

Fig. 5. Spectra of a uniform quadrature sampled bandpass signal.

$e^{j[n\pi - \pi/2]}$ . The time-domain description is the band-limited Hilbert transformer

$$s_Q(t) = \frac{2(-1)^n \sin^2(\pi Bt)}{\pi t} \quad (16)$$

Applying the method of Section III-B, it is found that the time delay in the sampling of the interpolant which ensures constructive aliasing in the spectra of the digital quadrature interpolant is  $\tau_Q = -k$ . This time-shift is equal and opposite to the initial time difference between the sample streams. Using this result to digitize  $s_Q(t)$ , the digital quadrature interpolant is found to be the delayed, digital Hilbert transformer

$$s_Q^{\delta 1}(t) = \frac{2(-1)^n}{p\pi + \pi/2} \quad (17)$$

The two post-interpolation sample streams are time-aligned thus, following Section III-C, the digitized frequency-shifting function  $y_Q^{\delta 1}(t) = (-1)^p$  applied to both sample streams will produce baseband  $I^{\delta 1}(t)$  and  $Q^{\delta 1}(t)$ .

This result corresponds to those described by Rice and Wu [6], and by Waters and Jarrett [7], but for half the latter sampling rate. The above derivation shows that the previous methods for producing uniform quadrature sampling are special cases of the above method for general second-order sampling. It is only in the special case of uniform quadrature sampling that the combined effect of second-order bandpass sampling, interpolant under-sampling, and digital frequency-shifting produces two time-aligned, digital sample streams.

### V. CONCLUSION

It is possible to frequency-shift a bandpass signal by using bandpass sampling and the appropriate interpolants. Where the signal is second-order sampled, the frequency-shifted signal is phase-shifted relative to the original signal. This phase-shift may be varied by changing the sample streams' separation.

It has been shown that interpolants for frequency-shifting a bandpass signal using second-order bandpass sampling can be implemented digitally. It has also been shown how previous digital implementations of quadrature interpolants are a special case of the general, second-order digital interpolants.

### REFERENCES

- [1] R. G. Vaughan, N. L. Scott, and D. R. White, "The theory of bandpass sampling," *IEEE Trans. Signal Processing*, vol. 39, no. 9, Sept. 1991.
- [2] A. Kohlenberg, "Exact interpolation of band-limited functions," *J. Appl. Phys.*, vol. 24, Dec. 1953.
- [3] D. A. Linden, "A discussion of sampling theorems," in *Proc. IRE*, vol. 47, 1959, pp. 1219-1226.
- [4] O. D. Grace and S. P. Pitt, "Sampling and interpolation of bandlimited signals by quadrature methods," *J. Acoust. Soc. Amer.*, vol. 48, no. 6, 1970.
- [5] M. A. Poletti and A. J. Coulson, "On quadrature and uniform bandpass sampling," under review.
- [6] D. W. Rice and K. H. Wu, "Quadrature sampling with high dynamic range," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-18, no. 4, Nov. 1982.
- [7] W. M. Waters and B. R. Jarrett, "Bandpass signal sampling and coherent detection," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-18, no. 4, Nov. 1982.
- [8] A. J. Coulson, R. G. Vaughan, and M. A. Poletti, "Interpolation in bandpass sampling," in *Proc. Int. Symp. Signal Processing, Applicat. (ISSPA-92)*, Aug. 1992, pp. 23-26.

### Polar Coordinate Quantizers That Minimize Mean-Squared Error

Stephen D. Voran and Louis L. Scharf

**Abstract**—A quantizer for complex data is defined by a partition of the complex plane and a representation point associated with each cell of the partition. A polar coordinate quantizer independently quantizes the magnitude and phase angle of complex data. We derive design equations for minimum mean-squared error polar coordinate quantizers and report some interesting theoretical results on their performance, including performance limits for "phase-only" representations. The results provide a concrete example of a biased estimator whose mean-squared error is smaller than that of any unbiased estimator. Quantizer design examples show the relative importance of magnitude and phase encoding.

### I. INTRODUCTION

A quantizer for complex data partitions a region of interest in the complex plane into a finite number of cells and assigns a representation point to each. The most general complex quantization problem is illustrated in Fig. 1. That is,  $\hat{z} = z_i$  whenever  $z \in C_i$ . The quantizer design problem is to determine the cells  $C_i$  and the quantized representations  $z_i$  so that  $E\epsilon$ , the expected value of the error function  $\epsilon$ , is minimized.

If the error function increases monotonically with  $|z - \hat{z}|$ , then it is clear that the  $C_i$  should form a nearest neighbor partition of the  $z_i$ . If the error function is further restricted to the squared error function  $\epsilon(z, \hat{z}) = |z - \hat{z}|^2$ , it is easy to show that each representation point  $z_i$  must be the conditional mean of  $z$  when  $z$  is in the corresponding

Manuscript received February 16, 1993; revised August 18, 1993. This work was supported by the Office of Naval Research under contract No. N00014-89-J-1070 and by the NSF Center for Optoelectronic Computing Systems at the University of Colorado, under contract No. 8622236. The associate editor coordinating the review of this paper and approving it for publication was Prof. Tamal Bose.

The authors are with the Department of Electrical and Computer Engineering, University of Colorado, Boulder, Colorado, USA.  
IEEE Log Number 9400039.

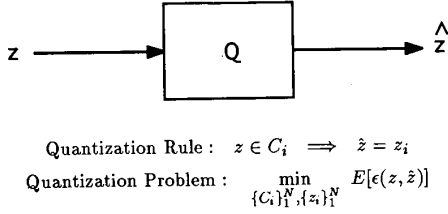


Fig. 1. Quantizer.

quantization cell  $C_i$ . This pair of results defines what is commonly called the *Lloyd-Max quantizer* [1], [2], although Lukaszewicz and Steinhaus deserve recognition [3]. The Lloyd-Max (LM) equations are  $\hat{z} = z_i$  when  $z \in C_i$ , where

$$C_i = \{z : |z - z_i|^2 \leq |z - z_j|^2\}, \quad j = 1 \text{ to } N, \quad j \neq i$$

$$z_i = E[z | z \in C_i]. \quad (1.1)$$

The conditional mean estimator (CME)  $\hat{z}$  is unbiased. That is,  $E[\hat{z}] = E[z]$ . The LM equations are necessary but not sufficient conditions for the minimization of  $E|z - \hat{z}|^2$ . In the case where  $z$  is a real scalar, Fleischer [4] has shown that, if the density function for  $z$ ,  $f_z(z)$ , satisfies the weak convexity condition

$$\frac{d^2}{dz^2} \ln(f_z(z)) < 0 \quad \forall z \quad (1.2)$$

then the LM equations are necessary and sufficient for the minimization of  $E|z - \hat{z}|^2$ . Because the LM equations are coupled, an iterative solution is necessary in all but the most trivial cases.

By constraining the cell shapes to be some regular shape and possibly constraining the representation points, we can find quantizers which are easy to implement but which have larger average error than does the LM quantizer. In this correspondence we investigate a "sector-annulus" partition of the complex plane which allows independent quantization of magnitude and phase angles in a polar representation of complex data. This representation may be advantageous for directional data and data from quadrature demodulators. Our results generalize the results of Bucklew and Gallagher [5], [6] to include arbitrary distributions and "nonasymptotic" bit rates. Our results also provide a concrete example of a biased estimator whose mean-squared error is smaller than that of any unbiased estimator.

## II. POLAR COORDINATE QUANTIZERS

By selecting an appropriate partition of the complex plane, it is possible to quantize the complex data  $z = re^{j\phi}$  by independently quantizing the magnitude (or radius) of  $z$ ,  $r = |z|$ , and the phase angle of  $z$ ,  $\phi = \arg(z)$ . The result is that the data  $re^{j\phi}$  is represented as  $\hat{r}e^{j\hat{\phi}}$ , where  $\hat{r}$  and  $\hat{\phi}$  are quantized versions of  $r$  and  $\phi$ , respectively, and  $\hat{r}e^{j\hat{\phi}}$  approximates the original data  $re^{j\phi}$  as closely as possible. In the development that follows, we assume a stochastic model for the data and design the polar coordinate quantizer that minimizes the average quantization noise energy, or mean-squared error, of the quantizer:

$$\xi_z^2 = E|re^{j\phi} - \hat{r}e^{j\hat{\phi}}|^2 \quad (2.1)$$

$$\hat{r} = r_i \text{ when } t_i < r \leq t_{i+1};$$

$$\hat{\phi} = \phi_i \text{ when } d_i < \phi \leq d_{i+1}.$$

Note that independent quantization of  $r$  and  $\phi$  permits us to replace the general cell  $C_i$  by the intervals  $(t_i, t_{i+1}]$  for  $r$  and  $(d_i, d_{i+1}]$  for  $\phi$ .

Note that in general,  $\xi_z^2$  is parameterized by both the number of phase representation points  $N_p = 2^p$  and the number of magnitude

representation points  $N_m = 2^m$ . When  $b$  bits are available, one should examine these error expressions for different  $(m, p)$  pairs that satisfy  $m + p = b$  to find the quantization scheme that makes the most efficient use of these  $b$  bits. Section IV provides two examples that demonstrate the relative importance of magnitude and phase encoding.

## III. MINIMUM MEAN-SQUARED ERROR POLAR QUANTIZER DESIGN

We assume that the data  $re^{j\phi}$  comes from a stochastic source, the probability density functions for  $r$  and  $\phi$  are known, and  $r$  and  $\phi$  are statistically independent. We then obtain

$$\xi_z^2 = E|re^{j\phi} - \hat{r}e^{j\hat{\phi}}|^2 = E(r^2) - 2E(r\hat{r})E\cos(\phi - \hat{\phi}) + E(\hat{r}^2)$$

$$= E(r^2) - 2\alpha E(r\hat{r}) + E(\hat{r}^2); \quad \alpha = E[\cos(\phi - \hat{\phi})]. \quad (3.1)$$

Since  $\hat{r}$  is a function only of  $r$  and since  $r$  and  $\hat{r}$  are positive quantities, the minimization of  $\xi_z^2$  requires the maximization of  $\alpha = E[\cos(\phi - \hat{\phi})]$ . The parameter  $\alpha$  depends only on  $\phi$  and  $\hat{\phi}$  and is the natural performance measure for the quantization of  $\phi$  and  $e^{j\phi}$  in the context of this problem. The estimator  $\hat{\phi}$  that maximizes  $\alpha = E[\cos(\phi - \hat{\phi})]$  is also the estimator that minimizes the mean-squared error (MSE) of a phasor quantizer. That is,

$$\xi_\phi^2 = E[|e^{j\phi} - e^{j\hat{\phi}}|^2] = E[2(1 - \cos(\phi - \hat{\phi}))] = 2(1 - \alpha). \quad (3.2)$$

Therefore, when  $\hat{\phi}$  maximizes  $\alpha = E[\cos(\phi - \hat{\phi})]$ , the corresponding phasor estimator  $e^{j\hat{\phi}}$  is the minimum mean-squared error (MMSE) estimator of the phasor  $e^{j\phi}$ . This means that  $e^{j\hat{\phi}}$  is a conditional mean estimator (CME). As every CME is unbiased, we see that  $e^{j\hat{\phi}}$  is an unbiased estimator of  $e^{j\phi}$ :

$$E[e^{j\hat{\phi}}] = E[e^{j\phi}]. \quad (3.3)$$

If the phase  $\phi$  happens to be uniformly distributed on  $(0, 2\pi]$ , then  $Ee^{j\phi} = 0$ . But, more generally, we can write  $\hat{z} = \hat{r}e^{j\hat{\phi}}$  and note that the mean of  $\hat{z}$  is

$$E[\hat{z}] = E[\hat{r}]E[e^{j\hat{\phi}}]. \quad (3.4)$$

This will equal  $E[z]$  iff  $E[\hat{r}] = E[r]$ . As we shall see,  $E[\hat{r}] \neq E[r]$  except in the case of infinitely fine phase quantization. So, generally, the estimator  $\hat{r}$  is a biased estimator of  $r$  and  $\hat{z}$  is a biased estimator of  $z$ , but these biased estimators have smaller variance and smaller MSE than unbiased estimators would have. When the phase quantizer is infinitely fine, then  $\alpha = 1$  and the MSE  $\xi_z^2$  is just  $\xi_r^2$ , the MSE of a magnitude quantizer:

$$\xi_z^2 = E[(r - \hat{r})^2] = \xi_r^2 \text{ when } \alpha = 1. \quad (3.5)$$

*Phase Quantizer Design:* We assume the quantizer design described by

$$\hat{\phi} = \phi_k \text{ whenever } d_k < \phi \leq d_{k+1} \quad (3.6)$$

and  $k = 0, 1, \dots, N_p - 1$ . The free parameters in this design are the thresholds  $\{d_k\}_0^{N_p-1}$  and the representation points  $\{\phi_k\}_0^{N_p-1}$ . The expected cosine error becomes

$$\alpha = E[\cos(\phi - \hat{\phi})] = \int_0^{2\pi} \cos(\phi - \hat{\phi}) f_\phi(\phi) d\phi$$

$$= \sum_{k=0}^{N_p-1} \left[ \left( \cos(\phi_k) \int_{d_k}^{d_{k+1}} \cos(\phi) f_\phi(\phi) d\phi \right) \right. \\ \left. + \left( \sin(\phi_k) \int_{d_k}^{d_{k+1}} \sin(\phi) f_\phi(\phi) d\phi \right) \right]. \quad (3.7)$$

A necessary condition for  $\alpha$  to be maximized is

$$\nabla_{\mathbf{q}} \alpha = 0, \quad \mathbf{q}^T = [d_0, d_1, \dots, d_{N_p-1}, \phi_0, \phi_1, \dots, \phi_{N_p-1}]. \quad (3.8)$$

This results in two sets of equations. The first is

$$0 = \frac{\partial \alpha}{\partial d_k} = f_\phi(d_k) [\cos(d_k - \phi_{k-1}) - \cos(d_k - \phi_k)].$$

If  $f_\phi(d_k) = 0$ , then  $d_k$  can be moved without consequence. Hence, we assume  $f_\phi(d_k) \neq 0$  and obtain

$$\begin{aligned} d_k &= \frac{\phi_{k-1} + \phi_k}{2}, \quad k = 1 \text{ to } N_p - 1 \\ d_0 &= \frac{\phi_0 + \phi_{N_p-1} - 2\pi}{2}, \quad d_{N_p} = d_0 + 2\pi. \end{aligned} \quad (3.9)$$

The second set of equations is

$$\begin{aligned} 0 &= \frac{\partial \alpha}{\partial \phi_k} \\ &= \cos(\phi_k) E[\sin(\phi) | d_k \leq \phi < d_{k+1}] P[d_k \leq \phi < d_{k+1}] \\ &\quad - \sin(\phi_k) E[\cos(\phi) | d_k \leq \phi < d_{k+1}] P[d_k \leq \phi < d_{k+1}] \\ \implies \phi_k &= \tan^{-1} \left\{ \frac{E[\sin(\phi) | d_k \leq \phi < d_{k+1}]}{E[\cos(\phi) | d_k \leq \phi < d_{k+1}]} \right\}; \\ &k = 0 \text{ to } N_p - 1 \end{aligned} \quad (3.10)$$

where  $\tan^{-1}$  denotes the four-quadrant arctangent function.

The quantizer that maximizes  $\alpha$  must satisfy equations (3.9) and (3.10) for all  $k$ . We compute the solution by an iterative technique. First, uniformly spaced  $\{d_k\}$  and  $\{\phi_k\}$  are assumed. We then alternately adjust  $\{\phi_k\}$  to satisfy (3.10) and  $\{d_k\}$  to satisfy (3.9). At each iteration, the vector  $\mathbf{q}$  moves closer to a critical point since one of the two optimality criteria is enforced. Thus the algorithm converges to a critical point and, in practice, this critical point maximizes  $\alpha$ .

*Phase-Only Quantizer:* If we quantize only phase and not magnitude, then the best prior choice for  $\hat{r}$  is the choice that minimizes

$$\xi_z^2 = E(r^2) - 2\alpha \hat{r} E(r) + \hat{r}^2. \quad (3.11)$$

That is,

$$\hat{r} = \alpha E(r). \quad (3.12)$$

Then the MSE of the phase-only quantizer is

$$\xi_z^2 = E(r^2) - 2\alpha^2 E^2(r) + \alpha^2 E^2(r) = E(r^2) - \alpha^2 E^2(r). \quad (3.13)$$

If the phase quantizer is infinitely fine, then  $\alpha = 1$  and  $\xi_z^2 = \text{var}(r)$ . This is as small as the MSE can be for phase-only quantization. If the phase is uniformly distributed on  $(0, 2\pi]$ , then the optimum phase quantizer is a uniform quantizer and  $\alpha = \text{sinc}(\pi/N_p)$ .

*Magnitude Quantizer Design* We have seen that the optimum phase quantizer design is independent of the magnitude distribution. We simply satisfy (3.9) and (3.10) and then calculate  $\alpha$  using (3.7). On the other hand, the design of the optimum magnitude quantizer is dependent on the phase quantizer design. This is evident from (3.1). We begin our study of the optimum magnitude quantizer by studying equation (3.1) in the special case where phase quantization is arbitrarily fine, meaning that  $\alpha = 1$  and also meaning that the MSE  $\xi_z^2$  is just the MSE  $\xi_r^2$  of a magnitude quantizer:

$$\xi_z^2 = \xi_r^2 = E(r^2) - 2E(r\hat{r}) + E(\hat{r}^2) = E(\hat{r} - r)^2. \quad (3.14)$$

(We are using  $\hat{r}$  in place of  $\tilde{r}$  to indicate that  $\hat{r}$  is a quantizer for  $r$  when  $\alpha = 1$ ; it is not the optimum quantizer for  $r$  when  $\alpha < 1$ .) In this case, the Lloyd-Max quantizer  $\hat{r}$  is found in the usual way: define  $\hat{r} = r_k$  whenever  $t_k < r \leq t_{k+1}$  for  $k = 0, 1, \dots, N_m - 1$ .

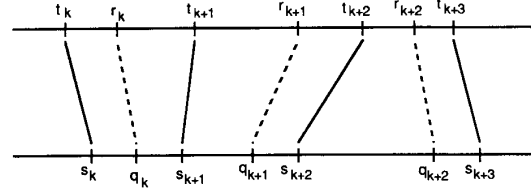


Fig. 2. Correspondence between the optimum quantizer and the LM quantizer.

Then following the procedure outlined for the phase quantizer, the LM equations are

$$\begin{aligned} t_k &= \frac{r_k + r_{k+1}}{2}; \quad k = 1 \text{ to } N_m - 2, \quad t_0 = 0, t_{N_m-1} = \infty \\ r_k &= E[r | t_k < r \leq t_{k+1}]. \end{aligned} \quad (3.15)$$

The solution for  $r_k$  is, of course, the CME of  $r$  given  $t_k < r \leq t_{k+1}$ . The CME estimator  $\hat{r}$  is unbiased:

$$E(\hat{r}) = E(r). \quad (3.16)$$

The design of an LM quantizer starts with uniformly spaced  $\{t_k\}$  and  $\{r_k\}$  and alternatively enforces the two conditions in (3.15). The parameters  $\{t_k\}$  and  $\{r_k\}$  converge so that both conditions are simultaneously satisfied. Fleisher's weak convexity condition (1.2) on  $f_r(r)$  guarantees that the LM conditions are also sufficient.

What relevance does the LM quantizer  $\hat{r}$  have to the problem at hand when  $\alpha < 1$ ? To answer this question, we rewrite the equation for  $\xi_z^2$  as

$$\xi_z^2 = E[r^2] - 2\alpha E[r\hat{r}] + E[\hat{r}^2] = E[(r - \alpha\hat{r})^2] + (1 - \alpha^2)E[\hat{r}^2]. \quad (3.17)$$

We are denoting the quantizer by  $\hat{r}$  to distinguish it from the LM quantizer  $\tilde{r}$ . In general,  $\hat{r}$  will have its own set of representations  $q_k$  and thresholds  $s_k$ . However, they may always be written in terms of the corresponding representations and thresholds for the LM quantizer. That is, as illustrated in Fig. 2, the  $(q_k, s_k)$  for the optimum quantizer may be placed in 1:1 correspondence with the  $(r_k, t_k)$  of the LM quantizer. This means that the quantizer  $\hat{r}$  may actually be referenced to the LM quantizer:

$$\hat{r} = q(r_k) \quad \text{whenever } s_k = h(t_k) < r \leq h(t_{k+1}) = s_{k+1}. \quad (3.18)$$

However, rather than test  $s_k < r \leq s_{k+1}$ , we may simply test  $t_k < r \leq t_{k+1}$  and transform  $r_k$ :

$$\hat{r} = q(r_k) = q(\tilde{r}) \quad \text{whenever } t_k < r \leq t_{k+1}. \quad (3.19)$$

This makes  $\hat{r}$  a nonlinear function of the LM quantizer  $\tilde{r}$ . Now we give and take  $\tilde{r}$  in the first term of  $\xi_z^2$ , exploit the orthogonality of  $r - \tilde{r}$  to any function of  $\tilde{r}$  (such as  $\hat{r}$ ), and write

$$\xi_z^2 = E[(r - \tilde{r})^2] + E[(\tilde{r} - \alpha\hat{r})^2] + (1 - \alpha^2)E[\hat{r}^2] \quad (3.20)$$

or

$$\xi_z^2(r, \hat{r}) = \xi_r^2 + \xi_z^2(\tilde{r}, \hat{r}). \quad (3.21)$$

We have used the arguments  $(r, \hat{r})$  and  $(\tilde{r}, \hat{r})$  to denote that in one case  $\xi_z^2$  is the MSE between  $re^{j\phi}$  and  $\hat{r}e^{j\phi}$  and in the other case  $\xi_z^2$  is the MSE between  $\tilde{r}e^{j\phi}$  and  $\hat{r}e^{j\phi}$ . The term  $\xi_r^2 = E[(r - \tilde{r})^2]$  is just the MSE of a LM magnitude quantizer.

It is not hard to show that the solution for  $\hat{r}$  that minimizes  $\xi_z^2(\tilde{r}, \hat{r})$  and hence minimizes  $\xi_z^2(r, \hat{r})$  is

$$\hat{r} = \alpha\tilde{r} = \alpha r_k \quad \text{whenever } t_k < r \leq t_{k+1}. \quad (3.22)$$

TABLE I  
QUANTIZER SNR IN dB

$m$ , bits of magnitude encoding	$\rho$ , bits of phase encoding						
	0	1	2	3	4	5	6
0	0.0*	1.7*	4.4*	5.9	6.5	6.6	6.7
1	0.0	2.0	6.0*	9.2*	10.7	11.2	11.3
2	0.0	2.2	6.6	10.8*	13.3*	14.2	14.5
3	0.0	2.2	6.9	11.7	15.0*	16.5*	17.0
4	0.0	2.2	7.0	11.9	15.6	17.5*	18.1

The quantizer  $\hat{r}$  is a worse approximation to  $r$  than the LM quantizer  $\hat{r}$ . Furthermore, it is biased, and the corresponding polar coordinate quantizer is biased:

$$E[\hat{r}] = \alpha E[r] = \alpha E[r], \quad E[\hat{z}] = E[\alpha \hat{r} e^{j\hat{\phi}}] = \alpha E[z]. \quad (3.23)$$

However, this bias in  $\hat{z}$  is compensated by low variance, which makes bias-squared plus variance for  $\hat{z}$  smaller than the variance of any competing unbiased estimator. The MSE of the quantizer  $\hat{z} = \hat{r} e^{j\hat{\phi}}$  is

$$\begin{aligned} \xi_z^2 &= E[|r e^{j\phi} - \hat{r} e^{j\hat{\phi}}|^2] = E[r^2] - 2\alpha E[r\alpha \hat{r}] + \alpha^2 E[\hat{r}^2] \\ &= E[r^2] - \alpha^2 E[\hat{r}^2] = E[r^2] - \alpha^2 \sum_{k=0}^{N_p-1} p_k r_k^2 \geq \xi_r^2. \end{aligned} \quad (3.24)$$

This shows that the polar coordinate quantizer cannot work better than a LM quantizer for magnitude used in conjunction with an infinitely fine phase quantizer. When phase quantization is finite, however, it outperforms the LM quantizer for magnitude.

#### IV. EXAMPLES

We have derived MMSE polar coordinate quantizer designs for specific complex data distributions. Two examples are given here. The case of complex data  $z$  with independent, identically distributed Gaussian real and imaginary parts (alternately,  $|z|$  is Rayleigh and is independent from  $\arg(z)$ , which is uniform) has been treated in [6], and our results concur. Table I shows MMSE quantizer signal-to-noise ratios (SNR's) for a range of  $p$  (bits of phase encoding) and  $m$  (bits of magnitude encoding) values. SNR is defined as

$$\text{SNR} = 10 \log_{10} \left\{ \frac{E|r e^{j\phi}|^2}{E|r e^{j\phi} - \hat{r} e^{j\hat{\phi}}|^2} \right\}. \quad (4.1)$$

The single asterisk on the  $(b+1)^{\text{st}}$  antidiagonal of the table indicates the best SNR given the constraint  $p+m=b$ .

For a second example, we assume a phase distribution of complex data clustered near the real line. The phase angles are drawn from a truncated Laplacian distribution. The mean of the Laplacian distribution takes the values  $0$  or  $\pi$  with equal probability. The resulting probability density function is

$$f_\phi(\phi) = \begin{cases} c \frac{1}{2\sigma} e^{-\frac{|\phi|}{\sigma}}, & -\frac{\pi}{2} \leq \phi < \frac{\pi}{2} \\ c \frac{1}{2\sigma} e^{-\frac{|\phi-\pi|}{\sigma}}, & \frac{\pi}{2} \leq \phi < \frac{3\pi}{2} \end{cases} \quad \text{where } c = \left[ 2(2 - e^{-\frac{\pi}{2\sigma}}) \right]^{-1}. \quad (4.2)$$

TABLE II  
QUANTIZER SNR IN dB

$m$ , bits of magnitude encoding	$\rho$ , bits of phase encoding						
	0	1	2	3	4	5	6
0	0.0*	2.9*	3.8	4.5	4.7	4.8	4.8
1	0.0	4.5*	6.4*	8.2*	9.0	9.2	9.2
2	0.0	5.3	8.0	11.4*	13.3*	14.0	14.2
3	0.0	5.6	8.7	12.2	16.9*	18.8	19.4
4	0.0	5.6	9.0	13.9	18.9*	22.5	24.1

We select  $\sigma = 1/2$ . We assume that the magnitude of the data is gamma distributed and independent of the phase angle distribution:

$$f_r(r) = \frac{r}{\sigma_r^2} e^{-\frac{r}{\sigma_r}}, \quad 0 \leq r, \quad E(r) = 2\sigma_r, \quad E(r^2) = 6\sigma_r^2. \quad (4.3)$$

We select  $\sigma_r = 1$ .

The results of this correspondence allow us to design first a phase quantizer and then a magnitude quantizer for the complex data  $z = r e^{j\phi}$ . The resulting quantizer SNRs for this data distribution are given in Table II. As in the previous example, the asterisks indicate the highest SNR for a fixed antidiagonal line,  $m+p=b$ . Because the phase distribution is tightly clustered around  $0$  and  $\pi$ , phase encoding is not as difficult as in the previous example. When only one bit is available, phase-only representation is optimum. When two bits are used, one bit of magnitude encoding along with one bit of phase encoding will save 0.7 dB of SNR compared with two bits of phase-only encoding. As the total bit constraint is increased, we trace out a somewhat irregular staircase towards the southeast corner of the table.

#### V. CONCLUSION

We have derived design equations for the polar coordinate quantizer that minimizes mean-squared error. This 2-D quantizer for complex data can be implemented as a pair of scalar quantizers which function independently. The phase angle quantizer is a LM, conditional mean quantizer which can be designed with no knowledge of the magnitude data. We have provided necessary conditions for the optimality of a phase quantizer. In practice, these conditions behave like necessary and sufficient conditions. The magnitude quantizer turns out to be a modified LM quantizer, and the modification is a function of the phase quantizer used. The design given is both necessary and sufficient for optimum performance. In general, the total complexity of the quantizer may be appropriately divided between the phase portion and the magnitude portion so that best performance is attained. The examples show that the appropriate division is dependent on the complex data distribution.

#### REFERENCES

- [1] S. P. Lloyd, "Least squares quantization in PCM," unpublished memorandum, Bell Laboratories, 1957.
- [2] J. Max, "Quantizing for minimum distortion," *IRE Trans. Inform. Theory*, vol. IT-6, pp. 7-12, 1960.
- [3] A. Gersho, "Principles of quantization," *IEEE Trans. Circuits, Syst.*, vol. CAS-25, pp. 427-436, 1978.
- [4] P. E. Fleischer, "Sufficient conditions for achieving minimum distortion in a quantizer," *IEEE Int. Convention. Rec.*, 1964, pt. 1, pp. 104-111.

- [5] Bucklew and Gallagher, "Quantization schemes for bivariate Gaussian random variables," *IEEE Trans Inform. Theory*, vol. IT-25, no. 3, Sept. 1979.
- [6] Bucklew and Gallagher, "Two-dimensional quantization of bivariate circularly symmetric densities," *IEEE Trans. Inform. Theory*, vol. IT-25, no. 6, Nov. 1979.

## A Recurrence Relation for the Product of the Nonzero Eigenvalues of Singular Symmetric Toeplitz Matrices

Jean Laroche

**Abstract**—This communication presents an extension of a well-known recurrence relation for Toeplitz symmetric matrices to the case of incomplete rank matrices. It is shown that the product of the nonzero eigenvalues of the matrix of order  $p+1$  can be obtained from the product of the non-zero eigenvalues of the matrix of order  $p$  and the so-called minimum-norm prediction vector introduced by Kumaresan and Tufts in the context of parameter estimation.

### I. INTRODUCTION

This correspondence presents an extension of a classical result concerning the determinant of symmetric nonnegative Toeplitz matrices to the case when the matrices are singular. Let us define the  $(p+1)$  by  $(p+1)$  nonnegative Toeplitz symmetric matrix:

$$\mathbf{M}_p = \begin{bmatrix} r_0 & r_1 & r_2 & \cdots & r_p \\ r_1 & r_0 & r_1 & \cdots & r_{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_p & r_{p-1} & r_{p-2} & \cdots & r_0 \end{bmatrix}.$$

It is possible [1] to associate to  $\mathbf{M}_p$  a linear prediction vector  $\mathbf{a}_p$  and a linear prediction error  $\sigma_p^2$  defined by

$$\begin{bmatrix} r_0 & r_1 & r_2 & \cdots & r_p \\ r_1 & r_0 & r_1 & \cdots & r_{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_p & r_{p-1} & r_{p-2} & \cdots & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1^p \\ a_2^p \\ \vdots \\ a_p^p \end{bmatrix} = \begin{bmatrix} \sigma_p^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (1)$$

with  $\mathbf{a}_p = (a_0^p, a_1^p, \dots, a_p^p)^t$  and  $a_0^p = 1$ . A well-known result on full-rank symmetric Toeplitz matrices is [2]

$$\frac{\text{Det}(\mathbf{M}_{p+1})}{\text{Det}(\mathbf{M}_p)} = \sigma_{p+1}^2. \quad (2)$$

This correspondence presents an extension of this result to the case of symmetric Toeplitz matrices of incomplete rank  $r \leq p$ . Symmetric Toeplitz matrices of incomplete rank arise in various situations, e.g., the autocorrelation matrix of a signal containing  $r$  constant amplitude complex sinusoids is a symmetric Toeplitz matrix of rank  $r$  [3]. In such cases,  $\text{Det}(\mathbf{M}_p)$  is null for  $r \leq p$ , and (2) cannot be applied.

We will show that in this case, the preceding relation becomes

$$\frac{\prod_{i=1}^r \lambda_i^{p+1}}{\prod_{i=1}^r \lambda_i^p} = \|\mathbf{a}_{p+1}^{mn}\|^2 \quad (3)$$

Manuscript received April 21, 1993; revised September 20, 1993. The associate editor coordinating the review of this paper and approving it for publication was Dr. James Zeidler.

The author is with Télécom Paris, Department Signal, Paris, France.  
IEEE Log Number 9400416.

in which  $\lambda_i^p$  is the  $i$ th nonzero eigenvalue of matrix  $\mathbf{M}_p$ , and  $\mathbf{a}_{p+1}^{mn}$  is the minimum norm vector in the null-space of  $\mathbf{M}_{p+1}$  under the constraint  $a_0^{p+1} = 1$ , as introduced by Kumaresan and Tufts [4] in the context of parameter estimation.

### II. DERIVATION OF THE RECURRENCE

In all that follows, we will suppose that the symmetric Toeplitz matrices  $\mathbf{M}_p$  have the same incomplete rank  $r$ , for every  $p \geq r$ ,  $\mathbf{M}_{p+1}$  being obtained by adding a new coefficient  $r_{p+1}$  and completing  $\mathbf{M}_p$  to a symmetric Toeplitz matrix.

Furthermore, for any scalar  $\delta$ , we will define  $\mathbf{M}_p(\delta) \triangleq \mathbf{M}_p + \delta \mathbf{I}_p$ ,  $\mathbf{I}_p$  being the  $p+1$  by  $p+1$  identity matrix. Because  $\mathbf{M}_p$  is supposed to be of rank  $r \leq p$ ,  $\text{Det}(\mathbf{M}_p(\delta))$  is a polynomial in  $\delta$  whose roots include the opposite of the  $r$  nonzero eigenvalues of  $\mathbf{M}_p$  along with  $p+1-r$  null roots. We have

$$\text{Det}(\mathbf{M}_p(\delta)) = \delta^{p+1-r} Q_p(\delta)$$

in which  $Q_p(\delta)$  is a polynomial of degree  $r$  whose roots are all nonzero. The constant term  $Q_p(0)$  is therefore the product of the  $r$  nonzero eigenvalues of matrix  $\mathbf{M}_p$ . We now proceed to determine  $Q_p(0)$ .

$\mathbf{M}_{p+1}(\delta)$  can be written as

$$\mathbf{M}_{p+1}(\delta) = \begin{bmatrix} & & & \vdots & & \\ & & & \vdots & & \\ & & & \vdots & & \\ & & & \vdots & & \\ & & & \vdots & & \\ \cdots & \cdots & \cdots & \vdots & \cdots & \\ & & & \mathbf{r}_p^t & & r_0 + \delta \end{bmatrix}$$

with  $\mathbf{r}_p \triangleq (r_{p+1}, r_p, r_{p-1}, \dots, r_1)^t$ .

A well-known result on bordered matrices [5] yields

$$\text{Det}(\mathbf{M}_{p+1}(\delta)) = \text{Det}(\mathbf{M}_p(\delta)) [(r_0 + \delta) - \mathbf{r}_p^t \mathbf{M}_p^{-1}(\delta) \mathbf{r}_p]$$

or equivalently

$$\delta Q_{p+1}(\delta) = Q_p(\delta) [(r_0 + \delta) - \mathbf{r}_p^t \mathbf{M}_p^{-1}(\delta) \mathbf{r}_p]. \quad (4)$$

Note that  $\mathbf{M}_p(\delta)$  is supposed to be full-rank (which means that  $-\delta$  should not be equal to any of the eigenvalues of  $\mathbf{M}_p$ ).

Because  $\mathbf{M}_{p+1}$  is singular, there exists at least one linear prediction vector  $\mathbf{a}_{p+1}$  satisfying (1) with  $\sigma_{p+1}^2 = 0$  [6]. We will denote  $\mathbf{u}_p$  the vector obtained by reversing the order of the  $p$  last coefficients of  $\mathbf{a}_{p+1}$ :

$$\mathbf{u}_p = (a_{p+1}^{p+1}, a_p^{p+1}, \dots, a_1^{p+1})^t. \quad (5)$$

Because  $\mathbf{M}_p$  is centro-symmetric, we have

$$\mathbf{M}_p \mathbf{u}_p = -\mathbf{r}_p. \quad (6)$$

Inserting (6) into  $\mathbf{r}_p^t \mathbf{M}_p^{-1}(\delta) \mathbf{r}_p$ , we obtain successively

$$\begin{aligned} \mathbf{r}_p^t \mathbf{M}_p^{-1}(\delta) \mathbf{r}_p &= -\mathbf{r}_p^t \mathbf{u}_p + \delta \mathbf{r}_p^t \mathbf{M}_p^{-1}(\delta) \mathbf{u}_p \\ &= r_0 - \delta \|\mathbf{u}_p\|^2 + \delta^2 \mathbf{u}_p^t \mathbf{M}_p^{-1}(\delta) \mathbf{u}_p \end{aligned} \quad (7)$$

in which we used the fact that  $\mathbf{r}_p^t \mathbf{u}_p = -r_0$ , and the symmetry of  $\mathbf{M}_p(\delta)$ .  $\|\cdot\|$  refers to the standard Euclidian norm. Inserting (7) into (4), we obtain

$$Q_{p+1}(\delta) = Q_p(\delta) [1 + \|\mathbf{u}_p\|^2 - \delta \mathbf{u}_p^t \mathbf{M}_p^{-1}(\delta) \mathbf{u}_p]. \quad (8)$$

Note that (6) is valid both if  $\mathbf{M}_p$  is singular or nonsingular. By contrast, the equality  $\mathbf{r}_p^t \mathbf{u}_p = -r_0$  is valid only if  $\mathbf{M}_p$  is singular. We now evaluate  $\delta \mathbf{u}_p^t \mathbf{M}_p^{-1}(\delta) \mathbf{u}_p$ .