Dissertation

Low Rank Representations of Matrices using Nuclear Norm Heuristics

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Abstract

Low Rank Representations of Matrices using Nuclear Norm Heuristics

The pursuit of low dimensional structure from high dimensional data leads in many instances to the finding the lowest rank matrix among a parameterized family of matrices. In its most general setting, this problem is NP-hard. Different heuristics have been introduced for approaching the problem. Among them is the nuclear norm heuristic for rank minimization. One aspect of this thesis is the application of the nuclear norm heuristic to the Euclidean distance matrix completion problem. As a special case, the approach is applied to the graph embedding problem. More generally, semi-definite programming, convex optimization, and the nuclear norm heuristic are applied to the graph embedding problem in order to extract invariants such as the chromatic number, $\mathbb{R}^n$-embeddability, and Borsuk-embeddability. In addition, we apply related techniques to decompose a matrix into components which simultaneously minimize a linear combination of the nuclear norm and the spectral norm. In the case when the Euclidean distance matrix is the distance matrix for a complete k-partite graph it is shown that the nuclear norm of the associated positive semidefinite matrix can be evaluated in terms of the second elementary symmetric polynomial evaluated at the partition. We prove that for k-partite graphs the maximum value of the nuclear norm of the associated positive semidefinite matrix is attained in the situation when we have equal number of vertices in each set of the partition. We use this result to determine a lower bound on the chromatic number of the graph. Finally, we describe a convex optimization approach to decomposition of a matrix into two components using the nuclear norm and spectral norm.

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CHAPTER 1

Optimization for Characterizing Data

1.1. Introduction

In many real world application we deal with high dimensional data. In many situations, real life data is highly concentrated on low-dimensional, sparse, or degenerate structures in a high-dimensional space. There is a high interest in learning and exploiting these low-dimensional structures as they allow us to characterize and ease the "visualization" of the high-dimensional data. In some situations, the real life application data, like digital images, video sequences or experimental data often contain missing observations, corruptions, or even malicious errors which adds to the challenge of learning the low dimensional structure. A central goal of dimensionality reduction is to obtain and exploit compact representations of the original data to enable classification, anomaly detection, and higher-level decision making. Many types of high dimensional data can be approached in this way:

Video Surveillance: Given a sequence of surveillance video frames, we would like to identify activities that stand out from the background. Due to the correlation between frames, the low rank component is usually identified with the background and the sparse component is associated with the moving objects [6], [7].

Face Recognition: Images of a human's face can be approximated by a low-dimensional manifold. In many applications, such as face recognition, it is helpful to exploit this low dimensional geometry. Basri and Jacobs, showed that convex, Lambertian objects, images taken under distant illumination lie near an approximately nine-dimensional linear subspace known as the harmonic plane [8]. But due to cast shadows, the images can be viewed as having a low rank component perturbed by errors that are large in magnitude, but sparse in

the spatial domain. The face recognition problem is one of the most active area of research in computer vision, see, for example [9],[10],[11], [12].

Latent Semantic Indexing: The term document matrix of interest in text processing also can be modeled using subspace geometry. In the situation when we are able to decompose a matrix as a sum of a low-rank component and a sparse component then it is possible to characterize common words using a low rank approximation and the unusual or rare words using a sparse component [13].

Ranking and Collaborative Filtering: The problem of anticipating user tastes is of significance for online commerce and advertisement. User rankings for various products, like movies, books, games are collected with the purpose of forecasting future behavior of the customers. Taking into account that sometimes there are incomplete rankings provided by the users, we would like in these situations to be able to predict the preference of any given user with respect to any of the products. This problem is typically cast as a low-rank matrix completion problem [14].

Euclidean Distance Matrix Completion: This problem amounts to finding the missing entries of a matrix where the entries represent the square distances between points in a Euclidean space. The goal is to complete the matrix such that it contains all pairwise squared distances between a collection of points in a Euclidean space. Euclidean distance matrices have applications that cover a large range of fields from wireless sensor network localization [15], [16], [17] to graph realizability [18], [19], [20] and molecular conformation [16].

Maximum clique in a graph, finding a maximum edge biclique in a bipartite graph: The maximum clique problem takes as input an undirected graph and asks for the largest clique (i.e., a subgraph of nodes that are completely interconnected). The maximum-edge biclique

takes as input a bipartite graph and asks for the subgraph that is a complete bipartite graph $K_{m,n}$ that maximizes the product mn. Both problems can be expressed as matrix rank minimization problems and can be approached using convex optimization programs [21].

In this thesis we are concerned with using the convex optimization framework for solving problems related to graph embeddings with constraints. In Chapter 2 we will introduce basic definitions and notations from linear algebra and convex analysis used throughout the thesis and we will illustrate the concept of the convex conjugate of a function through several examples. Also, we will provide a summary of the necessary and sufficient conditions for optimality together with a description of the main step of the central path interior point algorithm for solving semidefinite programs. Semidefinite programs contain important classes of problems as special cases, such as linear and quadratic programming (LP and QP). But also, important applications exist in combinatorial optimization and electrical engineering [22]. Due to work of Nesterov and Nemirovskii [23],[24], Alizadeh [25], [26], Vandenberghe and Boyd [27], and Todd [28] polynomial time solution strategies (interior point methods) have emerged that allow us to efficiently solve semidefinite programs. Nesterov and Nemirovskii have developed a general approach for using interior point methods for solving convex programming problems based on the concept of p-self concordant barrier functions. Convex optimization solvers are now widely available including (CVX, YALMIP, SeDuMI, SDPT3) for solving semidefinite programs. In the examples from this thesis we made use of the CVX solver [29]. CVX is designed and implemented by Michael Grant, with input from Stephen Boyd and Yinyu Ye. CVX is implemented in MATLAB with model specifications constructed using common MATLAB operations and functions and standard MATLAB code can be included with these specifications. Also, it can interface to other solvers, like SDPT3 or SeDuMI solvers.

In general, the minimization of the rank of a matrix i.e. the *Rank Minimization Problem* is a difficult problem to solve, in general, it is known to be computationally intractable (NP-hard) [27]. Different heuristics have been introduced that solve the problem approximately but efficiently.

The nuclear norm heuristic was proposed by Fazel, Hindi and Boyd in [30] and further explained in Fazel's Ph.D thesis [31]. The heuristic is to replace the (non-convex) rank objective with the sum of the singular values of the matrix, which can be proven to be the dual of the spectral norm.

The choice of this heuristic for the rank minimization problem is justified by the fact that the convex envelope of the rank function over a bounded set is the nuclear norm.

Chapter 3 introduces the nuclear norm heuristic and contains a complete and detailed proof of two facts: the nuclear norm is the convex envelope of the rank function over a bounded set and nuclear norm minimization admits a formulation as a semidefinite program. The first result offers a justification for the replacement of the rank function with the nuclear norm. The second result provides the tool to actually solve, in practice, optimization problems using the nuclear norm heuristic via algorithms for semidefinite programs. The information can be found in [30], [31], [32] and [27], [33] and was gathered in this chapter for an easier understanding of the heuristic. The nuclear norm minimization problem admits a formulation as a semidefinite program [30]. Chapter 4 contains an overview of the structure of the positive semidefinite cone of matrices. In Chapter 5 we will introduce the definition, notations and main properties of Euclidean distance matrices as well as the linear transformations that characterize their close connection with the cone of positive semidefinite matrices. Also, we will present the formulation of the Euclidean distance matrix completion problem and its semidefinite program formulation using the nuclear norm heuristic.

In Chapter 6 and Chapter 7 semi-definite programming, convex optimization, and the nuclear norm heuristic are applied to the graph embedding problem in order to extract invariants such as the chromatic number, $\mathbb{R}^n$-embeddability, and Borsuk-embeddability.We show the connection between the entries of a Euclidean distance matrix and the nuclear norm of the matrix in the positive semidefinite cone given by the one to one correspondence between the two cones. In the case the Euclidean distance matrix is the distance matrix for a complete k-partite graph, we prove the nuclear norm of the associated positive semidefinite matrix can be evaluated in terms of the second elementary symmetric polynomial evaluated at the partition. Also, we prove that for k-partite graphs the maximum value of the nuclear norm of the associated positive semidefinite matrix is attained in the situation when we have equal number of vertices in each set of the partition. We use this result to determine a lower bound on the chromatic number of the graph. We use nuclear norm minimization to find low dimensional graph embeddings. We show that it is possible to lower the dimension of the embedding obtained using the nuclear norm heuristic by considering larger graphs and looking at corresponding submatrices. Chapter 8 is dedicated to matrix decompositions, it contains a description of the low rank and sparse matrix decomposition problem presented by [34]. Also, in Chapter 8 we study the decomposition of a full rank matrix into two components resulting from the minimization of a linear combination of the nuclear norm and the spectral norm.

1.2. Convex optimization and data analysis

An optimization problem can be defined through three components: variables, constraints and an objective function. Between 1960 and 1990, optimization problems were analyzed based on linear/non-linear contraposition. Although the simplex method was widely used

for solving linear programs and quite an efficient algorithm in practice, in the worst case, the method has exponential complexity in the size of the problem[35]. As an alternative, Karmarkar's interior point method for linear programming [36], for which he reported polynomial time convergence, generated a high level of interest. Further, it was shown that there is a connection between interior point methods and the barrier methods which were widely used during the 1960s for problems with nonlinear constraints. As a consequence it was understood that barrier methods could be applied to other classes of optimization problems like semidefinite and cone programming. Driven by the large number of applications that can be formulated as a convex optimization problem, extremely efficient interior point methods, capable of handling nonlinear large scale problems with polynomial time complexity results, have emerged [24],[25], [26],[27], [28]. Thus, when looking at optimization problems the boundary shifted to convex optimization problems and non-convex optimization problems: "In fact the great watershed in optimization isn't between linearity and non-linearity but convexity and non-convexity", R.T Rockafellar(1993).

The importance of the convex optimization problems lies also in the fact that the local solutions are global and the set of optimal solutions is a convex set.

DEFINITION 1.2.1. A set $\boldsymbol{C} \subseteq \mathbb{R}^n$ is a *convex* set if for any $x, y \in \boldsymbol{C}$ and any scalar $\lambda \in [0,1]$ we have that $\lambda x + (1-\lambda)\, y \in \boldsymbol{C}$. A convex set $\boldsymbol{C}$ is also a *cone* if it closed under positive linear combinations.

DEFINITION 1.2.2. A real valued function f defined on a convex set $\boldsymbol{C}$ is said to be a *convex* function if for any x, $y \in \boldsymbol{C}$ and any scalar $\lambda \in [0,1]$ we have that

$$f\left(\lambda x + (1-\lambda)\, y\right) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1}$$

DEFINITION 1.2.3. A convex optimization problem is an optimization problem of the form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0,\ i = 1, \ldots, m \\ & h_i(x) = 0,\ i = 1, \ldots, p \end{aligned}$$

where the objective function f_0 and f_i, $i = 1, \ldots, m$ are convex functions and the functions which define the equality constraints h_i are affine.

Convex optimization problems display important properties: the local solutions are global, the set of solutions of a convex optimization problem is a convex set, duality theory and optimality conditions can be derived.

THEOREM 1.2.1. *For convex problems, any locally optimal point is globally optimal and the optimal set i.e the set of feasible points for which the objective function achieves the optimal value is a convex set.*

PROOF. Let x^* be a local minimizer of f_0 on the set X. Let $y \in X$. By definition $x^* \in \text{dom} f_0$. If $f_0(y) = \infty$ the statement is true. Now, if $y \in \text{dom} f_0$ let $x_\theta = \theta y + (1 - \theta)x^*$. The feasible set is convex set because it is the intersection of the domain of the problem $\mathcal{D} = \cap_{i=1}^m \text{dom} f_i$ which is a convex set with m convex sublevel sets $\{x : f_i(x) \leq 0\}$ and p hyperplanes $\{x : h_i(x) = 0\}$. f_0 is convex function. Therefore point $x_\theta = \theta y + (1 - \theta)x \in X$ and

$$f_0(x_\theta) - f_0(x^*) \leq \theta(f_0(y) - f_0(x^*) \tag{2}$$

θ can be chosen such that the point x_θ is in a small neighborhood of x^* and as a consequence the left side of the equation (2) is non-negative and $f_0(y) \geq f_0(x^*)$.

The optimal set is convex since it admits a formulation

$$X^{opt} = \{x \in \mathbb{R}^n : f_0(x) \leq p^*, x \in X\}$$

□

Convex optimization applications are numerous. In control theory, many examples are cataloged in [37]. Among them, to minimize the maximum eigenvalue of a matrix that depends affinely on a variable, minimization of the condition number of a positive semidefinite matrix, minimization of norm by scaling, and matrix completion problems.

The problem of determining the minimum volume ellipsoid that contains given points $x_1, \ldots, x_k \in \mathbb{R}^n$ and also the maximum volume ellipsoid contained in a given convex set can be formulated as convex optimization problems [38].

In circuit design, many examples including signal propagation delay, minimum area subject to bound on delay, minimum power dissipation subject to bound on delay, and minimum delay subject to area and power constraints can be cast as convex optimization problems [38].

Convex optimization has also emerged as an important signal processing tool. For design it is used to choose the weights or algorithm parameter which are then used in a signal processing algorithm. Also, it can be applied to sparse reconstruction of signals [39].

In geometry, convex optimization can be used to construct convex relaxations for Euclidean distance matrix completion problem [40].

In combinatorial optimization, semidefinite programming is used in the case of finding a maximum independent set and a maximum cut in a graph [33]. In finance, [41] applies conic optimization to the sampling error problem in portfolio optimization.

1.3. Contributions of this thesis

One aspect of this thesis is the application of the nuclear norm heuristic to the Euclidean distance matrix completion problem. As a special case, the approach is applied to the graph embedding problem. More generally, semi-definite programming, convex optimization, and the nuclear norm heuristic are applied to the graph embedding problem in order to extract invariants such as the chromatic number, $\mathbb{R}^n$-embeddability, and Borsuk-embeddability. In addition, we apply related techniques to decompose a matrix into components which simultaneously minimize a linear combination of the nuclear norm and the spectral norm. The specific contributions are as follows

- We show the connection between the entries of an Euclidean distance matrix and the nuclear norm of the matrix in the positive semidefinite cone given by the one to one correspondence between the two cones. In the case when the Euclidean distance matrix is the distance matrix for a complete k-partite graph, the nuclear norm of the associated positive semidefinite matrix can be evaluated in terms of the second elementary symmetric polynomial evaluated at the partition.
- We prove that for k-partite graphs the maximum value of the nuclear norm of the associated positive semidefinite matrix is attained in the situation when we have an equal number of vertices in each set of the partition. We use this result to determine a lower bound on the chromatic number of the graph.
- In the particular case of balanced partite graphs with edges of unit size, we can correctly identify the chromatic number in about 30% of the cases and we can find bounds on the chromatic number for the rest of the cases.
- We use nuclear norm minimization to find low dimensional graph embeddings. We show that it is possible to lower the dimension of the embedding obtained using the

nuclear norm heuristic by considering larger graphs and looking at corresponding submatrices.

- We describe a convex optimization approach to decomposition of a matrix into two components using the nuclear norm and spectral norm.

CHAPTER 2

The Convex optimization framework

In this Chapter we will define the concepts and terminology we will use in later sections. The next section will define concepts specific to vectors and matrices.

2.1. Definitions and Notations

We denote by $\mathbb{R}^n$ the set of n-dimensional real vectors. For any $x \in \mathbb{R}^n$ we use x_i to indicate the i^{th} component of the vector.

The space of real n-vectors is equipped with the inner product

$$\langle x, y \rangle = x^t y = \sum_{i=1}^n x_i y_i.$$

Definition 2.1.1 (Vector norms). The Euclidean norm of a vector, or 2-norm, $x = (x_1, \ldots, x_n)$ is defined as

$$\|x\|_2 = (x^t x)^{\frac{1}{2}} = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$$

The Maximum norm or l_∞-norm is defined as

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

and the vector $l_1$norm is

$$\|x\|_{l_1} = \sum_{i=1}^n |x_i|$$

We denote by $\mathbb{R}^{m \times n}$ the space of real m by n matrices. For any matrix A we use A_{ij} to denote its ij^{th} element. The transpose of matrix the A is written as A^t and is a matrix whose entries are given by the relation $A^t_{ij} = a_{ji}$, $j = 1, \ldots, n$, $i = 1, \ldots, m$. For any two matrices A and B of compatible dimensions, the transpose of the product matrix $(AB)^t = B^t A^t$.

Definition 2.1.2. Given a square matrix A is called a symmetric matrix if $A^t = A$.

DEFINITION 2.1.3 (Range, Nullspace). The range space of m by n matrix A, denoted by $\mathbf{R}(A)$ is represented by the set of all vectors $y \in \mathbb{R}^m$ such that $y = Ax$ for some $x \in \mathbb{R}^n$. The nullspace of A, denoted by $\mathbf{N}$(A) comprises all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.

DEFINITION 2.1.4. The *column (row) rank* of a matrix A is the dimension of the range space of A. The rank of A^t is equal to the maximum number of linearly independent column (row) vectors of A.

A matrix and its transpose have the same rank i.e the column rank and the row rank are the same. Also, it can be shown that given two $m \times n$ matrices A and B

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$$

G. Marsaglia and G. P. H. Styan [42] show that equality holds when the row and column spaces of the two matrices intersect only at the origin.

DEFINITION 2.1.5. Inner product on the space of m by n matrices A, $B \in \mathbb{R}^{m \times n}$ is defined as

$$\langle A, B \rangle = \text{trace}(A^t B) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}$$

DEFINITION 2.1.6. The Frobenius norm of a matrix

$$\|A\|_F = \sqrt{\langle A, A \rangle}$$

is also equal to the Euclidean norm of the vector of singular values.

DEFINITION 2.1.7. Spectral norm (operator norm or induced 2-norm) of a matrix is equal to its largest singular value $\|A\| = \sigma_1(A)$.

DEFINITION 2.1.8. The nuclear norm, also known as *trace norm*, *Ky Fan* norm or *Schatten* norm, is defined as

$$\|A\|_* = \sum_{i=1}^{r} \sigma_i(A).$$

The relation between these three norms is characterized by the following inequality

$$\|A\| \leq \|A\|_F \leq \|A\|_* \leq \sqrt{r}\,\|A\|_F \leq r\,\|A\| \tag{3}$$

for any matrix A of rank at most r.

LEMMA 2.1.1 (Fazel, 2007, *The additivity of nuclear norm*). *Let A and B be matrices of the same dimensions. If $AB^t = 0$ and $A^tB = 0$, then*

$$\|A + B\|_* = \|A\|_* + \|B\|_*$$

PROOF. We can express the singular value decomposition of A and B to reflect the zero and non-zero singular vectors

$$A = \begin{bmatrix} U_{A1} & U_{A2} \end{bmatrix} \begin{bmatrix} \Sigma_A & \\ & 0 \end{bmatrix} \begin{bmatrix} V_{A1} & V_{A2} \end{bmatrix}^t$$

and

$$B = \begin{bmatrix} U_{B1} & U_{B2} \end{bmatrix} \begin{bmatrix} \Sigma_B & \\ & 0 \end{bmatrix} \begin{bmatrix} V_{B1} & V_{B2} \end{bmatrix}^t$$

. Since $AB^t = 0$ then $V_{A1}^t V_{B1} = 0$. Similarly $A^tB = 0$ implies $U_{A1}^t U_{B1} = 0$. Hence, there exist matrices U_C and V_C such that $\begin{bmatrix} U_{A1} & U_{B1} & U_C \end{bmatrix}$ and $\begin{bmatrix} V_{A1} & V_{B1} & V_C \end{bmatrix}$ are orthogonal matrices. Thus, the following are valid singular value decompositions for A and B

$$A = \begin{bmatrix} U_{A1} & U_{B1} & U_C \end{bmatrix} \begin{bmatrix} \Sigma_A & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} V_{A1} & V_{B1} & V_C \end{bmatrix}^t$$

$$\text{and } B = \begin{bmatrix} U_{A1} & U_{B1} & U_C \end{bmatrix} \begin{bmatrix} 0 & & \\ & \Sigma_B & \\ & & 0 \end{bmatrix} \begin{bmatrix} V_{A1} & V_{B1} & V_C \end{bmatrix}^t$$

Then the sum of the matrices A and B can be written as

$$A + B = \begin{bmatrix} U_{A1} & U_{B1} \end{bmatrix} \begin{bmatrix} \Sigma_A & \\ & \Sigma_B \end{bmatrix} \begin{bmatrix} V_{A1} & V_{B1} \end{bmatrix}^t$$

This shows that the singular values of $A + B$ are equal to the union (with repetition) of the singular values of A and B leading to the relation

$$\|A + B\|_* = \|A\|_* + \|B\|_*$$

□

2.2. Positive semidefinite matrices

Definition 2.2.1. Let $\mathbb{S}^n$ be the set of real symmetric $n \times n$ matrices. Matrix $A \in \mathbb{S}^n$ is *positive semidefinite (PSD)* matrix if

$$x^T A x \geq 0 \text{ for any } x \in \mathbb{R}^n.$$

Theorem 2.2.1 (Spectral Theorem). *Let A by any $n \times n$ symmetric matrix. There exists a spectral decomposition of A into an orthogonal matrix U ($U^t U = I_n$) and a real diagonal matrix D such that*

$$A = UDU^t$$

Let u_i be the ith column of U and λ_i denote the ith diagonal entry of D. Then $\{u_1, \ldots, u_n\}$ is an orthonormal basis consisting of eigenvectors of A and λ_i is the eigenvalue corresponding to u_i.

The usual notation for a positive semidefinite matrix A is $A \succeq 0$ and the set of positive semidefinite matrices is denoted by $\mathbb{S}^n_+$. If the inequality above is strict, i.e if $x^T Ax > 0$ for all $x \in \mathbb{R}^n, x \neq 0$ then A is called a positive definite matrix.

Also, we introduce some terminology that shall be used later. We write for $\text{diag}(A) \in \mathbb{R}^n$ the vector whose components are the diagonal elements of the matrix A. The trace of a square matrix is given by the sum of the diagonal elements of the matrix i.e $\text{trace}(A) = \sum_i^n a_{ii}$. Important properties of the trace include

$$\text{trace}(AB) = \text{trace}(BA)$$
$$\text{and trace}(A) = \textstyle\sum_{i=1}^n \lambda_i$$

where λ_i, $i = 1, \ldots, n$ are the eigenvalues of the matrix $A \in \mathbb{S}^n$.

The positive semidefinite matrices can be characterized in several ways:

(1) $A \succeq 0$ if an only if $\lambda_{min}(A) \geq 0$, where $\lambda_{min}(A)$ represents the smallest eigenvalue of matrix A.

(2) $A \succeq 0$ if an only if $\lambda_i \geq 0$, $i = 1, \ldots, n$, where λ_is are the eigenvalues of matrix A.

(3) $A \succeq 0$ if and only if $\det(A_{I,I}) \geq 0$ for any $I \subset \{1, \ldots, n\}$, i.e., all principal sub-determinants are non-negative.

(4) $A \succeq 0$ if and only if $A = P^t P$, where $P \in \mathbb{R}^{r \times n}$ (r is the rank of matrix A).

(5) *Schur complement characterization*: Let

$$U = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix}$$

where A and C are symmetric matrices and $A \succ 0$. Then

$$U \succeq 0 \Leftrightarrow C - B^t A^{-1} B \succeq 0$$

Matrix $C - B^t A^{-1} B$ is called the Schur complement of A in U ([43]).

2.3. Basic Convex Analysis results

In this section we present a brief summary of basic definitions and notations from convex analysis. The Conjugacy that characterizes the behavior of the conjugate and biconjugate of a function is stated. We illustrate the definition of the conjugate of a function with two simple examples were included. Our presentation is based on Bertsekas [44].

Definition 2.3.1. A set $\boldsymbol{C} \subseteq \mathbb{R}^n$ is a *convex* set if for any $x, y \in \boldsymbol{C}$ and any scalar $\lambda \in [0,1]$ we have that $\lambda x + (1-\lambda)\, y \in \boldsymbol{C}$. A convex set $\boldsymbol{C}$ is also a *cone* if it closed under positive linear combinations.

Definition 2.3.2. A point of the form $\theta_1 x_1 + ... + \theta_k x_k$, where $\theta_1 + ... + \theta_k = 1$ and $\theta_i \geq 0$, $i = 1, ..., k$ is a *convex combination* of the points $x_1, ..., x_k$.

Definition 2.3.3. The *convex hull* of a set $\boldsymbol{C}$, denoted $\mathbf{conv}\,\boldsymbol{C}$, is the set of all convex combinations of points in $\boldsymbol{C}$

$$\mathbf{conv}\,\boldsymbol{C} = \{\theta_1 x_1 + ... + \theta_k x_k, x_i \in \boldsymbol{C}, \theta_i \geq 0, i = 1, ...k, \theta_1 + ... + \theta_k = 1\}$$

The convex hull $\mathbf{conv}\,\boldsymbol{C}$ is the smallest convex set that contains $\boldsymbol{C}$.
Convexity is preserved under intersection: if S_1 and S_2 are convex, then $S_1 \cap S_2$ is convex. This statement is also valid for the intersection of an infinite number of sets.

Definition 2.3.4. A real valued function f defined on a convex set $\boldsymbol{C}$ is said to be a *convex* function if for any $x, y \in \boldsymbol{C}$ and any scalar $\lambda \in [0,1]$ we have that

$$f\left(\lambda x + (1-\lambda)\, y\right) \leq \lambda f(x) + (1-\lambda) f(y) \tag{4}$$

Definition 2.3.5. The α *sublevel* set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$C_\alpha = \{x \in \mathbf{dom} f, f(x) \leq \alpha\}.$$

Sublevel sets of a convex function are convex, for any value of α, but the converse is false.

Definition 2.3.6. The graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is represented by the set

$$\{(x, f(x)), x \in \mathbf{dom} f\}$$

and is a subset of $\mathbb{R}^{n+1}$.

Definition 2.3.7. The *epigraph* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\mathbf{epi} f = \{(x, t), x \in \mathbf{dom} f, f(x) \leq t\}.$$

A function is convex if and only if its epigraph is a convex set. Also, a function is convex if and only if it is convex when restricted to any line that intersects its domain.

Example 2.3.1 (Convex functions). All linear and affine functions are convex. Every norm on $\mathbb{R}^n$ is convex function.

Among operations that preserve convexity are

- A non-negative weighted sum of convex functions is convex;
- Composition with an affine mapping is convex;
- If f_1,...f_m are convex functions, then their pointwise maximum is convex.

Definition 2.3.8. Let f be an extended real-valued function $f : \mathbb{R}^n \to [-\infty, \infty]$. The function f is called *closed* if its epigraph, **epi**(f) is a closed set.

Definition 2.3.9. The *convex closure* or convex envelope of f is the function that has as epigraph the closure of the convex hull of epi(f).

DEFINITION 2.3.10. For function $f : \mathbb{R}^n \to \mathbb{R}$, its *conjugate function* $f^* : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$f^*(y) = \sup_{x \in \mathbf{dom} f} \left(y^T x - f(x)\right)$$

The domain of the conjugate function contains all vectors $y \in \mathbb{R}^n$ such that the supremum is finite. The conjugate function f^* is a convex function as it is the pointwise supremum of a family of convex functions of y, no matter if f is a convex function or not.

From the definition of conjugate function we obtain the *Fenchel's inequality*

$$f(x) + f^*(y) \geq x^t y$$

We say that function f is proper if there exists at least one vector x such that $f(x) < \infty$ and $f(x) > -\infty$ for all values of x in the domain of f. The closure of the convex hull of the epigraph of f is the epigraph of some function, called the convex closure of f. The following theorem characterizes the relation between the function, its conjugate and its double conjugate.

THEOREM 2.3.1 (Conjugacy Theorem, Bertsekas, Section 1.6). *Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be a function, let f^* be its convex conjugate, and consider the conjugate of f^**

$$f^{**}(x) = sup_{\lambda \in \mathbb{R}^n} \left\{\lambda^T x - f^*(\lambda)\right\},\ x \in \mathbb{R}^n$$

(a) We have $f(x) \geq f^{**}(x)$, for any $x \in \mathbb{R}^n$;

(b) If f is convex, then the properness of any of the functions f, f^* and f^{**} implies the properness of the other two;

(c) If f is closed, proper and convex, then

$$f(x) = f^{**}(x), \text{ for any } x \in \mathbb{R}^n$$

(d) Let $\hat{f}$ be the convex closure of f. If $\hat{f}$ satisfies $\hat{f} \geq -\infty$ for all $x \in \mathbb{R}^n$, then

$$\hat{f} = f^{**}(x).$$

In the case that f is a convex and differentiable function with $\mathbf{dom} f = \mathbb{R}^n$, let f^* be its conjugate function (also called *Fenchel conjugate*) and let x^* be any maximizer of $y^t x - f(x)$. Then we have $y = \nabla f(x^*)$ and we can write the relation

$$f^*(y) = x^* \nabla f(x^*) - f(x^*).$$

To illustrate the notion of conjugacy let's consider some simple examples.

EXAMPLE 2.3.2 (Conjugate and biconjugate of parabola). Let f be a parabola defined on $x \in [-3, 2]$, $f(x) = -x^2$. Note f is a concave function. Then the conjugate is $g(\lambda) = \sup_{x \in [-3,2]} \{\lambda x + x^2\}$. We are interested in the behavior of the function $h(x) = \lambda x + x^2$ on $[-3, 2]$. It is differentiable and its derivative is $h'(x) = \lambda + 2x$. Then

$$g(\lambda) = \sup \left\{ h(-3), h(-\tfrac{\lambda}{2}), h(2) \right\}$$

$$g(\lambda) = \sup \left\{ -3\lambda + 9, h(-\tfrac{\lambda}{2}), 2\lambda + 4 \right\}$$

Therefore $g(\lambda) = -3\lambda + 9$ on $(-\infty, 1]$ and $g(\lambda) = 2\lambda + 4$ on $(1, \infty)$. Once we determine the

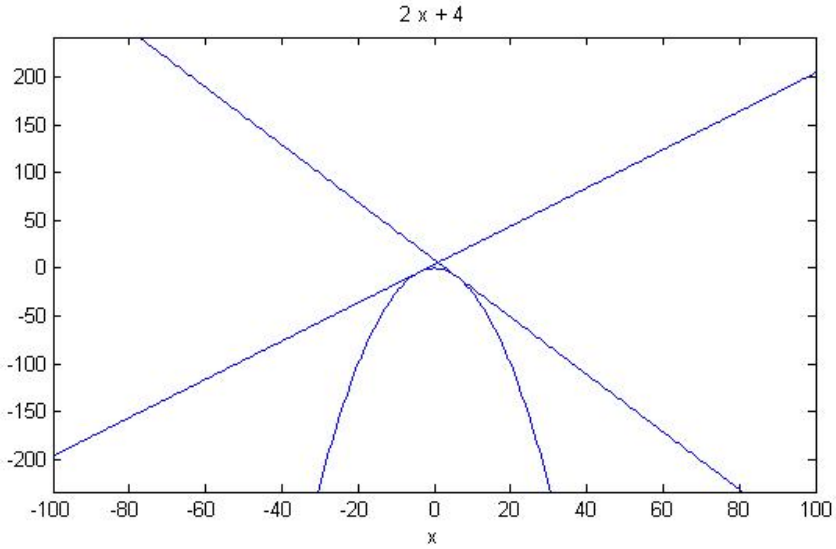

FIGURE 2.1. Conjugate function of concave parabola.

conjugate of f we can follow the same procedure to determine the biconjugate, i.e.,

$$f^{**}(p) = \sup_{\lambda \in R} \{p\lambda - g(\lambda)\}.$$

We have the monotonicity of function $s(\lambda) = p\lambda - g(\lambda)$ depends on the sign of its first derivative. $s^{'}(\lambda) = p + 3$ if $\lambda \in (\infty, 1)$ and $s^{'}(\lambda) = p - 3$ for $\lambda \in (1, \infty)$. When p ranges in $\mathbb{R}$ the supremum of s is finite only for values of $p \in [-3, 2]$ and its value is $p - 6$. Therefore the conjugate of the conjugate, i.e, biconjugate of f is $f^{**}(p) = p - 6$, $p \in [-3, 2]$.

Example 2.3.3. Let's consider now a simple convex function defined as

$$f(x) = x^2,\ x \in [-3, 2].$$

The conjugate of f is determined by

$$g(\lambda) = \sup_{x \in [-3,2]} \{\lambda x - x^2\}.$$

We have $g(\lambda) = -3\lambda - 9$ on $(-\infty, -6)$, $g(\lambda) = \frac{\lambda^2}{4}$ on $[-6, 4]$, $g(\lambda) = 2\lambda - 4$ on $(4, \infty)$
If we use the definition of the biconjugate and we follow a similar procedure as in the previous example, then the conjugate of the conjugate function can be expressed as $f^{**}(p) = p^2$ on $x \in [-3, 2]$ which is expected since the parabola in this case it is a convex function.

2.4. Overview of interior point method for linear programs

As mentioned in the Chapter 1, convex optimization problems are important and occur widely in applications in combinatorial optimization and electrical engineering [22]. Due to work of Nesterov and Nemirovskii [23],[24], Alizadeh [25], [26], Vandenberghe and Boyd [27], and Todd [28] polynomial time solution strategies (interior point methods) have emerged that allow us to efficiently solve semidefinite programs. Nesterov and Nemirovskii developed a general approach for using interior point methods for solving convex programming problems based on the concept of p-selfconcordant barrier functions. Alizadeh takes a specific interior point algorithm for linear programming (Ye's projective potential reduction method [45]) and extends it to semidefinite programs [25]. In what follows we provide a summary of the

necessary and sufficient conditions for optimality and the description of the main step of the central path interior point algorithm for solving semidefinite programs. Our discussion focuses on the primal dual path following interior point method. We begin our discussion by outlining the main ideas of the interior point methods for linear programs.

In linear programming the goal is to maximize (or minimize) a linear function which depends on n variables subject to linear equality and inequality constraints. The importance of linear programs is significant since many problems in the scientific world and economics can be cast as a linear program. Consequently, there has been a lot of interest in developing methods for solving linear programs. Simplex method and interior point methods are the main tools to solve linear programs. In case of simplex method the combinatorial analysis indicates it could be quite inefficient for some problems. In their paper [35] Klee and Minty show that, in the worst case, the method has exponential complexity in the size of the problem.

Interior point methods use Newton's method for solving non-linear equations, Lagrange's method of optimization with equality constraints, and Fiacco and McCormick's barrier method (1968) for optimization with inequality constraints.

Newton's method is used to compute the zero of a function $f(x) = 0$. In the case when x is a single variable, to find the zero of the function we can follow the steps: given an initial estimate x_0, a sequence is computed $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ for $k = 0, 1, 2, \ldots$. The stopping criteria is $|f(x_k)| < \epsilon$.

If x is a n-dimensional vector and f is a function $f : \mathbb{R}^n \to \mathbb{R}^n$, then the Jacobian is given by the expression $J(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)$ and the Newton's step which leads to finding the zero of the function is $x_{k+1} = x_k - [J(x_k)]^{-1} f(x_k)$.

Lagrange's method offers transforms a constrained optimization problem with equality constraints into a unconstrained problem. Thus, to solve the problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) = 0,\ i = 1, 2, \ldots, m. \end{aligned}$$

one forms a Lagrangian function

$$L(x, y) = f(x) - \textstyle\sum_{i=1}^{m} y_i g_i(x)$$

To minimize the function L we solve a system of $n + m$ equations in $n + m$ variables:

$$\tfrac{\partial L}{\partial x_j} = \tfrac{\partial f}{\partial x_j}(x) - \textstyle\sum_{i=1}^{m} y_i \tfrac{\partial g_i}{\partial x_j}(x) = 0, \text{ for } j = 1, \ldots, n$$

and

$$\tfrac{\partial L}{\partial y_i} = -g_i(x) = 0, \text{ for } i = 1, \ldots, m.$$

Newton's method can now be applied to solve these equations.

Fiacco and McCormick [46] indicate how to deal with inequality constraints. Consider a linear program in standard form

$$\text{minimize } c^t x, \text{ subject to } Ax = b,\ x \geq 0$$

where x is the vector of variables and matrix A and vector b are known. In case the linear programs have inequality constraints these can be converted into equations by adding non-negative slack variables. The inequalities that remain are the non-negativity conditions $x \geq 0$. The idea of the barrier function is to start from a point in the strict interior of the inequalities ($x_j^0 > 0$), for all j and construct a barrier that prevents any variable from reaching the boundary ($x_j^0 = 0$). One of the functions used in order to make the objective function to increase without bound as x_j approaches 0 is the function $-\log(x_j)$. But, since

the constrained optimum lies on the boundary then we have to use a barrier parameter to balance the contribution of the true objective function against that of the barrier function.

Therefore a minimization problem with non-negativity conditions can be written into a sequence of unconstrained minimization problems

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \geq 0. \end{aligned}$$

and replaced by

$$\text{minimize } B(x, \mu)$$

where $B(x, \mu) = f(x) - \mu \sum_{j=1}^{n} log(x_j)$. Fiacco and McCormick (1968) showed that the minimizer $x(\mu) \to x^*$, where x^* is the constrained minimizer, as $\mu \to 0$.

The main steps of the interior point algorithm for linear programs can be summarized as follows [47]:

(1) Choose $\mu_0 > 0$, $k = 0$;

(2) Find element $x_k(\mu_k)$, the minimizer of $B(x, \mu_k)$ using Newton's method;

(3) If $\mu_k < \epsilon$, stop. Otherwise choose $\mu_{k+1} < \mu_k$;

(4) Set $k = k + 1$ and go to step 2.

2.5. Semidefinite Programs

In *semidefinite programming* the space $\mathbb{R}^n$ gets replaced by the space of symmetric matrices, the inner product of two vectors is replaced by the inner product between two matrices. Let's denote standard inner product between two matrices with $X \bullet Y = \text{trace}\,(X^t Y)$. Also, instead of condition that a variable should be positive, it gets replaced with the condition that matrix X *positive semidefinite* ($X \succeq 0$).

Standard references for semidefinite programming are Nesterov and Nemirovskii [24] and Vandenberghe and Boyd [27] which also contains a large collection of examples and applications.

Several authors have discussed generalizations of interior-point algorithms for linear programming to the context of semidefinite programs. The landmark work in this direction is due to Nesterov and Nemirovskii [24] where a general approach for using interior-point methods for solving convex programs is proposed based on the notion of self-concordant functions. They show that the problem of minimizing a linear function over a convex set K can be solved in "polynomial time" as long as a selfconcordant barrier function for K is known. In particular, Nesterov and Nemirovskii show that linear programs, convex quadratic programs with convex quadratic constraints, and semidefinite programs all have explicit and easily computable selfconcordant functions, and hence can be solved in "'polynomial time".

Further we will state the strong duality result for semidefinite programs. The information in this section follows [33] which considers the standard form of the semidefinite program the maximization of the linear function. Other authors [27] consider the semidefinite programs in standard form given by the minimization of a linear function. The change from one form to the other can be easily done $\max f(x) = \min(-f(x))$.

2.6. Semidefinite programs and their dual

A semidefinite program can be written in standard form as:

$$\begin{aligned} &\text{maximize } \langle C, X\rangle \\ &\text{subject to } \langle A_1, X\rangle = b_1 \\ &\qquad\qquad \langle A_2, X\rangle = b_2 \\ &\qquad\qquad \dots \\ &\qquad\qquad \langle A_m, X\rangle = b_m, \\ &\qquad\qquad X \succeq 0 \end{aligned}$$

where C and A_k, $k = 1, \dots, m$ are symmetric matrices. A matrix X is called a feasible solution for the program if it is a symmetric matrix that verifies the constraints. The system of m linear constraints can be written as only one relation $\mathbb{A}(X) = b$ if we let $b = (b_1, \dots, b_m)$ and $\mathbb{A} : \mathbb{S}^n \to \mathbb{R}^m$, where $\mathbb{A}(X) = (A_1 \bullet X, \dots, A_m \bullet X)$. With these notations we obtain the formulation of the semidefinite program in equational form

$$\begin{aligned} &\underset{X}{\text{maximize}}\ C \bullet X \\ &\text{subject to } \mathbb{A}(X) = b \\ &\qquad\qquad X \succeq 0 \end{aligned}$$

An optimal solution is a feasible solution X^* such that $C \bullet X^* \geq C \bullet X$ for all feasible solutions X. We now include the theorem for strong duality for semidefinite programming without proof.

THEOREM 2.6.1 (Strong duality for semidefinite programming). *If the semidefinite program*

$$\underset{X}{\text{maximize}}\ C \bullet X$$

$$\text{subject to } \mathbb{A}(X) = b$$

$$X \succeq 0$$

is feasible and has a finite value γ *and if there is a positive definite matrix* $\tilde{X}$ *such that* $\mathbb{A}(\tilde{X}) = b$*, then the dual program*

$$\underset{y}{\text{minimize}}\ b^t y$$

$$\text{subject to } \sum_{i=1}^{m} y_i A_i - C \succeq 0$$

is feasible and has finite value $\beta = \gamma$*.*

For the proof please see [33].

2.7. PRIMAL DUAL CENTRAL PATH ALGORITHM FOR SEMIDEFINITE PROGRAMS

In this section we state the existence and the uniqueness of the central path under suitable conditions followed by a description of the algorithm main step.

As usual consider a semidefinite program in equational form

$$\begin{aligned} \text{maximize} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i, i = 1 \dots m \\ & X \succeq 0 \end{aligned}$$

where matrices C and A_i are symmetric matrices.

In central path interior point methods the constraint $X \succeq 0$ is dropped by modifying the objective function. In this case, it is added a barrier function such that the objective function tends to $-\infty$ as we approach the boundary of the set of positive semidefinite matrices $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n, X \geq 0\}$.

Let $\mu > 0$. We consider the auxiliary problem

$$\begin{aligned} \underset{X}{\text{maximize}} \quad & f_\mu(X) = C \bullet X + \mu \text{lndet} X \\ \text{subject to} \quad & A_i \bullet X = b_i, i = 1, \dots m \\ & X \succ 0. \end{aligned} \tag{5}$$

Uniqueness of the solution: If optimization problem (5) has an optimal solution, then it has a unique optimal solution $X^*(\mu)$. This follows from the fact that f_μ is strictly concave over the interior of $\mathbb{S}^n$. To obtain necessary conditions for optimality we use the method of Lagrange multipliers.

LEMMA 2.7.1. *If $X^*(\mu) > 0$ is the optimal solution of the optimization problem (5), then there is a vector $\tilde{y} \in R^m$ such that $X^*(\mu)$ and $\tilde{y}$ satisfy the equations*

$$A_i \bullet X = b_i, i = 1, 2 \dots m \tag{6}$$

$$C + \mu X^{-1} = \sum_{i=1}^{m} y_i A_i \tag{7}$$

The necessary conditions for optimality result from the method of Lagrange multipliers applied to function f_μ, the constraints $g_i(X) = A_i \bullet X - b_i$ and to linear constraints that impose the symmetry of variable X, $g_{ij}(X) = x_{ij} - x_{ji}$, $i, j = 1, 2, \dots, n$.

Equation (6) is a consequence of the fact that $X^*(\mu)$ is feasible.

The matrix $X \in \mathbb{R}^{m\times n}$ is positive definite and $\det(X) > 0$, $\nabla \text{lndet} X = (X^T)^{-1}$. Now (7) can be derived from the condition involving the Lagrange multipliers. A complete proof is found in [33].

Introducing a new variable $S = \sum_{i=1}^{m} y_i A_i - C = \mu X^{-1}$ where S symmetric matrix it follows that $X^*(\mu)$ satisfies the following *Lagrange system* for $y \in \mathbb{R}^m$ and $S = \sum_{i=1}^{m} y_i A_i - C = \mu X^{-1}$.

$$
\begin{aligned}
& A_i \bullet X = b_i, i = 1, \dots m \\
& \sum_{i=1}^{m} y_i A_i - S = C \\
& SX = \mu I_n \\
& S, X \succ 0.
\end{aligned}
\tag{8}
$$

2.7.1. A Primal Dual Interpretation. The equations (8) provide a primal feasible solution and a dual feasible solution. The *duality gap* represents the difference between dual and primal objective function value.

Lemma 2.7.2. *If $\tilde{X}$, $\tilde{S} \in \mathbb{R}^{n\times n}$, $\tilde{y} \in \mathbb{R}^m$ satisfy the Lagrange equations (8) for some $\mu > 0$, then the following statements hold:*

(i) The matrix $\tilde{X}$ is a strictly feasible solution ($X \succ 0$) of the primal semidefinite program

$$
\begin{aligned}
& \textit{maximize} \quad C \bullet X \\
& \textit{subject to} \quad A_i \bullet X = b_i, i = 1 \dots m \\
& X \succeq 0
\end{aligned}
\tag{9}
$$

(ii) The vector $\tilde{y}$ is a strictly feasible solution of the dual semidefinite program

$$\begin{aligned} &\textit{minimize} \quad b^T y \\ &\textit{subject to} \quad \sum_{i=1}^{m} y_i A_i - C \succeq 0 \end{aligned} \tag{10}$$

where strict feasibility means that $\sum_{i=1}^{m} \tilde{y}_i A_i - C \succ 0$.

(iii) The duality gap satisfies $b^T \tilde{y} - C \bullet \tilde{X} = n\mu$

PROOF. We know that $\tilde{X}$, $\tilde{S}$ satisfy the Lagrange system, then $\tilde{X}, \tilde{S} \succ 0$ and we obtain that $\tilde{X}$ is strictly feasible for the primal and $\tilde{y}$ is strictly feasible for the dual. For the third statement we will use the linearity of $\bullet$ in the first argument and $\tilde{S}\tilde{X} = \mu I_n$:

$$C \bullet \tilde{X} = \left(\sum_{i=1}^{m} \tilde{y}_i A_i - \tilde{S} \right) \bullet \tilde{X}$$

from which it follows

$$C \bullet \tilde{X} = \sum_{i=1}^{m} \tilde{y}_i \left(A_i \bullet \tilde{X} \right) - \tilde{S} \bullet \tilde{X}$$

and then we have

$$C \bullet \tilde{X} = \sum_{i=1}^{m} \tilde{y}_i b_i - \tilde{S} \bullet \tilde{X}$$

or

$$C \bullet \tilde{X} = \sum_{i=1}^{m} \tilde{y}_i b_i - \text{trace} \left(\tilde{S}\tilde{X} \right).$$

Taken into account the relation between S and X we obtain

$$C \bullet \tilde{X} = b^T \tilde{y} - n\mu.$$

□

We see that if we could compute $X(\mu)$ for a small value of μ then we would have an almost optimal solution of the semidefinite program. Sufficient conditions on the semidefinite program under which the Lagrange system is uniquely solvable and yields a maximum of f_μ are given by following Lemma

LEMMA 2.7.3. *Suppose that both the primal program (9) and the dual program (10) have strictly feasible solutions $\tilde{X}$ and $\tilde{y}$, respectively, and that the matrices A_i, $i = 1, \ldots, m$ are linearly independent as elements of the vector space SYM_n.*

Then for every $\mu > 0$, the Lagrange system (8) has a unique solution $X^* = X^*(\mu)$, $y^* = y^*(\mu)$, $S^* = S^*(\mu)$. Moreover, $X^*(\mu)$ is the unique maximizer of f_μ subject to $A_i \bullet X = b_i$, $i = 1, \ldots m$ and $X \succ 0$.

A complete proof of the Lemma can be found in [33].

2.7.2. CENTRAL PATH STEP. The Lagrange system (8) can be solved for small values of the parameter μ with the purpose to obtain the primal and dual solutions. The *primal-dual central path* of the semidefinite program (9) is defined as the set

$$\left\{(X^*(\mu), y^*(\mu), S^*(\mu)) \in \mathbb{S}^n_+ \times \mathbb{R}^m \times \mathbb{S}^n_+ : \mu > 0\right\}$$

If we consider a fixed value of μ then the *central path function* F defined below captures the deviation of a given triple (X, y, S) from the central path

$$F : \mathbb{S}^n \times \mathbb{R}^m \times \mathbb{S}^n \to \mathbb{R}^m \times \mathbb{S}^n \times \mathbb{S}^n$$

$$F(X, y, S) = \begin{bmatrix} P(X, y, S) \\ Q(X, y, S) \\ R(X, y, S) \end{bmatrix}$$

where P, Q and R are defined as follows

$$P\left(X,y,S\right)=\begin{bmatrix} A_1\bullet X-b_1 \\ A_1\bullet X-b_1 \\ \vdots \\ A_m\bullet X-b_m \end{bmatrix},$$

and

$$Q(X,y,S)=\sum_{i=1}^{m}y_iA_i-S-C$$

and

$$R(X,y,S)=SX-\mu I_n.$$

But we know that $F\left(X^*(\mu),y^*(\mu),S^*(\mu)\right)=(0,0,0)$. We would like to compute this zero for small μ, in order to obtain almost optimal solutions for the primal and dual programs. For this we use the Newton's method. We ignore the symmetry of X and S and write $F:\mathbb{R}^{2n^2+m}\rightarrow\mathbb{R}^{2n^2+m}$. Let's assume that we are at the step k and we denote $X^{(k)}=\tilde{X}$, $y^{(k)}=\tilde{y}$ and $S^{(k)}=\tilde{S}$. The goal is to compute the next iterate $X^{(k+1)}=\tilde{X}'$, $y^{(k+1)}=\tilde{y}'$ and $S^{(k+1)}=\tilde{S}'$.

As usual, let

$$\Delta X=\tilde{X}'-\tilde{X},\ \Delta y=\tilde{y}'-\tilde{y},\ \Delta S=\tilde{S}'-\tilde{S}$$

Then we have

$$DF(\tilde{X},\tilde{y},\tilde{S})\begin{bmatrix}\Delta X\\ \Delta y\\ \Delta S\end{bmatrix}=-F(\tilde{X},\tilde{y},\tilde{S})=\begin{bmatrix}0\\ 0\\ \mu I_n-\tilde{S}\tilde{X}\end{bmatrix}$$

Again, taking into account that function F is defined in terms of P, Q and R the matrix $DF(X,y,S)$ has the following block structure

$$DF(X,y,S) = \begin{bmatrix} DP_{y,S}(X) & 0 & 0 \\ 0 & DQ_{X,S}(y) & DQ_{X,S}(S) \\ DR_{y,S}(X) & 0 & DR_{X,y}(S) \end{bmatrix}$$

where the subscripts mean that the corresponding arguments are fixed. The functions that we need to differentiate in the blocks are linear functions and their Jacobian is equal with the function itself. Therefore

$$DP_{y,S}(X)(\Delta X) = \begin{bmatrix} A_1 \bullet \Delta X \\ A_2 \bullet \Delta X \\ \vdots \\ A_m \bullet \Delta X \end{bmatrix}$$

$$DQ_{X,S}(y)(\Delta y) + DQ_{X,y}(S)(\Delta S) = \sum_{i=1}^{m} (\Delta y)_i A_i - \Delta S$$

$$DR_{y,S}(X)(\Delta X) + DR_{X,y}(S)(\Delta S) = S(\Delta X) + (\Delta S)X$$

We solve for ΔX, Δy and ΔS and we obtain the following system of equations:

$$A_i \bullet (\Delta X) = 0 \tag{11}$$

$$\sum_{i=1}^{m} (\Delta y)_i A_i - (\Delta S) = 0 \tag{12}$$

$$\tilde{S}(\Delta X) + (\Delta S)\tilde{X} = \mu I_n - \tilde{S}\tilde{X} \tag{13}$$

This system has a unique solution [33].

In our search for the next iterate of the Newton method we need a valid next iterate which means that ΔX has to be symmetric. One choice that we have is to allow

$$\tilde{X}' = \tilde{X} + \tfrac{1}{2}(\Delta X + (\Delta X)^T)$$

In the paper of [48] it is shown that this *modified Newton step* also leads to theoretical convergence and good practical performance.

In order to start the algorithm we need a initial point. The initial point should be selected in such way that we start sufficiently close to the central path. In case of the primal-dual interior point algorithm, we need to embed the primal problem and the dual problem into a "larger" semidefinite program for which an interior point is readily available. Solving this larger problem using the algorithm also yields approximately optimal primal and dual solutions for the original problem. This topic is treated in [49].

To solve the system, we keep the equations (11) and (12) and equation (13) gets replaced with

$$\tilde{X}^{\frac{-1}{2}}(\tilde{X}\Delta S + \Delta X\tilde{S})\tilde{X}^{\frac{1}{2}} + \tilde{X}^{\frac{1}{2}}(\Delta S\tilde{X} + \tilde{S}\Delta X)\tilde{X}^{\frac{-1}{2}} = 2(\mu I_n - \tilde{X}^{\frac{1}{2}}\tilde{S}\tilde{X}^{\frac{1}{2}}) \tag{14}$$

where $\tilde{X}^{\frac{1}{2}}$ is the square root of $\tilde{X}$, the unique positive definite matrix whose square is $\tilde{X}$.

In this setting, a generic step of the *short step path following algorithm* displays as follows:

(1) Given the current iterate $X^{(k)}$, $y^{(k)}$, $S^{(k)}$ set

$$\mu_k = \frac{S^{(k)} \bullet X^{(k)}}{n}$$

If $X^{(k)}$, $y^{(k)}$, $S^{(k)}$ is on the central path then $X^{(k)} = X^*(\mu_i)$

(2) Perform one step of Newton method, where $\mu = \sigma\mu_k$ and $\sigma = 1 - \frac{0.3}{\sqrt{n}}$ is the *centrality parameter*. This means to solve the Equations (11), (12), (14) and set

$$X^{(k+1)} = X^{(k)} + \Delta X$$

$$y^{(k+1)} = y^{(k)} + \Delta y$$

$$S^{(k+1)} = S^{(k)} + \Delta S$$

to obtain the next iteration.

CHAPTER 3

Rank Minimization Problem

3.1. Introduction

This Chapter introduces the nuclear norm heuristic. It contains a complete and detailed proof of two results: the nuclear norm is the convex envelope of the rank function over a bounded set and the nuclear norm minimization admits a formulation as a semidefinite program. The first result offers a justification for the replacement of the rank function with the nuclear norm in optimization problems. The second result provides a tool to actually solve, in practice, optimization problems using the nuclear norm heuristic (i.e. using algorithms for semidefinite programming). Details for this chapter can be found in [30], [31], [32] and [27], [33]. Main results have been gathered in this chapter for an easier understanding of the heuristic.

The minimization of the rank of a matrix over a convex set is often encountered in cases where there is a need to look for a low dimensional matrix in high dimensional data. It is also called the **Rank Minimization Problem**. It can be formulated as

$$\begin{aligned} &\underset{X}{\text{minimize}} && \operatorname{rank}(X) \\ &\text{subject to} && X \in C \end{aligned} \tag{15}$$

where $X \in \mathbb{R}^{m \times n}$ is an optimization variable and C is a convex set. Probably the most known example of a rank minimization problem is approximating a given matrix with a low-rank matrix in the spectral or Frobenius norm. It can be solved via the singular value decomposition (SVD). In general, the **Rank Minimization Problem** problem is difficult

to solve. In its general formulation, it is known to be computationally intractable (NP-hard) [27]. Different heuristics lead to efficient but only approximate solutions to the Rank Minimization Problem. In [30] and [31] it is introduced the nuclear norm heuristic.

3.2. Nuclear norm heuristic

When a matrix X is a *positive semidefinite* matrix, the heuristic usually used for the Rank Minimization Problem is to ***replace the rank with the trace***. And consequently, to solve the following optimization problem

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & \text{trace}(X) \\ \text{subject to} \quad & X \in C, \\ & X \succeq 0. \end{aligned} \tag{16}$$

The replacement is motivated by the fact that for a positive semidefinite matrix the eigenvalues are non-negative. Thus, trace minimization is the same as the minimization in the l_1 norm of the vector of eigenvalues.

Minimization in the l_1 norm encourages many of the eigenvalues to be zero and the resulting matrix to be low-rank. The use of the l_1 norm instead of the l_0 norm of a vector is another heuristic used often in convex optimization. The l_0 norm of a vector is given by the number of nonzero components of the vector.

The problem of minimizing the number of nonzero elements of a vector x (subject to some constraints in x) arises in many different fields. However, except in very special cases, it is a very difficult problem to solve numerically. Instead, the l_1 norm of a vector is used as a proxy. The approach considered is to minimize the l_1 norm of x instead of minimizing its nonzero entries, where the l_1 norm of x is defined as $\|x\|_{l_1} = |x_1| + ... + |x_n|$.

For example, compressive sensing (CS) has been one of the hot topics in the signal processing and optimization communities. In CS theory, E. Candes in [50],[51] and Bruckstein, Donoho and Elad in [52] show that the minimum l_1 norm solution to an underdetermined system of linear equations is also the sparsest possible solution under quite general conditions. More specifically, suppose there exists an unknown signal $x_0 \in \mathbb{R}^n$, a measurement vector $b \in \mathbb{R}^d$ such that $(d < n)$, and a measurement matrix $A \in \mathbb{R}^{d \times n}$ such that A is full rank and $b = Ax_0$. Recovering x_0 given A and b, constitutes a non-trivial linear inversion problem, since the number of measurements in b is smaller than the number of unknowns in x_0. A conventional solution to this problem is the linear least squares, which finds the minimum l_2 norm solution (or the solution of least energy) to this system. However, if x_0 is sufficiently sparse and the sensing matrix A is incoherent with the basis under which x_0 is sparse, then x_0 can be exactly recovered by computing the minimum l_1 norm solution, as given by the following optimization problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \|x\|_{l_1} \\ \text{subject to} \quad & Ax = b \end{aligned}$$

The replacement of the rank with the trace can be applied only in the case where the matrix is positive semidefinite. The extension to matrices that are not positive semidefinite and also to non square matrices is made using the nuclear norm of the matrix. The nuclear norm heuristic for the Rank Minimization Problem reduces to solving

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & \|X\|_* \\ \text{subject to} \quad & X \in C \end{aligned} \tag{17}$$

The relation between the nuclear norm of a matrix and the rank of the matrix is further explained using the notion of conjugate function and convex envelope from convex analysis.

3.3. Convex Envelope of Matrix Rank

This section presents the biconjugate for the matrix rank function. It will be shown how the dual of the spectral norm is the convex envelope of the rank function on the set of matrices with norm less than one. The derivation follows [30], [31] and relies on the convex analysis and linear algebra results presented in the previous sections.

Theorem 3.3.1 (Fazel, 2002,[31]). *On the set* $S = \{X \in R^{m\times n}, \|X\| \leq 1\}$, *the convex envelope of function* $\phi(X) = rank(X)$ *is*

$$\phi_{env}(X) = \|X\|_* = \sum_{i=1}^{min\{m,n\}} \sigma_i(X) \tag{18}$$

Proof. **Step 1: Determine conjugate of the rank function**.

According to the definition of the conjugate function we have

$$\phi^*(Y) = \sup_{\|X\|\leq 1} (\text{trace}Y^t X - \phi(X))$$

Let $q = min\{m, n\}$. According to the Von Neumann trace theorem we have that

$$\text{trace}(Y^T X) \leq \textstyle\sum_{i=1}^{q} \sigma_i(Y)\sigma_i(X)$$

If we let $X = U_X \Sigma_X V_X^t$ and $Y = U_Y \Sigma_Y V_Y^t$ then in the relation above equality holds when choosing

$$U_X = U_Y,\ V_X = V_Y.$$

Function $\phi(X) = \text{rank}(X)$ is independent of U_X, V_X. Consider $U_X = U_Y$, $V_X = V_Y$ and then we can apply the Von Neumann trace theorem. Thus, the conjugate function of the matrix rank can be expressed as

$$\phi^*(Y) = \sup_{\|X\| \leq 1} (\sum_{i=1}^{q} \sigma_i(Y)\sigma_i(X) - \text{rank}(X))$$

In case X=0 then $\phi^*(Y) = 0$. For the particular case when $\text{rank}(X) = r$ the convex conjugate is given by

$$\phi^*(Y) = \sum_{i=1}^{r} \sigma_i(Y) - r.$$

Therefore we can express the conjugate of the matrix rank function as

$$\phi^*(Y) = \max\{0, \sigma_1(Y) - 1, .., \sum_{i=1}^{r} \sigma_i(Y) - r, .., \sum_{i=1}^{q} \sigma_i(Y) - q\}$$

In the set above the largest term is the one that sums all positive terms $\sigma_i(Y) - 1$. Therefore

$$\phi^*(Y) = 0 \text{ if } \|Y\| \leq 1,$$

$$\phi^*(Y) = \sum_{i=1}^{r} \sigma_i(Y) - r, \sigma_r(Y) > 1 \text{ and } \sigma_{r+1}(Y) \leq 1$$

or

$$\phi^*(Y) = \sum_{i=1}^{q} (\sigma_i(Y) - 1)_+$$

Step 2: Determine conjugate of the conjugate of rank function

To determine the biconjugate function we apply again the definition

$$\phi^{**}(Z) = \sup_{Y} (\text{trace} Z^t Y - \phi^*(Y)$$

Now choose $U_Y = U_Z$, $V_Y = V_Z$ and the biconjugate function is

$$\phi^{**}(Z) = \sup_{Y} (\sum_{i=1}^{q} \sigma_i(Z)\sigma_i(Y) - \phi^*(Y))$$

$$\phi^{**}(Z) = \sup_Y \left(\sum_{i=1}^{q} \sigma_i(Z)\sigma_i(Y) - \phi^*(Y)\right)$$

If $\|Z\| > 1$ then $\sigma_1(Y)$ can be chosen large enough such that $\phi^{**}(Z) \to \infty$ because the coefficient of $\sigma_1(Y)$ is positive

$$\phi^{**}(Z) = \sup_Y \left(\sum_{i=1}^{q} \sigma_i(Z)\sigma_i(Y) - \left(\sum_{i=1}^{r} \sigma_i(Y) - r\right)\right)$$

Let $\|Z\| \leq 1$. If $\|Y\| \leq 1$ then $\phi^*(Y) = 0$ and the supremum is

$$\phi^{**}(Z) = \sum_{i=1}^{q} \sigma_i(Z) = \|Z\|_*$$

If $\|Y\| > 1$ the expression above can be re-written as:

$$\phi^{**}(Z) = \sum_{i=1}^{q} \sigma_i(Y)\sigma_i(Z) - \sum_{i=1}^{r} \left(\sigma_i(Y) - 1\right)$$

Adding and subtracting the term $\sum_{i=1}^{q} \sigma_i(Z)$ and grouping the terms in a convenient way we obtain

$$\phi^{**}(Z) = \sum_{i=1}^{q} \sigma_i(Y)\sigma_i(Z) - \sum_{i=1}^{r} \left(\sigma_i(Y) - 1\right) - \sum_{i=1}^{q} \sigma_i(Z) + \sum_{i=1}^{q} \sigma_i(Z)$$

$$\phi^{**}(Z) = \sum_{i=1}^{r} (\sigma_i(Y) - 1)(\sigma_i(Z) - 1) + \sum_{i=r+1}^{q} (\sigma_i(Y) - 1)\sigma_i(Z) + \sum_{i=1}^{q} \sigma_i(Z)$$

which leads to

$$\phi^{**}(Z) < \sum_{i=1}^{q} \sigma_i(Z)$$

Therefore

$$\phi^{**}(Z) = \|Z\|_*$$

over the set $\{Z; \|Z\| \leq 1\}$. □

3.4. Nuclear norm heuristic as a semidefinite program

This section explains how the minimization of the nuclear norm can be formulated as a semidefinite program. In fact, solving the nuclear norm minimization problem

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & \|X\|_* \\ \text{subject to} \quad & X \in C \end{aligned}$$

leads to solving the semidefinite program

$$\begin{aligned} \underset{W_1,W_2}{\text{minimize}} \quad & \frac{1}{2}(\text{trace}(W_1) + \text{trace}(W_2)) \\ \text{subject to} \quad & \begin{bmatrix} W_1 & X \\ X^T & W_2 \end{bmatrix} \succeq 0 \\ & X \in C \end{aligned} \tag{19}$$

For any given norm $\|\|$ in an inner product space, there exists a dual norm $\|\|_d$ defined

$$\|X\|_d := \max_Y \{\langle X, Y\rangle, \|X\| \leq 1\}$$

For vectors from $\mathbb{R}^n$ the dual norm of the l_p norm is the l_q norm $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$. For matrices the dual norm of the Frobenius norm is the Frobenius norm and the dual norm of the spectral norm is the nuclear norm.

The proof of the duality between the spectral norm and the nuclear norm is presented in [32] in the context of the nuclear norm heuristic representation as a semidefinite program, it can also be found in [53].

Lemma 3.4.1 (Fazel, [32]). *The dual norm of the spectral norm $\|X\|$ in $\mathbb{R}^{m\times n}$ is the nuclear norm $\|X\|_*$.*

PROOF. *Step 1*: We express $\|X\| \leq t$ as a linear matrix inequality

$$\|X\| = \sigma_{max}(X)$$

$$\sigma_{min}(X)^2 I \succeq X^t X \succeq \sigma_{max}(X)^2 I$$

If $\|X\| \leq t$ then $t^2 I - X^t X \succeq 0$. We use Schur complement formula to further rewrite this inequality.

LEMMA 3.4.2 (Nonstrict Schur complement formula). *Let Q and R be symmetric matrices. Then the condition*

$$\begin{bmatrix} Q & S \\ S^t & R \end{bmatrix} \geq 0$$

is equivalent to the following $R \geq 0,\ \ Q - SR^+S^t \geq 0,\ \ S(I - RR^+) = 0.$

We apply the Schur complement formula for $t^2 I - X^t X \succeq 0$ and we have

$$\begin{bmatrix} tI_m & X \\ X^t & tI_n \end{bmatrix} \geq 0$$

We will prove the relation $\|X\|_d \geq \|X\|_*$. Let $X = U\Sigma V^t$ and $Y = UV^t$, then $\|Y\| = 1$.

$$\text{trace}(XY^t) = \textstyle\sum_{i=1}^{r} \sigma_i(X) = \|X\|_*$$

Therefore

$$\|X\|_d \geq \|X\|_* . \tag{20}$$

Having the norm inequality expressed as linear matrix inequality we can write:

$$\|X\|_d = \max_Y \{\langle X, Y\rangle ; \|Y\| \leq 1\}$$

as

$$\|X\|_d = \max_Y \text{trace}(X^t Y)$$
$$\text{such that } \begin{bmatrix} I_m & Y \\ Y^t & I_n \end{bmatrix} \geq 0$$

Now, the dual form of this program is given by

$$\underset{W_1,W_2}{\text{minimize}} \quad \frac{1}{2}(\text{trace}(W_1) + \text{trace}(W_2))$$
$$\text{subject to} \quad \begin{bmatrix} W_1 & X \\ X^T & W_2 \end{bmatrix} \succeq 0$$

Set $W_1 = U\Sigma U^T$ and $W_2 = V\Sigma V^T$. The triple (W_1, W_2, X) is feasible for the above minimization problem. Any feasible solution of the minimization problem provides an upper bound for the primal program, therefore we have that

$$\|X\|_d \leq \|X\|_* . \tag{21}$$

From relations (20) and (21) we can conclude there is equality between the dual norm and the nuclear norm. □

There is one thing left to show, the duality between the semidefinite programs. In order to justify it is necessary to show

LEMMA 3.4.3 (Fazel,2002, [31]). *For $X \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{R}$ we have $\|X\|_* \leq t$ if and only if there exist matrices $Y \in \mathbb{R}^{m\times m}$ and $Z \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} Y & X \\ X^t & Z \end{bmatrix} \succeq 0$$

and

$$trace(Y) + trace(Z) \leq 2t$$

PROOF. "$\Longleftarrow$" Suppose Y and Z satisfy the relation

$$\begin{bmatrix} Y & X \\ X^t & Z \end{bmatrix} \succeq 0$$

and

$$\text{trace}(Y) + \text{trace}(Z) \leq 2t$$

Consider $X = U\Sigma V^t$ and let r be the rank of X. Also, we have that the trace of the product of two positive semidefinite matrices is non-negative

$$\text{trace} \begin{bmatrix} UU^t & -UV^t \\ -VU^t & VV^t \end{bmatrix} \begin{bmatrix} Y & X \\ X^t & Z \end{bmatrix} \succeq 0.$$

Further

$$\text{trace}(UU^tY) - \text{trace}(UV^tX^t) - \text{trace}(VU^tX) + \text{trace}(VV^tZ) \geq 0 \tag{22}$$

U has orthonormal columns and we can add more to complete to a full basis. Therefore $\|UU^t\| \leq 1$. According to the Von Neumann Trace theorem

$$|\text{trace}(UU^tY)| \leq \textstyle\sum_i \sigma_i(UU^t)\sigma_i(Y) \leq \text{trace}(Y)$$

The same relation exists for the matrix V

$$\text{trace}(VV^tZ) \leq \text{trace}(Z) \text{ and } \text{trace}(VU^tX) = \text{trace}(V\Sigma V^t) = \text{trace}(\Sigma)$$

Then, using equation (22) we have

$$\text{trace}(Y) + \text{trace}(Z) - 2\text{trace}(\Sigma) \geq 0$$

$$\text{trace}(\Sigma) \leq \tfrac{1}{2}(\text{trace}(Y) + \text{trace}(Z))$$

$$\text{trace}(\Sigma) = \|X\|_* \leq t$$

"$\Longrightarrow$" Suppose $\|X\|_* \leq t$.

Define

$$Y = U\Sigma U^t + \gamma I$$

$$Z = V\Sigma V^t + \gamma I$$

$$\text{trace}(Y) + \text{trace}(Z) = 2\text{trace}(\Sigma) + \gamma(m+n)$$

Choose $\gamma = \frac{2(t-\|X\|_*)}{m+n}$. Then

$$\text{trace}(Y) + \text{trace}(Z) = 2t$$

$$\begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} = \begin{bmatrix} U\Sigma U^T & U\Sigma V^T \\ V\Sigma U^T & V\Sigma V^T \end{bmatrix} + \gamma \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Sigma \begin{bmatrix} U^T & V^T \end{bmatrix} + \gamma I$$

which is positive semidefinite.

□

CHAPTER 4

Positive Semidefinite Cone of Matrices

4.1. Closed convex sets in Euclidean space

We are interested in local properties of closed convex sets in Euclidean space. A finite dimensional closed convex set always has an interior when considered in a proper ambient space and, therefore, has a non-trivial boundary. In [54] Barvinok explores the structure of the boundary and defines and studies faces and extreme points. In particular the structure of faces and extreme points of the cone of positive semidefinite matrices is studied. The convex relaxation for the Euclidean distance matrix completion problem returns as solution points that are on the boundary of the positive semidefinite cone. Therefore we are interested in how to understand and make use of the structure of the boundary of the positive semidefinite cone. The structure and properties of the positive semidefinite cone are detailed in [55], [1], [56].

Definition 4.1.1 (Cone, Dual cone). A set K is a cone if for any $x \in K$ the element $tx \in K$, $t \in \mathbb{R}_+$. Let K be a cone. The set

$$K^* = \{y : x^t y \geq 0, \text{for all x} \in K\}$$

is called the dual cone of K.

K^* is a cone, and is always convex, even when the original cone K is not. $K = K^{**}$ if and only if K is a closed convex cone.

An element y belongs to the dual cone if $-y$ is the normal of a hyperplane that supports K at the origin.

EXAMPLE 4.1.1 (Positive semidefinite cone). The positive semidefinite cone $\mathbb{S}^n_+$ is self-dual. Given matrices X,$Y \in \mathbb{S}^n$ we have that

$$\text{trace}(XY) \geq 0 \text{ for all } X \succeq 0 \Leftrightarrow Y \succeq 0$$

If we consider a matrix Y such that $Y \notin \mathbb{S}^n_+$ we have that $q^tYq = \text{trace}(qq^tY) < 0$ for a vector $q \in \mathbb{R}^n$. Let $X = qq^t$ which is a rank 1 positive semidefinite matrix. Then $\text{trace}(XY) < 0$ and $Y \notin (\mathbb{S}^n_+)^*$.

Let $X, Y \in \mathbb{S}^n_+$. If $X = \sum_{i=1}^n \lambda_i q_i q_i^t$ where $\lambda_i \geq 0, i = 1, \ldots, n$ it can be easily seen that

$$\text{trace}(YX) = \sum_{i=1}^n \lambda_i q_i^t Y q_i \geq 0$$

which shows that Y belongs to the dual cone of the positive semidefinite cone.

If V is a vector space and we consider L a subspace of V then $A = L + u$ is an *affine subspace of V. The dimension of A* is the dimension of the subspace L. The affine subspaces of dimension 1 are straight lines. An affine hyperplane is the set of points in the space V such that $f(v) = \alpha$, where f is a linear functional $f : V \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. *The Isolation Theorem* offers a tool to explore the structure of a convex set.

THEOREM 4.1.1 (Isolation Theorem). *Let V be a vector space, let $A \subset V$ be an algebraically open convex set and let $u \notin A$ be a point. Then there exists an affine hyperplane H which contains u and strictly isolates A.*

DEFINITION 4.1.2. Let $A \subset \mathbb{R}^d$ be a set. A point $u \in A$ is called an interior point of A if there exists an $\epsilon > 0$ such that the open ball $B(u, \epsilon) = \{x : \|x - u < \epsilon\|\}$ is contained in A. The set of all interior points of A is called the interior of A and denoted $\text{int}(A)$. The set of all non-interior points of A is called the boundary of A and is denoted ∂A.

LEMMA 4.1.1. *Let $A \subset \mathbb{R}^d$ be a convex set and let $u_0 \in int(A)$ be an interior point of A. Then, for any point $u_1 \in A$ and any $0 \leq \alpha < 1$, the point $u_\alpha = (1-\alpha)u_0 + \alpha u_1$ is an interior point of A.*

COROLLARY 4.1.2. *Let $A \subset \mathbb{R}^d$ be a convex set. Then int(A) is a convex set.*

DEFINITION 4.1.3. The dimension of a convex set $A \subset \mathbb{R}^d$ is the dimension of the smallest affine subspace that contains A. By convention, the dimension of the empty set is -1.

DEFINITION 4.1.4. Let $K \subset \mathbb{R}^d$ be a closed convex set. A (possibly empty) set $F \subset K$ is called a face of K if there exists an affine hyperplane which isolates K and such that $F = K \cap H$. If F is a point, then F is called an exposed point of K. A non-empty face $F \neq K$ is called a proper face of K.

A face is a closed convex set and a face of a compact convex set is a compact convex set. Also, it can be shown that every compact convex set is the closure of the convex hull of the set of its exposed points. A boundary point lies in some face of a closed convex set.

THEOREM 4.1.3. *Let $K \subset \mathbb{R}^d$ be a convex set with a non-empty interior and let $u \in \partial K$ be a point. Then there exists an affine hyperplane H, called a support hyperplane at u, such that $u \in H$ and H isolates K.*

PROOF. If H is convex with non-empty interior then its interior is a convex set. We know that $u \notin int(K)$. We apply the Isolation Theorem and we obtain that there is an affine hyperplane H which contains u and isolates the int(K) and also isolates K. □

COROLLARY 4.1.4. *Let $K \subset \mathbb{R}^d$ be a closed convex set with a non-empty interior and let $u \in \partial K$ be a point. Then there is a proper face F of K such that $u \in F$.*

PROOF. Let H be a support hyperplane of K at u. Let $F = H \cap K$. $\square$

THEOREM 4.1.5 (Isolation Theorem for convex sets in $\mathbb{R}^d$). *Let $A \subset \mathbb{R}^d$ be a non-empty convex set and let $u \notin A$ be a point. Then there is an affine hyperplane $H \subset \mathbb{R}^d$ such that $u \in H$ and H isolates A.*

PROOF. Choose the minimal affine subspace $L \subset \mathbb{R}^d$ such that $A \subset L$. If $u \notin L$ then we can choose H disjoint from L. If $u \in L$ then int(A)$\neq \emptyset$ is in L and $u \in L$. Then exists a hyperplane $\hat{H}$ in L such that $u \in \hat{H}$ and $\hat{H}$ isolates A. Then we choose any hyperplane H such that $H \cap L = \hat{H}$. $\square$

DEFINITION 4.1.5. Let V be a vector space and let $A \subset V$ be a set. A point $a \in A$ is called an extreme point of A provided for any two points $b, c \in A$ such that $\frac{b+c}{2} = a$ we must have $b = c = a$. The set of all extreme points of A is denoted ex(A).

An extreme point is a point that is not an interior point of any line segment lying entirely in A. The extreme points of a line segment are its endpoints, while the extreme points of the unit circle together with its interior are the points from the boundary of the circle.

THEOREM 4.1.6 (Barvinok). *Let V be a vector space, let $A \subset V$ be a non-empty set and let $f : V \to \mathbb{R}$ be a linear functional.*

(1) Suppose that f attains its maximum (resp. minimum) on A at a unique point $u \in A$, that is, $f(u) > f(v)$ for all $v \neq u$, $v \in A$ (resp. $f(u) < f(v)$ for all $v \neq u$, $v \in A$). Then u is an extreme point of A.

(2) Suppose that f attains its maximum (minimum) α on A and suppose that $B = \{x \in A : f(x) = \alpha\}$ is the set where the maximum (minimum) is attained. Let u be an extreme point of B. Then u is an extreme point of A.

PROOF. Let's suppose f attains its maximum on A. For $u = \frac{a+b}{2}$ $f(u) = \frac{f(a)+f(b)}{2}$, where $f(a) \leq f(u)$ and $f(b) \leq f(u)$. Then we have $f(a) = f(b) = f(u)$ and $a = b = u$ from the uniqueness of the maximum point.

For $u = \frac{a+b}{2}$ for $a, b \in A$. Then $\alpha = f(u) = \frac{f(a)+f(b)}{2}$ and $f(a), f(b) \leq \alpha$. We must have then $f(a) = f(b) = \alpha$, so $a, b \in B$. Then $a = b = u$ since u is an extreme point of B. □

Note that when $K \subset \mathbb{R}^d$ is a compact set and $u \in K$ is a point such that $\|u\| \geq \|v\|$ for each $v \in K$ then u is an extreme point of K.

THEOREM 4.1.7 (M.G. Krein, D.P.Milman, 1940). *Let $K \in \mathbb{R}^d$ be a compact convex set. Then K is the convex hull of the set of its extreme points.*

PROOF. The proof can be obtained through induction. When $d = 0$ we have that K is a point and then the conclusion takes places. Now consider the case when $d > 0$. When $\text{int}(K) = \emptyset$, K lies in an affine subspace of a smaller dimension and the statement is true by the induction hypothesis.

When $\text{int}(K) \neq \emptyset$, we have to prove that for all points in the set K they can expressed as a convex combination of extreme points of K. For a point on the boundary of K we can apply corollary 4.1.4 and we have that exists a face F of K such that the point belongs to that face. But F lies in an affine subspace of a smaller dimension and we apply the induction hypothesis $u \in \text{conv}(\text{ex}(F))$ and $\text{ex}(F) \subset \text{ex}(K)$. When point $u \in \text{int}(K)$ consider the straight line denoted by L that contains u. The intersection between L and K is an interval $[a, b]$ where the points a, b are on the boundary of K and u is an interior point of this interval. For points on the boundary the statement takes place, therefore a and b can be expressed as a convex combination of extreme points. u is a convex combination of a and b therefore it is a convex combination of extreme points. □

A closed and convex set has at least one extreme point if and only if it does not contain a line.

EXAMPLE 4.1.2. Let $P = \{x \in \mathbb{R}^n : a_j^t x \leq b_j, j = 1, \ldots, r\}$, where a_j, b_j are given. A vector $v \in P$ is an extreme point of P if and only if the set $A_v = \{a_j; a_j^t v = b_j, j = 1, \ldots, r\}$ contains n linearly independent vectors.

4.2. Positive semidefinite cone: faces and extreme points

The positive semidefinite cone $\mathbb{S}_+^n$ can be expressed as

$$\bigcap_{z \neq 0} \{X \in \mathbb{S}^n : z^T X z \geq 0\} \tag{23}$$

For each $z \neq 0$, $z^T X z$ is (not identically zero) linear function of X so the sets $\{X \in \mathbb{S}^n : z^T X z \geq 0\}$ are half-spaces in $\mathbb{S}^n$. In this way the positive semidefinite cone is the intersection of an infinite number of half spaces, and so is convex.

The positive semidefinite cone of $n \times n$ matrices is isomorphic with the vector space $\mathbb{R}^{\frac{n(n+1)}{2}}$ (given by the number of free variables in a symmetric $n \times n$ matrix). There is an isometry T from the space of symmetric matrices to $\mathbb{R}^{\frac{n(n+1)}{2}}$ given by
$T(X) = \big(X_{11}, \sqrt{2}X_{12}, \ldots, \sqrt{2}X_{1n}, X_{22}, \sqrt{2}X_{23}, \ldots\big)$.

DEFINITION 4.2.1 (Proper cone). A cone K is called a proper cone if it satisfies the following

(1) K is convex

(2) K is closed

(3) K is solid, which means it has non-empty interior

(4) K is pointed, which means that it contains no line ($x \in K$, $-x \in K \Rightarrow x = 0$).

A proper cone K can be used to define a generalized inequality, which is a partial ordering on $\mathbb{R}^d$ defined by:

$$x \preceq_K y \Leftrightarrow y - x \in K$$

Similarly, it can be used to define an associated strict partial ordering.

The positive semidefinite cone is a proper cone in $\mathbb{S}^n$. The associated generalized inequality $\preceq_K$ is the matrix inequality $X \preceq Y$ which means $Y - X$ is a positive semidefinite matrix. The strict inequality would mean that the difference matrix is positive definite.

LEMMA 4.2.1 (Matousek, Lemma 4.7.4, page 67). *The Positive Semidefinite Cone is generated by rank one matrices.*

PROOF. Consider the matrix $M = \sum_{i=1}^{n} \lambda_i q_i q_i^t$, $\lambda_i \geq 0$. Matrices $q_i q_i^t$ are positive semidefinite and since the positive semidefinite cone is a convex cone, every non-negative linear combination of such matrices is also positive semidefinite. Given a positive semidefinite matrix M we can diagonalize it $M = SDS^t$. If we denote its eigenvalues with $\lambda_1, \ldots, \lambda_n$, these are non-negative. Then the matrix M can be written as

$$M = \sum_{i=1}^{n} \lambda_i q_i q_i^t.$$

where q_i is the ith column of S. □

EXAMPLE 4.2.1. Euclidean projection on a proper cone: To project a symmetric matrix onto the positive semidefinite cone, form its eigenvalue expansion and drop terms associated with the negative eigenvalues. This matrix is also the projection onto the positive semidefinite cone in the l_2, or spectral norm.

In $\mathbb{S}_+^2$ each and every ray on the boundary of the positive semidefinite cone in $\mathbb{R}^3$ corresponds to a symmetric rank 1 matrix, but that does not hold in any higher dimension.

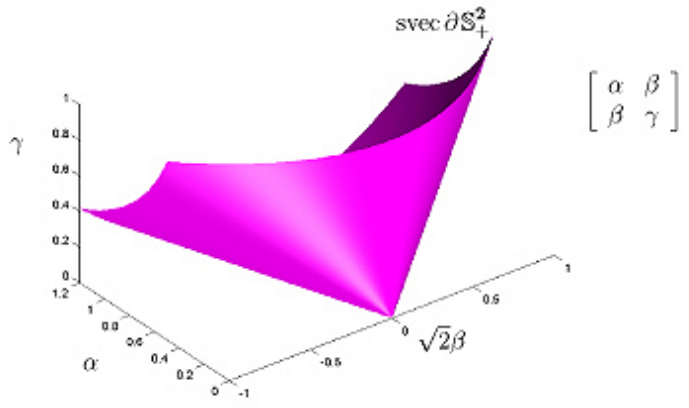

FIGURE 4.1. Boundary of the positive semidefinite cone S^2 [1].

Each and every face of the PSD cone contains the origin.

PROPOSITION 4.2.1. *Let A be an $n \times n$ positive semidefinite matrix of rank$A = r$. If $r = n$ then A is an interior point of the cone of positive semidefinite matrices. If $r < n$, then A is an interior point of a face F, where dim $F = \frac{r(r+1)}{2}$. There is a rank preserving isometry identifying the face F with the cone of positive semidefinite $r \times r$ matrices.*

PROOF. If rank of A is n then A is positive definite and belongs to the interior of $\mathbb{S}^n_+$.

Suppose that rank $A= r < n$. A hyperplane $H \subset \mathbb{S}^n$ which contains A and isolates the set $\mathbb{S}^n_+$ shall be constructed.

Let $\lambda_1, \ldots, \lambda_r > 0$ be the non-zero eigenvalues of A and U the orthogonal matrix such that

$$U^{-1}AU = D = diag(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$$

Let the matrix $C = \text{diag}(0, \ldots, 0, 1, \ldots, 1)$ be the matrix whose first r diagonal entries are 0 and the last $n - r$ diagonal entries are 1. Further, define $Q = UCU^{-1}$.

Then Q is a non-zero positive semidefinite matrix and the following relation takes place

$$\langle Q, A \rangle = \langle C, D \rangle = 0$$

For any positive semidefinite $n \times n$ matrix X, the matrix $Y = U^{-1}XU$ is positive semidefinite and

$$\langle Q, X \rangle = \langle UCU^{-1}, UYU^{-1} \rangle = \langle C, Y \rangle \geq 0$$

since the diagonal entries of Y must be non-negative. Therefore, the hyperplane

$$H = \{X \in \mathbb{S}^n : \langle Q, X \rangle = 0\}$$

isolates $\mathbb{S}^n_+$ and contains A.

Let us describe the corresponding face

$$F = \left\{X \in \mathbb{S}^n_+ : \langle Q, X \rangle = 0\right\}$$

The map $X \to Y = U^{-1}XU$ is a non-degenerate linear transformation which maps $\mathbb{S}^n_+$ onto itself, maps Q onto C and A onto D. Then the face F is mapped onto a face $F^{'}$, containing D and consisting of all positive semidefinite matrices Y such that $\langle Y, C \rangle = 0$.

$$F^{'} = \left\{Y \in \mathbb{S}^n_+ : \langle Y, C \rangle = 0\right\}$$

Clearly, Y must have the last $n - r$ diagonal entries equal to zero. Since Y is positive semidefinite, all entries in the last $n - r$ rows and last $n - r$ columns must be zero. The upper left $r \times r$ submatrix of Y can be an arbitrary positive semidefinite matrix.

Thus the face $F^{'}$ may be identified with the cone of all $r \times r$ positive semidefinite matrices and it contains D in its interior (in particular $\dim F^{'} = \frac{r(r+1)}{2}$).

Since $Y \to X = UYU^{-1}$ is a non-degenerate linear transformation, which maps D onto A and the face $F^{'}$ onto F we can conclude that $\dim F = \frac{r(r+1)}{2}$ and F contains A in its interior. $\square$

Therefore we have the following statements

(1) The dimensions of faces F of the cone of positive semidefinite matrices are $0, 1, 3, \ldots \frac{r(r+1)}{2} \ldots$. If F is a face and its dimension is $\frac{r(r+1)}{2}$ then there is a matrix $A \in intF$ such that rank A equals r.

(2) Let $F \in \mathbb{S}^n_+$ be a face and let r be a positive integer such that dim $F < \frac{r(r+1)}{2} \leq \frac{n(n+1)}{2}$. There is a face F' of $\mathbb{S}^n_+$ such that F is a face of F' and $\dim F' = \frac{r(r+1)}{2}$.

(3) If r and n are two positive integers $0 < r < n$ and $\mathbb{S}^n_+$ be the cone of positive semidefinite $n \times n$ and $\mathbb{S}^r_+$ the cone of positive semidefinite $r \times r$ matrices. Let $F \subset \mathbb{S}^n_+$ be a face such that $\dim F = \frac{r(r+1)}{2}$. There exists a isometry between $\mathbb{S}^r_+$ and F.

Using proposition 4.2.1 we obtain the following description of the facial structure of the cone of positive semidefinite matrices.

COROLLARY 4.2.1. *The faces of $\mathbb{S}^n_+ \subset \mathbb{S}^n$ are parametrized by the subspaces of $\mathbb{R}^n$. For a subspace $L \subset \mathbb{R}^n$ let*

$$F_L = \left\{ Y \in \mathbb{S}^n_+ : L \subset ker Y \right\}$$

Then F_L is a face of $\mathbb{S}^n_+$ and $dim F_L = \frac{r(r+1)}{2}$, where $r = codim L$. As L ranges over all subspaces of codimension r, F_L ranges over all faces of dimension $\frac{r(r+1)}{2}$.

PROOF. Given a subspace L of codimension r, let's choose the coordinates such that $L = \{0, \ldots, 0, \xi_{r+1}, \ldots, \xi_n\}$. Then F_L consists of all matrices Y having the last $n-r$ columns and rows 0 and the upper $r \times r$ corner positive semidefinite. The supporting hyperplane for F_L is $H = \{X : \langle C, X \rangle = 0\}$, where C was defined in proposition 4.2.1. If F is a face of S^n_+ then $F = F_L$, where $L = \ker A$ and A is a matrix in the interior of F. $\square$

4.3. Example of face of positive semidefinite cone

Let's consider a simple example. Consider 3 points such that the distance between the vertices 1 and 2 is 1 and the other two distances are unknown. In this case the distance matrix is:

$$D = \begin{bmatrix} 0 & 1 & x \\ 1 & 0 & y \\ x & y & 0 \end{bmatrix}$$

Let matrix

$$J = I_3 - \tfrac{1}{3}\text{ones}(3,3).$$

Matrix Y is given by

$$Y = -\tfrac{1}{2}TDT.$$

With the above notation the entries of matrix Y are

$$S = \begin{bmatrix} \frac{2x}{9} - \frac{y}{9} + \frac{2}{9} & \frac{x}{18} + \frac{y}{18} - \frac{5}{18} & \frac{y}{18} - \frac{5x}{18} + \frac{1}{18} \\ \frac{x}{18} + \frac{y}{18} - \frac{5}{18} & \frac{2y}{9} + \frac{x}{9} - \frac{2}{9} & \frac{x}{18} - \frac{5y}{18} + \frac{1}{18} \\ \frac{y}{18} - \frac{5x}{18} + \frac{1}{18} & \frac{x}{18} - \frac{5y}{18} + \frac{1}{18} & \frac{2x}{9} - \frac{2y}{18} - \frac{1}{9} \end{bmatrix}$$

The optimization problem

$$\begin{aligned} \underset{Y}{\text{minimize}} \quad & \|Y\|_* \\ \text{subject to} \quad & Y_{ii} - 2Y_{ij} + Y_{jj} = D_{ij},\ (i,j) \in \Omega, \\ & Y \succeq 0. \end{aligned}$$

returns as optimal solution the distance matrix

$$D = \begin{bmatrix} 0 & 1 & 0.25 \\ 1 & 0 & 0.25 \\ 0.25 & 0.25 & 0 \end{bmatrix}$$

i.e the points are collinear, the third one is the middle of the segment formed by the other two.

The optimal semidefinite matrix is $Y = \begin{bmatrix} 0.25 & -0.25 & 0 \\ -0.25 & 0.25 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ of rank 1.

Then Y is an interior point of a face F, where dim $F = \frac{1(1+1)}{2} = 1$. There is a rank preserving isometry identifying the face F with the cone of positive semidefinite 1×1 matrices. (i.e explicit description $\mathbb{S}^1_+$: $x \geq 0$). Consider the decomposition of the matrix Y

$$Y = Q\Lambda Q^t$$

where

$$Q = \begin{bmatrix} -0.7071 & -0.4082 & 0.5774 \\ 0.7071 & -0.4082 & 0.5774 \\ 0 & 0.8165 & 0.5773 \end{bmatrix}$$

The corresponding face is given by

$$F = \left\{ X \in \mathbb{S}^3_+ : \left\langle Q \left(I - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) Q^t, X \right\rangle = 0 \right\}$$

Let $B = Q \left(I - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) Q^t$ and let X be a symmetric matrix

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}$$

If X is rank 1 matrix then the following relations take place between the entries of matrix X: $x_2 = \alpha x_1$, $x_3 = \beta x_1$,$x_4 = \alpha^2 x_1$, $x_5 = \alpha\beta x_1$, $x_6 = \beta^2 x_1$.

$$X = \begin{bmatrix} x_1 & \alpha x_1 & \beta x_1 \\ \alpha x_1 & \alpha^2 x_1 & \alpha\beta x_1 \\ \beta x_1 & \alpha\beta x_1 & \beta^2 x_1 \end{bmatrix}$$

$x_1 \neq 0$ for $X \neq 0$

$$X = x_1 \begin{bmatrix} 1 & \alpha & \beta 1 \\ \alpha & \alpha^2 & \alpha\beta \\ \beta & \alpha\beta & \beta^2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.3334 & 0.3334 & 0.3333 \\ 0.3334 & 0.3334 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \end{bmatrix}$$

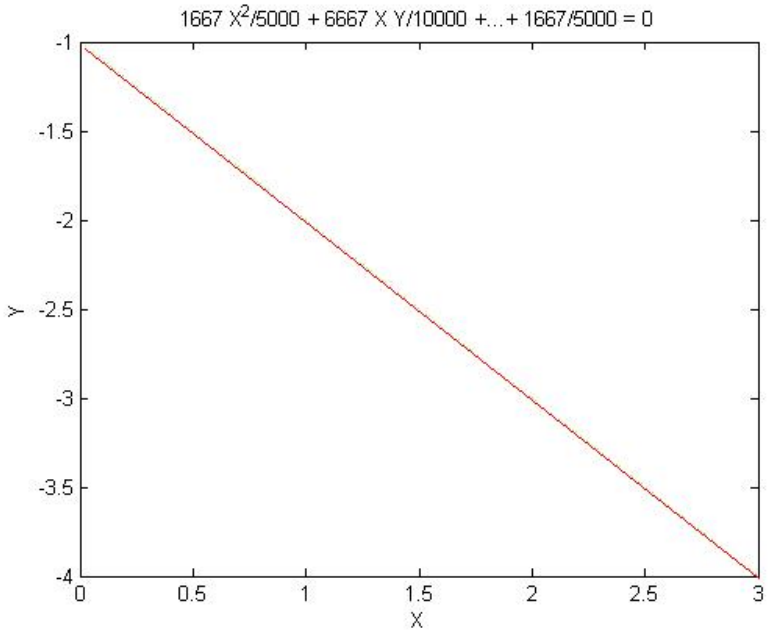

FIGURE 4.2. Face of positive semidefinite cone that contains rank 1 matrix.

The equation of the hyperplane, $\text{trace}(BX) = 0$, in $\mathbb{R}^6$ becomes

$$a_1 x_1 + a_2 \alpha x_1 + a_3 \beta x_1 + a_4 \alpha^2 x_1 + a_5 \alpha\beta x_1 + a_6 \beta^2 x_1 = 0$$

$$a_1 + a_2 \alpha + a_3 \beta + a_4 \alpha^2 + a_5 \alpha\beta + a_6 \beta^2 = 0$$

where $a_1 = 0.3334$, $a_2 = 0.6668$, $a_3 = 0.6667$,$a_4 = 0.3334$, $a_5 = 0.6667$,$a_6 = 0.3333$. in Figure 4.2

The faces of dimension 3, where lie the rank 2 matrices, they are the 2×2 semidefinite cone.

CHAPTER 5

Euclidean Distance Matrices

In this chapter we introduce the basic ideas behind the theory of Euclidean distance matrices and their characterization in terms of positive semidefinite matrices. Also, we will present the formulation of the low rank Euclidean distance matrix completion problem and its semidefinite program formulation using nuclear norm heuristic.

Definition 5.0.1. Let points $p_1, p_2, \ldots, p_n \in \mathbb{R}^r$. The matrix $D \in \mathbb{S}^n$ defined by $D_{ij} = \|p_i - p_j\|_2^2$, $i, j = 1, 2, \ldots, n$ is called a *Euclidean distance matrix*.

From its definition it can be easily seen that the elements of the diagonal of D are all zeros and $D_{ij} \geq 0$ for all $i, j = 1, 2, \ldots n$. Also, we have the triangle inequality: $\sqrt{D_{ik}} + \sqrt{D_{ij}} \geq \sqrt{D_{jk}}$. Schoenberg offers the following characterization of the Euclidean distance matrices [57]. This result is also found in a paper of Young and Householder [58].

Theorem 5.0.1 (Schoenberg). *A matrix $D \in \mathbb{S}^n$ with all diagonal elements equal to zero is a Euclidean distance matrix if and only if the matrix $-\frac{1}{2}JDJ$ is positive semidefinite, where e is the vector of all ones and $J = I_n - \frac{1}{n}ee^t$. Furthermore, if D is a Euclidean distance matrix, then the embedding dimension of D is equal with the rank of matrix $-\frac{1}{2}JDJ \leq n-1$.*

A proof of this theorem can be found in [17].

Now, given Euclidean distance matrix D we can find the doubly centered matrix $Y = -\frac{1}{2}JDJ$ and then recover the points that generate matrix D computing the factorization of $Y = PP^t$. The coordinates of points are given by the rows of matrix P. In the subspace of symmetric matrices, the set of all Euclidean distance matrices forms a pointed closed convex cone [1]. Cone of all Euclidean distance matrices $\mathcal{E}^n$ is the intersection of an infinite number

(when $n > 2$) of halfspaces about the origin and a finite number of hyperplanes through the origin in the vectorized variable $D = (D_{ij})$ [1]. Also [1], using the argument from [59] that any symmetric positive semidefinite matrix having a zero entry on its main diagonal must be zero along the entire row and column to which that zero entry belongs, concludes that there can be no positive or negative semidefinite Euclidean distance matrix except the zero matrix. (If a positive semidefinite matrix $A = (A_{ij}) \in \mathbb{R}^{n\times n}$ has a zero entry A_{ii} on its main diagonal, then $A_{ij} + A_{ji} = 0$, for all $j = 1, \ldots, n$.) Many authors have been interested to describe the cone of Euclidean distance matrices and its geometry [60], [61], [62], [63].

Given a graph G with n vertices, E the set of edges, and non-negative edge weights D_{ij}, $ij \in E$, we call a realization of G in $\mathbb{R}^d$ any placement of the vertices of G in $\mathbb{R}^d$ such that the Euclidean distance between the vertices connected by an edge is given by the weights D_{ij}. References on graph realizability include [18], [19], [20].

Applications of Euclidean Distance Matrices occur in a variety of fields. For example, a typical sensor network consists of a large number of sensors which are densely deployed in a geographical area. Sensors collect the local environmental information such as temperature or humidity and can communicate to each other. The sensor network localization problem is: assuming the accurate positions of some nodes (called anchors) are known, how to use them and partial pair-wise distance measurements to locate the positions of all sensor nodes in the network [15], [64], [65]. Another problem of interest relates to finding the structure of a protein given a (partial or complete) set of approximate pairwise distances between the atoms in the protein [16].

5.1. Relation with the Positive Semidefinite Cone of Matrices

In [66], [67], [68], [69], [40] the geometry of the Euclidean distance matrices has been presented in relation with the positive semidefinite cone of matrices whose geometry is understood. We will present a summary describing the mapping between the Euclidean distance cone and the positive semidefinite cone and their main properties.

The set of all Euclidean Distance Matrices is a pointed closed convex cone. Each and every principal submatrix of an Euclidean Distance Matrix is another Euclidean Distance Matrix. Suppose that matrix D is the distance matrix for the points $p_1, p_2, \ldots, p_n \in \mathbb{R}^r$. The matrix defined as $Y = PP^t$, where P is $n \times r$ matrix that contains the vectors p_i as its rows is usually called the Gram matrix of the points $p_1, p_2, \ldots, p_n$. For any $i, j \in \{1, 2, \ldots, n\}$ there exists a relation between the entries of the matrix D and the entries of the matrix Y, as follows

$$D_{ij} = \|p_i - p_j\|_2^2,$$

$$D_{ij} = p_i^t p_i + p_j^t p_j - 2p_i^t p_j,$$

$$D_{ij} = Y_{ii} + Y_{jj} - 2Y_{ij}.$$

Therefore we can define a linear operator $\mathcal{K} : \mathbb{S}^n \to \mathbb{S}^n$

$$\mathcal{K}(Y) = \text{diag}(Y)e^t + e\text{diag}(Y)^t - 2Y. \tag{24}$$

In the previous relation e denotes the vector of all ones. Componentwise, we have

$$\mathcal{K}(Y)_{ij} = Y_{ii} + Y_{jj} - 2Y_{ij},\ i, j = 1, \ldots, n.$$

Also, denote $\mathcal{D}_e(Y) = \text{diag}(Y)e^t + e\text{diag}(Y)^t$, for symmetric matrices $Y \in \mathbb{S}^n$.

In case of vectors, the same notation shall be used $\mathcal{D}_e(y) = \text{diag}(y)e^t + e\text{diag}(y)^t$, $y \in \mathbb{R}^n$. With this notation we can express

$$\mathcal{K}(Y) = \mathcal{D}_e(Y) - 2Y$$

and its Moore-Penrose pseudoinverse

$$\mathcal{K}^+(D) = -\tfrac{1}{2}J\text{offDiag}(D)J$$

where $J = I - \frac{1}{n}ee^t$, $\text{offDiag}(D) = D - \text{Diag}(\text{diag}(D))$. Multiplying a vector by J centers the vector by subtracting the mean of all coordinates from each coordinate, by shifting the origin to the centroid of points. Matrix J is also called double centered distance matrix [70]. Denote with $\mathcal{S}_H^n = \{D \in \mathbb{S}^n : \text{diag}(D) = 0\}$ and $\mathcal{S}_C^n = \{Y \in \mathbb{S}^n : Ye = 0\}$. In the literature $\mathcal{S}_H^n$ is called *hollow subspace* and $\mathcal{S}_C^n$ is called *centered subspace.* The matrix $J = I - \frac{1}{n}ee^t$ represents the orthogonal projection onto the subspace of hollow matrices, the matrices that are symmetric and have the diagonal elements equal zero [17].

In an Euclidean vector space equipped with the inner product the adjoint of a linear transformation T is usually denoted T^*. It satisfies the relation $\langle Tx, y\rangle = \langle x, T^*y\rangle$. In [67],[17] it is shown that the adjoints of these linear transformations are

$$\mathcal{D}_e^*(D) = 2\text{Diag}(De),$$

$$\mathcal{K}(D)^* = 2(\text{Diag}(De) - D)$$

and

$$\text{range}(\mathcal{K}^*) = \mathbb{S}_C^n,$$

$$\text{null}(\mathcal{K}^*) = \text{range}(\mathcal{D}_e).$$

The null space of adjoint of $\mathcal{K}^*$ consists of all diagonal matrices.

Then

$$\mathcal{K}\mathcal{K}^+ = \text{offDiag}(D),$$

$$\mathcal{K}^+\mathcal{K}(Y) = JYJ$$

PROPOSITION 5.1.1 (Al-Homidan, Wolkowicz). *The range space of the linear operator* $\mathcal{K} : \mathbb{S}^n \to \mathbb{S}^n$ *is given by*

$$range(\mathcal{K}) = \mathbb{S}^n_H$$

PROOF. Let $Y \in \mathbb{S}^n$. The elements on the main diagonal of $\mathcal{K}(Y)$ are $\mathcal{K}(Y)_{ii} = Y_{ii} + Y_{ii} - 2Y_{ii} = 0$ which concludes that $\mathcal{K}(Y) \in \mathbb{S}^n_H$. If we consider $D \in \mathbb{S}^n_H$ and $Y = -\frac{1}{2}D$, then $\mathcal{K}(Y) = D$. □

PROPOSITION 5.1.2 (Al-Homidan, Wolkowicz). *The null space of* $\mathcal{K}$ *is given by*

$$null(\mathcal{K}) = range(\mathcal{D}_e)$$

PROOF. $\mathcal{K}(Y)_{ij} = Y_{ii} + Y_{ii} - 2Y_{ij} = 0$ if and only if $Y_{ij} = \frac{1}{2}(Y_{ii} + Y_{jj})$ if and only if $Y = ye^t + ey^t = \mathcal{D}_e(y)$, where the vector is defined as $y = \frac{1}{2}\text{diag}(Y)$. □

If P is a matrix of points and $\hat{P}$ is the matrix obtained by translating every row of P by the vector v then these two matrices generate the same Euclidean distance matrix. Also, if we modify matrix P such that each row of P is rotated/reflected by the same orthogonal transformation $\hat{P} = PQ$, where Q orthogonal matrix then they generate the same Euclidean distance matrix and also we can notice that the Gram matrix of P is invariant under orthogonal transformation of points. The properties of the linear operators $\mathcal{K}$ and $\mathcal{K}^+$ and their link with the cone of Euclidean distance matrices are presented in detail in

[17]. For example, as shown in the following theorem, the operator $\mathcal{K}$ in (24) gives a relation between the entries of matrix Y and the entries of the Euclidean distance matrix D.

PROPOSITION 5.1.3 (N.Krislock [17]). *The range space and the null space of the linear maps $\mathcal{K}$ and $\mathcal{K}^+$ are given by*

$$range(\mathcal{K}^+) = \mathbb{S}_C^n \qquad null(\mathcal{K}^{+*}) = range(\mathcal{D}_e)$$

$$range(\mathcal{K}^{+*}) = \mathbb{S}_H^n \qquad null(\mathcal{K}^+) = range(Diag)$$

The map $\mathcal{K} : \mathbb{S}_C^n \to \mathbb{S}_H^n$ is a bijection and $\mathcal{K}^+ : \mathbb{S}_H^n \to \mathbb{S}_C^n$ is its inverse.

If $\mathcal{E}^n$ is the set of Euclidean distance matrices of dimension n, the map $\mathcal{K} : \mathbb{S}_+^n \cap \mathbb{S}_C^n \to \mathcal{E}^n$ is a bijection and $\mathcal{K}^+ : \mathcal{E}^n \to \mathbb{S}_+^n \cap \mathbb{S}_C^n$ is its inverse.

For a proof see [17].

5.2. EUCLIDEAN DISTANCE MATRIX COMPLETION PROBLEM

Euclidean distance matrix completion problems can be found in, e.g., [71], [72], [73], [40], [17]. Following [17] the *Euclidean distance matrix completion (EDMC) problem* means to find a Euclidean distance matrix which is a completion of a given matrix D, where D has all zero elements on the diagonal and some of the entries of D are known and nonnegative. D is also called in [66], [17] a *partial Euclidean distance matrix*.

We can associate the entries of partial distance matrix D with the edges of graph $G = (V, E, \omega)$.

The vertices of the graph V are given by the points that generate matrix D. Two points are connected by an edge if the entry D_{ij} is known. The edges weights are $\omega_{ij} = \sqrt{D_{ij}}$, for all $i, j \in E$.

In this context the Euclidean distance matrix completion problem can be formulated as

$$\begin{aligned} &\text{find} && \hat{D} \in \mathcal{E}^n \\ &\text{subject to} && \hat{D}_{ij} = D_{ij},\ \forall ij \in E \end{aligned} \tag{25}$$

Let $H \in \mathbb{S}^n$ be the adjacency matrix of G, the problem (25) can be expressed as

$$\begin{aligned} &\text{find} && \hat{D} \in \mathcal{E}^n \\ &\text{subject to} && H \circ \hat{D} = H \circ D,\ \forall ij \in E \end{aligned} \tag{26}$$

where $\circ$ is the Hadamard (component wise) matrix product [17], [66].

Now, using the relation between the Euclidean distance matrices and the positive semidefinite cone through the linear map $\mathcal{K}$ defined in (24) EDMC problem becomes

$$\begin{aligned} &\text{find} && Y \in \mathbb{S}^n_+ \cap \mathbb{S}^n_C \\ &\text{subject to} && H \circ \mathcal{K}(Y) = H \circ D \end{aligned} \tag{27}$$

5.3. Solving EDM via semidefinite programming

Using Schoenberg's characterization of Euclidean distance matrices [57] and the nuclear norm heuristic, the Euclidean Distance Matrix completion problem reduces to finding a low rank positive semidefinite matrix Y such that

$$\begin{aligned} &\underset{Y}{\text{minimize}} && \|Y\|_* \\ &\text{subject to} && Y_{ii} - 2Y_{ij} + Y_{jj} = D_{ij},\ (i,j) \in \Omega, \\ & && Y \succeq 0 \end{aligned} \tag{28}$$

where Ω is the set of indices corresponding to the known entries in the matrix D. If we treat the matrix D as the variable we can write this optimization problem as

$$\begin{aligned} \underset{\hat{D}}{\text{minimize}} \quad & \left\| -\frac{1}{2} J \hat{D} J \right\|_* \\ \text{subject to} \quad & \hat{D}_{ij} = D_{ij}, \ (i,j) \in \Omega, \\ & -\frac{1}{2} J \hat{D} J \succeq 0. \end{aligned} \tag{29}$$

The matrix $-\frac{1}{2} J \hat{D} J$ is positive semidefinite, therefore the nuclear norm is equal to the trace and an equivalent formulation of the optimization problem is

$$\begin{aligned} \underset{\hat{D}}{\text{minimize}} \quad & \text{trace}(-\frac{1}{2} J \hat{D} J) \\ \text{subject to} \quad & \hat{D}_{ij} = D_{ij}, \ (i,j) \in \Omega, \\ & -\frac{1}{2} J \hat{D} J \succeq 0. \end{aligned} \tag{30}$$

Semidefinite optimization for solving Euclidean distance matrix problems is studied in [71], [67],[66]. If we use the nuclear norm heuristic to search for the minimum rank solutions Y, it is equivalent to minimize the trace of matrix Y, since for positive semidefinite matrices the nuclear norm is equal with the trace of the matrix. Trace of a matrix is a linear function, therefore convex (and concave).

Given the trace of a square positive semidefinite matrix is both convex and concave we can minimize or maximize it, depending on the context. In this context of Euclidean distance matrix completion problem we will use it to look for low rank solutions as solutions to a minimization problem. In Maximum Variance Unfolding [74], [75] it is used as objective function that needs to be maximized. Given a set of points in $\mathbb{R}^n$ the goal of Maximum Variance Unfolding is to find same number of points in $\mathbb{R}^k$ such that the distances are

preserved for some given edge set. If this edge set is constrained to pairs of nearby points then maximum variance unfolding finds a non-linear embedding of the points. It is a two step procedure, first the manifold is stretched so that it is nearly linear by maximizing the trace of a covariance matrix. This is the step that can be formulated as a semidefinite program.

5.4. Relation between l_1 norm of matrix D and nuclear norm of matrix Y

In this section we develop the relationship between Euclidean distance matrix D and its double centered form by deriving an explicit formula for the nuclear norm of matrix Y and the entries of matrix D.

Proposition 5.4.1. *Let D be an Euclidean distance matrix and matrix $Y = -\frac{1}{2}JDJ$ where $J = I_n - \frac{1}{n}\mathbf{1}_{n\times n}$. Then*

$$\|Y\|_* = \frac{1}{2n}\|D\|_{l_1} \tag{31}$$

where $\|D\|_{l_1}$ is given by the sum of absolute values of entries of $D = (D_{ij})_{i,j=1,2,\ldots,n}$ and $\mathbf{1}_{n\times n}$ is matrix with all entries equal 1.

Proof. Let $Y_1 = JDJ$, we write the explicit relation that defines Y in terms of matrix D

$$Y = -\frac{1}{2}\left[I_n - \frac{1}{n}\mathbf{1}_{n\times n}\right] D \left[I_n - \frac{1}{n}\mathbf{1}_{n\times n}\right]$$

and Y_1 is given by

$$Y_1 = \left[I_n - \frac{1}{n}\mathbf{1}_{n\times n}\right] D \left[I_n - \frac{1}{n}\mathbf{1}_{n\times n}\right]$$

We'll use the following property of matrix Y: it is a positive semidefinite therefore its trace is equal with the nuclear norm $\|Y\|_* = \text{trace}(Y)$.

Further we show that the trace(Y) can be computed in terms of the l_1 norm of D.

$$\text{trace}(Y) = -\frac{1}{2}\text{trace}(\left[I_n - \frac{1}{n}\mathbf{1}_{n\times n}\right] D \left[I_n - \frac{1}{n}\mathbf{1}_{n\times n}\right])$$

$$\text{trace}(Y_1) = \text{trace}(\left[I_n - \frac{1}{n}\mathbf{1}_{n\times n}\right] D \left[I_n - \frac{1}{n}\mathbf{1}_{n\times n}\right])$$

$$\text{trace}(Y_1) = \text{trace}(D) - \frac{1}{n}\text{trace}(\mathbf{1}_{n\times n}D) - \frac{1}{n}\text{trace}(D\mathbf{1}_{n\times n}) + \frac{1}{n^2}\text{trace}(\mathbf{1}_{n\times n}D\mathbf{1}_{n\times n}) \tag{32}$$

We know that D is a distance matrix therefore $\text{trace}(D) = 0$. Further, we evaluate the other three terms from the relation (32) above

$$\mathbf{1}_{n\times n}D = \begin{bmatrix} \sum_{j=1}^{n} D_{j1} & \cdots & \cdots \\ \cdots & \sum_{j=1}^{n} D_{j2} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \sum_{j=1}^{n} D_{jn} \end{bmatrix}$$

$$\mathbf{1}_{n\times n}D\mathbf{1}_{n\times n} = \begin{bmatrix} n(\sum_{j=1}^{n} D_{1j}) & \cdots & \cdots \\ \cdots & n(\sum_{j=1}^{n} D_{2j}) & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & n(\sum_{j=1}^{n} D_{nj}) \end{bmatrix}$$

and $\text{trace}(\mathbf{1}_{n\times n}D) = \text{trace}(D\mathbf{1}_{n\times n})$.

Therefore, we obtain that

$$\text{trace}(\mathbf{1}_{n\times n}D) = \|D\|_{l_1}$$

$$\text{trace}(D\mathbf{1}_{n\times n}) = \|D\|_{l_1}$$

$$\text{trace}(\mathbf{1}_{n\times n}D\mathbf{1}_{n\times n}) = n\,\|D\|_{l_1}$$

and we use this to calculate the trace(Y)

$$\text{trace}(Y_1) = -\frac{2}{n}\|D\|_{l_1} + \frac{1}{n^2}n\|D\|_{l_1}$$
$$\text{trace}Y_1 = -\frac{1}{n}\|D\|_{l_1}$$

obtaining the relation between the nuclear norm of Y and entries of the matrix D

$$\|Y\|_* = \frac{1}{2n}\|D\|_{l_1} \tag{33}$$

□

CHAPTER 6

Bounding chromatic number using convex optimization

The chromatic number of a graph is the least number of colors required to color its vertices without using the same color for adjacent vertices. Finding the chromatic number of a graph is an NP-hard problem [76].

In general, a lower bound for the chromatic number of a graph is represented by the clique number of the graph (which is the largest set of mutually adjacent vertices in the graph). It is justified by the fact that all the vertices in the clique are mutually adjacent therefore they each must have a different color. In the case of complete graphs (all vertices connected with edges) with n vertices, the chromatic number is n. An upper bound for the chromatic number is given in terms of the maximum degree of the vertex i.e every graph can be colored with one more color than the maximum vertex degree.

In this chapter we will show that given a graph with all edges of size 1, we can determine bounds for the chromatic number using the information offered by the matrix $Y = -\frac{1}{2}J\hat{D}J$ and its nuclear norm, where $\hat{D}$ is an optimal solution of the convex optimization formulation of the Euclidean distance matrix completion problem

$$\begin{aligned} \underset{\hat{D}}{\text{minimize}} \quad & \left\| -\frac{1}{2}J\hat{D}J \right\|_* \\ \text{subject to} \quad & \hat{D}_{ij} = D_{ij},\ (i,j) \in E, \\ & -\frac{1}{2}J\hat{D}J \succeq 0. \end{aligned}$$

The partial distance matrix D is given by the squares of the edge lengths of the graph. All the graphs considered are unit graphs (i.e. all edge lengths are of length 1). In all the

examples considered, the rank of the matrix Y is counted as the number of singular values that are greater than 10^{-6}.

First, we will prove that in the case of complete $(a_1, a_2, \ldots, a_k)$ partite graphs, the nuclear norm of the matrix Y is given by the second elementary symmetric polynomial evaluated at the point $(a_1, a_2, \ldots, a_k)$ divided by the number of vertices in the graph. Further, it will be shown that the maximum value of the nuclear norm is attained in the case when each set in the partition has the same number of vertices $\frac{n}{k}$ and is equal to $n\frac{k-1}{2k}$. This information can be used to determine a lower bound for the chromatic number of a given graph. The criteria used to determine the lower bound is: if $\|Y\|_* > n\frac{k-1}{2k}$ then the graph is not k-partite, therefore its chromatic number is greater than k.

Also, for each of the examples considered we will determine the number of different points that generate matrix Y. The number of different points that generate matrix Y offers an upper bound for the chromatic number.

In cases where the graphs are balanced partite (same number of vertices in the partition) we can determine the chromatic number in a large number of cases. As we remove edges away from the complete graphs, we can still find the chromatic numbers for balanced k partite graphs keeping roughly 70% of all possible edges.

6.1. Nuclear norm of Y and l_1 norm of D for k-partite graphs

It was proven in Chapter 5

$$\|Y\|_* = \frac{1}{2n}\|D\|_{l_1}$$

where the l_1 norm of D is given by the sum of the absolute value of the entries of D. A k-partite graph is a graph whose vertices can be partitioned into k disjoint sets such that no vertices within the same set are adjacent. A complete k-partite graph is a graph whose

vertices are connected with all the others from the rest of $k-1$ sets. Any k-partite graph is a subgraph of a complete k-partite graph. In general the adjacency matrix of a graph is not a Euclidean distance matrix. But, for complete k-partite graphs the statement is true.

LEMMA 6.1.1. *The adjacency matrix of a complete k-partite graph is a Euclidean distance matrix.*

PROOF. The adjacency matrix of a complete k-partite graph admits the following block structure with a proper labelling of the nodes:

$$D = \begin{bmatrix} \mathbf{0}_{a_1 \times a_1} & \mathbf{1}_{a_1 \times a_2} & \dots & \mathbf{1}_{a_1 \times a_k} \\ \mathbf{1}_{a_2 \times a_1} & \mathbf{0}_{a_2 \times a_2} & \dots & \mathbf{1}_{a_2 \times a_k} \\ \dots & \dots & \dots & \dots \\ \mathbf{1}_{a_k \times a_1} & \mathbf{1} a_k \times a_2 & \dots & \mathbf{0}_{a_k \times a_k} \end{bmatrix} \tag{34}$$

We have a complete k- partite graph. The first a_1 nodes can be mapped to the same point since there is no edge connecting them in the graph. Same with the next a_2 points. Therefore we can look at the complete k partite graph as a graph that is formed with only k different points, each point connected with its $k-1$ neighbours. The distance matrix for this graph is exactly the adjacency matrix (34). □

PROPOSITION 6.1.1. *Let G be a k-partite graph with n vertices split into sets of cardinality $a_1, a_2, \dots, a_k$ and let D be its adjacency matrix. The nuclear norm of matrix $Y = -\frac{1}{2}JDJ$ is given by the second elementary symmetric polynomial evaluated at the point $(a_1, a_2, \dots, a_k)$ divided by the number of vertices in the graph*

$$\|Y\|_* = \frac{\sum_{i \neq j} a_i a_j}{n} \tag{35}$$

PROOF. Since G is a complete graph, the distance matrix D is

$$D = \begin{bmatrix} \mathbf{0}_{a_1 \times a_1} & \mathbf{1}_{a_1 \times a_2} & \dots & \mathbf{1}_{a_1 \times a_k} \\ \mathbf{1}_{a_2 \times a_1} & \mathbf{0}_{a_2 \times a_2} & \dots & \mathbf{1}_{a_2 \times a_k} \\ \dots & \dots & \dots & \dots \\ \mathbf{1}_{a_k \times a_1} & \mathbf{1}a_k \times a_2 & \dots & \mathbf{0}_{a_k \times a_k} \end{bmatrix}$$

As usual, let $J = I_n - \frac{1}{n}\mathbf{1}_{n \times n}$. Let $Y = -\frac{1}{2}JDJ$. The nuclear norm of Y is given by the trace of Y.

$$\begin{aligned} \text{trace}(Y) &= -\frac{1}{2}\text{trace}(Y_1) \\ Y_1 &= \left[I_n - \frac{1}{n}\mathbf{1}_{n \times n}\right] D \left[I_n - \frac{1}{n}\mathbf{1}_{n \times n}\right] \\ Y_1 &= D - \frac{1}{n}D\mathbf{1}_{n \times n} - \frac{1}{n}\mathbf{1}_{n \times n}D + \frac{1}{n^2}\mathbf{1}_{n \times n}D\mathbf{1}_{n \times n} \\ \text{trace}(Y_1) &= 0 - \frac{1}{n}\text{trace}(D\mathbf{1}_{n \times n}) - \frac{1}{n}\text{trace}(\mathbf{1}_{n \times n}D) + \frac{1}{n^2}\text{trace}(\mathbf{1}_{n \times n}D\mathbf{1}_{n \times n}) \end{aligned}$$

We evaluate each term in the relation above separately, for the products $D\mathbf{1}_{n \times n}$ and $\mathbf{1}_{n \times n}D$ we are interested in computing only the block matrices that contain the diagonal that allow us to compute the trace

$$\begin{bmatrix} \mathbf{0}_{a_1 \times a_1} & \mathbf{1}_{a_1 \times a_2} & \dots & \mathbf{1}_{a_1 \times a_k} \\ \mathbf{1}_{a_2 \times a_1} & \mathbf{0}_{a_2 \times a_2} & \dots & \mathbf{1}_{a_2 \times a_k} \\ \dots & \dots & \dots & \dots \\ \mathbf{1}_{a_k \times a_1} & \mathbf{1}a_k \times a_2 & \dots & \mathbf{0}_{a_k \times a_k} \end{bmatrix} \mathbf{1}_{n \times n} = \begin{bmatrix} \sum_{j \neq 1} a_j \mathbf{1}_{a_1 \times a_1} & \dots & \dots \\ \dots & \sum_{j \neq 2} a_j \mathbf{1}_{a_2 \times a_2} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \sum_{j \neq k} a_j \mathbf{1}_{a_k \times a_k} \end{bmatrix}$$

Therefore

$$\text{trace}(D\mathbf{1}_{n \times n}) = 2\sum_{i \neq j} a_i a_j$$

and

$$\text{trace}(\mathbf{1}_{n\times n}D) = 2\sum_{i\neq j} a_i a_j$$

Now we evaluate the product $\mathbf{1}_{n\times n}D\mathbf{1}_{n\times n}$.

$$\mathbf{1}_{n\times n}D\mathbf{1}_{n\times n} = \begin{bmatrix} t\mathbf{1}_{a_1\times a_1} & \dots & \dots \\ \dots & t\mathbf{1}a_2 \times a_2 & \dots \\ \dots & \dots & \dots \\ \dots & \dots & t\mathbf{1}a_k \times a_k \end{bmatrix} \text{ where } t = 2(\sum_{i\neq j} a_i a_j).$$

Therefore

$$\text{trace}(\mathbf{1}_{n\times n}D\mathbf{1}_{n\times n}) = t(a_1 + a_2 + \ldots + a_k)$$

and it follows that

$$\begin{aligned} \text{trace}(Y_1) &= -\frac{t}{n} - \frac{t}{n} + \frac{1}{n^2}(a_1 + \ldots + a_k)t \\ \text{trace}(Y) &= \frac{t}{2n} + \frac{t}{2n} - \frac{1}{2n^2}nt \\ \text{trace}(Y) &= \frac{t}{2n} \end{aligned}$$

Thus concluding that the nuclear norm of matrix Y for the case of complete k-partite graphs is given by

$$\|Y\|_* = \frac{\sum_{i\neq j} a_i a_j}{n}$$

□

PROPOSITION 6.1.2. *Let G be a k-partite graph, vertices split into sets of cardinality $a_1, a_2, \ldots, a_k$. Let n be the number of vertices of the graph. The maximum value of the nuclear norm of Y*

$$\|Y\|_* = \frac{\sum_{i\neq j} a_i a_j}{n}$$

is $n\frac{k-1}{2k}$ *attained for* $a_j = \frac{n}{k}$, $\forall j = 1, \ldots, k$.

PROOF. We know that $a_1 + a_2 + \ldots + a_k = n$. Let $f(a_1, \ldots, a_k) = \sum_{i\neq j} a_i a_j$. We express f as function of $k-1$ variables using the fact that $a_k = n - a_1 - a_2 - \ldots - a_{k-1}$. Thus

$$\frac{\partial f}{\partial a_j} = n - \sum_{i=1}^{k-1} a_i - a_j,\ j = 1, \ldots, k-1.$$

We set all the partial derivatives equal to 0. If we add all the $k-1$ equations given by the partial derivatives equal to 0 we obtain that $\sum_{i=1}^{k-1} a_i = \frac{n(k-1)}{k}$ and therefore $a_j = n - \frac{n(k-1)}{k} = \frac{n}{k}$. Value of the nuclear norm in this case is $\|Y\|_* = \frac{\sum_{i\neq j} a_i a_j}{n} = \frac{n^2}{k^2}(k-1+k-2+\ldots+1)\frac{1}{n} = n\frac{k-1}{2k}$. Now given $f(a_1, \ldots, a_k) = \sum_{i\neq j} a_i a_j$ we can express it as

$$f(a_1, \ldots, a_k) = (\sum_{i=1}^{k} a_i)^2 - \sum_{i=1}^{k} a_i^2$$

$$f(a_1, \ldots, a_k) = n^2 - \sum_{i=1}^{k} a_i^2 \tag{36}$$

Let $u = \left(\frac{n}{k}, \ldots, \frac{n}{k}\right)$. Consider the point $u + \epsilon v = \frac{n}{k}(1, \ldots, 1) + \epsilon v$, where v is a k dimensional vector with sum of its entries equal to 0. Then $v^t u = 0$ and $\|v+u\|^2 = \|v\|^2 + \|u\|^2$. Therefore

$$f(u+\epsilon v) = n^2 - \|u+\epsilon v\|^2 = n^2 - \|u\|^2 - \|\epsilon v\|^2 \leq n^2 - \|u\|^2 \leq f(u)$$

We can conclude that the maximum value of f is attained at the point $u = \left(\frac{n}{k}, \ldots, \frac{n}{k}\right)$. □

In the next section we will show how the behavior of the nuclear norm, which was characterized above, will lead to finding bounds for the chromatic number.

Example 6.2.1. Consider a tripartite graph with 4 vertices in each set, $a_k = 4$, $k = 1, 2, 3$. We generate a partial distance matrix D with the number of edges variable. To obtain it, consider a matrix $D = \text{rand}(12, 12)$. Set the diagonal elements equal to 0 and also 4×4 matrices around the diagonal to be $\mathbf{0}$. Let $p \in (0, 1)$ be a density number. The remaining entries of D greater than p were set to 0 and all those less than p were set to one. For various values of p we solve the optimization problem (29).

The results are shown in Table 6.1. In all the cases the nuclear norm of Y ranges between 3 and 4. We compare the nuclear norm of Y with the maximum value of the nuclear norm for bipartite graphs and we notice that we have $\|Y\|_* > \frac{n}{4} = 3$ therefore we can say that the graph is not bipartite and its chromatic number is greater than or equal to 3. The upper bound for the chromatic number is given by the number in the last column. When the graph is a complete tripartite graph we have 48 edges. In this case the upper bound for the chromatic number is 3 which coincides with the lower bound and we can conclude the chromatic number of the graph is 3.

As we remove edges we can still identify that we have a tripartite graph until we remove about 30% of the edges. After this we obtain a lower bound for the chromatic number via the nuclear norm and an upper bound given by the number of different points that generate the matrix Y.

Example 6.2.2. Consider a graph with 25 vertices. If it were a complete graph on 25 vertices then the number of edges would be $\frac{25(25-1)}{2} = 300$. Choose at random 162 edges. Use this partial matrix as input for the optimization problem 29. We obtain $\|Y\|_* = 9.93$. For $k = 4$ we have $\frac{n(k-1)}{2k} = 9.37$ therefore we can say that the graph its not 4-partite. The

TABLE 6.1. Norm and rank behavior for 4, 4, 4 tripartite graph

No of edges	l_1 norm of D	$\|Y\|_*$	Rank of Y	No of different points
48	96.000000	4.000000	2	3
47	96.000000	4.000000	2	3
46	96.000000	4.000000	2	3
45	96.000000	4.000000	2	3
44	96.000000	4.000000	2	3
43	96.000000	4.000000	2	3
42	96.000000	4.000000	2	3
41	96.000000	4.000000	2	3
38	96.000000	4.000000	2	3
37	96.000000	4.000000	2	3
36	96.000000	4.000000	2	3
35	96.000000	4.000000	2	3
34	96.000000	4.000000	2	3
33	91.668950	3.819540	2	4
31	91.668950	3.819540	2	6
29	88.000000	3.666667	3	7
28	96.000000	4.000000	3	12
27	90.996299	3.791512	4	12
26	94.166580	3.923607	6	12
24	87.804896	3.658537	4	12
23	83.217096	3.467379	4	12
22	87.710791	3.654616	4	9
21	90.096996	3.754041	5	12
20	80.571847	3.357160	4	12
19	84.870147	3.536256	5	12
18	76.078345	3.169931	2	8
17	72.430781	3.017949	2	5
16	74.510304	3.104596	4	12
15	76.054493	3.168937	5	12
14	73.097142	3.045714	4	12

matrix Y is generated by 25 different points. Now, if we increase the number of edges to 232 and we solve the semidefinite program we obtain the nuclear norm $10.92 > 25\frac{6-1}{12} = 10.41$ therefore the graph is not 6-partite.

EXAMPLE 6.2.3. Consider a tripartite graph, with 3, 3, and 4 vertices in each disjoint set. In Table 6.2 we show the evolution of the nuclear norm of Y and l_1 norm of D when we increase the number of edges one by one starting with a minimum configuration of edges

that makes the graph tripartite until we reach a complete tripartite graph. $\|Y\|_*$ ranges between 3.15 and 3.3. Since $\|Y\|_* > 10\frac{1}{4} = 2.5$ we can say that the graph is not bipartite in all cases.

We can identify that the graph is tripartite as we increase the number of edges and we approach the situation when the graph is complete tripartite. When rank of Y is equal 2 we have only 3 different points that generate Y and we can conclude that the graph is tripartite.

TABLE 6.2. Norms behaviour for 3,3,4 partite graph

Number of edges	$\|D\|_{l_1}$	$\|Y\|_*$	Iterations	Precision	Rank Y	Edge added
11	63.06	3.15	13	$5.8 * 10^{-11}$	5	
12	63.06	3.15	15	$1.2 * 10^{-10}$	5	$D_{15} = 1$
13	64.54	3.23	15	$2.2 * 10^{-10}$	6	$D_{16} = 1$
14	64.54	3.23	16	$7 * 10^{-11}$	6	$D_{18} = 1$
15	64.54	3.23	16	$1.6 * 10^{-10}$	6	$D_{19} = 1$
16	64.54	3.23	16	$6.6 * 10^{-11}$	6	$D_{24} = 1$
17	64.54	3.23	16	$5.8 * 10^{-11}$	6	$D_{26} = 1$
18	64.91	3.24	15	$2.5 * 10^{-10}$	5	$D_{27} = 1$
19	64.91	3.24	15	$5.6 * 10^{-11}$	5	$D_{29} = 1$
20	64.91	3.24	15	$2.2 * 10^{-11}$	5	$D_{2,10} = 1$
21	64.91	3.24	14	$4.5 * 10^{-10}$	5	$D_{34} = 1$
22	64.91	3.24	16	$3.7 * 10^{-11}$	5	$D_{35} = 1$
23	65.33	3.27	15	$1.8 * 10^{-11}$	4	$D_{37} = 1$
24	65.33	3.27	14	$1.4 * 10^{-11}$	4	$D_{38} = 1$
25	65.33	3.27	14	$6.9 * 10^{-12}$	4	$D_{3,10} = 1$
26	65.33	3.27	23	$6.9 * 10^{-11}$	4	$D_{48} = 1$
27	65.8	3.29	19	$8.1 * 10^{-14}$	3	$D_{49} = 1$
28	65.8	3.29	21	$6.1 * 10^{-15}$	3	$D_{4,10} = 1$
29	66	3.3	17	$1.5 * 10^{-15}$	2	$D_{57} = 1$
30	66	3.3	15	$1.1 * 10^{-15}$	2	$D_{59} = 1$
31	66	3.3	17	$1.8 * 10^{-15}$	2	$D_{5,10} = 1$
32	66	3.3	16	$3.3 * 10^{-15}$	2	$D_{67} = 1$
33	66	3.3	6	$7.1 * 10^{-16}$	2	$D_{68} = 1$
34	69.01	3.45	15	$1.5 * 10^{-15}$	3	not tripartite anymore

EXAMPLE 6.2.4. Consider an eight-partite graph with 6 vertices in each set, $n = 48$. We generate a partial distance matrix D with the number of edges variables. To obtain it

consider a matrix $D = \text{rand}(48, 48)$. Set the diagonal elements equal to 0 and also 6×6 matrices around the main diagonal be $\mathbf{0}_{6\times 6}$. Let $p \in (0, 1)$ be a density number. The remaining entries of D greater than p were set to 0 and all those less than p were set to 1.

For various values of p we solve the optimization problem (29). The results are presented in Table 6.3. The complete 8-partite graph with 6 vertices in each set of the partition has 1008 edges. We can identify that it is an 8-partite graph: matrix Y is generated by 8 different points therefore we have an upper bound for the chromatic number and since $\|Y\|_* = 21 > 48\frac{7-1}{14} = 20.57$ it is not a 7-partite graph.

For the graph with 778 edges we have that $\|Y\|_* = 20.94 > 48\frac{7-1}{14} = 20.57$ therefore it is not a 7-partite graph. An upper bound for the chromatic number in this case is 9.

TABLE 6.3. Norm and rank behavior for 8-partite graph

No of edges	l_1 norm of D	$\|Y\|_*$	Rank of Y	No of different points
1008.000000	2016.000000	21.000000	7	8
999.000000	2016.000000	21.000000	7	8
993.000000	2016.000000	21.000000	7	8
978.000000	2016.000000	21.000000	7	8
971.000000	2016.000000	21.000000	7	8
970.000000	2016.000000	21.000000	7	8
956.000000	2016.000000	21.000000	7	8
940.000000	2016.000000	21.000000	7	8
934.000000	2016.000000	21.000000	7	8
915.000000	2016.000000	21.000000	7	8
902.000000	2016.000000	21.000000	8	10
876.000000	2016.000000	21.000000	7	8
862.000000	2016.000000	21.000000	7	8
846.000000	2016.000000	21.000000	8	14
841.000000	2016.000000	21.000000	7	8
840.000000	2016.000000	21.000000	7	8
838.000000	2016.000000	21.000000	8	10
824.000000	2016.000000	21.000000	8	10
809.000000	2016.000000	21.000000	7	8
798.000000	2016.000000	21.000000	9	16
795.000000	2016.000000	21.000000	8	10
778.000000	2010.204453	20.939630	7	9
777.000000	2016.000000	21.000000	9	15
757.000000	2015.997394	20.999973	30	48
756.000000	2009.678647	20.934153	8	15
726.000000	2016.000000	21.000000	8	23
707.000000	2015.969498	20.999682	28	48
699.000000	2011.225968	20.950270	28	48
684.000000	2003.285580	20.867558	27	48
652.000000	2007.905434	20.915682	27	48
666.000000	2001.012876	20.843884	27	48
680.000000	2007.120782	20.907508	27	48
636.000000	1989.123562	20.720037	26	48
665.000000	1998.637279	20.819138	27	48
621.000000	1991.571369	20.745535	26	48
604.000000	1981.442624	20.640027	25	48
608.000000	1975.434746	20.577445	25	48
595.000000	1974.399113	20.566657	25	48
574.000000	1954.548251	20.359878	23	48
555.000000	1955.254664	20.367236	24	48
532.000000	1938.948665	20.197382	22	48
509.000000	1925.111398	20.053244	21	48
517.000000	1922.759725	20.028747	22	48

CHAPTER 7

Finding graph embeddings using the nuclear norm heuristic

7.1. Graph embeddings for unit distance graphs

Given a graph $G = (V, E)$ consisting of a set V of vertices, a set of edges E and a set of distances associated with each edge, the graph realization problem is to assign to each vertex coordinates in $\mathbb{R}^k$ such that the Euclidean distance between any two neighboring nodes matches the size of that edge. Graph realization is equivalent to the Euclidean distance matrix completion problem [18], [19], [20].

In this chapter we look for graph embeddings using the semidefinite optimization formulation of the Euclidean distance matrix completion problem. We aim to find low rank solutions using the nuclear norm heuristic i.e. minimization of the nuclear norm of the double centered matrix D such that the points that generate the distance matrix are all different. The examples considered partial distance matrices with the known entries equal to one, i.e., the edges of the graph are all equal one.

Let D be a partial distance matrix of dimension $n \times n$ with known entries that corresponds to the square distances between edges of a graph $G = (V, E)$. Then we use the semidefinite program

$$\begin{aligned} \underset{Y}{\text{minimize}} \quad & \|Y\|_* \\ \text{subject to} \quad & Y_{ii} - 2Y_{ij} + Y_{jj} = D_{ij},\ (i,j) \in \Omega, \\ & Y \succeq 0. \end{aligned} \tag{37}$$

to find the embedding dimension and the distances between the rest of the vertices. If a solution Y of the problem (37) has rank k and is generated by n distinct points, then we have a Euclidean distance matrix completion of D with embedding dimension k; this is possible even if D has a completion with a lower embedding dimension r. If the solution Y of (37) is not generated by distinct points we will add an additional constraint to the problem given by $D_{ij} \geq 0.01$, $\forall\ i \neq j$ which guarantees that the solution is generated by different points. The rank of matrix Y reflects the dimension of the embedding. So, basically we look for solutions of the following optimization problem

$$\begin{aligned} \underset{Y}{\text{minimize}} \quad & \|Y\|_* \\ \text{subject to} \quad & Y_{ii} - 2Y_{ij} + Y_{jj} = D_{ij},\ (i,j) \in \Omega, \\ & Y_{ii} - 2Y_{ij} + Y_{jj} > 0.01, \forall i \neq j \\ & Y \succeq 0. \end{aligned} \tag{38}$$

We evaluate the rank of matrix Y as the number of singular values of matrix Y that are greater than 10^{-6}. The Euclidean distance matrix D resulted from the optimal solution Y is given by the relation $D = \text{diag}(Y)\mathbf{1}_{1\times n} + \mathbf{1}_{n\times 1}\text{diag}(Y)^t - 2Y$.

Several convex optimization solvers are available (CVX, YALMIP, SeDuMI, SDPT3) to solve semidefinite programs. In our examples we made use of existing solver software i.e CVX solver [29].

Example 7.1.1. **The Moser Spindle** As a unit distance graph, the Moser Spindle is formed by two rhombi, so that the sides and short diagonals of the rhombi form equilateral triangles. The eleven edges of the graph are the eight rhombus sides, the two short diagonals of the rhombi, and the edge between the unit-distance pair of acute-angled vertices. We

build the partial distance matrix associated with its edges $D_{12} = D_{16} = D_{17} = 1$, $D_{23} = D_{25} = D_{26} = 1$, $D_{34} = D_{35} = 1$, $D_{45} = D_{47} = 1$, $D_{67} = 1$. The optimization problem (37)

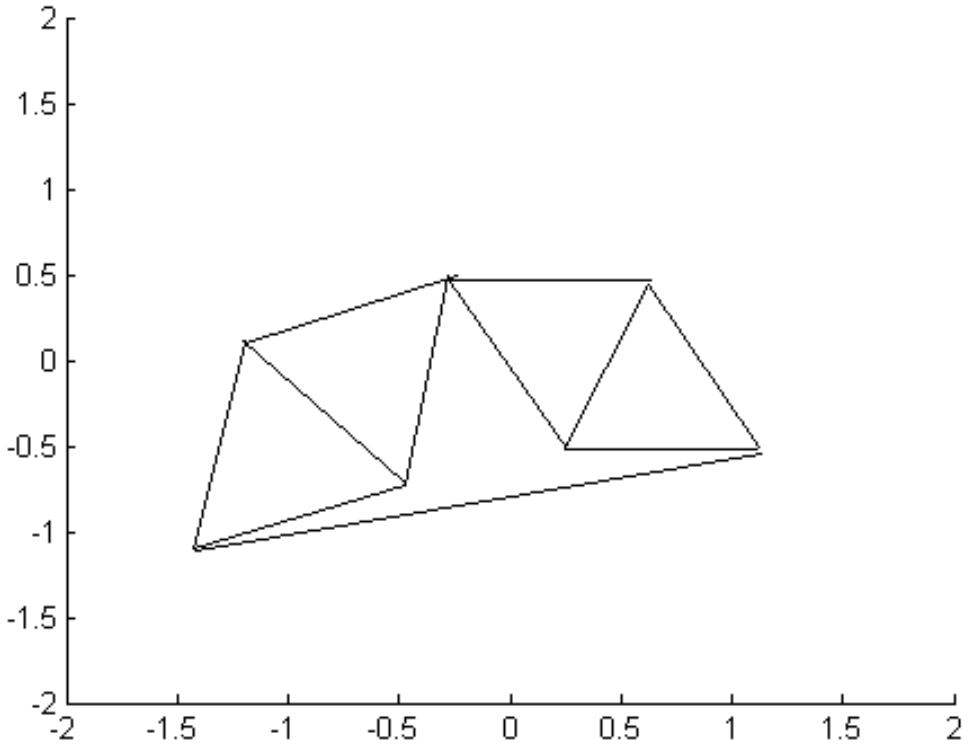

FIGURE 7.1. Moser Spindle graph.

has a solution Y of rank 4 generated by 7 different points in $\mathbb{R}^4$, therefore we obtain an embedding of the graph into $\mathbb{R}^4$.

EXAMPLE 7.1.2 (Butterfly graph). The butterfly graph is a graph with 5 vertices and edges $D_{12} = D_{13} = D_{14} = D_{15} = 1$, $D_{25} = 1$, $D_{34} = 1$. The solution of the problem (37) has a positive semidefinite matrix $Y = -\frac{1}{2}JDJ$ of rank 3, generated by 5 different points. Therefore we obtain an embedding of the graph into $\mathbb{R}^3$ (Figure 7.2).

EXAMPLE 7.1.3 (Möbius Kantor graph). We consider the partial distance matrix associated with the Möbius Kantor graph, a graph with 16 vertices and 24 edges (Figure 7.3).

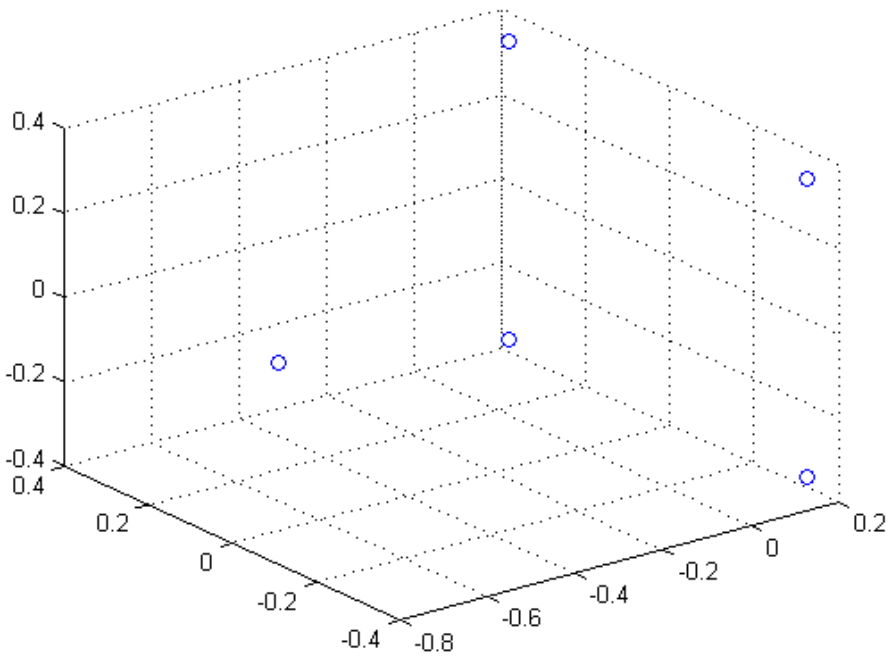

FIGURE 7.2. Butterfly graph embedding.

The solution Y of the optimization problem (37) is rank one, generated only by two different

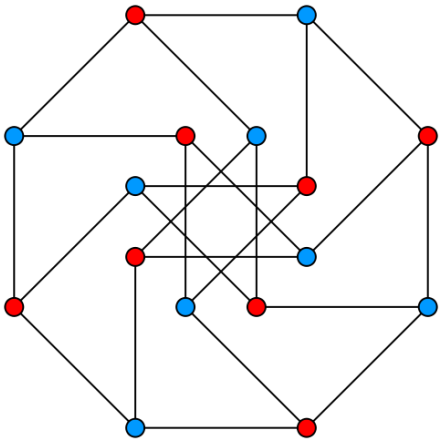

FIGURE 7.3. Möbius Kantor graph [2].

points confirming that this graph is a bipartite graph.

If we impose the condition that $D_{ij} > 0.01$, i.e., points to be non-overlapping for all $i \neq j$ we obtain an embedding of the graph into $\mathbb{R}^5$.

EXAMPLE 7.1.4. Consider 4 points in the plane such that the points form a square and each of the sides of the square has length 1 as in the Figure 7.4 below. The distance matrix

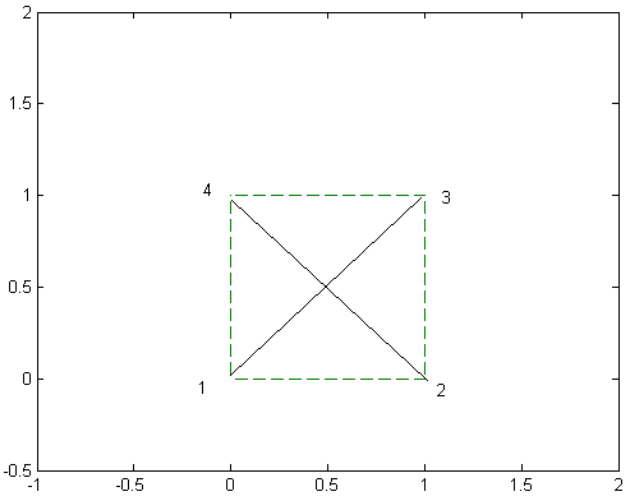

FIGURE 7.4. Example: 4 points in the plane.

for this case is

$$D = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}.$$

Then we set the entry that corresponds to the distance between vertex 1 and vertex 3 to be variable and we use this distance matrix as a input for the optimization problem.

Schoenberg results states that given a symmetric matrix with elements on the main diagonal equal zero, this matrix is a Euclidean distance matrix if and and only if matrix defined as $Y = -\frac{1}{2}JDJ$ is positive semidefinite matrix, where $J = I_n - \frac{1}{n}\mathbf{1}_{n\times n}$. In this case

distance matrix

$$D = \begin{bmatrix} 0 & 1 & x & 1 \\ 1 & 0 & 1 & 2 \\ x & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

gives us a corresponding matrix Y. We solve the optimization problem (37). We obtain that the rank of Y is 2, $x = 0$. We plot the points in the plane and we obtain points in Figure 7.5, therefore without imposing the condition that the entries in the matrix D should be strictly positive outside the main diagonal we don't obtain an embedding into $\mathbb{R}^2$, but a mapping. Nuclear norm of Y for variable $x = 0$ is 1.84. Nuclear norm of matrix Y for the initial distance matrix, with 4 different points arranged in the square is 2.4. Now, let's choose another point in the plane as in Figure 7.6 such that the distance matrix for these 5 points has the following entries

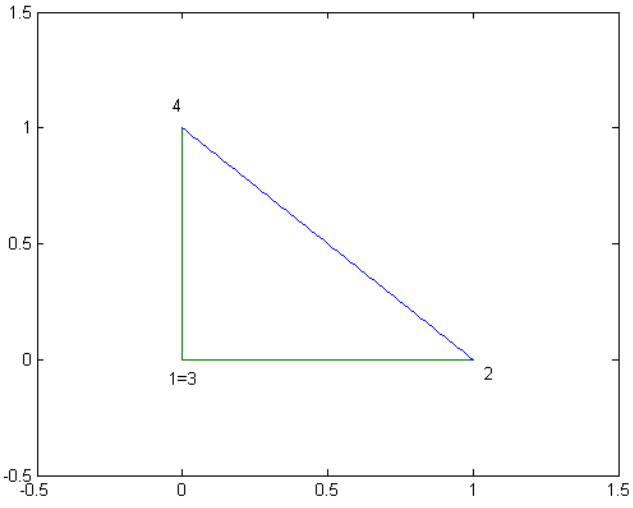

FIGURE 7.5. Configuration of points obtained via nuclear norm minimization.

$$D = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 5 \\ 1 & 2 & 1 & 0 & 4 \\ 1 & 2 & 5 & 4 & 0 \end{bmatrix}$$

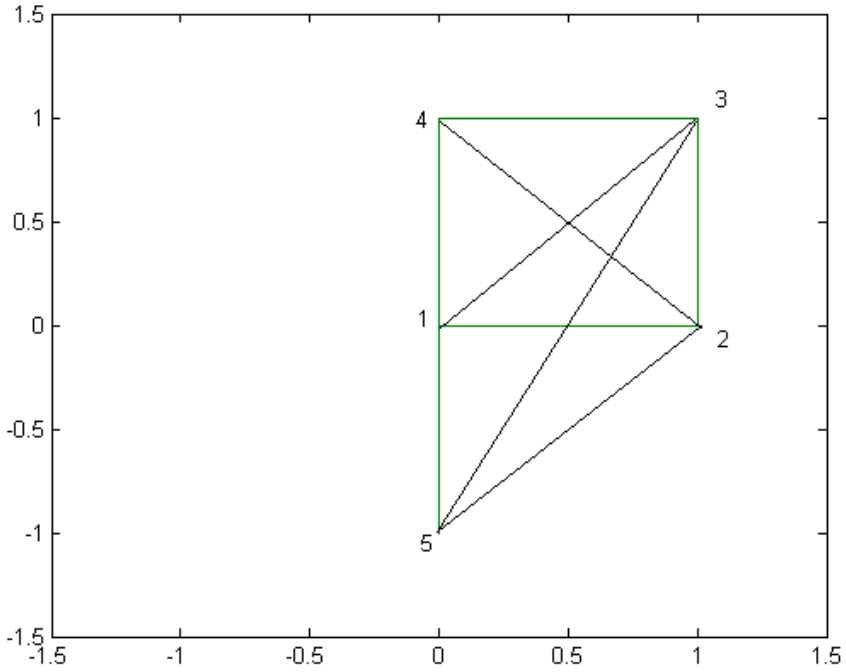

FIGURE 7.6. Example: 5 points in the 2-dimensional plane.

Let of one the entries be unknown

$$D = \begin{bmatrix} 0 & 1 & x & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ x & 1 & 0 & 1 & 5 \\ 1 & 2 & 1 & 0 & 4 \\ 1 & 2 & 5 & 4 & 0 \end{bmatrix}$$

The optimization problem 37 returns as optimal value $x = 2$. Same, if we let a second distance be unknown

$$D = \begin{bmatrix} 0 & 1 & x & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ x & 1 & 0 & 1 & 5 \\ 1 & 2 & 1 & 0 & y \\ 1 & 2 & 5 & y & 0 \end{bmatrix}$$

we get as optimal solution $x = 2$, $y = 4$. If we let a third distance be variable, with the notation

$$D = \begin{bmatrix} 0 & 1 & x & 1 & 1 \\ 1 & 0 & 1 & z & 2 \\ x & 1 & 0 & 1 & 5 \\ 1 & z & 1 & 0 & y \\ 1 & 2 & 5 & y & 0 \end{bmatrix}$$

the optimal solution gives $x = 2$, $y = 2$, $z = 0$. If we plot the points that generate this distance matrix we obtain Figure 7.7, where point 5 gets mapped as a vertex of the square and points 2 and 4 coincide.

The nuclear norm of matrix Y in this case is 3.2 while nuclear norm for matrix Y for our initial configuration of points is 4.

EXAMPLE 7.1.5. **McGee unit graph** is a graph with 24 vertices and 36 edges. Its chromatic number is 3, i.e., we can color its vertices with three colors such that no two adjacent vertices have the same color. Its vertices can be split into three disjoint sets of cardinality 4,10,10. We write the partial distance matrix D such that the known entries are equal with one. Using the CVX solver [29] for the optimization problem with the constraints given by the length of the edges, we obtain as solution a positive semidefinite matrix of rank

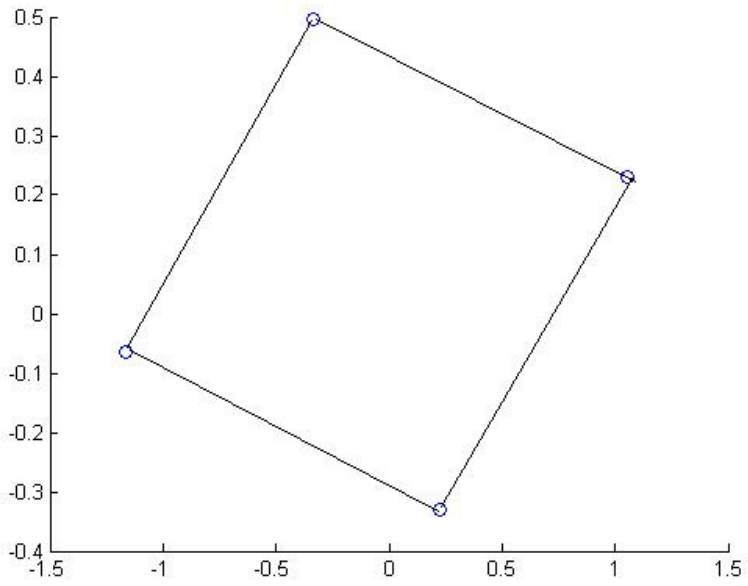

FIGURE 7.7. Solution of the optimization problem for the 5 points example.

5, generated by 24 different points in $\mathbb{R}^5$, we obtain an embedding into $\mathbb{R}^5$. Nuclear norm of matrix Y is 6.64 and l_1 norm of matrix D is 318.5.

EXAMPLE 7.1.6. **The Heawood graph** is an undirected graph with 14 vertices and 21 edges (Figure 7.8). The Heawood graph is a unit distance graph: it can be embedded in the plane such that adjacent vertices are exactly at distance one apart, with no two vertices embedded to the same point and no vertex embedded into a point within an edge. The chromatic number of the Heawood graph is 2. It is a bipartite graph, its vertices can be divided into two disjoint sets such that every edge connects a vertex in one set to one in the other set. We follow a similar procedure, using the partial distance matrix given by the edges of the graph as input for the optimization problem (37), the constraints given by the length of the edges, we obtain as solution a positive semidefinite matrix of rank 1, generated only by 2 different points in $\mathbb{R}$, we obtain an mapping into $\mathbb{R}$, which confirms that the chromatic number of the graph is 2. In this case, nuclear norm of matrix Y is 3.5 and l_1 norm of

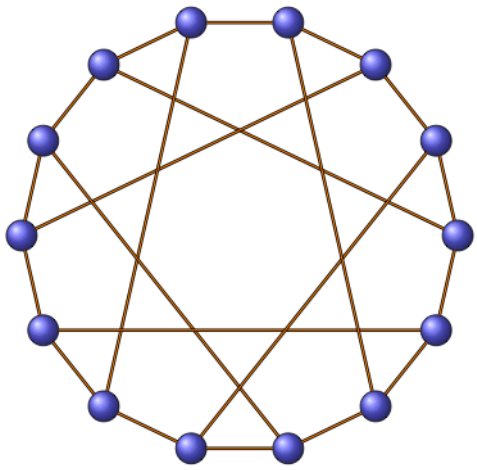

FIGURE 7.8. The Heawood graph [3].

matrix D is 98. Now, if we are looking to find an embedding we need to impose that the only zero elements of matrix D should be the ones on the diagonal. With the condition that $D_{ij} > 0.01$ for all $i \neq j$ we obtain a solution Y for problem (37) of rank 7, therefore an embedding into $\mathbb{R}^7$.

EXAMPLE 7.1.7. **The Cubical graph**: For this graph (Figure 7.9) the optimal solution of (37) has rank 1. We determine the points that generate this distance matrix and we obtain 2 different points, therefore the chromatic number of the graph is 2.

If we impose the condition that all entries of matrix D to be greater equal that 0.01 we obtain a matrix of rank 4 which gives us an embedding of the graph into $\mathbb{R}^4$.

EXAMPLE 7.1.8. **Claw graph**: We consider the partial distance matrix D for the claw graph in Figure 7.10 and the optimization problem (37) which minimizes the nuclear norm of the associated positive definite matrix Y. If we let the other entries of the matrix D as free variables we obtain a rank 1 matrix Y that is generated by 2 distinct points, chromatic number of the graph is 2. If we impose the condition that all entries of matrix D to be

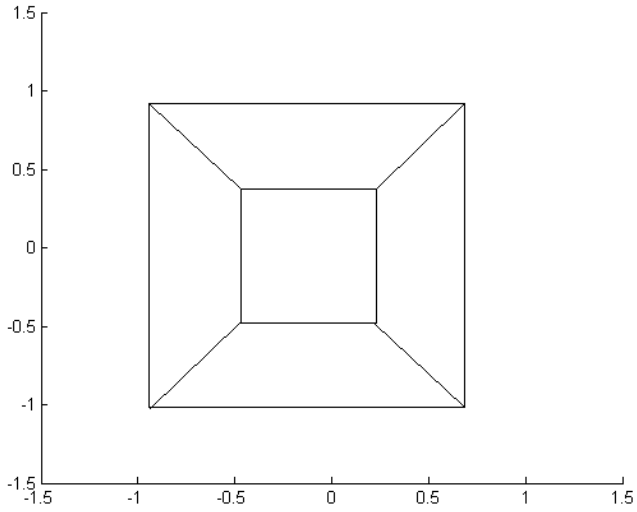

FIGURE 7.9. Cubical graph.

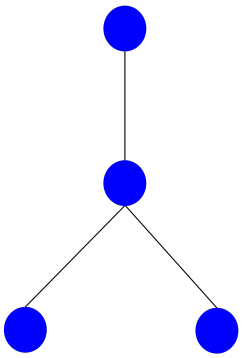

FIGURE 7.10. Claw graph [4].

greater equal 0.01 which implies that we have distinct points we obtain a matrix of rank 3. All entries in matrix D are less or equal 1.

EXAMPLE 7.1.9. **The Utility graph**: We consider the partial distance matrix for the utility graph, also known as Thomsen graph or $K_{3,3}$. It is a complete bipartite graph.

When allowing for overlapping points we obtain a rank 1 matrix S that is generated by 2 distinct points and therefore we have a bipartite graph. If we impose the condition that all entries of matrix D to be greater equal that 0.01 we obtain a matrix S of rank 5, an embedding into $\mathbb{R}^5$. All entries in matrix D are less or equal 1.

EXAMPLE 7.1.10. **The Franklin graph**: We consider the partial distance matrix D for

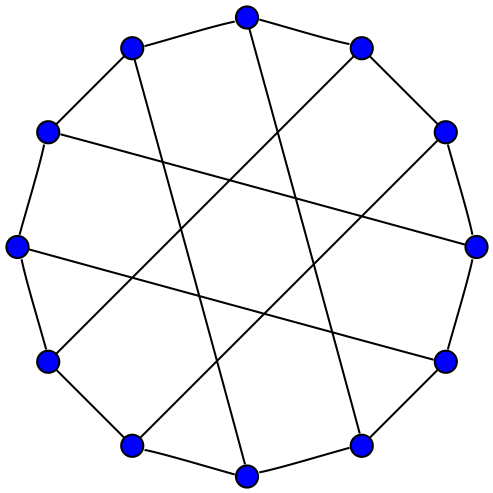

FIGURE 7.11. Franklin graph [5].

Franklin graph in Figure 7.11. When allowing for overlapping points we obtain a rank 1 matrix that is generated by 2 distinct points, therefore chromatic number 2. If we impose the condition that all entries of matrix D to be greater equal that 0.01 we obtain a matrix of rank 6. All entries in matrix D are less or equal 1.

7.2. Techniques for lowering the dimension of the embeddings

In this section we will show how to obtain a lower rank embedding for graphs when using the following heuristic: we have the partial distance matrix given by the edges of a graph, we consider two copies of the graph which we connect by a long edge and we use this partial distance matrix as input for the optimization problem. Then we analyze the information given by the submatrices corresponding to one copy of the graph. We compare the dimension of the embeddings with the dimension of the embedding resulted from the convex optimization problem corresponding to the partial distance matrix of the initial graph. In this way we can lower the dimension of the embedding by at least 1.

EXAMPLE 7.2.1. **McGee unit graph** Previously we obtained an embedding of the McGee graph into $\mathbb{R}^5$ as a unit distance graph. This graph with 24 nodes, 36 edges and chromatic number 3 can be embedded as a unit distance graph in the plane. Now consider two copies of the McGee unit graph connected with a long edge, 10 times longer than the edges of the graph. We build the partial distance matrix for this new graph that has 48 vertices and 73 edges and we use it as input for the optimization problem (7.1.5). We obtain that rank of Y is 8 and the euclidean distance matrix D is generated by 48 different points. Consider the distance matrices corresponding to one copy of the McGee graph D_1 and D_2 and $T = I_{24} - \frac{1}{24}\mathbf{1}_{24\times 24}$. Matrices $Y_i = -\frac{1}{2}TD_iT$, $i = 1, 2$ have rank 5 and 4. Rank 4 matrix Y_1 is generated by 24 different points therefore we obtain an embedding of the McGee graph into $\mathbb{R}^4$.

EXAMPLE 7.2.2. **The Heawood graph.** The Heawood graph is among the graphs that admit an embedding in the Euclidean plane in such a way that vertices correspond to points in the plane and adjacent vertices are connected with edges of size 1. It was only in 2008 that Gerbracht [77] presented 11 unit distance embeddings of the Heawood graph in the plane. Between 1972 and 2008 it was thought that such an embedding is not possible. In 1972 Chvatal [78] presented 37 open research problems from combinatorics. In [78] one of the open problems was related to the graphs obtained by assigning vertices to all the points and lines of a projective plane by joining a point-vertex to a line vertex if and only if the line passes through the point. The smallest example of these graphs is the Heawood graph. The Heawood graph is a point-line incidence graph of the Fano plane. An incidence graph is a bipartite graph associated with an incidence structure. An incidence structure consists of a set P of points and a set L of lines along with an incidence relation consisting of ordered pairs of points and lines. The graph has two types of vertices consisting of the points (one

color) and lines (the second color) with a point joined to a line if it is incident with it in S. The relation of incidence can be used in relation with points, lines or planes. Instead of saying that "'two lines meet" we may express the situation as "two lines are both incident with the same point", "'two points determine a line"' may be expressed as "'two points are both incident with some line". The Fano plane is defined as finite projective plane of order 2 with the smallest possible number of points and lines, 7 each, with 3 points on every line and 3 lines through every point. Chvatal's conjecture was that these graphs were not unit distance embeddable in the plane.

Previously we obtained an embedding of the Heawood graph into $\mathbb{R}^7$. Now we consider Heawood graph and one copy of it. We form a graph with 28 vertices. In addition we connect vertices 1 and 22 with a long edge of size 10. Therefore we have one known entry of matrix D 100 times larger than the rest of the other entries while all the other edges are of size 1. We are looking for Euclidean distance matrix completion of matrix D. 37 in this case returns a postive semidefinite matrix Y of size 28 by 28 of rank 3, generated by 18 distinct points, therefore a mapping into $\mathbb{R}^3$. The submatrices corresponding to one copy of the Heawood graph have rank 3 and is generated by 14 different points and rank 1, generated by 4 different points. Therefore we obtain an embedding into $\mathbb{R}^3$ for the Heawood graph. A plot of the points that generate the rank three submatrix is presented in Figure 7.12. If we impose the condition that all entries in the matrix $D_{ij} \geq 0.01$ i.e edges all greater than 0.1 we obtain an embedding for this bigger graph in $\mathbb{R}^8$. We analyze the rank of the positive semidefinite submatrices corresponding to each copy of the graph. They have rank 3 and rank 6. The condition $D_{ij} \geq 0.01$ ensures that we have no overlapping points and therefore we obtain also in this case an embedding of the Heawood graph in $\mathbb{R}^3$.

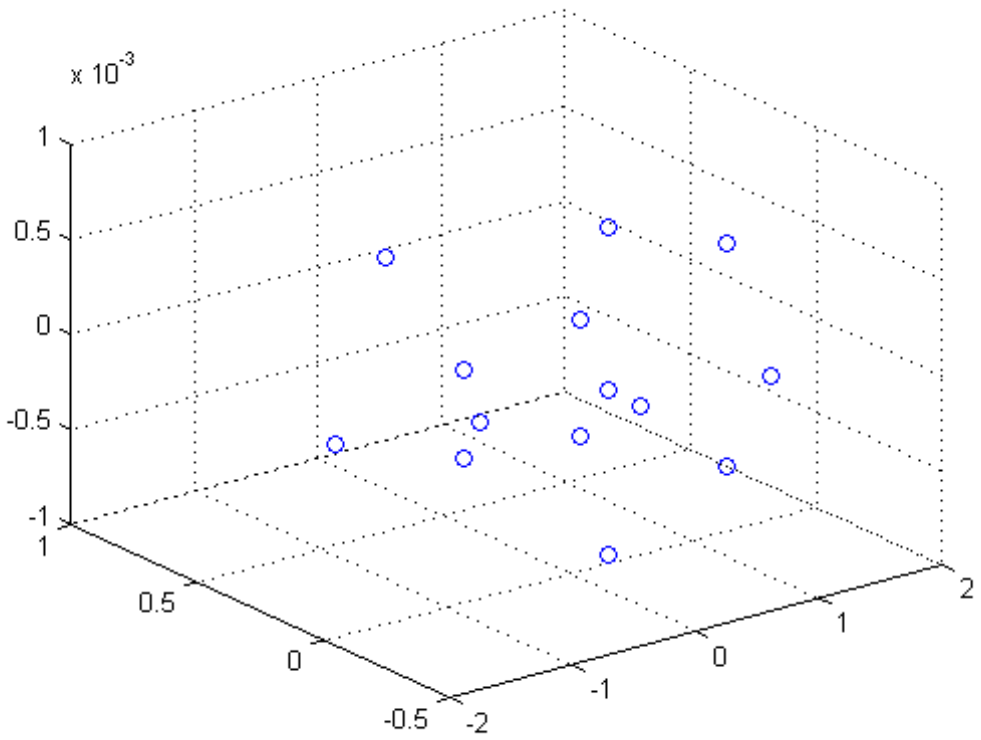

FIGURE 7.12. Heawood graph embedding in $\mathbb{R}^3$.

EXAMPLE 7.2.3. Consider a tripartite graph with 15 vertices, same number of vertices in the partition and 58 edges chosen at random.

We are looking for an embedding using (37). We impose the additional condition that $D_{i,j} \geq 0.01$ for all $i \neq j$. The solution Y has rank 10, therefore we obtain an embedding into $\mathbb{R}^{10}$.

Consider two copies of this graph connected by a long edge. We build the partial distance matrix 30×30, $D^* = [D\ \mathbf{0}_{15\times 15}; \mathbf{0}_{15\times 15}\ D]$ and then we connect the copies with a long edge $D^*_{12,20} = D^*_{20,12} = 100$. We impose the condition that $D^*_{ij} \geq 0.01$ and we solve the Euclidean distance matrix completion problem 37. The positive semidefinite submatrices corresponding to one copy of the graph have both rank 9, therefore we obtain an embedding into $\mathbb{R}^9$.

7.3. Borsuk Embeddings

In this section we formulate the problem of finding graph embeddings for unit diameter graphs establishing in this way a connection with the Borsuk problem which asks if a set of points in $\mathbb{R}^n$ can be split into $n+1$ parts of smaller diameter.

Definition 7.3.1. The diameter of a set $S \subset \mathbb{R}^n$ is $\sup_{x,y \in S} \|x-y\|$

Let $b(n)$ be the minimal number such that any bounded set in $\mathbb{R}^n$ consisting of at least 2 points can be partitioned into $b(n)$ parts of smaller diameter. In 1933 Borsuk conjectured that every bounded set in the n dimensional Euclidean space is the union of $n+1$ sets of smaller diameter [79], therefore $b(n) = n+1$. The Borsuk conjecture is true in dimensions 2 and 3. In 1993 Kalai and Kahn [80] presented an counterexample showing that $b(n) > 1.2^{\sqrt{n}}$ for large n. Their construction implied that $b(n) > n+1$ for $n = 1325$ and for all $n > 2014$ ([81]). In the next years improvements on the smaller dimension were reached $b(n) > n+1$ for $n = 946$ [82], $n = 561$ [83], $n = 560$ [84], $n = 323$ [85] and $n = 321$ [86]. In 2003 Hinrichs and Richter constructed finite sets in $\mathbb{R}^n$, $n \geq 298$, which cannot be partitioned into $n+11$ parts of smaller diameter thus decreasing the smallest dimension in which Borsuk conjecture is known to be false [87]. In 2013 it was shown that $b(65) \geq 84$ [81].

Theorem 7.3.1 (Bondarenko, 2013). *There is a two-distance subset $\{x_1, \ldots, x_{416}\}$ of the unit sphere $S^{64} \subset \mathbb{R}^{65}$ such that $\langle x_i, x_j \rangle = \frac{1}{5}$ or $-\frac{1}{15}$ for $i \neq j$ which cannot be partitioned into 83 parts of smaller diameter.*

A two distance set is a set of points in $\mathbb{R}^n$ for which all pairwise distances there are only two possible distances. Brouwer uses the example from [81] and shows that the conjecture is false in dimension $n = 64$ [88]. The set considered is a 64 dimensional set with 352 points

which cannot be split into fewer than 71 parts of smaller diameter. The image of the set of points can be seen as an image of a unit diameter graph in $\mathbb{R}^n$.

DEFINITION 7.3.2 (Unit diameter graph). A unit diameter graph is a graph $G = (V, E)$ where V is a set of points from $\mathbb{R}^n$, the set of the edges $E = \{(i, j) : \|x_i - x_j\| = 1\}$ and Euclidean distance between non-adjacent vertices is strictly less than 1.

All edges of the unit diameter graph have the same length and are given by the maximum distance between the points in the set V. The rule that defines the edges is given by the largest distance, i.e., we connect vertices in the graph with an edge only if their distance is equal to one, the maximum distance between the points in the set V. Therefore all non-edges have smaller length. The minimum number of parts of smaller diameter into which the set V can be partitioned equals the chromatic number of the graph G. We can generalize this definition to quasi-unit diameter graphs

DEFINITION 7.3.3 (Quasi-unit diameter graph). Given V a set of points, the quasi-unit diameter distance graph G is the graph that has the points in V as vertices and all the edges of distance 1 and for the non-adjacent vertices the Euclidean distance between them may be less or equal 1.

DEFINITION 7.3.4 (Borsuk embedding). A Borsuk embedding of a graph $G = (V, E)$ in $\mathbb{R}^n$ is a one to one mapping such that each vertex of the graph is mapped into $\mathbb{R}^n$ and the resulted graph is a unit diameter graph.

With this setting, Borsuk's problem can be also posed as follows: let the graph $G = (V, E)$ and its chromatic number $n + 2$, is there a Borsuk embedding into $\mathbb{R}^n$ such that G is a unit diameter graph? To be more concrete, we are interested in assigning to each node of the

graph a coordinate in $\mathbb{R}^n$, such that nodes that are neighbors in the graph have Euclidean distance 1 and nodes that are not neighbors have the Euclidean distance strictly less than 1.

Let G be unit distance graph. If we build the distance matrix between vertices of G we have that the Euclidean distance between the adjacent vertices in G is equal to one. Therefore $D_{ij} = 1,\ i \neq j,\ (i,j) \in E$. Now we can look for a completion of the distance matrix D such that the rest of the entries are strictly less than 1 when $Y = -\frac{1}{2}JDJ$ is low rank.

$$
\begin{aligned}
\underset{\hat{D}}{\text{minimize}} \quad & \left\| -\frac{1}{2} J\hat{D}J \right\|_* \\
\text{subject to} \quad & \hat{D}_{ij} = D_{ij},\ (i,j) \in E, \\
& -\frac{1}{2} J\hat{D}J \succeq 0, \\
& \hat{D}_{ii} = 0,\ i = 1, \ldots, n \\
& \hat{D}_{ij} < 1,\ (i,j) \notin E, \\
& \hat{D}_{ij} \geq 0.01 \forall i \neq j
\end{aligned}
\tag{39}
$$

Example 7.3.1. Consider a graph with 15 vertices and 52 edges of size 1, edges placed random. With the optimization problem (39) we can determine a Borsuk embedding into $\mathbb{R}^7$. The graph has lower bound 2 for the chromatic number and the upper bound is 15.

Example 7.3.2. Let G be a eight-partite graph with 6 vertices in each set of the partition and 503 edges. This graph has chromatic number 8. We can obtain a Borsuk embedding into $\mathbb{R}^{23}$.

CHAPTER 8

Variations on the Matrix Completion Problem

Another class of applications where the nuclear norm heuristic reveals its use is in decomposing a matrix into two components, one of the components being low rank. In Section 8.1 we illustrate the decomposition into a low rank and sparse component. This decomposition was studied in detail in [89]. In Section 8.2 we study the decomposition of a full rank matrix into two components using a convex optimization problem which minimizes a linear combination of the nuclear norm and the spectral norm.

8.1. Low rank and Sparse Matrix Decomposition

This section outlines the recovery of the low rank and sparse components from a given matrix. The references [34] and [89] provide a convex optimization formulation for splitting a matrix into its low rank and sparse components and also sufficient conditions that guarantee exact recovery of the components.

In many important applications when the input data has a significant size we wish to learn some condensed information from it. The case is that in many instances the data can be seen as being a sum of two components, a low rank and a sparse component [34], [89].

Intuitively, the problem of recovering the low rank and the sparse component is hard because of the ambiguity given by the fact that some very sparse matrices are also low rank.

For a low rank matrix with entries "perturbed" by a sparse matrix and also with the entries of arbitrary magnitude, the recovery of the initial low rank matrix is in general NP-hard. But, using convex relaxations and formulating the problem as a semidefinite program, under certain conditions, we can obtain exact recovery of the components by solving a semidefinite program.

In formulating the convex optimization problem for the recovery of the sparse component the l_1 norm is used, while for the recovery of the low rank component the nuclear norm [31] is used. The problem of recovering a sparse vector, by minimizing the l_1 norm under linear equality constraints, has received much attention, especially in the work of Candes and his collaborators [50], [90], [91], [92], Donoho [93], and Boyd and Vandenberghe [53]. The use of the l_1 norm as a heuristic for sparsity is justified by the fact that the l_1 norm is also the convex envelope of the cardinality function over the set $\{x \in R^n : \|x\|_\infty \leq 1\}$. The cardinality function of the vector x, denoted by card(x), represents the number of non-zero entries in the vector x.

Relying on the nuclear norm heuristic and on the l_1 norm heuristic Chandrasekaran [34] considers the convex problem which minimizes a mixture of nuclear norm and l_1 norm to obtain a split of a given matrix C into an unknown sparse matrix and an unknown low rank matrix. Thus the paper [34] analyzes the convex optimization problem which minimizes a combination of the l_1 norm and the nuclear norm of the components.

Optimization problems involving the rank of a matrix determined the necessity to develop convex relaxations for these problems. As it was shown in the Chapter 5, Chapter 6 and Chapter 7 rank minimization is also of interest because of its variety of applications (including the matrix completion problem and embedding in Euclidean spaces).

This Section illustrates the convex optimization problem presented by Chandrasekaran in [34]. Suppose that we are given a matrix $C = A^* + B^*$ with A^* an unknown sparse matrix and B^* an unknown low rank matrix.

The convex relaxation used in practice to solve this problem can be stated as a minimization problem. We would like to decompose a matrix C into a sparse and a low rank component. Therefore the objective function of this problem is represented by a combination

between the l_1 norm and the nuclear norm of the matrix variables A and B.

$$\begin{aligned} \underset{A,B}{\text{minimize}} \quad & \gamma \|A\|_{l_1} + \|B\|_* \\ \text{subject to} \quad & C = A + B \end{aligned} \tag{40}$$

In this case γ represents a regularization parameter that provides a trade-off between the components. More, this convex optimization problem can be formulated as a semidefinite program. As shown in Chapter 3 the minimization of the nuclear norm can be written as a semidefinite program

$$\begin{aligned} \underset{W_1,W_2}{\text{minimize}} \quad & \frac{1}{2}\left(\text{trace}W_1 + \text{trace}W_2\right) \\ \text{subject to} \quad & \begin{bmatrix} W_1 & B \\ B^T & W_2 \end{bmatrix} \succeq 0 \end{aligned} \tag{41}$$

The l_1 norm minimization in (40) can be formulated as a linear program. As usual, the l_1 norm of a matrix A is represented by the sum of the absolute values of all entries of A. Therefore, the low rank and sparse decomposition problem can be written as a convex optimization problem i.e as a semidefinite program in the following form:

$$\begin{aligned} \underset{A,B,W_1,W_2}{\text{minimize}} \quad & \frac{1}{2}\text{trace}(W_1) + \frac{1}{2}\text{trace}(W_2) + \gamma\text{trace}(Z1_{n\times n}) \\ \text{subject to} \quad & \begin{bmatrix} W_1 & B \\ B^T & W_2 \end{bmatrix} \succeq 0 \\ & -Z_{i,j} \leq A_{i,j} \leq Z_{i,j}, \text{for any i,j} \\ & A + B = C \end{aligned} \tag{42}$$

where the third constraint is written component-wise and it must be satisfied for all entries of A. Further we'll show how the l_1 minimization problem admits a formulation as a linear program. Thus, let A be a matrix and b a vector. The optimization problem

$$\text{minimize } \|Ax - b\|_1$$

admits an equivalent formulation

$$\begin{aligned} &\underset{y}{\text{minimize}} && \sum_i y_i \\ &\text{subject to} && -y \leq Ax - b \leq y \end{aligned} \tag{43}$$

or using the standard form for the linear program

$$\begin{aligned} &\text{minimize} && \tilde{c}^T \tilde{x} \\ &\text{subject to} && \tilde{A}\tilde{x} \leq \tilde{b} \end{aligned} \tag{44}$$

where we have the following notations $\tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix}$, $\tilde{b} = \begin{bmatrix} b \\ -b \end{bmatrix}$.

For the low rank and sparse minimization problem, the l_1 norm of a matrix can be viewed as the l_1 norm of the vector obtained by stacking the columns of the matrix on top of each other. Thus, we obtain the semidefinite formulation used in equation (42). The references [34] and [89] study the additional conditions that are required in order to ensure that there exists a unique decomposition into sparse and low-rank matrices. The characterization of the unique decomposition from [34] involves the development of the notion of *rank sparsity incoherence* to ensure that the low rank matrix is not too sparse. The condition is based on quantities involving the tangent spaces to the algebraic variety of sparse matrices and the algebraic variety of low rank matrices. An algebraic variety is the solution set of a system

of polynomial equations. The set of matrices of rank less than or equal to a fixed number k is an algebraic variety defined by the vanishing of all $(k+1) \times (k+1)$ minors.

There are classes of random matrices such that when matrices are chosen from these classes we have exact recovery with very high probability. Following [34] we illustrate how we can find the decomposition in case the matrices have incoherent row/column spaces. All simulations were produced using the CVX solver [29]. Consider a slightly modified version of equation (40):

$$\begin{aligned} \underset{A,B}{\text{minimize}} \quad & t\,\|A\|_{l_1} + (1-t)\,\|B\|_* \\ \text{subject to} \quad & C = A + B \end{aligned} \tag{45}$$

There is a one to one correspondence between equation (40) and and (45). The sparse component is generated as a 25×25 matrix with 25 sparse entries. Support(A^*) is chosen uniformly at random from the collection of all support sets of size 25. Matrix B^* is chosen to be rank 2 such that $B = XY^T$, where $X, Y \in \mathbb{R}^{25\times 2}$ with independent and identically distributed Gaussian entries. To determine the stopping criteria we need to define

$$\text{tol}_\gamma = \frac{\left\|\hat{A}-A^*\right\|_F}{\|A^*\|} + \frac{\left\|\hat{B}-B^*\right\|_F}{\|B^*\|}$$

We consider a parameter $t \in [0,1]$ and choose a small step ϵ. We compute the difference between solutions at step t and $t-\epsilon$ as follows:

$$\text{diff}_t = \left\|\hat{A}_{t-\epsilon} - \hat{A}_t\right\|_F + \left\|\hat{B}_{t-\epsilon} - \hat{B}_t\right\|_F$$

Whenever the recovery is successful i.e the tol_t is small, the value of diff_t also tends to be zero. If a good guess for the parameter value is not available, one could solve the optimization problem for a range of γ and choose a solution corresponding to a value in the range where the diff_γ is stable and near zero.

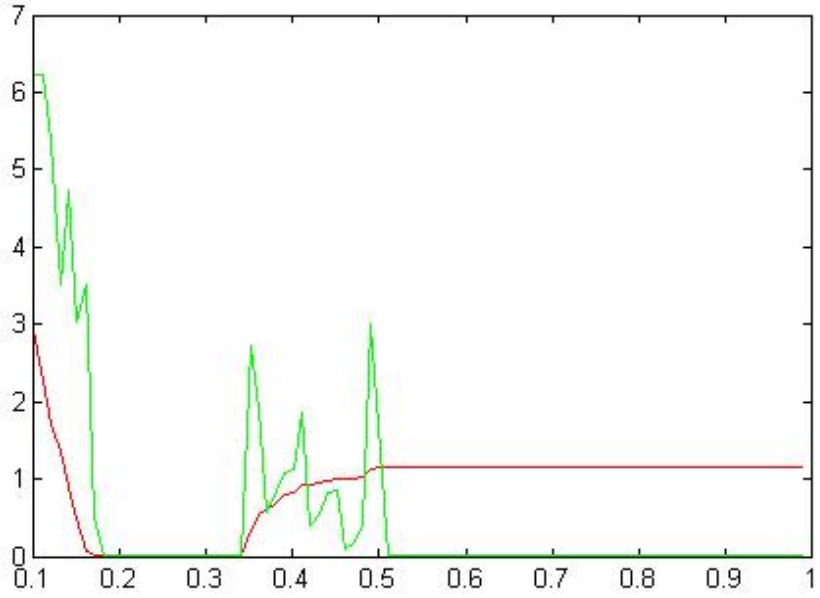

FIGURE 8.1. Comparison between toll ('r') and diff ('g') for a randomly generated example.

8.2. Decompositions using nuclear norm and spectral norm

Let M be a square n dimensional matrix, with singular decomposition $M = U\Sigma V^T$ and singular values ordered in decreasing order: $\sigma_1 > \sigma_2 > \ldots > \sigma_n$ where it is assumed that all singular values have multiplicity one.

Let $\lambda \in (0,1)$ be a positive real number. Consider the nuclear norm $\|\|_*$ and the spectral norm $\|\|$.

Consider the convex optimization problem where the objective function is a convex combination between the nuclear norm and the spectral norm

$$\begin{aligned} \underset{A,B}{\text{minimize}} \quad & \lambda \|A\|_* + (1-\lambda)\|B\| \\ \text{subject to} \quad & M = A + B \end{aligned} \tag{46}$$

The semidefinite formulation of the problem is

$$\begin{aligned} &\underset{A,B,t,W_1,W_2}{\text{minimize}} \quad \lambda\frac{1}{2}\left(\text{trace}W_1 + \text{trace}W_2\right) + (1-\lambda)t \\ &\text{subject to} \quad \begin{bmatrix} W_1 & A \\ A^T & W_2 \end{bmatrix}, \begin{bmatrix} tI_n & B \\ B^t & tI_n \end{bmatrix} \succeq 0 \\ &\qquad\qquad M = A + B \end{aligned} \tag{47}$$

Given matrices A and B, $A+B$ and their singular values $\sigma_i(A)$, $\sigma_i(B)$, $\sigma_i(A+B)$ in general the statement $\sigma_i(A+B) = \sigma_i(A) + \sigma_i(B)$, $i = 1, 2, \ldots$ is false.

To illustrate it we may consider matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 0 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 9 & 3 \\ 2 & 0 & 5 \\ 9 & 1 & 0 \end{bmatrix}$

The singular values of matrix A are 10.55, 1.47 and 0.71. The singular values of B are 9.75, 9.25 and 4.61. The sum $A+B$ has singular values 18.60, 7.82, 6.54. But $10.55+9.75 \neq 18.60$.

Example 8.2.1. Consider the full rank matrix

$$M = \begin{bmatrix} 4.3377 & 8.3897 & 0.9421 & 0.3532 & 0.6491 \\ 0.9001 & 8.2417 & 28.9561 & 1.8212 & 0.7317 \\ 1.3692 & 0.4039 & 4.5752 & 0.0154 & 0.6477 \\ 0.1112 & 8.0965 & 12.0598 & 20.0430 & 0.4509 \\ 2.7803 & 2.1320 & 6.2348 & 9.1690 & 20.5470 \end{bmatrix}$$

with singular values 37.7371, 21.2944, 15.5373, 8.6418, 1.3318. We solve the optimization problem 46 for different values of parameter λ. For values of the parameter λ between 0 and 1 we plot the singular values of the matrix A, the component corresponding to the nuclear norm term in Figure 8.2.

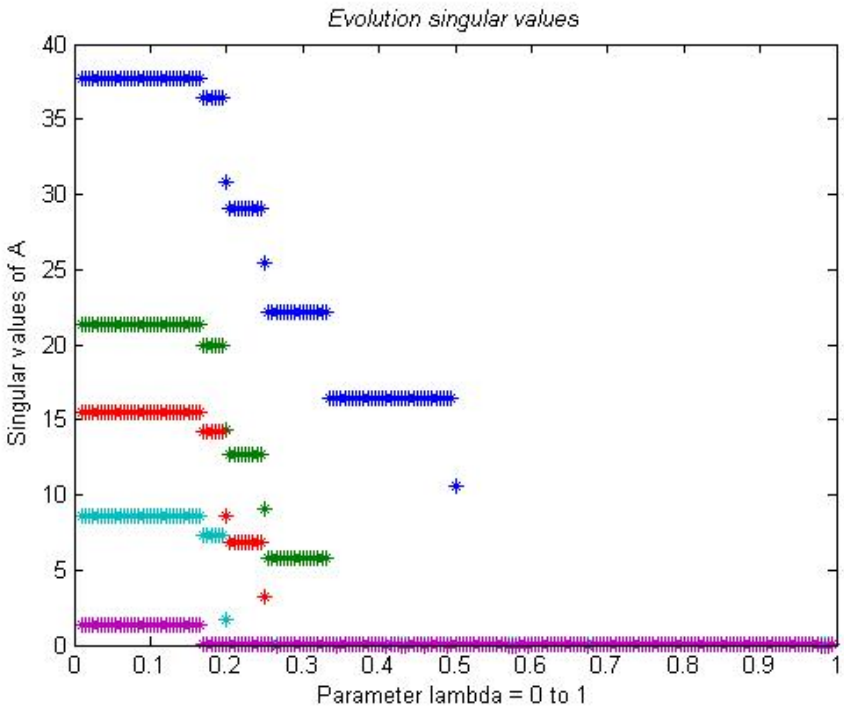

FIGURE 8.2. Singular values of matrix A, optimal solution for 46.

EXAMPLE 8.2.2. Let matrix $T_1 = \begin{bmatrix} 10.3377 & 0.3897 & 0.9421 & 0.3532 & 0.6491 \\ 0.9001 & 10.2417 & 0.9561 & 0.8212 & 0.7317 \\ 0.3692 & 0.4039 & 10.5752 & 0.0154 & 0.6477 \\ 0.1112 & 0.0965 & 0.0598 & 10.0430 & 0.4509 \end{bmatrix}$

In Figure 8.3 we present an evolution of the singular values of the optimal solutions for various values of the parameter λ. We obtain non-zero solutions A only for values of the parameter $\lambda \leq 0.5$. As the parameter λ ranges between 0 and 0.5 the solution B is always full rank while the rank of non-zero solutions A ranges between 4 and 1. These solutions display the following property $\sigma_i(A+B) = \sigma_i(A) + \sigma_i(B)$, $i = 1, 2, \ldots, 5$

EXAMPLE 8.2.3. Consider matrices of different dimensions, all full rank up to rank 13, singular values with multiplicity 1. For values of the parameter between $(0, 0.5]$ we obtain non-zero solutions A. When we count the number of different solutions we obtain 2 solutions

step 0.005/ initial point 0.01	All SVs have multiplicity one			
Matrix T1				
Parameter/198 steps	Singular Values of A	Singular Values of B	Singular values T1	svd(A)+Svd(B)
0.17-0.1950	2.8824	9.4645	12.3469	12.3469
	0.793	9.4645	10.2575	10.2575
	0.6083	9.4645	10.0728	10.0728
	0.1882	9.4645	9.6527	9.6527
	0	9.4645	9.4645	9.4645
0.2	2.8027	9.5442	12.3469	12.3469
	0.7133	9.5442	10.2575	10.2575
	0.5286	9.5442	10.0728	10.0728
	0.1085	9.5442	9.6527	9.6527
	0	9.4645	9.4645	9.4645
0.205-0.2450	2.6942	9.6527	12.3469	12.3469
	0.6048	9.6527	10.2575	10.2575
	0.4201	9.6527	10.0728	10.0728
	0	9.6527	9.6527	9.6527
	0	9.4645	9.4645	9.4645
0.25	2.4701	9.8768	12.3469	12.3469
	0.3807	9.8768	10.2575	10.2575
	0.196	9.8768	10.0728	10.0728
	0	9.6527	9.6527	9.6527
	0	9.4645	9.4645	9.4645
0.255-0.33	2.274	10.0728	12.3469	12.3468
	0.1847	10.0728	10.2575	10.2575
	0	10.0728	10.0728	10.0728
	0	9.6527	9.6527	9.6527
	0	9.4645	9.4645	9.4645
0.335-0.495	2.0893	10.2575	12.3469	12.3468
	0	10.2575	10.2575	10.2575
	0	10.0728	10.0728	10.0728
	0	9.6527	9.6527	9.6527
	0	9.4645	9.4645	9.4645
0.5	0.7039	11.643	12.3469	12.3469
	0	10.2575	10.2575	10.2575
	0	10.0728	10.0728	10.0728
	0	9.6527	9.6527	9.6527
	0	9.4645	9.4645	9.4645

FIGURE 8.3. An example of evolution of singular values of the optimal solutions A and B.

of rank 1, 1 solution of rank 2, 2 solutions of rank 3, 2 solutions of rank 4, 1 solution of rank 5, 1 solution of rank 6. We sampled the interval and took up to 2000 values for the parameter. The rank of the solutions A display the following pattern $2, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 1 \ldots$

Conjecture: Let M be a full rank $n \times n$ matrix, all singular values having multiplicity 1. Consider the optimization problem

$$\begin{aligned} \underset{A,B}{\text{minimize}} \quad & \lambda \|A\|_* + (1-\lambda)\|B\| \\ \text{subject to} \quad & M = A + B \end{aligned}$$

Consider the non-zero solutions A of the optimization problem and the corresponding matrices B. The following statements take place

(1) $\sigma_i(M) = \sigma_i(A) + \sigma_i(B)$, $i = 1, \ldots, n$ where $\sigma_i(M)$, $\sigma_i(A)$, $\sigma_i(B)$ are singular values of matrix M, respectively A and B.

(2) As the values of the parameter vary, there is a pattern between the number of solutions: 2,1,2,2,1,1,2,1,2,2,1,1 and the rank of the solution A:1,2,3,4,5,.....There is a unique solution of rank k, $k \leq n$ when the remainder of the division by 6 of k is 0, 2 or 5. For the rest of the cases, when the remainder is 1,3 and 4 we have 2 different solutions of rank k.

(3) There exists a value of the parameter that returns solution A of rank 1 such that the singular values of corresponding matrix B satisfy the relation $\sigma_i(B) = \sigma_i(M)$, $i = 2, ..n$ and $\sigma_1(B) = \sigma_2(B)$.

(4) There exists a value of the parameter that returns solution A of rank k such that the singular values of corresponding matrix B satisfy the relation: $\sigma_i(B) = \sigma_i(M)$, $i = k+1, ..n$ and $\sigma_1(B) = \sigma_2(B) = ... = \sigma_{k+1}(B)$.

CHAPTER 9

Conclusions and Future Work

In this dissertation we studied the application of the nuclear norm heuristic to the problem of finding low rank representations of matrices, and, in particular, for solving the Euclidean distance matrix completion problem. As a special case, we considered the low-rank embedding of incomplete graphs, again using the nuclear norm heuristic. We demonstrated how a number of interesting variations of the graph embedding problem could be explored using convex optimization, i.e., semidefinite programming. We have proposed two formulations of the graph embedding problem as constrained semidefinite programs. In this setting the nuclear norm heuristic leads to closed form optimal solutions for k-partite graphs on n nodes. These results are shown to provide useful insights into the computation of the chromatic number. This framework also lends itself naturally to the construction of unit distance graphs. Further, the Borsuk problem is formulated and examples of embeddings found numerically. Lastly, we applied related techniques to decompose a matrix into components which simultaneously minimize a linear combination of the nuclear norm and the spectral norm. The results of this thesis have opened up possibilities for future work:

We have demonstrated that low rank graph embeddings may be found using the nuclear norm heuristic and improved by exploring additional constraints. We have illustrated how one can use convex optimization to provide information concerning the properties of a graph, including chromatic number. Preliminary results have been presented to suggest that convex optimization may be applied to open problems in graph theory, such as the Borsuk conjecture related to unit diameter graphs.

We have observed that the use of the nuclear norm heuristic has some unanticipated consequences as a proxy for rank minimization. In particular, we have found specific examples where the nuclear norm approach fails to find solutions of minimum rank and that additional constraints are required to overcome this. We have explored possible solutions to this problem such as maximizing the distance between subsets sets of points on the graph. It has also proven effective to add carefully selected virtual points to further reduce the rank of the embedding. Note that although we use the minimum nuclear norm heuristic as a proxy to to minimize the rank of the configuration of points, the explicit goal of this objective function is to minimize the variance of the points in the graph subject to auxiliary constraints. As a result this approach has a tendency to collapse data points in the embedded configuration. While we have seen that this is a positive feature for the computation of chromatic number, it runs counter the goal of the graph embedding problem. Future work needs to be done to address the potentially conflicting goals of the objective function in the optimization problem.

The framework of convex optimization has been shown to be quite powerful for addressing questions relating to graph embeddings and the low rank completion of Euclidean distance matrices. We would like to further investigate this framework to determine mechanisms for further reducing rank. We are particularly interesting in exploring the case of graphs with high chromatic number and high girth. It is possible that the log-det heuristic, introduced in [31], could prove to be a useful alternative to the solution of the rank minimization problem. It would be interesting to explore the graph embedding problems described in this dissertation using this approach. In addition, other combinations of convex objective criteria could prove useful.

Although in this dissertation we have focused on the decomposition of matrices, clearly there are opportunities to extend this work to the construction of low rank tensor decompositions. The problem of tensor completion appears to be an interesting direction; see also [94]. Progress may be made by exploring unfoldings of tensors to matrices. Alternatively, one can search for convex envelopes related to definitions of the rank of a tensor.

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