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FLOW AROUND A CYLINDER WITH NON-UNIFORM APPROACH VELOCITY  
NUMERICAL TECHNIQUES OF INTEGRATION  
(Preliminary results)

By  
Giampaolo Di Silvio  
and  
Sylvester Petryk

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## 1. INTRODUCTION

While an enormous amount of literature (especially in aerodynamics) is concerned with the flow created by an obstacle immersed in a uniform stream, a relatively little amount of work has been done on flow about obstacles immersed in a non-uniform stream, such as a boundary layer near the walls of a duct.

In this case, in addition to the "primary flow" (namely the flow that should take place if the approach velocity was uniform), a transversal "secondary flow" arises which is the cause of several interesting phenomena. The secondary flow is responsible for the displacement of the stagnation streamlines in the direction of the lower velocity, with a consequent alteration of the drag coefficient of the immersed body; in the case that the immersed body is a Pitot tube, this "shift effect" is the cause of errors in the velocity evaluation. Another important feature related to the secondary flow is the formation of the "horseshoe vortex" at the foot of piers and abutments in rivers, responsible for the scour (fig. 1).

The purpose of this research is to determine the pressure and velocity distribution in the vicinity of a circular cylinder placed vertically in a channel with a non-uniform vertical distribution of the approach velocity. Numerical techniques for the integration of the flow equation have been considered.

No definitive solution has been achieved as yet, but some preliminary results are given with a suggested program for research continuation.

## 2. REVIEW OF THE LITERATURE

The non-uniform velocity distribution of the undisturbed stream approaching the obstacle is generally that corresponding to a boundary layer: in other words, it is generated by frictional effects. However, the presence of shear stresses in the vicinity of the obstacle is not at all, in itself, a necessary factor for the cropping-up of the secondary flow: direct cause of the secondary velocity component is rather the lower dynamic pressure near the wall of the channel, due, in its turn, to the lower approach velocity. In front of the obstacle, the friction forces can be neglected in comparison with the pressure forces; therefore, once the approach velocity distribution is prescribed far upstream from the obstacle, the flow field should be described with sufficient accuracy by the equations of the ideal (inviscid) fluid, even if, of course, the flow cannot be treated as irrotational [1].

Because of the complexity of the inviscid flow equations, approximations are always required if an analytical solution is sought. According to a classification of Hawthorne [2], the approximate solutions of secondary flow may be divided into three types:

1. Flow with small shear (i.e. small velocity gradient) and small disturbance (i.e. small stagnation-pressure variation);
2. Flow with small shear but large disturbance;
3. Flow with large shear but small disturbance.

Only a limited number of solutions, in very particular cases of two-dimensional or axi-symmetrical flow, are available with large shear and large disturbance.

It is noted that solutions of types (1) and (3) do not allow for the presence of stagnation points (the disturbance is assumed to be small); therefore, they do not apply but for thin airfoils and slender bodies.

Solutions of type (2) (known as "secondary flow" approximation) may be applied to the flow around thick bodies like cylinders, provided that the approach velocity distribution is only weakly sheared.

Lightill [3] has calculated the velocity field about an infinitely long circular cylinder immersed in a stream having a weak linear velocity gradient in the spanwise direction: Lightill's solution is reproduced in fig. 2. In the assumed hypothesis the velocity field results from the superposition of:

- i) a primary flow, contained in each plane perpendicular to the cylinder and coinciding with the two-dimensional potential-flow solution;

- ii) a secondary flow, consisting in a velocity distribution exclusively in spanwise direction.

This approximate solution presents two major limitations. The first is that the magnitude of the secondary (spanwise) velocity tends to infinity on the surface of the body and on the plane of symmetry behind it; as a consequence, the drag of the body results to be infinite. The second is that the cylinder is considered infinitely long; no information is given about the flow at its end, where the vertical velocity gives rise to the horseshoe vortex. Although the first limitation may be somehow overcome by introducing a boundary layer close to the surface of the body [4], the approximate hypothesis of a uniformly and weakly sheared oncoming flow does not lead, in practice, to a satisfactory description of the phenomenon.

Therefore, instead of looking for an analytical but approximate solution, it is thought that more information could be obtained by the numerical integration of the inviscid flow equations (Euler's and continuity equations).

A large number of articles are available about numerical solutions of the flow equations, both in the inviscid and viscous flow case, especially in axisymmetric situation. The aim of most of the articles, however, is to predict the evolution of the flow field during the time, starting from known initial conditions prescribed all over a certain region of space. (The first important contribution in this direction has probably been the paper by

Richardson [5] on meteorological forecasting). The problem being considered in this paper, instead, is to find the steady flow field over a certain region of space with specified conditions on its boundaries.

This latter type of problem has received a rather intensive investigation with reference to the flow within the turbomachines. In a recent paper [6], Boyd and Rice apply the three-dimensional Navier-Stokes' equations to determine the steady flow between parallel co-rotating disks; a finite-difference technique for solving these equations (in cylindrical coordinates) is developed which will be reconsidered later.

Finite difference integration of the three-dimensional Euler's equations (in rectangular coordinates) has also been performed by Stephan [7] to predict the (steady) free-surface shape of a jet issuing from a culvert outlet.

In both of these problems velocity and pressure distributions were prescribed at the upstream boundary, and the computation was carried out toward successively downstream positions, where pressure and velocity components were determined by their known values upstream. In the case of the co-rotating disks [6], the velocity distribution of the entering flow was arbitrarily prescribed on the outer periphery, while the external pressure was assumed constant; the velocity distribution and the (constant) value of the pressure was calculated successively for each inner radius (fig. 3a). In the case of the free jet issuing from a culvert [7], the velocity and pressure distribution were pres-



cribed in the outlet cross section, based on experimental results; the velocity and pressure distribution, as well as the jet-shape, was successively calculated for each downstream section (fig. 3b).

In our problem, however, the situation is still quite different. Our boundary conditions are theoretically defined only at infinite distance upstream from the cylinder, where the streamlines are supposed to be parallel (no transversal velocity component and a hydrostatic pressure-distribution exists); at any other finite distance from the cylinder the velocity and pressure distribution is unknown; (a secondary flow, no matter how small it is, is certainly present). The lack of knowledge of the boundary conditions at any finite distance led Dalton and Mash [8] to think that finite difference method is unfit to obtain a solution for this problem; (on the other hand, an attempt to solve it by a perturbation technique, referred to in the same paper, was also unsuccessful).

However, starting from a tentative condition at finite distance from the cylinder, some iterative procedure could probably be found able to reach a solution which is consistent with the asymptotic behaviour of the flows at the upstream infinity. Let us observe, by the way, that also the flow studied by Stephan [7] may be conveniently described by its asymptotic behaviour in the duct at infinite distance upstream from the outlet. In this case, the pressure and velocity distribution on the outlet cross section should ensue from the computation, instead of being prescribed on the base of experiments.

A question arises whether the asymptotic behaviour at the upstream infinity, assumed as a boundary condition, is able to guarantee the existence and uniqueness of the solution. In other words, the question is whether the problem is mathematically "well posed" [9].

In his interesting paper on secondary flow [1], Kronauer stated that the uniqueness of rotational flow solutions from given boundary values is complicated by the possibility of having closed interior regions of circulating fluid; however, this possibility is eliminated if one is interested in a solution where each particle traverses the entire duct length.

The horseshoe vortex at the foot of the cylinder is not, very likely, a closed region of circulating fluid; nevertheless, it is not sure that the solution of the inviscid flow equations (even if unique) will reveal the presence of the horseshoe vortex, since its formation may perhaps be due to some instability of the motion or to some effect neglected by the equations. An answer to the question cannot be given at this stage.

In the same paper, Kronauer suggests an iterative technique for calculating the flow field past a body with a non-uniform approach velocity. The technique is based on the repeated integration of Cauchy's vorticity equations, and it has been applied to the flow about a cascade of symmetric airfoils. As the above application is based upon some assumptions valid only for slender bodies, an eventual rearrangement of this technique for bluff bodies will require further investigation.

In the next paragraph an iterative procedure based on the repeated integration by finite differences of Euler's and continuity equations is suggested; the fluid is supposed to be incompressible.

### 3. FORMULATION OF THE PROBLEM

A vertical cylinder of radius  $r_0$  is placed between two horizontal planes, constituting the bottom and the top of a infinitely wide duct; the lower half of the arrangement is shown in fig. 4.

The undisturbed oncoming flow at the upstream infinity is assumed to be parallel to the axis  $y$ , uniform in each horizontal plane, but with a prescribed velocity distribution along the vertical axis  $z$ ; let  $v_0(z)$  be the undisturbed velocity in each horizontal plane and  $v_{00}$  the maximum value of  $v_0(z)$ .

As the plane  $(x,y)$  is a plane of symmetry, it will be chosen as the upper boundary of the flow region; (neglecting the effects near the surface, it may also approximately represent the water surface of an open channel). The lower boundary is set on the bottom of the flume, at a depth  $z=h_0$  below the plane  $(x,y)$ . Because of symmetry, a lateral boundary has been set at  $\theta = 0^\circ$ ; another one has been set at  $\theta = 90^\circ$ . The computations are not carried beyond  $\theta = 90^\circ$  since the inviscid flow equations do not apply because of separation effects. The outer boundary has been fixed at an arbitrary distance  $r_F$  from the cylinder axis.

Cylindrical coordinates have been employed, after some unsatisfactory results with rectangular coordinates. In cylindrical coordinates, the Euler and continuity equations describing the flow are

$$u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial (p + \gamma h)}{\partial r} ;$$

$$u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} + w \frac{\partial v}{\partial z} = - \frac{1}{\rho r} \frac{\partial (p + \gamma h)}{\partial \theta} ;$$

$$u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial (p + \gamma h)}{\partial z} ;$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 ;$$

where  $u, v, w$  are the velocity components on the  $r, \theta, z$  directions,  $\rho$  is the fluid density and  $(p + \gamma h)$  is the piezometric pressure.

After having defined the following non-dimensional ratios

$$R = \frac{r}{r_0} ; Z = \frac{z}{z_0} ; H_0 = \frac{h_0}{r_0} ;$$

$$U = \frac{u}{v_{00}} ; V = \frac{v}{v_{00}} ; W = \frac{w}{v_{00}} ;$$

$$P = \frac{p + \gamma h}{v_{00}^2} ;$$

the flow equations become

$$U \frac{\partial U}{\partial R} + \frac{V}{R} \frac{\partial U}{\partial \theta} - \frac{V^2}{R} + W \frac{\partial U}{\partial Z} = - \frac{\partial P}{\partial R} ; \quad (1)$$

$$U \frac{\partial V}{\partial R} + \frac{V}{R} \frac{\partial U}{\partial \theta} - \frac{UV}{R} + W \frac{\partial V}{\partial Z} = - \frac{1}{R} \frac{\partial P}{\partial \theta} ; \quad (2)$$

$$U \frac{\partial W}{\partial R} + \frac{V}{R} \frac{\partial W}{\partial \theta} + W \frac{\partial W}{\partial Z} = \frac{\partial P}{\partial Z} ; \quad (3)$$

$$\frac{\partial U}{\partial R} + \frac{U}{R} + \frac{1}{R} \frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial Z} = 0 ; \quad (4)$$

#### 4. DIFFERENCE FORM OF THE FLOW EQUATIONS

In order to write the above equations in finite difference form, the region around the cylinder has been marked by a grid, as shown in fig. 5. Grid points were designated by  $i, j, k$  subscripts, respectively at the  $R, \theta, Z$  coordinate.

If the subscripts  $i_o, i_F, j_o, j_F, k_o, k_F$  are respectively used for designating the points at  $R = R_o = 1, R = R_F, \theta = 0, \theta = \pi/2, Z = 0, Z = H_o$ , the mesh sizes in  $R, \theta, Z$  directions respectively are

$$D = \frac{1}{i_o} = \frac{R_F - 1}{i_F - i_o} \quad ;$$

$$T = \frac{\pi/2}{j_F - j_o} \quad ;$$

$$C = \frac{H_o}{k_F - k_o} \quad ;$$

and the coordinates

$$R = iD \quad ;$$

$$\theta = (j - j_o)T \quad ;$$

$$Z = (k - k_o)C \quad ;$$

At any point  $i, j, k$ , not lying on the boundaries, the derivatives with respect to  $R, \theta, Z$  of any quantity  $Q$  may be replaced by the following "central differences" [10] [11] :

$$\frac{\partial Q}{\partial R} \Big|_{i,j,k} \cong \frac{Q_{i+1,j,k} - Q_{i-1,j,k}}{2D} ;$$

$$\frac{\partial Q}{\partial \theta} \Big|_{i,j,k} \cong \frac{Q_{i,j+1,k} - Q_{i,j-1,k}}{2T} ;$$

$$\frac{\partial Q}{\partial Z} \Big|_{i,j,k} \cong \frac{Q_{i,j,k+1} - Q_{i,j,k-1}}{2C} .$$

Therefore, equations (1) (2) (3) and (4), written in "difference form", are:

$$U_{i,j,k} \frac{(U_{i+1,j,k} - U_{i-1,j,k})}{2D} + \frac{V_{i,j,k}}{iD} \cdot \frac{(U_{i,j+1,k} - U_{i,j-1,k})}{2T} - \frac{V_{i,j,k}^2}{iD} + W_{i,j,k} \frac{(U_{i,j,k+1} - U_{i,j,k-1})}{2C} = - \frac{(P_{i+1,j,k} - P_{i-1,j,k})}{2D} ; \quad (1a)$$

$$U_{i,j,k} \frac{(V_{i+1,j,k} - V_{i-1,j,k})}{2D} + \frac{V_{i,j,k}}{iD} \cdot \frac{(V_{i,j+1,k} - V_{i,j-1,k})}{2T} + \frac{U_{i,j,k} V_{i,j,k}}{iD} + W_{i,j,k} \frac{(V_{i,j,k+1} - V_{i,j,k-1})}{2C} = - \frac{1}{iD} \frac{(P_{i,j+1,k} - P_{i,j-1,k})}{2T} ; \quad (2a)$$

$$U_{i,j,k} \frac{(W_{i+1,j,k} - W_{i-1,j,k})}{2D} + \frac{V_{i,j,k}}{iD} \cdot \frac{(W_{i,j+1,k} - W_{i,j-1,k})}{2T} + W_{i,j,k} \frac{(W_{i,j,k+1} - W_{i,j,k-1})}{2C} = - \frac{(P_{i,j,k+1} - P_{i,j,k-1})}{2C} ; \quad (3a)$$

$$\frac{(U_{i+1,j,k} - U_{i-1,j,k})}{2D} + \frac{U_{i,j,k}}{iD} + \frac{1}{iD} \frac{(V_{i,j+1,k} - V_{i,j-1,k})}{2T} + \frac{(W_{i,j,k+1} - W_{i,j,k-1})}{2C} = 0. \quad (4a)$$

Solving successively eqs. (4a) (2a) (3a) (1a) for the unknown quantities, the following system of equations follows:

$$U_{i+1,j,k} = U_{i-1,j,k} - \frac{(V_{i,j+1,k} - V_{i,j-1,k})}{iT} - \frac{D}{C} (W_{i,j,k+1} - W_{i,j,k-1}) - \frac{2U_{i,j,k}}{i}; \quad (5)$$

$$V_{i+1,j,k} = V_{i-1,j,k} - \frac{V_{i,j,k}}{U_{i,j,k}} \frac{(V_{i,j+1,k} - V_{i,j-1,k})}{iT} - \frac{D}{C} \frac{W_{i,j,k}}{U_{i,j,k}} (W_{i,j,k+1} - W_{i,j,k-1}) - \frac{(P_{i,j+1,k} - P_{i,j-1,k})}{iTU_{i,j,k}} - \frac{2V_{i,j,k}}{i}; \quad (6)$$

$$W_{i+1,j,k} = W_{i-1,j,k} - \frac{V_{i,j,k}}{U_{i,j,k}} \frac{(W_{i,j+1,k} - W_{i,j-1,k})}{iT} - \frac{D}{C} \frac{W_{i,j,k}}{U_{i,j,k}} (W_{i,j,k+1} - W_{i,j,k-1}) - \frac{D}{C} \frac{(P_{i,j,k+1} - P_{i,j,k-1})}{U_{i,j,k}}; \quad (7)$$

$$\begin{aligned}
P_{i+1,j,k} = & P_{i-1,j,k} - U_{i,j,k} (U_{i+1,j,k} - U_{i-1,j,k}) - \\
V_{i,j,k} & \frac{(U_{i,j+1,k} - U_{i,j-1,k})}{iT} - \frac{D}{C} W_{i,j,k} (U_{i,j,k+1} - U_{i,j,k-1}) + \\
\frac{2 V_{i,j,k}^2}{i} & .
\end{aligned} \tag{8}$$

The above system allows the computation of the velocity components and pressure at the radius  $(i+1)$ , from the known values at the previous two radii  $(i)$  and  $(i-1)$ . The procedure is then repeated for the next ring, and so on.

This scheme, called "direct difference" scheme, failed to yield satisfactory results in the problem of Boyd and Rice [6]; in that case the indirect or "inverse-difference" scheme was then used and proved successful. Unlike the direct difference scheme, the indirect scheme does not allow one to calculate separately the field values at different positions on the ring  $(i+1)$ ; instead, all the values on the ring  $(i+1)$  are simultaneously provided by the solution of a system of  $(3n + 1)$  equations, where  $n$  is the number of grid points in the ring.

This second procedure requires much more computation time than the direct scheme, but it does not seem necessary to be applied to our problem. It is necessary to point out that the Navier-Stokes equations solved by Boyd and Rice involve second-order derivatives which are



probably more sensitive to accumulative errors. On the other hand, for the first-order equations of the inviscid fluid, the direct scheme was proved succesful by Stephan [7], even if he used the less precise "forward difference" instead of the "central difference" as has been done here; satisfactory results have also been achieved in some preliminary computations in this research (see paragraph 6). At any rate, the inverse difference scheme may be kept in mind if an alternative computational method is to be tried.

##### 5. SPECIAL CONDITIONS ON THE BOUNDARIES

The system (5) (6) (7) (8) cannot be directly employed on the boundaries because some of the quantities in the equations turn out to be unknown. These quantities will be replaced in the equations as it follows.

On the plane  $\Theta = 0$  ( $j=j_0$ ), because the symmetry, it is

$$\begin{aligned} V_{i,j-1,k} &= - V_{i,j+1,k} & , \\ U_{i,j-1,k} &= U_{i,j+1,k} & , \\ W_{i,j-1,k} &= W_{i,j+1,k} & , \\ P_{i,j-1,k} &= P_{i,j+1,k} & . \end{aligned}$$

On the plane  $Z = 0$  ( $k=k_0$ ), because the symmetry, it is

$$\begin{aligned}
 W_{i,j,k-1} &= -W_{i,j,k+1} \quad , \\
 U_{i,j,k-1} &= U_{i,j,k+1} \quad , \\
 V_{i,j,k-1} &= V_{i,j,k+1} \quad , \\
 P_{i,j,k-1} &= P_{i,j,k+1} \quad .
 \end{aligned}$$

On the plane  $\theta = \pi/2$  ( $j=j_F$ ), which is not a plan of symmetry, the central difference does not apply anymore and it is replaced by the backward difference; it is equivalent to put in the equations, for each quantity  $Q$ ,

$$Q_{i,j+1,k} = 2Q_{i,j,k} - Q_{i,j-1,k} \quad .$$

The same is it on the plane  $Z = H_0$  ( $k = k_F$ ):

$$Q_{i,j,k+1} = 2Q_{i,j,k} - Q_{i,j,k-1} \quad .$$

The equations (6) and (7) also do not apply where the quantity  $U_{i,j,k}$  is equal to zero. It happens on the plane  $\theta = \pi/2$  ( $j=j_F$ ), where the velocity is supposed to be tangential. In this case eqs. (6) and (7) will be replaced by the following ones, obtained by the eqs. (2) and (3) written in difference form at the point  $(i-1, j-1, k)$  ; (the backward difference is used for the derivatives with respect to  $R$  here).

$$\nabla_{i+1,j,k} = V_{i+1,j-2,k} - \frac{TD}{C} (i+1) \frac{W_{i+1,j-1,k}}{V_{i+1,j-1,k}}$$

$$\cdot (V_{i+1,j-1,k+1} - V_{i+1,j-1,k-1}) - 2T (i+1) \frac{U_{i+1,j-1,k}}{V_{i+1,j-1,k}}$$

$$\cdot (V_{i+1,j-1,k} - V_{i,j-1,k}) - 2T (U_{i+1,j-1,k}) -$$

$$\frac{(P_{i+1,j,k} - P_{i+1,j-2,k})}{V_{i+1,j-1,k}} ; \quad (6a)$$

$$W_{i+1,j,k} = W_{i+1,j-2,k} - \frac{TD(i+1)}{C} \frac{W_{i+1,j-1,k}}{V_{i+1,j-1,k}}$$

$$\cdot (W_{i+1,j-1,k+1} - W_{i+1,j-1,k-1}) - 2T(i+1) \frac{U_{i+1,j-1,k}}{V_{i+1,j-1,k}}$$

$$\cdot (W_{i+1,j-1,k} - W_{i,j-1,k}) -$$

$$\frac{DT}{C}(i+1) \frac{(P_{i+1,j-1,k+1} - P_{i+1,j-1,k-1})}{V_{i+1,j-1,k}} ; \quad (7a)$$

Finally, the geometric boundary conditions have to be stated for the velocity components, on the cylinder surface ( $i=i_0$ ) and on the planes  $j=j_0$ ,  $k=k_0$ , and  $k=k_F$ ; they will be respectively

$$U_{i,j,k} = 0 \quad , \quad (9)$$

$$V_{i,j,k} = 0 \quad , \quad (10)$$

$$W_{i,j,k} = 0 \quad . \quad (11)$$

## 6. CALIBRATION OF THE PROGRAM - PRELIMINARY RESULTS

A computational program has been written for Eqs. (5) (6) (7) (8) and for all the special conditions on the boundaries mentioned above.

This program is now being calibrated in the simple case of uniform approach velocity, comparing the computed results with the wellknown values of the two-dimensional potential flow theory. The velocity components and the pressure distribution corresponding to a uniform approach velocity,  $v_{\infty}$ , are given, according to the potential flow theory, by the following expressions:

$$U = - \cos \theta \cdot \left(1 - \frac{1}{R^2}\right)$$

$$V = \sin \theta \cdot \left(1 + \frac{1}{R^2}\right)$$

$$W = 0$$

$$P = \frac{1}{2} (1 - U^2 - V^2).$$

The main purpose of these calibration tests is to determine the more appropriate mesh sizes of the grid. In the first tests performed, computation has been carried out starting from the cylinder surface up to a distance of  $n = 3$  radii from the cylinder axis ( $R_F = 3$ ). The mesh size in the radial direction has been assumed uniform and equal to  $1/m = 1/4$  of the radius; therefore, it follows:

$$i_0 = m = 4 \quad (\text{subscript on the cylinder surface});$$

$$i_F = n \cdot m = 12 \quad (\text{subscript on the outer boundary}).$$

The number of steps in radial direction is then

$$i_F - i_0 = 8$$

It has been then assumed that the same number of steps will also be in tangential and vertical directions, namely:

$$k_F - k_O = j_F - j_O = i_F - i_O = 8$$

Putting  $j_O = 2$  and  $k_O = 2$  (for computer requirements,  $j_O$  and  $k_O$  must be larger than 1), it follows:

$$j_F = (n-1) m + 2 = 10$$

$$k_F = (n-1) m + 2 = 10$$

The computation has then been started from prescribed values of pressure and velocity components on the cylinder surface ( $i=i_O$ ) and on the next outer ring ( $i=i_O+1$ ), computed by the two-dimensional potential flow; the unknown values in the successive outer rings have then been computed by the finite difference equations. Comparison with the potential-flow values is satisfactory, but it seems that a still finer mesh should be used.

## 7. RESEARCH CONTINUATION

The calibration of the program, by means of potential flow theory, will be continued until a suitable mesh size of the grid will be singled out.

The mesh sizes, as well as the distance from the cylinder up to which the computation is carried on, may easily be varied on the program. This can be done either by changing the values  $n$  and  $m$  (that is, keeping an equal number of steps in the three directions) or, directly, by changing the values  $i_O$ ,  $i_F$ ,  $j_F$ ,  $k_F$  (if different numbers of steps in each direction are required).

We will mention that Boyd and Rice [6] performed their computation using 21 steps in the vertical direction and 1500 in the radial direction (the flow was axi-symmetric), but in our case a much coarser grid can probably be used.

The next stage, and the most important one, will be to find a suitable iterative procedure for solving the problem in the case of a non-uniform approach velocity.

It has already been said that, with a non-uniform approach velocity, the flow conditions are unknown everywhere but at the infinite distance upstream (parallel flow to the axis  $y$  and hydrostatic pressure distribution).

We will tentatively assume that, at a sufficient distance from the cylinder, the undisturbed flow will be affected only by lateral displacement of streamline. In other words, we will suppose that, far enough from the cylinder, the pressure and velocity values are still provided, in each horizontal plan, by the two-dimensional potential flow theory. Under this assumption, if the approach-velocity has a vertical distribution,  $v_o(z)$ , the velocity components and the pressure in each plane  $z$  are given by the following expressions:

$$U = - \cos \theta \cdot \left(1 - \frac{1}{R^2}\right) \cdot \left(\frac{v_o(z)}{v_{oo}}\right)$$

$$V = \sin \theta \cdot \left(1 + \frac{1}{R^2}\right) \cdot \left(\frac{v_o(z)}{v_{oo}}\right)$$

$$W = 0$$

$$P = \frac{1}{2} \left[ \left( \frac{v_o(z)}{v_{oo}} \right)^2 - U^2 - v^2 - W^2 \right] .$$

In particular, if the approach-velocity distribution is linear, the above starting conditions far from the cylinder could probably be improved adding a nonzero vertical component,  $W$ , according the results of Lighthill (fig. 2). It is recalled, however, that the Lighthill solution refers to an infinitely long cylinder and, consequently, the value of  $W$  does not vary in the spanwise direction; in our case, as the vertical velocity component must be zero both on the top and bottom planes, an appropriate variation of  $W$  versus  $z$  has to be assumed (an exponential function is probably the most proper:

$$W = (1 - e^{-Z}) \cdot W_L \quad \text{if } 0 < Z < \frac{H_o}{2}$$

$$W = (1 - e^{-(Z-H_o)}) \cdot W_L \quad \text{if } \frac{H_o}{2} < Z < H_o$$

where  $W_L(R, \theta)$  is the value provided by Lighthill).

In any case, after having fixed the starting conditions, we can begin our computation from the known values at the outermost two rings ( $i_F$  and  $i_F-1$ ) and go on toward the cylinder. As our starting conditions were not completely correct, we will very likely obtain an incorrect pressure and velocity distribution on the cylinder surface; (for instance, if we should not impose a zero-radial velocity according to eq. (9), we may find from eq. (5) a value of  $U$  different from zero). Anyway, the field

values obtained on the innermost rings ( $i_0$ ) and ( $i_0+1$ ) will be employed again as starting conditions for a new outward computation. If the values resulting from this last computation show an acceptable asymptotic behaviour (that is, they tend to the desired parallel flow), this may be considered the final solution; otherwise the computed values on the outermost rings will be used for correcting the assumed starting values, and the procedure will be repeated. A question arises about the convergence and the stability of this procedure, but an answer can probably be given only after having tested it.

If the suggested procedure is unsuccessful, another possibility is to assume a reasonable velocity and pressure distribution on the two innermost rings near the cylinder surface, possibly with the aid of some experimental information (see, for instance, ref. [8]). The outward computation will then supply the upstream oncoming flow corresponding to the assumed pressure and velocity distribution on the cylinder surface. Although the resulting oncoming flow will not possess the desired asymptotic characteristics, tentative alterations of the starting conditions may always provide some useful insight on the consequence of different types of approach velocity on the flow field near the cylinder.



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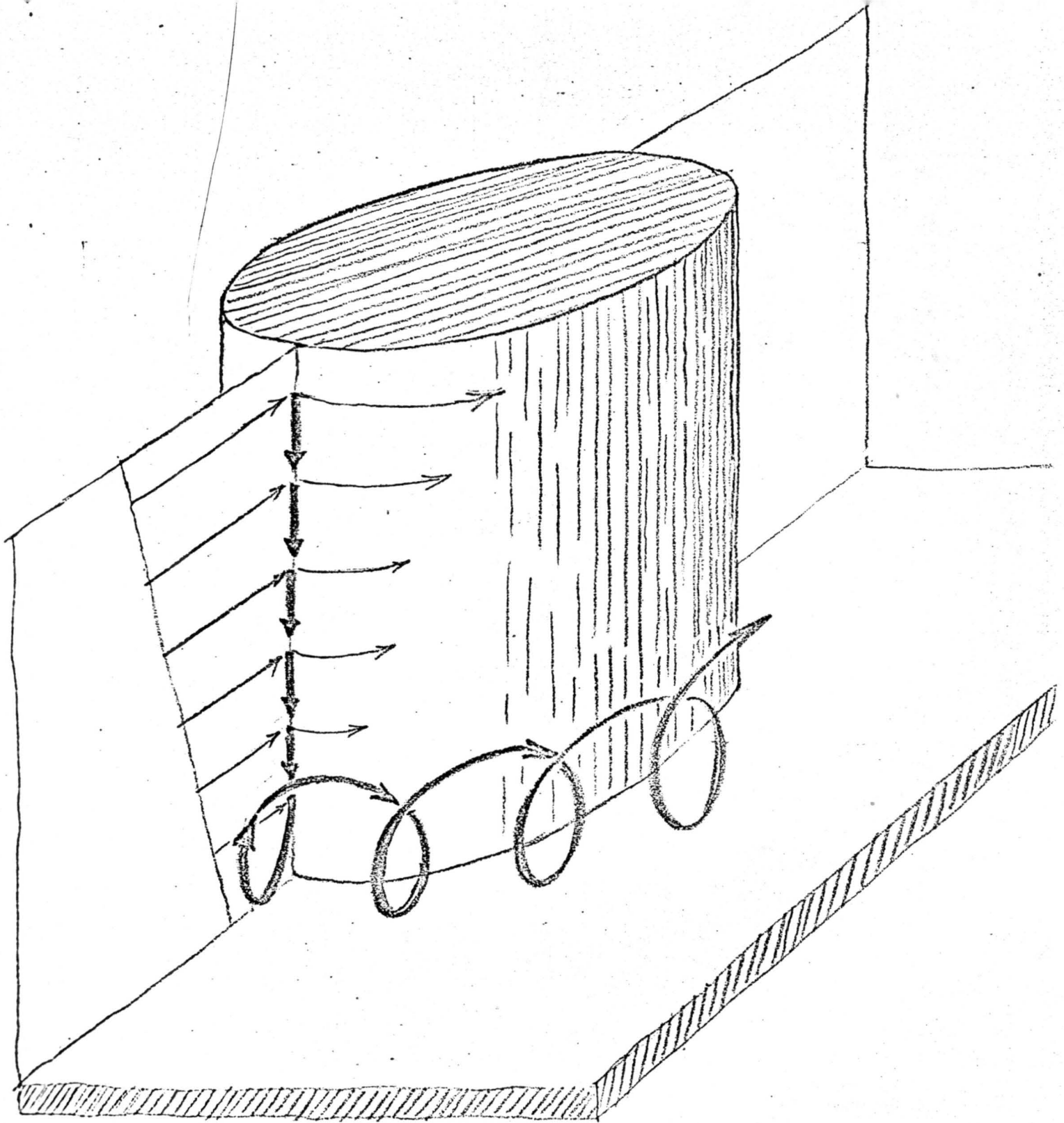
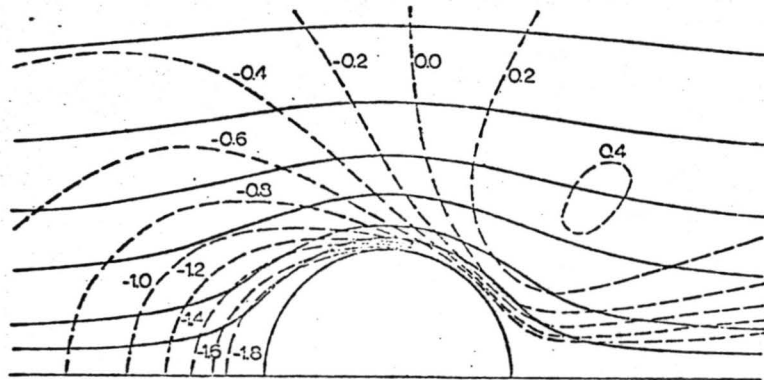


FIG.1 PRIMARY (→) AND SECONDARY (↪) FLOW AROUND  
A PIER WITH NON-UNIFORM APPROACH VELOCITY



**Figure 2.** Primary and secondary flow about a circular cylinder of radius  $a$ , with axis the  $y$ -axis, when the upstream velocity is  $(U + Ay, 0, 0)$ .  
 — Streamlines of the primary flow (note that the velocity components in the  $x$  and  $z$  directions follow these streamlines even when the secondary flow is included).  
 - - - Contours of constant  $v/Aa$  (note that the negative values, which predominate, denote secondary flow in the direction of decreasing primary flow velocity).  
 (Flow is from left to right and only the upper half is shown.) Lighthill (1956a).

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(FIGURE DRAWN FROM REF. [2] )

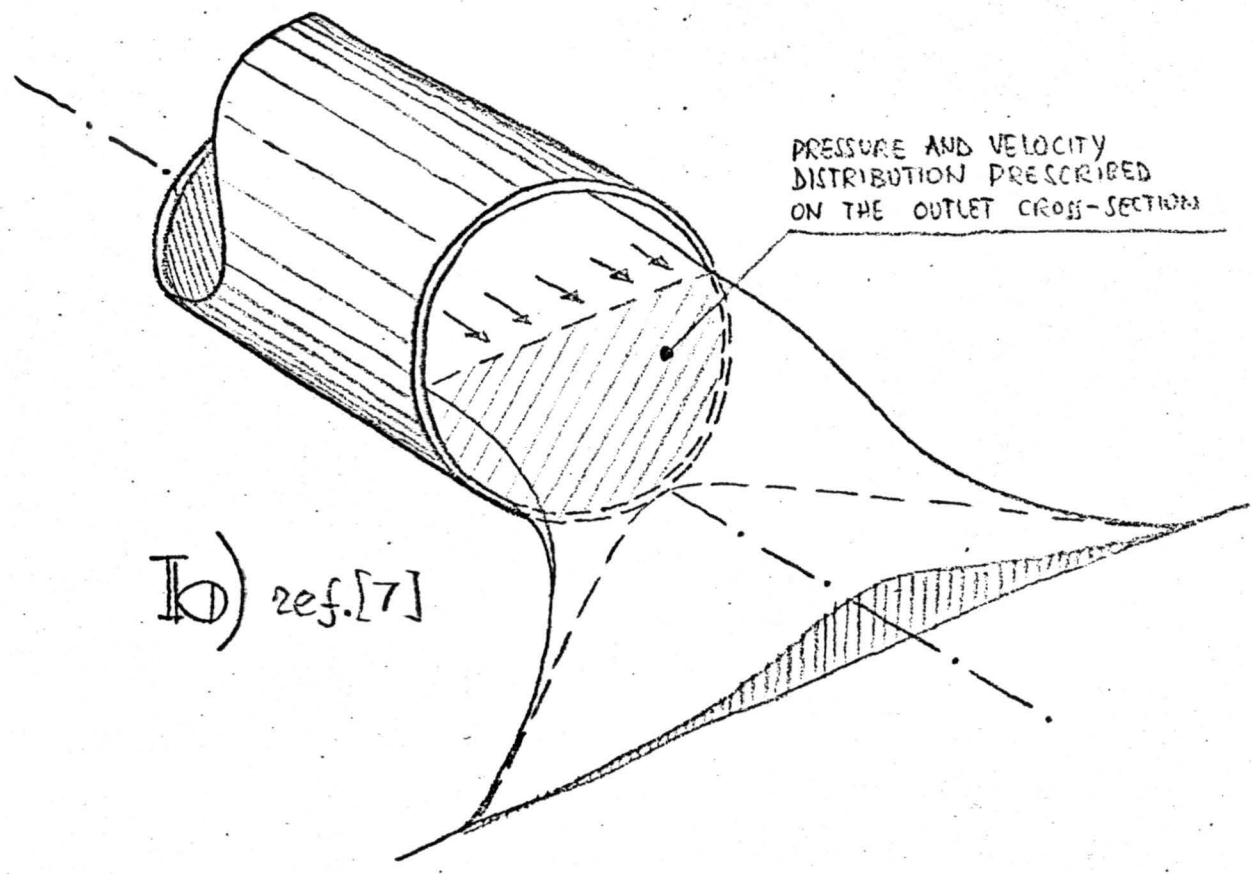
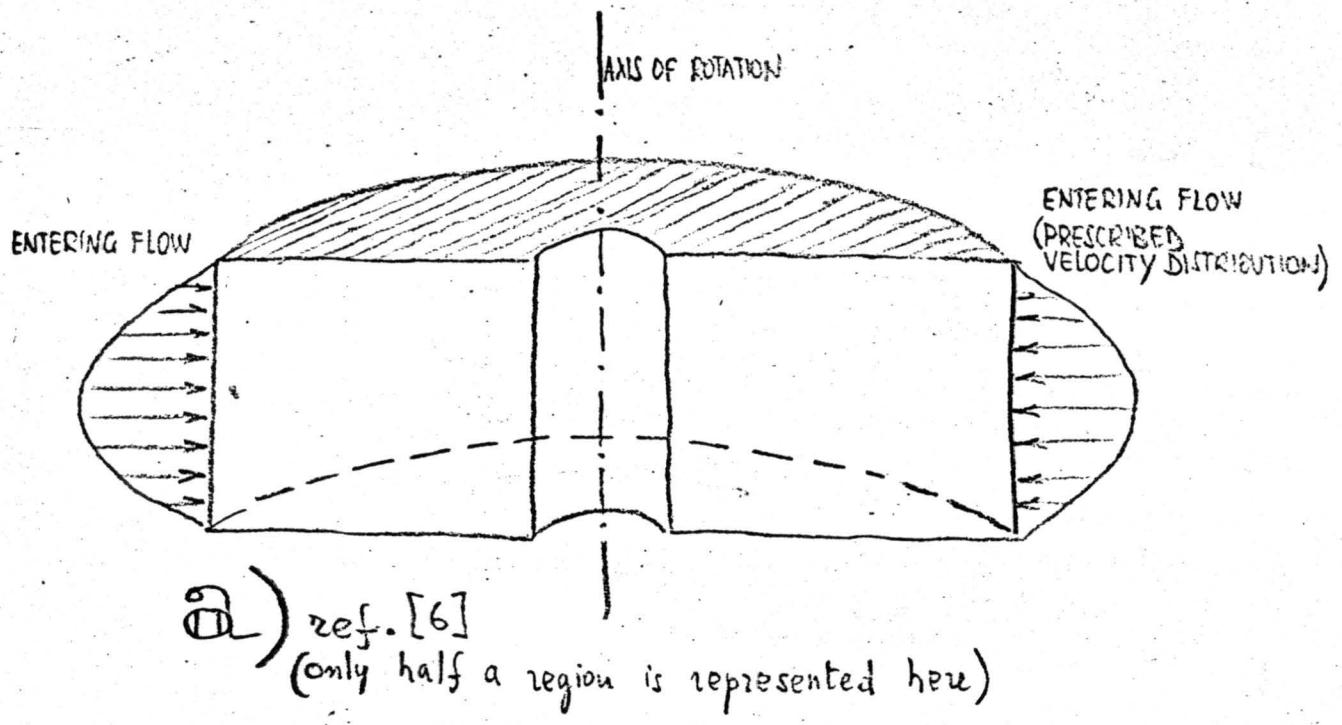


FIG. 3 - EXAMPLES OF NUMERICAL INTEGRATION OF THE FLOW EQUATIONS

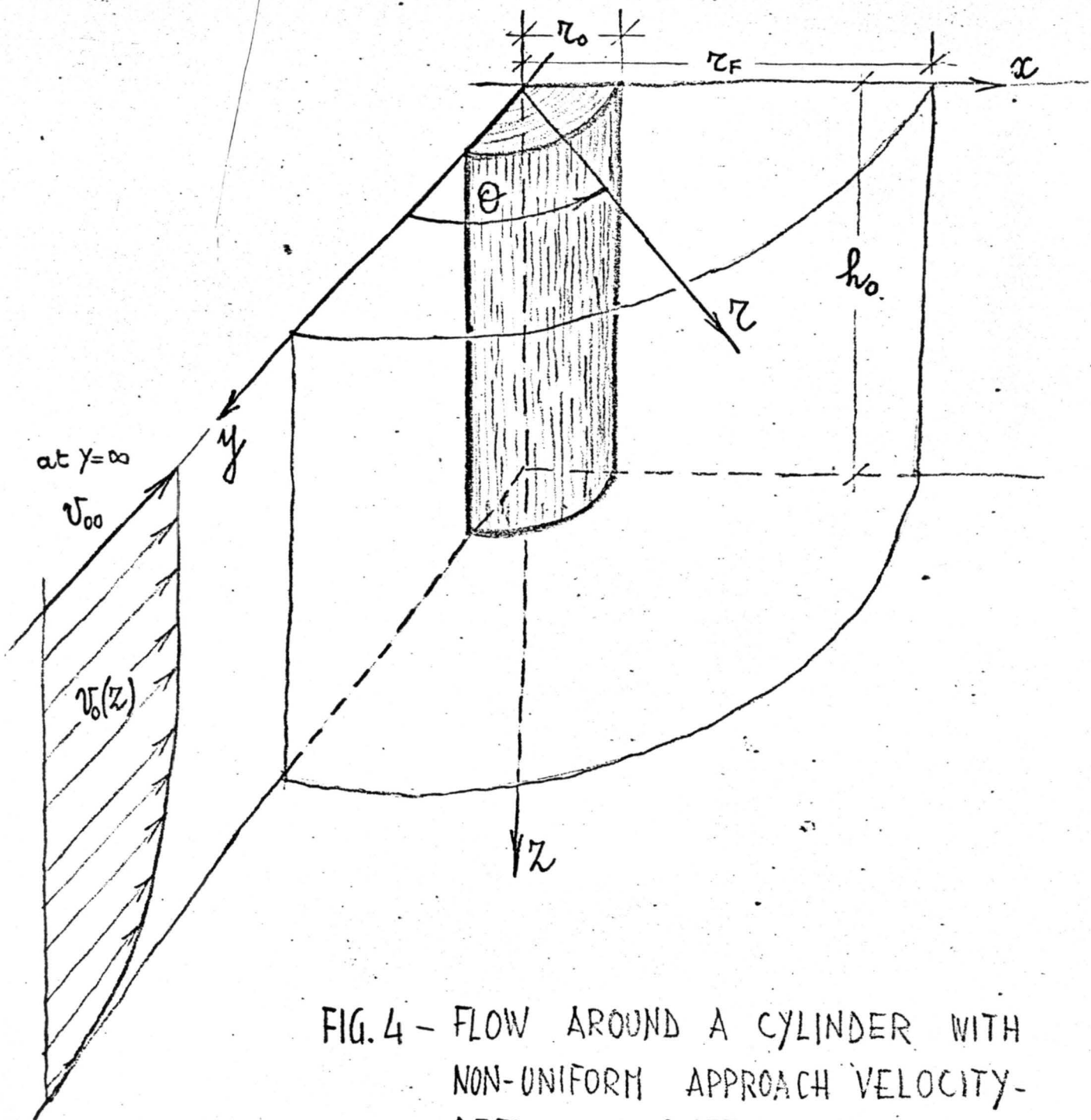


FIG. 4 - FLOW AROUND A CYLINDER WITH NON-UNIFORM APPROACH VELOCITY-DEFINITION SKETCH.