Baroclinic Instability and Atmospheric Development

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ABSTRACT

The baroclinic instability problem is formulated as an initial value problem in order to evaluate the effects of the initial configuration of the wave perturbation. The vertical shape of the initial perturbation is found to be as important as its wave length in determining the energy conversions during the early stages of its development. The general character of the solution of the initial value problem is compared with normal mode studies of baroclinic instability. It is concluded that the initial value formulation bridges the gap between normal mode solutions and non-linear studies.

1. Introduction.

Theoretical studies of cyclone development in the atmosphere took a significant turn, when attention shifted from unstable perturbations at frontal surfaces to instabilities arising from general shearing motions in the free atmosphere. Charney (1947) presented the first mathematical treatment of cyclone waves in terms of the properties of the baroclinic westerlies without reference to the interface between two airmasses of different density and velocity. Subsequent instability studies have mostly followed the guidelines set by Charney and, although some were directly concerned with the explanation of a physical phenomenon (e.g., Charney and Stern, 1962), the majority focused attention on the mathematical character of the problem.

The mathematical procedure followed dates back to investigations of the stability of fluid motions by Thomson, Rayleigh, and Helmholtz. The criterion for instability is determined by finding the conditions under which the phase speed of small periodic disturbances in the fluid becomes a complex number yielding an exponential increase of the amplitude with time. The marked distinction between the exponentially growing unstable disturbances and the neutral periodic perturbations is a consequence of the linear character of the equation describing the development of the perturbation with time. This distinction is not present if allowance is made for the nonlinear terms in the general hydrodynamic equations.

Even in the linear formulation of the atmospheric instability problem the transition from stable to unstable perturbations is smoothed out if the normal mode approach is replaced by a solution of the initial value problem, in particular if the solution is restricted to time periods consistent with the time scale of cyclone development. A more interesting aspect of the initial value formulation is the addition of a new parameter to the stability problem, ie., the initial structure of the perturbation. A new complication may be added to the stability problem if the normal modes do not form a complete set, which implies that an arbitrary initial perturbation cannot be represented in terms of only the normal modes. Thus, Case (1960) has stressed the relevance of the so-called continuous spectrum for a stability problem of the type here discussed.

The present paper summarizes the results of preliminary computations towards a theoretical study of atmospheric development including barotropic and baroclinic energy exchanges, the spherical shape of the earth, the nonlinear effects and the interaction between the developing wave and an idealized quasi-steady long wave pattern in the free atmosphere. The nonlinear formulation of the problem necessitates an initial value approach and the time-scale of the observed cyclone developments restricts the periods of integration to a few days. The main purpose of the future study is to investigate whether the inclusion of the initial value parameter and the wave interactions may render some information in addition to that obtained from the classical stability studies, in order to determine whether an observed small perturbation on the weather map grows into a mature cyclone.

The present preliminary study tries to clarify the role of a few basic parameters entering into the general problem as outlined above. In order not to encumber the interpretation of the results, the model is kept relatively simple and is formulated essentially as suggested by Charney (1947). The results obtained here are either explicitly or implicitly present in previous studies on atmospheric instability, but the interpretation is directed towards a better understanding of the relationship between the mathematical aspect of the stability problem and the physical problem at hand.

2. Basic Equations.

The time-scale of the problem justifies the use of an adiabatic, frictionless model. Considering a quasi-static, quasi-geostrophic atmosphere and using the beta-plane approximation, we obtain the following system of equations

$$\left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla\right) \nabla^2 \psi + \beta_0 \frac{\partial \psi}{\partial \chi} - f_0 \frac{\partial \omega}{\partial p} = 0 \qquad (1)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla\right) \frac{\partial \psi}{\partial p} + \frac{\sigma}{f_0} \omega = 0 \qquad (2)$$

where ψ is the stream function, $\omega \doteq dp/dt$ is a measure of the vertical velocity, p is pressure, t is time, and $\mathbb{V} = \mathbb{k} \times \nabla\psi$ where $\mathbb{k}$ is the vertical unit vector. The Coriolis parameter, f_0, and its derivative $\beta_0 \equiv df/dy$, are treated as constants. The static stability, $\sigma = -(\alpha/\theta)\, \partial\theta/\partial p$, where α is specific volume and θ is potential temperature, is considered a function of pressure only in order to preserve the energy consistency of the system.

The boundary conditions for this system of equations are

$$\omega = 0 \quad \text{for } p = 0 \quad \text{and } p = p_0 = 1000 \text{ mb} \tag{3}$$

Since the flow field must be periodic in longitudinal direction we may write

$$\begin{aligned} \psi(x,y,p,t) &= \overline{\psi}(y,p,t) + \psi'(x,y,p,t) \\ \omega(x,y,p,t) &= \overline{\omega}(y,p,t) + \omega'(x,y,p,t) \end{aligned} \tag{4}$$

where the bar denotes a zonal average and the prime denotes the deviation from this average, which is not necessarily small. Further, the deviation may be represented by a trigonometric series, the coefficients of which are functions of y, p, and t. Assuming that initially the perturbation on the zonal flow consists of one single wave of given zonal wavelength, we may write

$$\begin{aligned} \psi'(x,y,p,t) &= \Psi(y,p,t)e^{ikx} + \Psi^*(y,p,t)e^{-ikx} \\ \omega'(x,y,p,t) &= \Omega(y,p,t)e^{ikx} + \Omega^*(y,p,t)e^{-ikx} \end{aligned} \tag{5}$$

where k is the wavenumber and the asterisk denotes the complex conjugate, such that ψ' and ω' are real functions.

After substitution of (4) into (1) and (2), the equations will involve products of the perturbations, which, however, cannot contribute to the time rate of change of the wave of wavenumber k as long as (5) is satisfied. Thus, independent of the magnitude of the perturbation, we obtain for the initial time

$$\left(\frac{\partial}{\partial t} + ik\overline{u}\right)\left(k^2\Psi - \frac{\partial^2\Psi}{\partial y^2}\right) + ik\left(\beta_0 - \frac{\partial^2\overline{u}}{\partial y^2}\right)\Psi + f_0\,\frac{\partial\Omega}{\partial p} = 0 \tag{6}$$

$$\left(\frac{\partial}{\partial t} + ik\overline{u}\right)\frac{\partial\Psi}{\partial p} - ik\,\frac{\partial\overline{u}}{\partial p}\,\Psi + \frac{\sigma}{f_0}\,\Omega = 0, \qquad \overline{u} = -\frac{\partial\overline{\Psi}}{\partial y} \tag{7}$$

On the other hand, the square of the initial perturbations does contribute toward a change of the zonal flow and will also generate a wave of wavenumber 2k. In the usual terminology these effects are of second order. Thereafter the number of possible interactions increases continuously and a complete spectrum of waves is generated which implies that (6) and (7) will also involve products of the perturbations. The latter are of third or higher order.

Consider first the effect of the time-variation of the zonal wind alone. The system of equations consists then of (6) and (7), supplemented by two equations for the time rate of change of the zonal wind $\bar{u}$ (y,p,t), which involve products of Ψ and Ψ^*. Now if the perturbation grows with time, the shear of the zonal wind decreases, which in turn reduces the growth of the perturbation. Numerical integrations of such highly truncated (low-order) systems show that the amplitude of the perturbation tends to return to its initial value after reaching a maximum value. Exact solutions of simple nonlinear low-order systems such as that pertaining to a baroclinic two-layer model have been obtained by Baer (1968). They show that the solution is periodic in time for all values of the parameters, which implies that the distinction between neutral and unstable perturbations is a purely mathematical one, introduced by linearizing the equations.

The system becomes much more complex if the wave of wavenumber 2k is retained in the system. This effect has been studied in detail by R. King for a purely barotropic model and the results of his study will be available soon. The general baroclinic-barotropic study referred to in the Introduction includes both nonlinear effects here discussed. The present study, however, is confined to the purely baroclinic system obtained by ignoring the y-dependence of the zonal wind and the initial perturbation. Clearly then, the nonlinear contributions from the perturbation disappear and the usual linear system is obtained. The same still holds if the initial perturbation is periodic in y.

The system of equations (6) and (7) may be written in terms of the streamfunction only. Specified for the baroclinic problem the equation becomes

$$[k^2 - f_0^2 \frac{\partial}{\partial p}(\frac{1}{\sigma}\frac{\partial}{\partial p})]\frac{\partial \Psi}{\partial t} = ik[\beta_0 - k^2\bar{u} - f_0^2 \frac{d}{dp}(\frac{1}{\sigma}\frac{d\bar{u}}{dp})]\Psi + ik\bar{u} f_0^2 \frac{\partial}{\partial p}(\frac{1}{\sigma}\frac{\partial \Psi}{\partial p}) \tag{8}$$

where now Ψ is a function of t and p, and σ and $\bar{u}$ are functions of p only. Alternatively, one might derive a diagnostic omega-equation to be used together with either (6) or (7). For later reference, this equation follows below

$$[\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2} - k^2]\Omega = ik\frac{f_0}{\sigma}[\beta_0 \frac{\partial \Psi}{\partial p} - 2k^2 \frac{d\bar{u}}{dp}\Psi] \tag{9}$$

The boundary conditions become

$$\Omega = \frac{1}{\sigma}\frac{\partial}{\partial p}(\frac{\partial \Psi}{\partial t}) + \frac{ik}{\sigma}[\bar{u}\frac{\partial \Psi}{\partial p} - \frac{d\bar{u}}{dp}\Psi] = 0 \text{ for } p = o, p_0 \tag{10}$$

Various methods of obtaining the solution to this system of equations will be discussed in the last part of this paper. In the next section, the solutions will be presented and discussed from a less technical viewpoint. It may be noted here, that most of these solutions have been obtained from a 20-layer numerical model without time-truncation. As discussed in the last section, representative solutions have been tested for convergence in order to eliminate the possibility of errors due to truncation.

3. Results of Numerical Integration.

Consider first the classical method to determine the stability of the perturbations described by Eqs. (8) and (10). Since the coefficients of the equation are independent of time, a solution $\Psi \propto \exp(-ikct)$ can be assumed, which will grow indefinitely for complex values of the wavespeed c. Actually, in that case a growing and a decaying mode will exist simultaneuously, since complex phase speeds can only be found as complex conjugate pairs. Assuming this type of solution, Eq (8) becomes

$$(c-\bar{u})[k^2 - f_0^2 \frac{\partial}{\partial p}(\frac{1}{\sigma}\frac{\partial}{\partial p})]\Psi + [\beta_0 - f_0^2 \frac{d}{dp}(\frac{1}{\sigma}\frac{d\bar{u}}{dp})]\ \Psi = 0 \qquad (11)$$

and (10) becomes

$$\frac{1}{\sigma}[(c - \bar{u})\ \frac{\partial\Psi}{\partial p} + \frac{d\bar{u}}{dp}\ \Psi] = 0 \text{ for } p = 0,\ p_0 \qquad (12)$$

This equation has been solved numerically for the profiles of zonal wind and static stability shown in Fig. 1. The inverse static stability shown is obtained by averaging the values for summer and winter presented by Gates (1961). The zonal wind is an observed mean wind profile for middle latitudes used by Brown (1968).

Fig. 2 shows the real and imaginary part of the wavespeed of the unstable mode. The imaginary part has been multiplied by the wavenumber to obtain the growth rate, kc_i. Results are shown for numerical models with low and high vertical resolution, respectively. The two-layer model shows the familiar short-wave cut-off to instability, which is markedly reduced for the three-layer model and tends to disappear for high vertical resolution. This increase of instability with vertical resolution was suggested by Kuo (1953) and has been discussed in more detail by Brown (1968). Weak instabilities were also found for

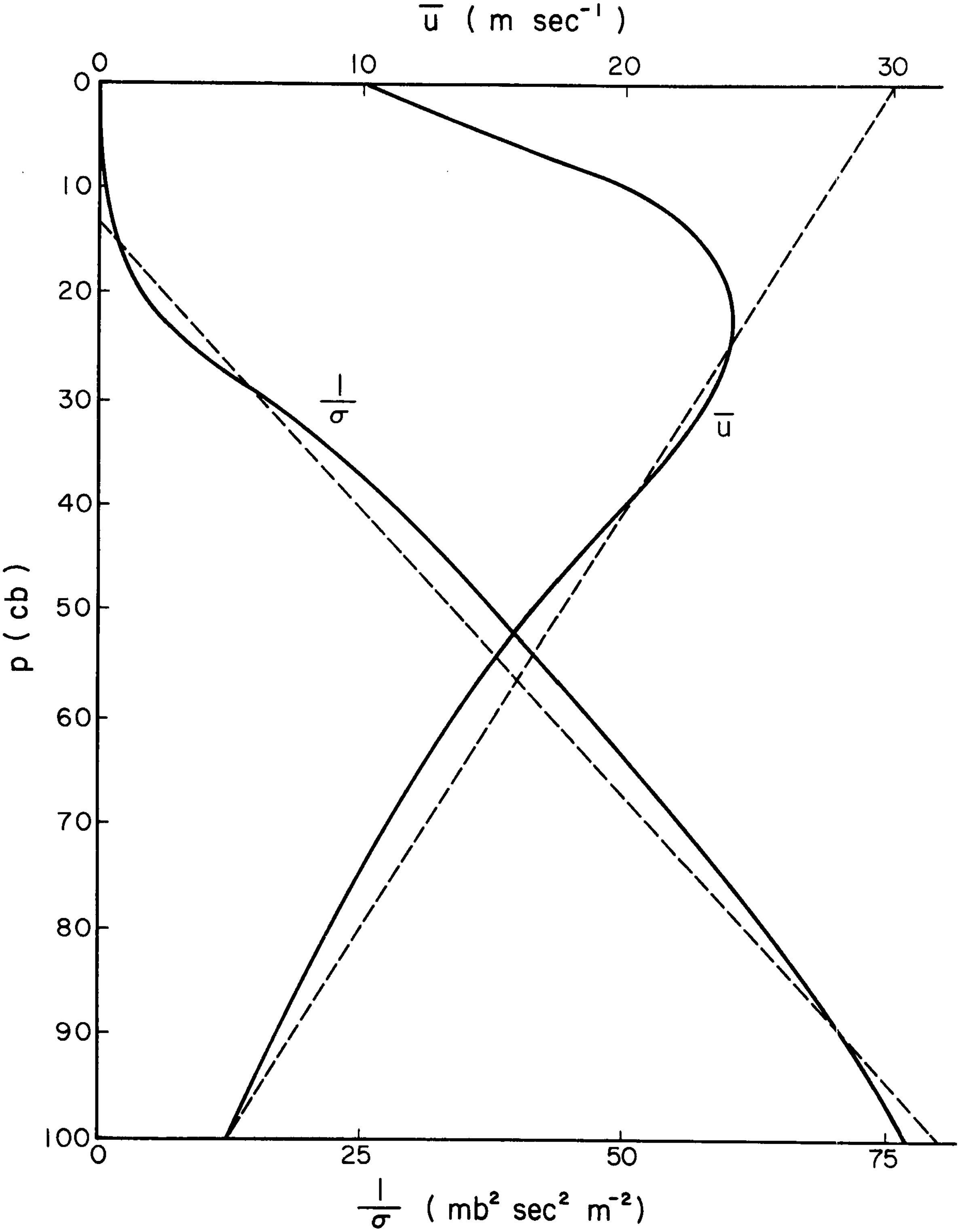


Fig. 1. Basic state zonal flow and inverse static stability. Profiles shown by solid lines used in all calculations unless stated otherwise.

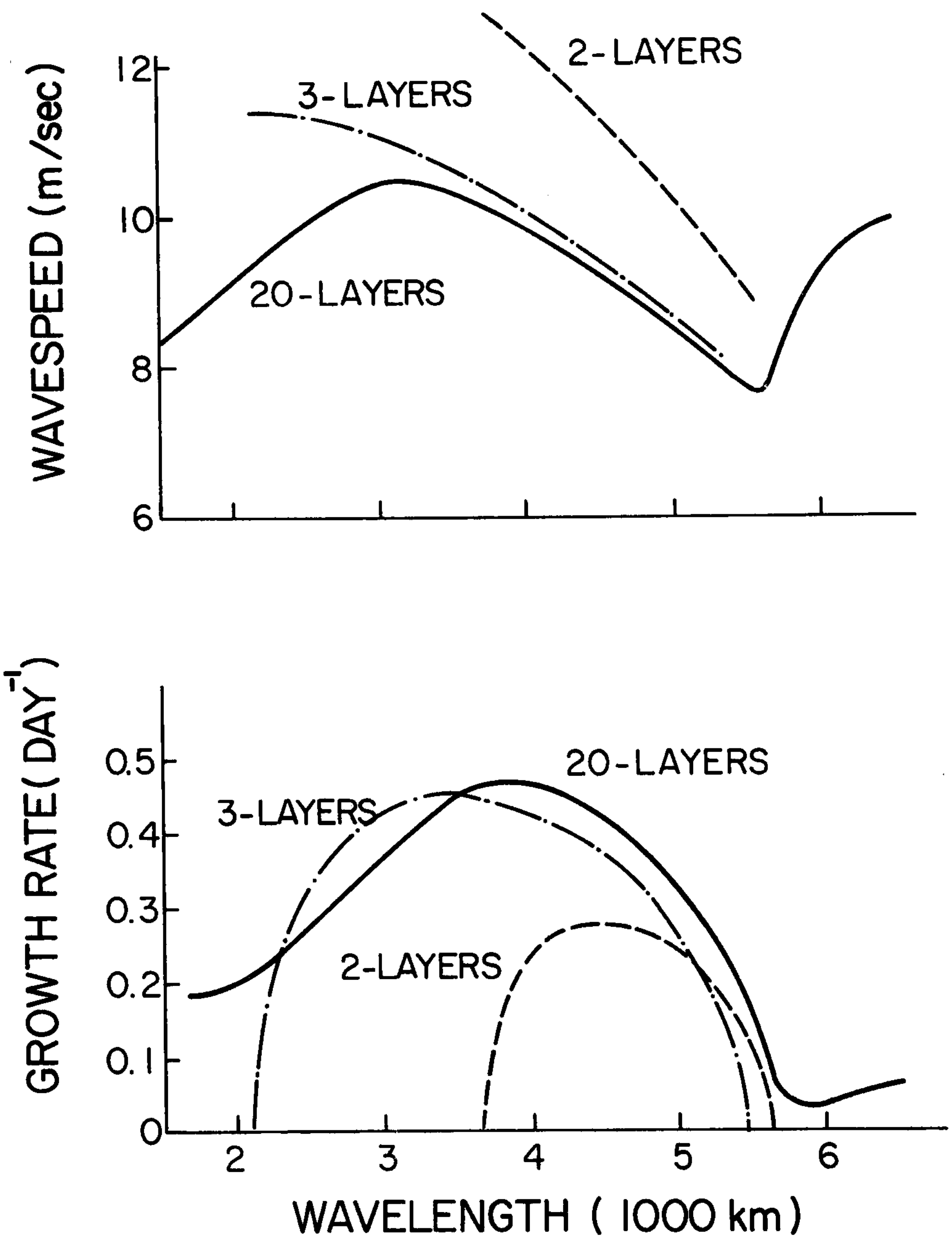
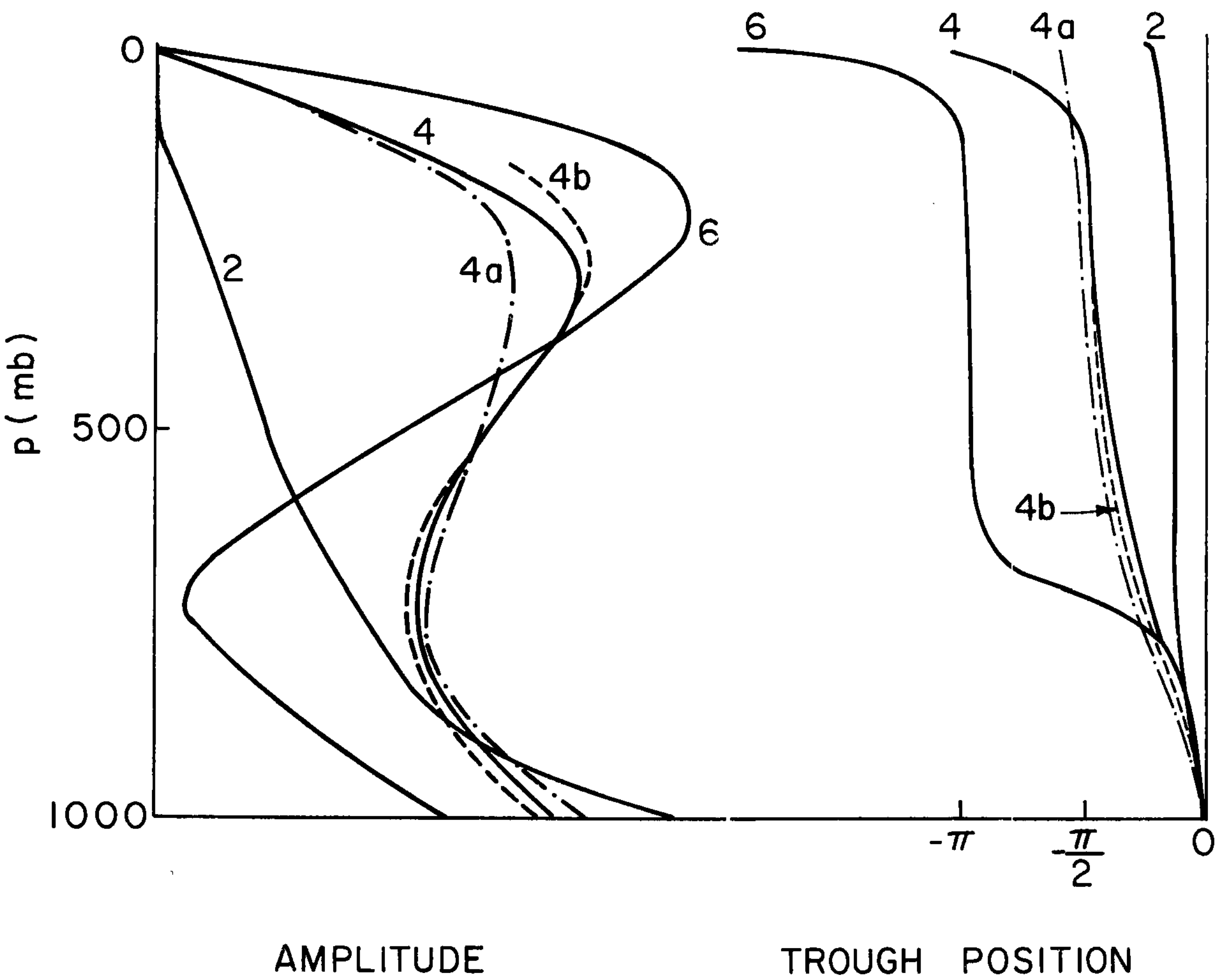


Fig. 2. Wavespeeds and growth rates of unstable waves as a function of wavelength and vertical resolution of the numerical model.



Perturbation	2	4	4a	4b	6
Wavespeed (m/sec)	9.2	9.9	10.6	9.8	9.4
Growth rate (day^{-1})	.20	.46	.50	.46	.04

Fig. 3. Structure of unstable mode for wavelengths of 2000, 4000, and 6000 km. Curve 4a is for dashed wind profile, 4b for dashed static stability profile shown in Fig.1.

long waves in models of sufficiently high resolution. In order to complete this part of Fig. 2, computations were carried out for 10 wavelengths between 5500 and 6500 km. Similar weak instabilities outside the region of major instability have been observed in earlier studies, e.g., Green (1960) and Gary (1965).

The vertical structure of the unstable modes is shown in Fig. 3 for wavelengths of 2000, 4000, and 6000 km. The relative amplitudes are, of course, arbitrary. The shorter disturbances are shallow waves without variations of phase with height. The longer waves develop in the upper atmosphere and tilt upward and westward. The weakly unstable long waves consist of a lower level wave and an upper level wave which are effectively uncoupled and are 180 degrees out of phase. This general structure of the unstable disturbances as a function of wavelength was obtained as early as 1953 by Kuo.

Fig. 3 includes also the structure of the 4000 km wave for the linear zonal wind represented by the dashed line in Fig. 1. Since the perturbation still decreases with height in the stratosphere although the zonal wind increases, this must be attributed completely to the correct representation of the static stability. A glance at the variation of inverse static stability shown in Fig.1, together with the governing equations (8) and (9), show that the stratospheric motion is nearly horizontal and non-divergent. This would suggest that the upper boundary condition may be applied at a lower level. The curve denoted by 4b in Fig. 3 represents the 4000-km disturbance if the static stability parameter assumes an infinitely large value above the 125-mb level as shown by the dashed stability curve of Fig. 1.

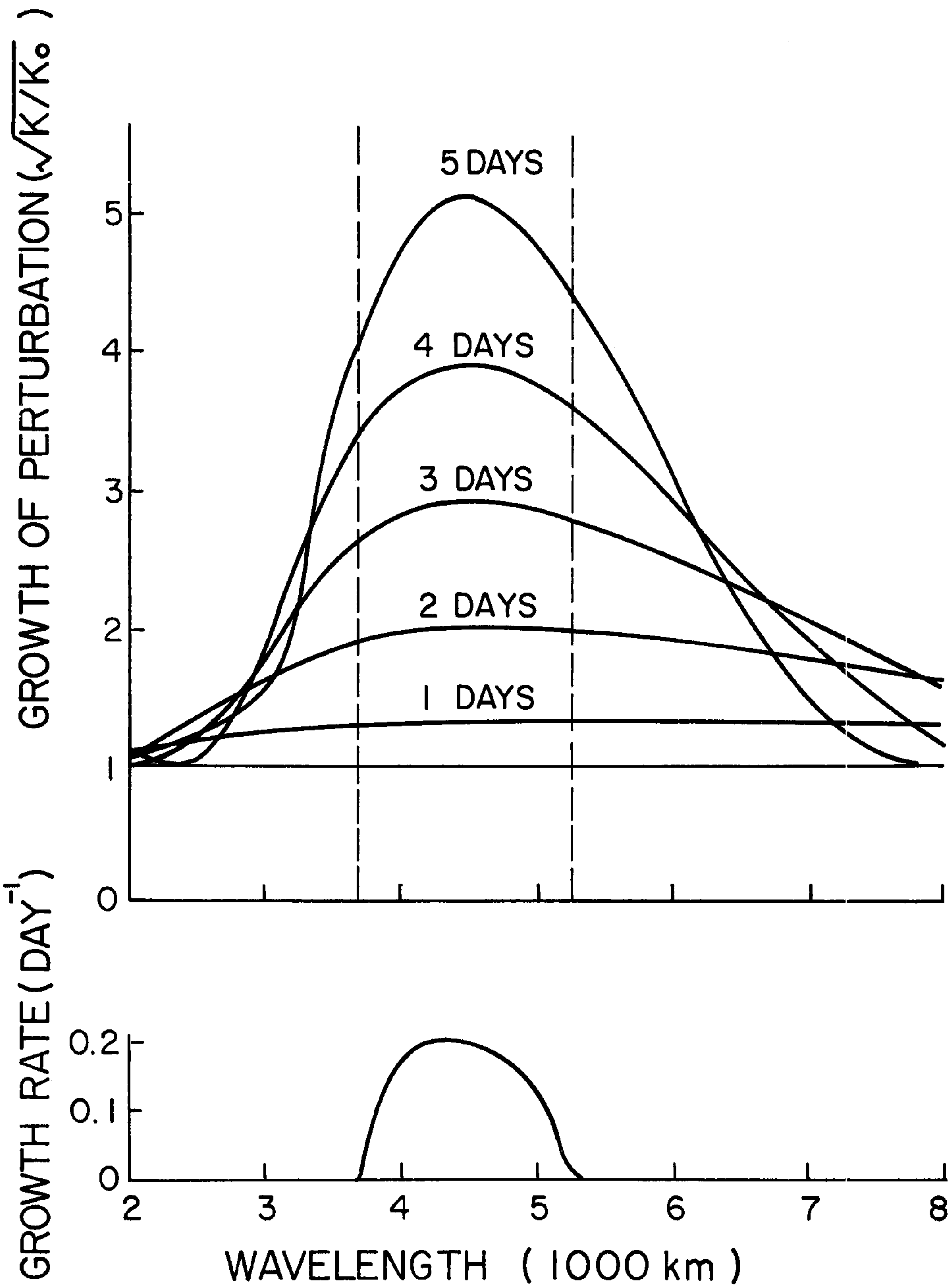


Fig. 4. Growth rate of unstable mode and actual growth of lower-layer perturbation in a two-layer model. Values of zonal wind taken from dashed curve of Fig. 1. Growth rate measured by kc_i, actual growth by $\sqrt{K/K_0}$ where K is the perturbation kinetic energy.

The preceding review of the normal mode instabilities is restricted to those results which may serve as a basis of comparison for the subsequent discussion. Far more elaborate computations have been carried out recently by Gary (1965). Returning now to the initial value problem, we must interpret the results above with reference to the time scales of atmospheric cyclones. Clearly then the emphasis is on the development of a perturbation during a relatively short time period, rather than the asymptotically different behavior of neutral and unstable waves. Fig. 4 illustrates how the time scale enters into the instability problem. The lower part of Fig. 4 shows the normal mode instability in a two-layer model for the zonal wind shear corresponding to the dashed line of Fig. 1. The actual growth of any perturbation introduced in the lower layer can be read off from the upper part of Fig. 4. Here the perturbation is measured by the square root of the vertically averaged kinetic energy of the wave. The pronounced distinction between neutral and unstable waves is not visible in the upper figure, quite similar to results obtained by nonlinear computations. The growth of the waves for real values of the wavespeeds is due to the fact that any perturbation is made up of more than one normal mode. The growth rates of the individual modes are equal to zero but the different phase speeds cause the amplitude variations of the sum of the modes.

It may be of interest to recall here that unstable waves with large growth rates may actually lose kinetic energy if their initial vertical configuration is not favorable for development. A typical example is the case of the short waves in a model of high vertical resolution. The growth rates of these waves are sufficiently large as

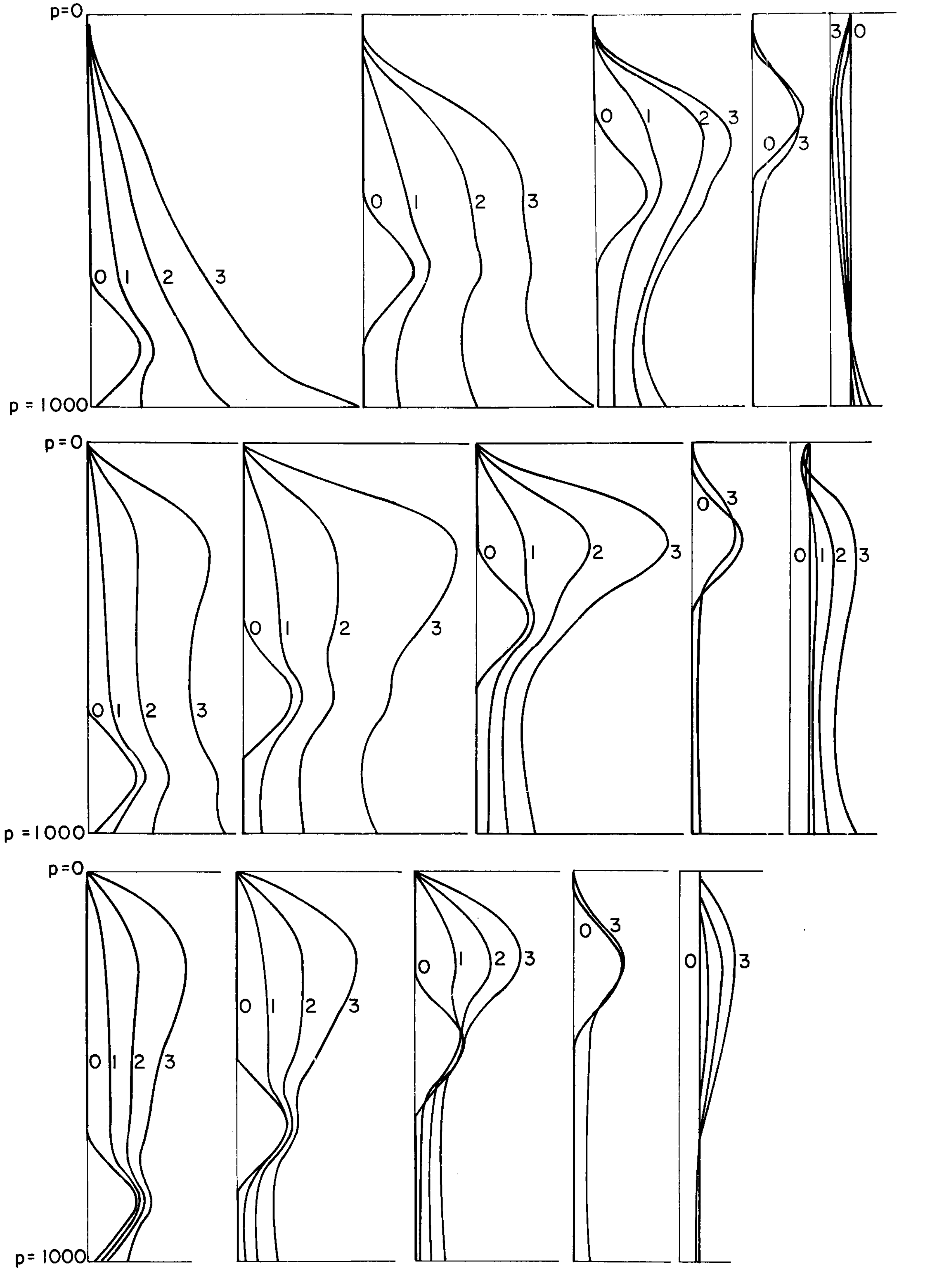


Fig. 5. Amplitudes of perturbations after 1, 2, and 3 days as a function of the initial vertical configuration (denoted by 0). From top to bottom the wavelengths are L = 2000 km, L = 4000 km, and L = 6000 km.

shown by Fig. 2. Nevertheless, the waves will not develop for many days if the initial perturbations extend through the whole depth of the atmosphere. This was also indicated by Wiin-Nielsen's (1962) calculations of initial energy tendencies for a vertically continuous model.

It should be stressed that the preceding does not contradict the general information obtained from the normal mode studies. Thus it is seen from Fig. 4 that these studies give an excellent indication of the most unstable waves, even for the time periods here considered. However, since any nonlinear study is essentially an initial value problem with finite initial perturbations, a meaningful evaluation of nonlinear effects can only be made if the inherent properties of the linear initial value problem are borne in mind.

The vertical shape of the initial perturbation was briefly mentioned before. This is the most interesting aspect of the initial value formulation of the stability problem. The normal mode instabilities are determined completely by the basic state parameters together with the horizontal scale of the perturbation. The following discussion shows that the initial configuration of the perturbation is just as important, if not more important, in a complete study of cyclone development.

Fig. 5 shows the growth of a wave disturbance in the atmosphere as a function of the level where the perturbation is introduced. The initial perturbations shown here are rather shallow, but similar patterns of development have been obtained for deeper perturbations. Also included is a perturbation extending initially through the whole depth of the atmosphere. All computations employed the basic state

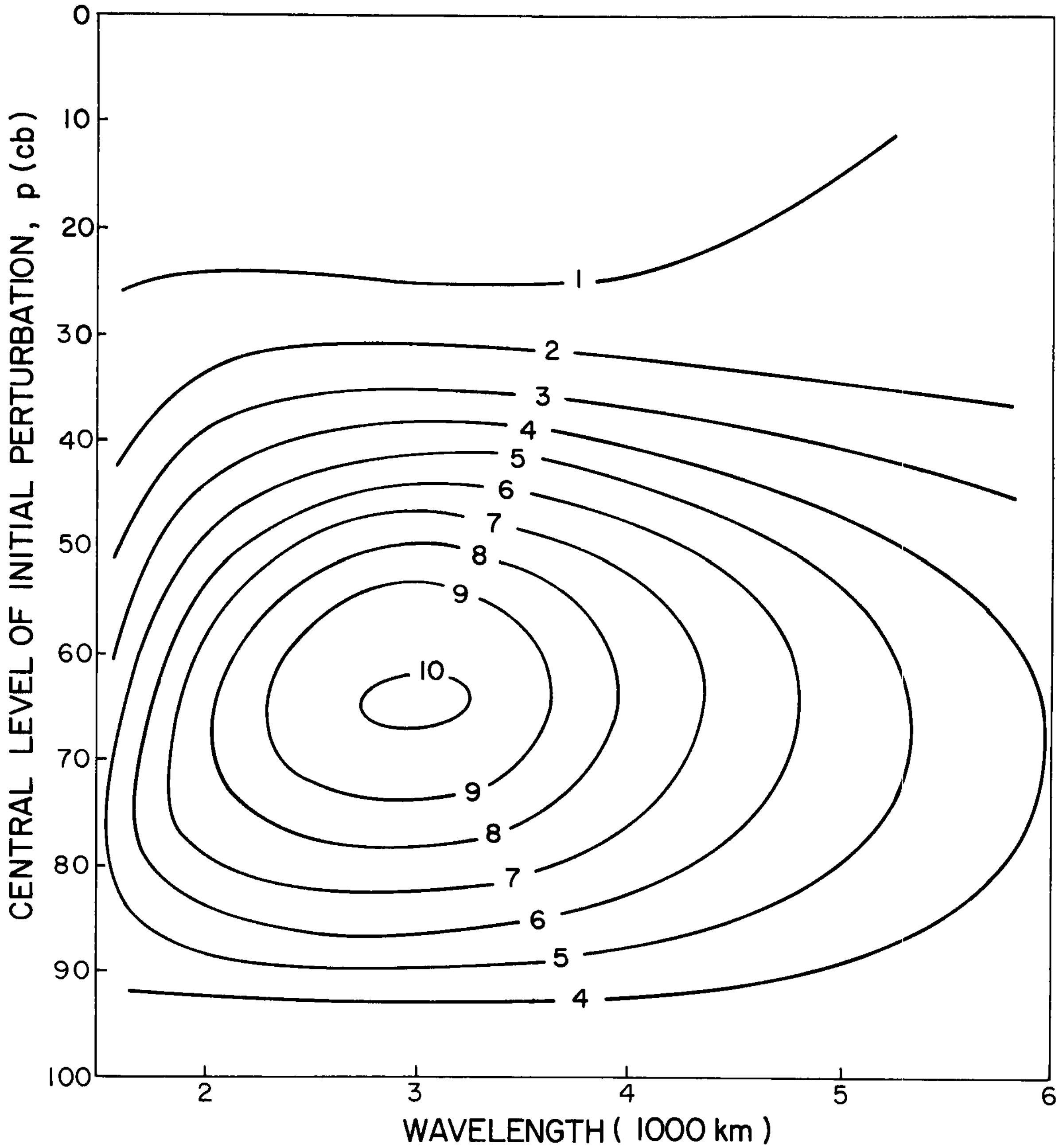


Fig. 6. Growth of wave perturbations after 3 days as a function of wavelength and initial perturbation level. Depths of initial perturbations are 300 mb as in Fig.5. Growth measured by $\sqrt{K_3/K_0}$ where K_n is the perturbation kinetic energy after n days.

parameters shown by the solid lines in Fig. 1. By comparison with the structure of the unstable mode shown in Fig. 3, it is seen that the quasi-continuous spectrum of neutral modes determines the shape of the perturbations to a large extent for our time scales.

The results of similar calculations for various wavelengths have been combined into Fig. 6, where the growth of a perturbation after three days is shown as a function of the wavelength and the level at which the perturbation is introduced. The initial shape of the perturbations is as shown in Fig. 5 and the growth is measured again by the square root of the kinetic energy of the perturbation. It is noteworthy that a wave perturbation at the jetstream level does not amplify but a disturbance introduced at mid-levels feeds immediately into the jetstream as seen from Fig. 5. It may also be noted that the short waves and the low-level disturbances develop strongly at the surface and therefore will be more sensitive to frictional effects. For that reason one must expect the maximum in Fig. 6 to move upward and to the right for the actual atmosphere.

So far the computations are restricted to a quasi-geostrophic atmosphere. The quasi-geostrophic approximation has been discussed extensively in the literature, but nevertheless it is of interest to make a quantitative comparison of the foregoing results with some computations performed on a primitive model. The primitive model is based on the divergence equation, the vorticity equation, and the thermodynamic equation for adiabatic flow, together with the continuity equation, the hydrostatic equation, and the equation of state. The pressure is the independent variable which implies that the continuity equation is a diagnostic equation. The height of a pressure surface is eliminated completely from the system by differentiating the divergence equation with respect to pressure and substituting the hydrostatic equation. By virtue of the equation of state we finally have three prediction equations and the diagnostic continuity equation to predict the three velocity components and the temperature.

The primitive equations have been solved for purely baroclinic flow by neglecting the latitudinal dependence of the coefficients in the final equations. Finite differences were used to approximate the vertical derivatives and the boundary conditions were those applied in the geostrophic model. After elimination of one dependent variable by virtue of the continuity equation, the computational problem is not essentially different from the geostrophic problem which will be reviewed in the next section.

All the normal mode calculations shown in Fig. 2 for the geostrophic model were repeated for the primitive model, but the differences were found to be negligible, independent of the vertical resolution. These results may be compared with similar comparisons made by Arnason (1963), and Derome and Wiin-Nielsen (1966). These authors conclude that the main role of the non-geostrophic effect is to stabilize the geostrophically unstable short waves. The effect is most pronounced for high shear and low static stability, that is, for small values of the Richardson number. Since the normal static stability and vertical wind shear used in the present calculations correspond to a relatively large Richardson number, the quasi-geostrophic approximation is well justified and it was to be expected that the non-geostrophic effects in our model are rather small. Further, the non-geostrophic effects might be found to be more important if the perturbations were allowed to vary in latitudinal direction, as suggested by Phillips (1964). This will be discussed in more detail in a subsequent paper.

4. Method of Solution

To solve the non-linear equations used in numerical weather prediction it is necessary to replace the equations by a large set of ordinary differential equation with time as the only independent variable. Even the linear baroclinic initial value problem has not been solved in its general form, but Case (1960,1962) and Pedlosky (1964) have discussed somewhat simpler and very similar initial value problems. Since the main purpose of the present study is to gain a basic understanding of the inherent properties of the general initial value problem, these exact solutions are not of primary importance and we will investigate methods of solution which can be applied to the non-linear equations. Therefore, we will reduce the governing equations to a system of ordinary differential equations.

The most common way to accomplish this is to replace the derivatives by finite differences. In order to apply this technique to Eq. (8), the interval $0 \leq p \leq p_0$ is divided into N layers of depth $\Delta p = p_0/N$ each. Eq. (8) is applied at the centers of these layers, the vertical derivatives are replaced by the usual centered differences, and the boundary conditions (10) are incorporated into the equations for the lower and upper layer. The result is a system of N equations for the variables $\Psi_n(t) = \Psi(t,p_n)$ where $p_n = p_0 - (n-\frac{1}{2})\Delta p$ and n ranges from 1 to N. The new variables are only time dependent and the system can be written in matrix notation as follows

$$\mathbb{B}\frac{d\vec{P}}{dt} = - ik\,\mathbb{A}\vec{P} \qquad (13)$$

where $\vec{P}$ is the vector consisting of elements Ψ_n, and $\mathbb{A}$ and $\mathbb{B}$ are square matrices of order N. The matrix elements are real and they are in this linear system determined completely by the basic state parameters and the wavenumber k. Inverting the matrix $\mathbb{B}$ we obtain

$$\frac{d\vec{P}}{dt} = - ik\ \mathbb{B}^{-1}\mathbb{A}\vec{P} \qquad (14)$$

Before proceeding to solve this system of coupled linear equations, we should investigate the vertical truncation error resulting from the replacement of (8) by (14). For that purpose equations (8) and (10) may be solved exactly for the time-derivative of the perturbation in terms of the perturbation at a given time, and the solution may be compared to the values obtained from (14). For a given profile of the zonal wind, the stability, and the perturbation, Eq. (8) becomes a non-homogeneous ordinary differential equation with pressure as the independent variable and the perturbation tendency as the dependent variable. The equation can be easily solved if the inverse static stability is represented by a linear function of pressure, an example of which is shown by the dashed line in Fig.1. Considering the high variability of the static stability this approximation is justified for many purposes, and for a quantitative estimate of this effect, Fig. 3 includes the structure and the growth rate of the perturbation under these conditions.

In order to solve Eq. (8) for the perturbation tendency, let the perturbation at a given time, together with the zonal wind, be represented by a power series in pressure. It follows then from (8) that the perturbation tendency will be made up of another power series plus the solution of the homogeneous equation. The coefficients of the latter series are found in terms of the coefficients of the given power series for the zonal wind and the streamfunction by equating the coefficients of equal powers in pressure. The series for the perturbation tendency together with Eq. (10) determine the boundary conditions to be satisfied by the homogeneous solution. If the static stability is inversely proportional to pressure, the homogeneous equation may be reduced to Bessel's equation and the solution which satisfies the boundary conditions is Bessel's function of the first kind of order zero with argument proportional to the square root of pressure. The same solution may of course be obtained by solving first for the vertical motion from (9) and then using the vorticity equation (6) to obtain the time-derivative of the streamfunction.

Solutions have been obtained for a wavelength of 4000 km and for relatively simple vertical variations of zonal wind and perturbations. These solutions were compared to solutions computed from (14). For a 20-layer model the error is of the order of one percent. This error must be expected to increase if the perturbation varies irregularly with height such as observed for short and long waves. Similar calculations have been made by Wiin-Nielsen (1962) to estimate the truncation error of prediction models with low vertical resolution. There it was assumed that the static stability varies as the inverse of the square of the pressure, but in another section of the same paper the linear variation of the inverse static stability was utilized.

Let us turn now to the solution of the system of equations given by (14). In the case of the general non-linear problem it would be

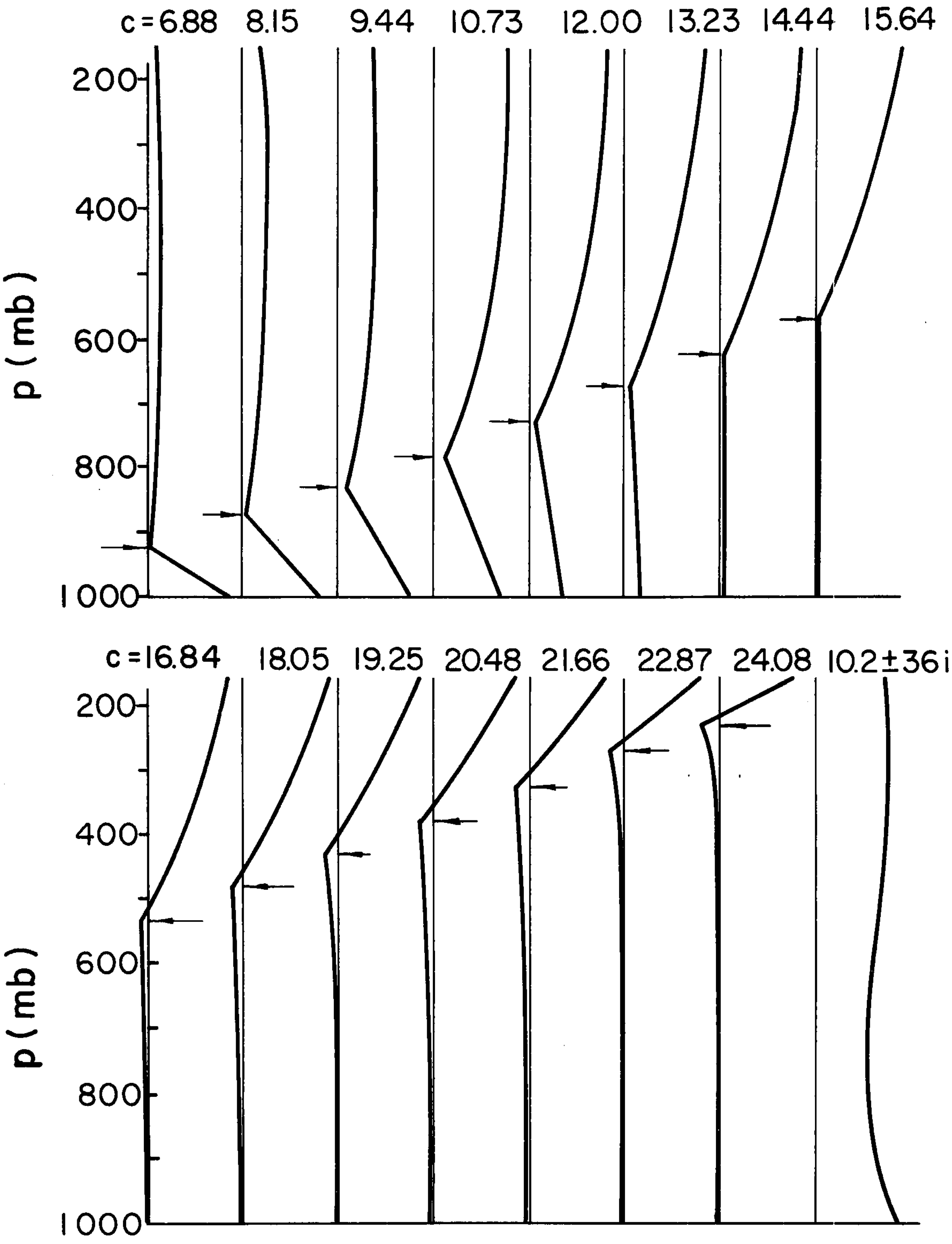


Fig. 7. Eigenvectors for quasi-continuous spectrum and amplitude of normal mode eigenvector. Basic state parameters given by dashed lines of Fig. 1, wavelength L = 4000 km. Truncation of numerical model N = 20, but upper three layers uncoupled ($1/\sigma = 0$) resulting into 17 eigenvectors.

integrations for which the convergence of the numerical solution is found to be satisfactory.

To conclude this discussion of the finite difference models, Fig. 7 shows the character of the eigenvectors obtained from (15). Here we have chosen the model with linear profiles of zonal wind and inverse static stability, where the singularity $c = \bar{u}$ occurs at only one level for a given root and the roots are equally spaced. The arrows in the figure indicate the location of the singularity, where the eigenvector shows a discontinuous first derivative as expected from (11). The singularity never coincides with a gridpoint except for the Eady model, where the real roots always equal the values of the zonal wind in all but the first and the last gridpoints. The amplitude and the phase angle of the unstable normal mode were shown by the dashed lines in Fig. 3. Recently, some eigenvectors were displayed by Yanai and Nitta (1968) for the purely barotropic instability problem which is quite similar from a computational viewpoint.

So far, we have discussed the method of replacing vertical derivatives by finite differences in order to reduce the continuous equations to a system of ordinary differential equations. This is not necessarily the best method and another technique often utilized to accomplish the same goal is to expand the dependent variable in a series with time-dependent coefficients. Such spectral models have been applied with success in the field of numerical weather prediction. The remaining of this section is concerned with spectral solutions and the convergence of these solutions as compared to the convergence of the finite difference solutions.

The boundary conditions (10) suggest a trigonometric expansion for the vertical motion, Ω, in terms of sines only. Therefore Ω is an odd function of pressure and it follows from (6) and (7) that the perturbation streamfunction Ψ should be an even function of pressure if $\bar{u}$ and σ were constant, and hence could be represented by a series of cosines. It is assumed that the same representations hold if the zonal wind and the static stability vary with height. The spectral prediction equations for the time-dependent coefficients of the series expansion are obtained by substituting the expansion into (8) and applying the orthogonality relationships. In order to obtain a unique system of equations, the mathematical half-plane $p<0$ must be taken into account.

The resulting system of ordinary differential equations will again be of the form

$$\frac{d\vec{P}}{dt} = -ik\mathbb{A}\vec{P} \qquad (17)$$

where $\vec{P}$ is the vector consisting of the time-dependent expansion coefficients (usually called parameters), and $\mathbb{A}$ is again a square matrix determined by the basic state parameters and the wavenumber. The solution of (17) is obtained by the methods applied before in solving (14).

The spectral model was first tested, together with the finite difference model, on the Eady problem, i.e., $\beta = 0$, constant static stability, and linear zonal wind. Both the 20-layer model and the 20-parameter model always found the unstable mode as determined from the exact solution. However, in addition, the spectral model found usually quite a few complex roots, the imaginary parts of which were about an order of magnitude smaller but in a few cases as large as 30 percent of the imaginary part of the normal mode. By virtue of the exact solution, the instability corresponding to these roots must be considered spurious. The same overall picture was observed after intoducing a non-zero value for β.

The performance of the spectral model greatly improved when the linear zonal wind was replaced by various non-linear wind profiles. Although the spectral model still tended to exaggerate the instabilities somewhat in comparison to the layered model, both models usually produced the same complex roots. On the other hand, when the constant static stability was also replaced by various pressure dependent functions, the spectral model again produced the spectrum of unstable roots with small growth rates, in addition to the primary modes found by both models. In these experiments the wavelength was 4000 km and the profiles of the zonal wind and the static stability were obtained as follows. First we computed the expansion coefficients of trigonometric series for the zonal wind and the inverse static stability shown by the curves in Fig.1. These expansion coefficients appear directly in the spectral prediction equations. Various profiles were then obtained by dropping part of the coefficient spectrum and summing the series again for use in the finite difference equations.

Since the layered model and the spectral model produce the same primary unstable modes for sufficiently large vertical resolution, it is of interest to investigate whether the convergence of these roots for increased resolution is the same for both models. Again a wavelength of 4000 km was chosen and various basic state profiles were considered. The vertical resolution was gradually increased till the normal modes seemed to reach their asymptotic values. In all cases the convergence of the layered model was at least as good or better than for the spectral model.

The results above do not agree with the computations made by Rao and Simons (1969) concerning a two-layer non-geostrophic model with lateral boundaries. The computational aspects of that problem are very similar, and the computer program was essentially the same as the one used in the present calculations. Nevertheless, there it was found that the spectral model always converged much faster than the finite difference model and furthermore, both models always found the same number of unstable roots.

The computations on the Eady model where the exact solution is known, seem to indicate that the spectral model tends to find more unstable modes than present in a continuous model. It must be noted, however, that the linear zonal wind profile is far more favorable for a finite difference model than for a spectral model. The finite difference approximation to the vertical derivative is exact for a linear profile while the same profile requires an infinite number of expansion coefficients in the spectral representation. In general, the problem of the spurious roots seems to become more serious if the static stability is a function of height and if the zonal wind requires a large number of expansion coefficients. This would indicate that the trigonometric functions are not appropriate for a vertical representation of the atmosphere. A vertical series representation was suggested elsewhere (Simons, 1968) in terms of the eigenfunctions of the differential operators in the homogeneous parts of Eqs. (8) and (9), for average values of static stability. Only a few of these functions are required to represent average atmospheric zonal wind profiles, which may indicate that such a spectral model converges faster than the models discussed above.

5. Conclusions.

It is concluded that the atmospheric instability problem formulated as an initial value problem has many interesting aspects which have been only partly explored in the past. Since the non-linear stability study is an initial value problem, the linear stability problem must be approached in the same manner in order to allow an evaluation of the non-linear effects. The most interesting aspect of the baroclinic initial value problem is the initial configuration of the perturbation. The vertical shape of the initial perturbation has been found to be as important as its wave length for determining its growth over a time period which is consistent with the time scale of meteorological development.

The meteorological equations have been solved for both quasi-geostrophic and non-geostrophic flow. The difference was found to be negligible. The solutions were obtained by replacing the continuous atmosphere by a layered model and also by using spectral methods. Comparison of asymptotic solutions obtained from the continuous model and the layered models showed that the initial value problem requires far better vertical resolution than the normal mode problem. The performance of the spectral model in terms of trigonometric functions was disappointing, but this may be due to the choice of the functions.

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Appendix.

The characteristic equation (11) for the baroclinic instability problem is singular and its solution is made up of a continuous spectrum in addition to the normal modes. The vertical profiles of the perturbations shown in Fig. 5 are in many cases quite different from the unstable mode profiles which emphasizes the importance of the contributions due to the continuous spectrum. It was shown by Case (1960, 1962) for a similar problem, that the continuum modes do not correspond to instability, and that the contributions due to the continuous spectrum vanish as the time approaches infinity. Pedlosky (1964) reached the same conclusions for the Eady (1949) instability problem, but noted that for the range of wavelengths for which the normal modes are stable, the asymptotic contributions of the continuous spectrum and the normal modes cannot be separated completely.

It is of some interest to investigate whether the numerical model can reproduce this asymptotic behavior of a continuous atmosphere with a zonal shear. Consider the Eady-type model obtained by setting $\beta = 0$ and σ = constant in Eq. (8) and using the linear zonal wind profile shown by the dashed line in Fig. 1. The inverse static stability is $1/\sigma = 37.5$, which is the value at the 500-mb level in Fig. 1. A perturbation of wavelength 6000 km is unstable with a growth rate of .42 per day and a phase speed equal to the mean wind, as follows from the exact solution for the continuous model. Actually, the same normal mode solution was found by our numerical models, the vertical resolution of which varied from 5 to 30 layers. We now consider the solution of the initial value problem for an initial perturbation which is independent of height, and compute the perturbation kinetic energy contributed by the quasi-continuous spectrum of real eigenvalues. Fig. 8 shows the square root of this kinetic energy expressed in terms of the total initial perturbation kinetic energy, for a 10-, 20-, and 30-layer model respectively.

It seems tempting to conclude from Fig. 8 that the asymptotic solution of the layered models approaches that of the continuous model if the vertical resolution increases, but this would indeed require an infinite number of layers. Certainly, the convergence is much less for this initial value problem than the convergence found in computing the normal modes. Of course, the solution shown in Fig. 8 will be soon overshadowed by the contributions of the unstable mode, which reduces its practical significance. For comparison we have therefore computed the evolution of the same initial perturbation but with a wavelength of 4000 km, which is stable in this model. The total perturbation kinetic energy as a function of time is shown in Fig. 9. The contributions from the individual roots may be evaluated from the coefficients d_n in the vector $\vec{E}(t)$ appearing in Eq. (16). Since the coefficients associated with the normal modes differ by two orders of magnitude for the 30-layer model as compared to the 20-layer model, while the total solution is quite similar for a long time, it is indeed indicated that the continuous spectrum must be included for a correct asymptotic solution.

Since the Eady-model is not very realistic from the viewpoint of atmospheric flow, similar calculations were performed on the general equation (8), where β is non-zero and the basic state parameters are those shown by the dashed lines in Fig. 1. The matrix $\mathbb{B}^{-1}\mathbb{A}$ has one pair of complex roots, supposedly the normal modes, and the growing

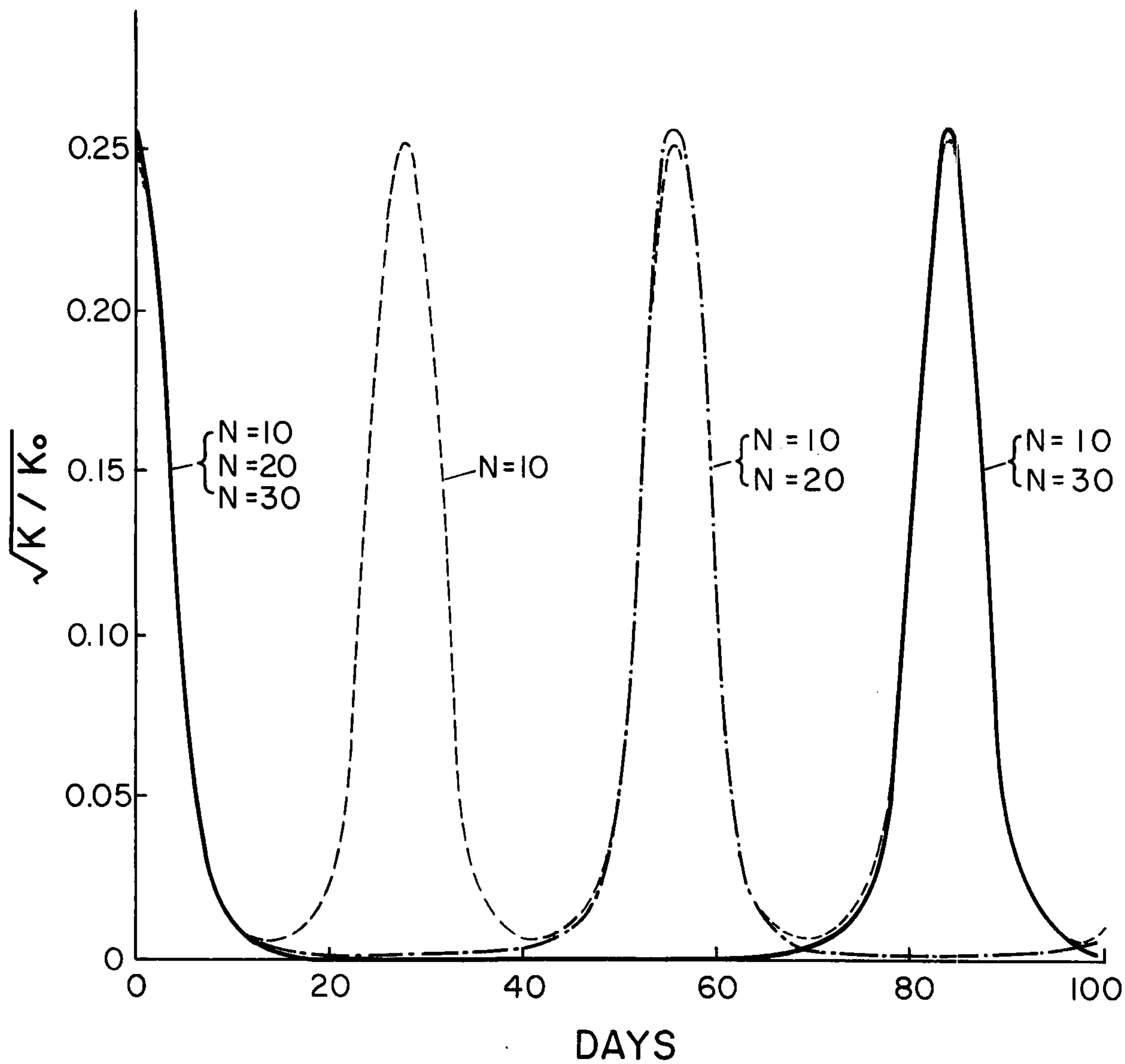

Fig. 8. Square root of perturbation kinetic energy contributed by quasi-continuous spectrum for case of unstable normal modes. Eady type model with $\beta = 0$, constant static stability, linear zonal wind, wavelength L = 6000 km, initial perturbation independent of height. N denotes the number of layers of the numerical models.

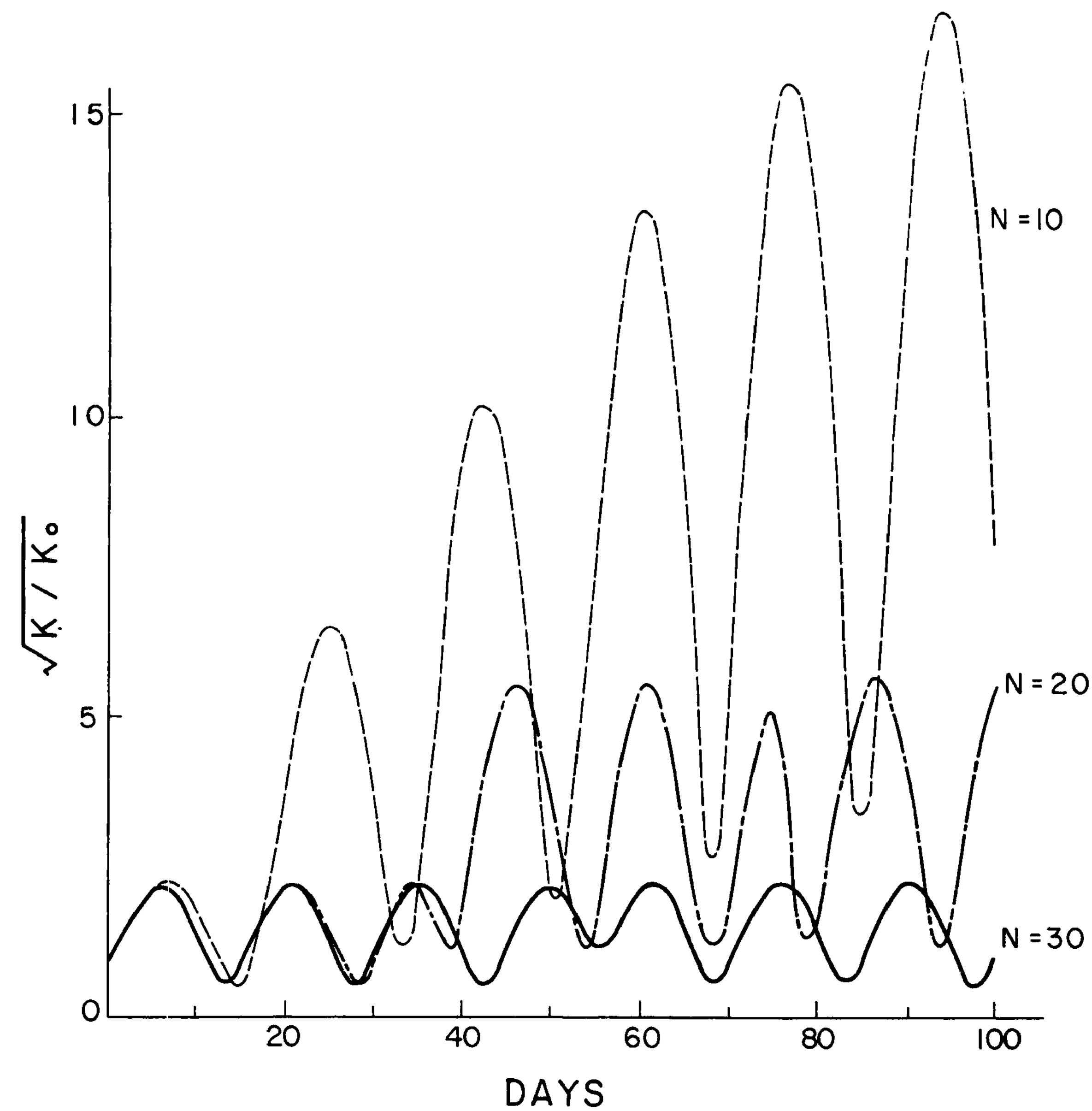

Fig. 9. Square root of perturbation kinetic energy for stable wave of length L = 4000 km. Other parameters as in Fig. 8.

mode has a phase speed of 10.2 m/sec and a growth rate of .48 per day, as found by all models independent of the vertical resolution. The amplitude of the continuous spectrum shows the same general character as before.

It is appropriate here to make a few notes concerning the accuracy of the solution of equation (14). The usual methods of testing the eigenvalues and eigenvectors were applied, but in addition the accuracy of the final solution was evaluated at all times by differentiating the exact solution (16) with respect to time and comparing the perturbation tendencies obtained from that equation with those computed from (14). For the waves of primary interest, i.e., the most unstable waves, this difference expressed in terms of the tendency itself was of the order of 10^{-8}. For simple models such as those used for the asymptotic integrations, the accuracy was better. We may therefore conclude that the system of equations (14) has been solved without error and the only truncation errors are made in going from the continuous model to the layered model.