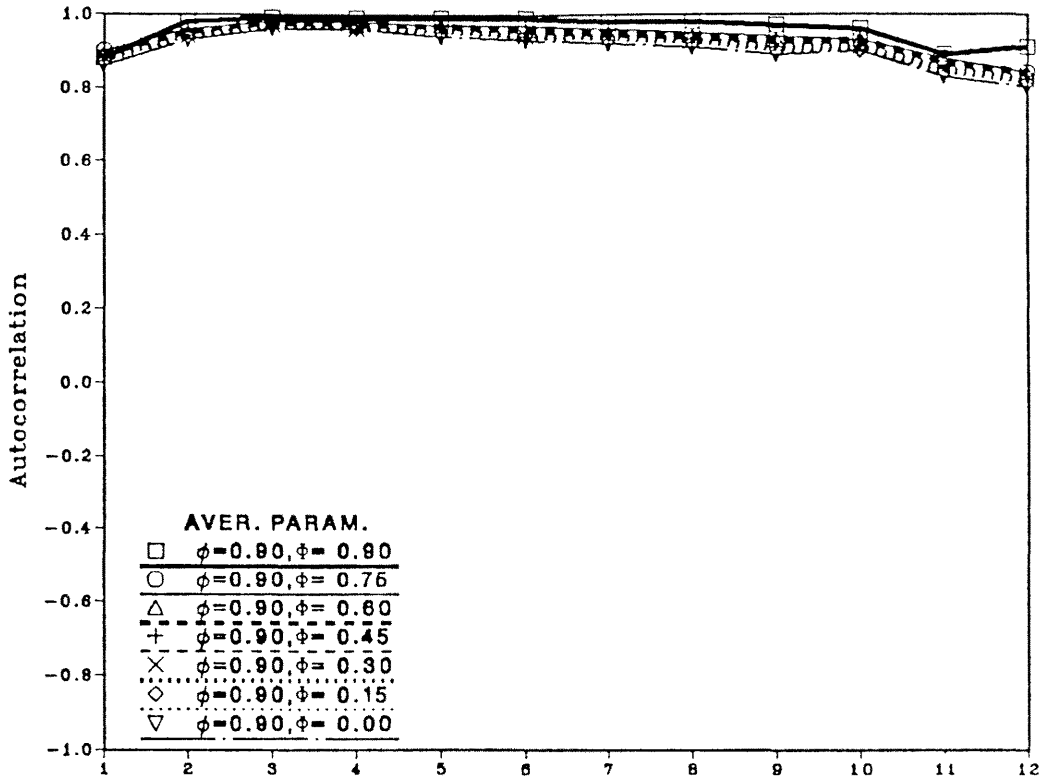

Figure 3.42 - Lag 1 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models - Average $\phi_{1,\tau}$ =0.9

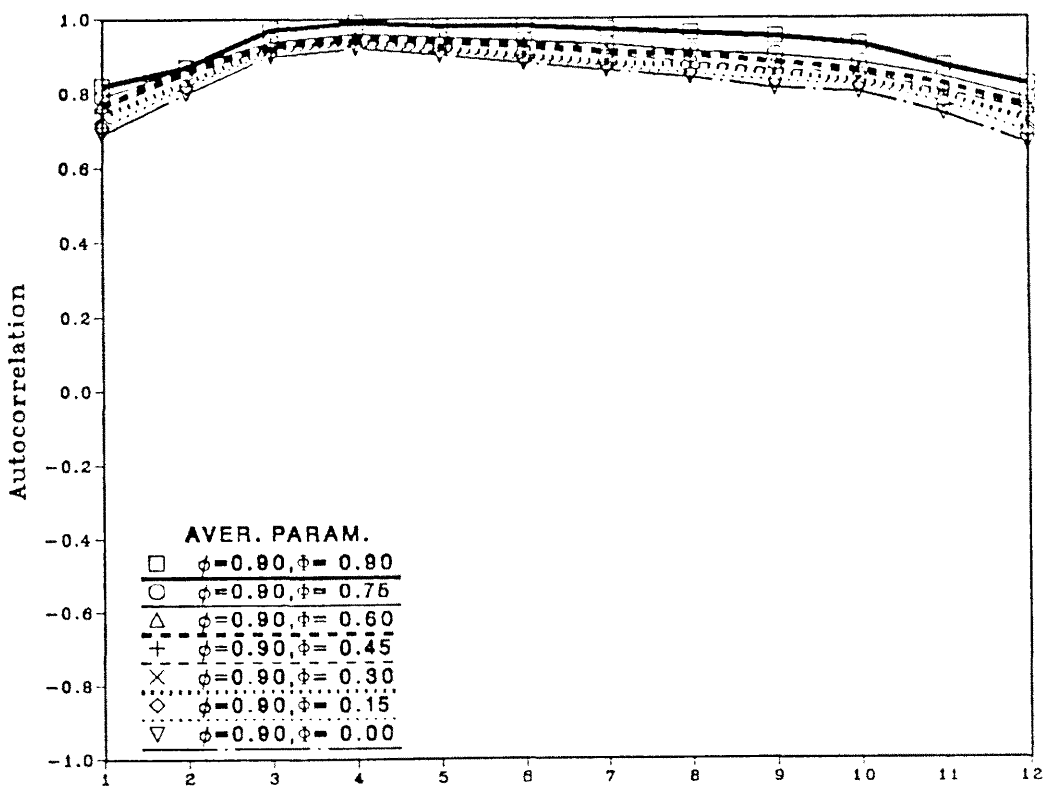

Figure 3.43 - Lag 2 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models - Average $\phi_{1,\tau}$ =0.9

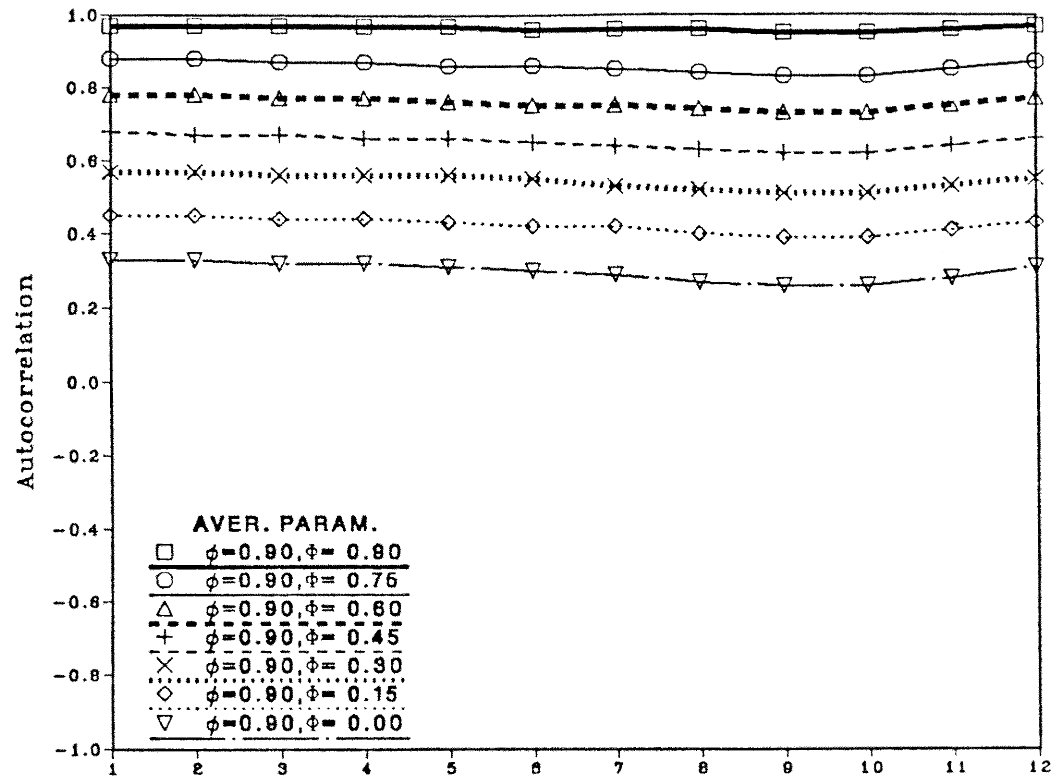

Figure 3.44 - Lag 12 seasonal autocorrelograms for Multiplicative PARMA(1,0)x(1,0) models - Average $\phi_{1,\tau}$ =0.9

Figures 3.6, 3.11, 3.16, 3.21 and 3.41 for annual autocorrelograms corresponding to Multiplicative PARMA(1,0)x(1,0) models show that the seasonal parameters $\Phi_{1,\tau}$ can greatly increase the values of autocorrelations, for all lags, over those obtained by using only the lag 1 seasonal autocorrelation parameters $\phi_{1,\tau}$. As an example, the Figure 3.21 shows that for average parameter $\phi_{1,\tau}$ =0.0 the lag 1 annual autocorrelation increases from zero, for average of parameters $\Phi_{1,\tau}$ equal to zero, to 0.95 for the average of the same parameters equal to 0.9. Figure 3.41 also shows that the annual autocorrelations for the PARMA(1,0)x(1,0) models remain high for monthly samples.

Figure 3.26 shows that the parameters $\underline{\Theta}_{1,\tau}$ are only able to increase the annual autocorrelations slightly. Figures 3.31 and 3.36 show that the same happens for the parameters $\theta_{1,\tau}$.

Figures for the seasonal autocorrelations of Multiplicative PARMA(1,0)x(1,0) models show that the parameters $\Phi_{1,\tau}$ can increase the seasonal autocorrelations considerably, and the effects are larger for larger lags. Figure 3.25, for average of parameters $\phi_{1,\tau}$ equal to zero, shows average lag 4, (between the same season in consecutive years), increasing from average zero to average 0.90. Figures 3.25 through 3.28 show that the parameters $\underline{\overline{\theta}}_{1,\tau}$ can increase the seasonal autocorrelations only slightly. The differences are higher for higher lags.

Figures 3.32 and 3.33 show that parameters $\Theta_{1,\tau}$ can increase the seasonal autocorrelations for small lags.

In general, it can be concluded that the parameters $\Phi_{1,\tau}$ are those able to cause larger increments both to annual and to seasonal autocorrelations. However, it does not mean that other parameters are not important. They can allow better fit to particular data. It is widely recognized that, for example, the PARMA(1,1) gives better fitting to more hydrologic samples than the PAR(1) model does.

It can be concluded that Multiplicative PARMA models presenting parameters $\Phi_{1,\tau}$ represent good alternatives in the selection of models when preservation of annual and large lag seasonal autocorrelations are important.

3.9 Parsimony and Analysis of Fitting

3.9.1 Remarks

Before reporting on application of the developed technique to real hydrologic data, discussion about parsimony seems important due to the fact that most Multiplicative PARMA models present more parameters than other existing models for seasonal series.

The selection of the historical statistical properties of a sample to be preserved by a stochastic model depends on the variability of the series and on the future uses of the model (Salas et al. ,1980). The model should preserve the statistics that are necessary for reproducing the variability of the series in analysis and that are important for the solution of the hydrological problem being solved. After deciding which properties are important, one must select a model that could reproduce these properties with some degree of accuracy. A balance between preservation of statistics and complexity of models must be maintained. By the principle of parsimony, the best model is theone that can reproduce the important statistics with a minimum number of parameters. It must be remarked here that the important statistics are different for each particular case. A model that is satisfactory for one modeling purpose may not be for other objectives.

Models that are too simple can fail to preserve important statistics. Models that are too complex can present difficult application and sometimes reproduce only sampling variability of historical series rather than real properties.

For seasonal hydrologic series modeling, ARIMA models have fewer parameters compared with models presenting varying parameters. However, the ARIMA models are not adequate for simulation. Using these models, it is common to simulate streamflows without floods and droughts present in historical series(Bras and Rodriguez-Iturbe,1985) . This results from the fact that ARIMA models are not capable of reproducing seasonal variances and autocorrelations.

For the design of reservoirs, failure to represent multi-lag correlations among the various months or seasons can cause serious mistakes. The reservoirs may be underdesigned if seasonal correlations higher than one are significant but not considered. Inadequate use of PAR(1) models (Thomas-Fiering) may result in this kind of error. Correct representation of annual dependence is also important in many cases for water resources planning because it affects the occurrence of large periods of floods and droughts.

Previously in this chapter, the development of a class of models, Multiplicative PARMA, that generally present more parameters than those seasonal models presently utilized in practice of stochastic hydrology was described. However, the use of this model can be justified if it is judged by the modeler that the preservation of annual and seasonal autocorrelations of the historic sample by models with fewer parameters is not adequate. The following table presents the number of parameters for different models for a series of 12 seasons, including some disaggregation models:

Table 3.24 - Number of parameters for various models for monthly series

$ARIMA(p,d,q)x(P,D,Q)_{12}$	p+d+q+P+Q+D+1
$ARIMA(1,0,1)x(1,0,1)_{12}$	5
$ARIMA(2,0,1)x(2,0,1)_{12}$	7
PARMA(p,q)	(p+q+1)x12
PAR(1)	24
PAR(2)	36
PAR(3)	48
PARMA(1,1)	36
PARMA(2,1)	48
Multiplicative PARMA(p,q)x(P,Q)	(p+q+P+Q+1)x12
Multiplicative PARMA(1,0)x(1,0)	36
Multiplicative PARMA(1,1)x(1,0)	48
Multiplicative PARMA(1,1)x(1,1)	60
Multiplicative PARMA(1,0)x(2,0)	48
Multiplicative PARMA(2,1)x(1,0)	60
Multiplicative PARMA(3,1)x(1,1)	84
Lane (Disaggregation)	36
Valencia and Schaake (Disaggregation)	90
Mejia and Rousselle (Disaggregation)	102

The number of parameters for the disaggregation models on Table 3.24 does not include the necessary parameters for fitting the annual series. Although disaggregation models present many more parameters than the ARIMA and PARMA models, they have been used extensively in practical hydrology. This utilization is justified by hydrologists on the necessity of preserving long term autocorrelation of historical samples in the solution of many water resource problems.

The utilization of Multiplicative PARMA models can also be justified using the same concept if they are able to improve the preservation of this long-term dependence over the other models with fewer parameters. The Multiplicative PARMA models also present an important advantage in comparison with the disaggregation models; that is, they deal directly with the seasonal series, avoiding the aggregation of this series and the estimation from annual series that is subjected to problems described previously. Existing tests of goodness of fit and selection criteria of models for seasonal series are dicussed in the next subchapter.

3.9.2 Model Testing and Selection

3.9.2.1 Tests of Goodness of Fit

Various statistical tests have been utilized for the analysis of fitting of models, by verifying if residuals are normally and independently distributed. One of the most common tests of normality is made by verifying if the skewness coefficient is not significantly different from zero. If residuals have periodic variance, the tests of normality may be applied to each season individually or to the complete series by scaling the seasonal residuals as $\epsilon'_{v,\tau} = \epsilon_{v,\tau} / \sigma_\tau(\epsilon)$ (Salas et al.,1980). Thompstone(1984) indicated that the normality assumption is usuallly much less important than the independence assumption.

A method for verifying the independence of residuals is to plot the autocorrelogram of the residuals along with the upper and lower confidence limits that can be calculated using the expression

$$CL_{\alpha}(r_k) = \frac{-1 \pm u_{\alpha}\sqrt{N-k-1}}{N-k} \tag{3.90}$$

where u_{α} is the standard normal deviate corresponding to a confidence level α, N is the sample size, k is the lag and r_k is the autocorrelation at lag k.

The most accepted tests for independence of residuals are those called portmanteau lack-of-fit tests.

The first of these kinds of tests is based on the scaled residuals defined above. The corresponding statistic is :

$$Q_1 = Nw \sum_{k=1}^{L} r_k^2(\epsilon') \tag{3.91}$$

where N=number of years, w= number of seasons, $r_k(\epsilon')$ is the estimated serial lag k correlation of residuals and L is the maximum number of lags considered. Commonly, L = 0.25 Nw. The distribuition for Q_1 was assumed approximately chi-square with L - N_p degrees of freedom by Tao and Delleur(1976). N_p symbolizes the number of the parameters of the model.

Tao and Delleur(1976) also suggested the use of another statistic, computed directly from the seasonal residuals, for comparison among models:

$$Q_2 = N \sum_{\tau=1}^{w} \sum_{k=1}^{L} r^2{}_{k,\tau}(\epsilon) \tag{3.92}$$

where $r_{k,\tau}(\epsilon)$ is the estimated seasonal lag k correlation of residuals, L is the maximum number of lags considered, commonly 0.25 N.

McLeod and Hipel(1983) proposed another statistic :

$$Q_3 = N(N+2) \sum_{\tau=1}^{w} \sum_{k=1}^{L} r^2_{k,\tau}(\epsilon) \tag{3.93}$$

Another portmanteau test statistic was proposed by Thompstone(1984):

$$Q_4 = Q_2 + w\ L(L+1)/2N \tag{3.94}$$

All elements from equations (3.96) and (3.97) were previously defined. For comparison among tests using the statistics Q_1, Q_2, Q_3 and Q_4, the assumption of chi-square distributions, with L-Np degrees of freedom for Q_1 and $w(L-N_p)$ degrees of freedom for Q_2, Q_3 and Q_4, was utilized.

Some simulations of Multiplicative PARMA model samples were made for analysis of the performance of the statistics for testing the goodness of fit for Multiplicative PARMA models. Firstly, large values of the parameters $\Phi_{1,\tau}$ and $\underline{\overline{\Theta}}_{1,\tau}$ were assumed to check if the tests would accept models with $\Phi_{1,\tau}$ and $\underline{\overline{\Theta}}_{1,\tau}$ null (PAR(1), PARMA (1,1) and PAR(2)) as better models only because the number of parameters affects the number or degrees of freedom and consequently the critical value of the chi-square distribuition. For all cases, $N_p = (p+q+P+Q+1)w$ and a 95% confidence limit were used.

Parameters :

$\phi_1=\phi_4= 0.9$ $\phi_2=\phi_3= 0.6$ $\Phi_1=\Phi_4= 0.9$ $\Phi_2=\Phi_3= 0.6$

$\theta_1=\theta_2=\theta_3=\theta_4= 0.75$ $\underline{\overline{\theta}}_1=\underline{\overline{\theta}}_2=\underline{\overline{\theta}}_3=\underline{\overline{\theta}}_4= 0.75$

$\sigma_1=\sigma_2=0.25$ $\sigma_3=\sigma_4=0.1$

Table 3.25 shows the average portmanteau statistics, critical chi-square values and number of samples passing each test as a PAR(1), PAR(2), PARMA(1,1) or Multiplicative PARMA(1,1)x(1,1) model for one hundred 100-year generated samples.

Table 3.26 presents the same characteristics for 50-year samples.

Table 3.25 - Portmanteau statistics - 100 samples
100 years

Model	Aver. Q_1	Crit. Value	Cases Passing	Aver. Q_2	Crit. Value	Cases Pass.
PAR(1)	183.4	118.7	14	210.9	115.3	0
PAR(2)	146.8	114.2	22	176.7	110.9	1
PARMA(1,1)	125.3	114.2	15	172.6	110.9	0
PARMA(1,1)x(1,1)	84.9	105.3	97	95.6	101.9	73
No model passing			03			27

Model	Aver. Q_3	Crit. Value	Cases Passing	Aver. Q_4	Crit. Value	Cases Pass.
PAR(1)	244.3	115.3	0	223.5	115.3	0
PAR(2)	205.1	110.9	0	188.7	110.9	0
PARMA(1,1)	197.8	110.9	0	184.7	110.9	0
PARMA(1,1)x(1,1)	111.6	101.9	32	107.9	101.9	36
No model passing			68			64

Table 3.26 - Portmanteau statistics - 100 samples
50 years

Model	Aver. Q_1	Crit. Value	Cases Pass.	Aver. Q_2	Crit. Value	Cases Pass.
PAR(1)	77.8	62.8	32	72.5	60.5	25
PAR(2)	57.8	58.1	65	53.3	55.8	58
PARMA(1,1)	77.4	58.1	72	75.1	55.8	68
PARMA(1,1)x(1,1)	48.5	48.6	61	48.0	46.0	50
No model passing			20			24

Model	Aver. Q_3	Crit. Value	Cases Pass.	Aver. Q_4	Crit. Value	Cases Pass.
PAR(1)	85.7	60.5	3	78.7	60.5	15
PAR(2)	64.3	55.8	32	81.3	55.8	44
PARMA(1,1)	89.4	55.8	37	81.3	55.8	53
PARMA(1,1)x(1,1)	58.0	46.0	21	54.3	46.0	29
No model passing			51			39

Tables 3.25 and 3.26 indicate that the number of years has a great influence on the tests for smaller samples. For these samples and various other simulations, the statistic Q_1 did the best job of accepting the correct model. Hence, only this statistic will be computed for other samples throughout this dissertation.

3.9.2.2 - Selection Criteria -

Common procedures for selecting among different models are those based on information criteria, called Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The AIC was proposed by Akaike(1974) and the BIC was proposed by Schwarz (1978).

The AIC statistic may be defined as (Thompstone,1983)

$$AIC= \left(\sum_{\tau=1}^{w} - 2 \log L_\tau \right) + 2 \left[w(N_{p+2}) + \delta \right] \qquad (3.95)$$

where $\log L_\tau = - N \log \sigma_\tau(\epsilon) + (\alpha-1) \sum_{v=1}^{N} \log z_{v,\tau}$

in which α is the parameter of the Box-Cox transformation defined by

$$z_{v,\tau}^{(\alpha)} =(1/\alpha) \; (z^{\alpha}_{v,\tau} - 1) \qquad , \alpha \neq 0$$

$$=\log z_{v,\tau} \qquad , \alpha = 0$$

$\delta=0$ if $\alpha=1$ and $\delta=1$ if $\alpha\neq1$.

$\sigma_\tau(\epsilon)$ is the estimated standard deviation for residuals,N is the number of years and w the number of seasons.

The Bayesian Information Criterion statistic may be expressed as (Thompstone, 1983)

$$BIC = \left(\sum_{\tau=1}^{w} -2 \log L_\tau \right) + \left[w(N_p+2)+ \delta\right] \log(Nw) \qquad (3.96)$$

All terms in expression (3.96) were previously defined and the selection criterion is to choose the model which has the smallest AIC or BIC. Throughout this dissertation only logarithm transformations were employed.

The same simulated samples, utilized for analysis of portmanteau statistics, were employed for analyzing the reliability of the criteria for selecting among different PARMA models.

Results for the application of the two criteria are shown in Tables 3.27 and 3.28.

Table 3.27 -Number of samples for which each model was selected as best, by AIC and BIC - 100 years

Model	AIC	BIC
PAR(1)	0	6
PAR(2)	0	0
PARMA(1,1)	3	7
Mult.PARMA(1,1)x(1,1)	97	87

Table 3.28 -Number of samples for which each model was selected as best, by AIC and BIC - 50 years

Model	AIC	BIC
PAR(1)	30	96
PAR(2)	6	2
PARMA(1,1)	6	2
Mult.PARMA(1,1)x(1,1)	58	0

As the above numbers indicate, the AIC and BIC statistics are also greatly influenced by the number of parameters, and this influence increases for smaller samples. The Akaike Information Criterion in both cases does the better job of selecting the correct model as the best. So, only this criterion will be utilized for future simulations.

3.9.2.3 Autocorrelation and Selection

Data generation was conducted to observe how the portmanteau statistics and Akaike Information Criteria combine with the analysis of preservation of annual and seasonal correlations. Samples presenting 50 years and 100 years were simulated by using three different sets of parameters.

a) Parameters :

$\phi_{1,1} = \phi_{1,4} = 0.85$ $\qquad$ $\phi_{1,2} = \phi_{1,3} = 0.65$

$\Phi_{1,1} = \Phi_{1,4} = 0.85$ $\qquad$ $\Phi_{1,2} = \Phi_{1,3} = 0.65$

$\theta_{1,1} = \theta_{1,2} = \theta_{1,3} = \theta_{1,4} = 0.75$

Table 3.29 - Average portmanteau and AIC statistics - 100 simulations - 100 years

Model	Aver. Q_1	Crit. Q_1	Samp. Pass.	Aver. AIC	Best Model
PAR(1)	727.5	118.7	0	-1247	0
PAR(2)	598.4	114.2	0	-1280	0
PARMA(1,1)	1357.8	114.2	0	-1128	0
Mult.PARMA(1,1)x(1,1)	83.1	105.3	97	-1645	97

Table 3.30 - Average portmanteau and AIC statistics - 100 simulations - 50 years

Model	Aver. Q_1	Crit. Q_1	Samp. Pass.	Aver. AIC	Best Model
PAR(1)	264.7	62.8	0	-646	0
PAR(2)	224.5	58.1	0	-660	0
PARMA(1,1)	330.2	58.1	0	-402	0
Mult.PARMA(1,1)x(1,1)	45.1	48.6	81	-830	81

Tables 3.29 and 3.30 show that in this case the portmanteau statistic Q_1 and the Akaike Information Criteria correctly reject the models PAR(1), PAR(2) and PARMA(1,1) in all cases and select the PARMA(1,1)x(1,1) as the best model.

Figure 3.45 shows annual autocorrelograms obtained from simulations utilizing average parameters estimated by the different models for 100-year samples for comparison with the autocorrelograms for the correct model and parameters.

Figures 3.46 and 3.47 show average lag 1 and lag 4 seasonal autocorrelations for the same purpose.

Figure 3.48 shows annual autocorrelograms, and figures 3.49 and 3.50 seasonal lag 1 and lag 4 autocorrelations corresponding to the purposes of figures 3.45 through 3.47 but for parameters from 50-year samples.

Figures 3.45 through 3.50 indicate that the PAR(1), PAR(2) and PARMA(1,1) models are not able to preserve the annual and lag 4 seasonal autocorrelations for the original Mult. PARMA(1,1)x(1,1) model and were correctly rejected by the portmanteau test and AIC criteria.

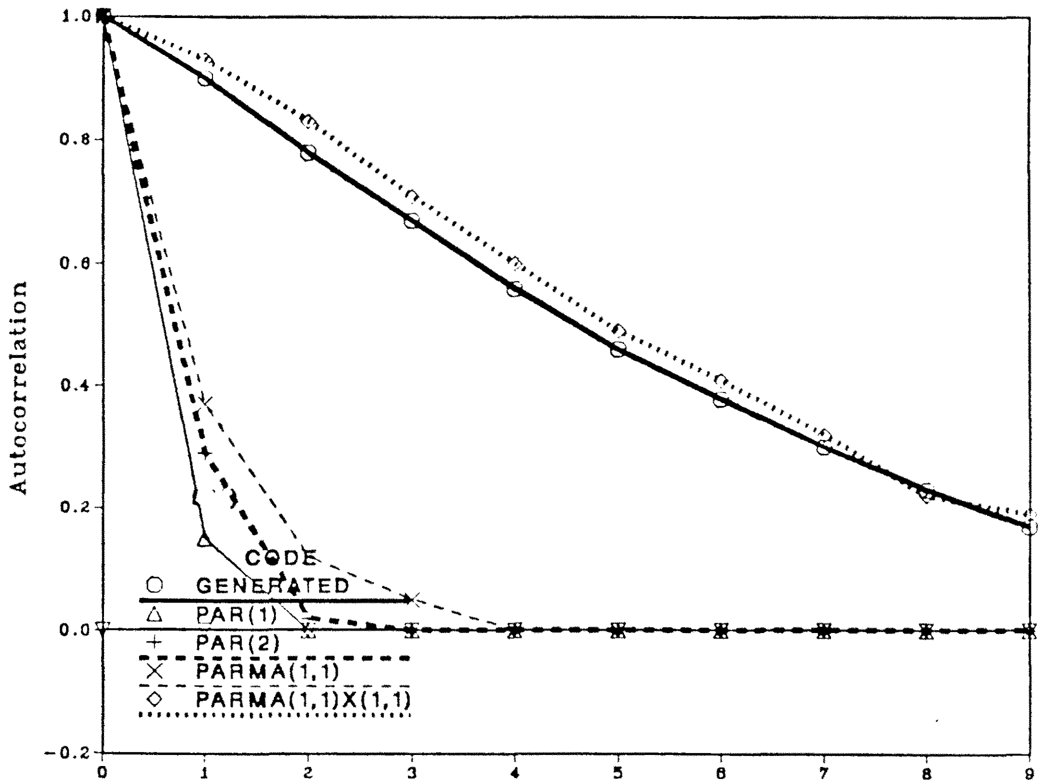

Fig.3.45-Annual autocorrelograms-For Mult.PARMA(1,1)x(1,1) and for average estimated parameters- 100 years

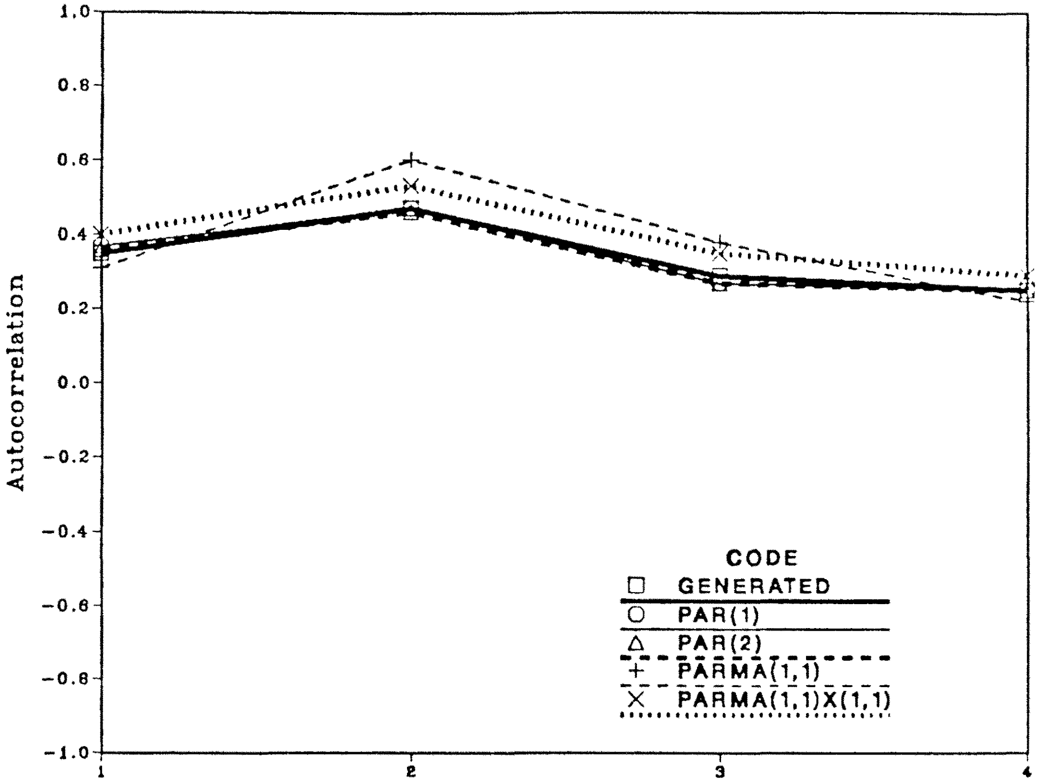

Fig.3.46-Lag 1 autocorrelations-For Mult.PARMA(1,1)x(1,1) and for average estimated parameters - 100 years

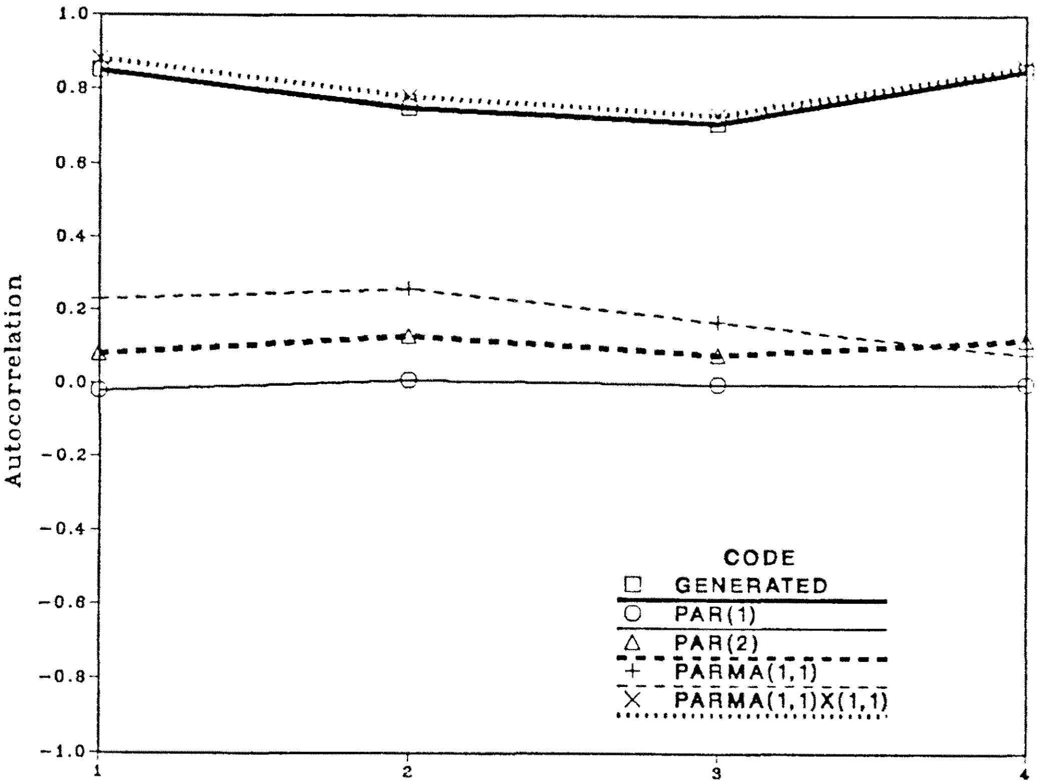

Fig.3.47-Lag 4 autocorrelations- For Mult.PARMA(1,1)x(1,1) and for average estimated parameters - 100 years

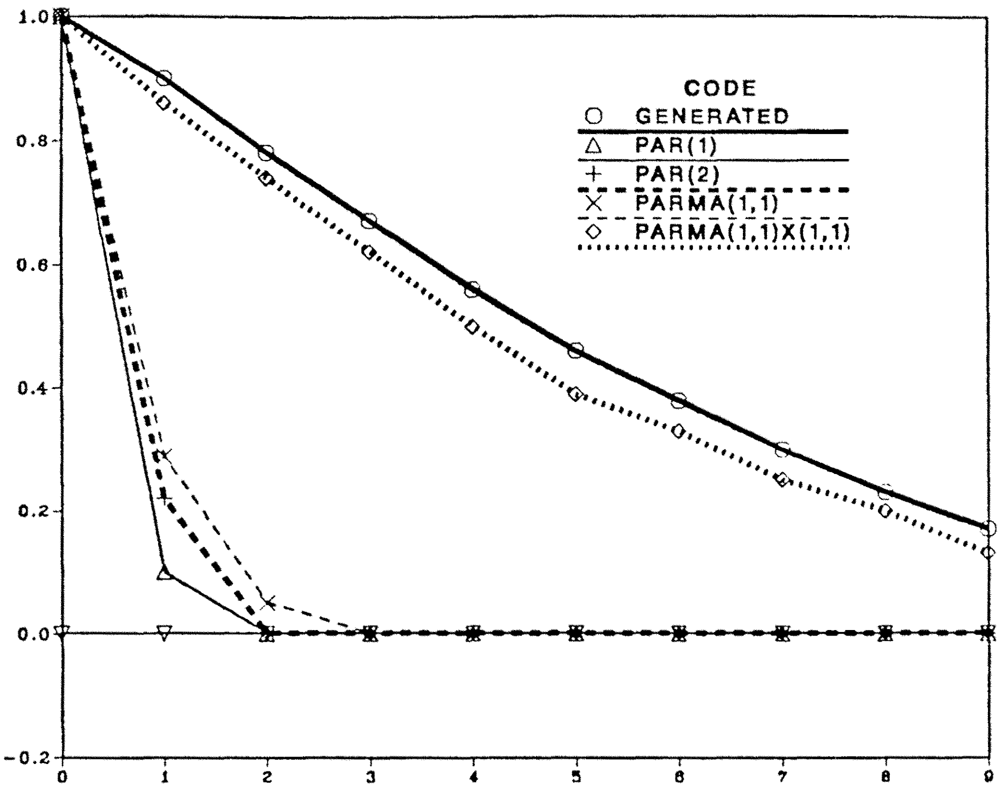

Fig.3.48-Annual autocorrelograms-For Mult.PARMA(1,1)x(1,1
and for average estimated parameters - 50 years

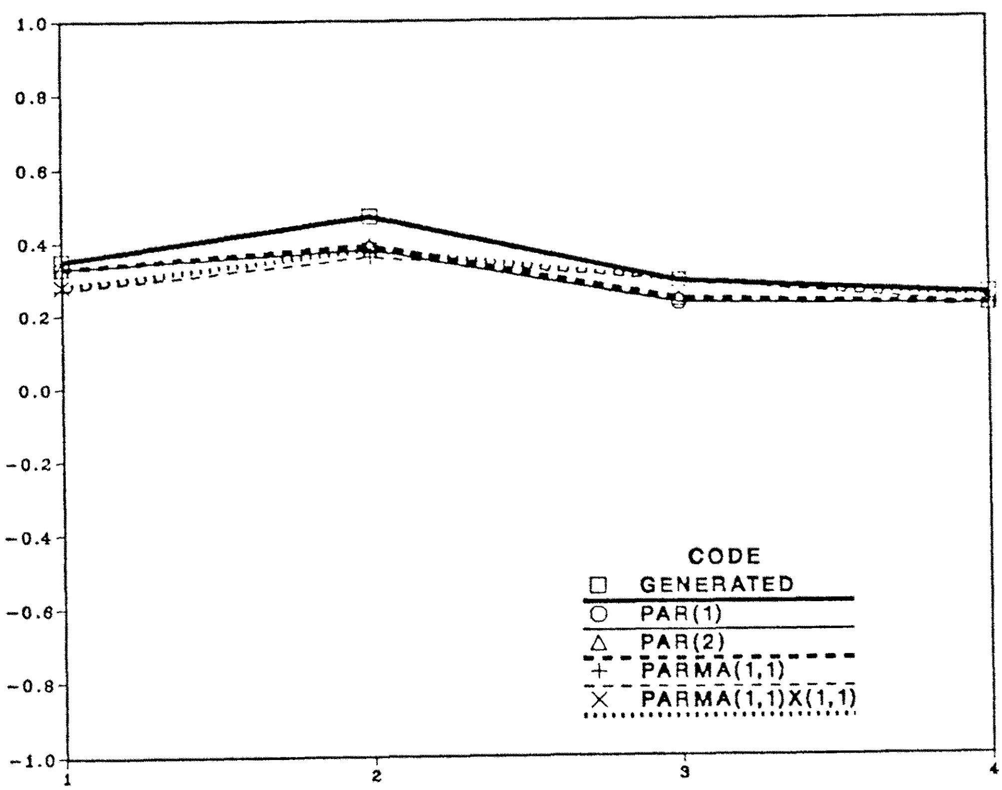

Fig.3.49-Lag 1 autocorrelations-For Mult.PARMA(1,1)x(1,1)
and for average estimated parameters - 50 years

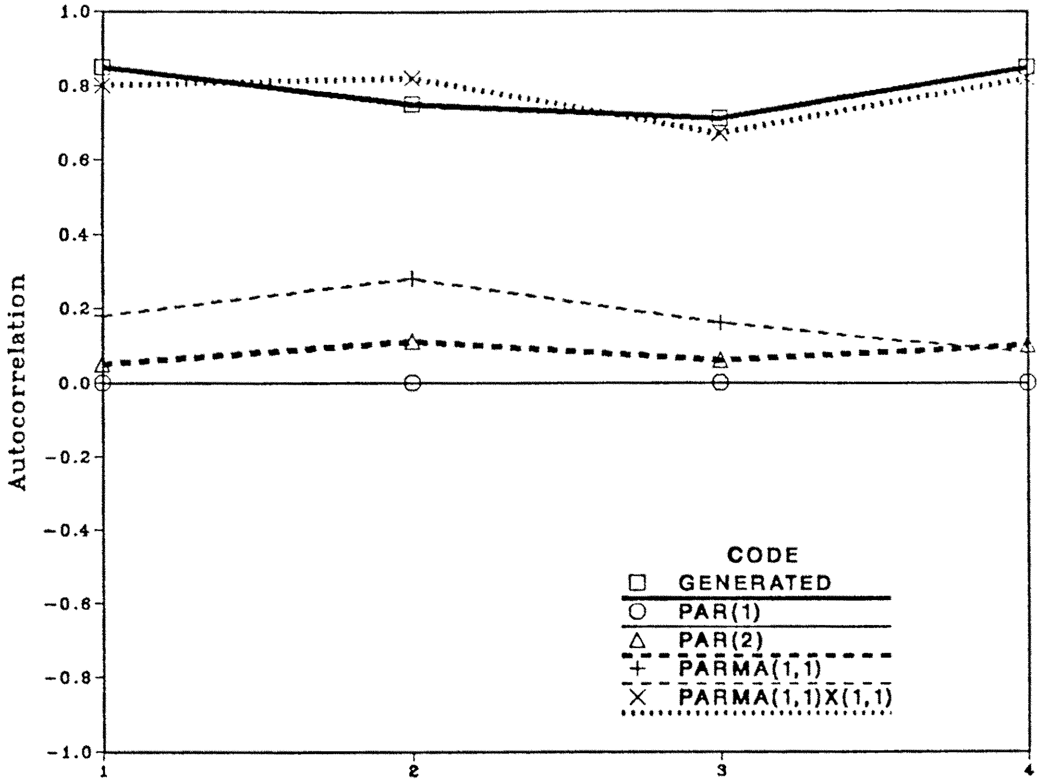

Fig.3.50-Lag 4 autocorrelograms-For Mult.PARMA(1,1)x(1,1) and for average estimated parameters - 50 years

b) Parameters :

$\phi_{1,1} = \phi_{1,4} = 0.40$ $\qquad\qquad$ $\phi_{1,2} = \phi_{1,3} = 0.20$

$\Phi_{1,1} = \Phi_{1,4} = 0.85$ $\qquad\qquad$ $\Phi_{1,2} = \Phi_{1,3} = 0.65$

$\theta_{1,1} = \theta_{1,2} = \theta_{1,3} = \theta_{1,4} = 0.75$

Table 3.31 - Average portmanteau and AIC statistics - 100 simulations - 100 years

Model	Aver. Q_1	Crit. Q_1	Samp. Pass.	Aver. AIC	Best Model
PAR(1)	711.4	118.7	0	-1285	0
PAR(2)	568.5	114.2	0	-1315	0
PARMA(1,1)	833.4	114.2	0	-824	0
Mult.PARMA(1,1)x(1,1)	80.8	105.3	92	-1648	92

Table 3.32 - Average portmanteau and AIC statistics - 100 simulations - 50 years

Model	Aver. Q_1	Crit. Q_1	Samp. Pass.	Aver. AIC	Best Model
PAR(1)	249.3	62.8	0	-667	0
PAR(2)	180.0	58.1	0	-689	0
PARMA(1,1)	259.8	58.1	0	-416	0
Mult.PARMA(1,1)x(1,1)	44.6	48.6	78	-815	78

Tables 3.31 and 3.32 show that also in this case, the portmanteau statistic Q_1 and the Akaike Information Criteria correctly reject, in all cases, the models PAR(1), PAR(2) and PARMA(1,1) and select the Multiplicative PARMA(1,1)x(1,1) as the best model.

Figure 3.51 shows annual autocorrelograms obtained from simulations using average parameters estimated by the different models for 100-year samples for comparison with the autocorrelograms for the correct model and parameters.

Figures 3.52 and 3.53 show average lag 1 and lag 4 seasonal autocorrelations for the same purpose.

Figure 3.54 shows annual autocorrelograms, and figures 3.55 and 3.56 present seasonal lag 1 and lag 4 autocorrelations with the same objective described for the case of 100-year samples but now using the 50-year samples

Analysis of figures 3.51 through 3.56 indicates that the PAR(1), PAR(2) and PARMA(1,1) models also for this case are not able to preserve the annual and lag 4 seasonal autocorrelations for the original Multiplicative PARMA(1,1)x(1,1) model and were correctly rejected by the portmanteau test and the AIC.

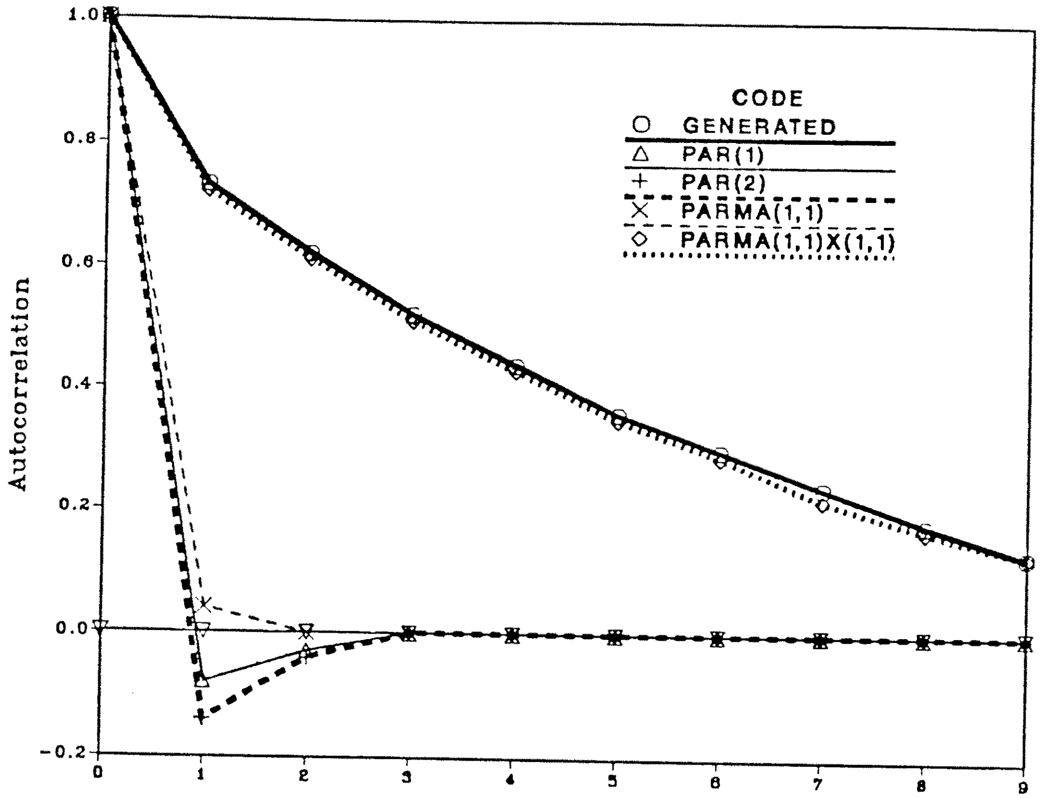

Fig.3.51-Annual autocorrelograms-For Mult.PARMA(1,1)x(1,1) and for average estimated parameters- 100 years

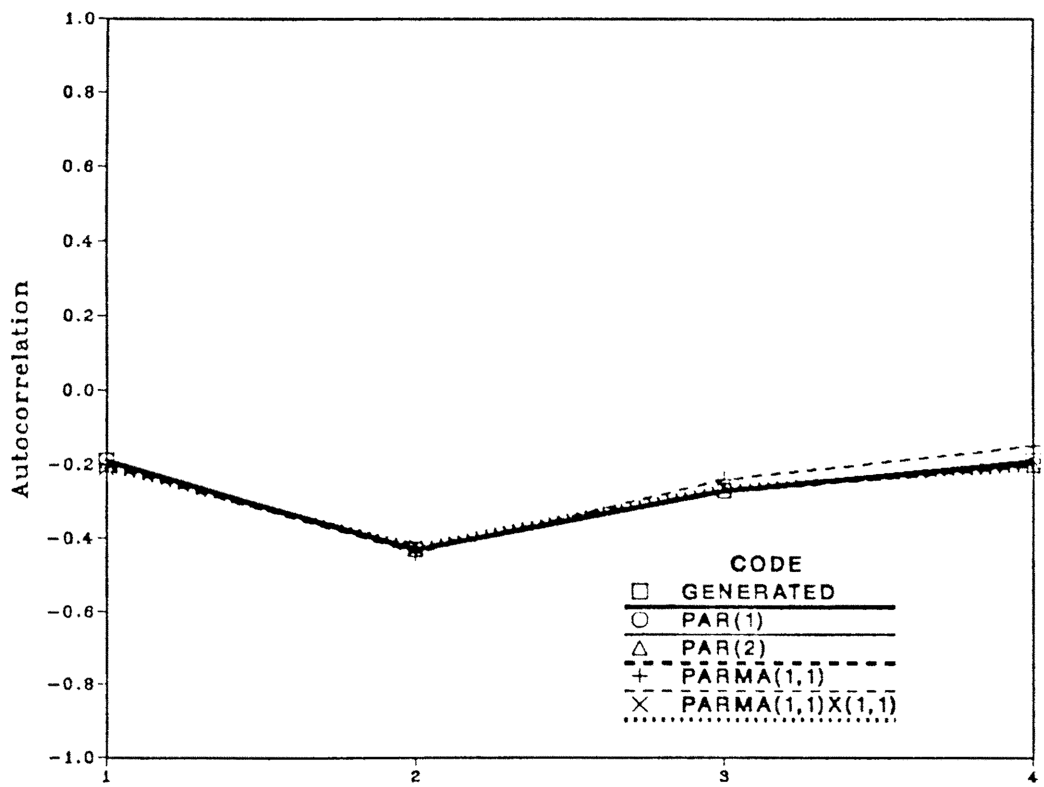

Fig.3.52 -Lag 1 autocorrelations-For Mult.PARMA(1,1)x(1,1) and for average estimated parameters- 100 years

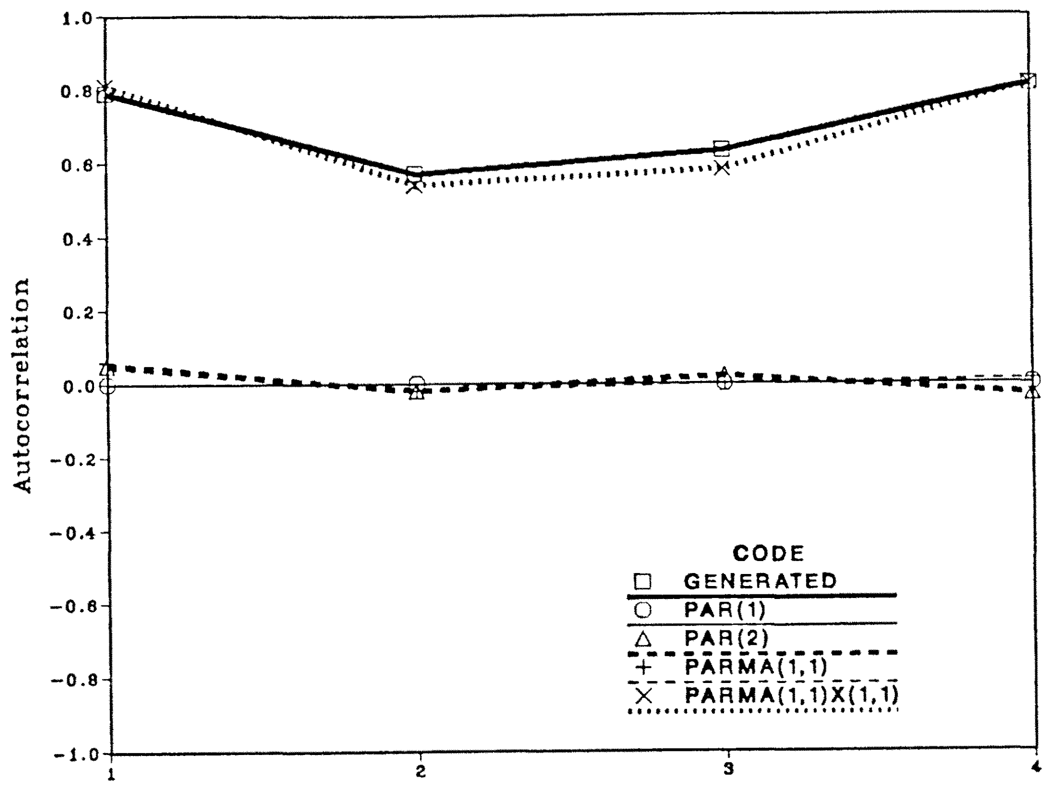

Fig.3.53 -Lag 4 autocorrelations-For Mult.PARMA(1,1)x(1,1) and for average estimated parameters - 100 years

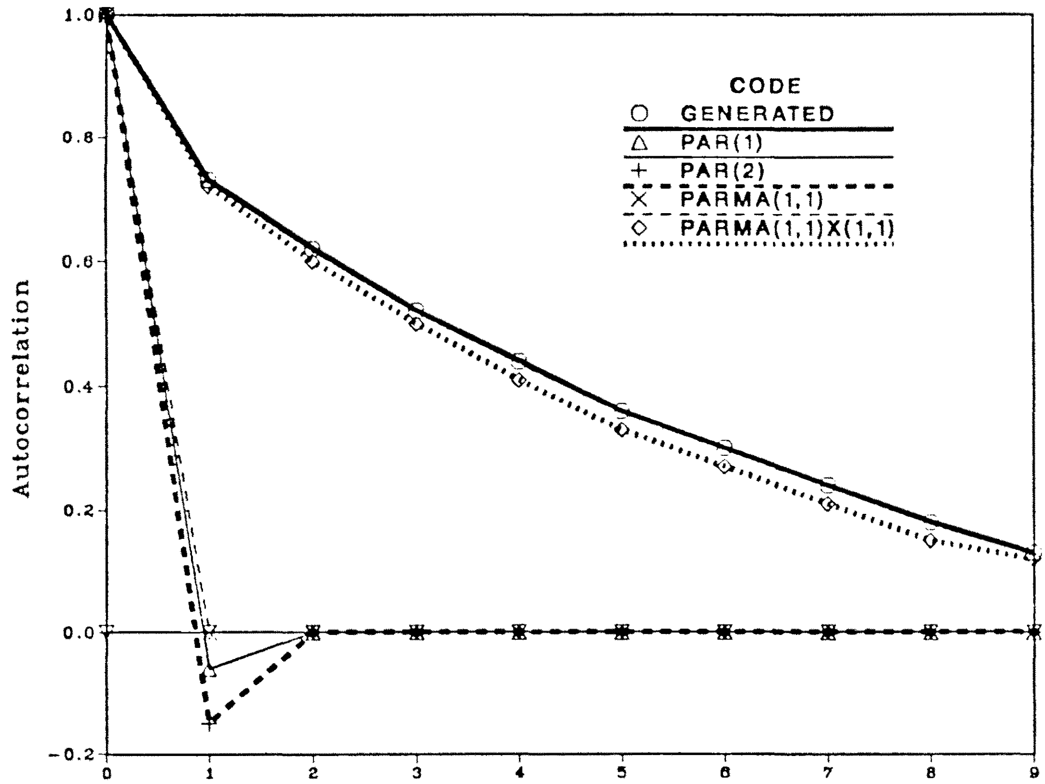

Fig.3.54-Annual autocorrelograms-For Mult.PARMA(1,1)x(1,1) and for average estimated parameters - 50 years

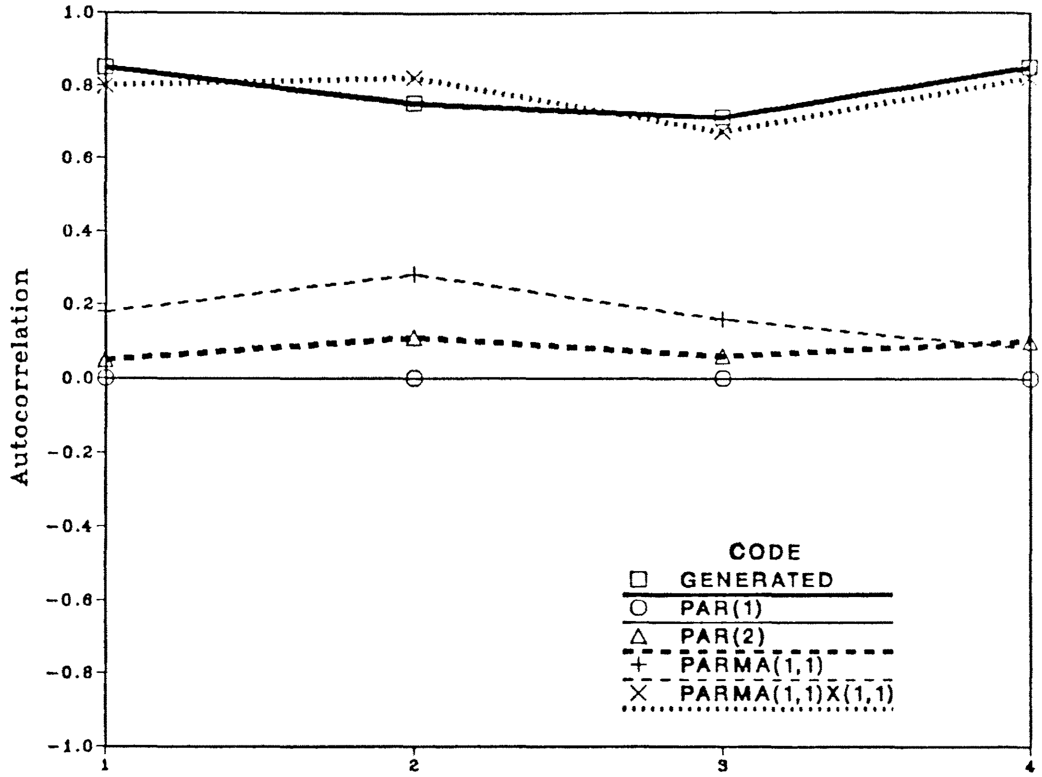

Fig.3.55-Lag 1 autocorrelations- For Mult.PARMA(1,1)x(1,1) and for average estimated parameters - 50 years

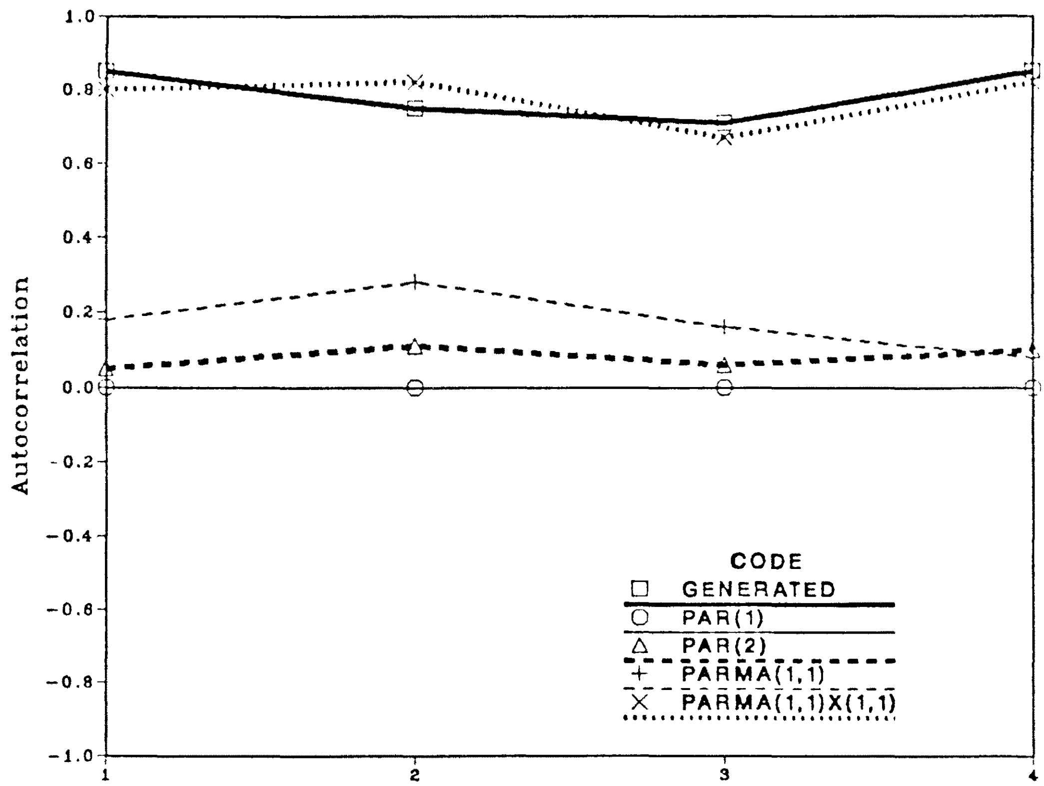

Fig.3.56-Lag 4 autocorrelograms-For Mult.PARMA(1,1)x(1,1) and for average estimated parameters - 50 years

c) Parameters :

$\phi_{1,1} = \phi_{1,4} = 0.85$ $\phi_{1,2} = \phi_{1,3} = 0.65$

$\Phi_{1,1} = \Phi_{1,4} = 0.10$ $\Phi_{1,2} = \Phi_{1,3} = -0.10$

$\theta_{1,1} = \theta_{1,2} = \theta_{1,3} = \theta_{1,4} = 0.75$

Table 3.33 - Average portmanteau and AIC statistics - 100 simulations - 100 years

Model	Aver. Q_1	Crit. Q_1	Samp. Pass.	Aver. AIC	Best Model
PAR(1)	87.0	118.7	92	-1657	14
PAR(2)	85.8	114.2	94	-1659	41
PARMA(1,1)	87.8	114.2	93	-1486	33
Mult.PARMA(1,1)x(1,1)	88.0	105.3	89	-1647	12

Table 3.34 - Average portmanteau and AIC statistics - 100 simulations - 50 years

Model	Aver. Q_1	Crit. Q_1	Samp. Pass.	Aver. AIC	Best Model
PAR(1)	53.1	62.8	95	-820	41
PAR(2)	50.5	58.1	93	-817	27
PARMA(1,1)	53.1	58.1	94	-795	29
Mult.PARMA(1,1)x(1,1)	45.1	45.6	72	-811	07

In this case, the true model is approximately a PARMA(1,1) model and the results differ greatly from those obtained for cases (a) and (b), where the true model was a Multiplicative PARMA with very high $\Phi_{1,\tau}$ parameters.

The acceptance of the models here is correct because in this case all models showed annual and seasonal autocorrelations very close to those for the original parameters.

The analysis of results for the simulations indicates that the use of AIC and Q_1 statistics are useful for preliminary tests of fitting and comparison among competing models.

For final acceptance of a model, it is recommended that data be generated and that the annual and seasonal autocorrelograms be plotted for comparison. Many authors recognize the importance of the preservation of multi-lag autocorrelations for the reproduction of long-term hydrologic characteristics such as floods and droughts.

CHAPTER IV

APPLICATION TO HYDROLOGIC DATA

4.1 General Remarks

In order to verify the applicability of the developed model and its estimation procedure in practical hydrology, discharge data from various watersheds were utilized. Here, the results from application to four watersheds with distinct statistical characteristics are reported.

The developed technique was applied to a 90-year sample for the St. Lawrence River at Ogdensburg, (1861-1950); to a 50-year sample for the Niger River at Kaulikoro, (1908-1957); to a 42-year sample for the Elkhorn River at Waterloo(1929-1970); and to a 60-year sample for the Yellowstone River at Corwing Springs(1911-1970).

Various Multiplicative PARMA models were fitted to logarithms of 4-season and 6-season samples from the discharge data for analysis of the influence of the number of seasons on the performance of different models.

PAR(1), PAR(2) and PARMA(1,1) models, which are the most common models with periodic parameters in hydrology, were fitted to the data by the proposed method of least-squares, for comparing their performance with those from other Multiplicative PARMA models fitted by the same technique.

Statistical characteristics for the residuals and for generated data were computed and utilized for checking the fit and for comparing models.

Method-of-moments estimations of parameters for PAR(1), PAR(2) and PARMA(1,1) models also were made for the logarithm of 6-year samples for all of the stations in order to check if there was significant difference from estimates by the method of least-squares.

4.1.1 St. Lawrence River

4.1.1.1 Four-season sample

a) Residual characteristics -

Seasonal means of residuals were very close to zero for every model. The seasonal standard deviations of residuals are listed below.

Table 4.1 - Seasonal standard deviations of residuals
Logarithm of 4-season discharges- St. Lawrence

Model	σ_1	σ_2	σ_3	σ_4
PAR(1)	0.031	0.066	0.059	0.035
PAR(2)	0.030	0.062	0.052	0.036
PARMA(1,1)	0.029	0.063	0.051	0.035
PARMA(2,1)	0.028	0.061	0.048	0.035
Mult.PARMA(2,1)x(1,0)	0.028	0.060	0.049	0.035
Mult.PARMA(1,1)x(1,1)	0.030	0.060	0.049	0.034
Mult.PARMA(3,1)x(1,1)	0.029	0.056	0.049	0.034
PAR(3)	0.029	0.062	0.050	0.034
Mult.PARMA(1,0)x(2,0)	0.030	0.063	0.057	0.034

Seasonal autocorrelations of residuals are shown in Figures 4.1 through 4.8. Portmanteau statistics and Akaike Information Criteria are shown in Table 4.2.

Table 4.2 Portmanteau and AIC statistics- St.Lawrence Four seasons

Model	Q_1	AIC
PAR(1)	77.9	2548
PAR(2)	82.1	2513
PARMA(1,1)	88.3	2505
PARMA(2,1)	80.4	2485
Mult. PARMA(2,1)x(1,0)	80.3	2490
Mult. PARMA(1,1)x(1,1)	93.1	2487
Mult. PARMA(3,1)x(1,1)	89.4	2502
PAR(3)	84.6	2497
Mult. PARMA(1,0)x(2,0)	77.6	2532

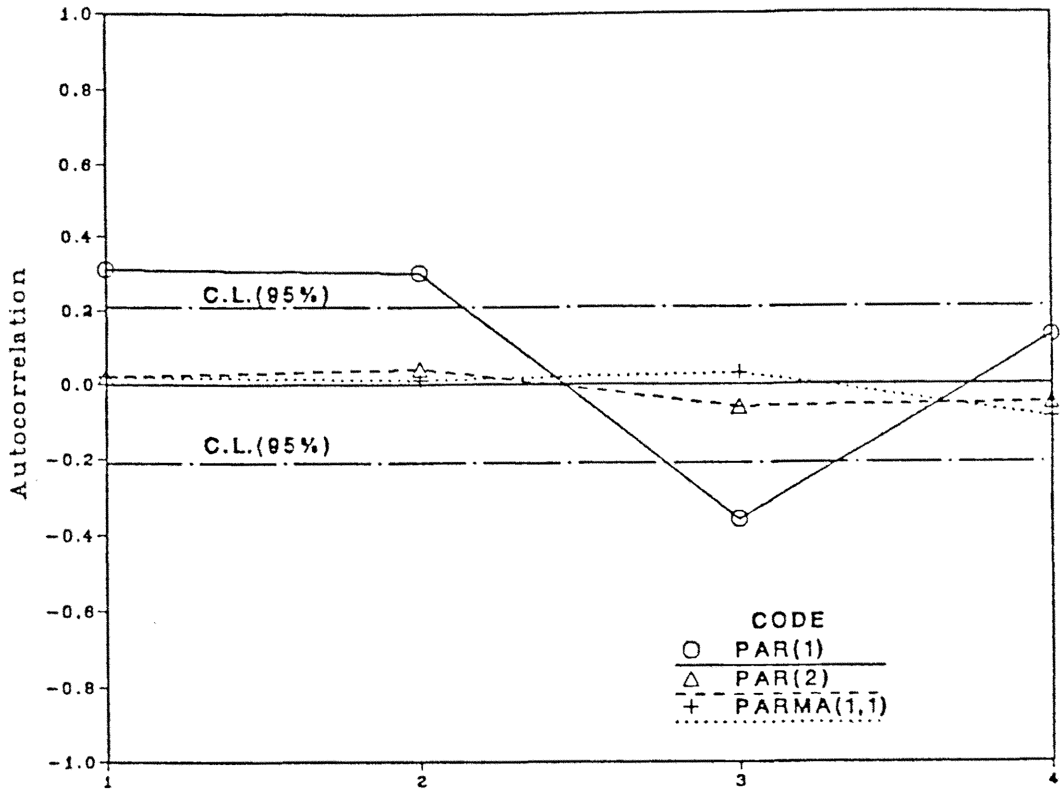

Figure 4.1- Lag 1 seasonal autocorrelation of residuals
Least-squares estimation - St. Lawrence

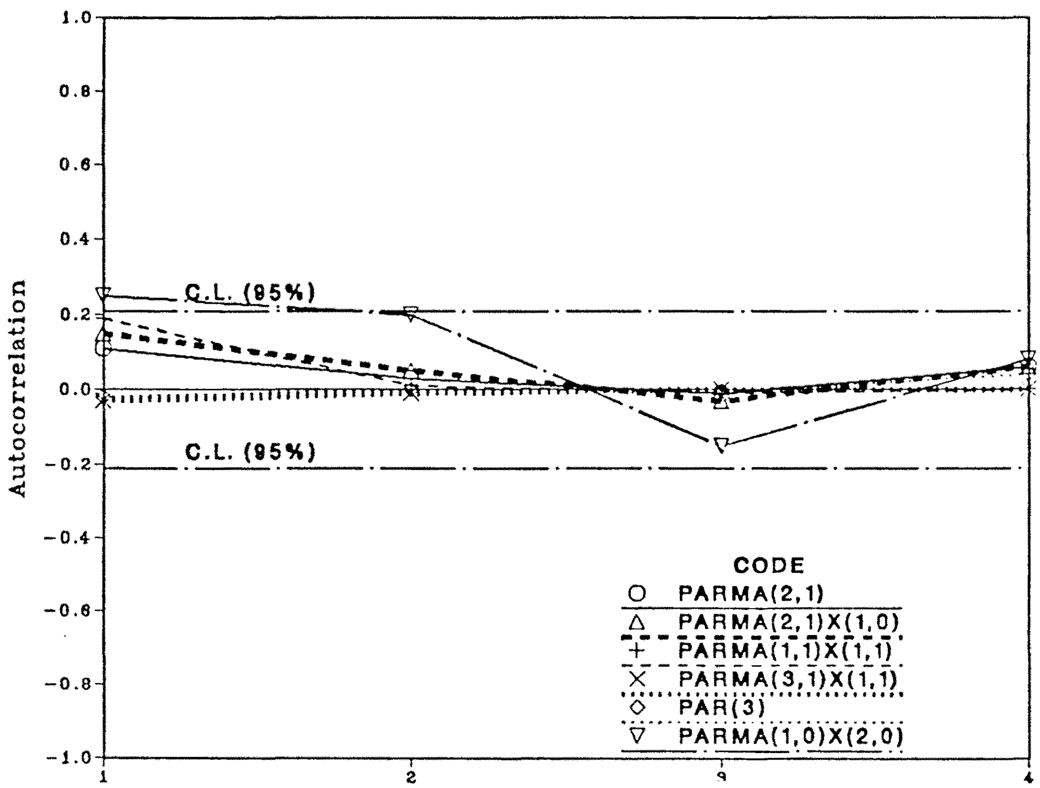

Figure 4.2 - Lag 1 autocorrelation of residuals
Least-squares estimation - St. Lawrence

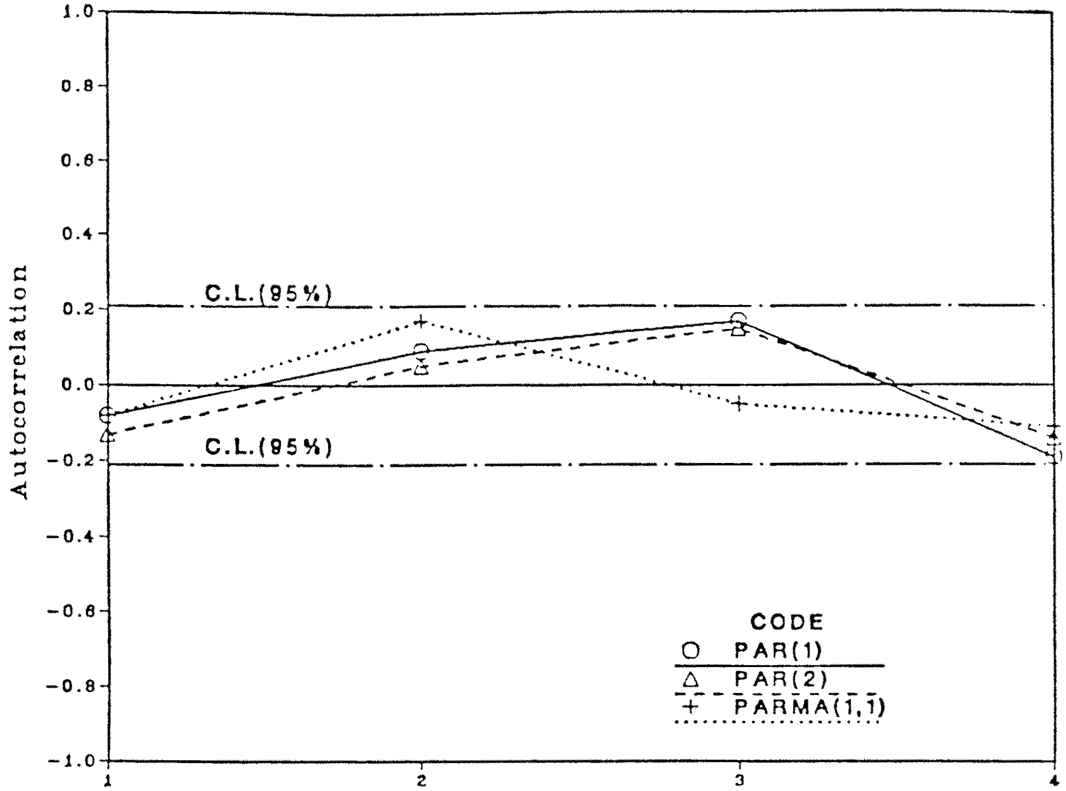

Figure 4.3 - Lag 2 seasonal autocorrelation
Least-squares estimation - St. Lawrence

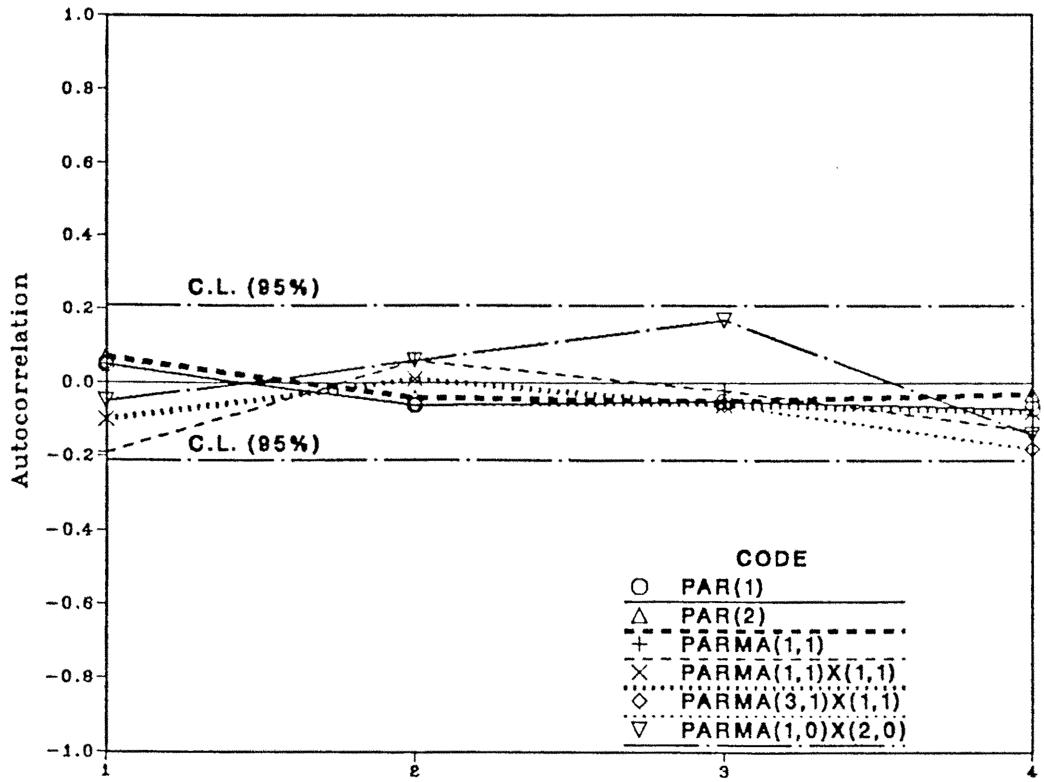

Figure 4.4 - Lag 2 autocorrelation of residuals
Least-squares estimation - St. Lawrence

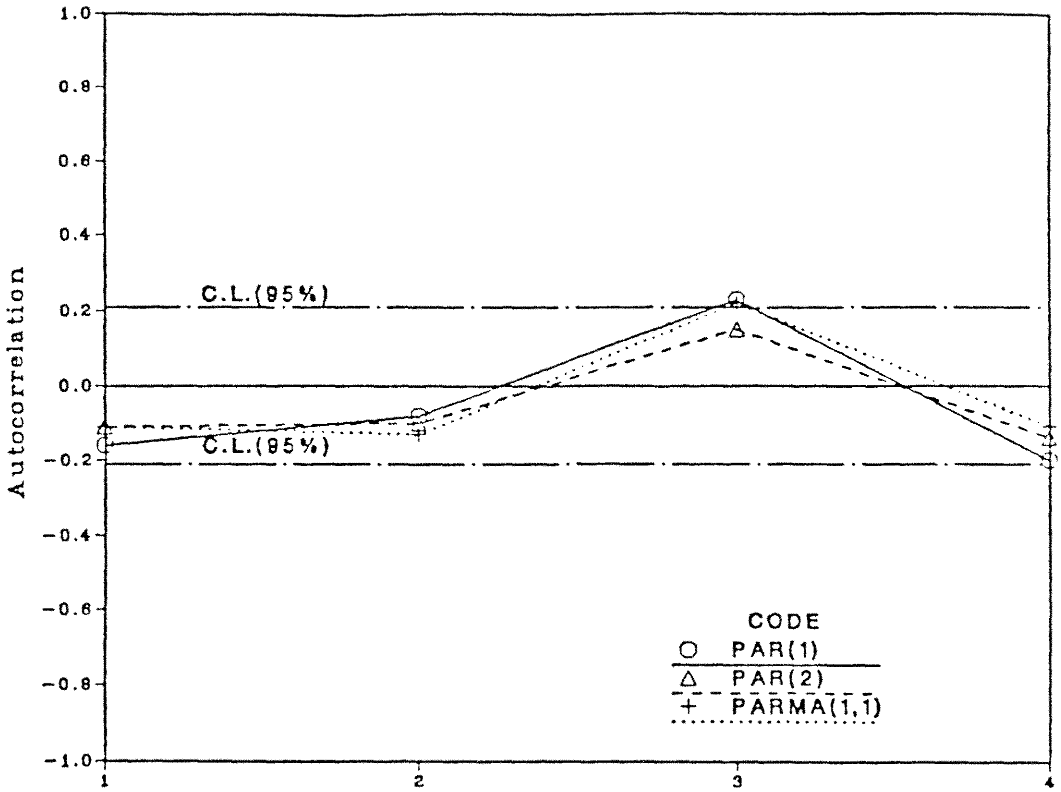

Figure 4.5 - Lag 3 autocorrelation of rediduals
Least-squares estimation - St. Lawrence

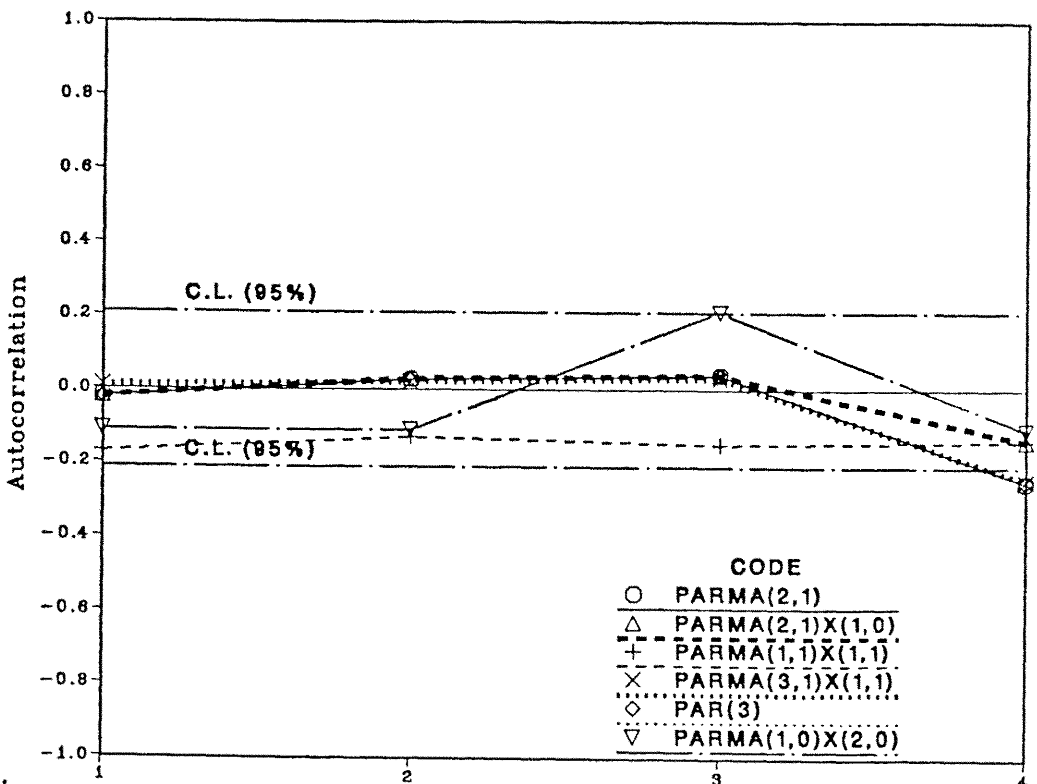

Figure 4.6 - Lag 3 autocorrelation of residuals
Least-squares estimation - St. Lawrence

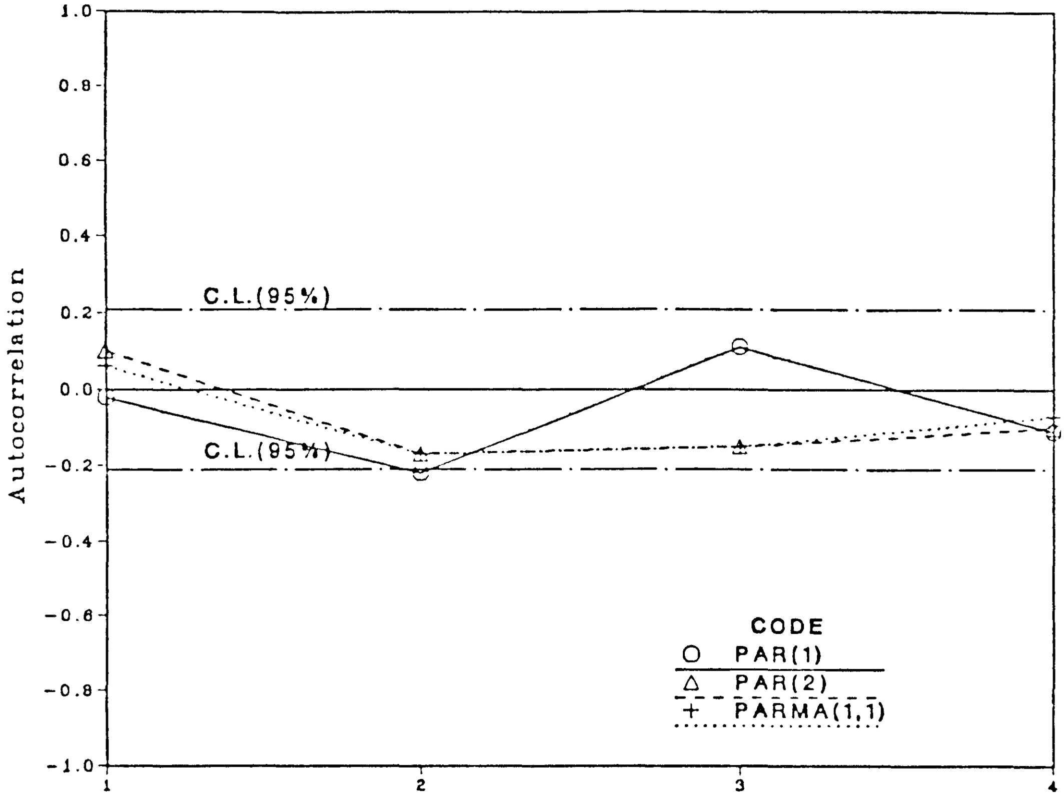

Figure 4.7 - Lag 4 autocorrelation of residuals
Least-squares estimation - St. Lawrence

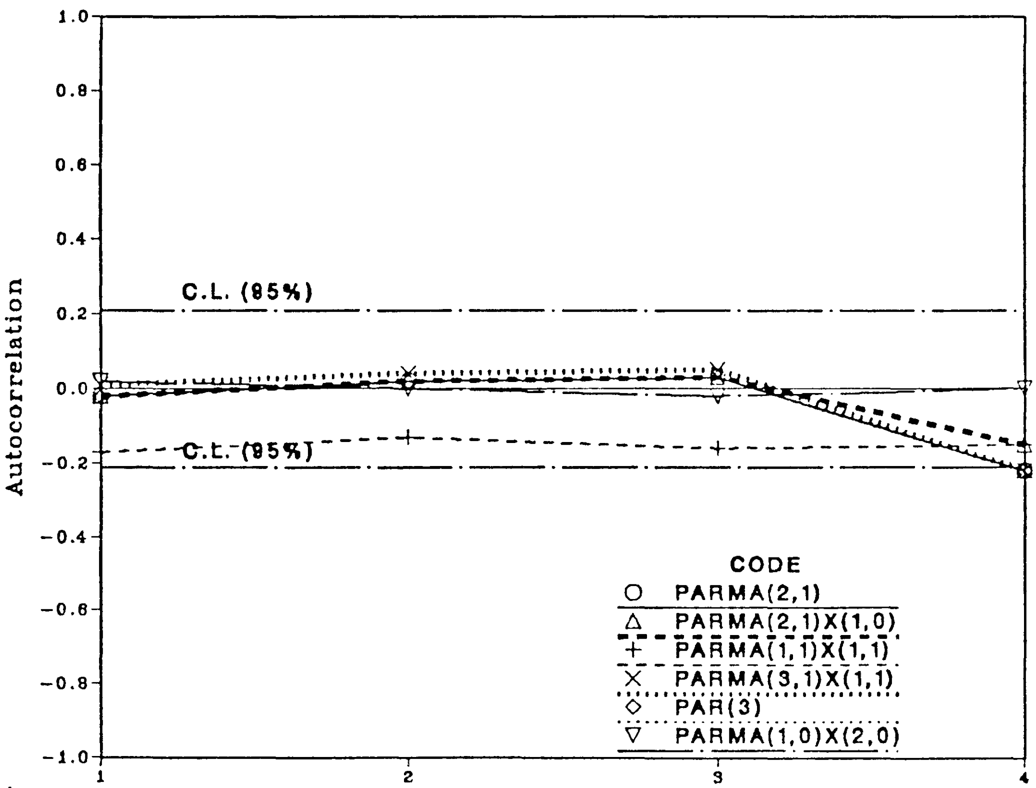

Figure 4.8 - Lag 4 autocorrelation of residuals
Least-squares estimation - St. Lawrence

b) Preservation of statistics -

Data generation was utilized to verify and compare the reproduction of statistics by different fitted models. Utilizing the parameters estimated from the historic sample for the models, 40 samples presenting 250 years each(total of 10,000 years) were generated. Various statistics were estimated from each sample and results were averaged. Large samples were simulated because they present smaller sample variability and allow computation of statistics closer to the true value, an, here, the objective is to see how close the true statistics corresponding to the models are to the statistics from the historical samples. It was also verified that the chosen number of samples was enough to obtain the statistics by analyzing the standard deviations of estimations and by verifying that averages of statistics for groups of 20 samples were very close.

Averages for seasonal means for the 4-season samples for the St. Lawrence River are presented by Figures A.1 and A.2 in the Appendix. Averages for seasonal standard deviations are shown in Figures A.3 and A.4 in the Appendix. Averages for seasonal skewness are shown in Figures A.5 and A.6 in the same Appendix.

Averages of annual autocorrelations are presented by Figures 4.9 and 4.10.

Averages of seasonal autocorrelations are shown in Figures (4.11) through (4.18).

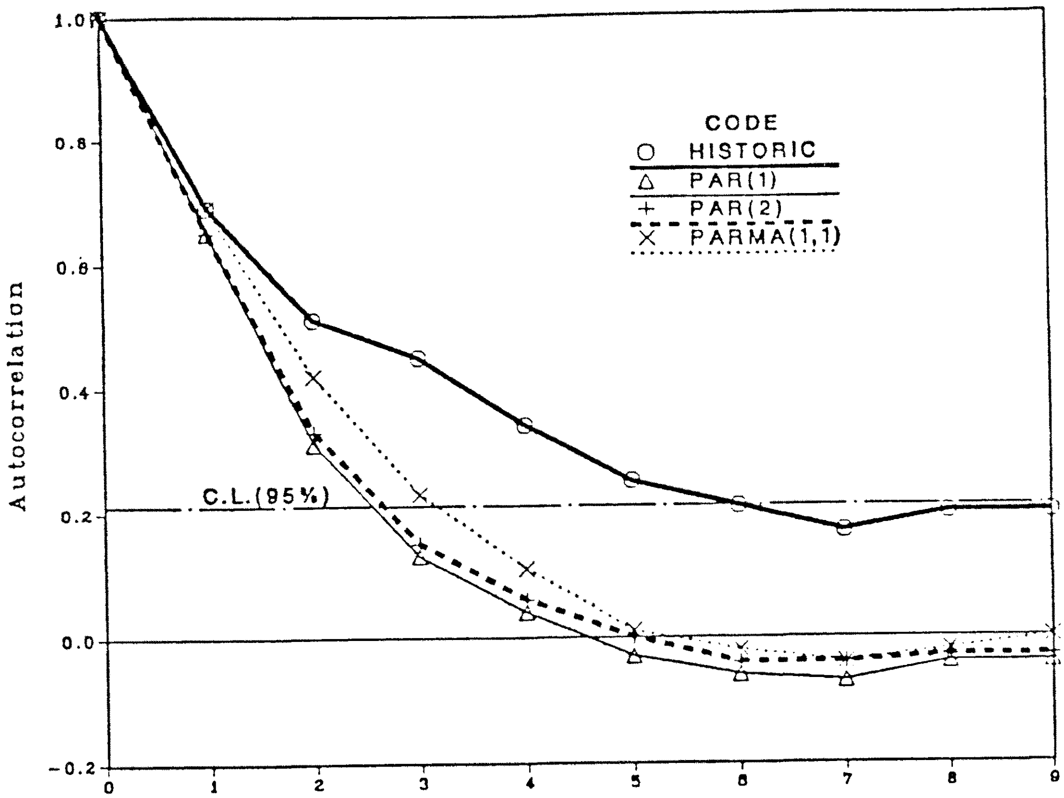

Figure 4.9 - Annual autocorrelation - 4 seasons
Least-squares estimation - St. Lawrence

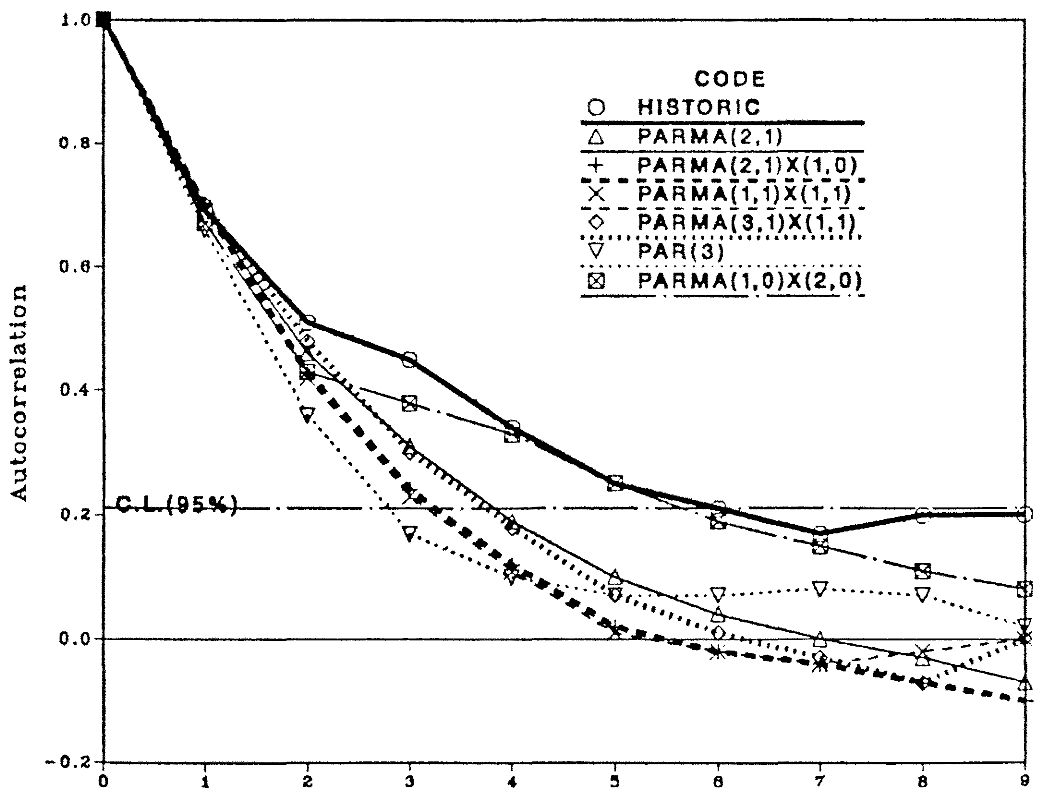

Figure 4.10 - Annual autocorrelation - 4 seasons
Least-squares estimation - St. Lawrence

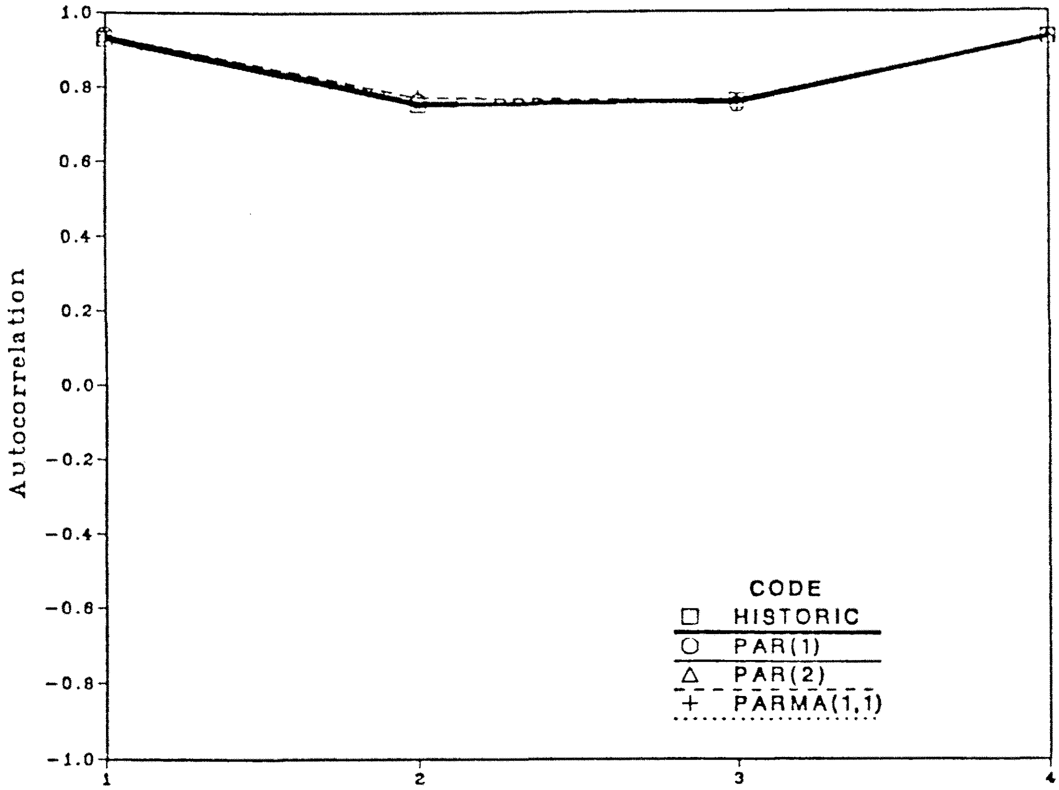

Figure 4.11 - Lag 1 seasonal autocorrelation
Least-squares estimation - St. Lawrence

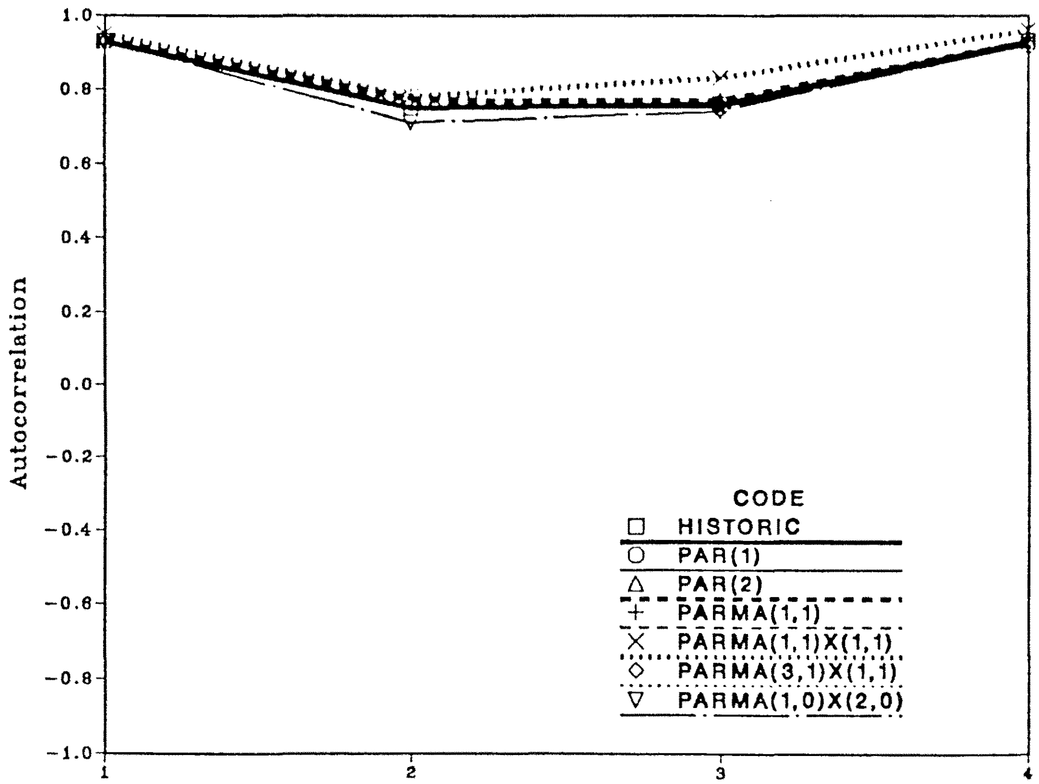

Figure 4.12 - Lag 1 seasonal autocorrelation
Least-square estimation - St. Lawrence

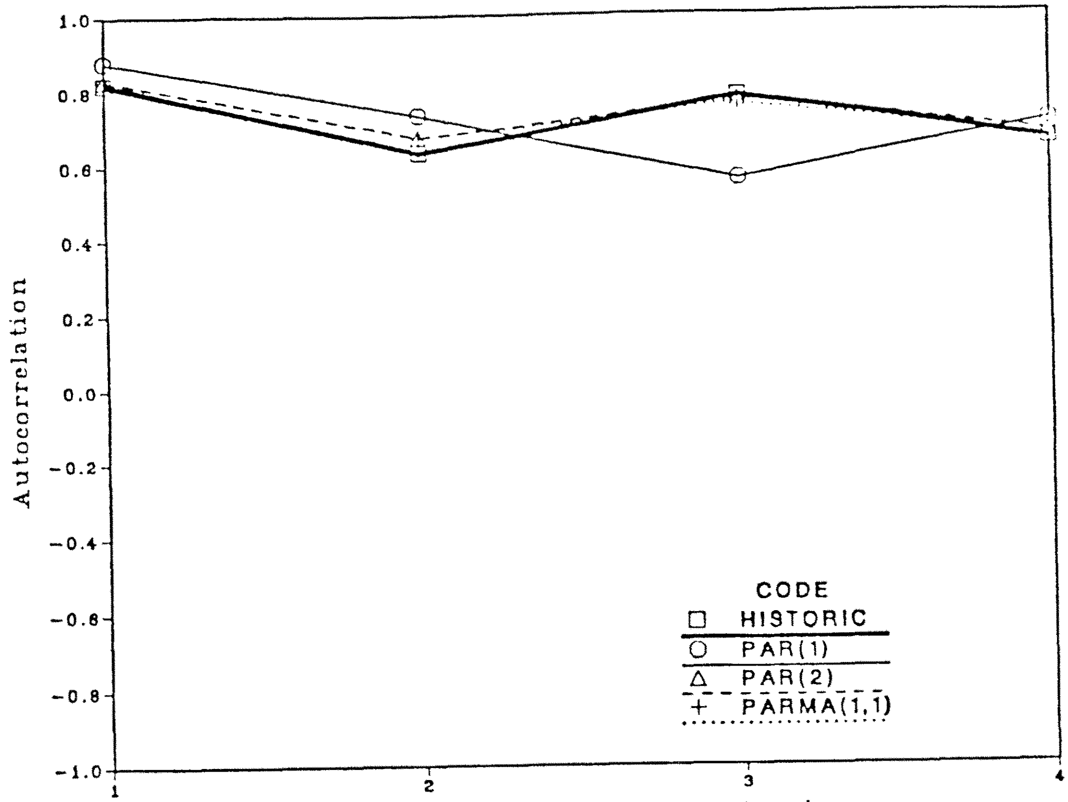

Figure 4.13 - Lag 2 seasonal autocorrelation
Least-squares estimation - St. Lawrence

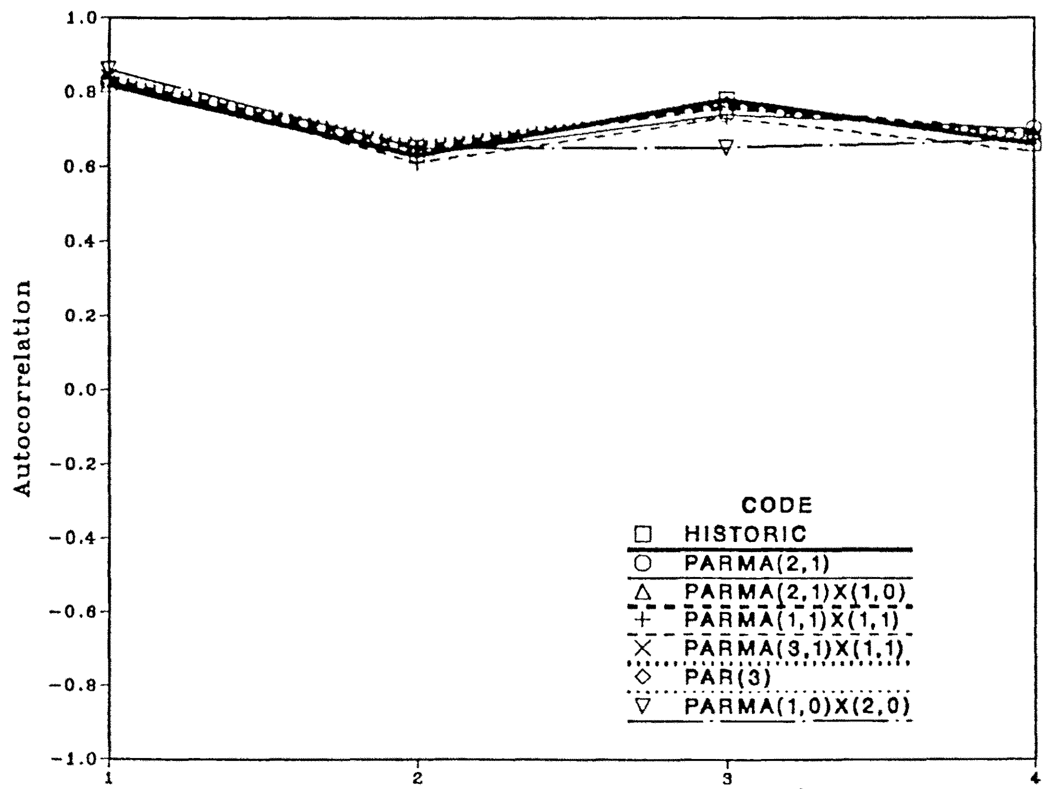

Figure 4.14 - Lag 2 seasonal autocorrelation
Least-squares estimation - St. Lawrence

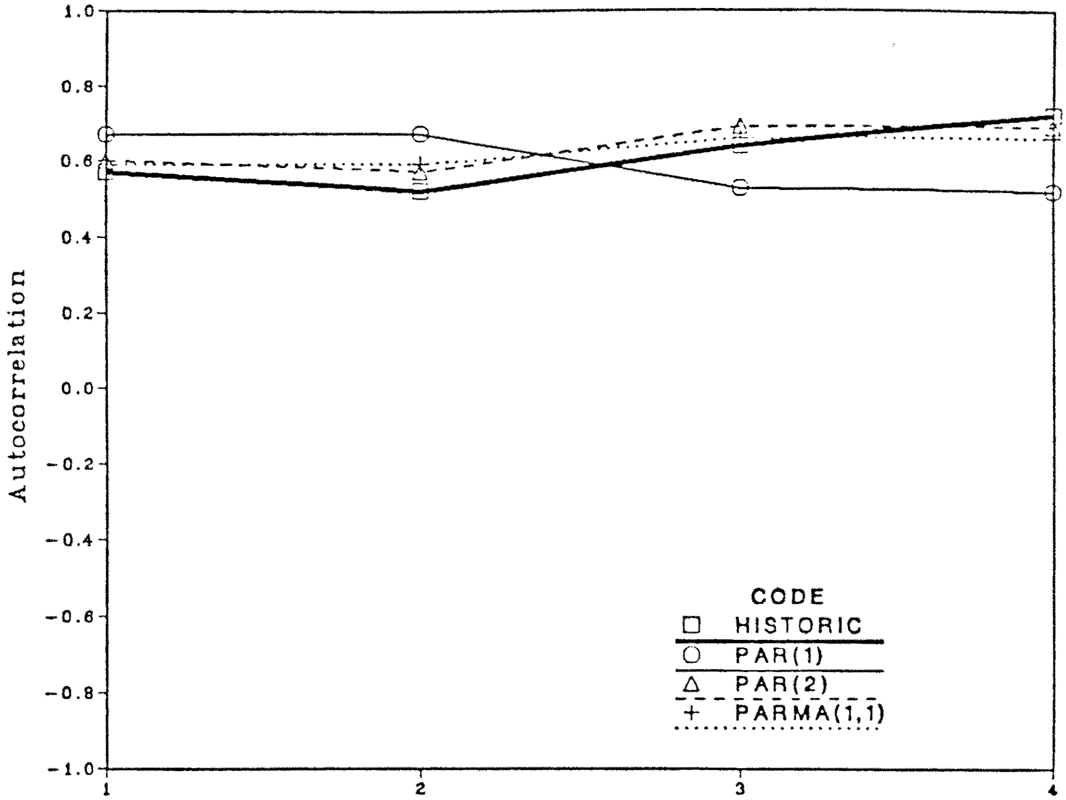

Figure 4.15 - Lag 3 seasonal autocorrelation
Least-squares estimation - St. Lawrence

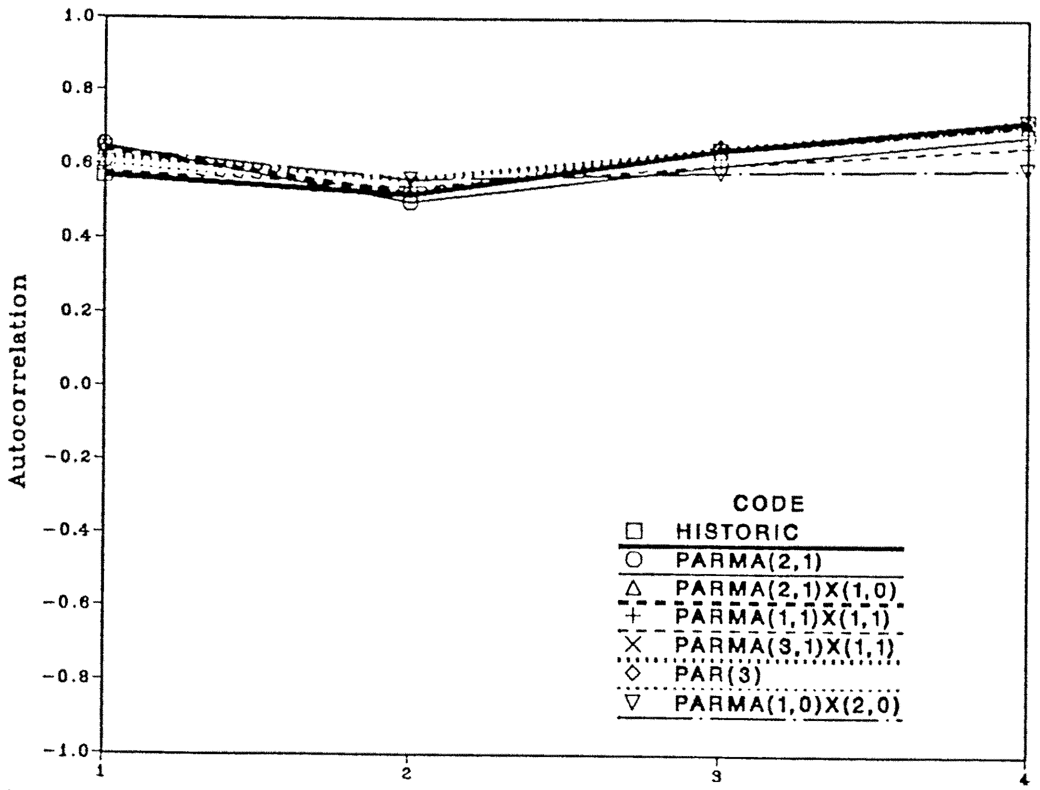

Figure 4.16 - Lag 3 seasonal autocorrelation
Least-squares estimation - St. Lawrence

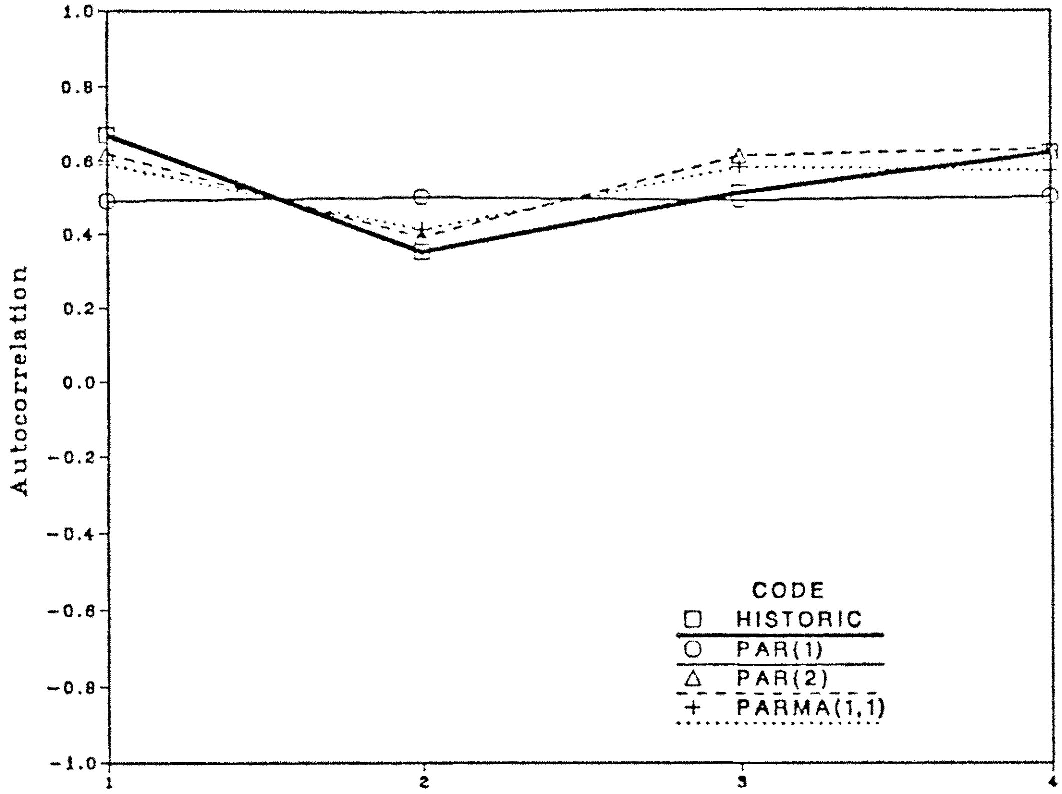

Figure 4.17 - Lag 4 seasonal autocorrelation
Least-squares estimation - St. Lawrence

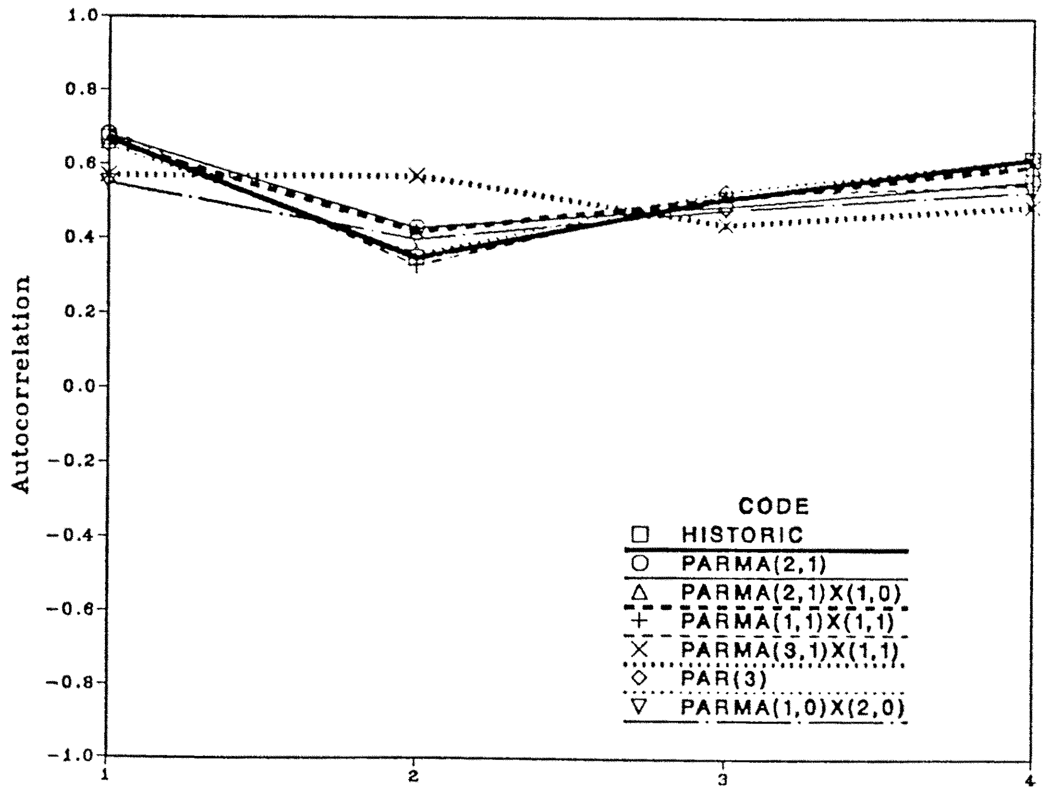

Figure 4.18 - Lag 4 seasonal autocorrelation
Least-squares estimation - St. Lawrence

4.1.1.2 <u>Six-season sample</u>

The same models applied to the St. Lawrence four-year sample were also applied to the six-year sample for the same discharge data for comparison between performances for a different number of seasons by the Multiplicative PARMA models.

Logarithms of a six-season series were utilized for fit.

a) Residuals characteristics -

Seasonal means of residuals for these series were also very close to zero. The seasonal standard deviations of residuals from different models are listed in Table 4.3.

Table 4.3 - Seasonal standard deviations of residuals Logarithm of six-season discharges

Model	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
PAR(1)	0.024	0.051	0.077	0.058	0.033	0.023
PAR(2)	0.022	0.050	0.076	0.051	0.032	0.021
PARMA(1,1)	0.022	0.050	0.075	0.049	0.033	0.021
PARMA(2,1)	0.021	0.050	0.073	0.047	0.033	0.021
Mult. PARMA(2,1)x(1,0)	0.022	0.047	0.074	0.048	0.032	0.021
Mult. PARMA(1,1)x(1,1)	0.023	0.048	0.074	0.048	0.030	0.021
Mult. PARMA(3,1)x(1,1)	0.021	0.048	0.073	0.047	0.028	0.021
PAR(3)	0.022	0.048	0.074	0.049	0.029	0.020
Mult. PARMA(1,0)x(2,0)	0.022	0.050	0.075	0.055	0.032	0.022

Seasonal autocorrelations of residuals are plotted in Figures 4.19 through 4.30.

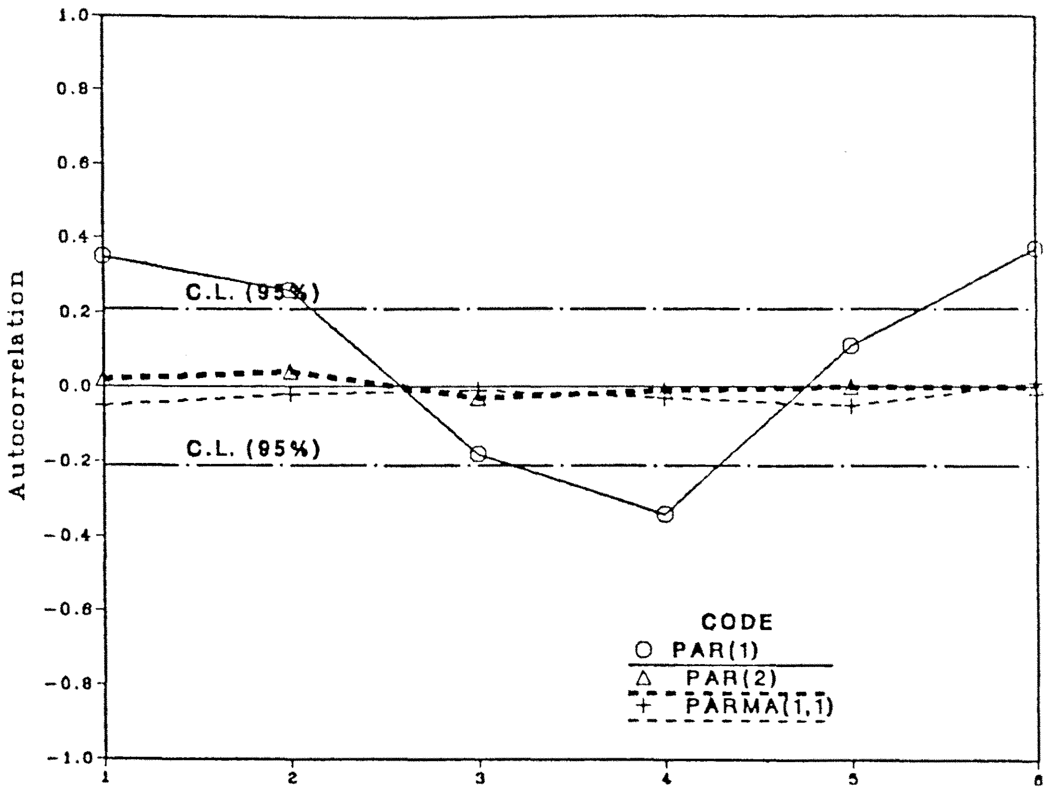

Figure 4.19 - Lag 1 seasonal autocorrelation of residuals
Least squares estimation - St. Lawrence

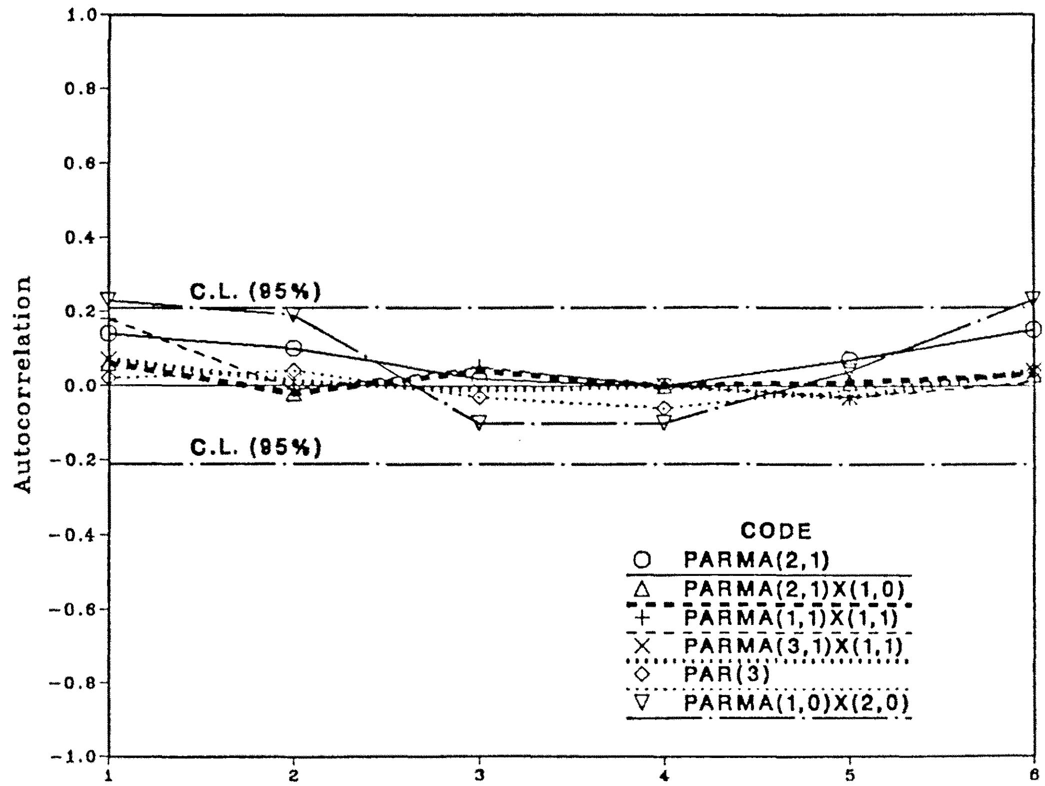

Figure 4.20 - Lag 1 seasonal autocorrelation of residuals
Least-squares estimation - St. Lawrence

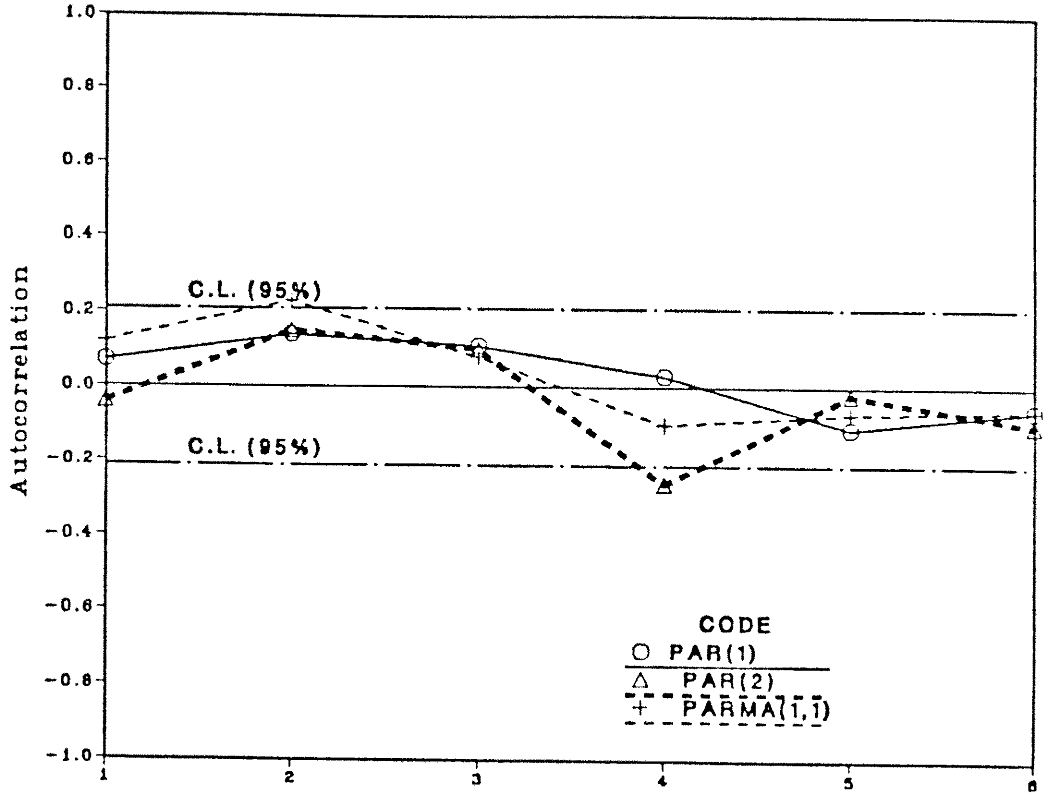

Figure 4.21 - Lag 2 autocorrelation of residuals
Least-squares estimation - St. Lawrence

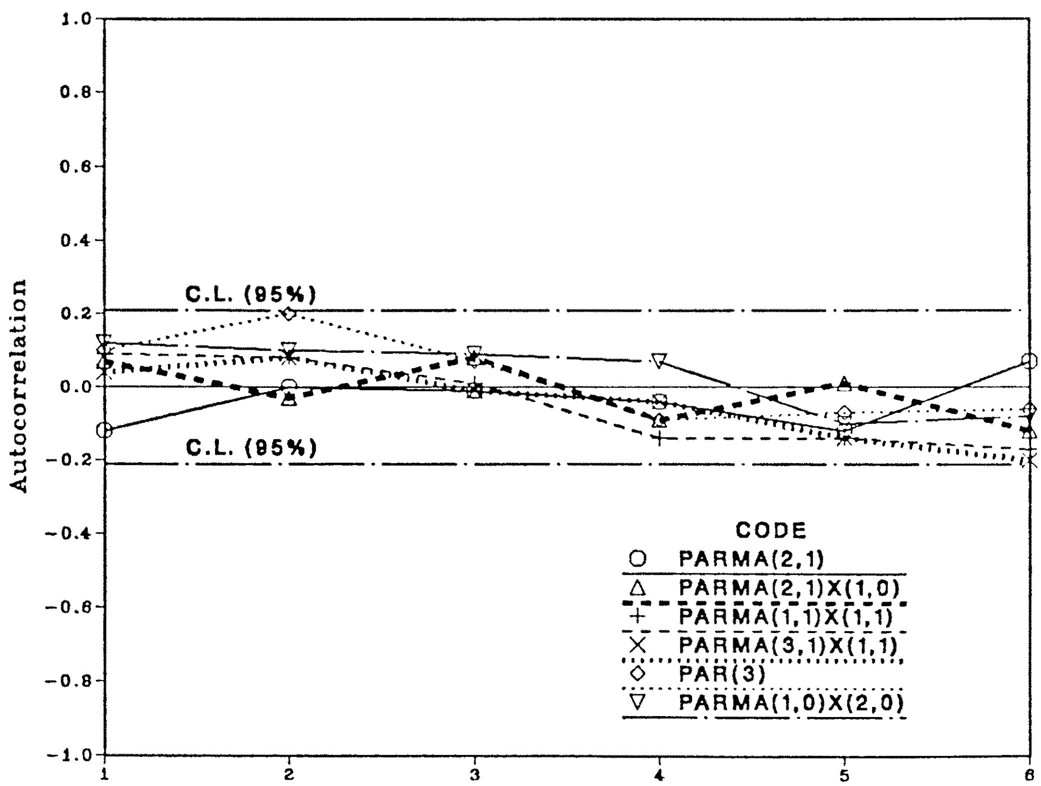

Figure 4.22 - Lag 2 autocorrelation of residuals
Least-squares estimation - St. Lawrence

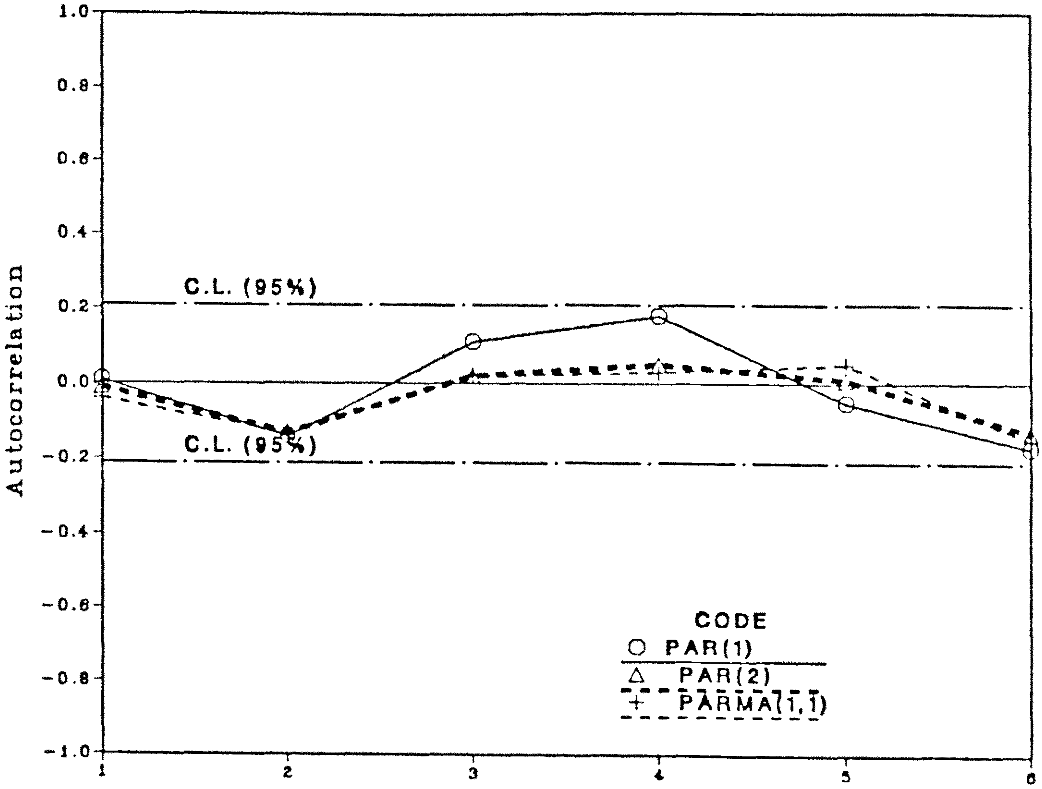

Figure 4.23 - Lag 3 autocorrelation of residuals
Least-squares estimation - St. Lawrence

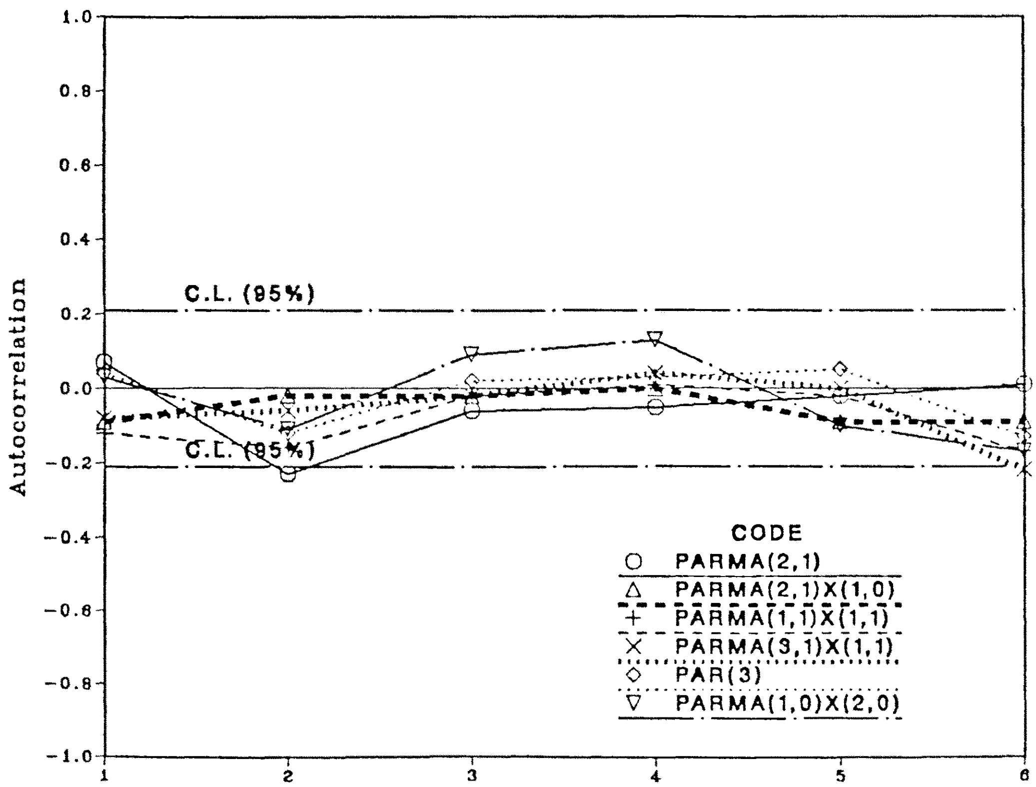

Figure 4.24 - Lag 3 autocorrelation of residuals
Least-squares estimation - St. Lawrence

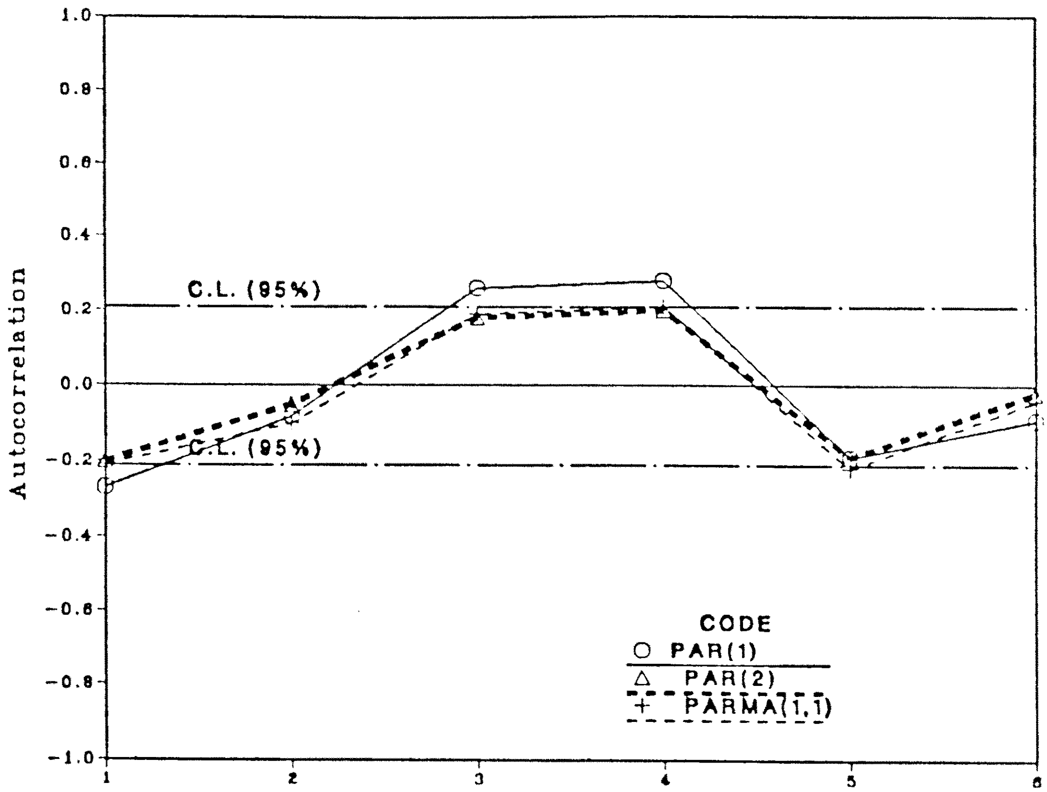

Figure 4.25 - Lag 4 seasonal autocorrelation of residuals
Least-squares - St. Lawrence

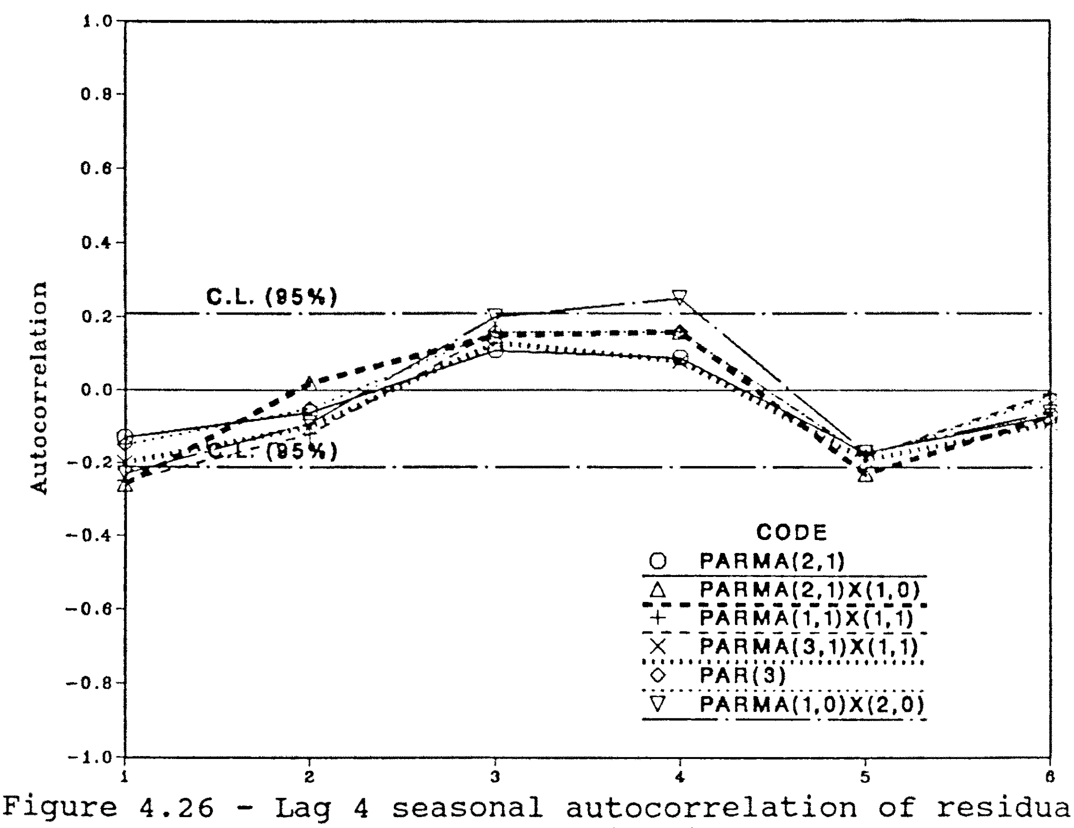

Figure 4.26 - Lag 4 seasonal autocorrelation of residuals
Least-squares estimation - St. Lawrence

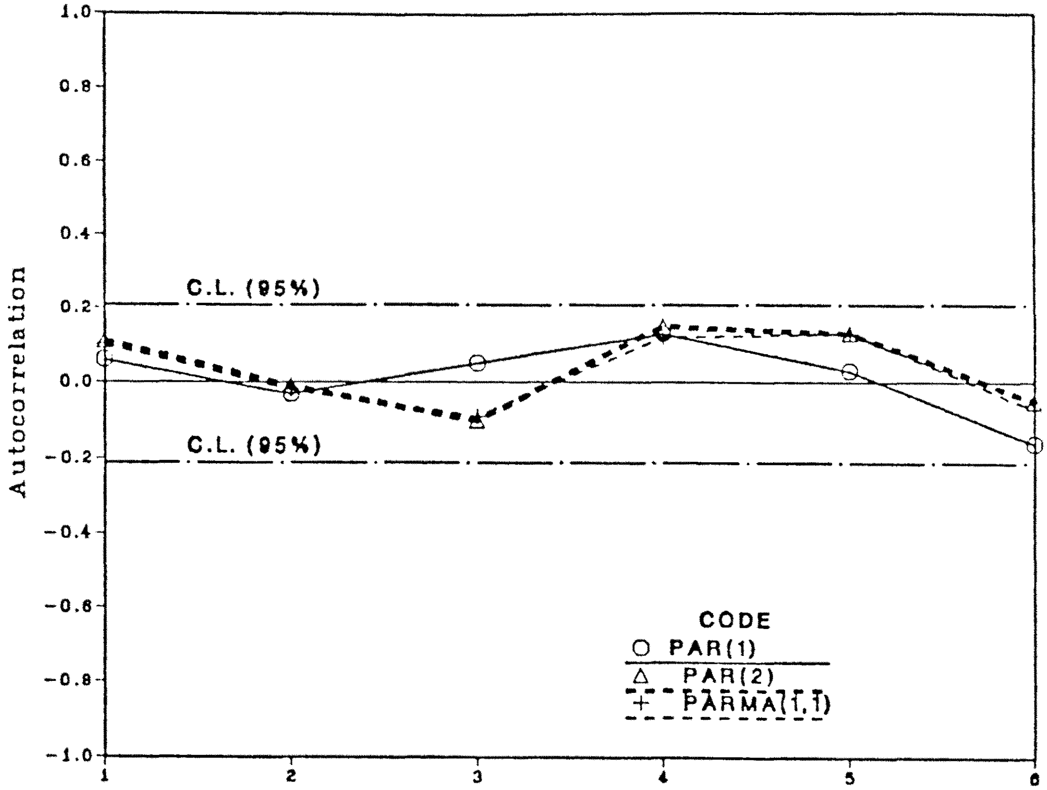

Figure 4.27 - Lag 5 seasonal autocorrelation of residuals Least-squares estimation - St. Lawrence

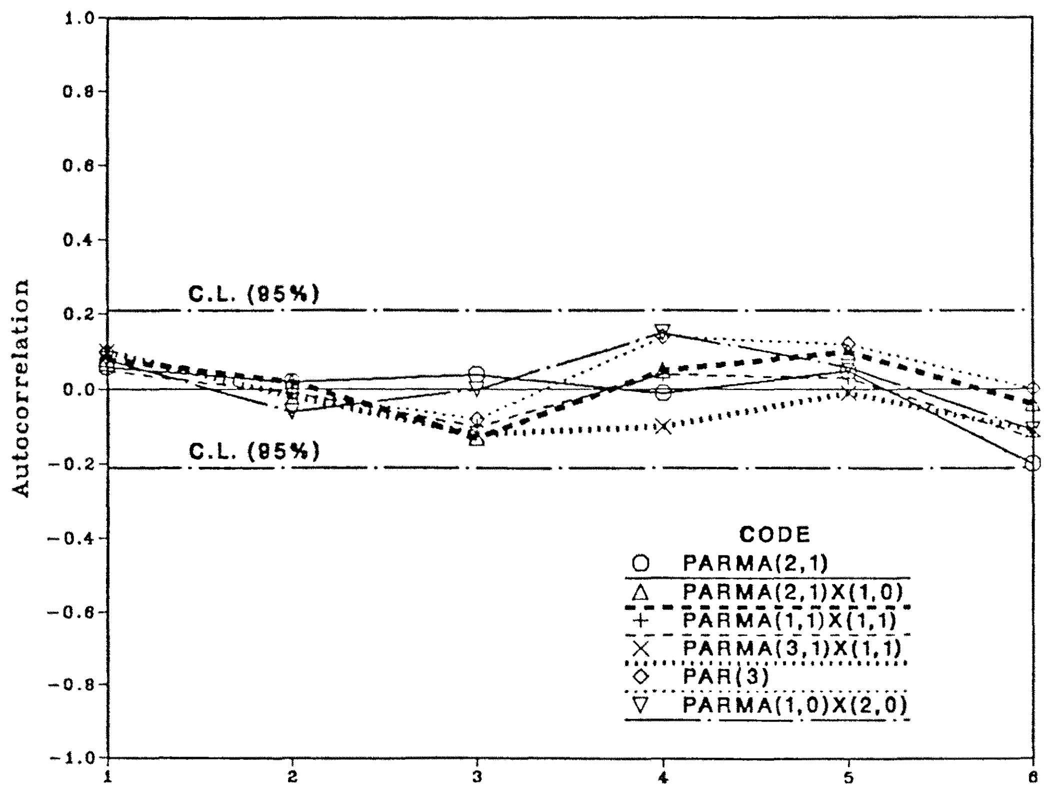

Figure 4.28 - Lag 5 seasonal autocorrelation of residuals Least-squares estimation - St. Lawrence

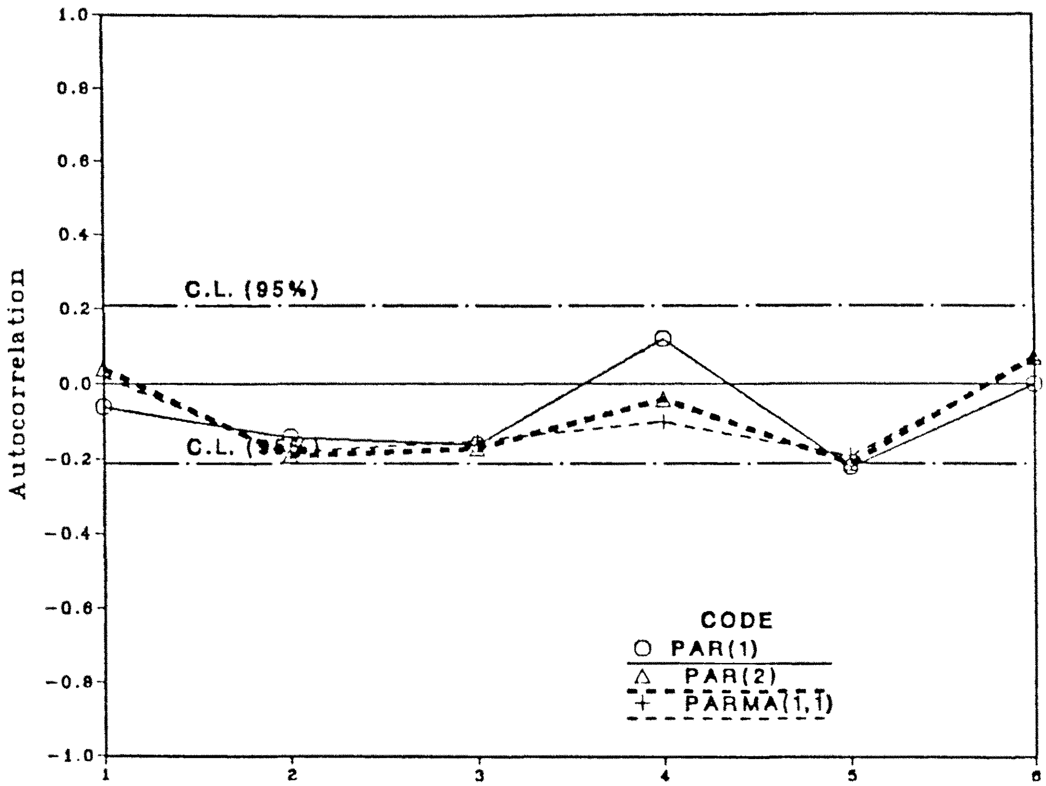

Figure 4.29 - Lag 6 seasonal autocorrelation of residuals
Least-squares estimation - St. Lawrence

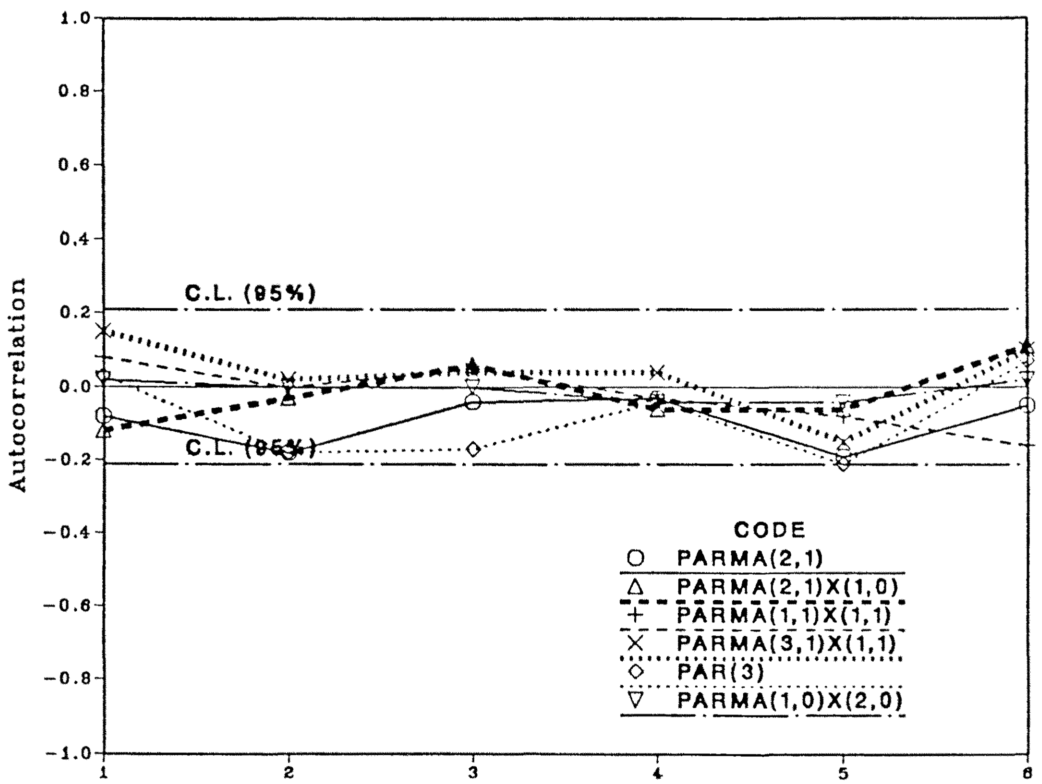

Figure 4.30 - Lag 6 seasonal autocorrelation of residuals
Least squares estimation - St. Lawrence

The portmanteau and Akaike Information Criterion statistics computed from the seasonal autocorrelation of residuals for the various models are listed in the following Table 4.4.

Table 4.4 - Portmanteau and AIC statistics - St. Lawrence River - Six seasons

Model	Q_1	AIC
PAR(1)	118.0	3243
PAR(2)	120.4	3198
PARMA(1,1)	115.8	3190
PARMA(2,1)	112.1	3179
Mult. PARMA(2,1)x(1,0)	118.9	3164
Mult. PARMA(1,1)x(1,1)	122.3	3169
Mult. PARMA(3,1)x(1,1)	119.4	3148
PAR(3)	121.8	3178
Mult. PARMA(1,0)x(2,0)	113.7	3225

b) Preservation of statistics -

The data generation and computation of statistics for the different models were made in the same way as those made for four-season samples (forty 250-year samples).

Seasonal means are shown in Figures A.7 and A.8 in the Appendix.

Seasonal standard deviations are plotted in Figures A.9 and A.10.

Figures A.11 and A.12 present the seasonal skewness, while figures 4.31 and 4.32 present the annual autocorrelograms.

The seasonal autocorrelograms are plotted in figures 4.33 through 4.44.

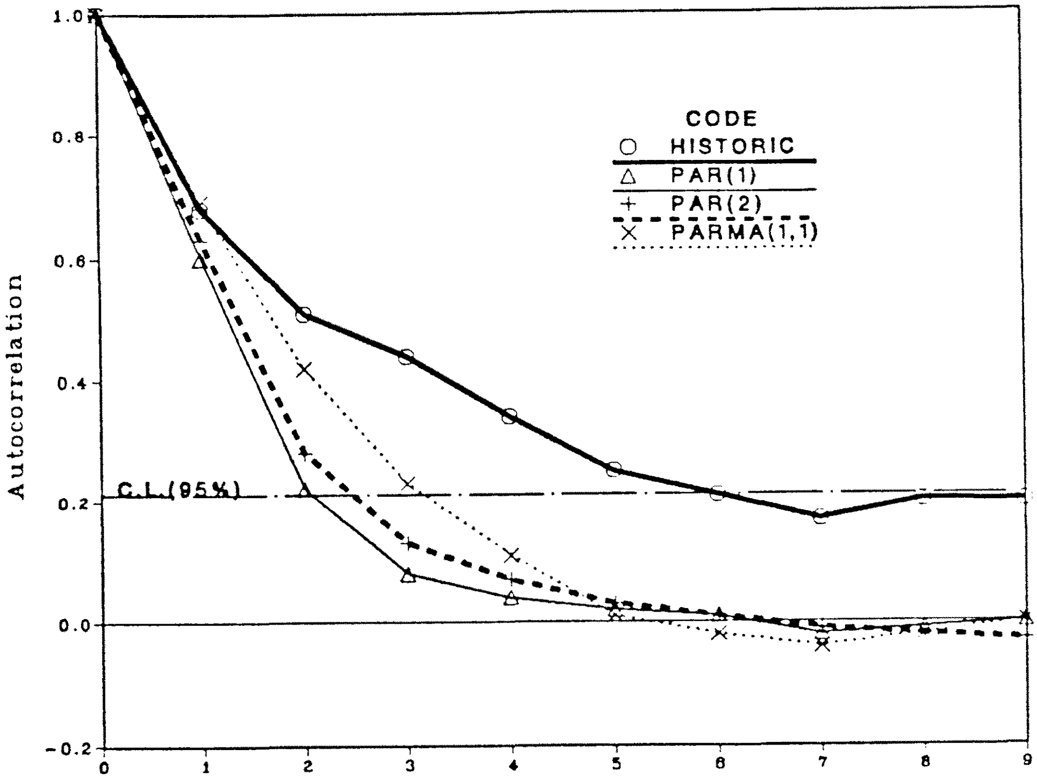

Figure 4.31 - Annual autocorrelations - 6 seasons
Least-squares estimation - St. Lawrence

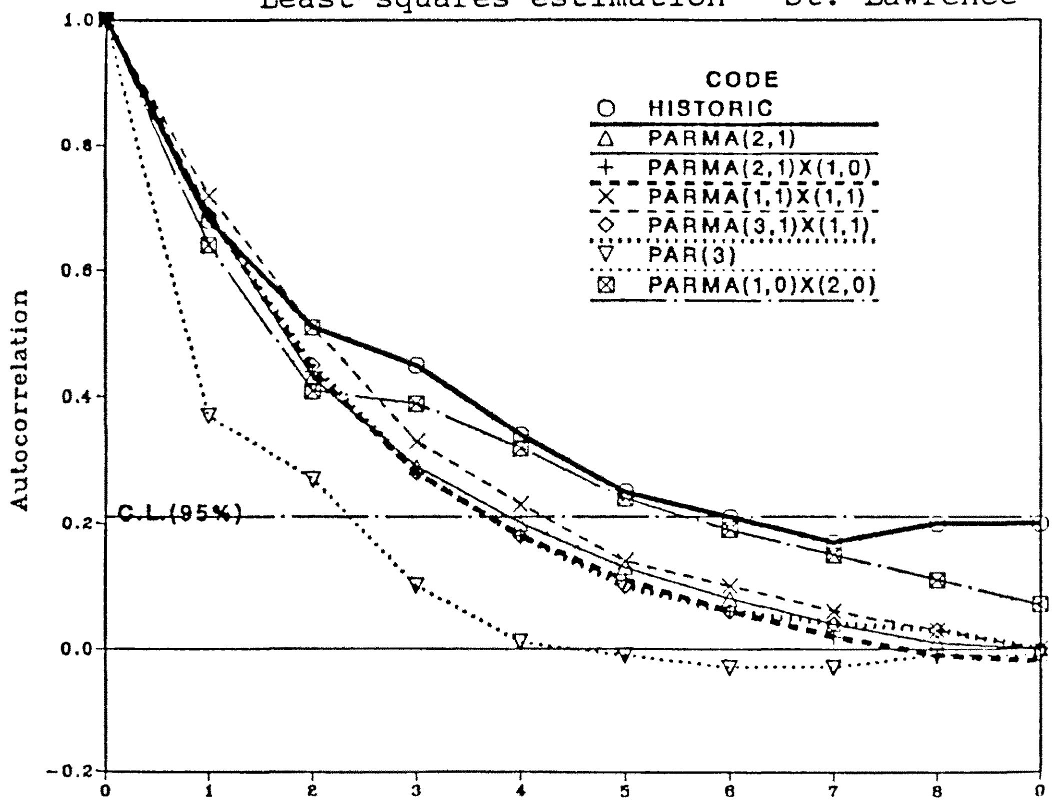

Figure 4.32 - Annual autocorrelations - 6 seasons
Least-squares estimation - St. Lawrence

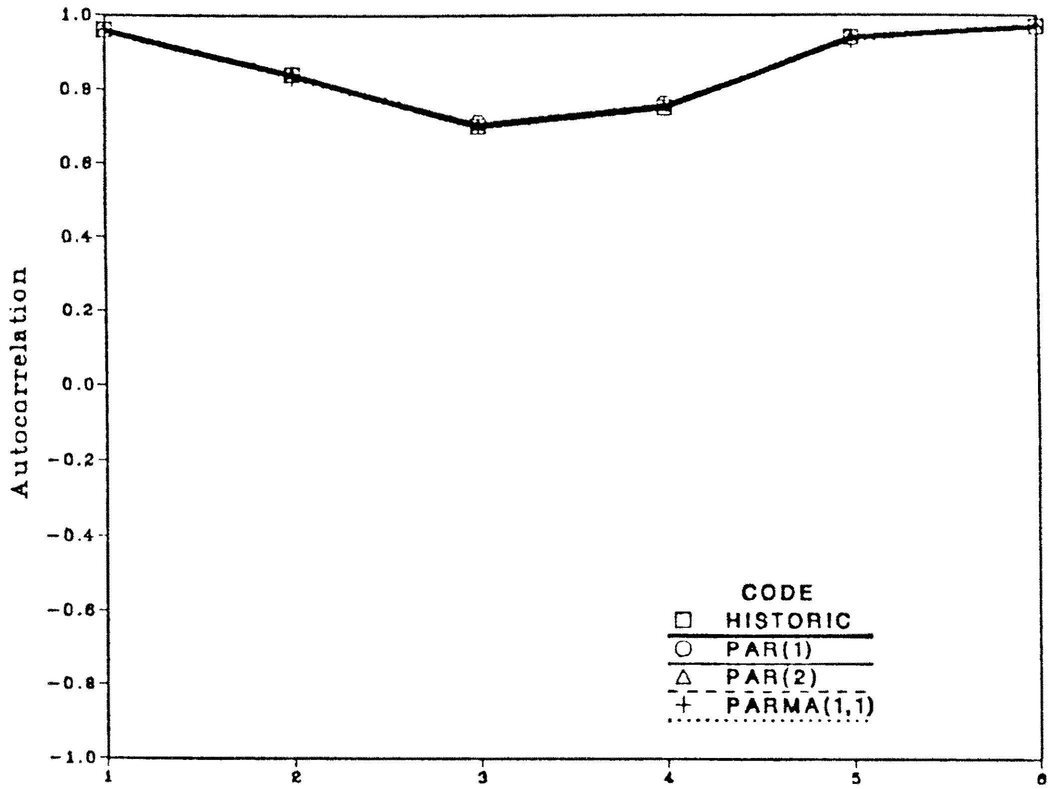

Figure 4.33 - Lag 1 seasonal autocorrelations
Least-squares estimation - St. Lawrence

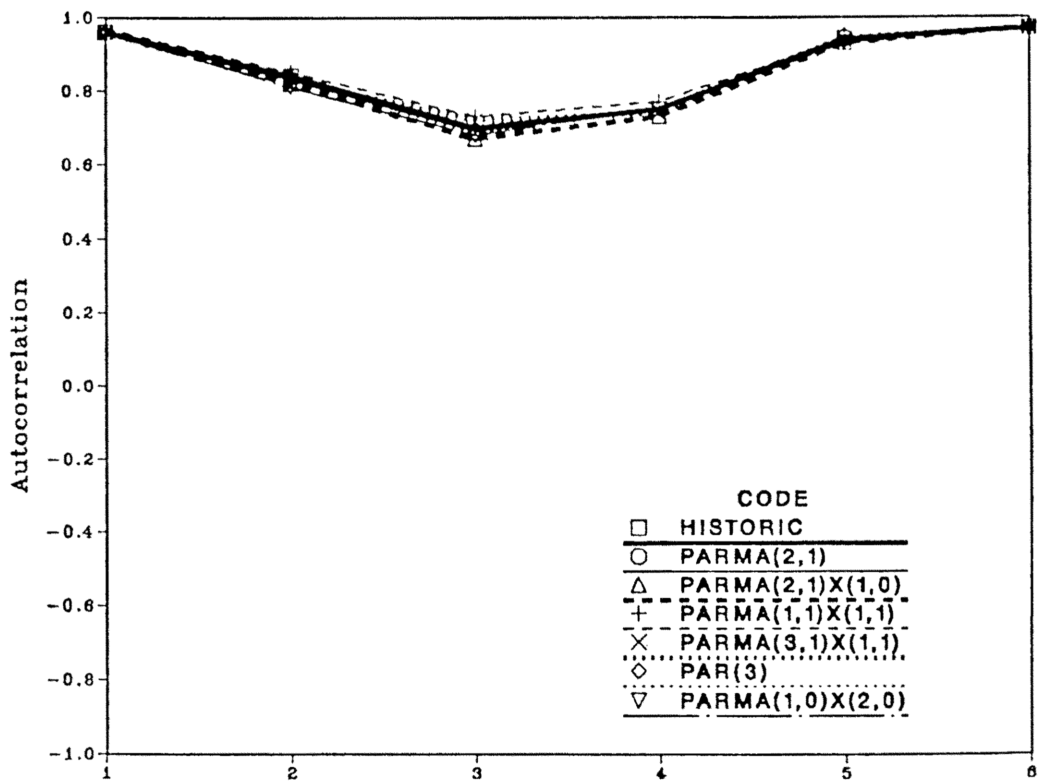

Figure 4.34 - Lag 1 seasonal autocorrelations
Least-squares estimation - St. Lawrence

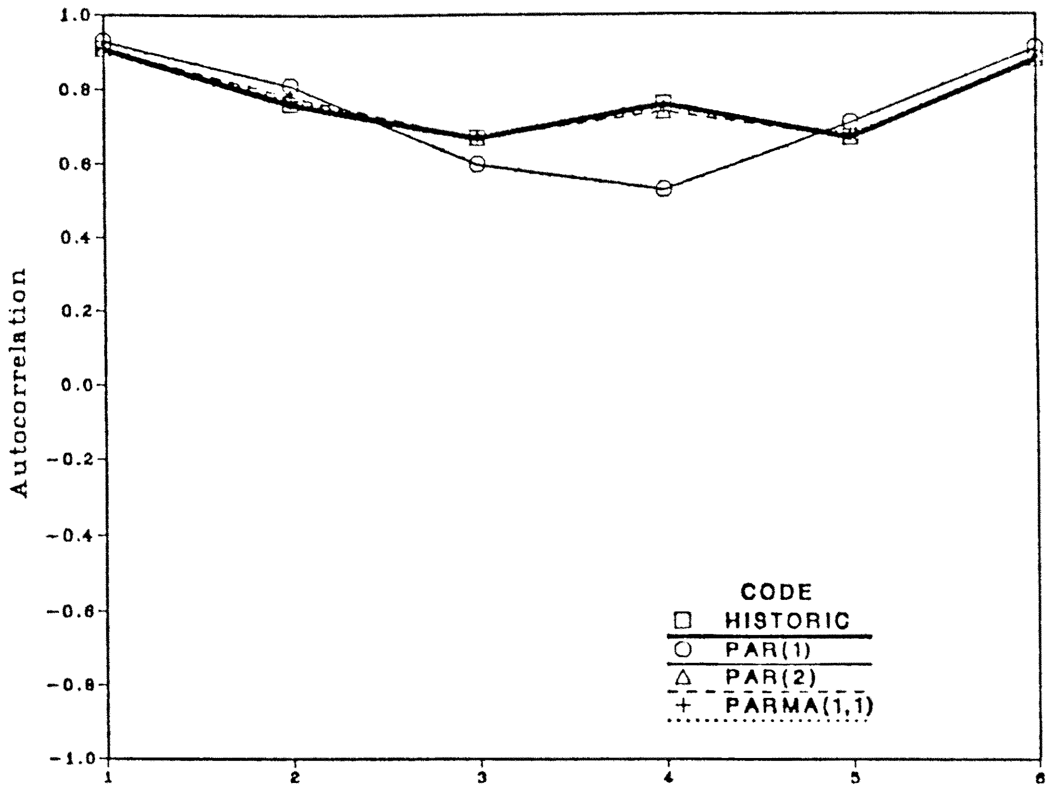

Figure 4.35 - Lag 2 seasonal autocorrelations
Least-squares estimation - St. Lawrence

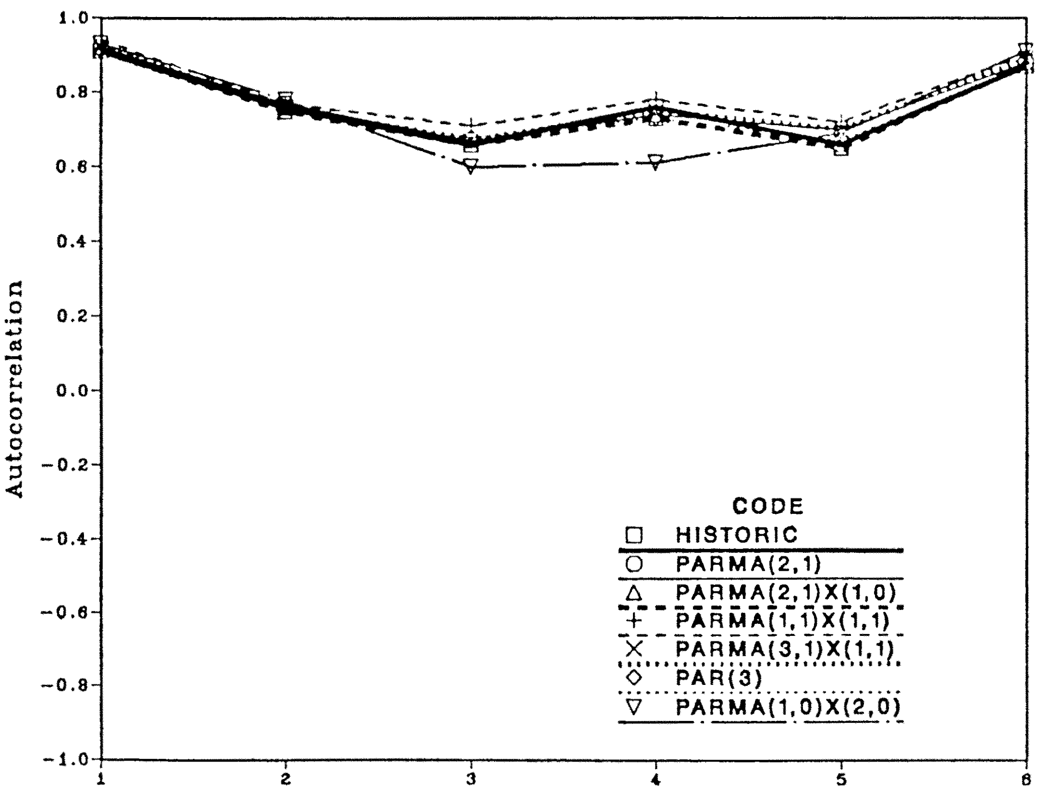

Figure 4.36 - Lag 2 seasonal autocorrelations
Least-squares estimation - St. Lawrence

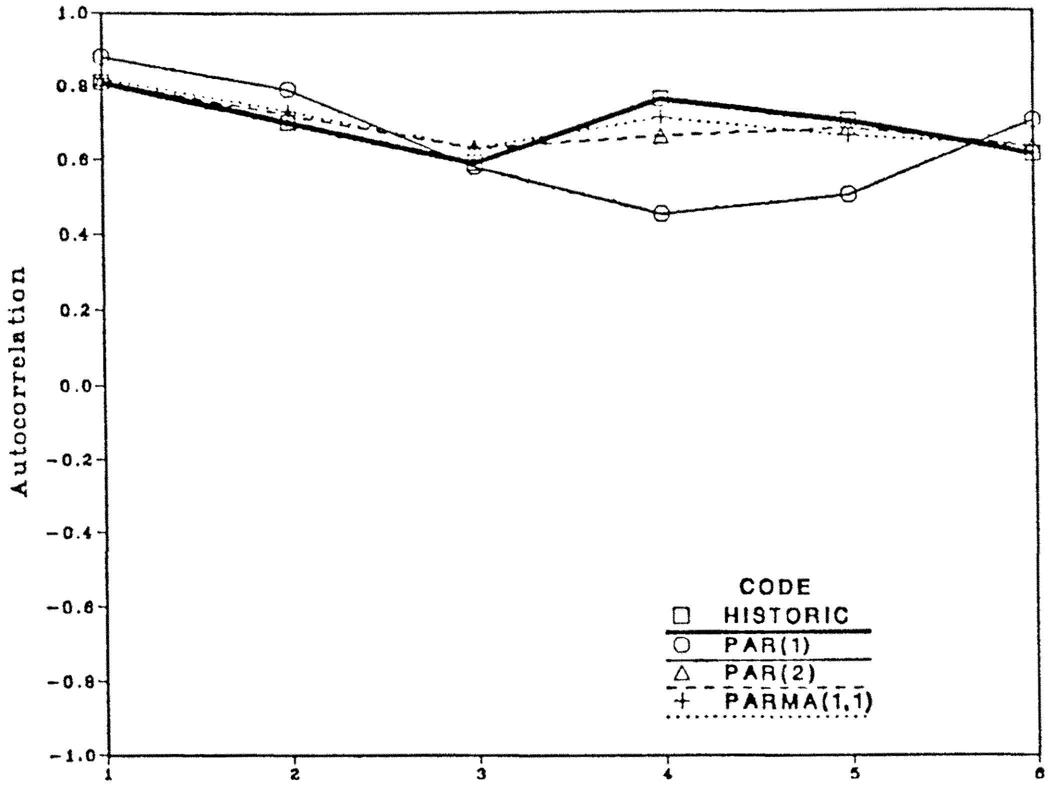

Figure 4.37 - Lag 3 seasonal autocorrelations
Least-squares estimation - St. Lawrence

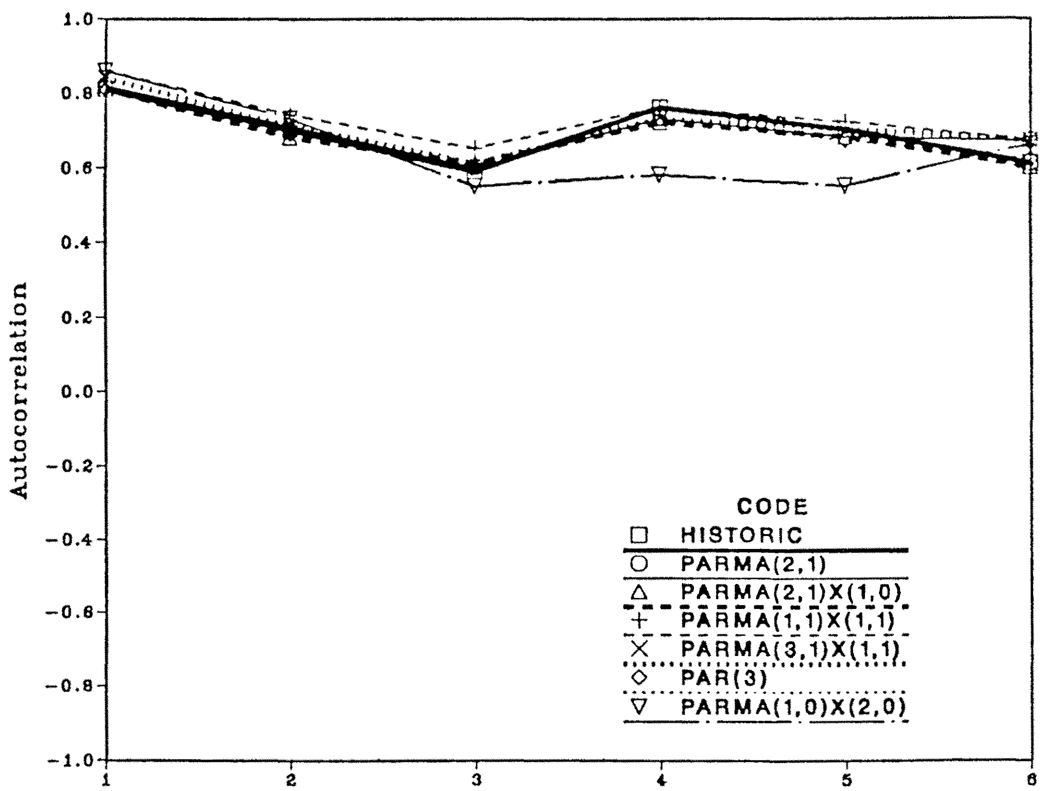

Figure 4.38 - Lag 3 seasonal autocorrelations
Least-squares estimation - St. Lawrence

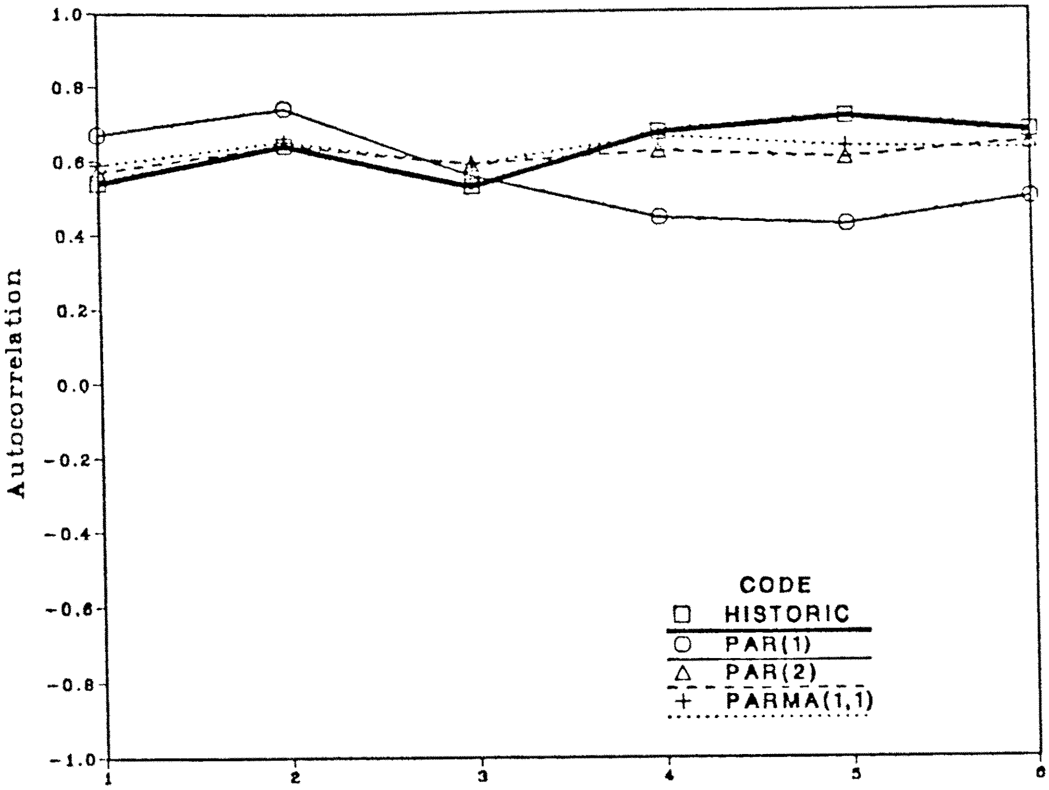

Figure 4.39 - Lag 4 seasonal autocorrelation
Least-squares estimation - St.Lawrence

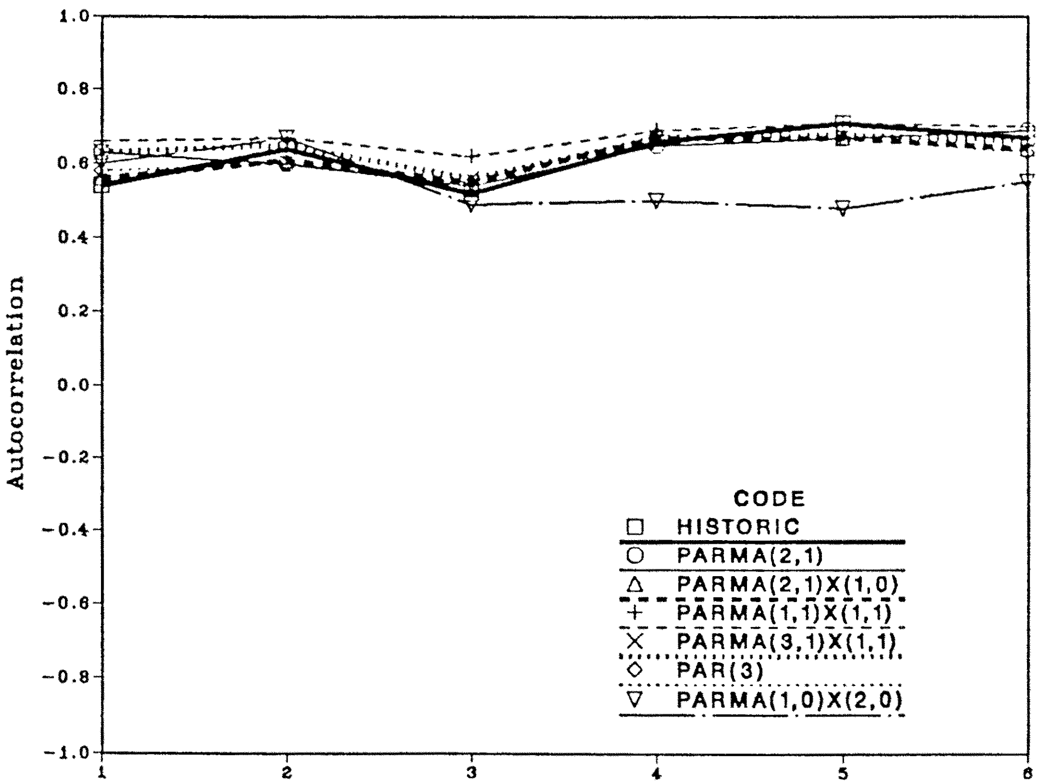

Figure 4.40 - Lag 4 seasonal autocorrelation
Least-squares estimation - St. Lawrence

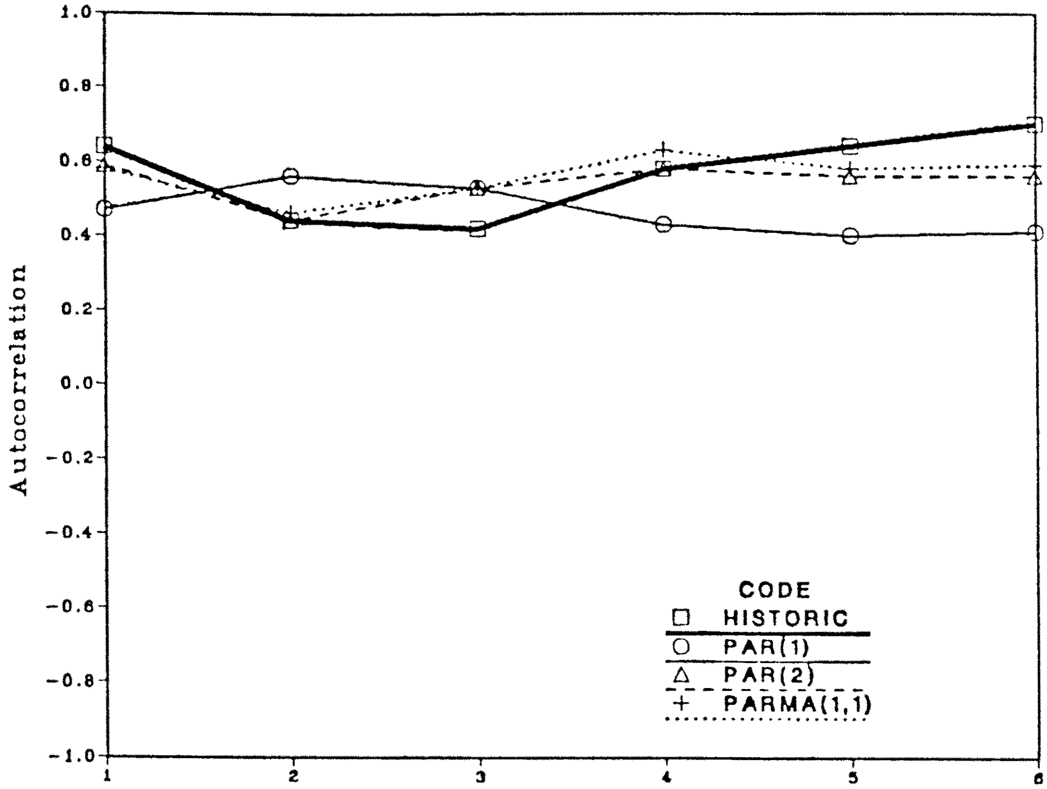

Figure 4.41 - Lag 5 seasonal autocorrelation
Least-squares estimation - St. Lawrence

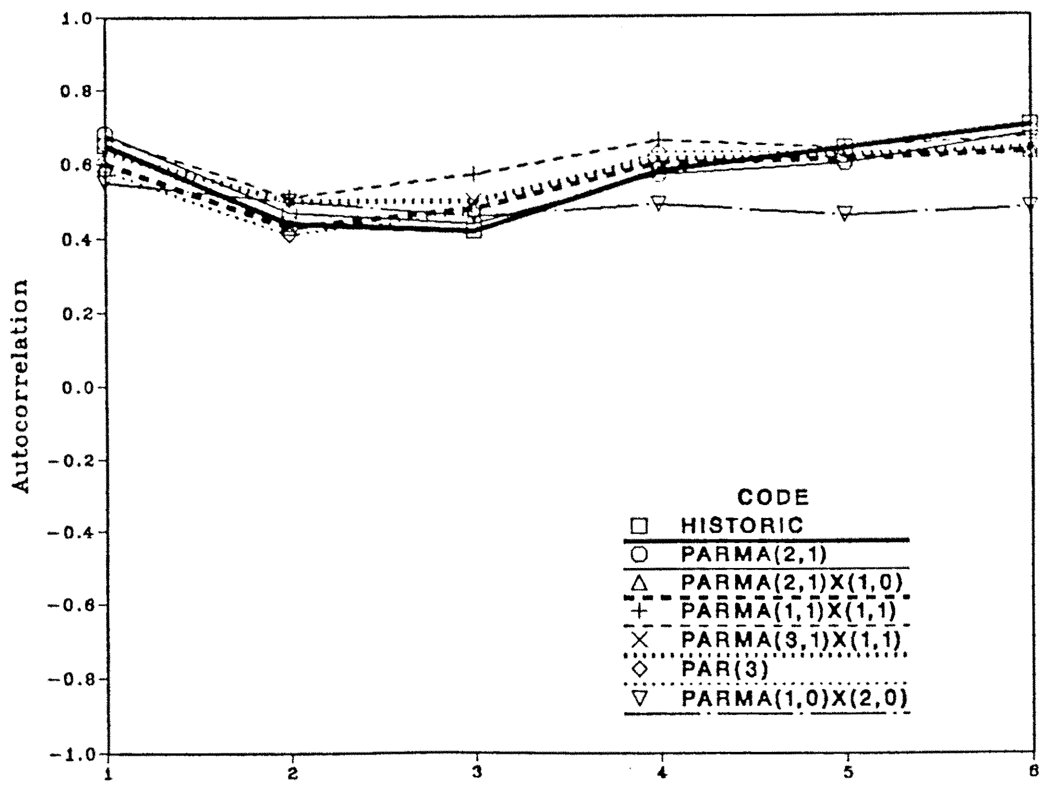

Figure 4.42 - Lag 5 seasonal autocorrelations
Least-squares estimation - St. Lawrence

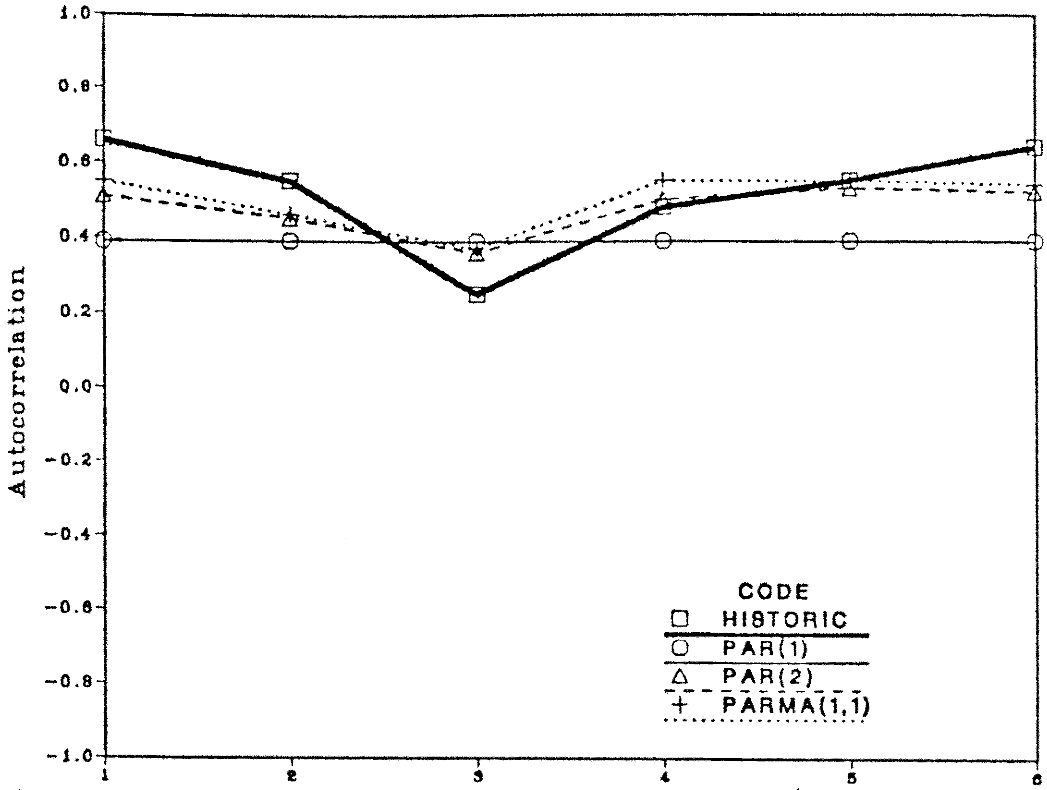

Figure 4.43 - Lag 6 seasonal autocorrelations
Least-squares estimation - St. Lawrence

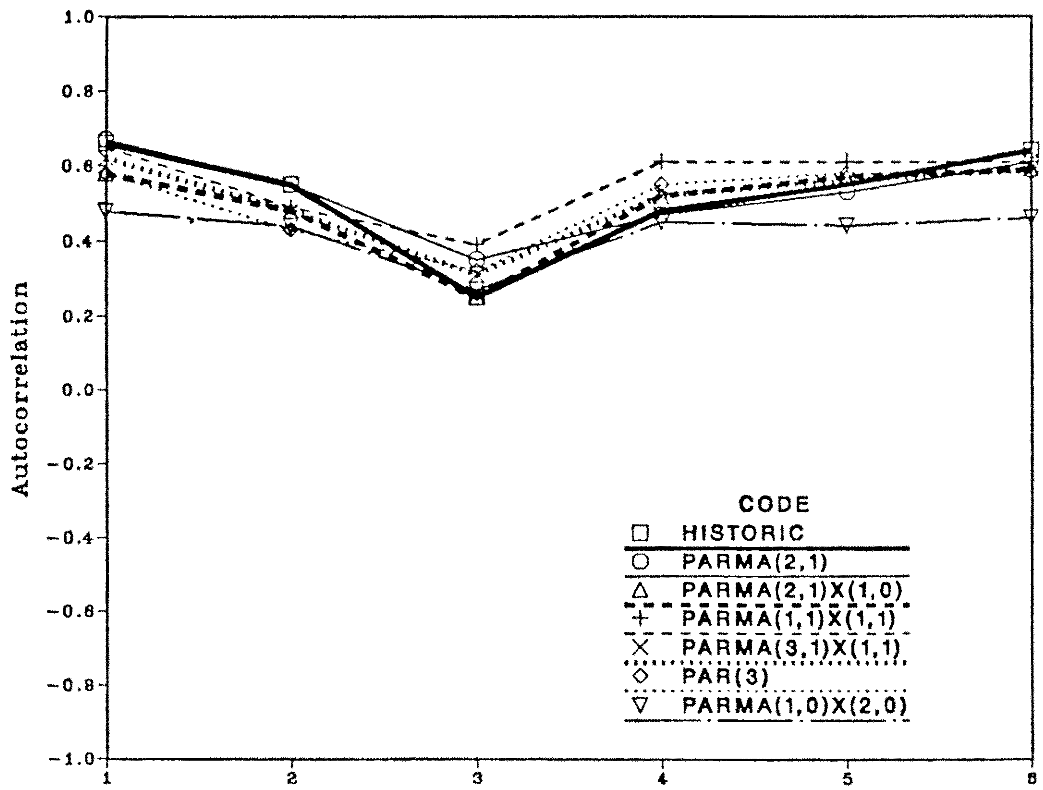

Figure 4.44 - Lag 6 seasonal autocorrelations
Least-squares estimation - St. Lawrence

4.1.2 Niger River

4.1.2.1 Four-season sample

a) Residuals Characteristics -

The seasonal means of the residuals for every fitted model were approximately zero. Table 4.5 shows the seasonal standard deviations for the residuals:

Table 4.5 - Seasonal standard deviations of residuals - Logarithms of 4-season discharges - Niger

Model	σ_1	σ_2	σ_3	σ_4
PAR(1)	0.21	0.53	0.19	0.21
PAR(2)	0.18	0.52	0.18	0.21
PARMA(1,1)	0.17	0.53	0.18	0.21
PARMA(2,1)	0.16	0.50	0.17	0.21
Mult. PARMA(2,1)x(1,0)	0.16	0.46	0.20	0.20
Mult. PARMA(1,1)x(1,1)	0.17	0.45	0.17	0.18
Mult. PARMA(3,1)x(1,1)	0.16	0.35	0.14	0.21
PAR(3)	0.17	0.46	0.19	0.21
PARMA(1,0)x(2,0)	0.19	0.50	0.17	0.21

Seasonal autocorrelations of residuals are shown in Figures 4.45 through 4.52 ; portmanteau and AIC statistics are shown in Table 4.6

Table 4.6 - Portmanteau and AIC statistics - Niger River Four seasons

Model	Q_1	AIC
PAR(1)	88.8	2552
PAR(2)	59.7	2538
PARMA(1,1)	55.2	2537
PARMA(2,1)	56.6	2528
Mult. PARMA(2,1)x(1,0)	56.6	2526
Mult. PARMA(1,1)x(1,1)	58.8	2530
Mult. PARMA(3,1)x(1,1)	51.0	2528
PAR(3)	59.5	2535
Mult. PARMA(1,0)x(2,0)	77.6	2540

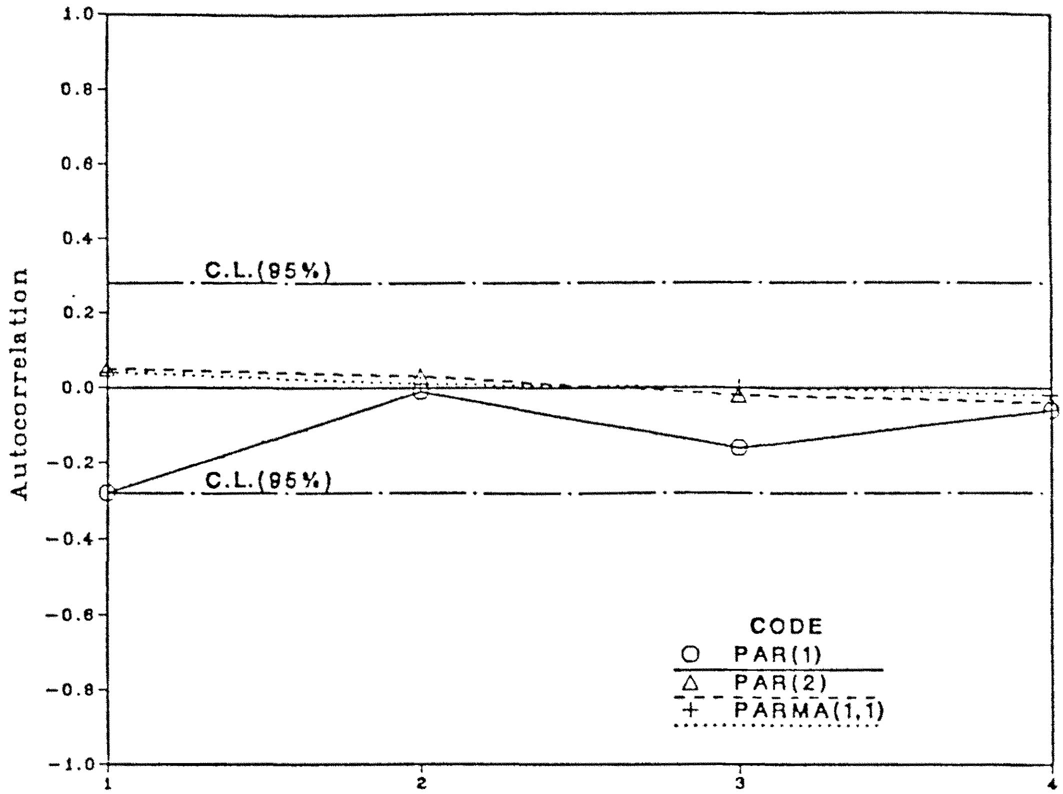

Figure 4.45 Lag 1 autocorrelation of residuals
Least-squares estimation - Niger River

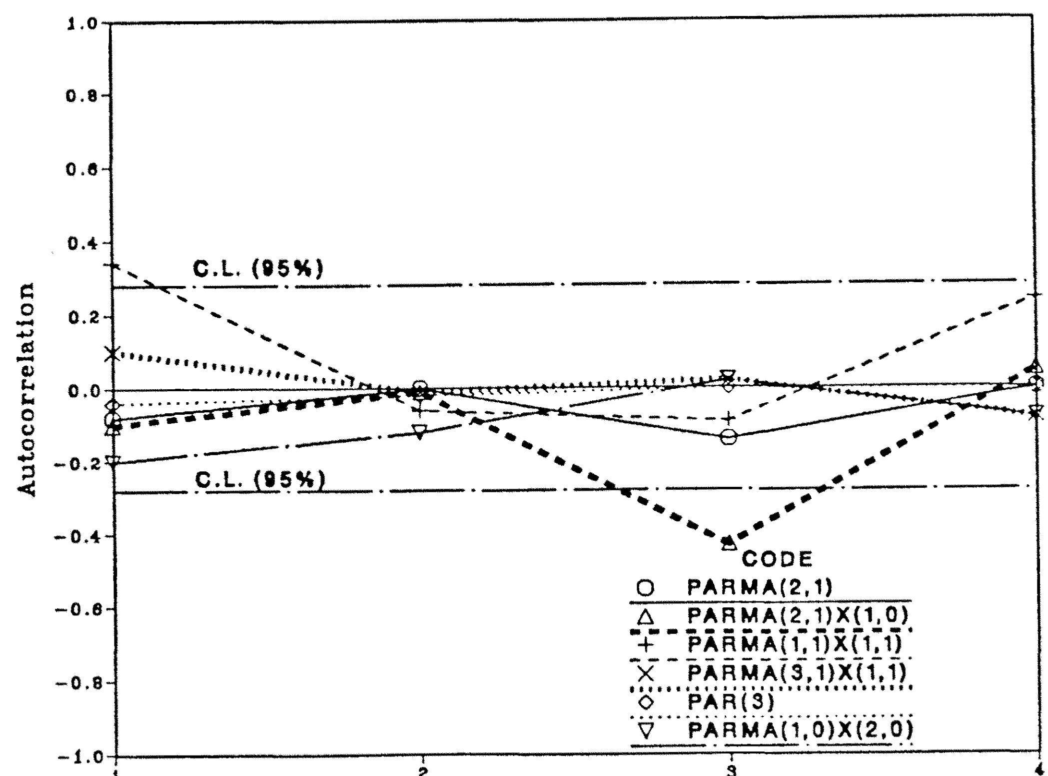

Figure 4.46 - Lag 1 autocorrelation of residuals
Least-squares estimation - Niger River

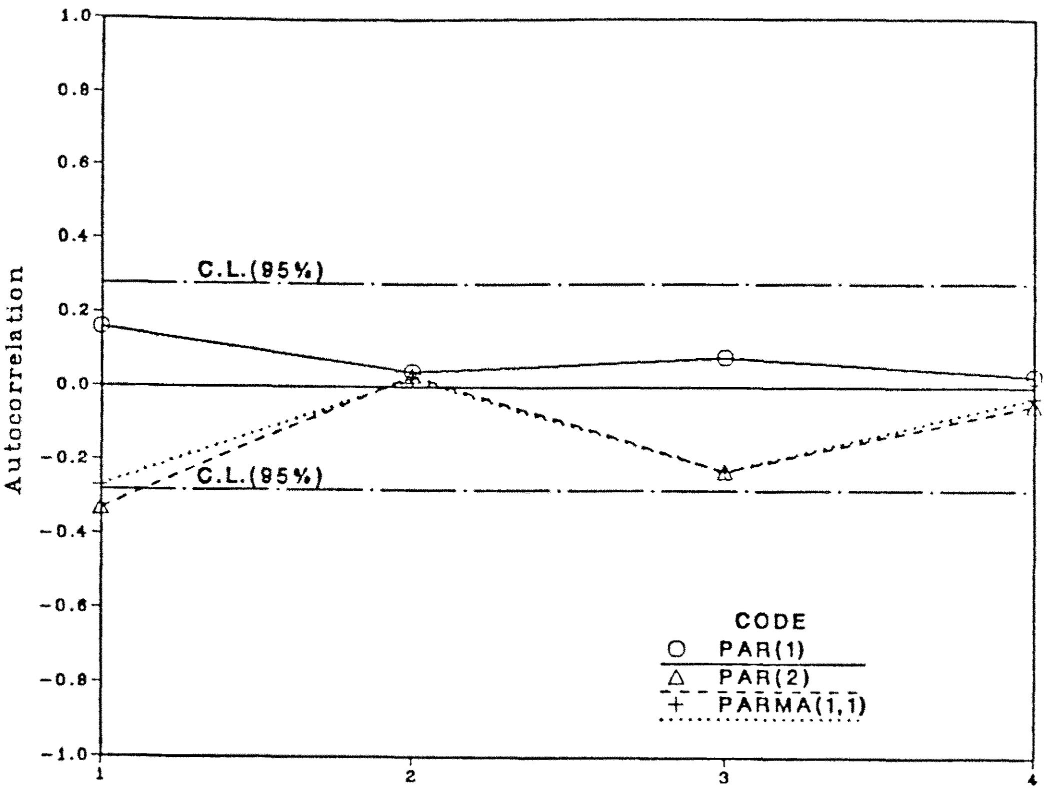

Figure 4.47 - Lag 2 autocorrelation of residuals
Least-squares estimation - Niger River

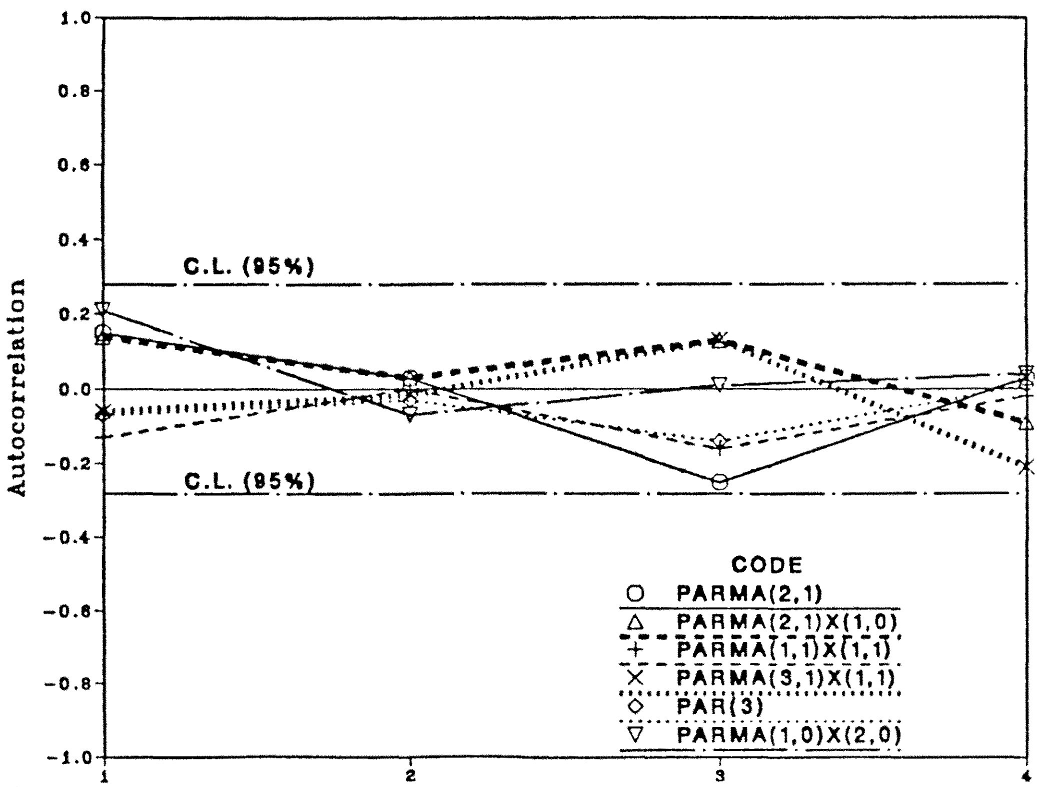

Figure 4.48 - Lag 2 autocorrelation of residuals
Least-squares estimation - Niger River

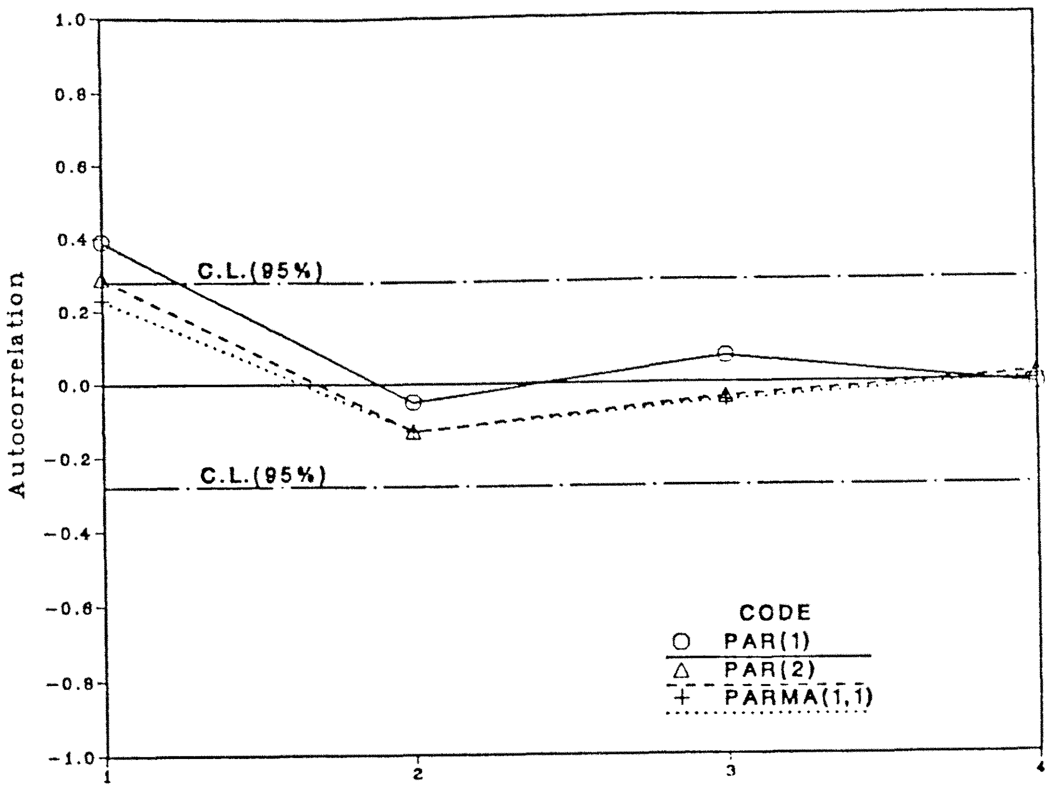

Figure 4.49 - Lag 3 autocorrelation of residuals
Least-squares estimation - Niger River

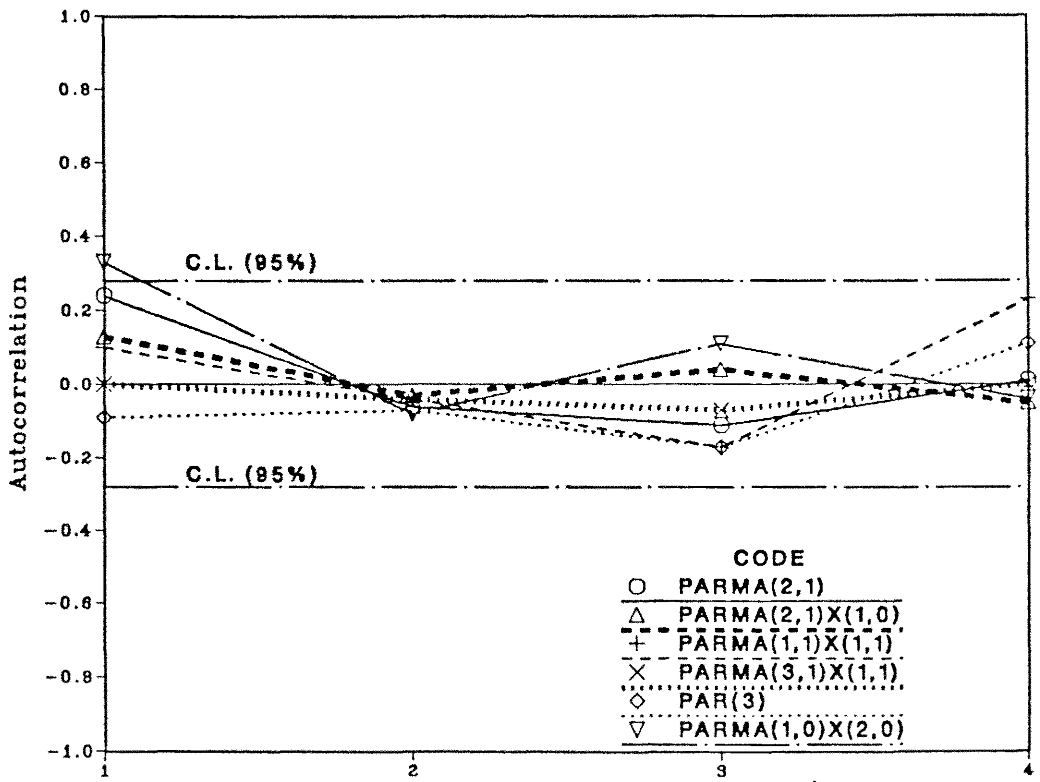

Figure 4.50 - Lag 3 autocorrelation of residuals
Least-squares estimation - Niger River

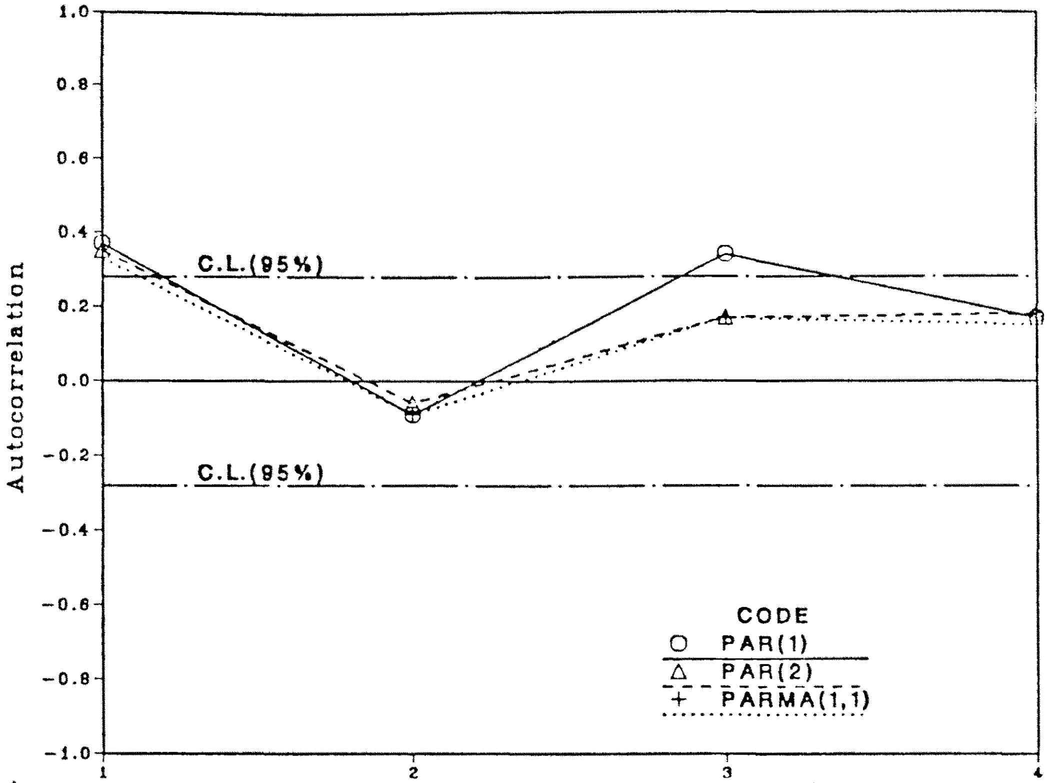

Figure 4.51 - Lag 4 autocorrelation of residuals
Least-squares estimation - Niger River

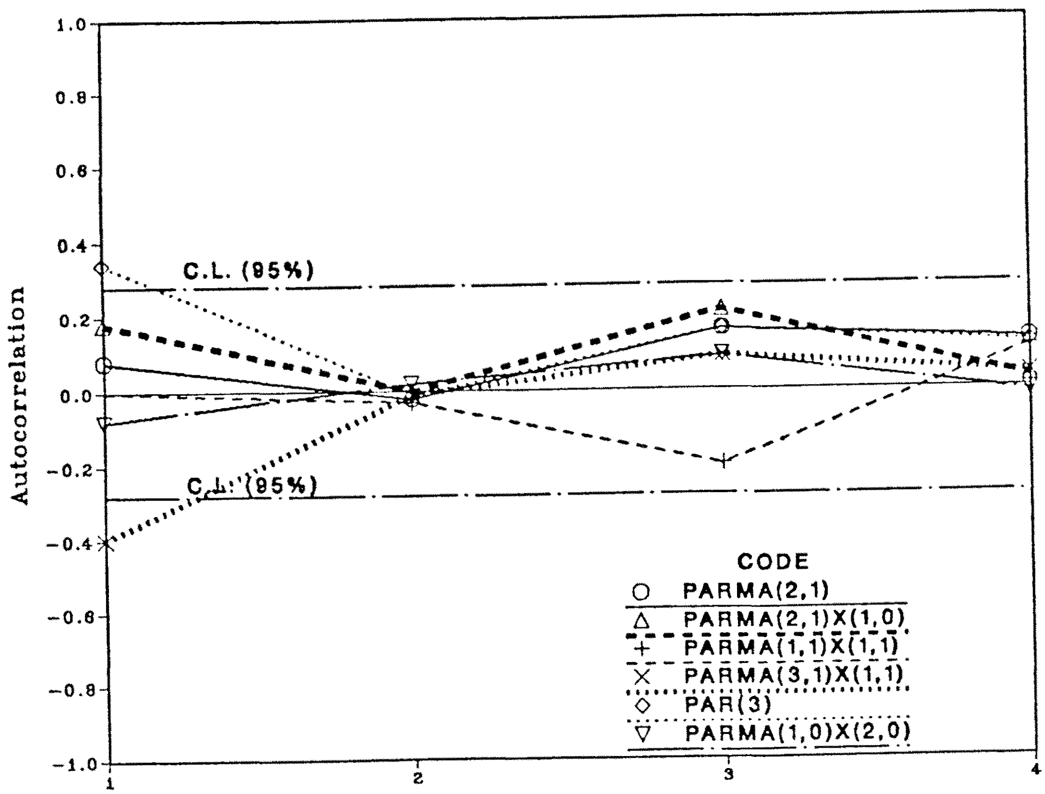

Figure 4.52 - Lag 4 autocorrelation of residuals
Least-squares estimation - Niger River

b) Preservation of statistics

The same number of samples presenting the same size as for the St. Lawrence River were generated using parameters estimated from the 4-season historic data from the Niger River at Kaulikoro. Estimations of statistics from simulated samples were made using the proposed least-squares technique for PAR(1), PAR(2) and PARMA(1,1), Multiplicative PARMA((2,1)x(0,0), or PARMA(2,1), Multiplicative PARMA(2,1)x(1,0), Multiplicative PARMA(1,1)x(1,1), Multiplicative PARMA(3,1)x(1,1), PAR(3) and Multiplicative PARMA(1,0)x(2,0) models.

Averages for seasonal means are presented in Figures A.13 and A.14 in Appendix.

Averages for seasonal standard deviations are shown in Figures A.15 and A.16, while averages for seasonal skewness are shown in Figures A.17 and A.18.

Averages annual autocorrelations are presented in Figures 4.53 and 4.54. Figures 4.55 through 4.62 show the averages for seasonal autocorrelations.

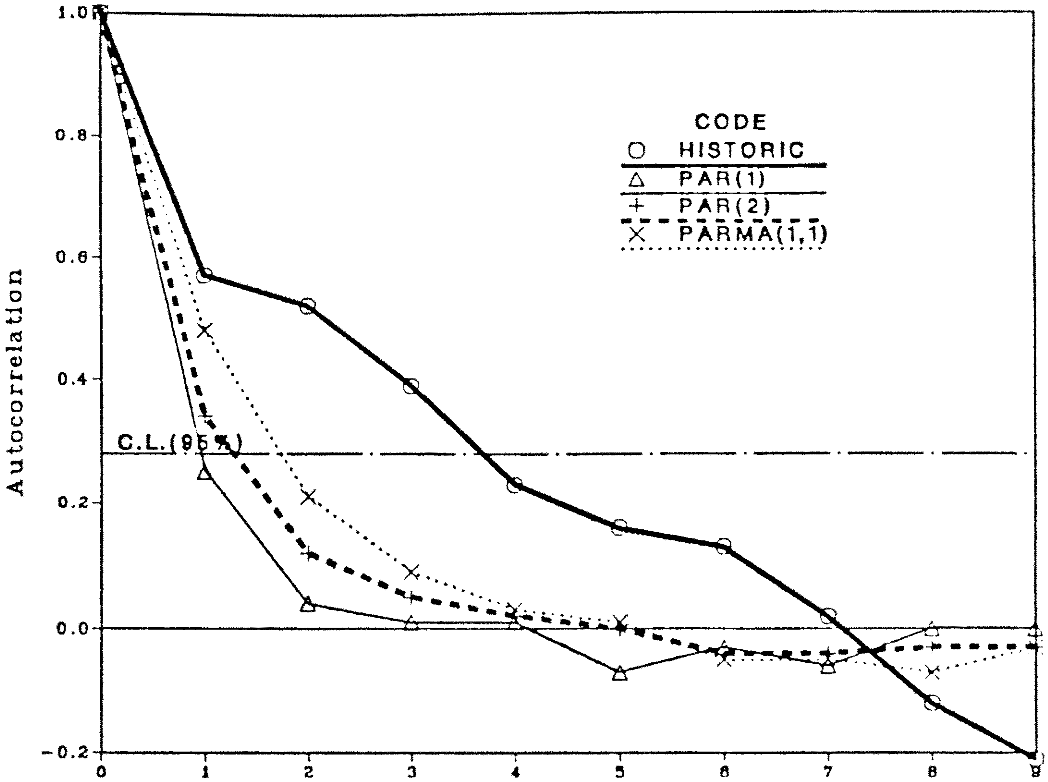

Fig. 4.53 - Annual autocorrelogram - 4 seasons
Least-squares estimation - Niger River

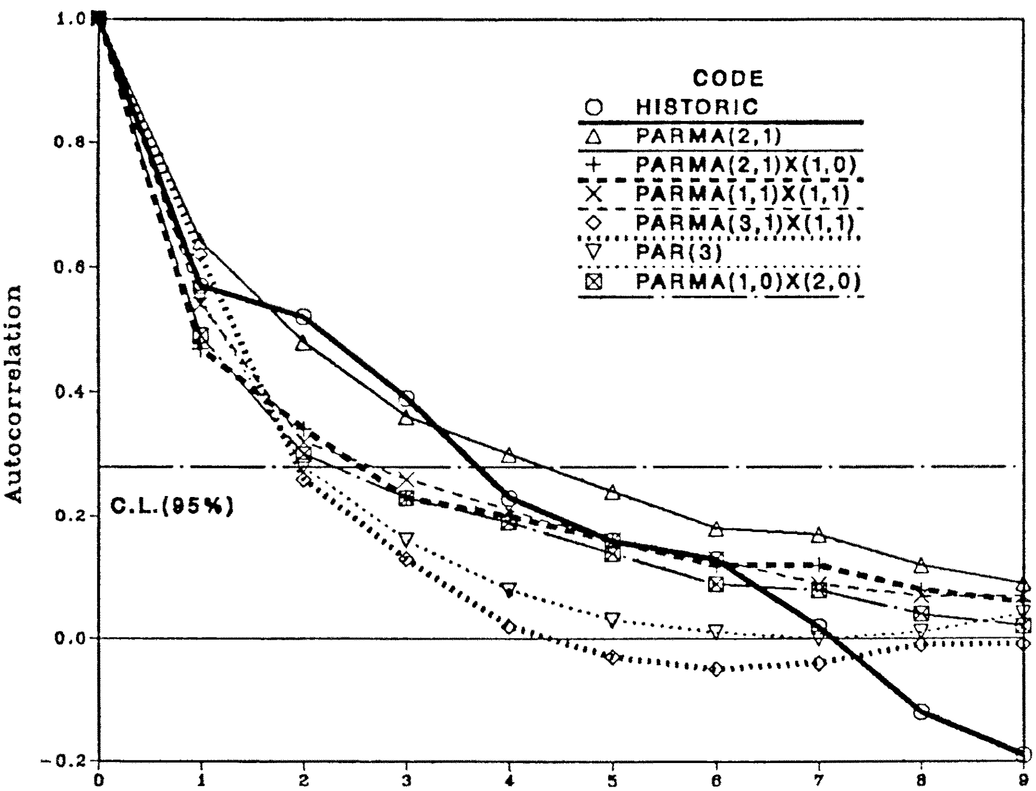

Fig. 4.54 - Annual autocorrelogram - 4 seasons
Least-squares estimation - Niger River

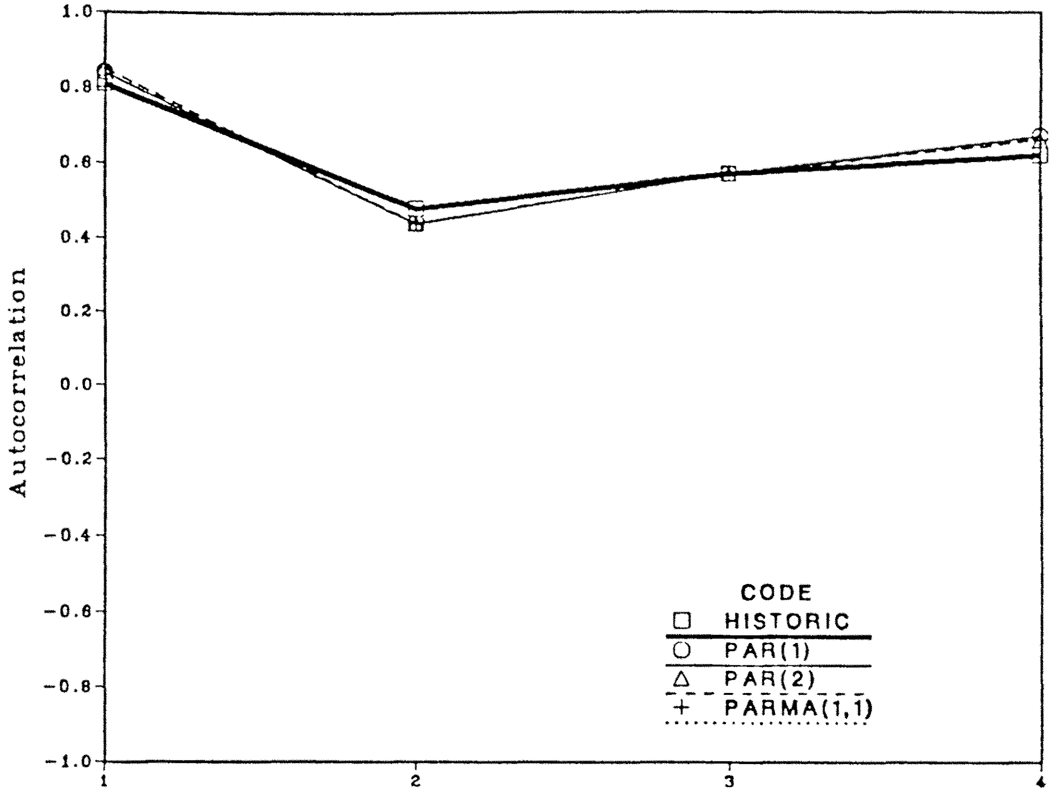

Figure 4.55 - Lag 1 seasonal autocorrelation
Least-squares estimation - Niger River

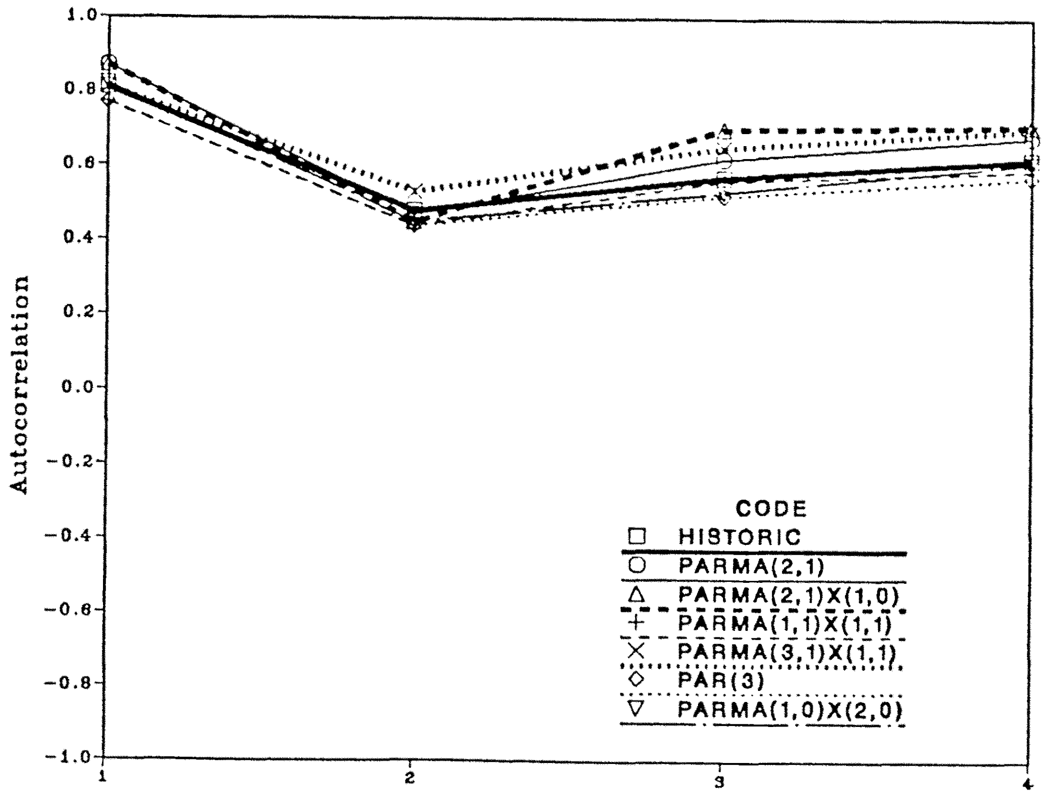

Figure 4.56 - Lag 1 seasonal autocorrelation
Least-squares estimation - Niger River

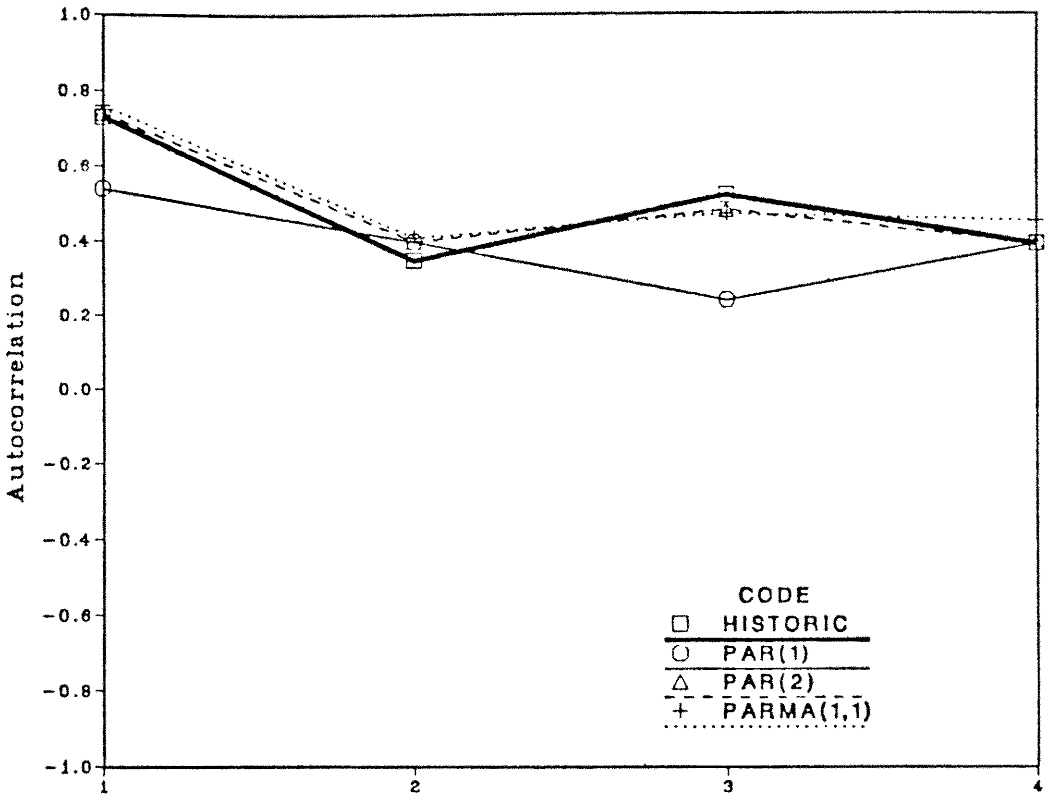

Figure 4.57 - Lag 2 seasonal autocorrelation
Least-squares estimation - Niger River

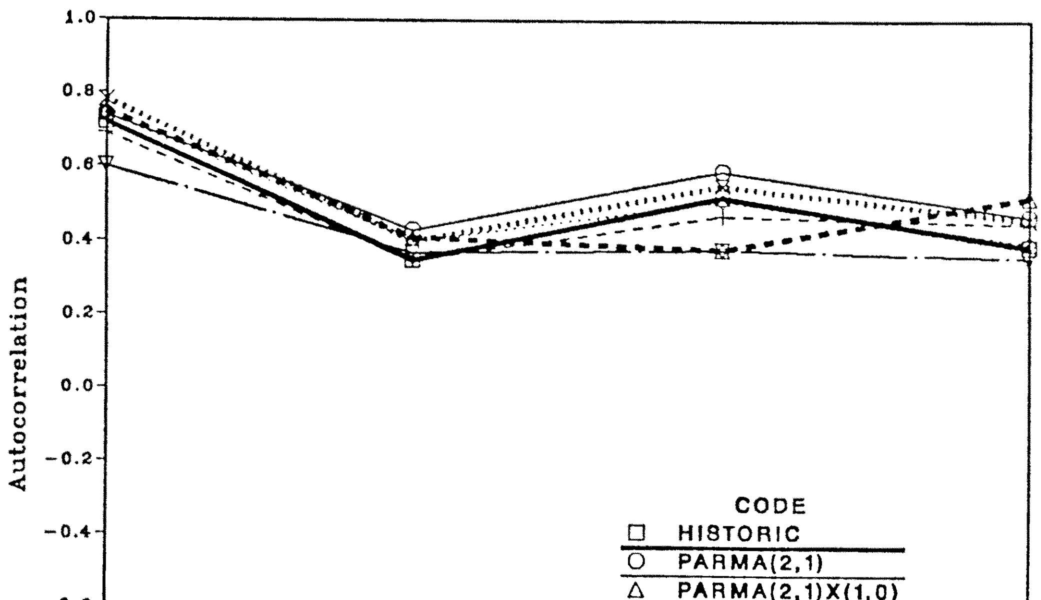

Figure 4.58 - Lag 2 seasonal autocorrelation
Least-squares estimation - Niger River

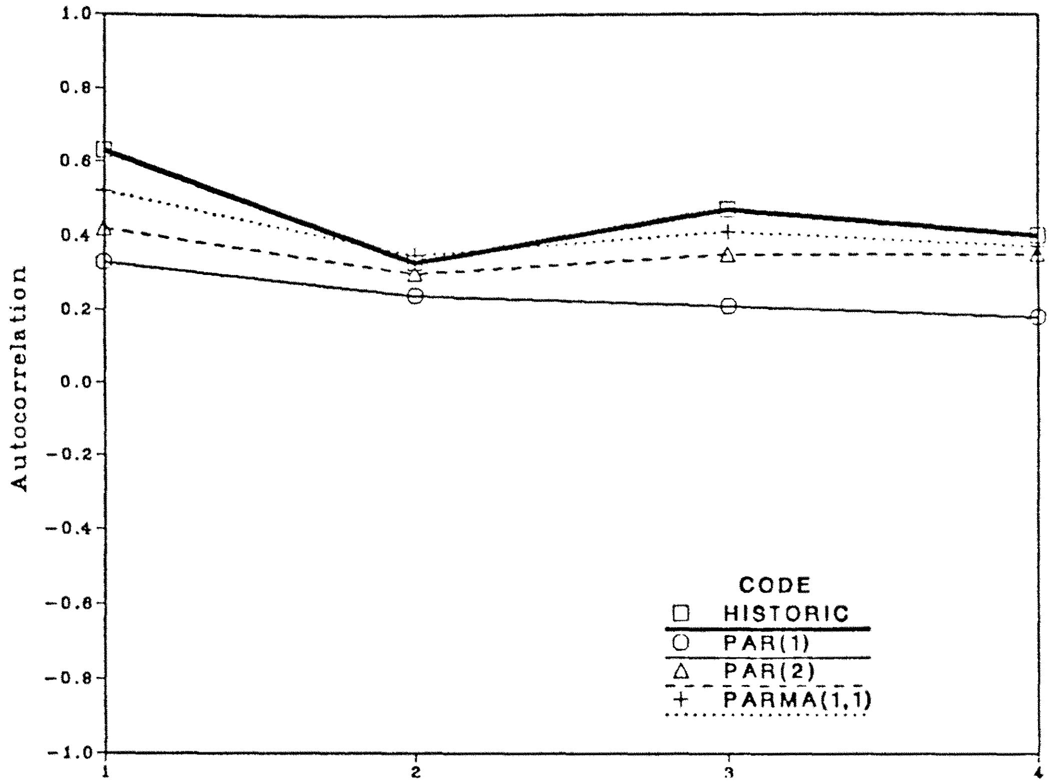

Figure 4.59 - Lag 3 seasonal autocorrelation
Least-squares estimation - Niger River

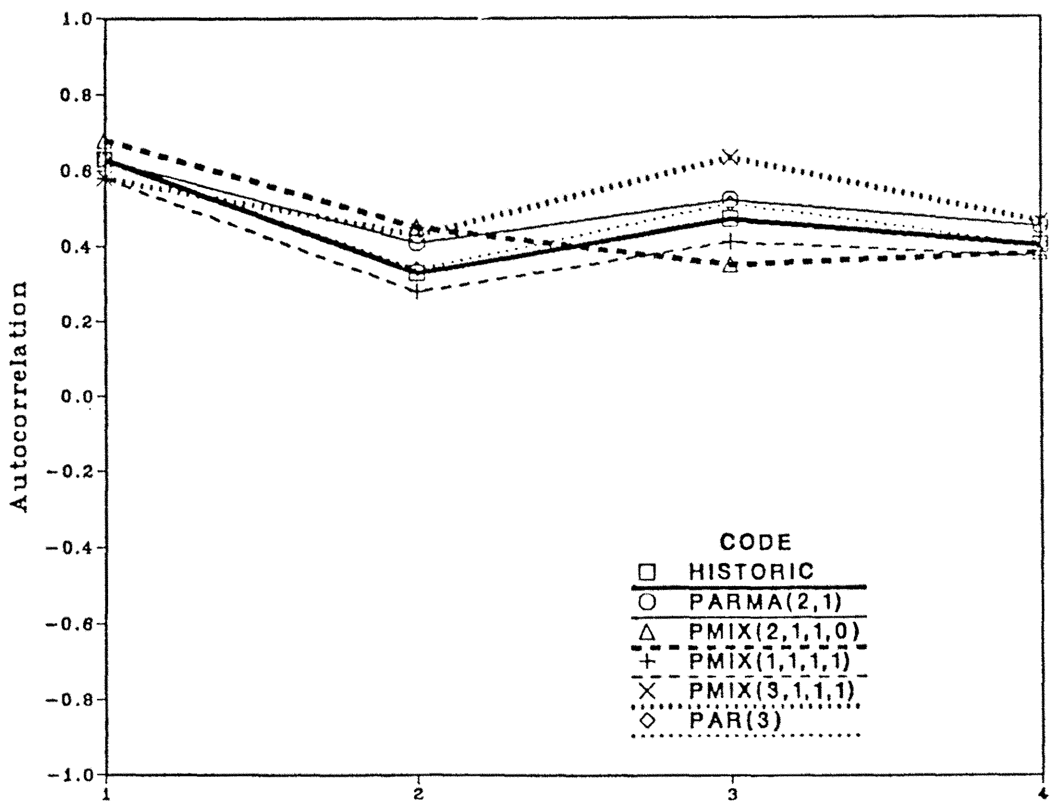

Figure 4.60 - Lag 3 seasonal autocorrelation
Least-squares estimation - Niger River

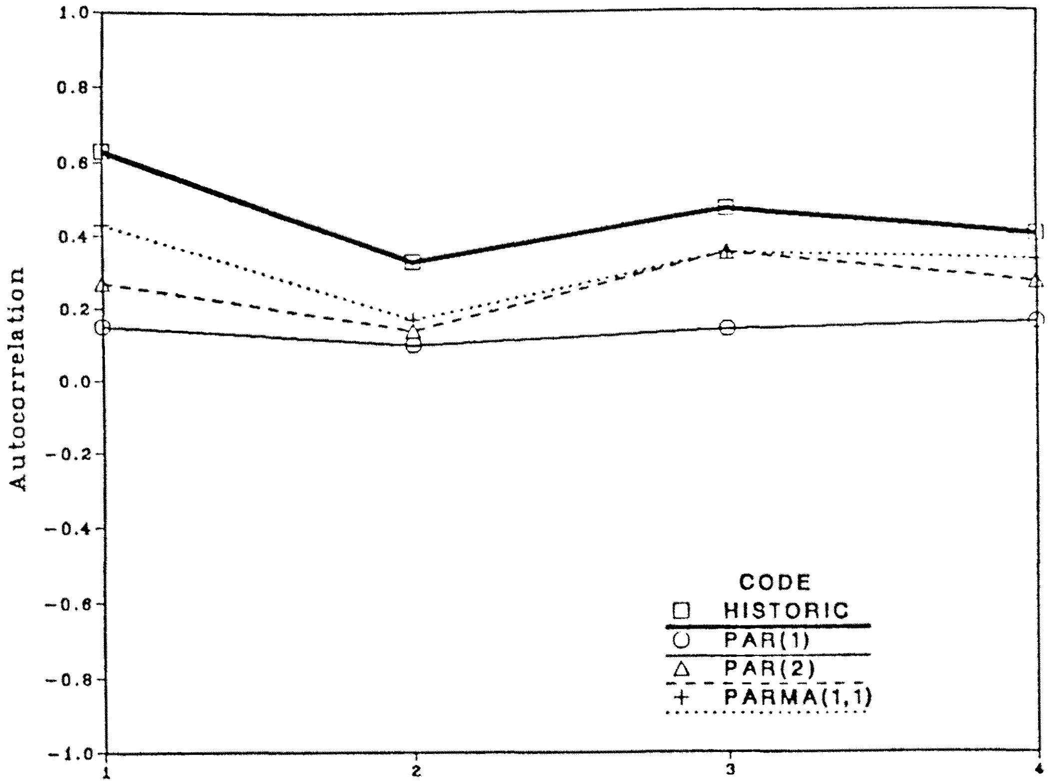

Figure 4.61 - Lag 4 seasonal autocorrelation
Least-squares estimation - Niger River

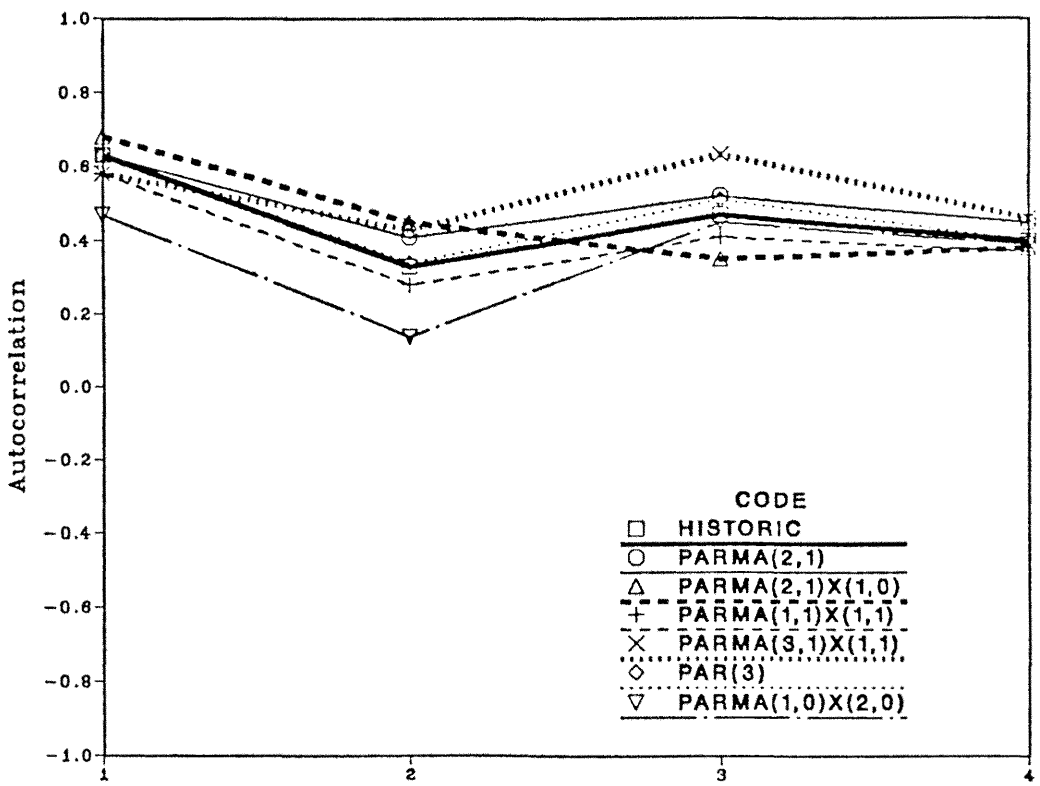

Figure 4.62 - Lag 4 seasonal autocorrelations
Least-squares estimation - Niger River

4.1.2.2 Six-season sample

a) Residuals characteristics

As for all previous samples, the seasonal means of residuals were approximately zero.

The seasonal standard deviations of residuals are shown in Table 4.7.

Table 4.7 - Seasonal standard deviations of residuals
Logarithms of six-season discharges

Model	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
PAR(1)	0.18	0.23	0.54	0.25	0.14	0.21
PAR(2)	0.16	0.23	0.52	0.24	0.14	0.21
PARMA(1,1)	0.16	0.23	0.52	0.25	0.14	0.21
PARMA(2,1)	0.14	0.22	0.48	0.24	0.14	0.20
Mult. PARMA(2,1)x(1,0)	0.16	0.22	0.44	0.23	0.13	0.20
Mult. PARMA(1,1)x(1,1)	0.13	0.24	0.48	0.22	0.14	0.20
Mult. PARMA(3,1)x(1,1)	0.17	0.25	0.40	0.17	0.12	0.18
Mult. PARMA(1,0)x(2,0)	0.15	0.21	0.51	0.23	0.13	0.21

Seasonal autocorrelations of residuals are plotted in Figures 4.63 through 4.74. The portmanteau and AIC statistics computed from residuals are shown in Table 4.8.

Table 4.8 - Portmanteau statistics - Niger River
Six seasons

Model	Q_1	AIC
PAR(1)	108.0	3442
PAR(2)	89.8	3435
PARMA(1,1)	95.9	3433
PARMA(2,1)	76.1	3423
Mult. PARMA(2,1)x(1,0)	78.7	3412
Mult. PARMA(1,1)x(1,1)	75.5	3421
Mult. PARMA(3,1)x(1,1)	81.5	3374
PAR(3)	80.9	3420
Mult. PARMA(1,0)x(2,0)	80.3	3433

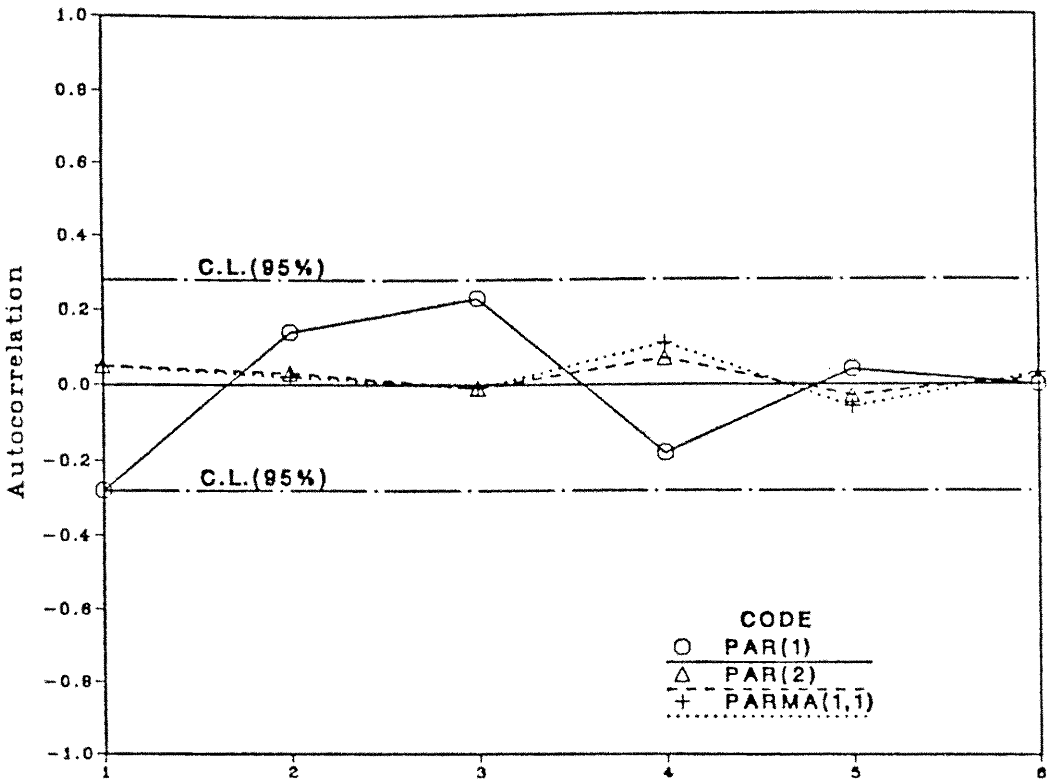

Figure 4.63 - Lag 1 autocorrelation of residuals
Least-squares estimation - Niger River

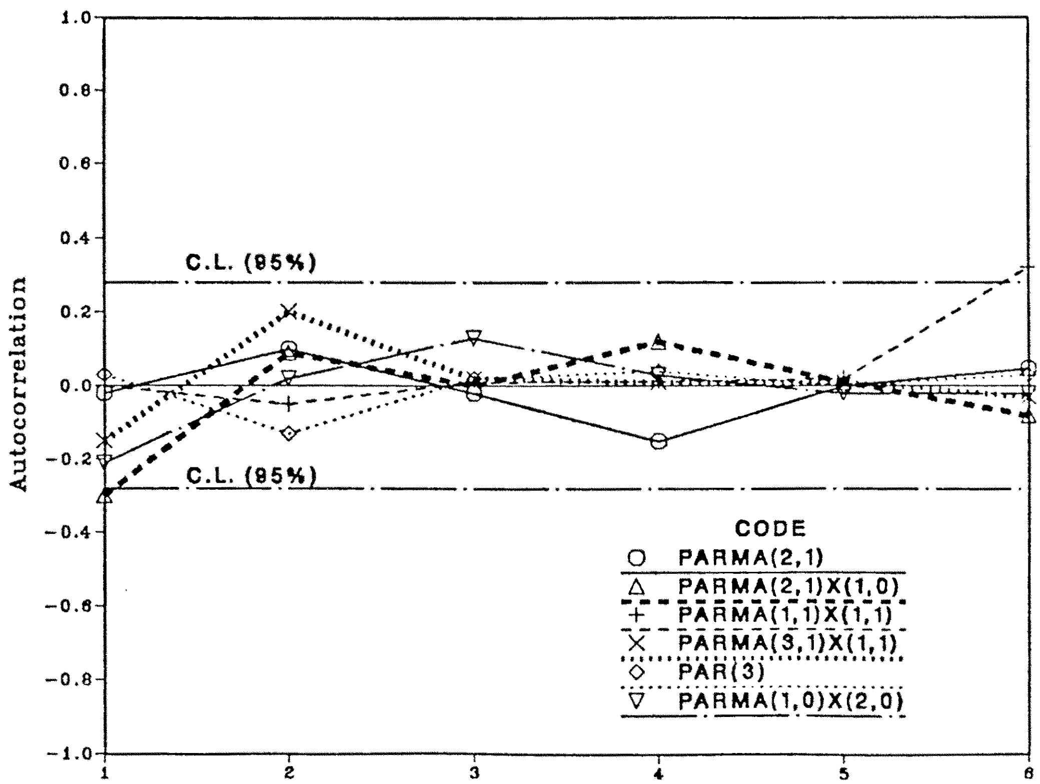

Figure 4.64 - Lag 1 autocorrelation of residuals
Least-squares estimation - Niger River

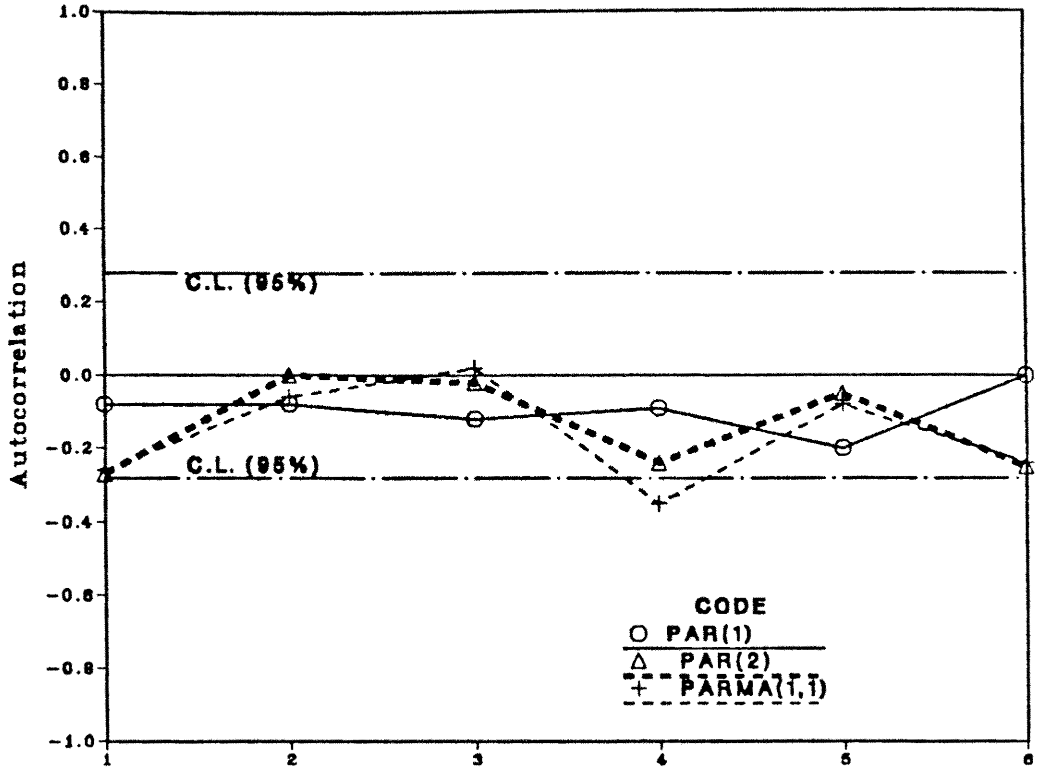

Figure 4.65 - Lag 2 autocorrelation of residuals
Least-squares estimation - Niger River

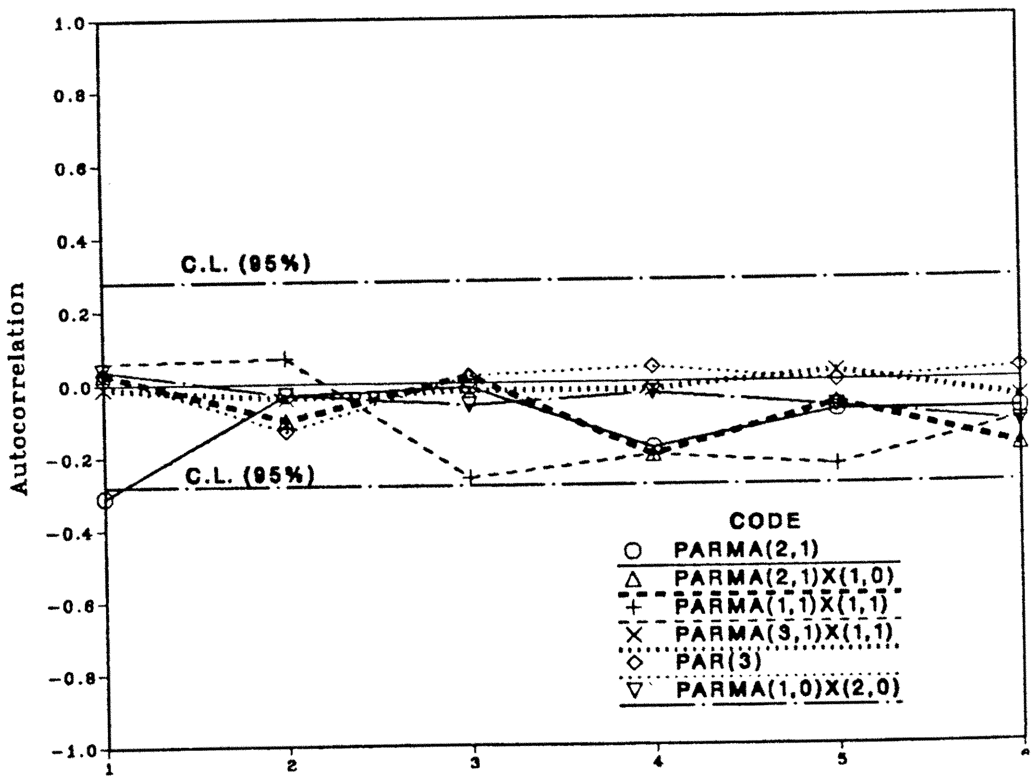

Figure 4.66 - Lag 2 autocorrelation of residuals
Least-squares estimation - Niger River

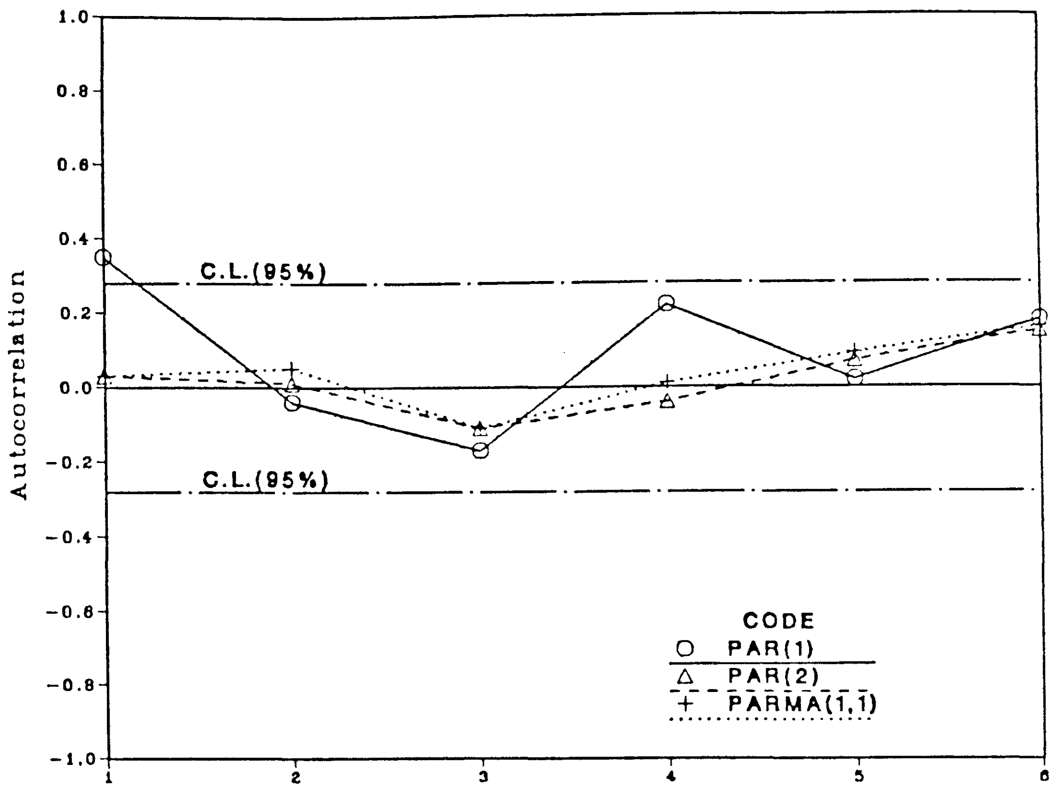

Figure 4.67 - Lag 3 autocorrelation of residuals
Least-squares estimation - Niger River

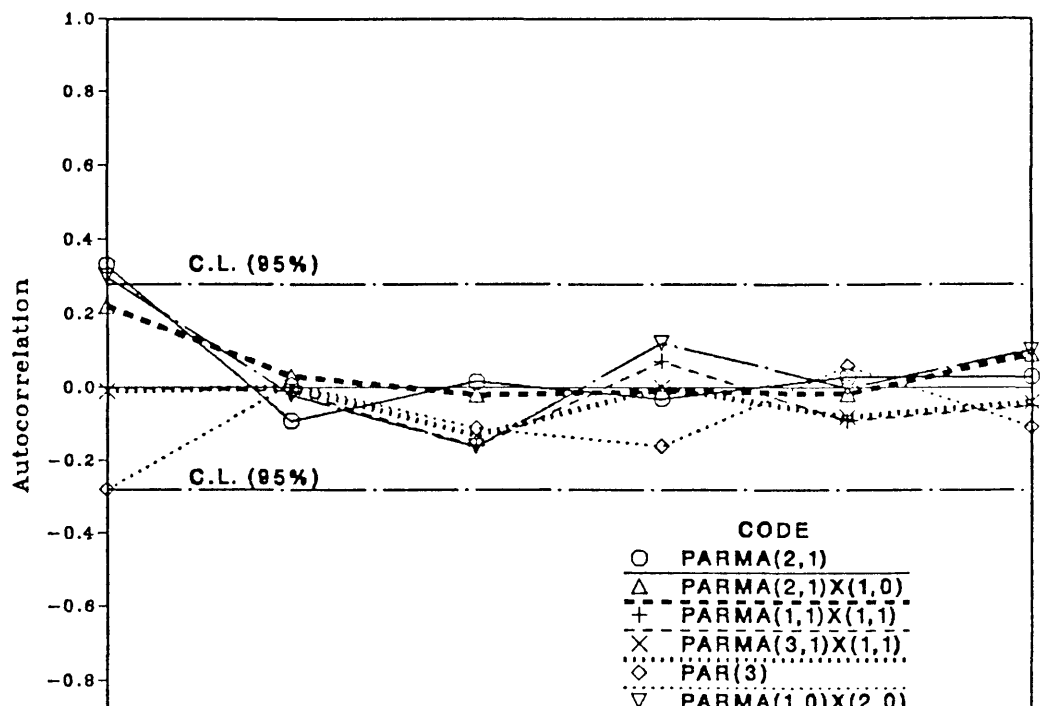

Figure 4.68 - Lag 3 autocorrelation of residuals
Least-squares estimation - Niger River

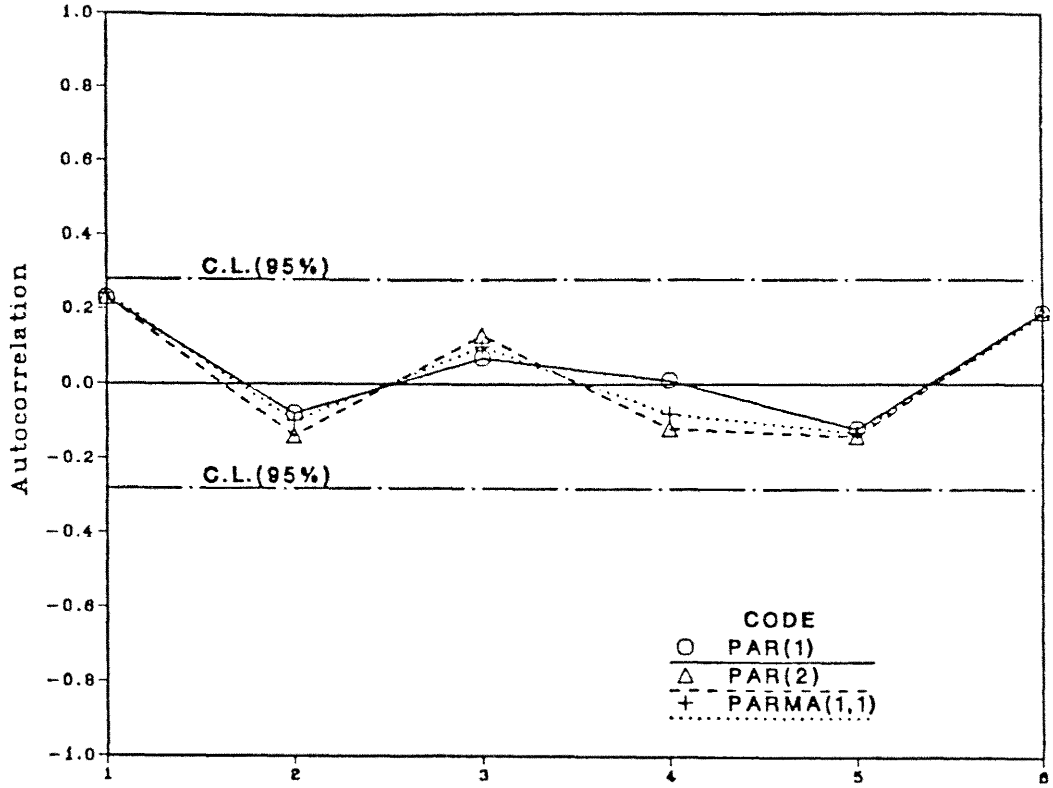

Figure 4.69 - Lag 4 autocorrelation of residuals
Least-squares estimation - Niger River

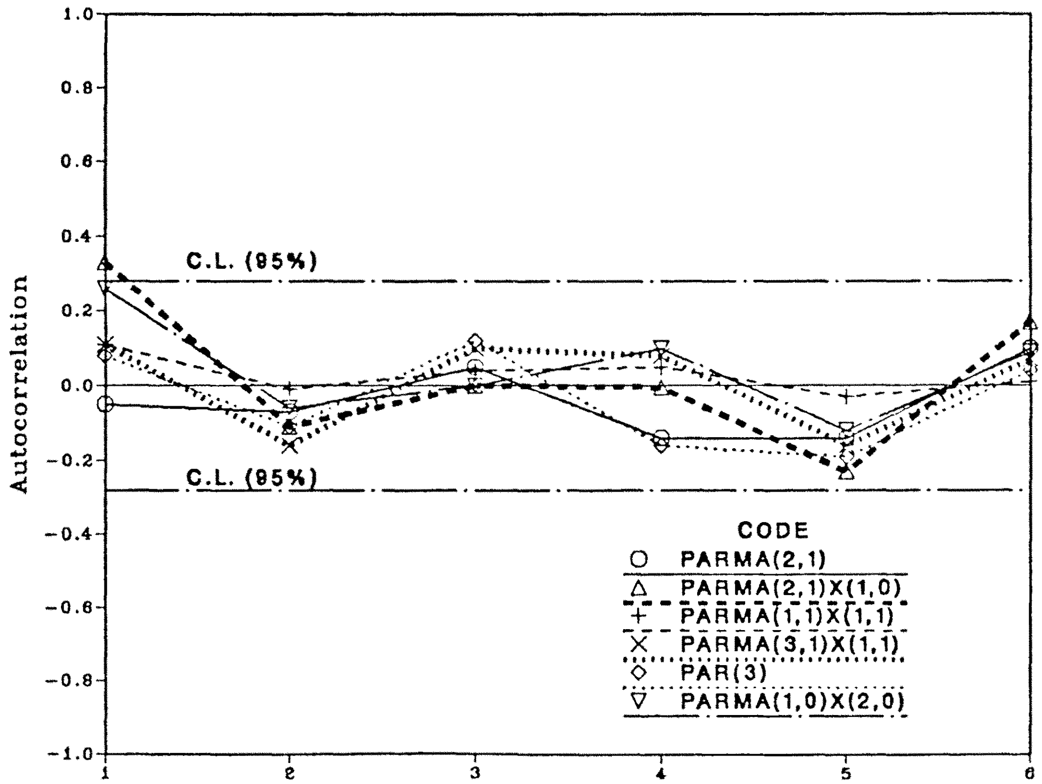

Figure 4.70 - Lag 4 autocorrelation of residuals
Least-squares estimates - Niger River

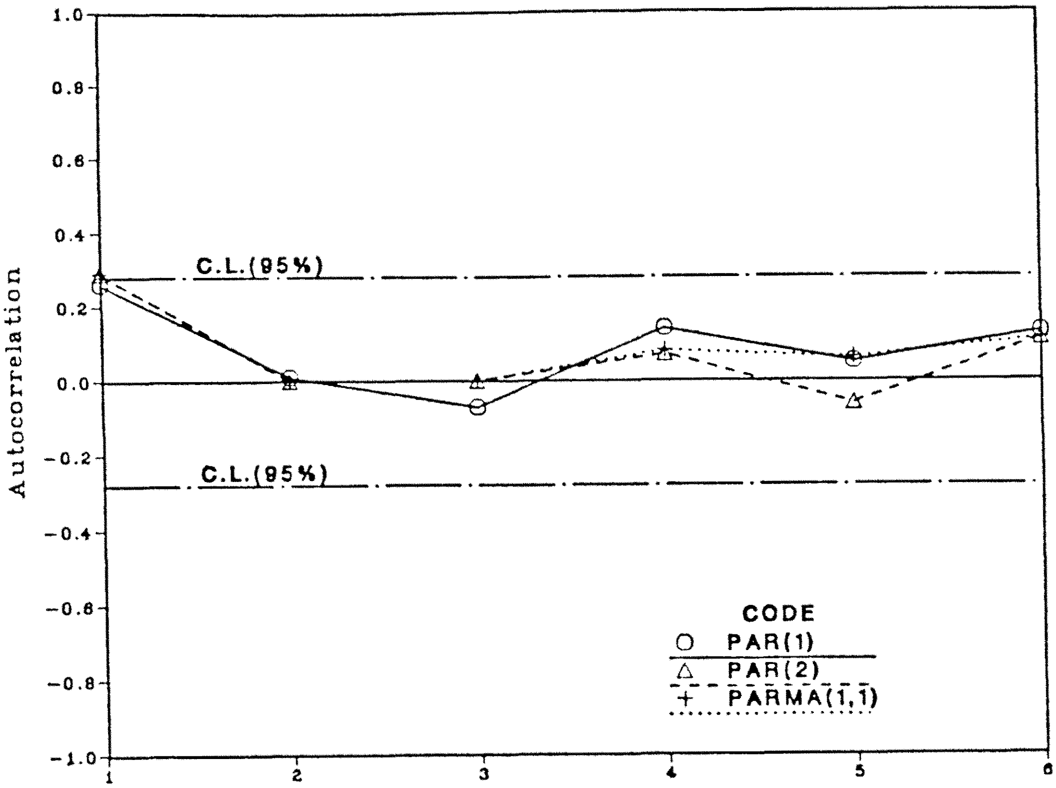

Figure 4.71 - Lag 5 autocorrelation of residuals
Least-squares estimation - Niger River

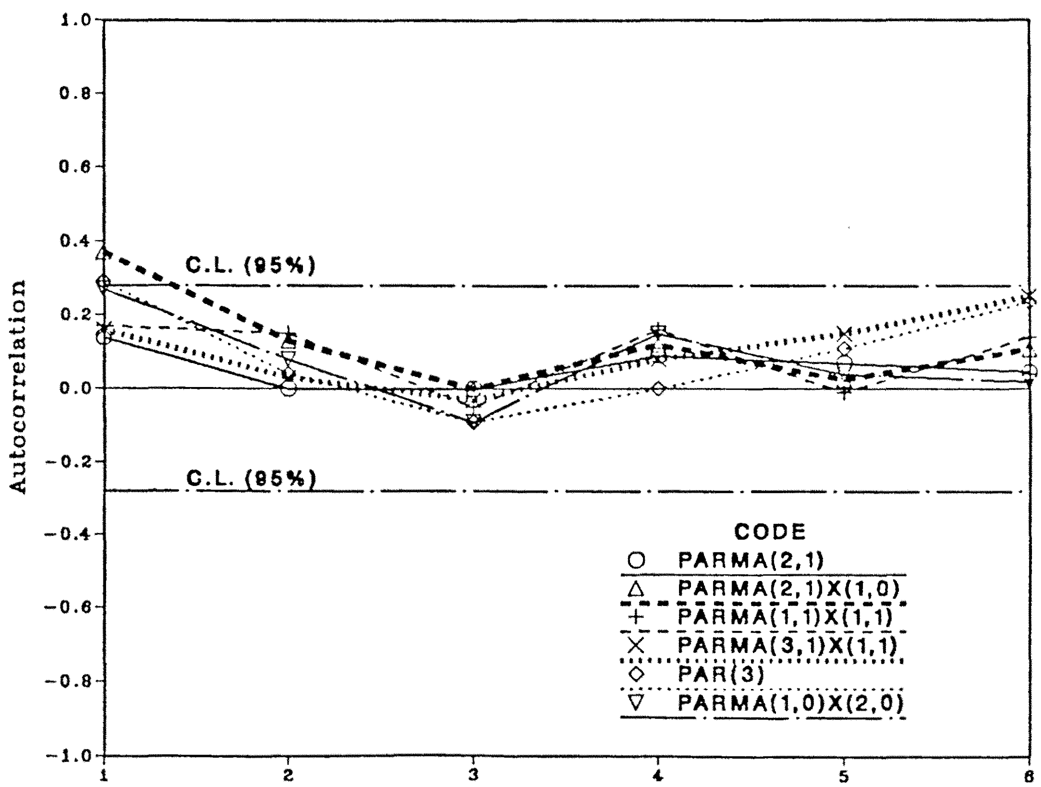

Figure 4.72 - Lag 5 autocorrelation of residuals
Least-squares estimation - Niger River

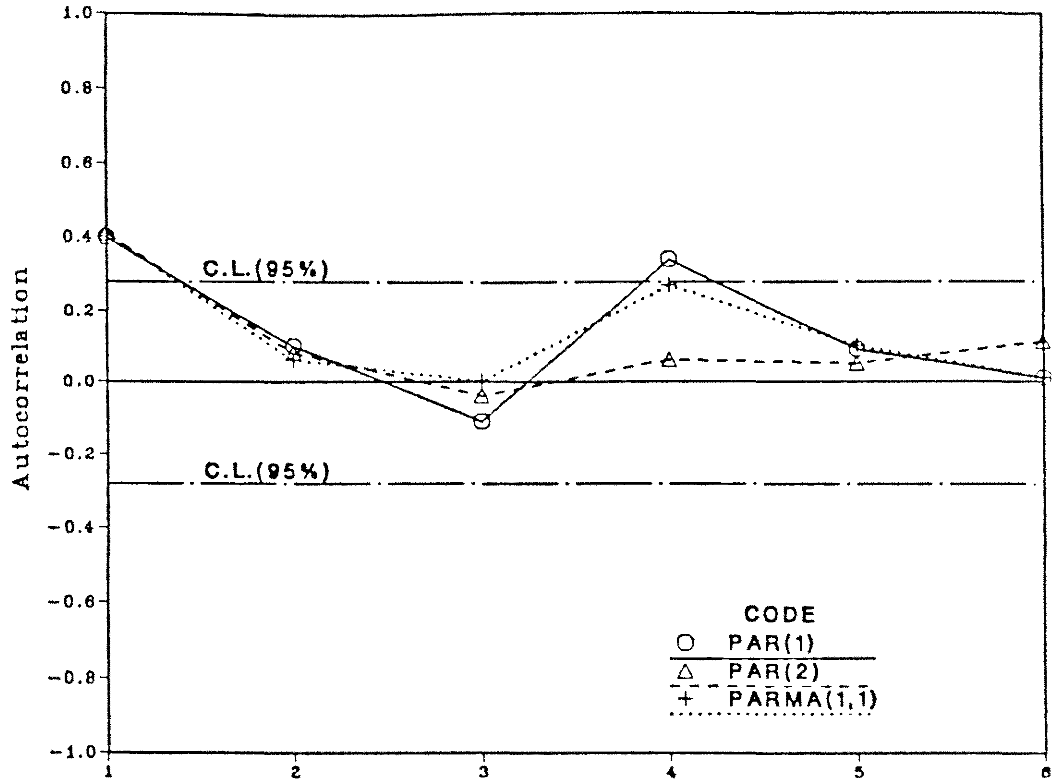

Figure 4.73 - Lag 6 autocorrelation of residuals
Least-squares estimation - Niger River

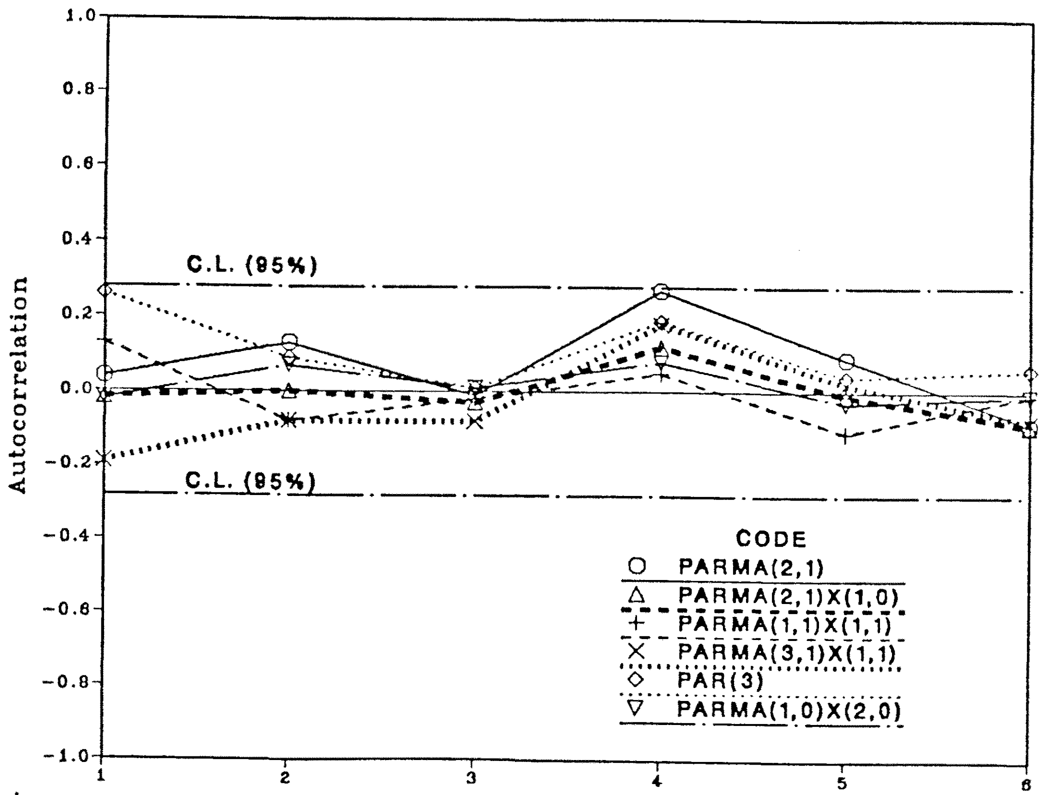

Figure 4.74 - Lag 6 autocorrelation of residuals
Least-squares estimation - Niger River

b) Preservation of statistics

Forty 250-year samples were also generated here utilizing parameters estimated for the same models applied to four-season samples. Computed statistics were averaged for analysis of reproduction of statistics by comparison with corresponding historical statistics as done previously for the St. Lawrence River.

Seasonal average discharges are shown in Figures A.19 and A.20 in Appendix. Seasonal standard deviations are shown in Figures A.21 and A.22. Figures A.23 and A.24 present seasonal skewness.

Annual autocorrelograms are plotted in Figures 4.75 and 4.76, while seasonal autocorrelograms are plotted in Figures 4.77 through 4.88.

4.1.3 <u>Elkhorn River</u>

4.1.3.1 <u>Four-season sample</u>

For the Elkhorn and Yellowstone River only the models PAR(1), PAR(2), PARMA(1,1), PARMA(1,1)x(1,1) were employed since it was seen that most of them did a good job in reproducing statistical autocorrelations. Parameters were computed by using the least-squares technique.

a) Residuals characteristics

Seasonal means of residuals were approximately equal to zero, as computed for all previously fitted models and samples.

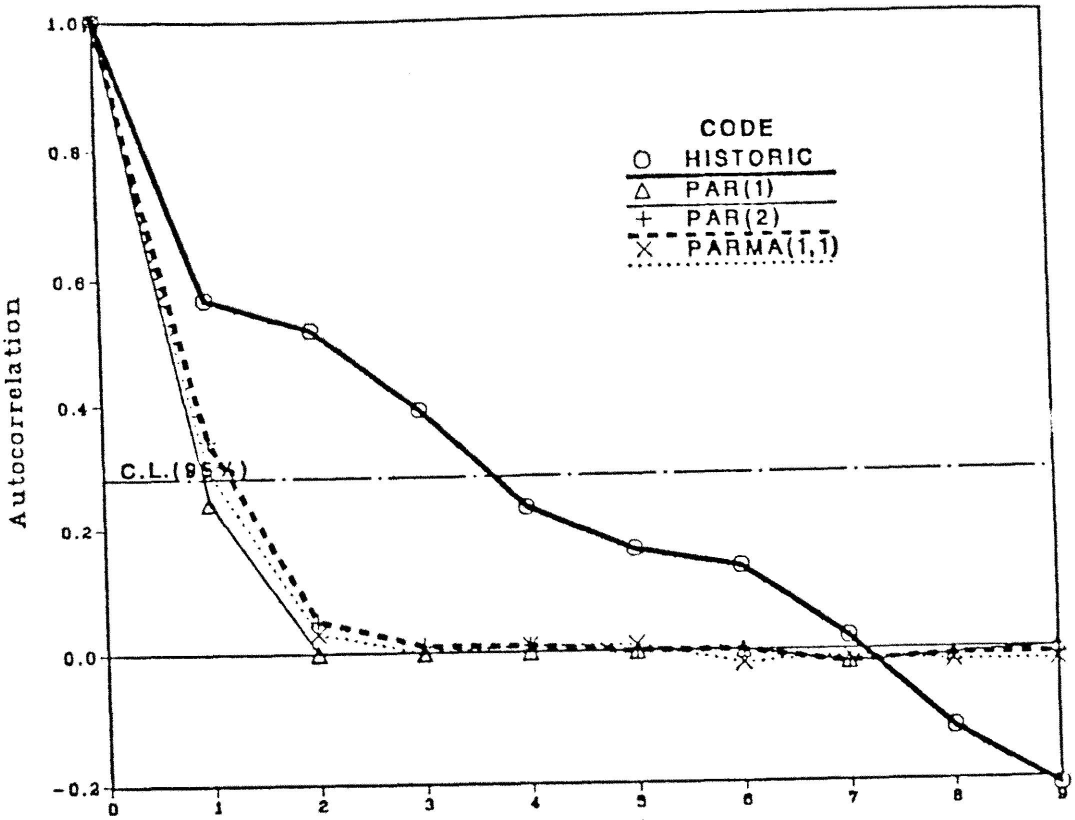

Figure 4.75 - Annual autocorrelations - 6 seasons
Least-squares estimation - Niger River

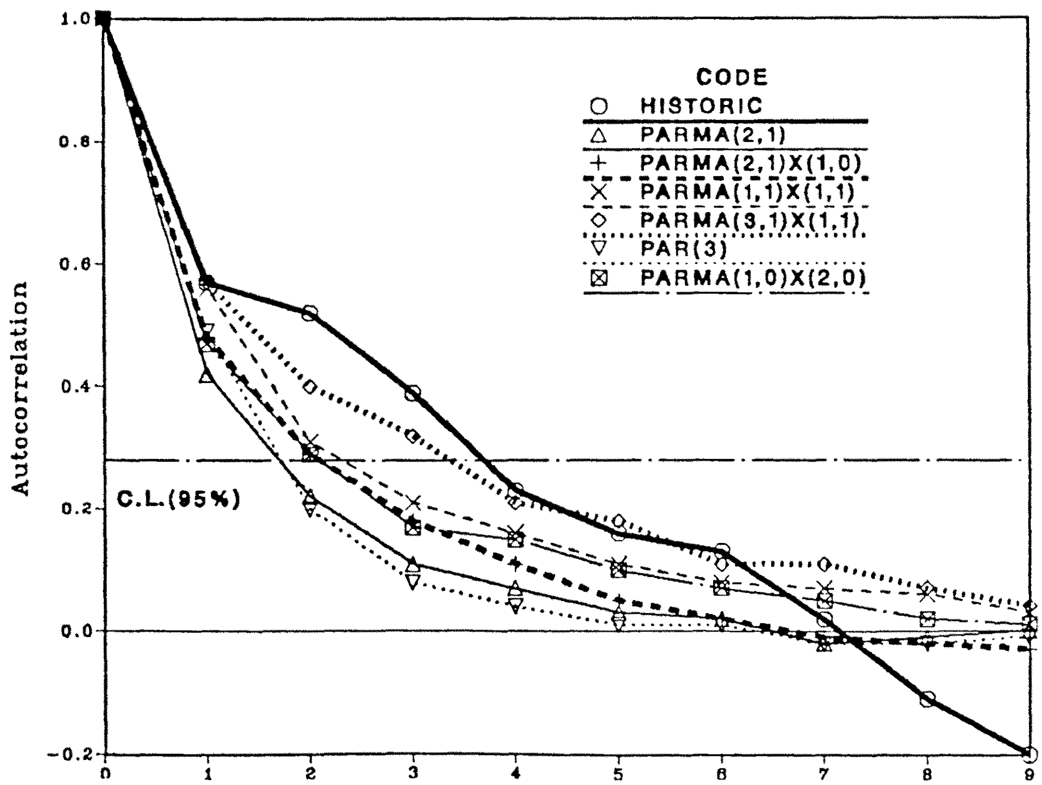

Figure 4.76 - Annual autocorrelations - 6 seasons
Least-squares estimation - Niger River

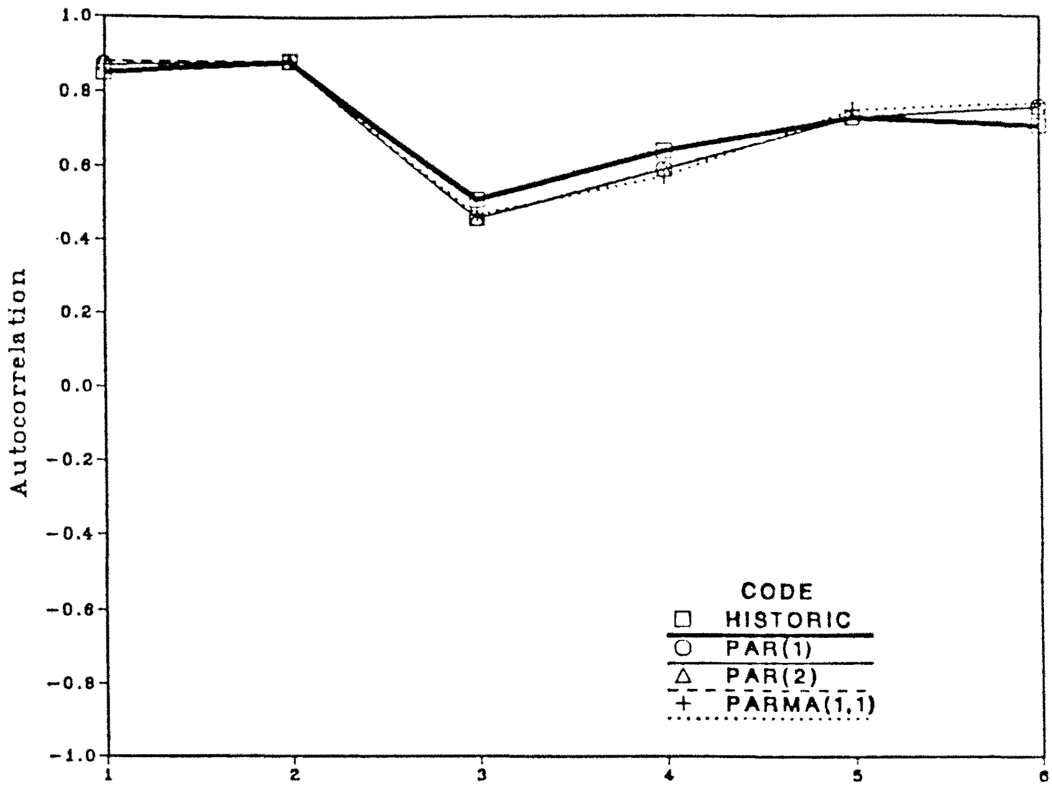

Figure 4.77 - Lag 1 seasonal autocorrelation
Least-squares estimation - Niger River

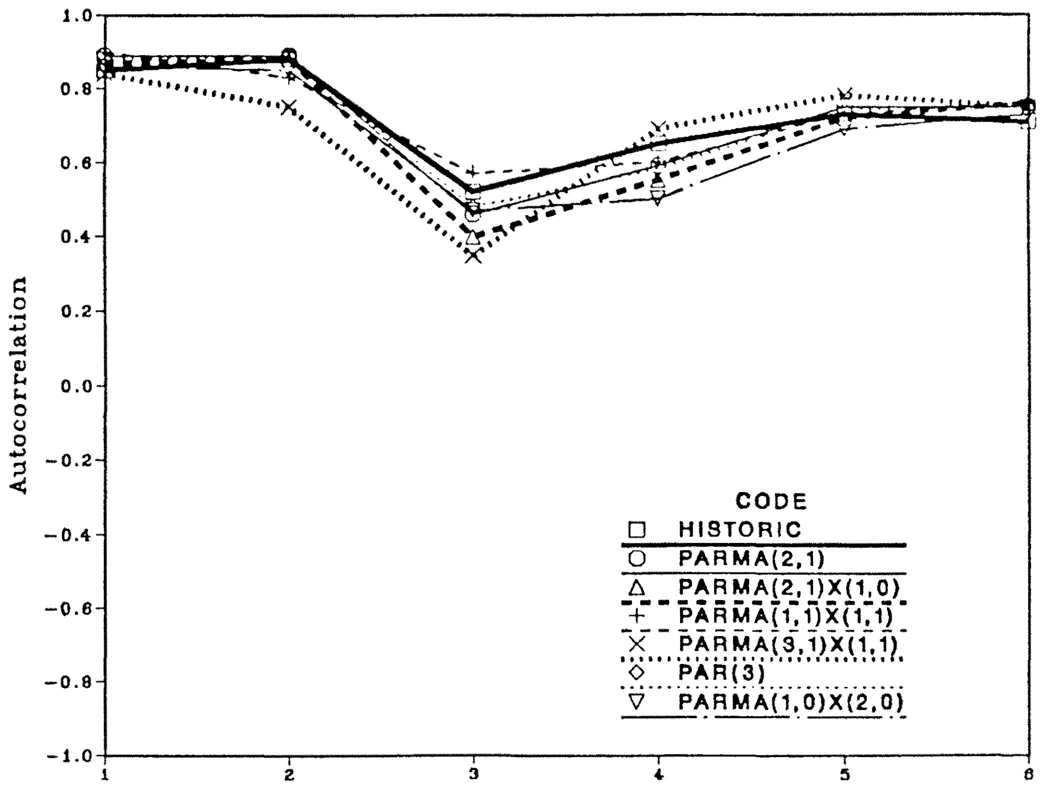

Figure 4.78 - Lag 1 seasonal autocorrelation
Least-squares estimation - Niger River

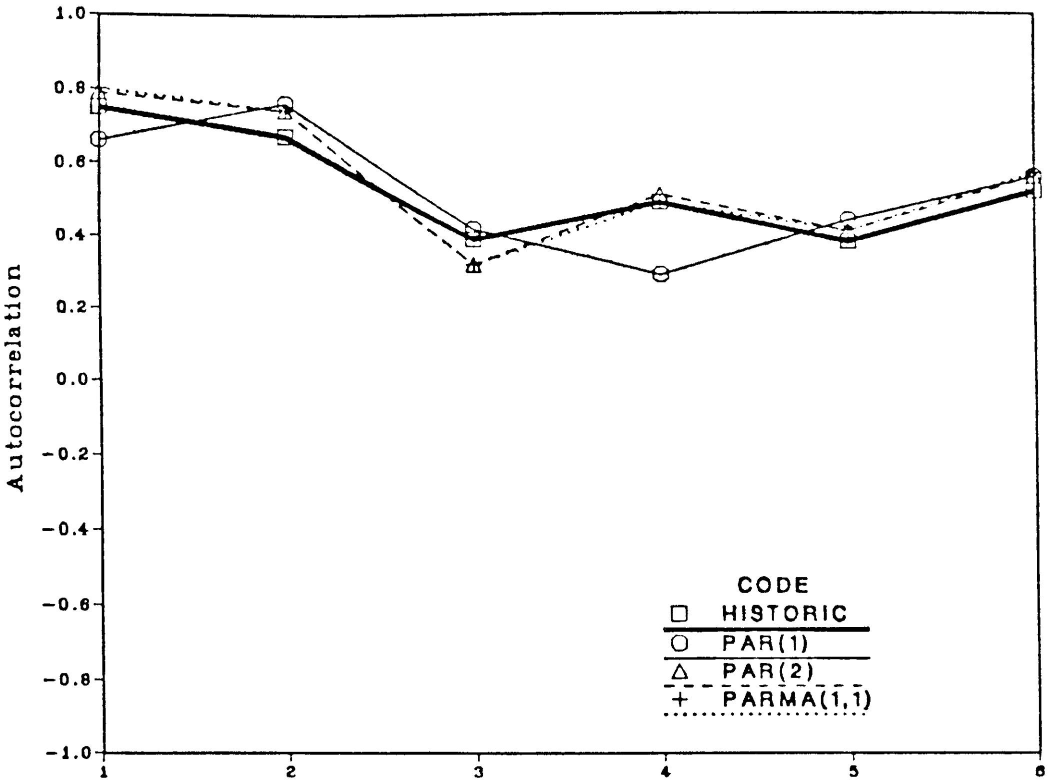

Figure 4.79 - Lag 2 seasonal autocorrelation
Least-squares estimation - Niger River

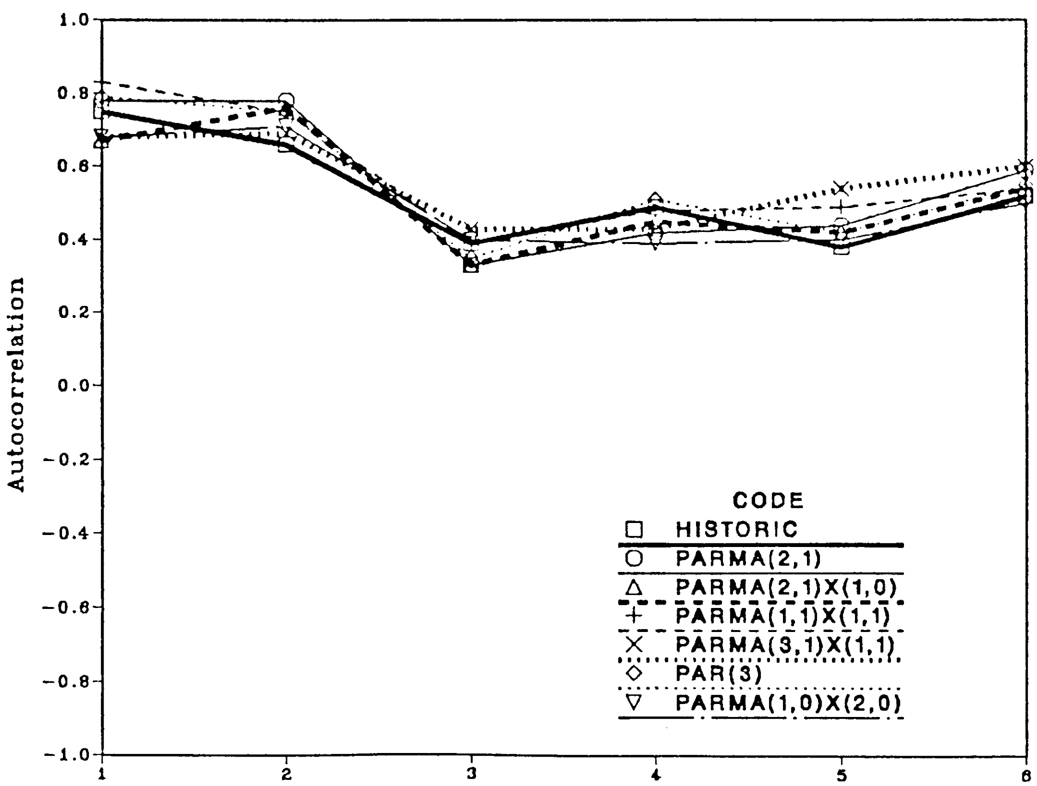

Figure 4.80 - Lag 2 seassonal autocorrelations
Least-squares estimation - Niger River

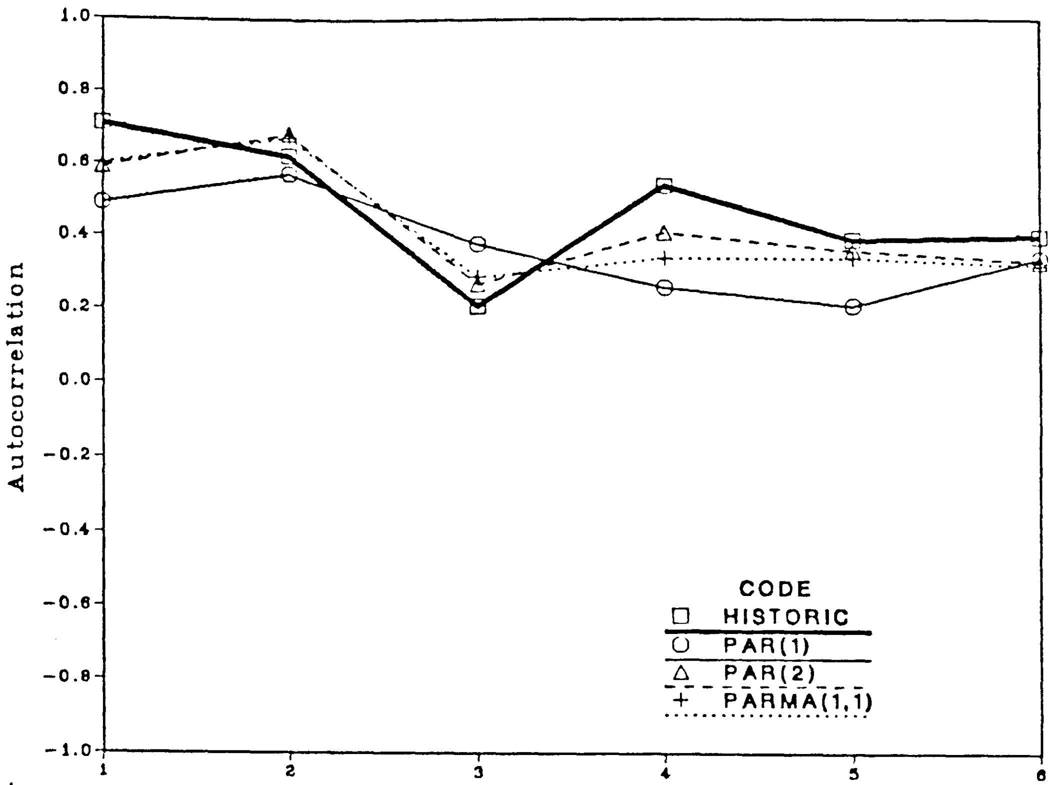

Figure 4.81 - Lag 3 seasonal autocorrelations
Least-squares estimation - Niger River

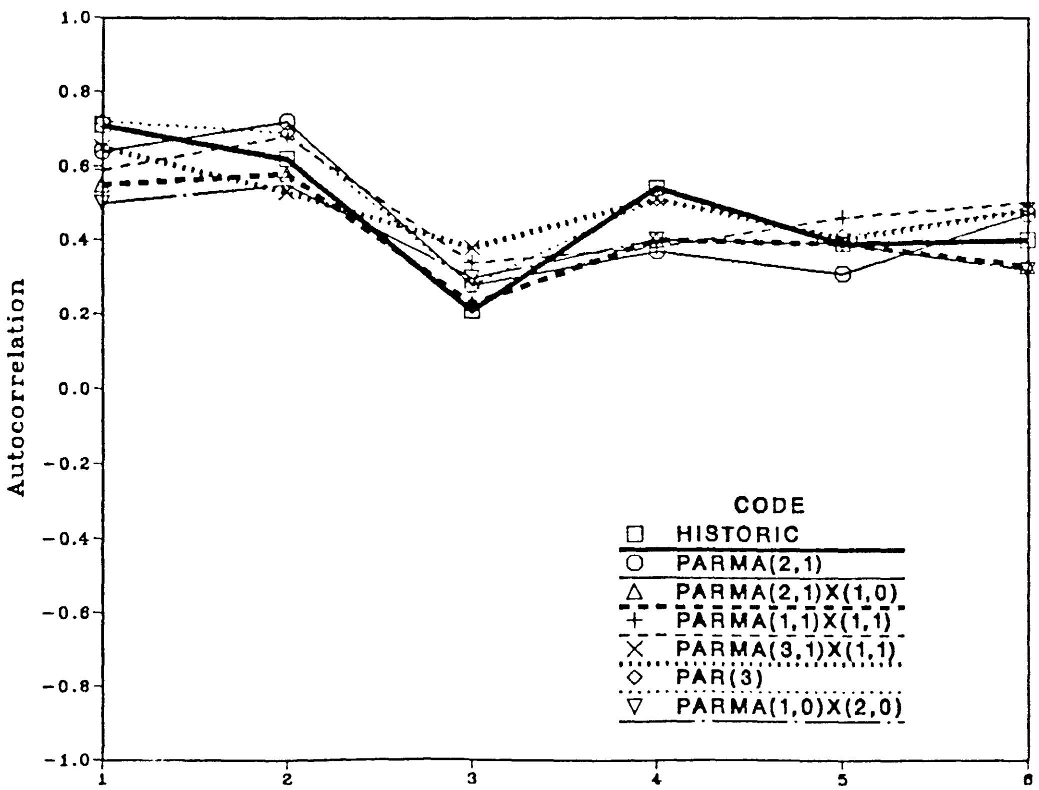

Figure 4.82 - Lag 3 seasonal autocorrelations
Least-squares estimation - Niger River

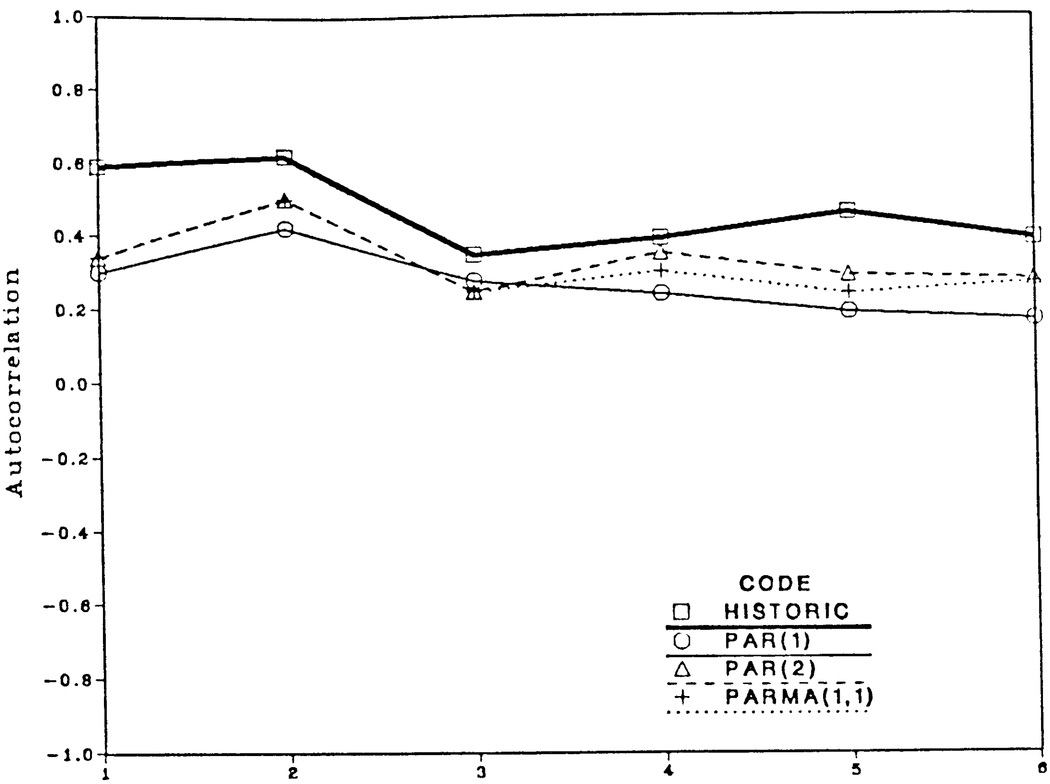

Figure 4.83 - Lag 4 seasonal autocorrelations
Least-squares estimation - Niger River

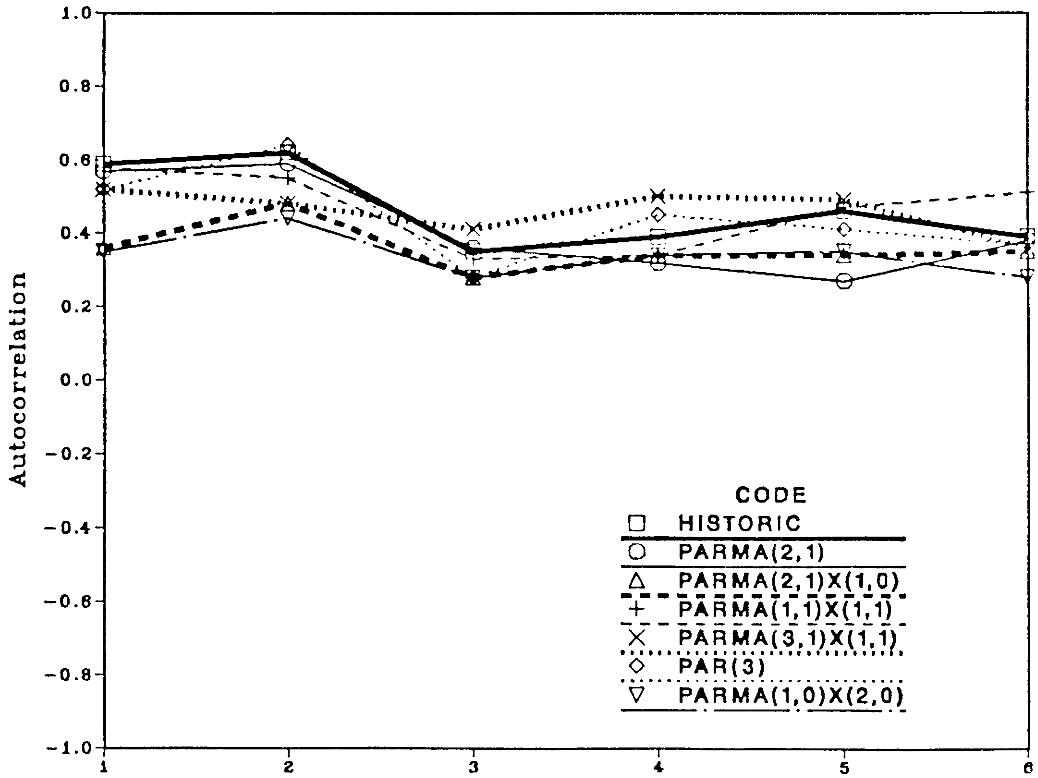

Figure 4.84 - Lag 4 seasonal autocorrelations
Least-squares estimation - Niger River

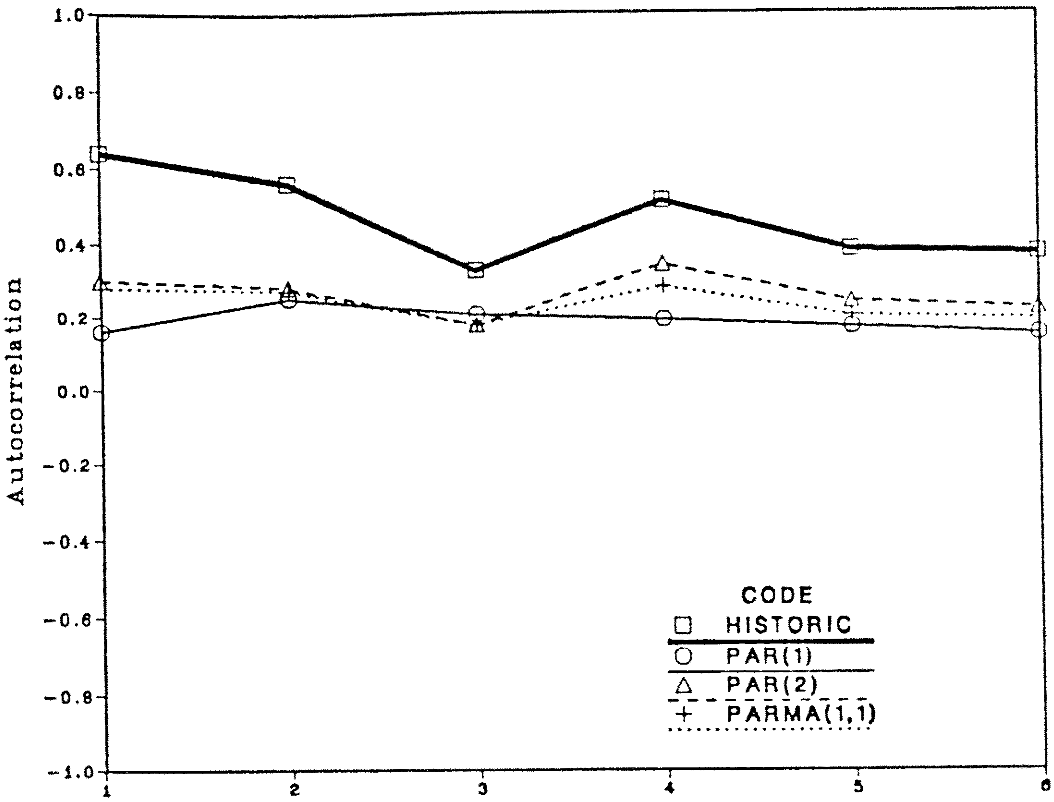

Figure 4.85 - Lag 5 seasonal autocorrelations
Least-squares estimation - Niger River

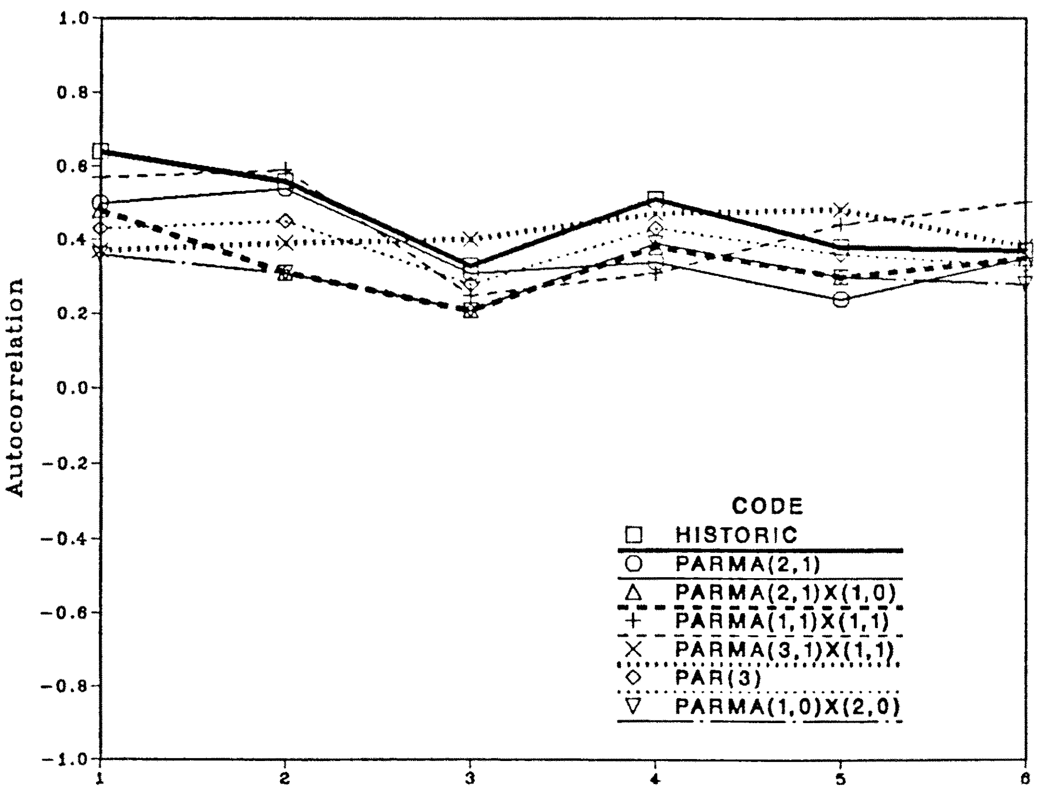

Figure 4.86 - Lag 5 seasonal autocorrelations
Least-squares estimation - Niger River

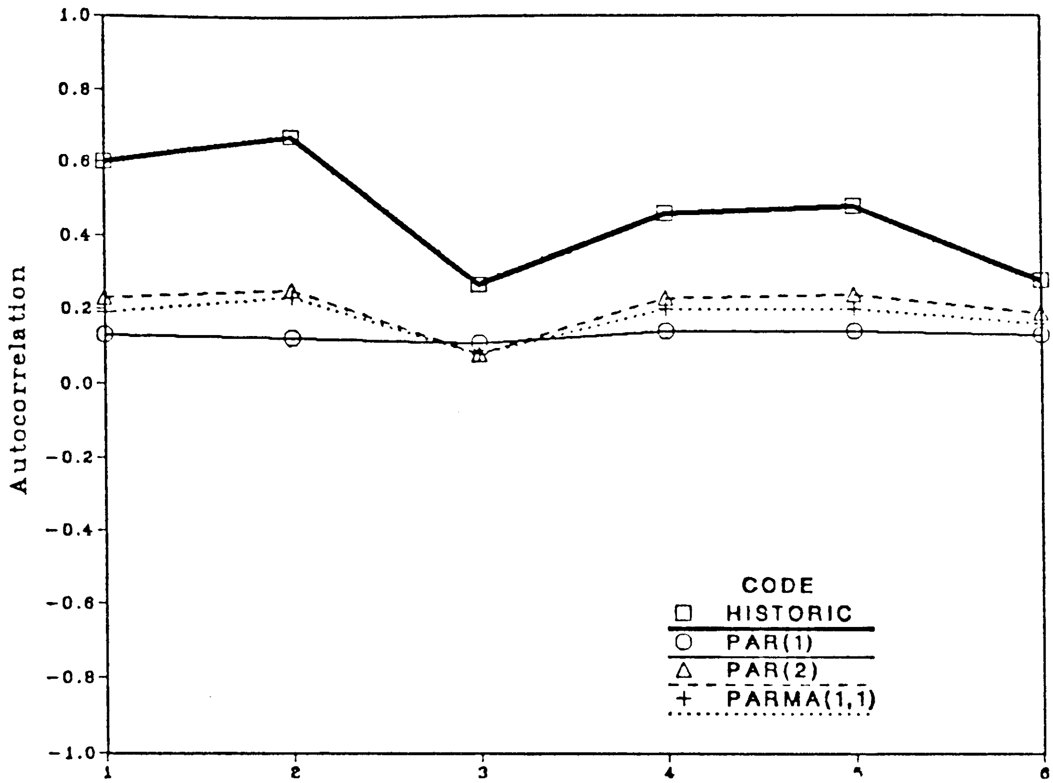

Figure 4.87 - Lag 6 seasonal autocorrelations
Least-squares estimation - Niger River

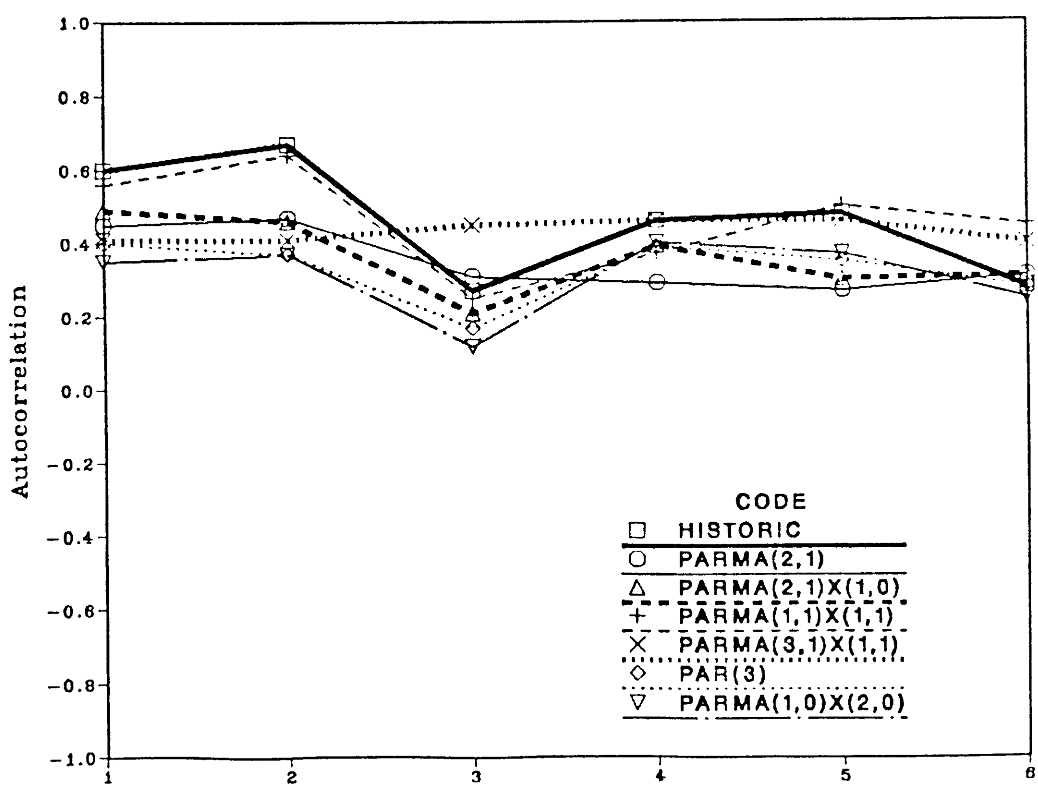

Figure 4.88 - Lag 6 seasonal autocorrelations
Least-squares estimation - Niger River

Seasonal standard deviations of residuals for the Elkhorn River are shown in Table 4.9.

Table 4.9 - Seasonal standard deviations of residuals - Logarithms of four-seasons discharges -Elkhorn

τ	PAR(1)	PAR(2)	PARMA(1,1)	Mult. PARMA(1,1)(1,1)	Mult. PARMA(3,1)(1,1)
1	0.74	0.74	0.72	0.72	0.62
2	0.73	0.69	0.71	0.70	0.72
3	0.89	0.88	0.88	0.86	0.77
4	0.72	0.66	0.67	0.68	0.57

Portmanteau and AIC statistics calculated from the seasonal autocorrelation of residuals are listed in Table 4.10.

Table 4.10- Portmanteau statistics - Elkhorn River Four seasons

Model	Q_1	AIC
PAR(1)	48.2	1625
PAR(2)	37.6	1622
PARMA(1,1)	33.0	1621
Mult. PARMA(1,1)x(1,1)	29.2	1634
Mult. PARMA(3,1)x(1,1)	30.1	1598

b) Preservation of statistics

Seasonal average discharges, seasonal standard deviations and seasonal skewness are shown in Figures A.19, A.20 and A.21, respectively, in Appendix.

Annual autocorrelations are plotted in Figure 4.89 and seasonal autocorrelations are plotted in Figures 4.90 through 4.93.

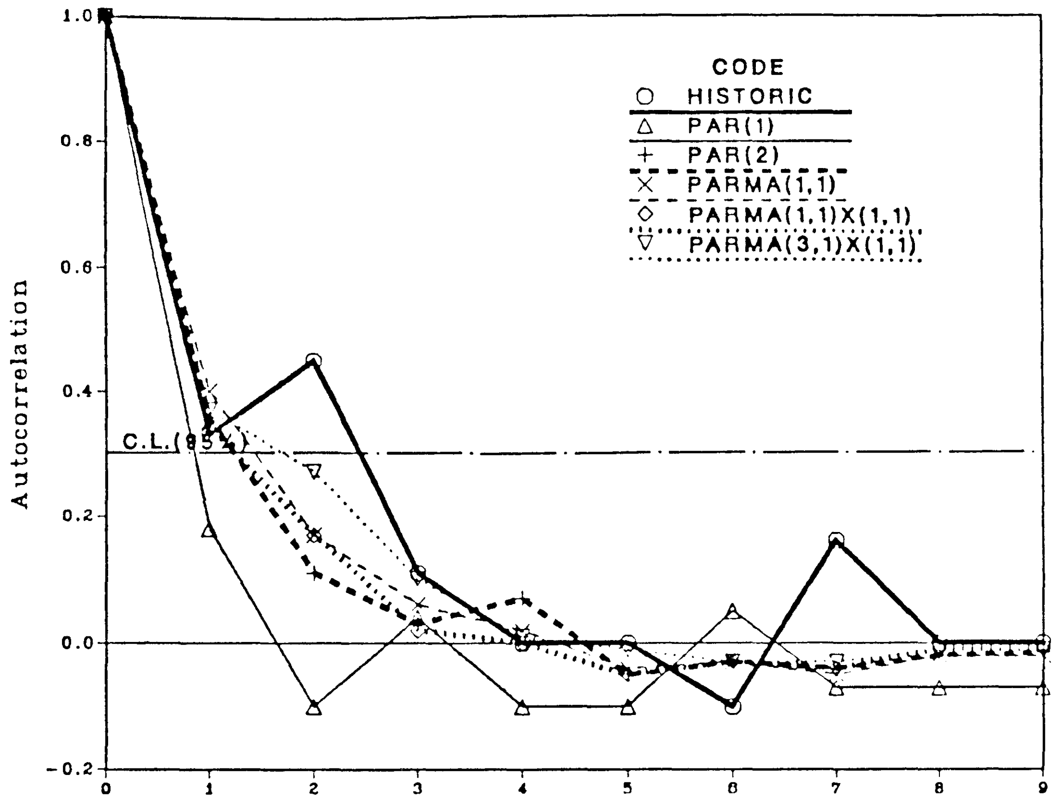

Figure 4.89 Annual autocorrelations - 4 seasons
Least-squares estimation - Elkhorn River

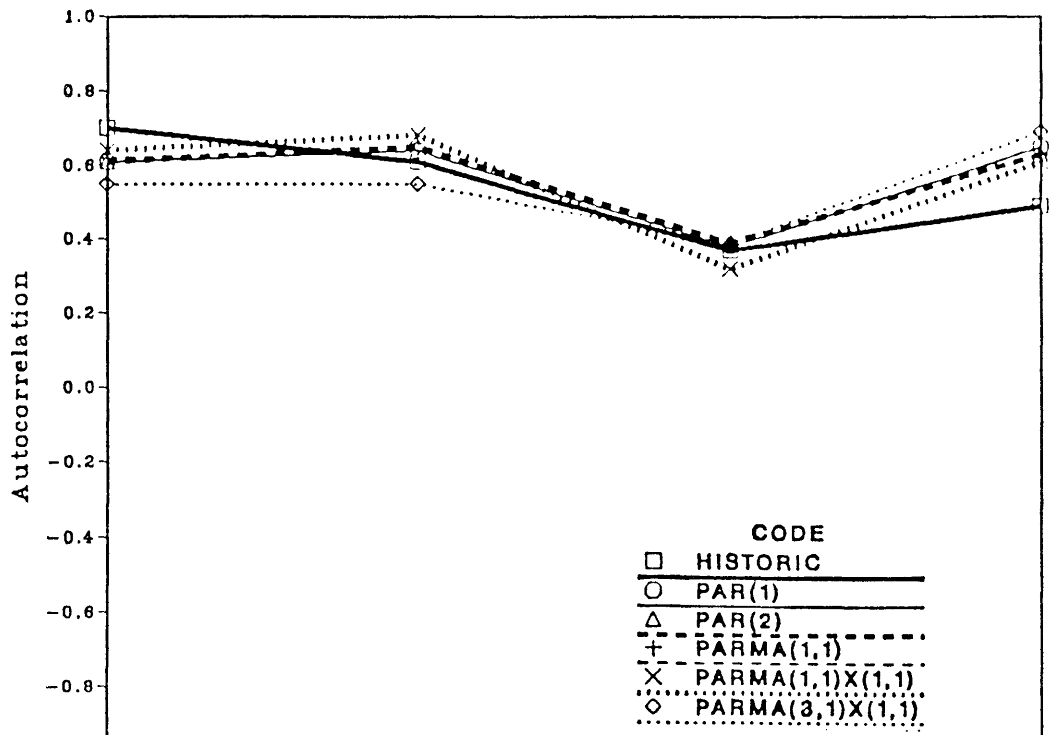

Figure 4.90 Lag 1 seasonal autocorrelations
Least-squares estimation- Elkhorn River

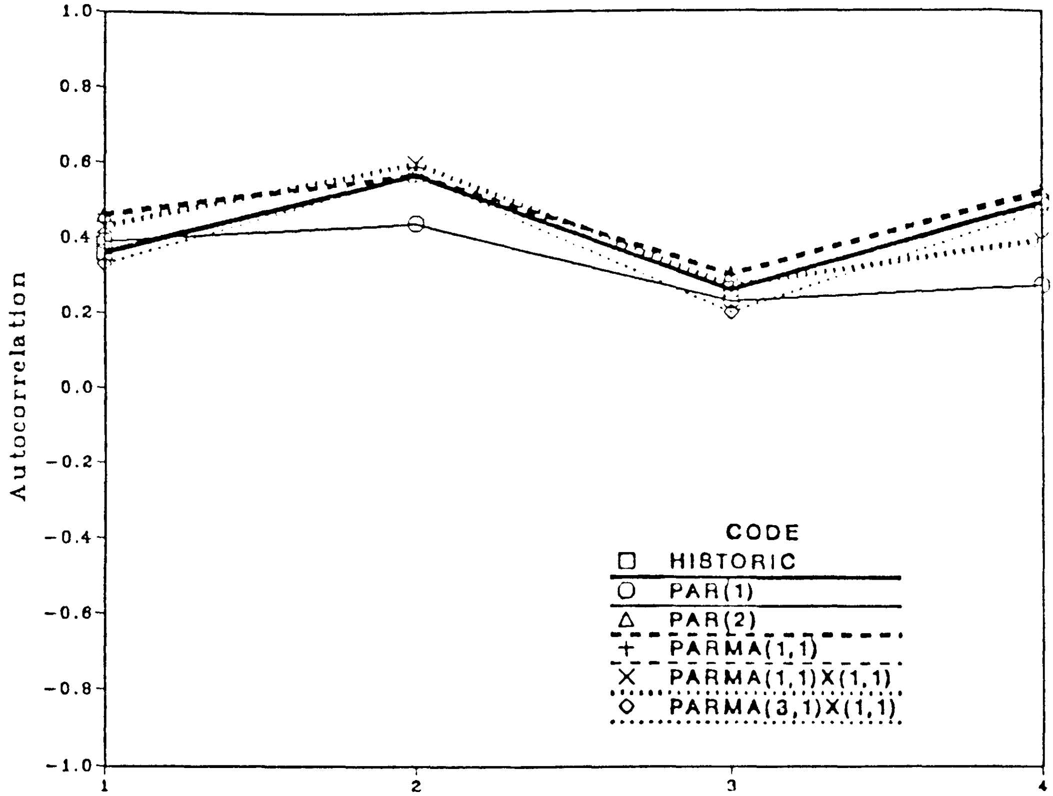

Figure 4.91 Lag 2 seasonal autocorrelations
Least-squares estimation - Elkhorn River

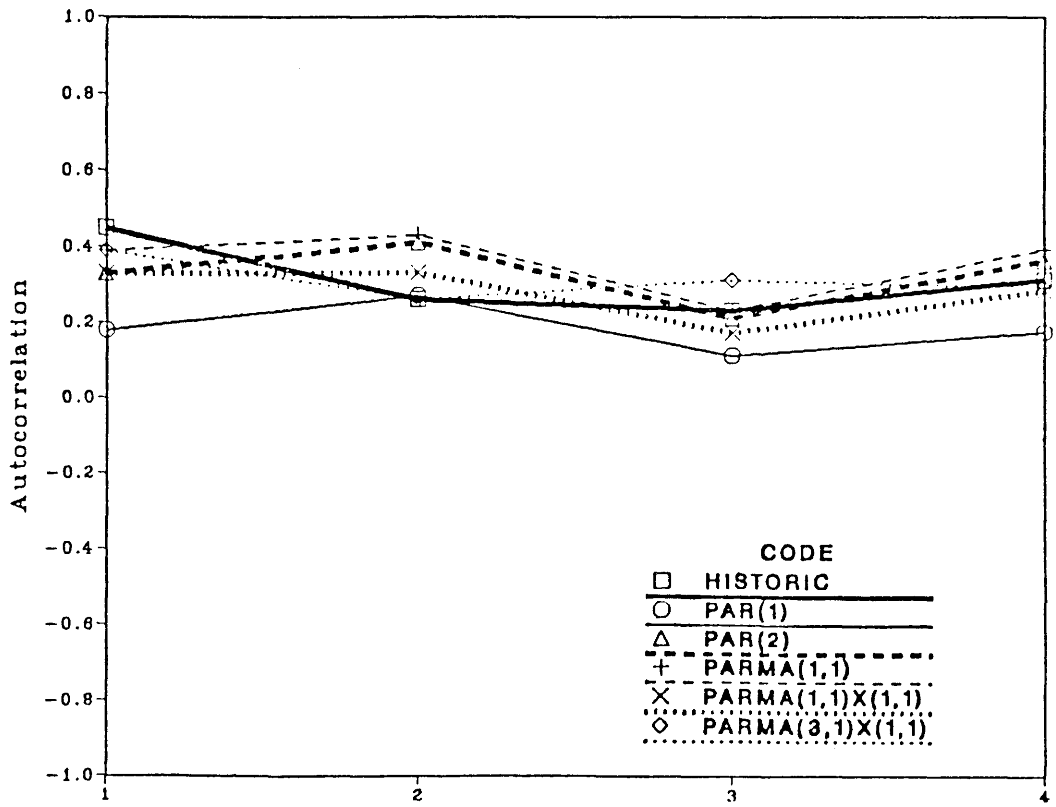

Figure 4.92 - Lag 3 seasonal autocorrelations
Least-squares estimation -Elkhorn River

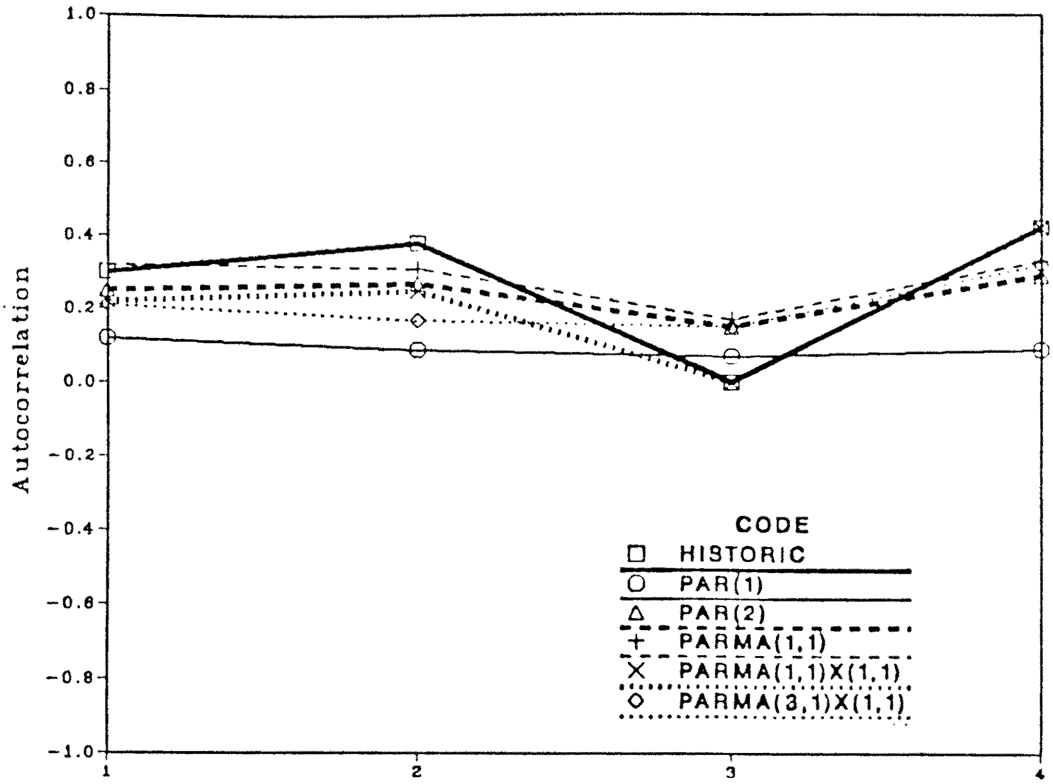

Figure 4.93 Lag 4 seasonal autocorrelations
Least-squares estimation - Elkhorn River

4.1.3.2 Six-season sample

The same models applied to four-year samples were also applied to six-year samples for a the same discharge data for comparison between performances for different number of seasons by the same Multiplicative PARMA models, estimating parameters through the proposed least-squares technique. Logarithms of the series were also utilized here for transformation of the original series.

a) Residuals characteristics -

Seasonal means of residuals for these series were also very close to zero. The seasonal standard deviations of residuals from different models are listed in Table 4.11.

Table 4.11 - Seasonal standard deviations of residuals
Logarithms of six-season discharges

τ	PAR(1)	PAR(2)	PARMA(1,1)	Mult. PARMA(1,1)(1,1)	Mult. PARMA(3,1)(1,1)
1	0.31	0.31	0.30	0.28	0.28
2	0.20	0.20	0.20	0.15	0.19
3	0.45	0.44	0.43	0.38	0.40
4	0.49	0.49	0.50	0.49	0.39
5	0.60	0.60	0.60	0.60	0.51
6	0.54	0.45	0.47	0.42	0.40

The portmanteau and AIC statistics computed from the seasonal autocorrelation of residuals from the various models are listed in Table 4.12.

Table 4.12 - Portmanteau and AIC statistics - Elkhorn River - Six seasons

Model	Q_1	AIC
PAR(1)	61.3	1891
PAR(2)	55.3	1884
PARMA(1,1)	54.6	1882
Mult. PARMA(1,1)x(1,1)	47.2	1871
Mult. PARMA(3,1)x(1,1)	55.5	1846

b) Preservation of statistics -

The data generation and computation of statistics for the different models were made in the same way as those made for four-season samples (forty 250-year samples).

Seasonal means, standard deviations and skewness are shown in Figures A.22, A.23 and A.24, in Appendix.

Figure 4.94 presents the annual autocorrelograms.

The seasonal autocorrelograms are plotted in Figures 4.95 through 4.100.

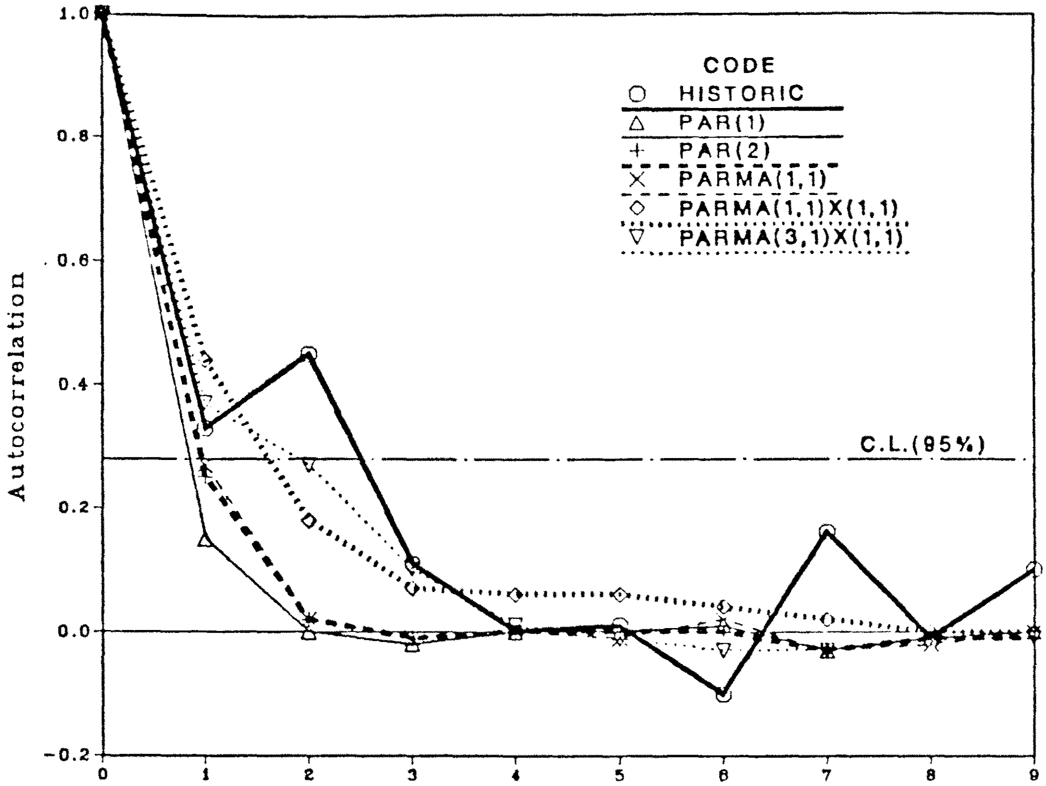

Figure 4.94 - Annual autocorrelations - Six seasons
Least-squares estimation -Elkhorn River

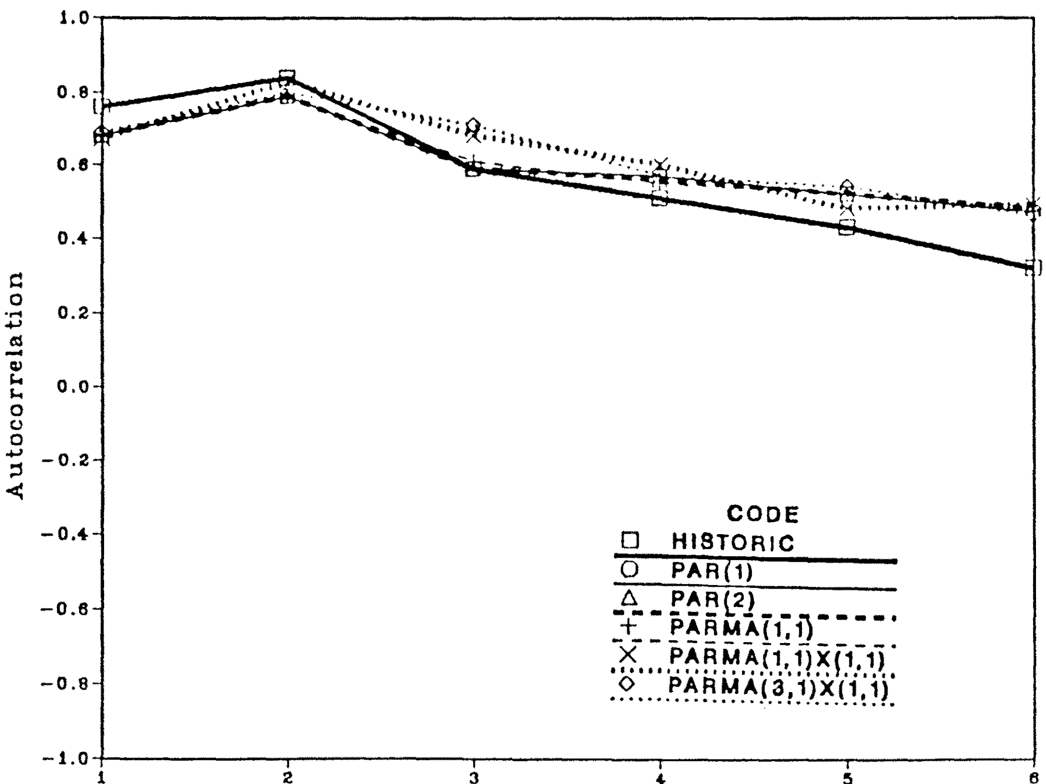

Figure 4.95 - Lag 1 seasonal autocorrelations
Least-squares estimation - Elkhorn River

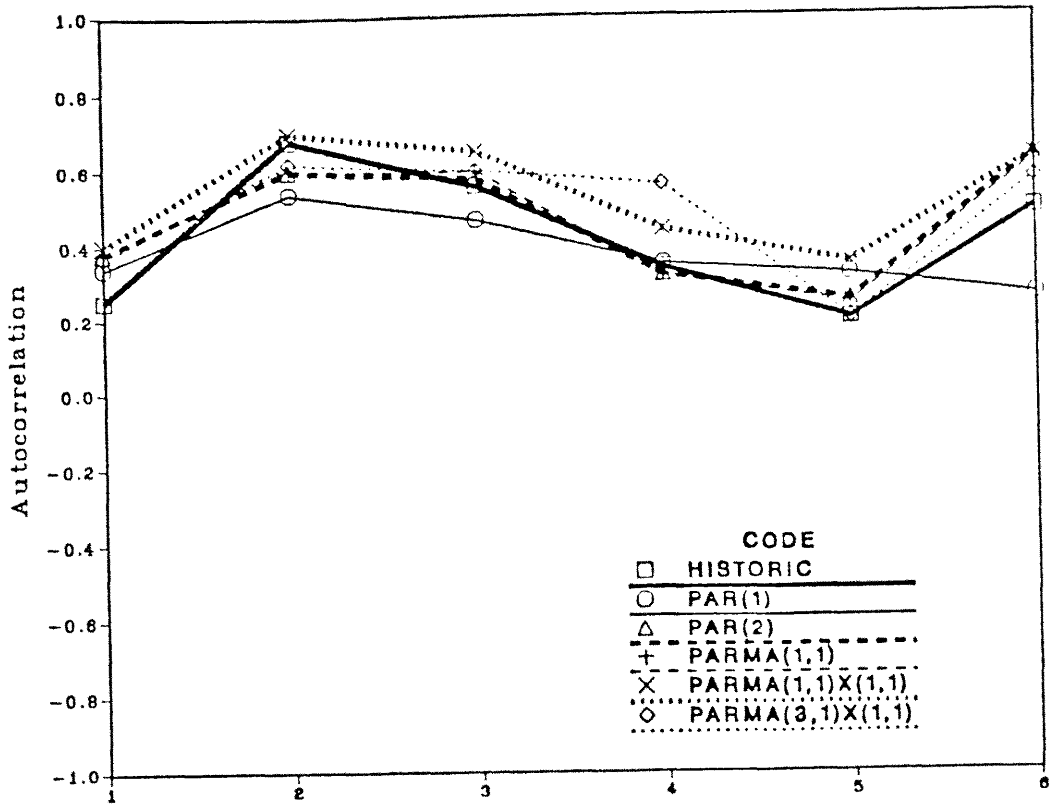

Figure 4.96 - Lag 2 seasonal autocorrelations
Least-squares estimation - Elkhorn River

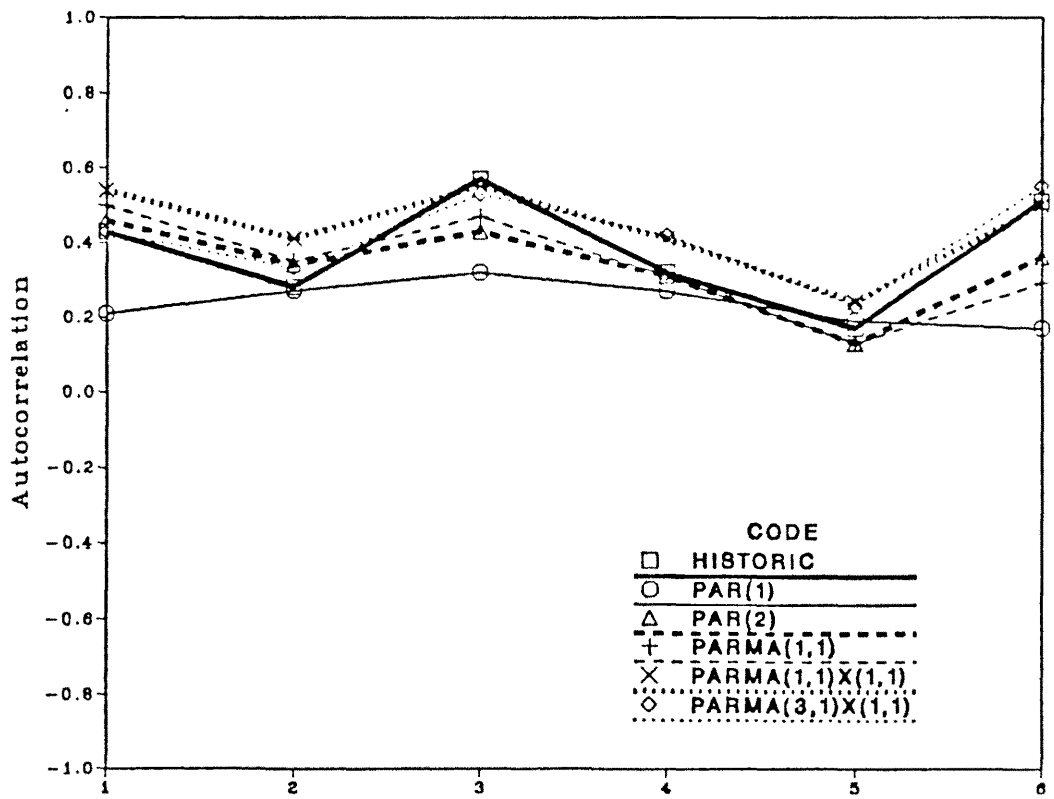

Figure 4.97 - Lag 3 seasonal autocorrelations
Least-squares estimation - Elkhorn River

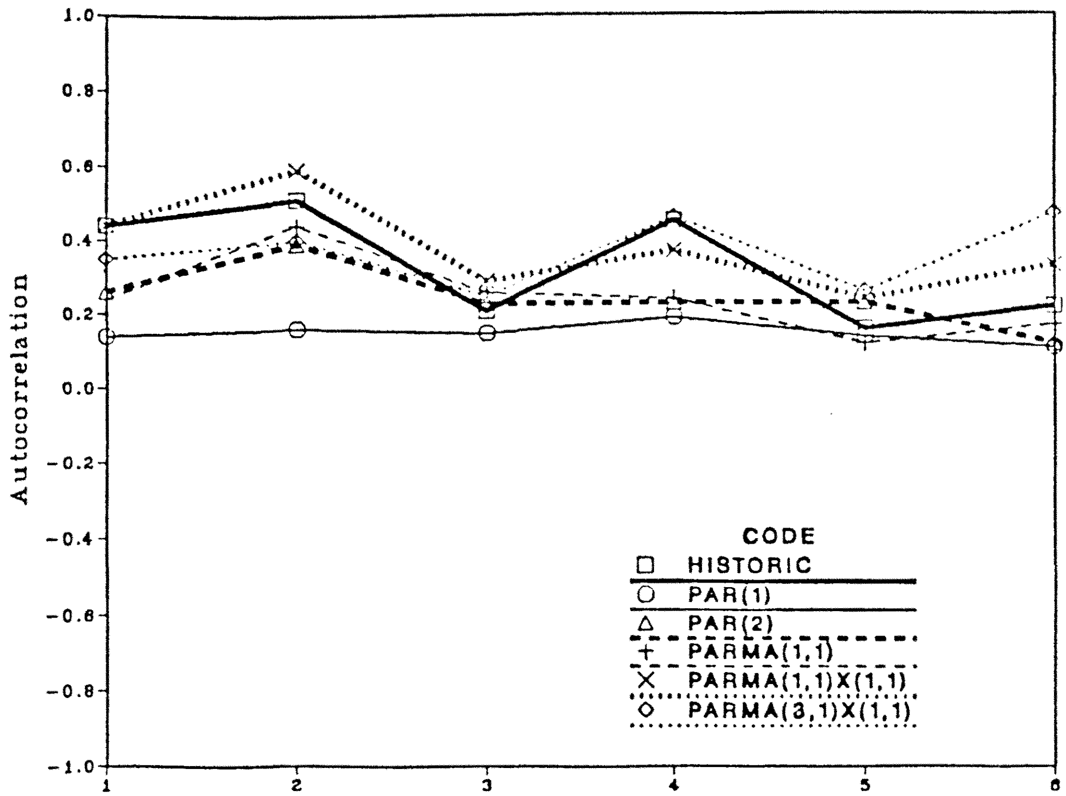

Figure 4.98 - Lag 4 seasonal autocorrelations
Least-squares estimation - Elkhorn River

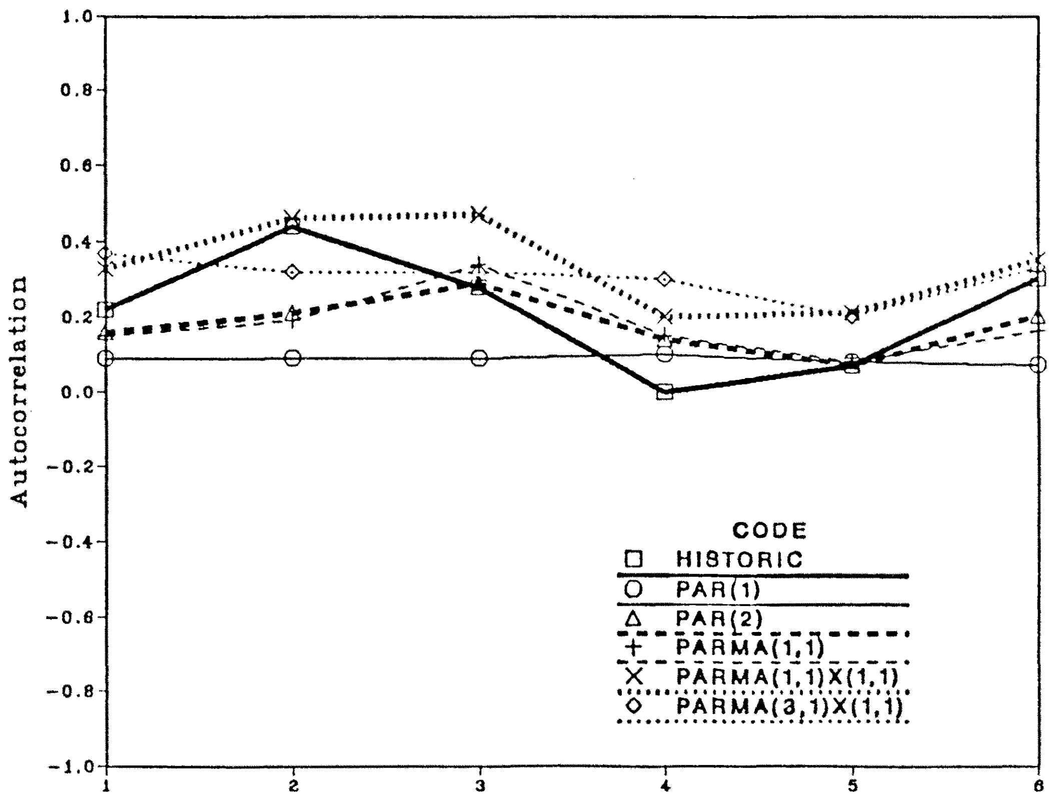

Figure 4.99 - Lag 5 seasonal autocorrelations
Least-squares estimation - Elkhorn River

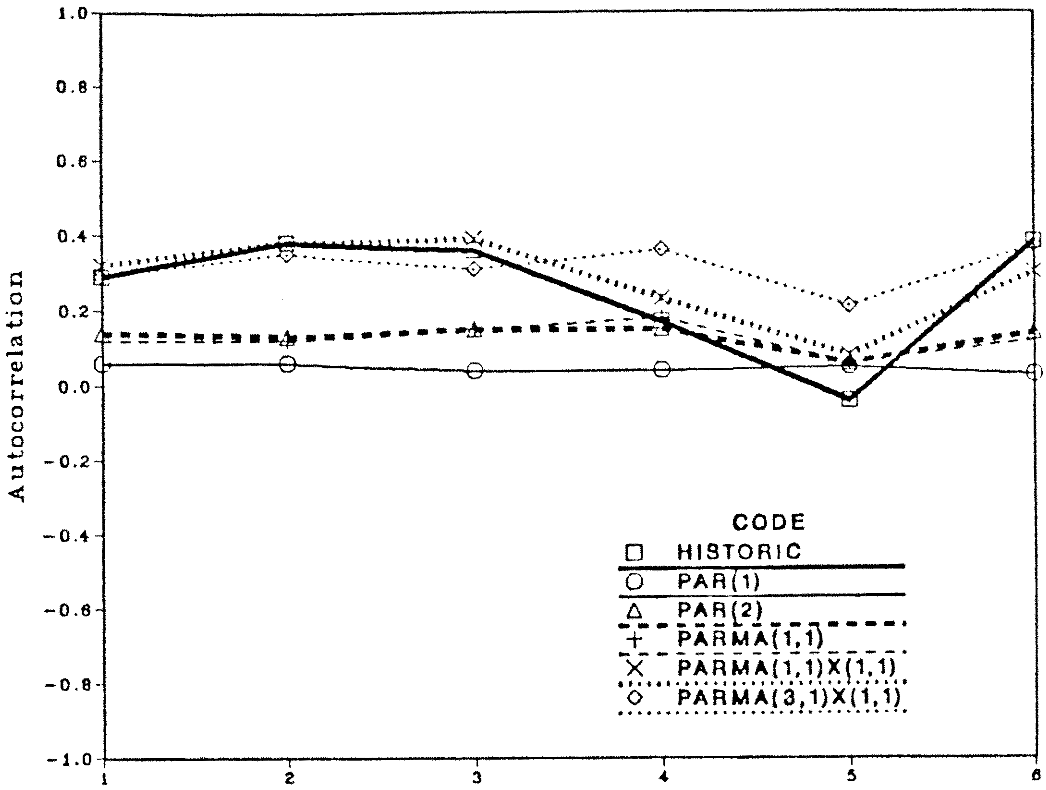

Figure 4.100 - Lag 6 seasonal autocorrelations
Least-squares estimation - Elkhorn River

4.1.4 Yellowstone River

4.1.4.1 Four-season sample

For the Yellowstone River, the same models as those for the Elkhorn River were estimated. Generation and computation of statistics and averages were made in the same way.

a) Residuals characteristics -

Seasonal means of residuals were very close to zero. The seasonal standard deviations of residuals are listed in Table 4.13.

Table 4.13 - Seasonal standard deviations for residuals Logarithm of 4-season discharges -Yellowstone

τ	PAR(1)	PAR(2)	PARMA(1,1)	Mult. PARMA(1,1)(1,1)	Mult. PARMA(3,1)(1,1)
1	0.16	0.16	0.16	0.15	0.12
2	0.13	0.13	0.13	0.12	0.12
3	0.21	0.21	0.21	0.21	0.19
4	0.26	0.26	0.26	0.25	0.25

Portmanteau and AIC statistics calculated from the residuals for the fitted models are presented by Table 4.14.

Table 4.14 - Portmanteau and AIC statistics - Yellowstone River - Four seasons

Model	Q_1	AIC
PAR(1)	48.5	2053
PAR(2)	50.6	2058
PARMA(1,1)	51.4	2058
Mult. PARMA(1,1)x(1,1)	40.4	2053
Mult. PARMA(3,1)x(1,1)	39.8	2020

b) Preservation of statistics

Seasonal average discharges, seasonal standard deviations and seasonal skewnesses are shown in Figures A.25, A.26 and A.28, in Appendix.

The annual autocorrelograms are shown in Figure 4.101 and the seasonal autocorrelations are in Figures 4.102 through 4.105.

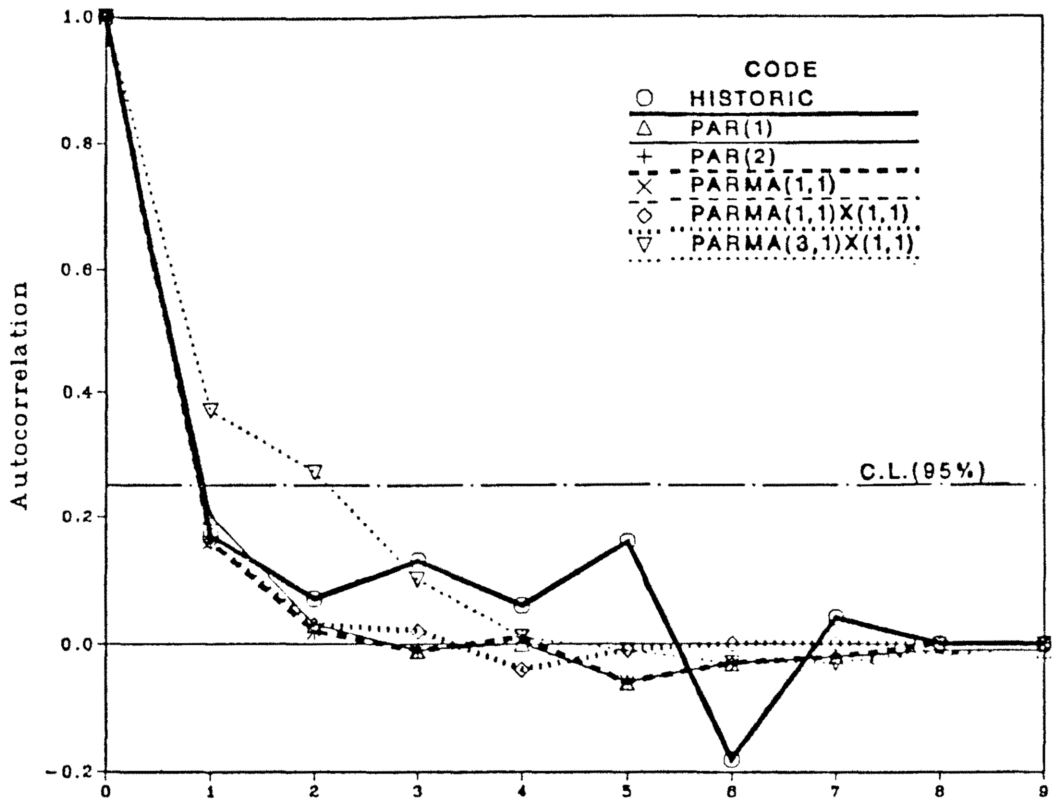

Figure 4.101-Annual autocorrelations - Four seasons
Least-squares estimation - Yellowstone River

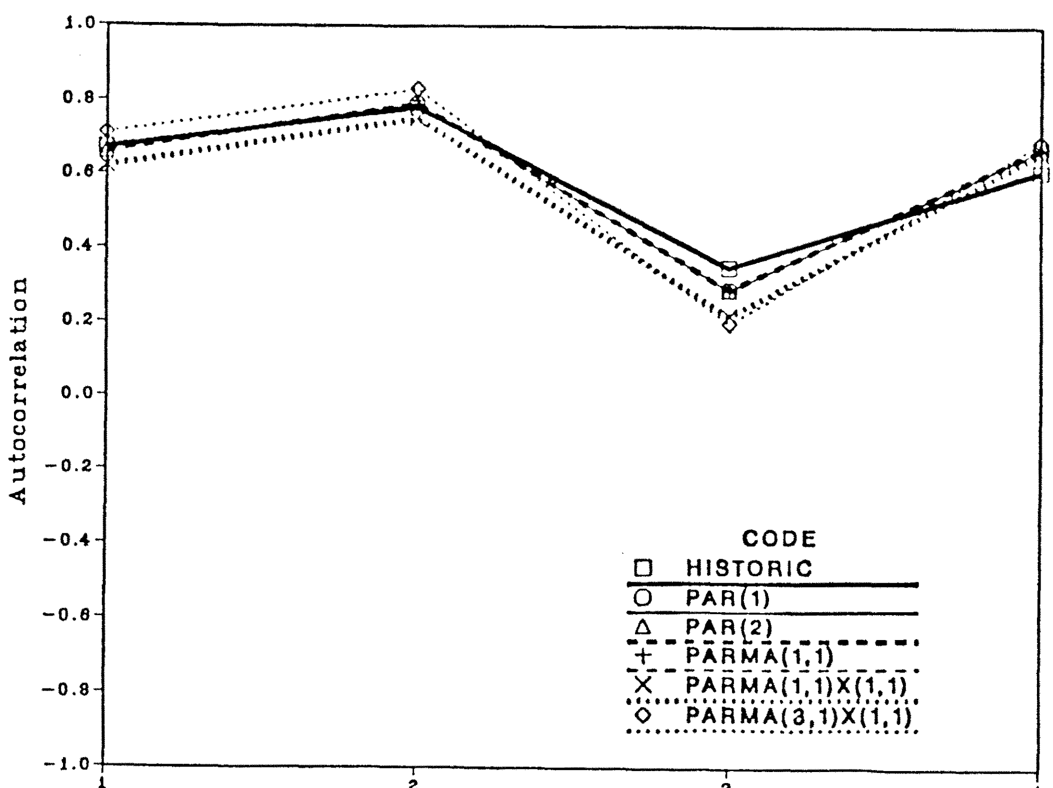

Figure 4.102-Lag 1 seasonal autocorrelations
Least-squares estimation - Yellowstone River

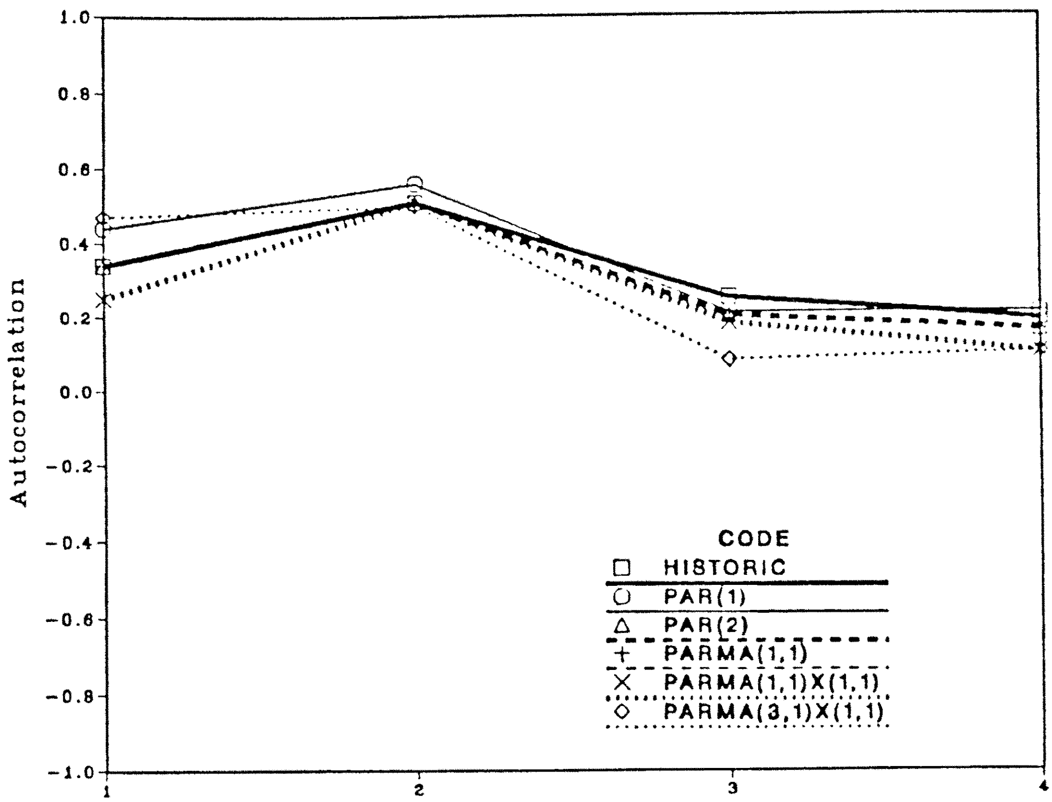

Figure 4.103-Lag 2 seasonal autocorrelations
Least-squares estimation - Yellowstone River

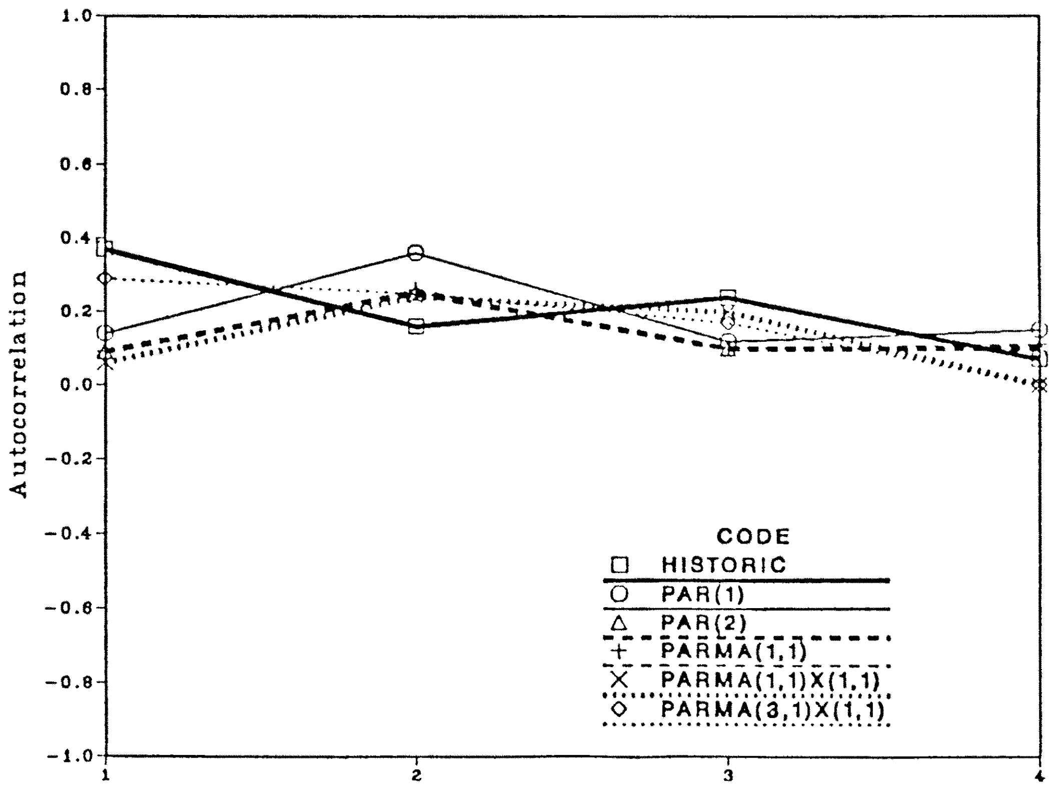

Figure 4.104 - Lag 3 seasonal autocorrelations
Least-squares estimation - Yellostone River

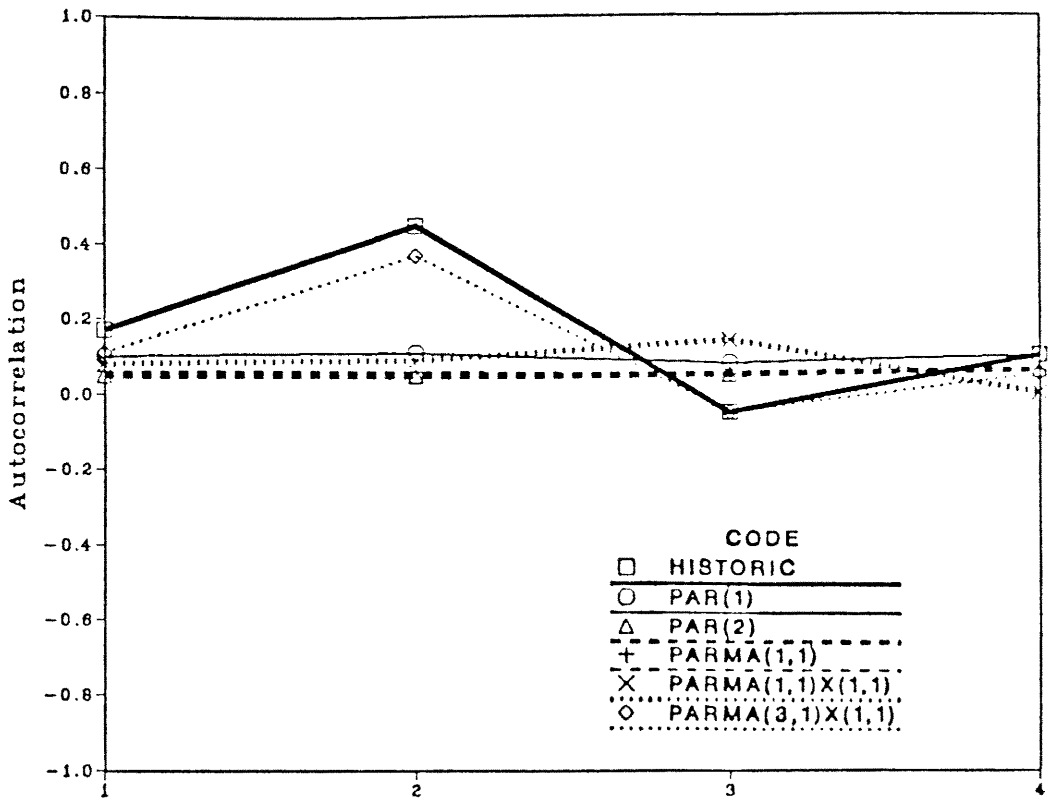

Figure 4.105-Lag 4 seasonal autocorrelations
Least-squares estimation - Yellowstone River

4.1.4.2 - Six-season sample

a) Residuals characteristics

Seasonal means of residuals from all models were approximately equal to zero.

Seasonal standard deviations of residuals for the Yellowstone River from various models are shown in Table 4.15.

Table 4.15 - Seasonal standard deviations of residuals-
Logarithm of six-season discharges
Yellowstone River

τ	PAR(1)	PAR(2)	PARMA(1,1)	Mult. PARMA(1,1)(1,1)	Mult. PARMA(3,1)(1,1)
1	0.15	0.14	0.14	0.14	0.14
2	0.10	0.10	0.10	0.09	0.09
3	0.11	0.10	0.10	0.10	0.10
4	0.28	0.28	0.28	0.28	0.28
5	0.34	0.34	0.34	0.32	0.32
6	0.14	0.14	0.14	0.13	0.13

Portmanteau and AIC statistics calculated from the seasonal autocorrelations of residuals are shown on Table 4.16.

Table 4.16 - Portmanteau and AIC statistics - Yellowstone River - Six seasons

Model	Q_1	AIC
PAR(1)	81.6	2674
PAR(2)	81.5	2676
PARMA(1,1)	80.6	2676
Mult. PARMA(1,1)x(1,1)	67.5	2680
Mult. PARMA(3,1)x(1,1)	69.2	2683

b) Preservation of statistics

Seasonal average discharges, seasonal standard deviations and seasonal skewness are shown, respectively, in Figures A.28, A.29 and A.30, in Appendix.

Figure 4.106 shows the annual autocorrelations and Figures 4.107 through 4.112 show seasonal autocorrelations for lags 1 to 6.

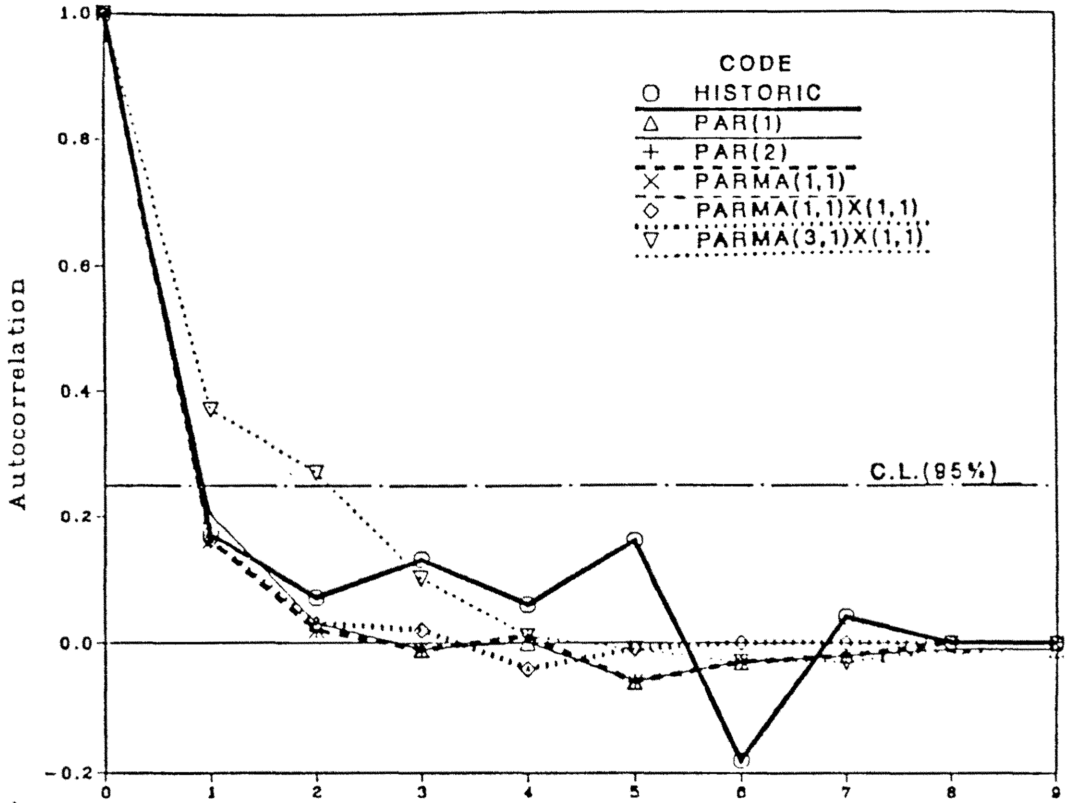

Figure 4.106 Annual autocorrelations - Six seasons
Least-squares estimation - Yellowstone River

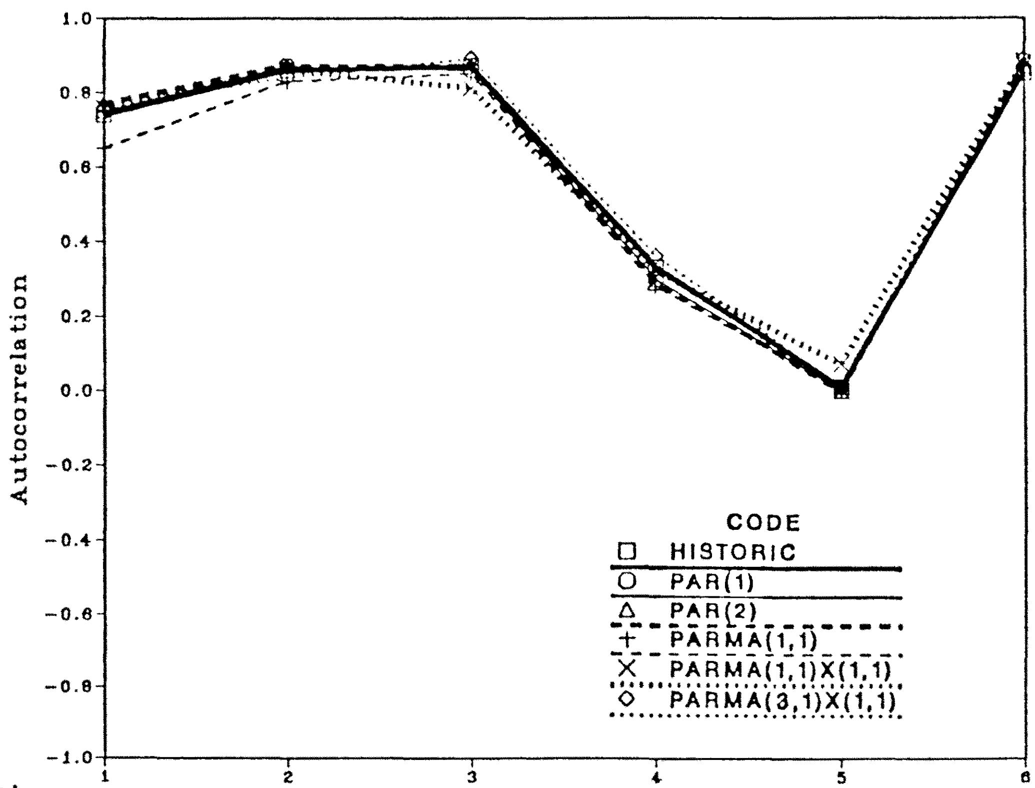

Figure 4.107-Lag 1 seasonal autocorrelations
Least-squares estimation - Yellowstone River

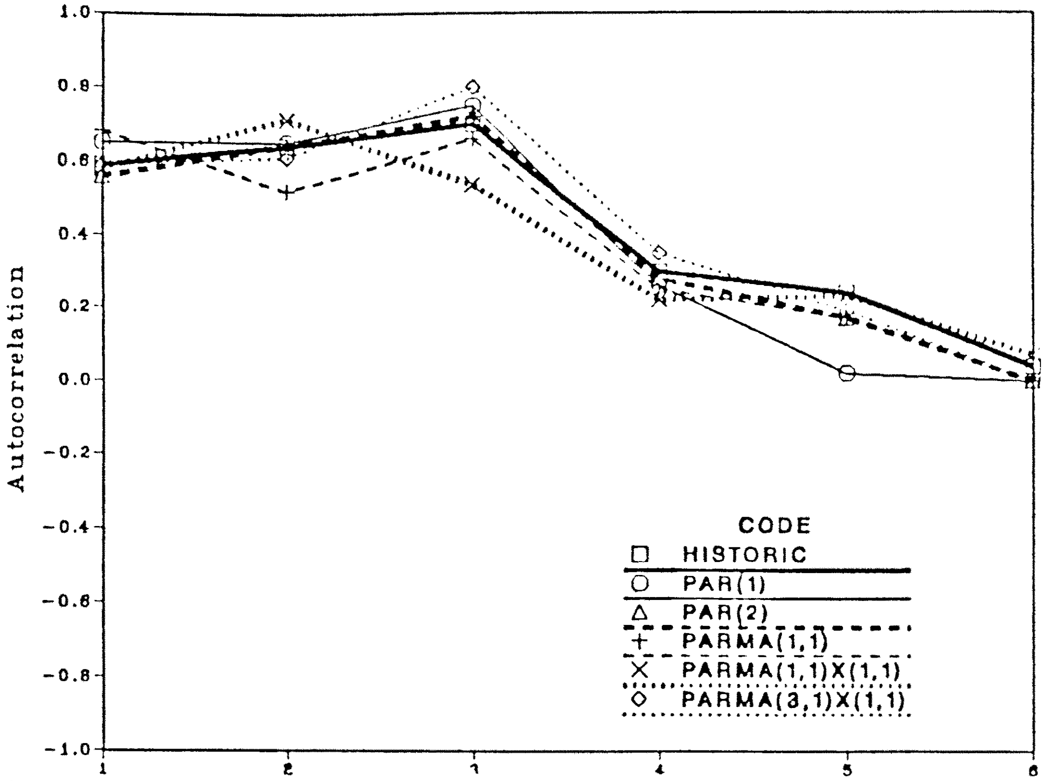

Figure 4.108-Lag 2 seasonal autocorrelations
Least-squares estimation - Yellowstone River

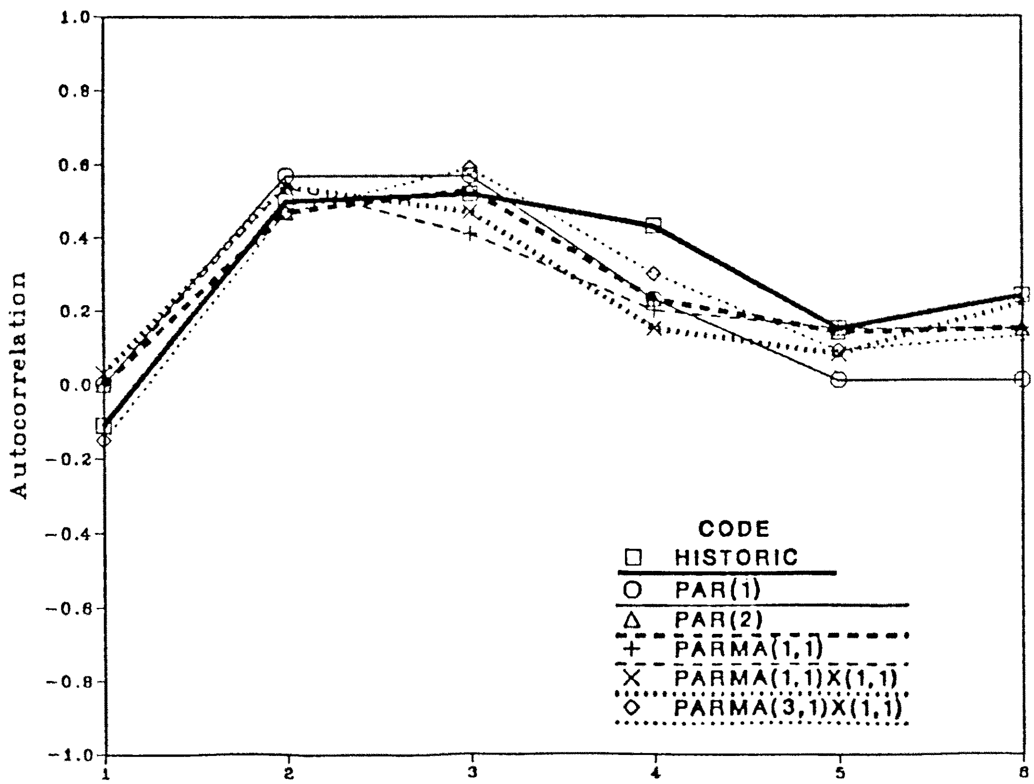

Figure 4.109-Lag 3 seasonal autocorrelations
Least-squares estimation - Yellowstone River

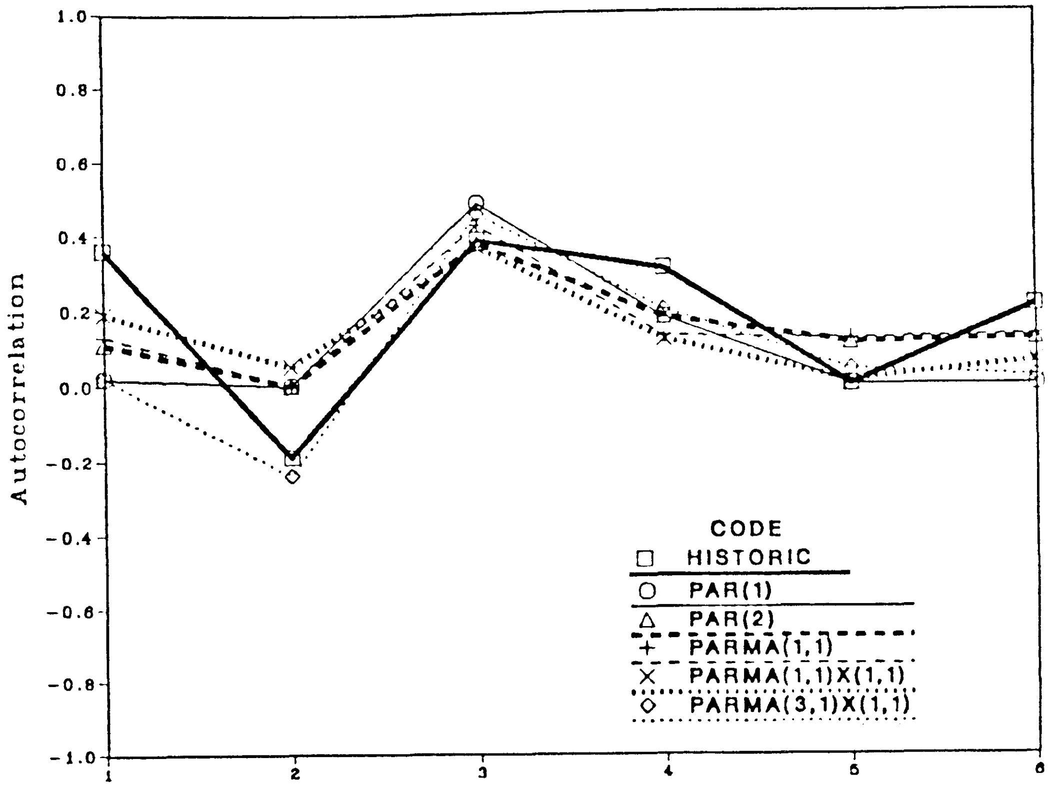

Figure 4.110-Lag 4 seasonal autocorrelations
Least-squares estimation - Yellowstone River

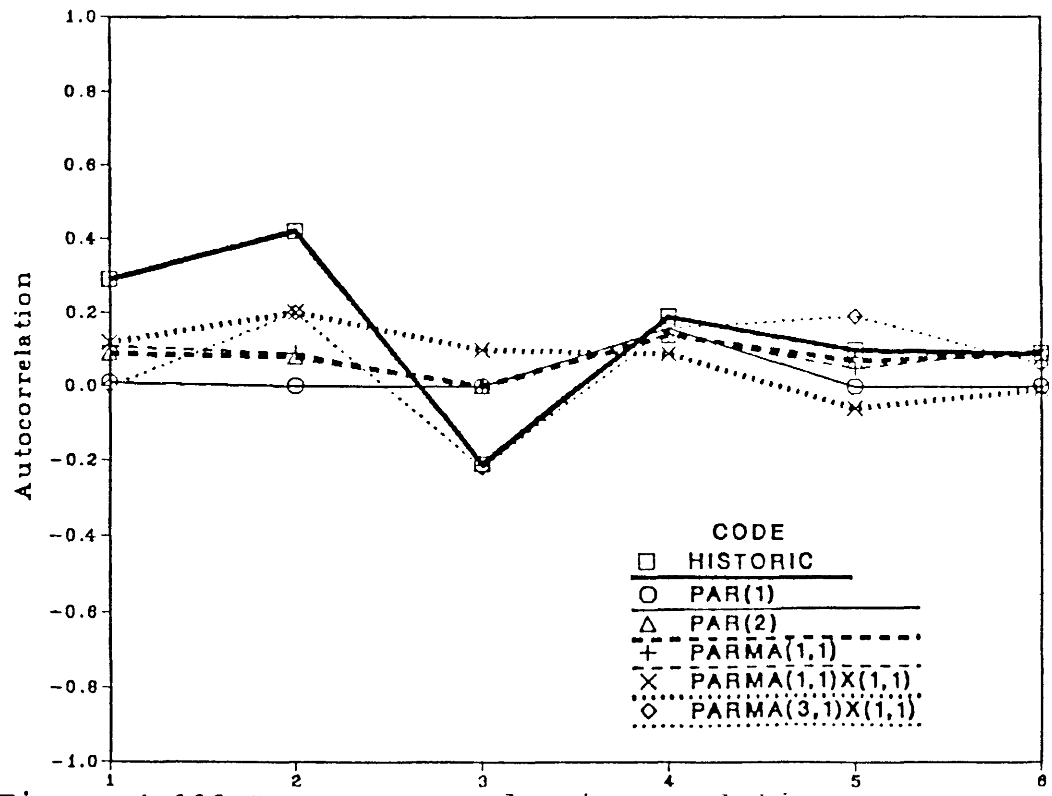

Figure 4.111-Lag 5 seasonal autocorrelations
Least-squares estimation - Yellowstone River

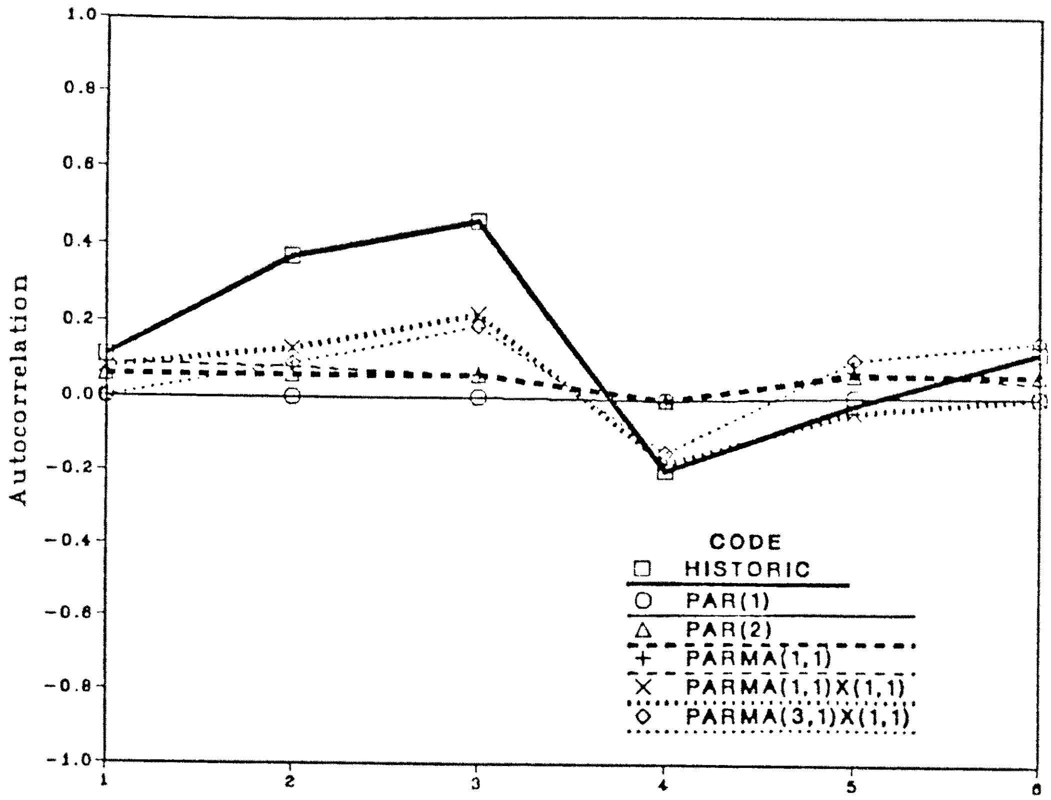

Figure 4.112-Lag 6 seasonal autocorrelations
Least-squares estimation - Yellowstone River

4.2 Estimation Techniques

The methods of moments and of least-squares for estimation of parameters for PAR(1), PAR(2) and PARMA(1,1) were applied (Chapter III) to synthetic 25-year, 50-year and 100-year samples .Bias, standard deviation and Root Mean Square Errors were compared showing that for 50- and 100-year samples results are very close. For 25-year samples, the method of moments presented a higher bias but a lower variability of estimation. Here, the same methods are applied to the logarithm of six-season samples of discharge for the St. Lawrence, Niger, Elkhorn and Yellowstone rivers to compare estimated parameters and the reproduction of annual autocorrelograms.

4.2.1 St. Lawrence River

Table 4.17 presents the parameters for the PAR(1) model estimated from the logarithms of 90 years of bimonthly discharge for the St. Lawrence River. Table 4.18 shows the average annual autocorrelograms from forty 250-year samples (10,000 years), generated utilizing the sets of parameter from Table 4.17.

Table 4.17 - Estimated parameters
St. Lawrence River - PAR(1)

τ	$\phi_{1,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS
1	0.96	0.95	0.02	0.02
2	0.85	0.85	0.05	0.05
3	0.84	0.84	0.08	0.08
4	0.62	0.61	0.06	0.06
5	1.00	1.00	0.03	0.03
6	0.97	0.97	0.02	0.02

Table 4.18 - Annual Autocorrelograms :

LAG	MOM	MOLS
0	1.00	1.00
1	0.60	0.60
2	0.23	0.22
3	0.08	0.08
4	0.04	0.04
5	0.00	0.02
6	-0.01	0.01
7	-0.01	-0.02
8	-0.00	-0.01
9	0.00	0.00

Table 4.19 presents estimated parameters for the PAR(2) model and Table 4.20 shows the corresponding annual autocorrelograms:

Table 4.19 - Estimated Parameters
St. Lawrence River - PAR(2)

τ	$\phi_{1,\tau}$		$\phi_{2,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS	MOM	MOLS
1	1.32	1.32	-0.37	-0.38	0.02	0.02
2	1.29	1.43	-0.48	-0.59	0.05	0.05
3	0.56	0.57	0.33	0.34	0.08	0.08
4	0.36	0.34	0.40	0.40	0.05	0.05
5	1.07	1.06	-0.07	-0.06	0.03	0.03
6	1.22	1.22	-0.29	-0.29	0.02	0.02

Table 4.20 - Annual autocorrelograms :
St. Lawrence River - PAR(2)

LAG	MOM	MOLS
0	1.00	1.00
1	0.64	0.63
2	0.30	0.28
3	0.14	0.13
4	0.07	0.07
5	0.04	0.03
6	0.02	0.01
7	-0.01	-0.01
8	-0.02	-0.02
9	-0.03	-0.03

Parameters for the PARMA(1,1) model, estimated through the two techniques are shown in Table 4.21. Table 4.22 shows the average annual autocorrelations from generated samples.

Table 4.21 - Estimated Parameters
St. Lawrence River - PARMA(1,1)

τ	$\phi_{1,\tau}$		$\theta_{1,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS	MOM	MOLS
1	0.94	0.95	-0.46	-0.08	0.02	0.02
2	0.81	0.79	-0.54	-0.75	0.05	0.05
3	0.95	0.98	0.42	0.53	0.08	0.08
4	0.84	0.85	0.50	0.54	0.05	0.05
5	0.96	0.97	-0.14	-0.11	0.03	0.03
6	0.94	0.93	-0.30	-0.30	0.02	0.02

Table 4.22 -Annual Autocorrelogramm :
St. Lawrence River - PARMA(1,1)

LAG	MOM	MOLS
0	1.00	1.00
1	0.67	0.69
2	0.34	0.37
3	0.18	0.20
4	0.09	0.11
5	0.05	0.06
6	0.02	0.03
7	-0.01	0.00
8	-0.02	-0.01
9	-0.02	-0.02

Figure 4.113 shows the average autocorrelograms for all the models, from samples generated by using the method-of-moments estimates.

Figure 4.114 shows average autocorrelograms obtained from estimates by the least-squares method.

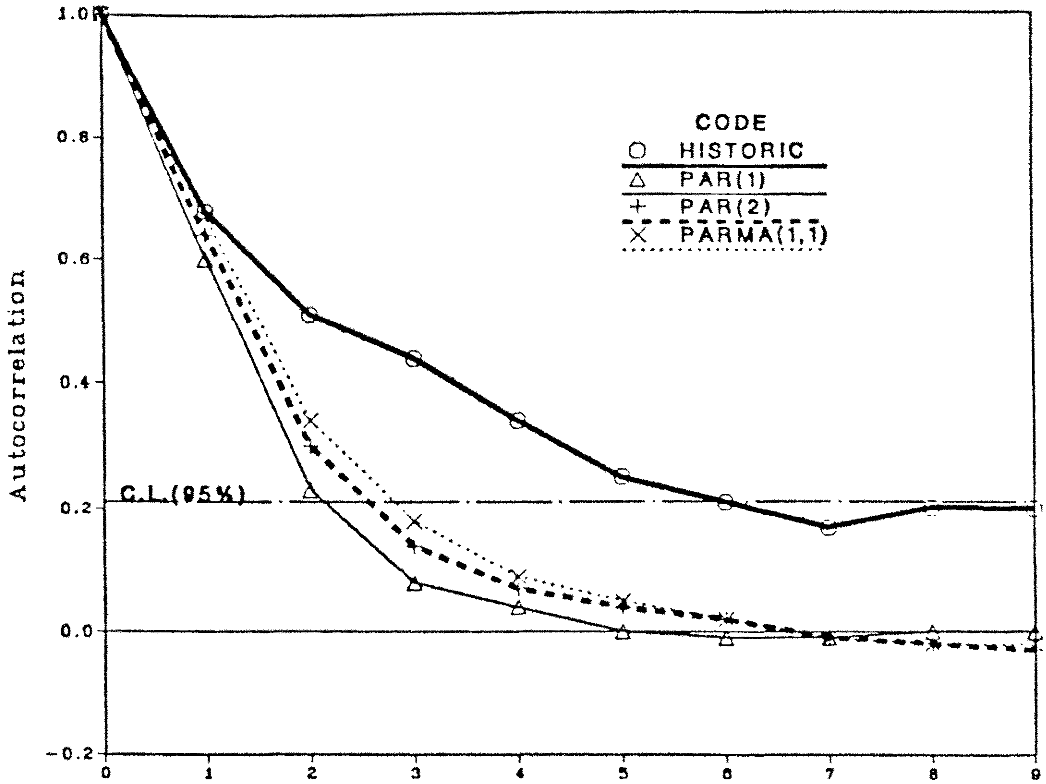

Figure 4.113-Annual autocorrelograms - St. Lawrence River
Method of Moments estimation - Six seasons

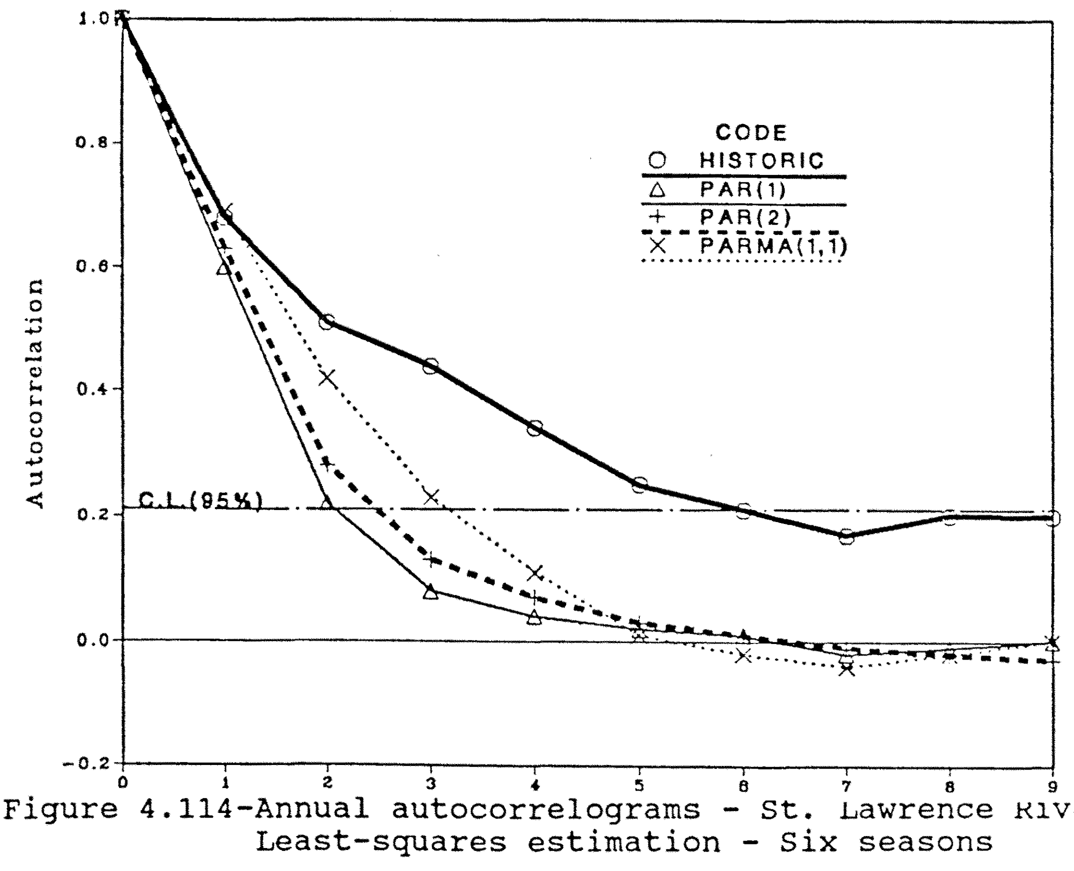

Figure 4.114-Annual autocorrelograms - St. Lawrence River
Least-squares estimation - Six seasons

4.2.2 Niger River

Table 4.23 shows parameters for PAR(1) model estimation by the two methods, and Table 4.24 presents the corresponding average generated annual autocorrelations.

Table 4.23 - Estimated parameters
Niger River - PAR(1)

τ	$\phi_{1,\tau}$		$\sigma_{1,\tau}$	
	MOM	MOLS	MOM	MOLS
1	1.04	1.10	0.18	0.18
2	1.15	1.15	0.23	0.23
3	0.66	0.65	0.54	0.52
4	0.33	0.32	0.25	0.25
5	0.47	0.47	0.14	0.14
6	1.24	1.24	0.21	0.21

Table 4.24 -Annual Autocorrelograms :
Niger River - PAR(1)

LAG	MOM	MOLS
0	1.00	1.00
1	0.24	0.25
2	0.01	0.01
3	-0.01	-0.01
4	0.01	0.00
5	0.00	0.00
6	0.00	0.00
7	-0.03	-0.03
8	-0.01	-0.01
9	0.00	0.00

Table 4.25 shows the estimated PAR(2) parameters while Table 4.26 shows the corresponding average generated annual autocorrelograms.

Table 4.25 - Estimated parameters
Niger River - PAR(2)

τ	$\phi_{1,\tau}$		$\phi_{2,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS	MOM	MOLS
1	0.77	0.81	0.56	0.61	0.16	0.16
2	1.30	1.33	-0.21	-0.26	0.23	0.23
3	1.21	1.12	-0.78	-0.68	0.52	0.51
4	0.24	0.24	0.20	0.20	0.24	0.24
5	0.49	0.49	-0.02	-0.02	0.14	0.14
6	1.24	1.23	0.00	0.01	0.21	0.21

Table 4.26 - Annual autocorrelograms
Niger River - PAR(2)

LAG	MOM	MOLS
1	0.35	0.37
2	0.06	0.07
3	0.00	0.01
4	0.01	0.01
5	0.00	0.00
6	0.00	0.00
7	-0.02	-0.02
8	-0.02	-0.02
9	-0.02	-0.02

Table 4.27 presents the parameters for the PARMA(1,1) estimated by the method of moments and by the least-squares method.

Table 4.28 shows the corresponding average generated annual autocorrelograms.

Table 4.27 - Estimated parameters
Niger River - PARMA(1,1)

τ	$\phi_{1,\tau}$		$\theta_{1,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS	MOM	MOLS
1	1.23	1.38	0.46	0.70	0.16	0.14
2	1.10	1.11	-0.24	-0.27	0.23	0.23
3	0.52	0.57	-0.70	-0.51	0.52	0.51
4	0.54	0.46	0.32	0.23	0.25	0.25
5	0.44	0.58	-0.06	0.26	0.14	0.15
6	1.24	1.52	0.01	0.63	0.21	0.21

Table 4.28 -Annual Autocorrelograms

LAG	MOM	MOLS
0	1.00	1.00
1	0.30	0.38
2	0.03	0.11
3	0.00	0.03
4	0.00	0.02
5	0.00	0.01
6	0.00	0.01
7	-0.02	-0.03
8	-0.01	-0.02
9	-0.02	-0.01

Figure 4.115 presents the average generated annual autocorrelograms for all the models. These autocorrelograms were computed from samples generated utilizing the moments estimates of parameters.

Figure 4.116 shows the same autocorrelograms, but from least-squares estimates of parameters

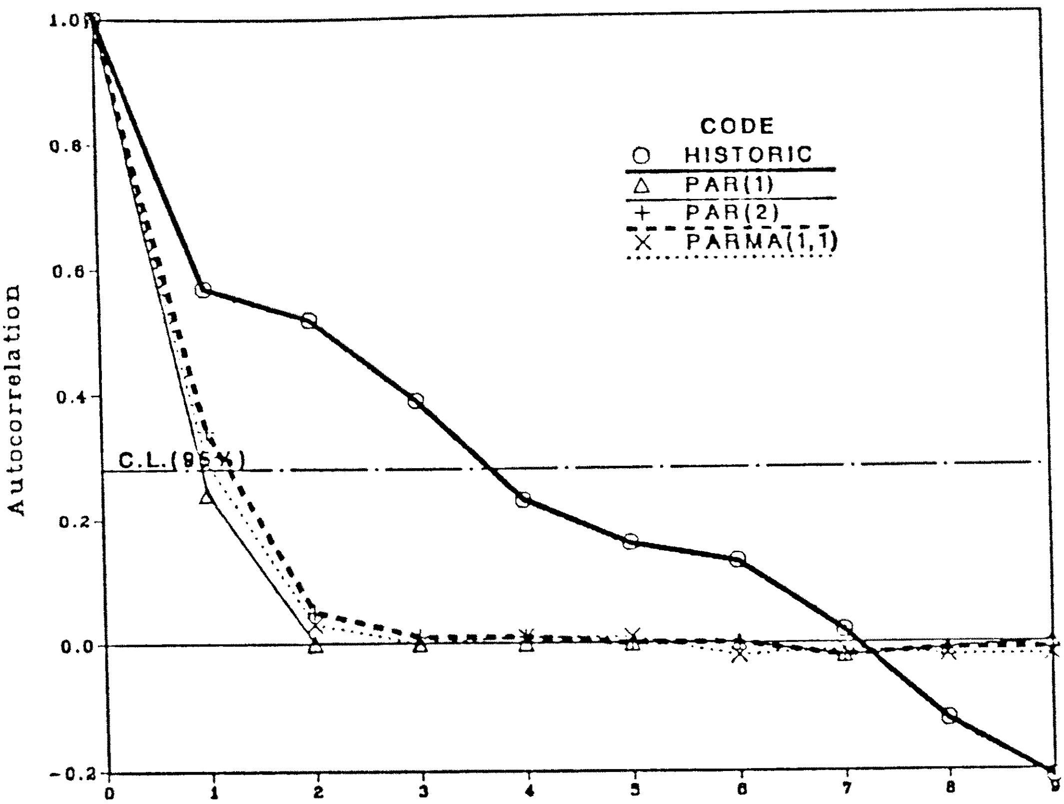

Figure 4.115 - Annual autocorrelograms - Niger River
Method of moments estimation

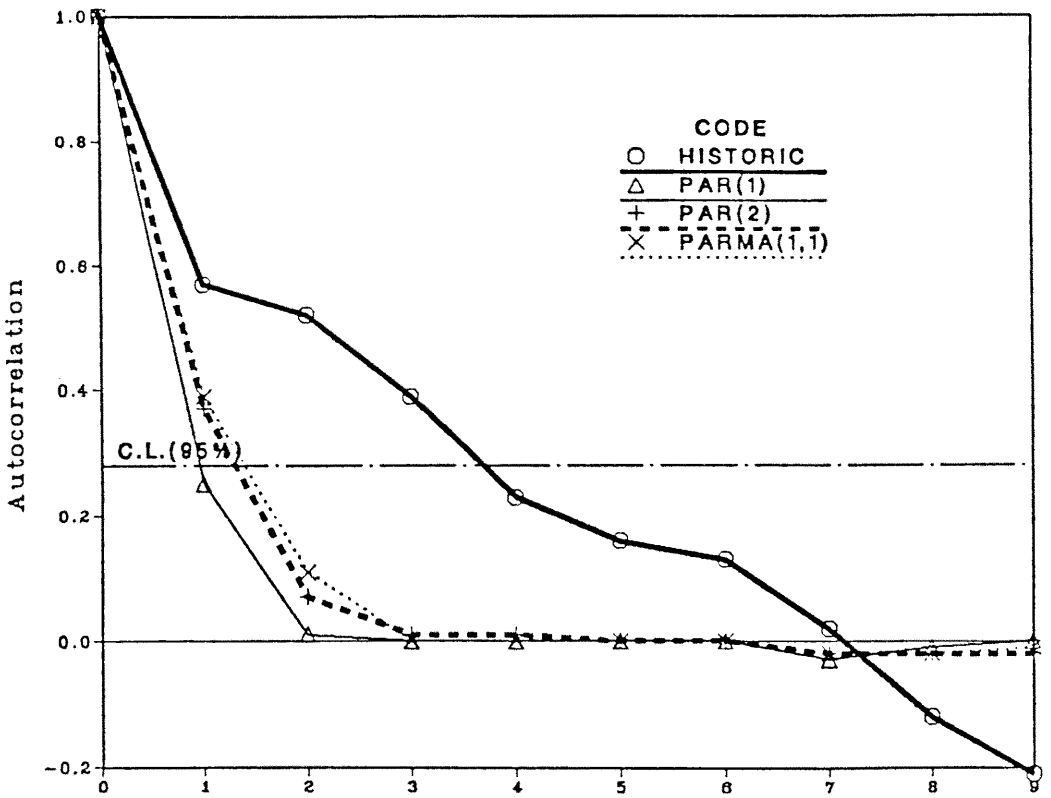

Figure 4.116 - Annual autocorrelograms - Niger River
Least-squares estimation - Six seasons

4.2.3 Elkhorn River

Tables 4.29, 4.31 and 4.33 show the estimated parameters for PAR(1), PAR(2) and PARMA(1,1) models, estimated by the method-of-moments and by the least-squares techniques.

Tables 4.30, 4.32 and 4.34 present the average generated autocorrelations corresponding to estimated parameters.

Table 4.29 - Estimated parameters
Elkhorn River - PAR(1)

τ	$\phi_{1,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS
1	0.48	0.49	0.31	0.32
2	0.61	0.62	0.20	0.15
3	1.10	1.25	0.45	0.43
4	0.66	0.66	0.49	0.49
5	0.64	0.64	0.60	0.60
6	0.48	0.48	0.54	0.54

Table 4.30 - Annual autocorrelograms
Elkhorn River - PAR(1)

LAG	MOM	MOLS
0	1.00	1.00
1	0.17	0.17
2	-0.10	-0.02
3	0.04	-0.01
4	-0.01	0.00
5	-0.01	0.00
6	0.05	0.00
7	-0.07	-0.03
8	-0.07	-0.02
9	-0.02	-0.02

Figure 4.31 - Estimated parameters
Elkhorn River - PAR(2)

τ	$\phi_{1,\tau}$		$\phi_{2,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS	MOM	MOLS
1	0.46	0.46	0.04	0.04	0.31	0.31
2	0.53	0.56	0.07	0.06	0.20	0.15
3	0.67	0.94	0.41	0.26	0.44	0.43
4	0.69	0.70	-0.07	-0.11	0.49	0.49
5	0.71	0.70	-0.13	-0.12	0.60	0.60
6	0.21	0.21	0.56	0.56	0.45	0.45

Figure 4.32 - Annual autocorrelograms
Elkhorn River - PAR(2)

LAG	MOM	MOLS
0	1.00	1.00
1	0.35	0.27
2	0.11	0.02
3	0.03	-0.01
4	0.07	-0.01
5	-0.05	-0.00
6	-0.03	0.00
7	-0.03	-0.04
8	-0.02	-0.02
9	-0.01	-0.01

Figure 4.33 - Estimated parameters
Elkhorn River - PARMA(1,1)

τ	$\phi_{1,\tau}$		$\theta_{1,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS	MOM	MOLS
1	0.54	0.58	0.13	0.21	0.31	0.31
2	0.69	0.75	0.16	0.31	0.20	0.15
3	1.34	1.43	0.70	0.73	0.43	0.42
4	0.62	0.71	-0.07	0.09	0.50	0.50
5	0.52	0.72	-0.20	0.19	0.60	0.61
6	1.09	1.17	0.89	0.97	0.47	0.44

Table 4.34 -Annual autocorrelograms :
Elkhorn River - PARMA(1,1)

LAG	MOM	MOLS
0	1.00	1.00
1	0.40	0.40
2	0.17	0.14
3	0.06	0.04
4	0.02	0.01
5	-0.04	-0.01
6	-0.03	-0.01
7	-0.05	-0.03
8	-0.02	-0.02
9	-0.02	-0.02

Figure 4.117 shows the average autocorrelograms for all the models,from samples generated by using the method-of-moments estimates of parameters for the Elkhorn River.

Figure 4.118 shows average autocorrelograms obtained from estimates by the least-squares method.

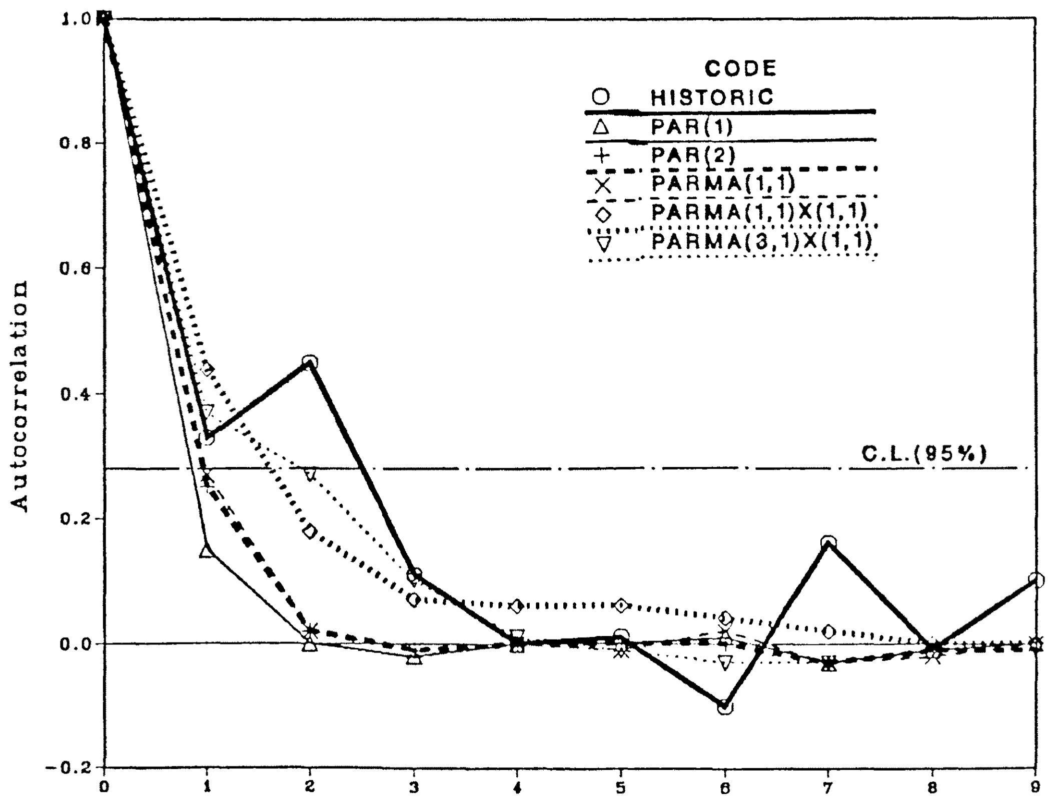

Figure 4.117 - Annual autocorrelograms - Elkhorn River
Method-of-moments estimation - Six seasons

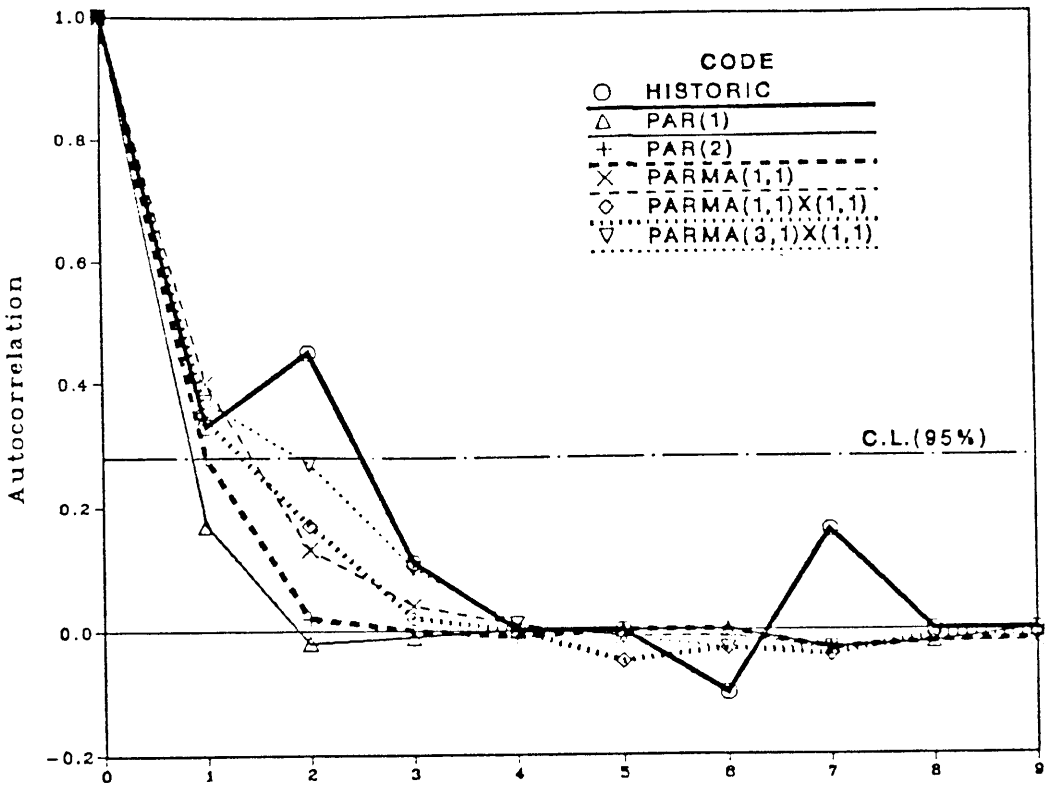

Figure 4.114 - Annual autocorrelograms - Elkhorn River
Least-squares estimation - Six seasons

4.2.4 Yellowstone River

Table 4.35 shows parameters for the PAR(1) model estimated by the two methods, and Table 4.36 presents the corresponding average generated annual autocorrelations.

Table 4.35 - Estimated parameters
Yellowstone River - PAR(1)

τ	$\phi_{1,\tau}$		$\sigma_{1,\tau}$	
	MOM	MOLS	MOM	MOLS
1	0.62	0.62	0.15	0.15
2	0.76	0.75	0.10	0.10
3	0.94	0.93	0.11	0.11
4	0.41	0.43	0.28	0.28
5	-0.03	-0.02	0.34	0.34
6	0.74	0.75	0.14	0.14

Table 4.36 -Annual autocorrelograms :
Yellowstone River - PAR(1)

LAG	MOM	MOLS
0	1.00	1.00
1	0.13	0.14
2	0.00	-0.01
3	0.01	0.01
4	0.00	0.00
5	-0.01	-0.01
6	0.00	0.00
7	0.00	0.00
8	-0.01	-0.01
9	-0.02	-0.02

Table 4.37 shows the estimated PAR(2) parameters estimated by the method-of-moments and by the least-squares methods while Table 4.38 shows the corresponding average generated annual autocorrelograms.

Table 4.37 - Estimated parameters
Yellowstone River - PAR(2)

τ	$\phi_{1,\tau}$		$\phi_{2,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS	MOM	MOLS
1	0.91	0.92	-0.28	-0.29	0.14	0.14
2	0.80	0.81	-0.04	-0.06	0.10	0.10
3	1.08	1.08	-0.15	-0.15	0.10	0.10
4	0.32	0.35	0.12	0.09	0.28	0.34
5	-0.10	-0.08	0.29	0.26	0.34	0.34
6	0.74	0.75	0.00	-0.01	0.14	0.14

Table 4.38 - Annual autocorrelograms
Yellowstone River - PAR(2)

LAG	MOM	MOLS
0	1.00	1.00
1	0.17	0.16
2	0.00	0.02
3	0.01	0.01
4	0.00	0.00
5	-0.01	-0.01
6	0.00	0.00
7	0.00	0.00
8	-0.01	-0.01
9	0.00	0.00

Table 4.39 presents the parameters for the PARMA(1,1) estimated by the methods of moments and least-squares.

Table 4.40 shows the corresponding average generated annual autocorrelograms.

Table 4.39 - Estimated parameters
Yellowstone River - PARMA(1,1)

τ	$\phi_{1,\tau}$		$\theta_{1,\tau}$		σ_τ	
	MOM	MOLS	MOM	MOLS	MOM	MOLS
1	0.53	0.53	-0.37	-0.39	0.14	0.14
2	0.73	0.75	-0.08	-0.01	0.10	0.10
3	0.88	0.86	-0.20	-0.25	0.10	0.10
4	0.44	0.41	0.13	0.28	0.28	0.28
5	0.61	0.54	0.71	0.62	0.34	0.34
6	0.75	0.73	0.00	-0.02	0.14	0.14

Table 4.40 -Annual autocorrelograms
Yellowstone River - PARMA(1,1)

LAG	MOM	MOLS
0	1.00	1.00
1	0.17	0.17
2	0.01	0.00
3	0.01	0.01
4	0.00	0.00
5	-0.01	-0.01
6	0.00	0.00
7	0.00	0.00
8	-0.01	-0.01
9	0.00	0.00

Figure 4.119 presents the average generated annual autocorrelograms for all the models obtained by using moment estimates, while Figure 4.120 shows autocorrelograms corresponding to least-squares estimates of parameters.

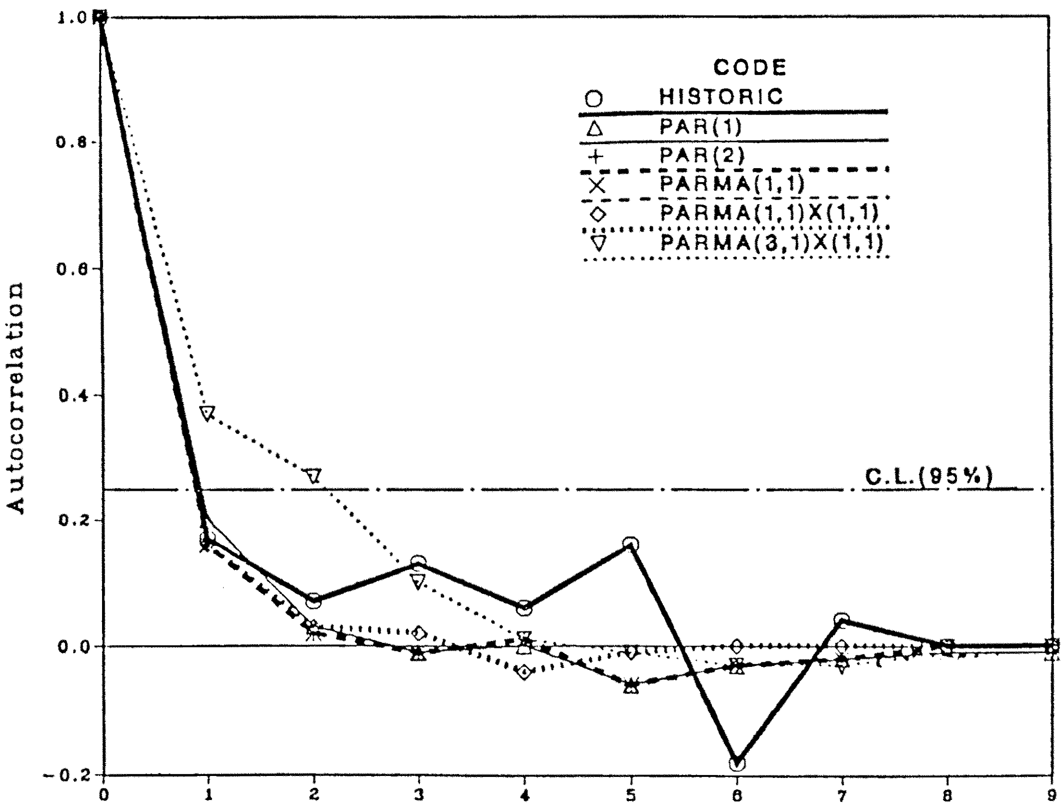

Figure 4.119 - Annual autocorrelograms - Yellowstone River
Method-of-moments estimation

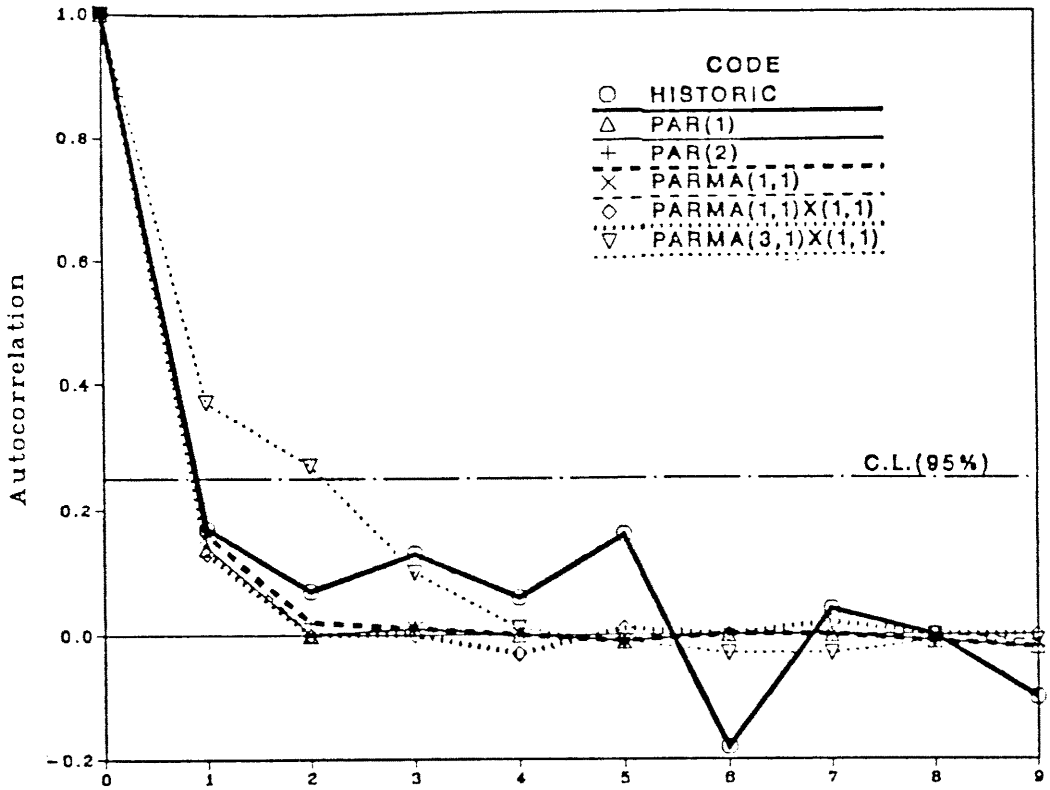

Figure 4.120 - Annual autocorrelograms - Yellowstone River
Least-squares estimation - Six seasons

Tables 4.17 through 4.40 show that estimates of all parameters by the two methods for PAR(1) and PAR(2) models are almost exactly equal and that, consequently, the annual autocorrelograms are very close . Significant differences are present only in the estimates of a small group of moving average parameters for the PARMA(1,1) model. Figures 4.113 through 4.120 show that small differences happened for only the lag 1 annual autocorrelation for the Niger and Elkhorn rivers.

Hence, it may be concluded that the two techniques are almost equivalent. However, the estimation by the method of moments presents the advantage of requiring less computer time. However, if computer time is not a limitation, the parameters for the PARMA(1,1) could be estimated by the least-squares method with estimations starting from the method-of-moments estimates as a form of verification and refinement.

4.3 Discussion

a) Residuals characteristics

Average residuals are very close to zero for all samples and models. Seasonal standard deviations for the PAR(1) model are generally higher than those for other models. Models presenting parameters relating data directly from the same season for consecutive years(Φs and $\underline{\bar{\theta}}$s) present standard deviation of residuals slightly lower than the other models not presenting these parameters.

On the average, models presenting more parameters show lower portmanteau statistics. Graphs for the seasonal autocorrelation of residuals do not show much difference between models for lower lags. For higher lags, the models Multiplicative PARMA(2,1)X(1,0), Multiplicative PARMA(1,1)x(1,1), and Multiplicative PARMA(3,1)x(1,1), on the average, present significatively lower autocorrelations. These models break the yearly autocorrelations better (between the same season for consecutive years) probably because of the parameters Φ and $\underline{\bar{\theta}}$.

b) Preservation of statistics -

All models preserve well the seasonal means and standard deviations for all samples well. Averages for seasonal skewness for generated series are significantly different from the historic values for every model and samples. However, the average skewness generated by different model, with parameters estimated from the same samples, are very close. For the cases of the St. Lawrence and Niger rivers, the shapes of the periodicity for skewness of generated samples and for skewness of historic samples are similar. In general, the differences could be explained by the large variability for skewness estimation for samples presenting the size of analyzed historic samples (Salas et al., 1980).

Lag 1 seasonal autocorrelations were, in general, well preserved by all models for every sample. However, significant differences can be seen for large orders. Models that do not present parameters relating data for the same season in consecutive years tended to present flat correlograms for higher orders. Mainly for the six-season samples these models were not able to preserve the high-order autocorrelations. PAR(1) model presented the worst reproductions. The Multiplicative PARMA(3,1)x(1,1) model sometimes produced autocorrelations higher than the historic values. It may be concluded that the models presenting Φs better preserve high order autocorrelations, comfirming results from analysis of synthetic samples.

The annual autocorrelations produced by Multiplicative PARMA(1,0)x(2,0), Multiplicative PARMA(1,1)x(1,1), Multiplicative PARMA(2,1)x(1,0) and Multiplicative PARMA(3,1)x(1,1) were significantly closer to the historic autocorrelations for samples presenting higher (St. Lawrence and Niger rivers) and moderate (Elkhorn river) long-term autocorrelation. The other models showed a tendency to present a decreasing autocorrelation for an increasing number of seasons, confirming results from Obeysekera and Salas(1986) for the Niger River. The Multiplicative PARMA(3,1)x(1,1) model showed higher autocorrelations than the historical for the Yellowstone River. For the Niger River it presented higher autocorrelations for lags greater than two.

For the Yellowstone river, which presents low long-term dependence, the reproduction of statistics by models presenting fewer parameters was similar to that for the models presenting parameters Φs and $\underline{\bar{\theta}}$s.

In many simulations the portmanteau statistic Q_1 and the AIC proved to be better than the other statistics for preliminary analysis of goodness of fit and for comparison of models. Data generation and plotting of autocorrelograms is suggested for final analysis of the preservation of dependence and acceptance of a model.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

This research has focused primarily on seasonal time-series modeling. A class of models, called Multiplicative Periodic Autoregressive Moving-Average (Multiplicative PARMA) was developed. The modeling technique utilizes the Powell minimization algorithm for a least-squares search of the parameters. The model is a generalization for PARMA models and for some multiplicative ARIMA models and presents periodic parameters relating data for consecutive seasons from the same year and periodic parameters relating data for the same season in consecutive years.

Analysis of the sensitivity of the annual autocorrelograms to the different parameters showed that the proposed model is able to reproduce high long term annual and seasonal autocorrelations well.

Data generation showed that the developed estimation procedure is able to find good estimates of parameters for the Multiplicative PARMA models. It also showed that good estimation of low-order PARMA models can be obtained by the technique and that the least-squares method can also be utilized for refinement of estimations from the method-of-moments, by utilizing these moments estimates as starting points. For estimation of the Multiplicative PARMA models it was also verified that the utilization of low-order PARMA parameters as the starting point saves computer time and guarantees better fitting than that given by the starting parameters. If PARMA parameters are not available, it is recommended that more than one estimation be made, starting from different points, to avoid accepting local optimal points that can result in a bad fitting of the sample. Application to real historic data showed that the Multiplicative PARMA models presenting parameters directly relating information for the same season in consecutive years are able to best preserve high annual autocorrelations and seasonal autocorrelations for lags greater than one. The yearly autoregressive parameters, Φs, proved to be the most important for this purpose. This ability to reproduce long-term autocorrelations by models is important for the solution of many hydrologic problems because failure to preserve multi-lag correlations may cause generation of sequences presenting floods and droughts of lower magnitude than those presented by the historic series and, consequently, the undersizing of hydraulic structures.

For samples presenting low annual dependence, it was verified that the low-order PARMA models are capable of preserving the most important statistics well. Series obtained by adding any number of consecutive seasons from these models were shown to follow a regular ARMA(1,1) model with an autoregressive parameter equal to the product of all seasonal autoregressive parameters. Relationships between parameters for the PARMA models and for the aggregated ARMA(1,1) parameters were derived.

The portmanteau, Q_1, and the Akaike Information Criterion statistics showed better performance, in preliminary analysis of goodness of fitting and comparison among competing models than the competing Q_2, Q_3, Q_4 and Bayesian Information Criterion statistics. Data generation and comparison between historic and generated correlograms is recommended for final acceptance of models.

Important areas for future research related to the subject of this dissertation are very ample. One of the possible topics for future research could be the search for relationships between parameters of the Multiplicative PARMA model and characteristics of watersheds. Another possible area for research could be related to the application of the model to forecasting. Only a few particular cases of the proposed family of models were applied throughout this research. However, the developed estimation procedure is general, and application to other particular models is possible. The estimation procedure can be adapted to multivariate series. The test of other optimimization algorithms, in substitution for the Powell algorithm, could be made to verify if it is possible to decrease computer time spent for searching parameters.

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APPENDIX

GRAPHICAL REPRESENTATION OF HISTORIC AND GENERATED SEASONAL MEANS, STANDARD DEVIATIONS AND SKEWNESS.

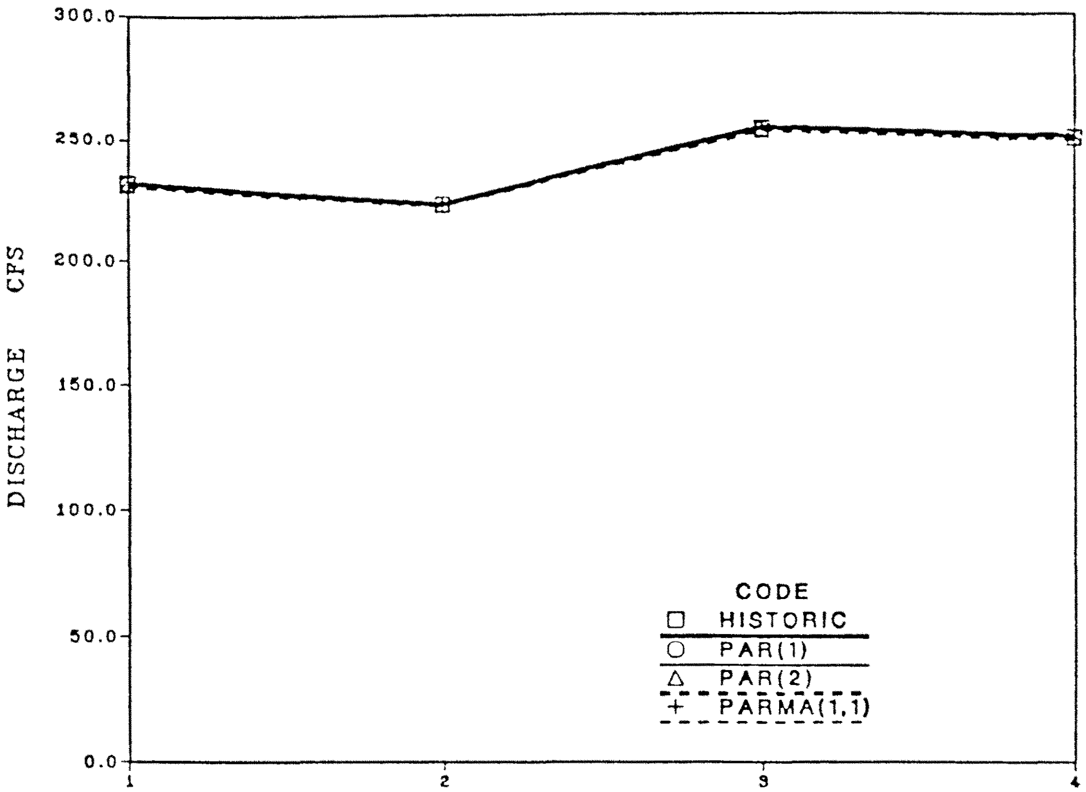


Figure A.1 - Seasonal average discharge (cfs)
St. Lawrence River - Four seasons

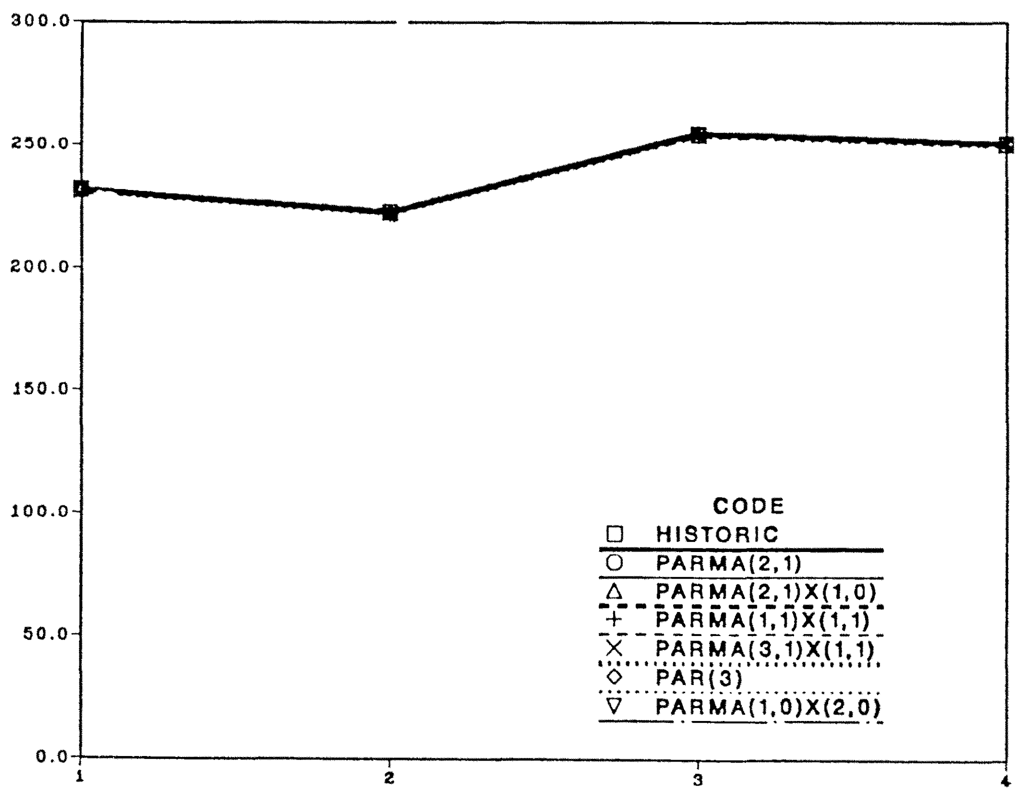


Figure A.2 - Seasonal average discharge (cfs)
St. Lawrence River - Four seasons

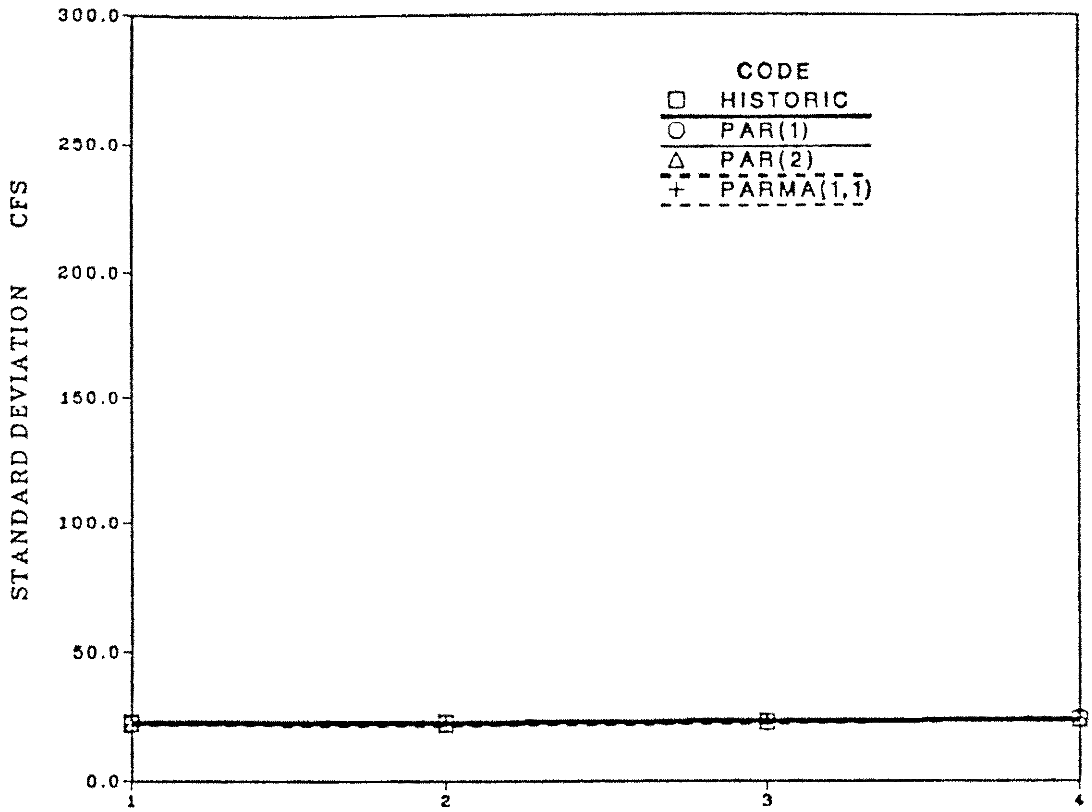


Figure A.3 - Seasonal standard deviation (cfs)
St. Lawrence River - Four seasons

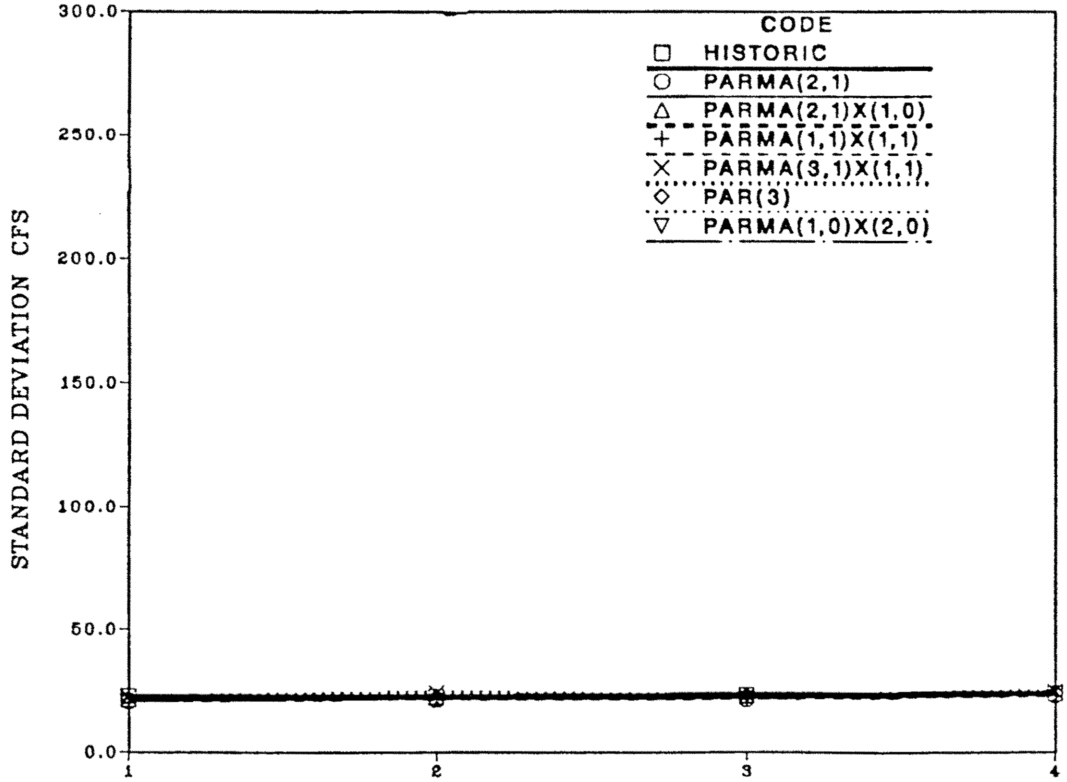


Figure A.4 - Seasonal standard deviation (cfs)
St. Lawrence River - Four seasons

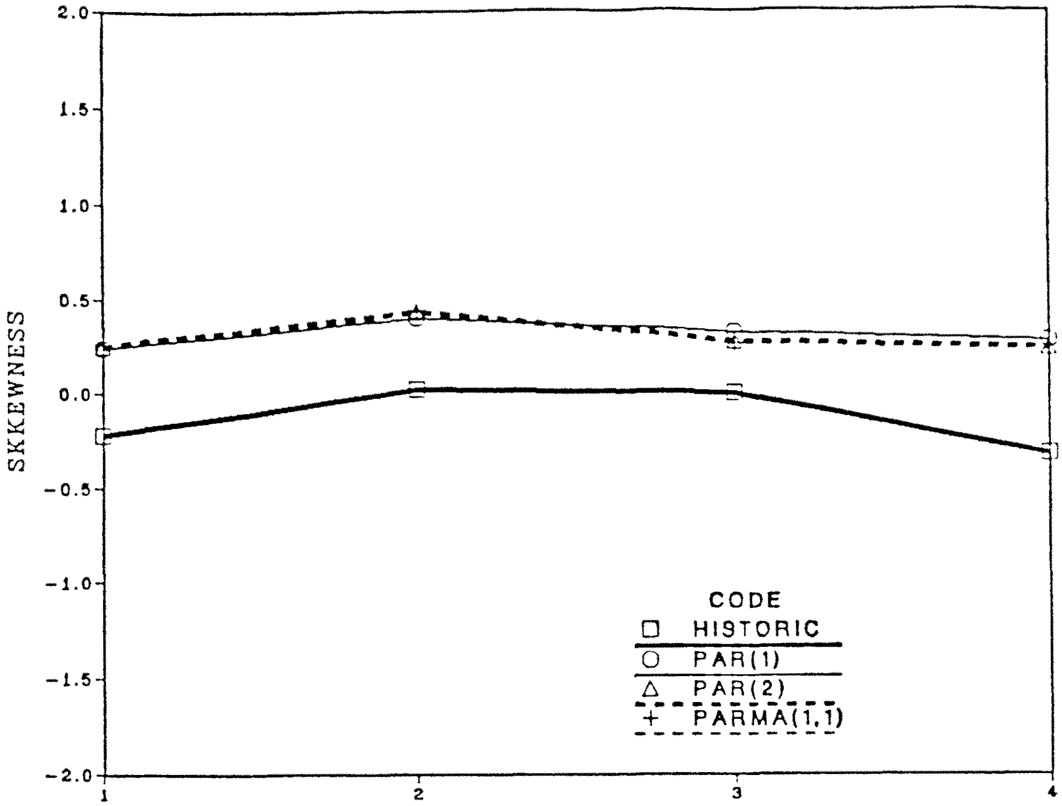


Figure A.5 - Seasonal skewness
St. Lawrence River - Four seasons

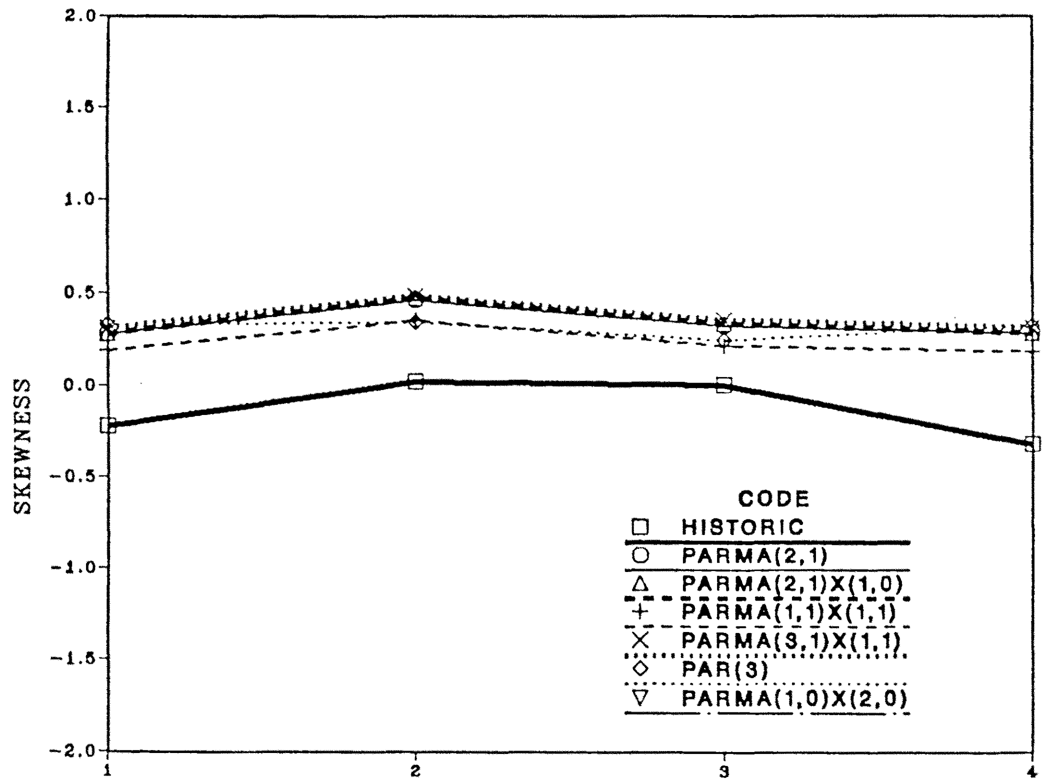


Figure A.6 - Seasonal skewness
St. Lawrence River - Four seasons

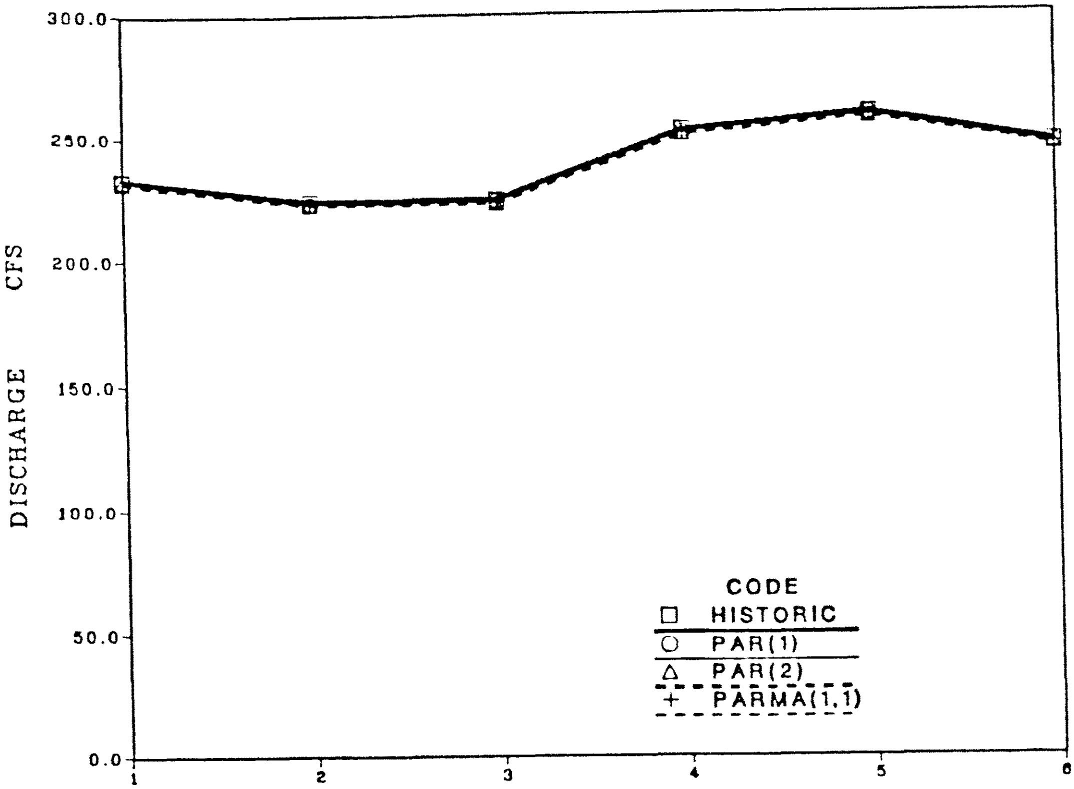


Figure A.7 - Seasonal average discharge (cfs)
St. Lawrence River - Six seasons

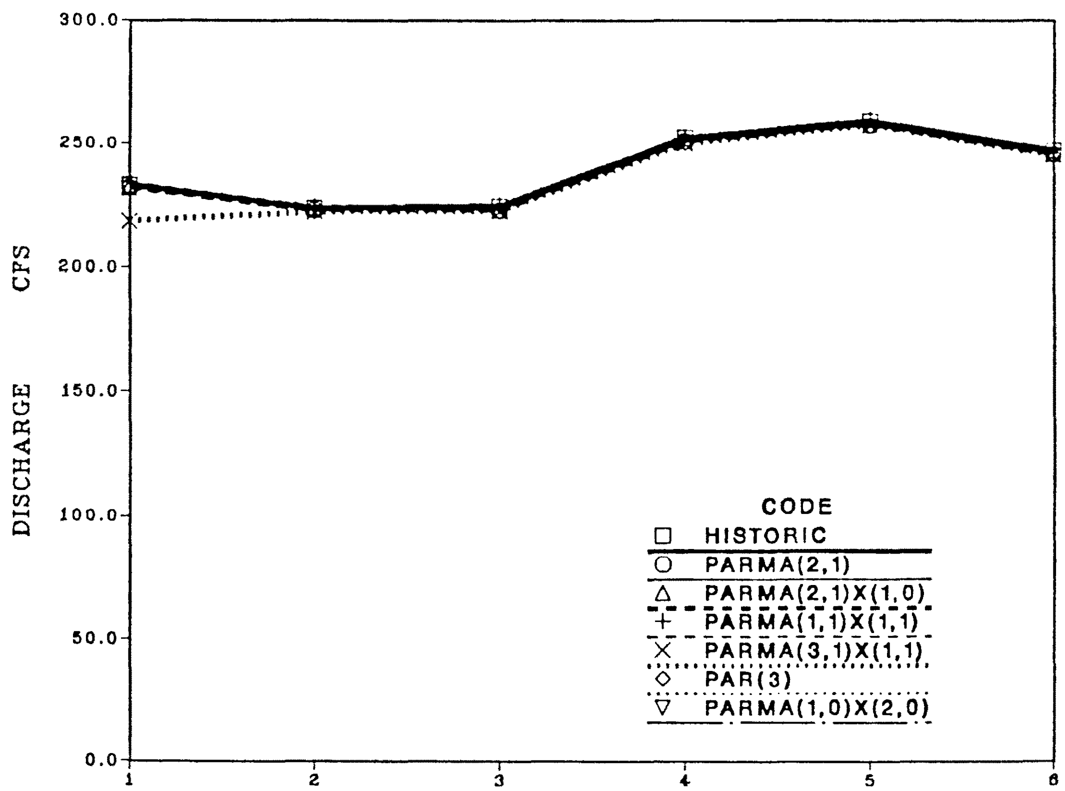


Figure A.8 - Seasonal average discharge (cfs)
St. Lawrence River - Six seasons

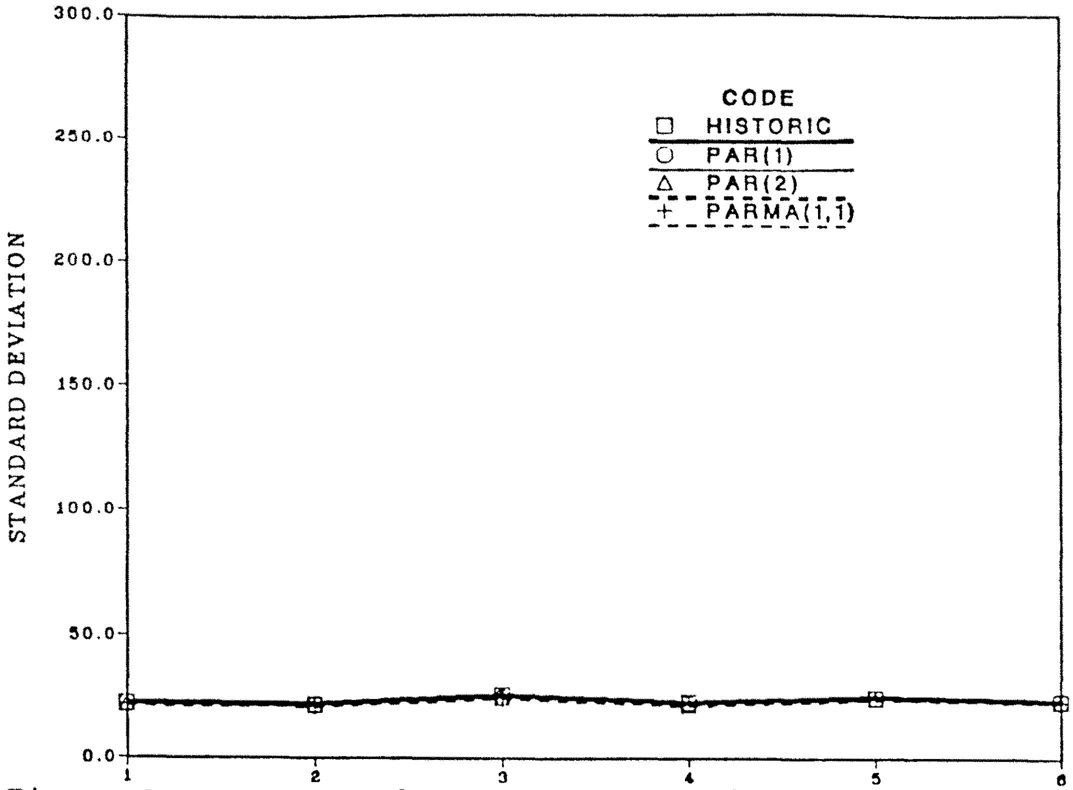


Figure A.9 - Seasonal standard deviation (cfs)
St. Lawrence River - Six seasons

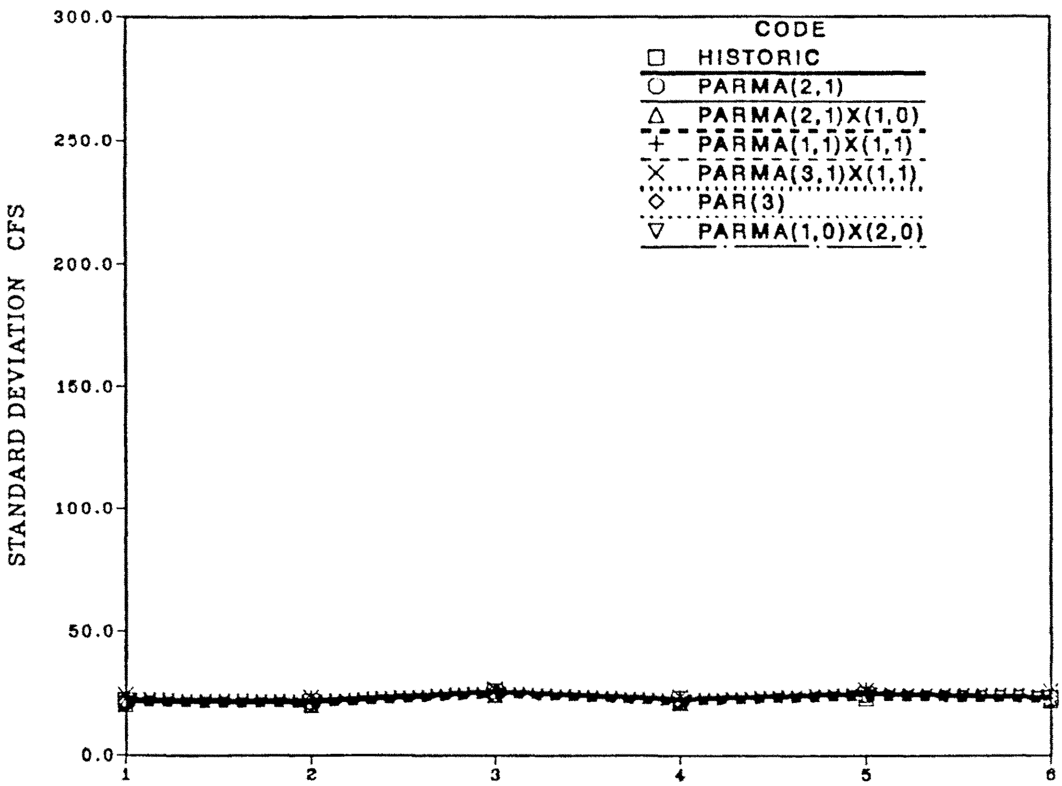


Figure A.10 - Seasonal standard deviation (cfs)
St. Lawrence River - Six seasons

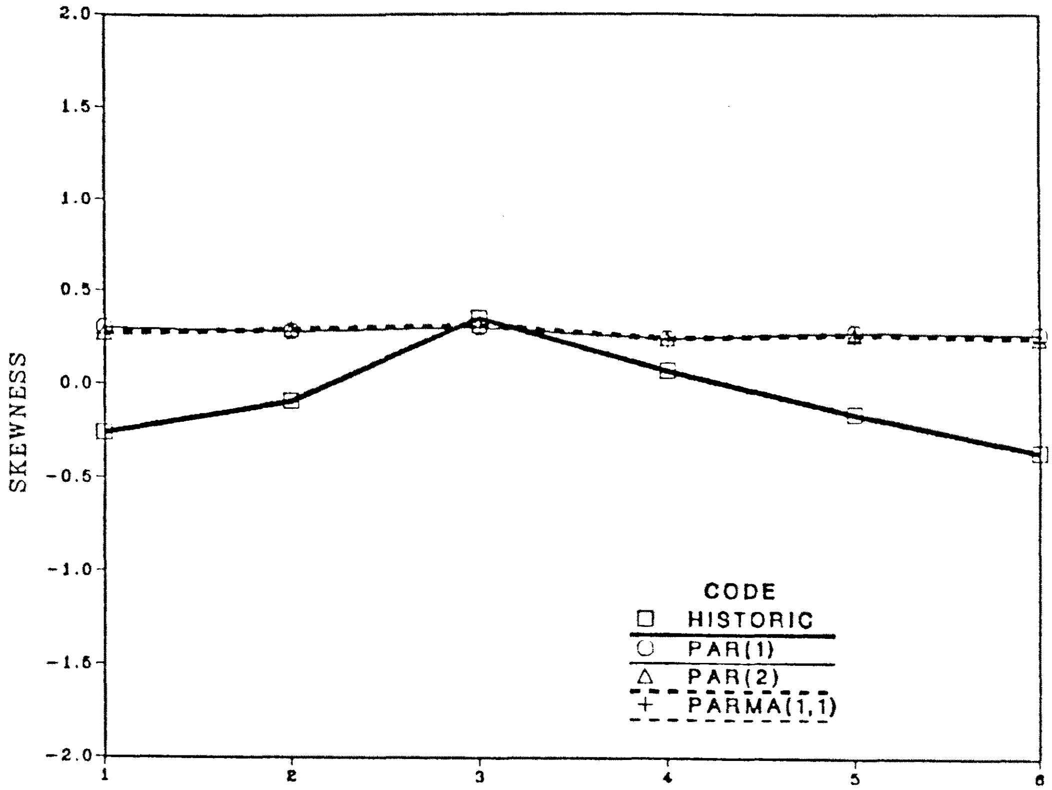


Figure A.11 - Seasonal skewness
St. Lawrence River - Six seasons

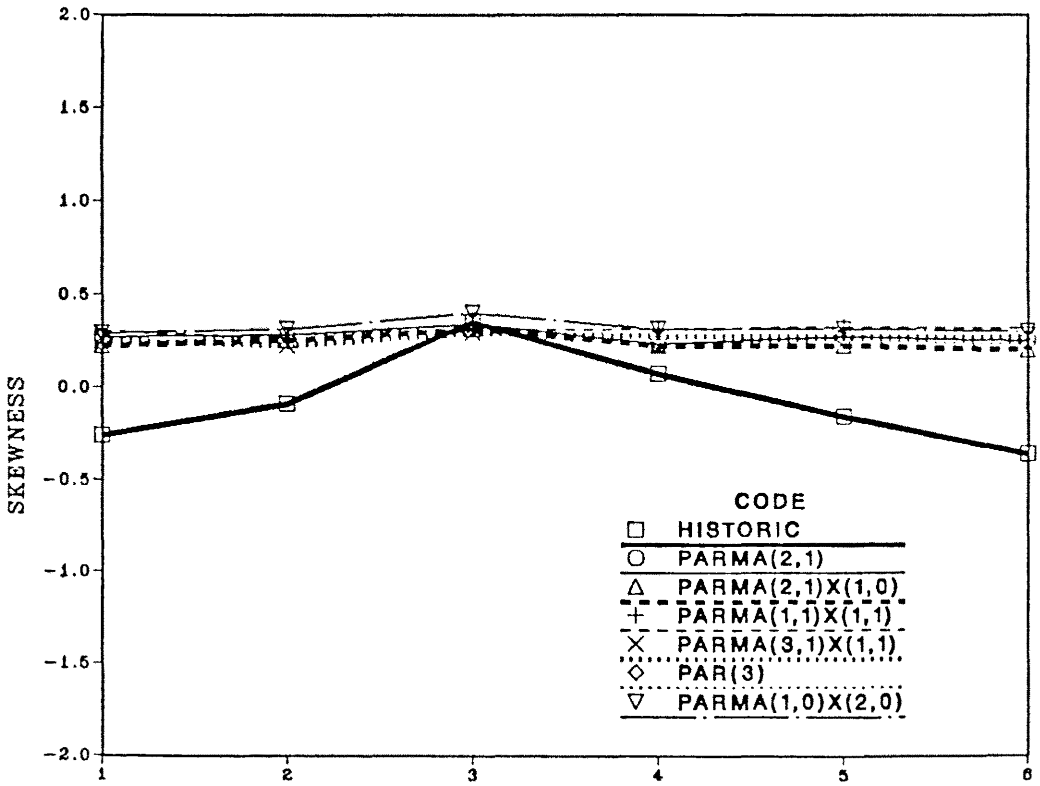


Figure A.12 - Seasonal skewness
St. Lawrence River - Six seasons

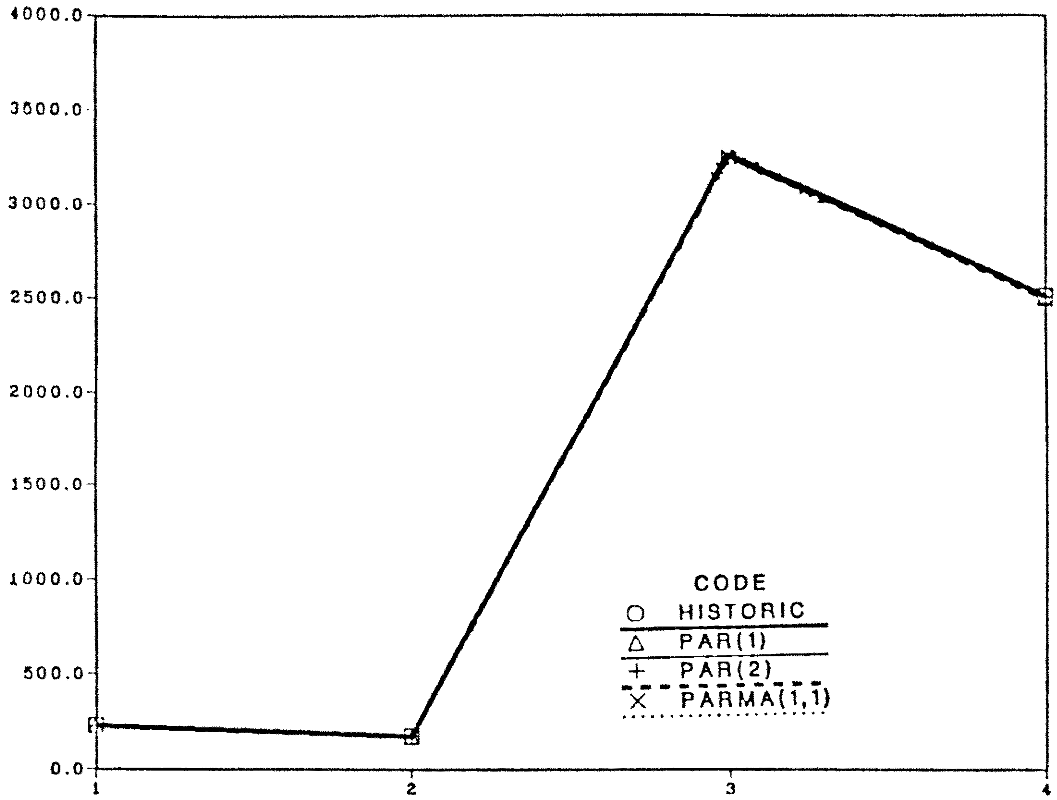


Figure A.13 - Seasonal average discharge (cms)
Niger River - Four seasons

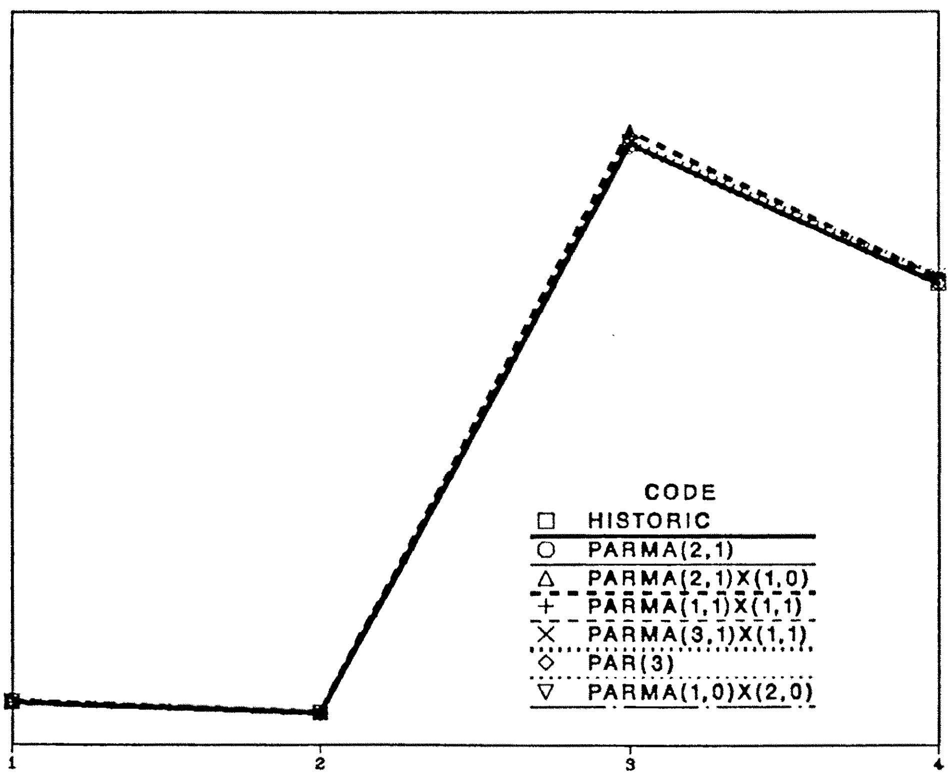


Figure A.14 - Seasonal average discharge (cms)
Niger River - Four seasons

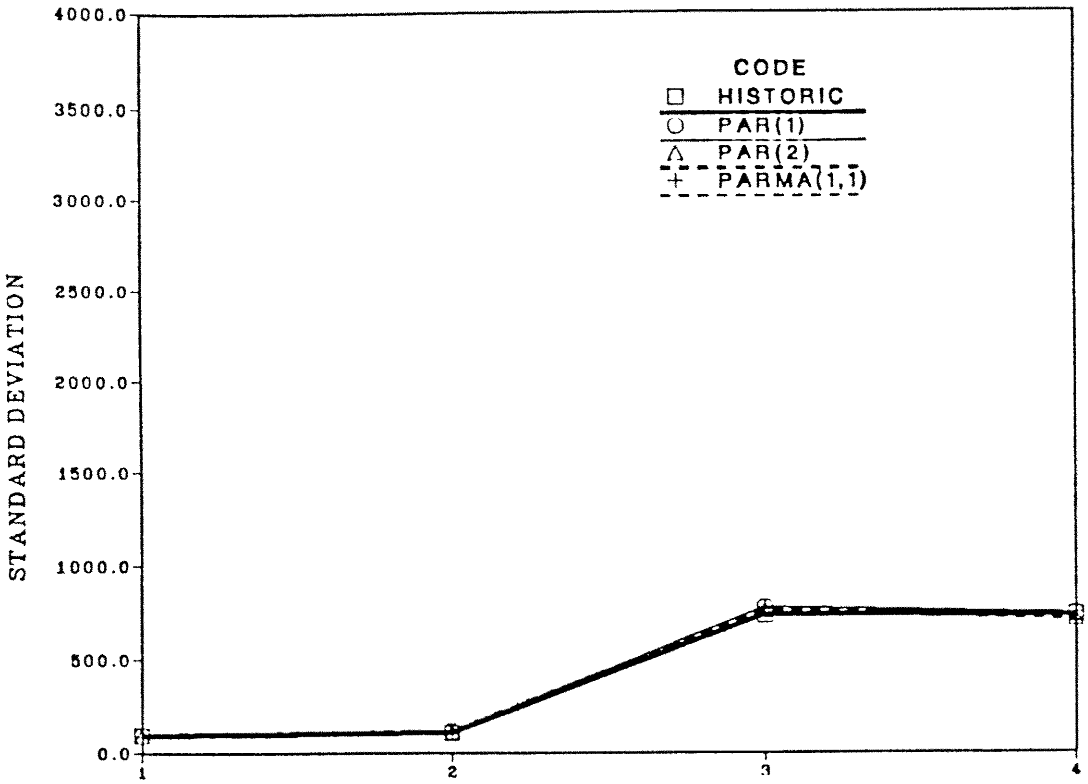


Figure A.15 - Seasonal standard deviation (cms)
Niger River - Four seasons

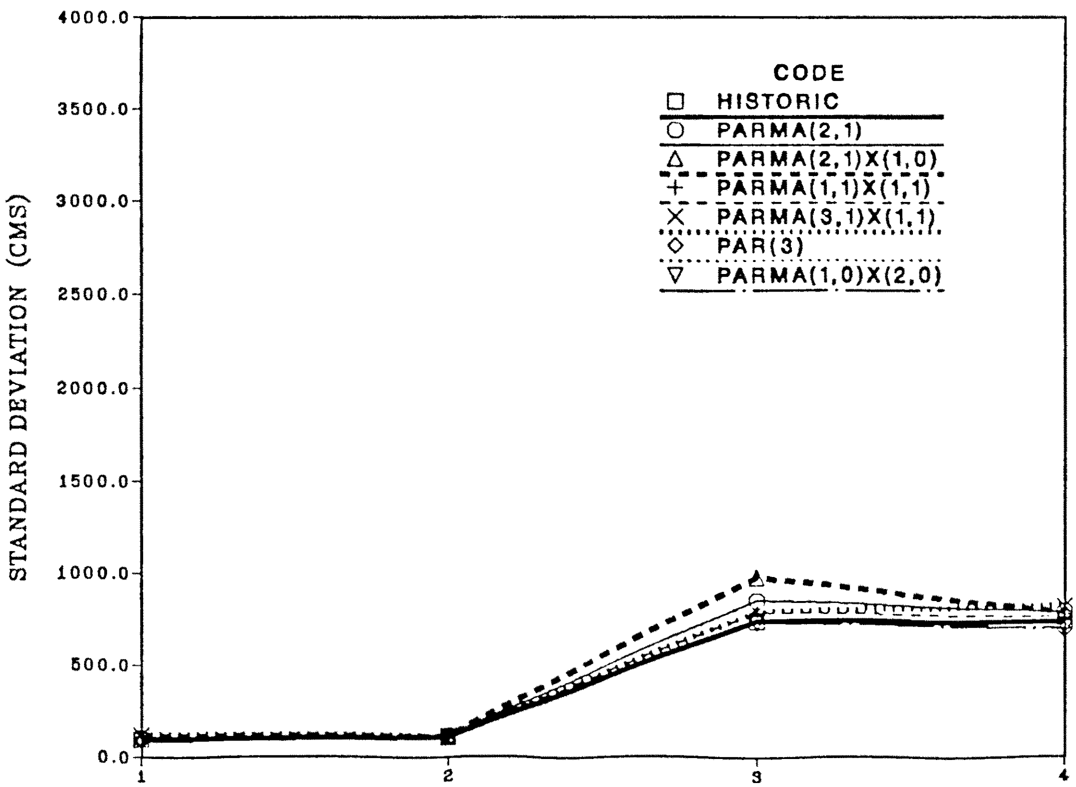


Figure A.16 - Seasonal standard deviation (cms)
Niger River - Four seasons

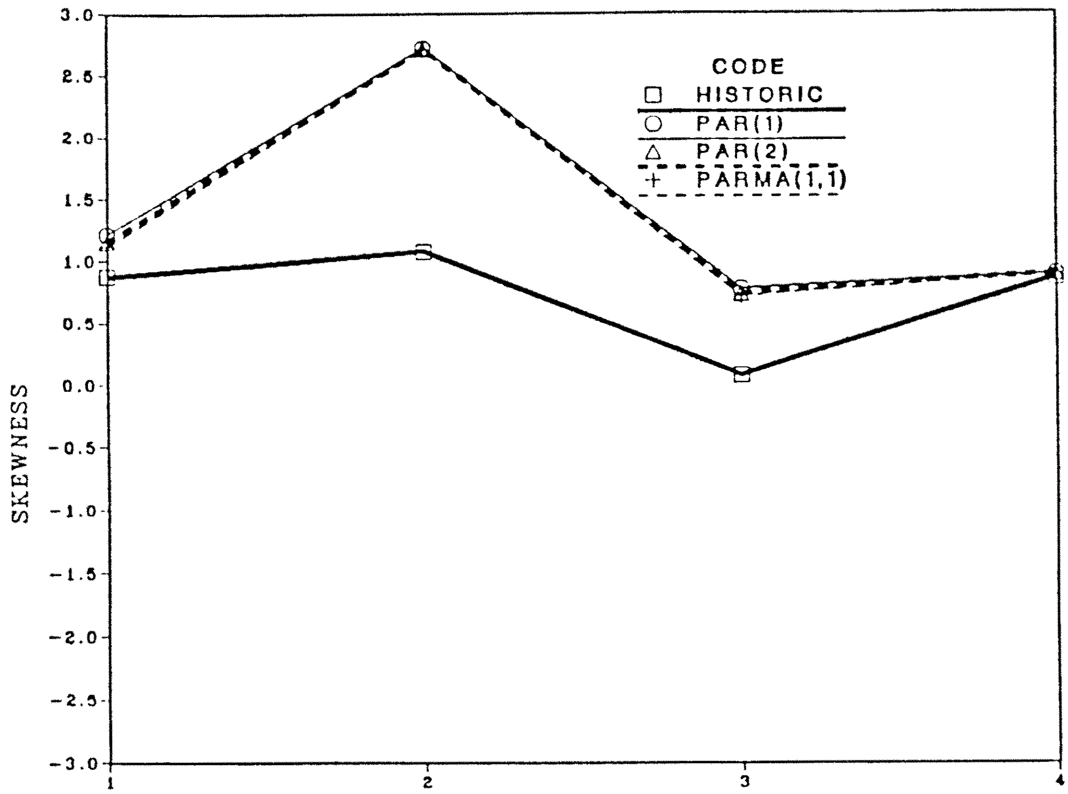


Figure A.17 - Seasonal skewness
Niger River - Six seasons

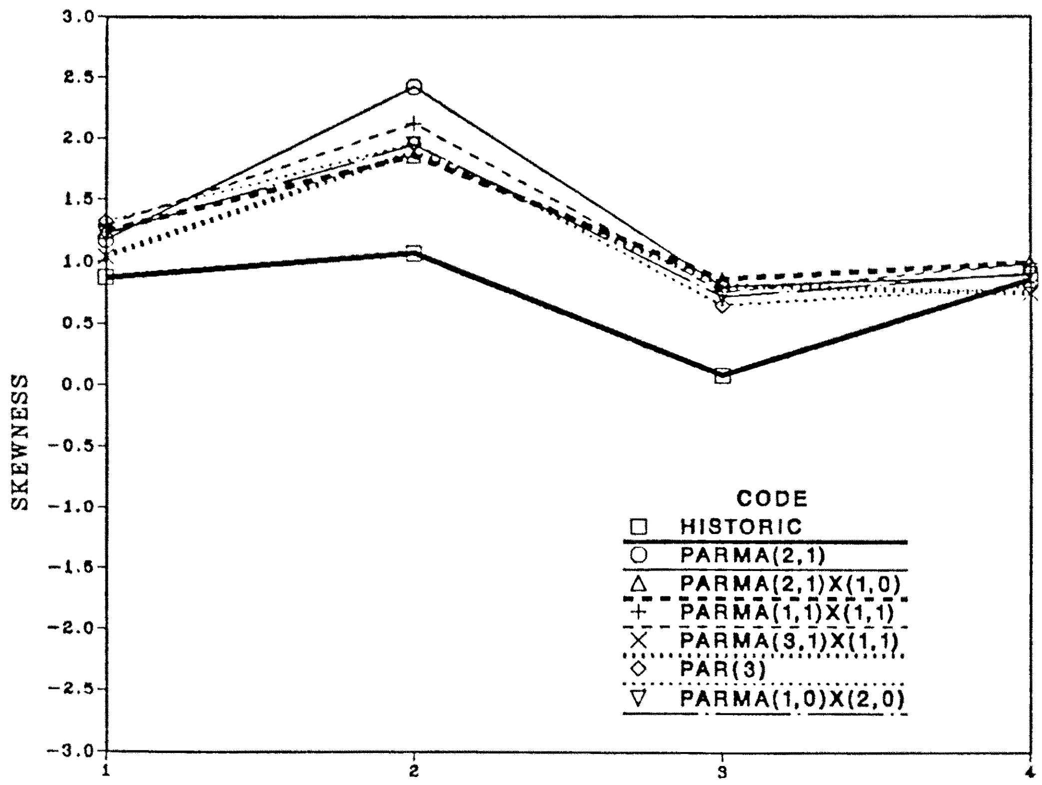


Figure A.18 - Seasonal skewness
Niger River - Four seasons

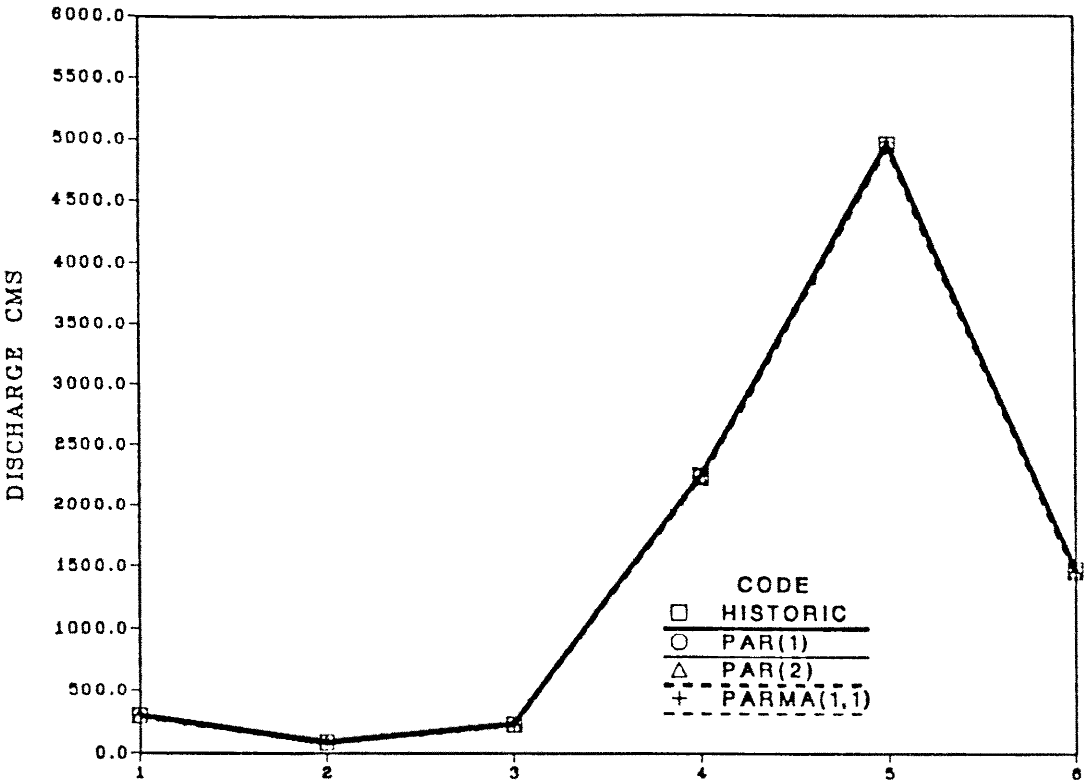


Figure A.19 - Seasonal average discharge (cms)
Niger River - Six seasons

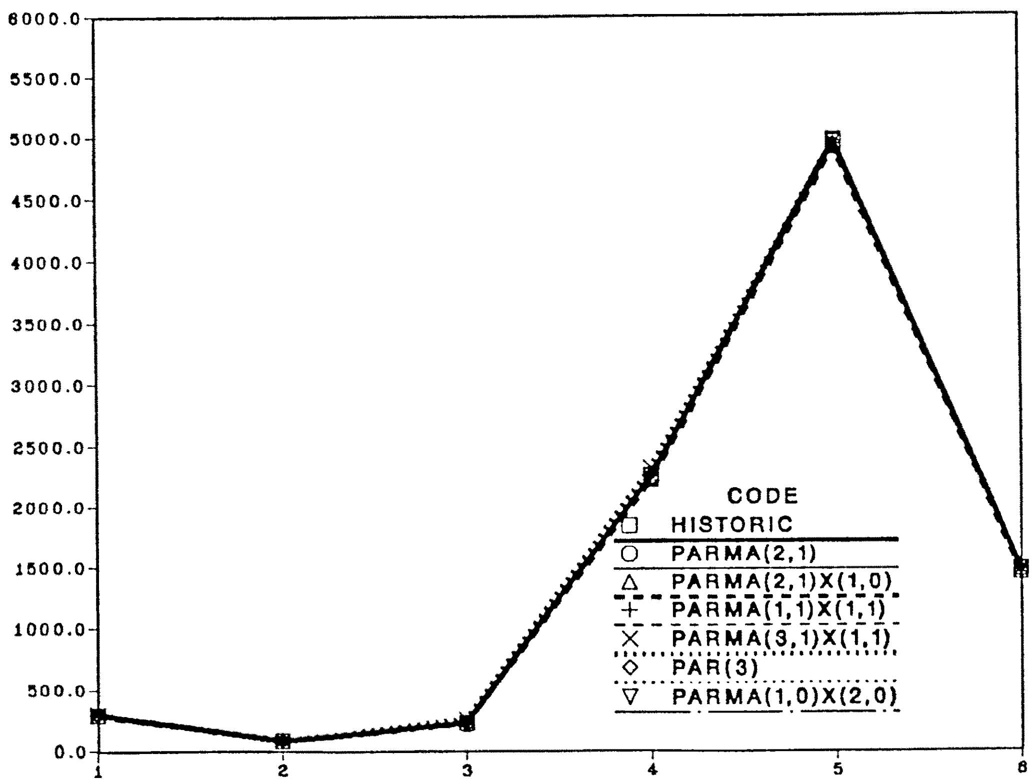


Figure A.20 - Seasonal average discharge (cms)
Niger River - Six seasons

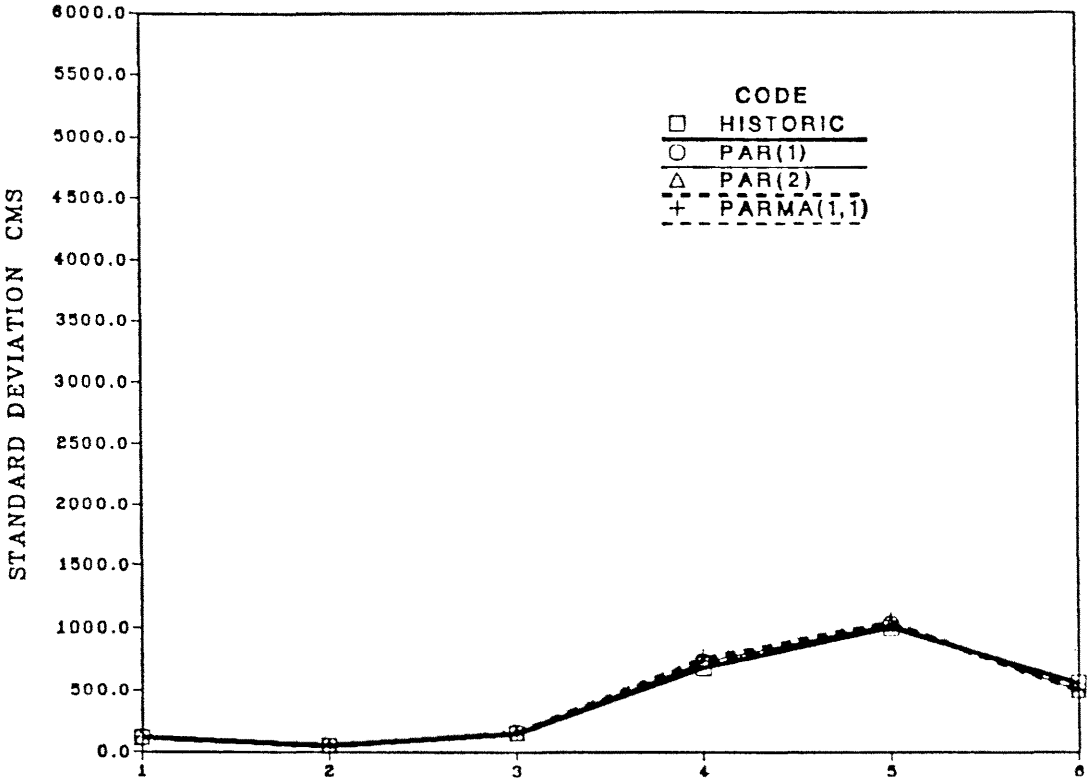


Figure A.21 - Seasonal standard deviation (cms)
Niger River - Six seasons

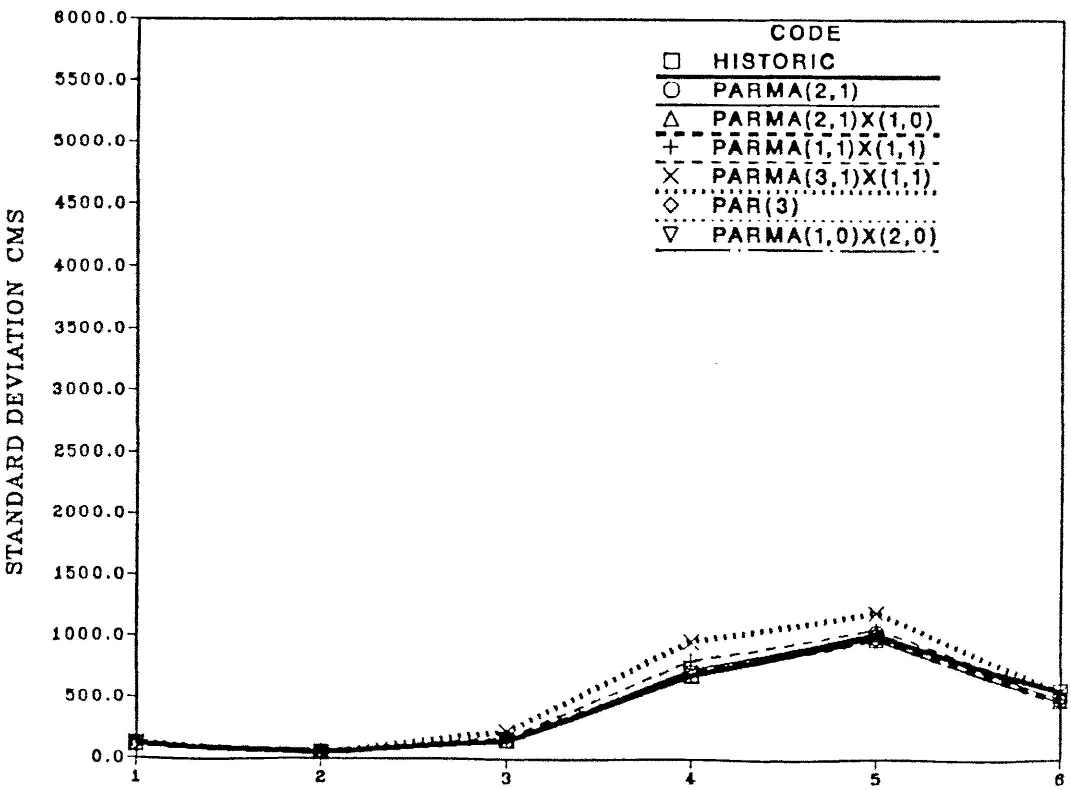


Figure A.22 - Seasonal standard deviation (cms)
Niger River - Six seasons

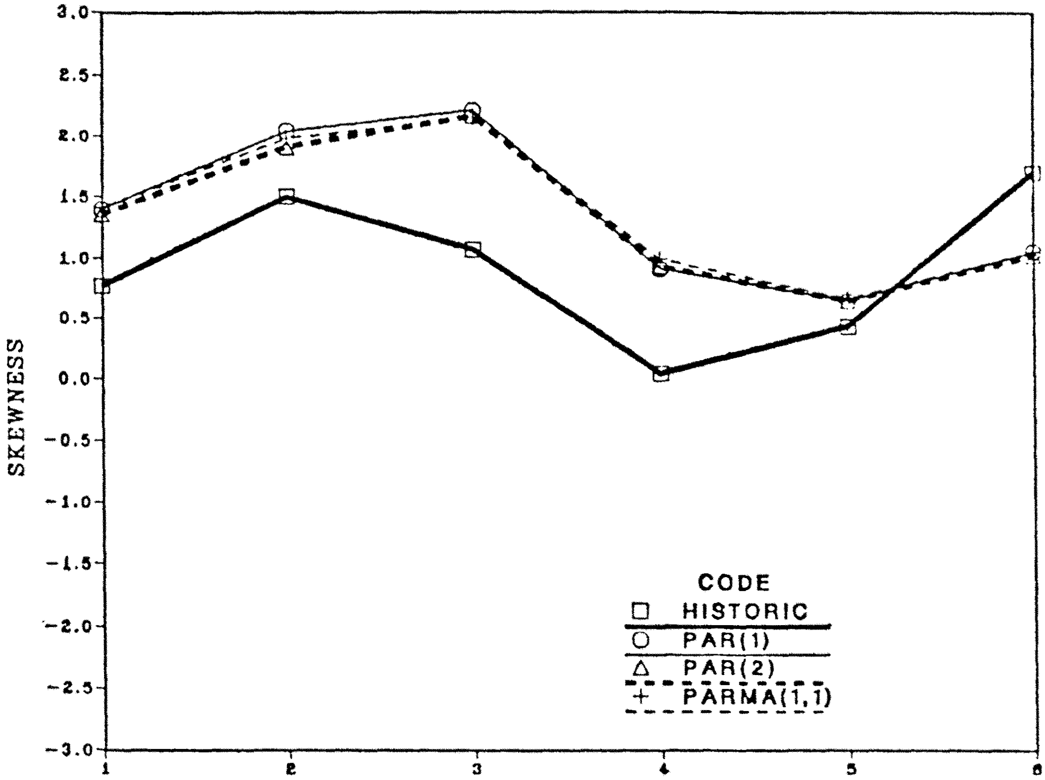


Figure A.23 - Seasonal skewness
Niger River - Six seasons

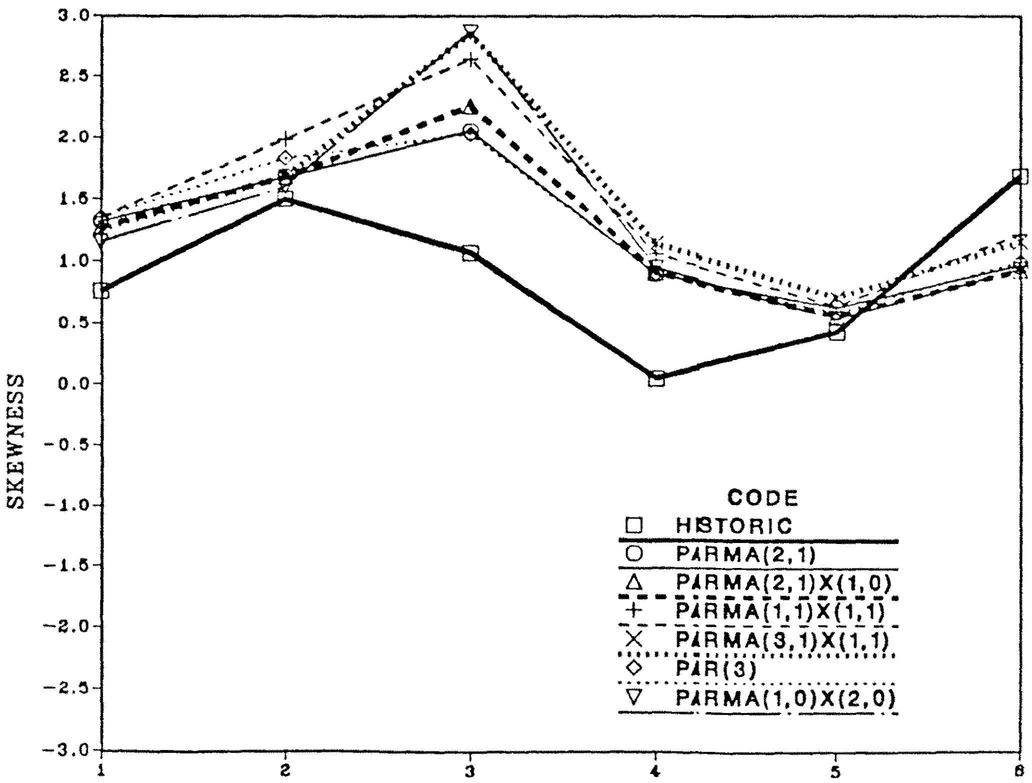


Figure A.24 - Seasonal skewness
Niger River - Six seasons

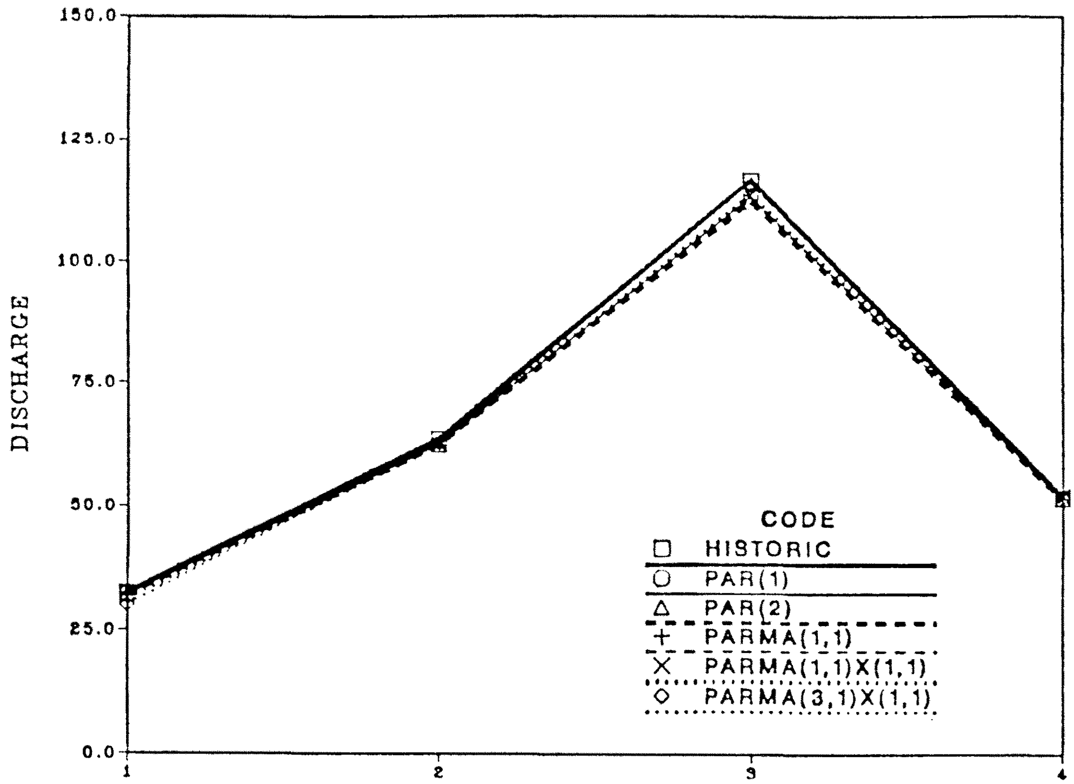


Figure A.25 - Seasonal average discharge (cfs)
Elkhorn River - Four seasons

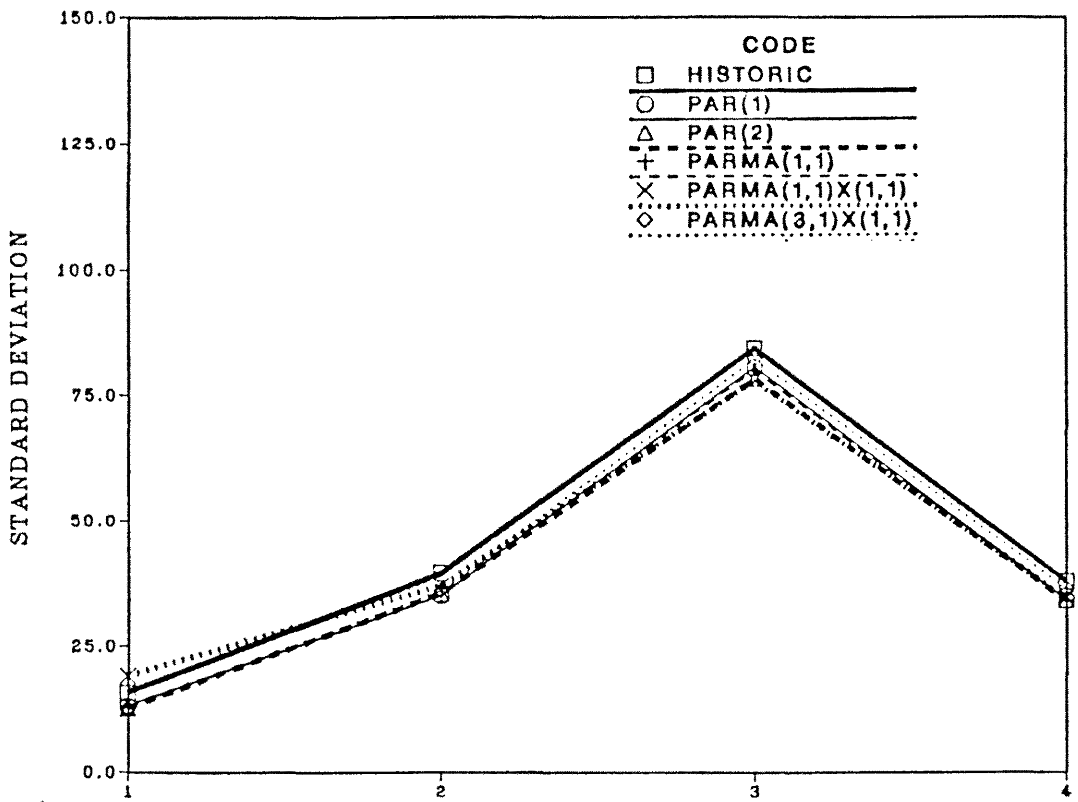


Figure A.26 - Seasonal standard deviation (cfs)
Elkhorn River - Four seasons

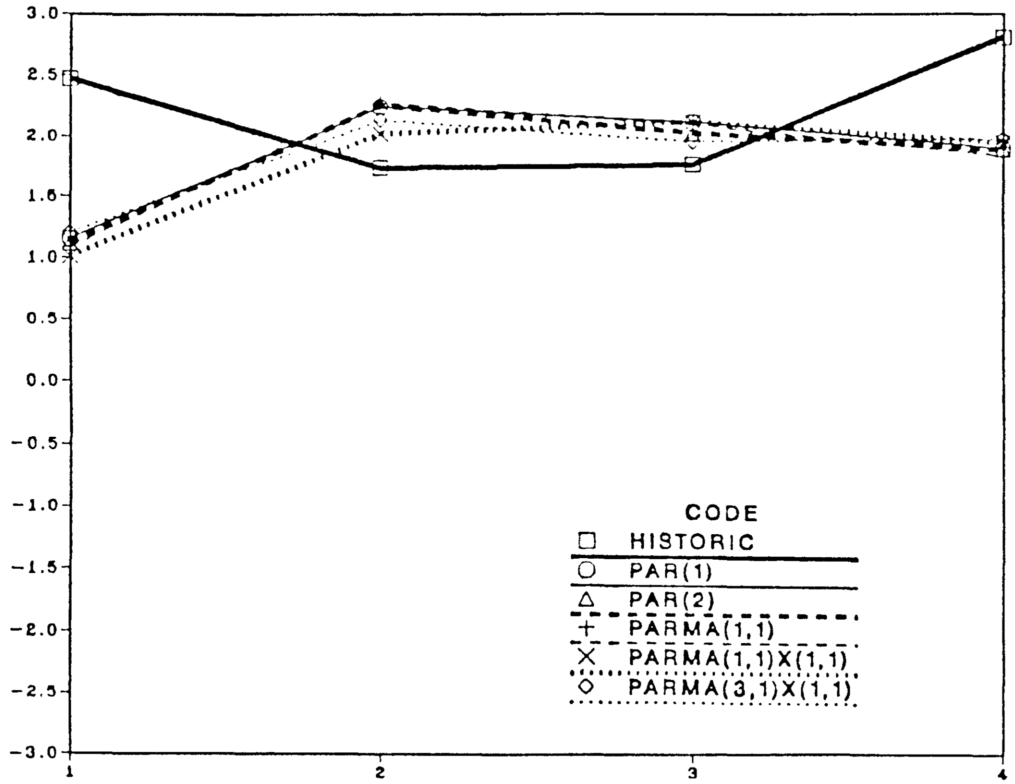


Figure A.27 - Seasonal skewness
Elkhorn River - Four seasons

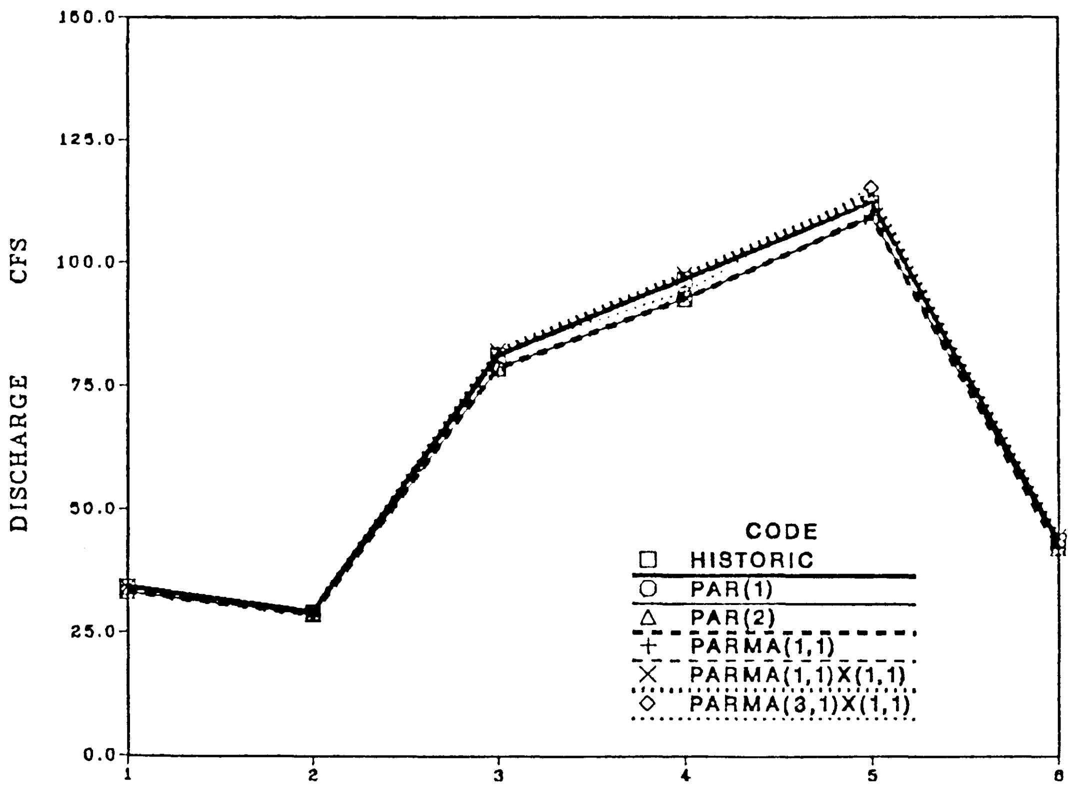


Figure A.28 - Seasonal average discharge (cfs)
Elkhorn River - Six seasons

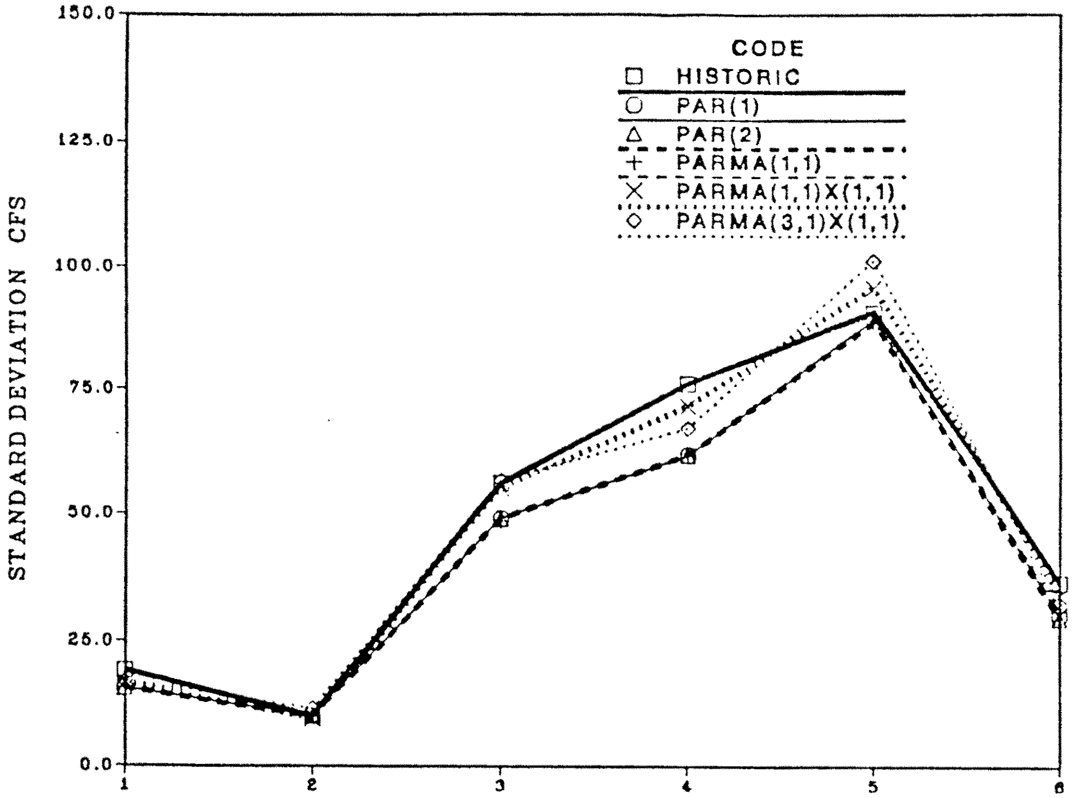


Figure A.29 - Seasonal standard deviation (cfs)
Elkhorn River - Six seasons

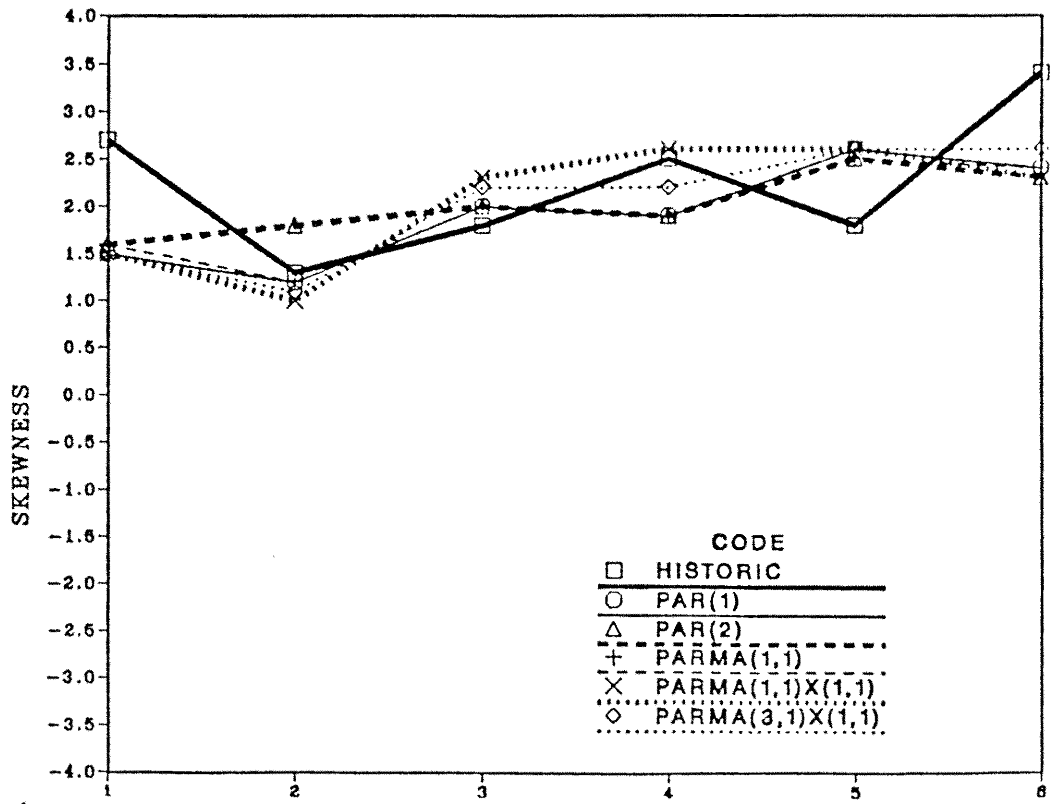


Figure A.30 - Seasonal skewness
Elkhorn River - Six seasons

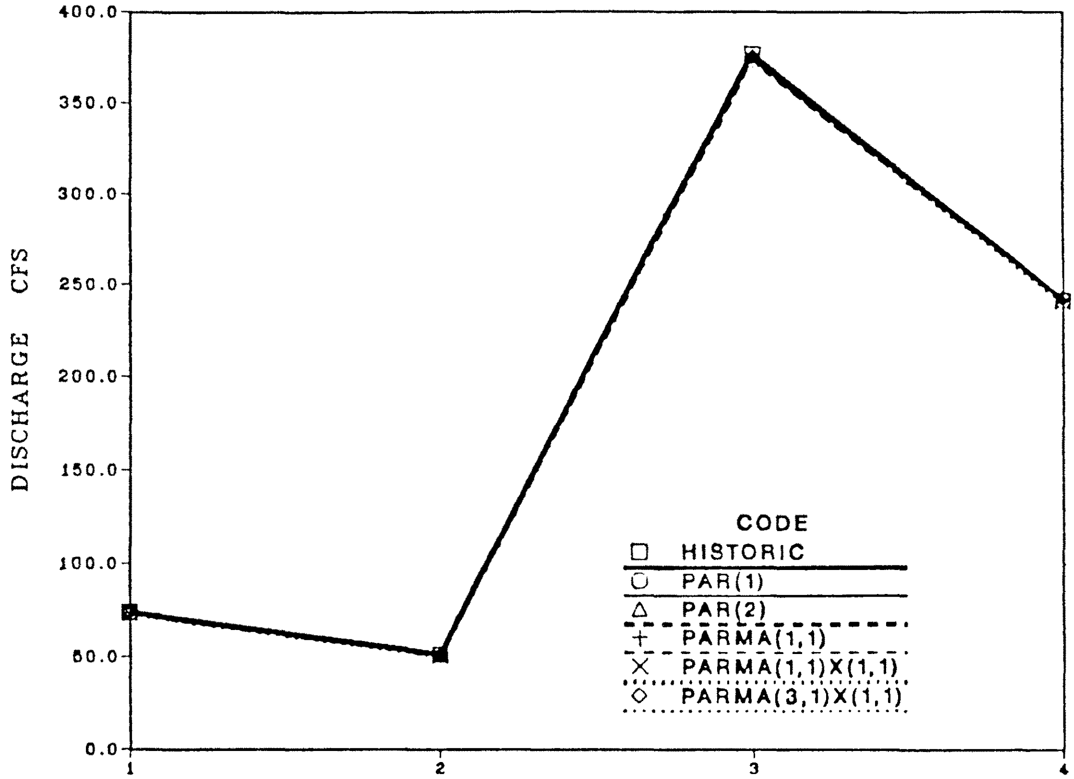


Figure A.31 - Seasonal average discharge (cfs)
Yellowstone River - Four seasons

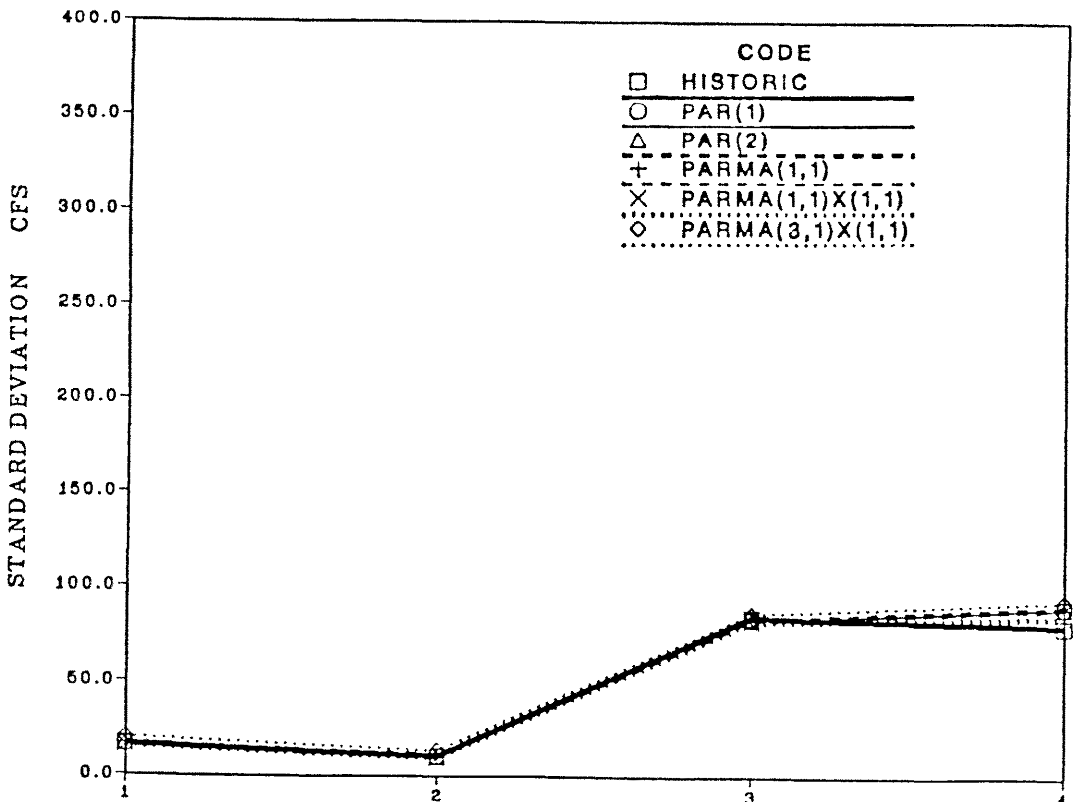


Figure A.32 - Seasonal standard deviation (cfs)
Yellowstone River - Four seasons

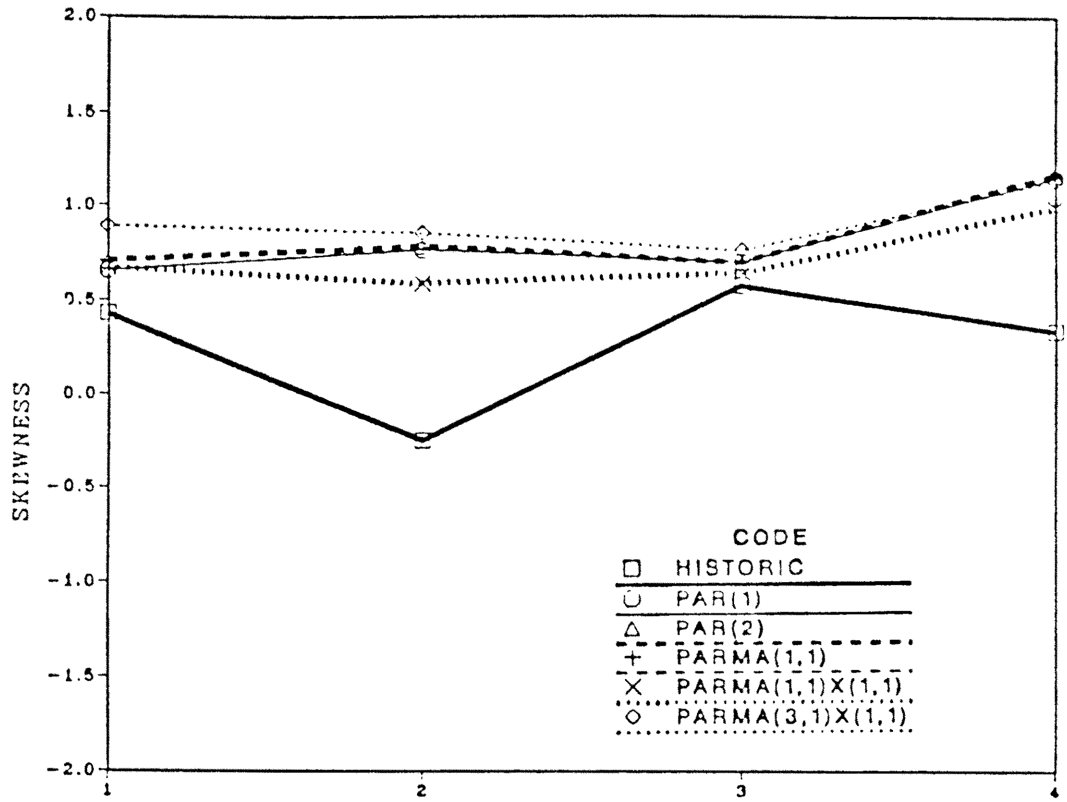
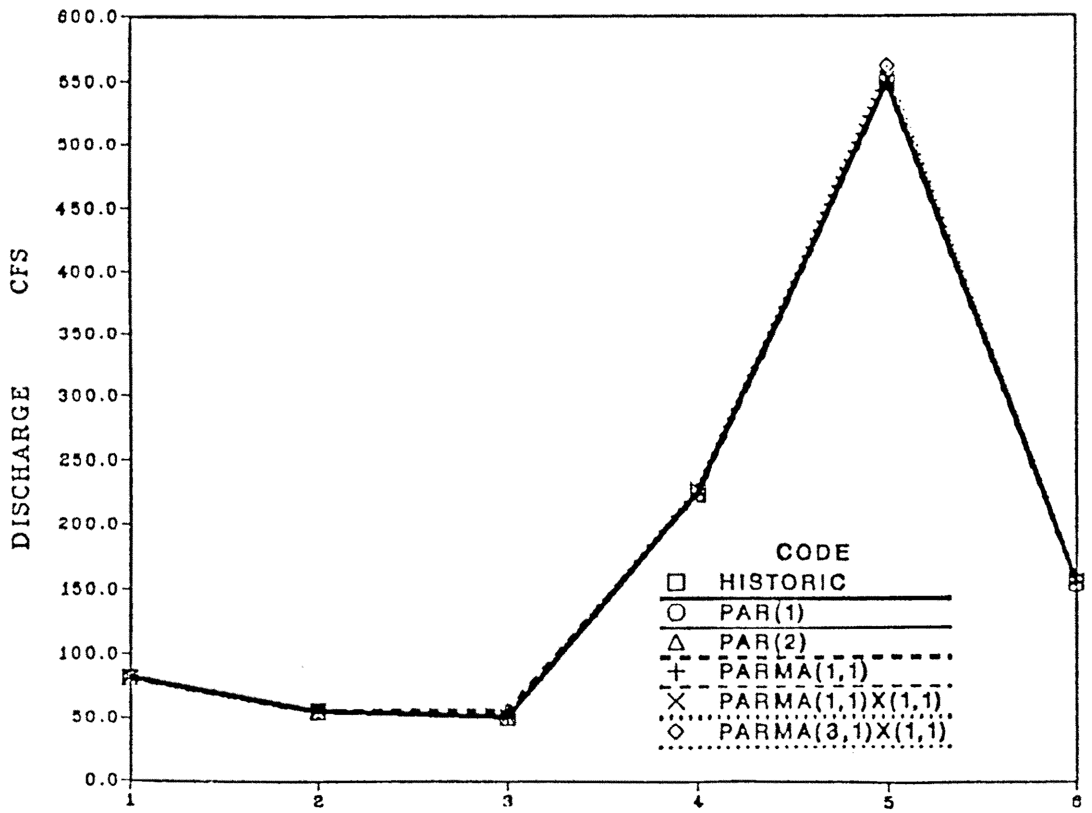


Figure A.33 - Seasonal skewness
Yellowstone River - Four seasons



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Yellowstone River - Six seasons

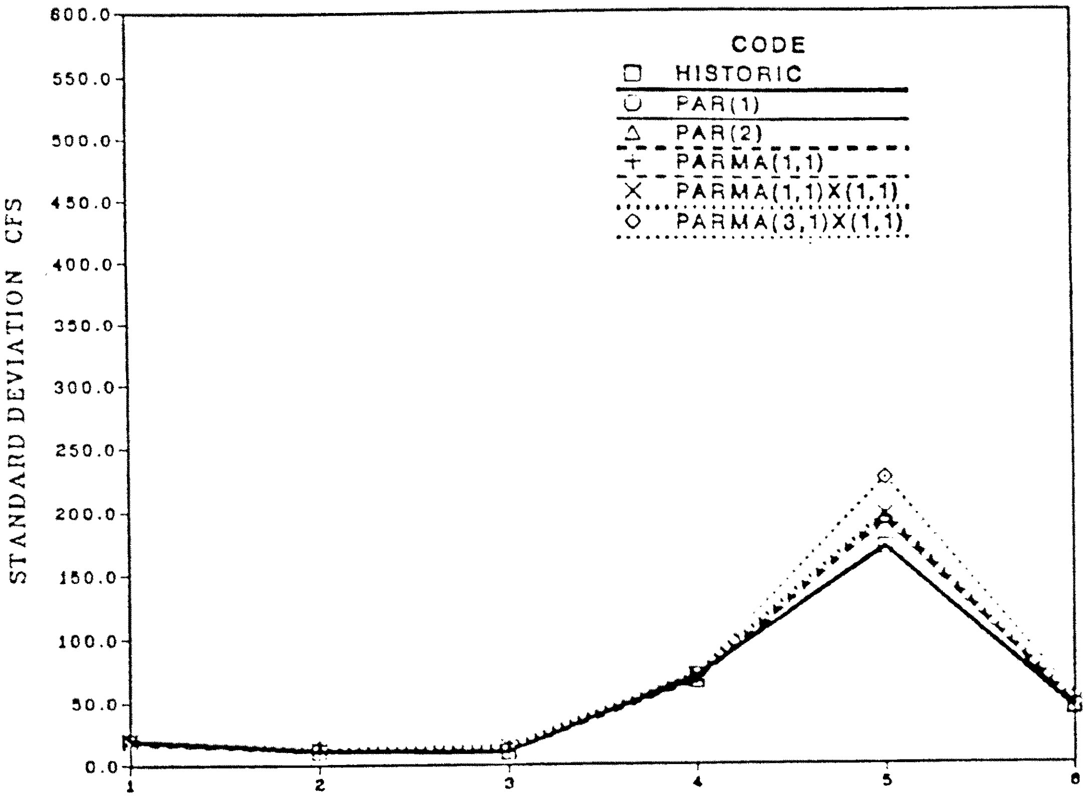

Figure A.35 - Seasonal standard deviation (cfs)
Yellowstone River - Six seasons

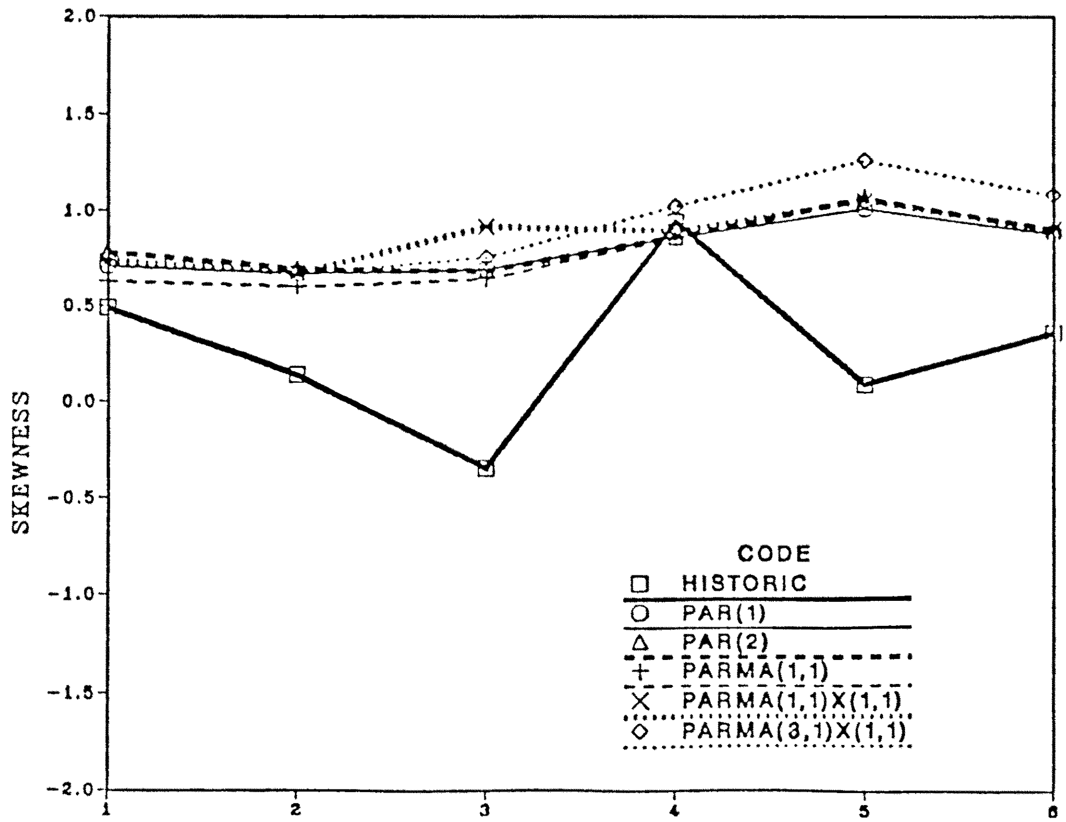

Figure A.36 - Seasonal skewness
Yellowstone River - Six seasons