Cumulus Parameterization Theory in terms of Feedback and Control

By
Wayne H. Schubert

Department of Atmospheric Science
Colorado State University
Fort Collins, Colorado

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ABSTRACT

The cumulus parameterization theory presented by Arakawa and Schubert (1974) describes the mutual interaction of a cumulus cloud ensemble with its large-scale environment. This mutual interaction can be subdivided into three interaction loops: feedback, static control, dynamic control. The mathematical formulation of each of these interaction loops is discussed. The feedback loop describes how the cumulus scale transport terms and source terms modify the large-scale temperature and moisture fields. The static control loop describes the normalized mass flux and the thermodynamic properties of each cloud type in terms of the large-scale temperature and moisture fields. The dynamic control loop describes how the large-scale fields control the total cloud ensemble vertical mass flux and its distribution among the various cloud types. The feedback, static control, and dynamic control loops constitute a closed parameterization theory.

A simple formalism for the diagnostic use of the feedback and static control portions of the theory is presented. The relation of the subgrid scale flux forms and the detrainment forms of the large-scale heat and moisture budgets is also discussed.

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1. INTRODUCTION

A cumulus parameterization theory must describe the mutual interaction of a cumulus cloud ensemble with the large-scale environment. This mutual interaction is shown schematically in Figure 1. It consists of feedback and control loops.[1] The feedback loop describes how the cumulus scale transport terms and source terms modify the large-scale temperature and moisture fields. The control loops describe how the properties of the cloud ensemble are controlled by the large-scale fields.

A cumulus parameterization theory describing the mutual interaction of a cloud ensemble with the large-scale environment was recently given by Arakawa and Schubert (1974) and Schubert and Arakawa (1974).[2] In this paper we shall conceptually group the equations given in I and II into three categories: feedback, static control, dynamic control.[3] The mathematical formulation of the feedback loop is discussed in Section 2, the static control loop in Section 3, and the dynamic control loop in Section 4. Although the theory has been designed for use in large-scale prognostic models, the feedback and static control portions of the theory can be used in diagnostic studies. This is discussed in Section 5. The relation of the 'subgrid scale' flux forms and the detrainment forms of the heat and moisture budget equations is discussed in Section 6.

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[1]The terms feedback and control were first used in this context by Betts (1974).

[2]Hereafter referred to as I and II, respectively.

[3]The feedback and static control portions are conceptually similar to those given by Ooyama (1971).

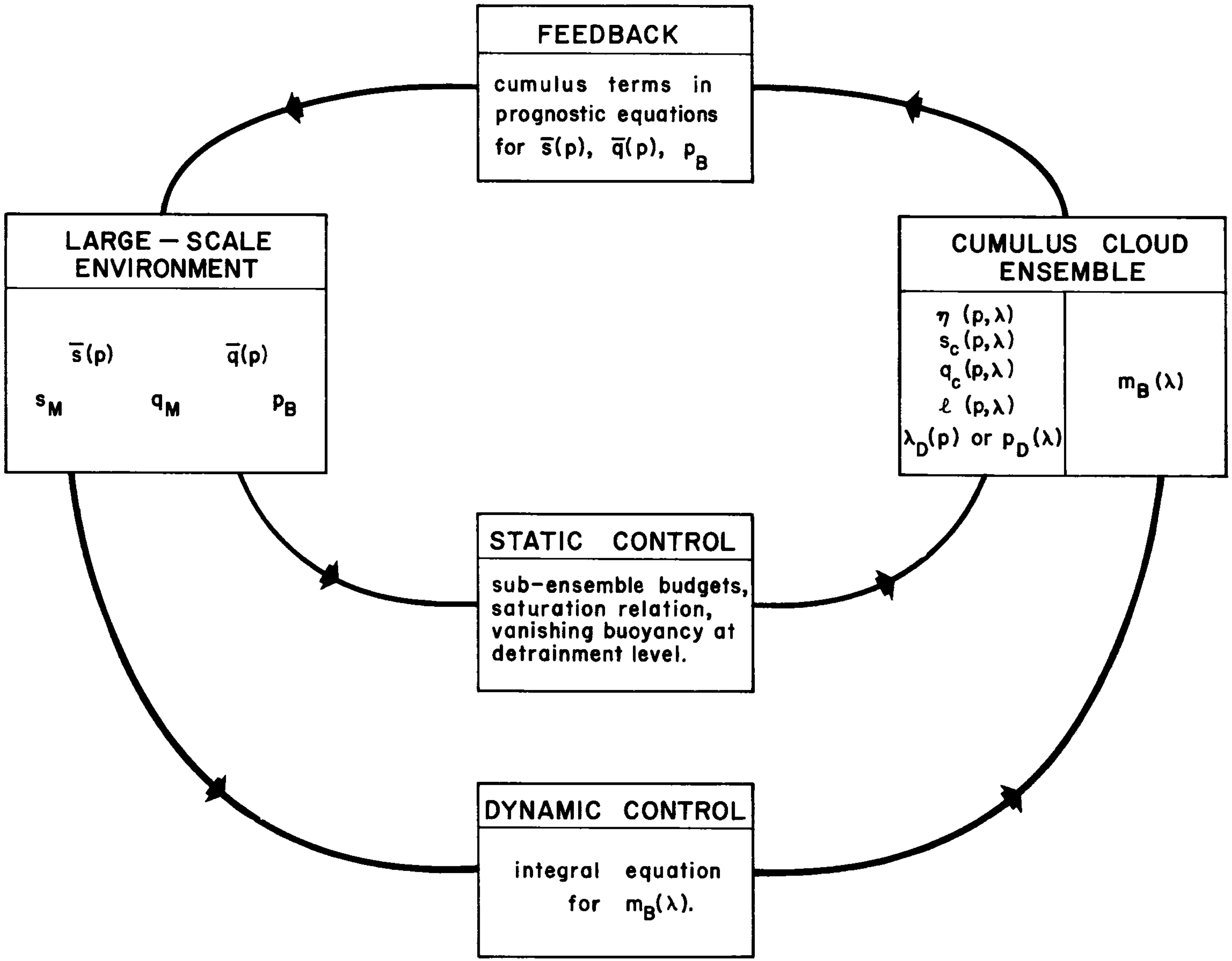

Fig. 1. Schematic representation of the mutual interaction of a cumulus cloud ensemble with the large-scale environment.

2. FEEDBACK

Consider a horizontal area large enough to contain an 'ensemble' of clouds but small enough to cover only a fraction of a large-scale disturbance. We shall refer to the vertical transports caused by motions on a scale smaller than this area as the subgrid scale transports.

Let the large-scale environment of the cloud ensemble be divided into the subcloud mixed layer, the infinitisimally thin transition layer, and the region above (see Figure 2). In the subcloud mixed layer the dry static energy s, water vapor mixing ratio q, and therefore the moist static energy h, are constant with height, having the respective values s_M, q_M, and h_M. The top of the subcloud mixed layer p_B is usually somewhat below cloud base p_C. Below p_B subgrid scale transports are accomplished by the turbulence of the mixed layer. This turbulence is confined below p_B by the stable and infinitesimally thin transition layer. Across the transition layer there can be discontinuities in temperature and moisture, and also discontinuities in the subgrid scale fluxes. Above p_B the subgrid scale transports are accomplished by the cloud ensemble, which is spectrally divided into 'sub-ensembles' according to the fractional entrainment rate λ, small λ corresponding to deep clouds and large λ corresponding to shallow clouds.

Let us define the subgrid scale fluxes of dry static energy, water vapor, and liquid water as

$$F_s(p) \equiv \begin{cases} \displaystyle\int_0^{\lambda_D(p)} \eta(p,\lambda)\ [s_c(p,\lambda) - \bar{s}(p)]\ m_B(\lambda)d\lambda & p_B > p \qquad (1a) \\[2ex] \displaystyle (F_s)_o + [(F_s)_B - (F_s)_o]\ \frac{p_o - p}{p_o - p_B} & p_o \geq p > p_B \qquad (1b) \end{cases}$$

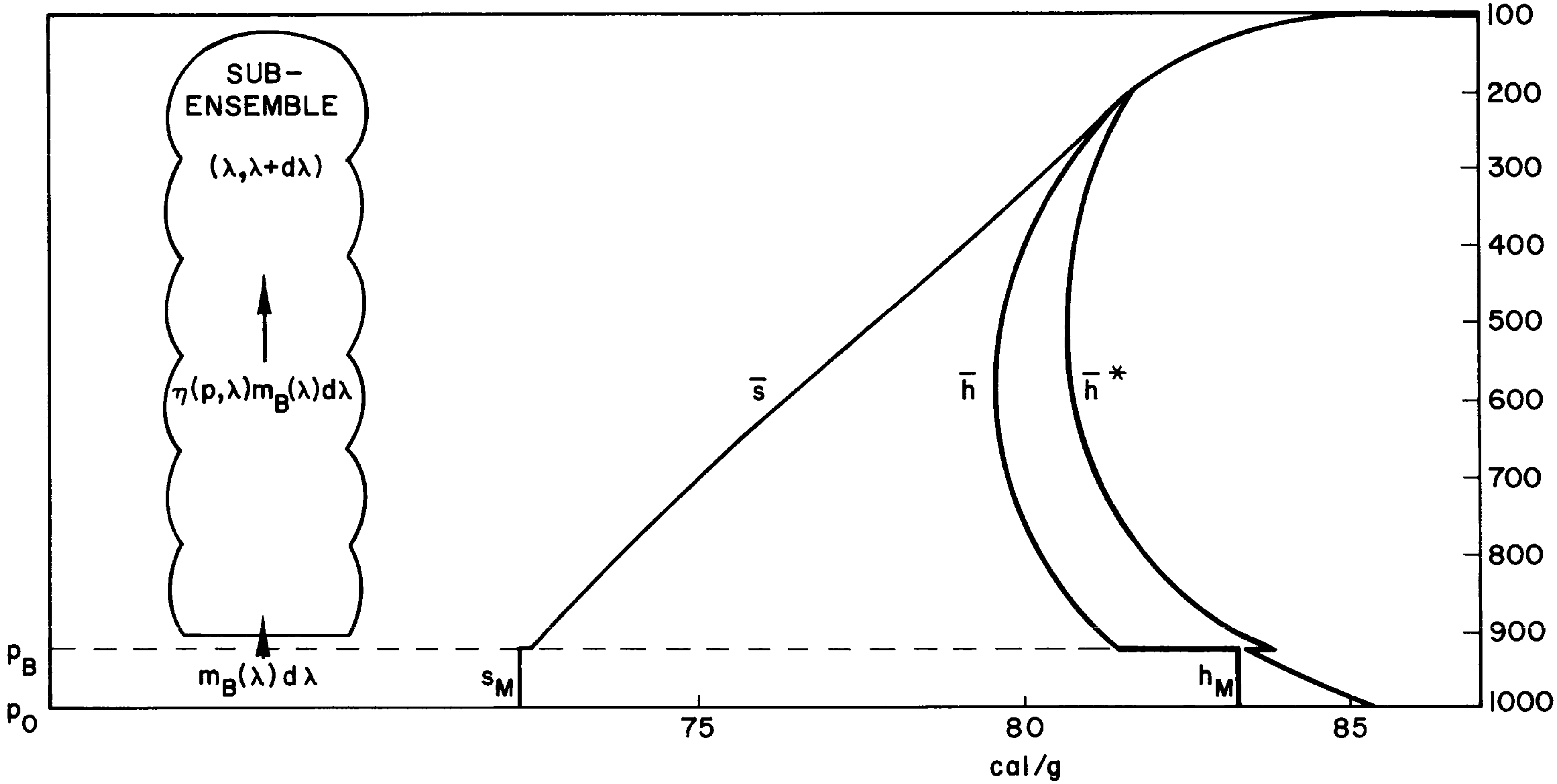

Fig. 2. Typical ITCZ profiles of $\bar{s}$, $\bar{h}$, and $\bar{h}^*$. Above p_B these profiles are those of Yanai, Esbensen and Chu (1973). The schematic sub-ensemble has cloud base p_c slightly above p_B. The mass flux at p is $\eta(p,\lambda)m_B(\lambda)d\lambda$, while the mass flux at p_B is $m_B(\lambda)d\lambda$.

$$F_q(p) \equiv \begin{cases} \int_o^{\lambda_D(p)} \eta(p,\lambda)[q_c(p,\lambda) - \bar{q}(p)]\ m_B(\lambda)d\lambda & p_B>p \quad (2a) \\ (F_q)_o + [(F_q)_B - (F_q)_o] \dfrac{p_o - p}{p_o - p_B} & p_o\geq p>p_B \quad (2b) \end{cases}$$

$$F_\ell(p) \equiv \begin{cases} \int_o^{\lambda_D(p)} \eta(p,\lambda)\ell(p,\lambda)\ m_B(\lambda)d\lambda & p_B>p \quad (3a) \\ 0 & p_o\geq p>p_B \quad (3b) \end{cases}$$

Below p_B the subgrid scale fluxes of s and q are linear in p with the values $(F_s)_o$ and $(F_q)_o$ at the surface p_o and the values $(F_s)_B$ and $(F_q)_B$ just below p_B. The subgrid scale flux of ℓ is zero everywhere below p_B.

Above p_B the subgrid scale fluxes are accomplished by the cloud ensemble. Let $s_c(p,\lambda)$ be the dry static energy at level p inside sub-ensemble λ and $\eta(p,\lambda)m_B(\lambda)d\lambda$ be the vertical mass flux at level p due to sub-ensemble λ . Let $\eta(p,\lambda)$ be the normalized mass flux, having the value unity at p_B. Then $m_B(\lambda)d\lambda$ is the sub-ensemble mass flux at p_B. We shall refer to $m_B(\lambda)$ as the mass flux distribution function since it gives the distribution of mass flux in λ space. The upward flux of dry static energy inside sub-ensemble λ at level p is $\eta(p,\lambda)s_c(p,\lambda)m_B(\lambda)d\lambda$. The downward flux in the environment at level p, caused by the induced subsidence of sub-ensemble λ, is given by $\eta(p,\lambda)\bar{s}(p)m_B(\lambda)d\lambda$. Thus, the total upward flux at level p due to sub-ensemble λ is $\eta(p,\lambda)[s_c(p,\lambda) - \bar{s}(p)]\ m_B(\lambda)d\lambda$. The total ensemble flux at level p is an integral over all sub-ensembles which penetrate level p. Sub-ensembles which penetrate level p have fractional entrainment rates in the interval $o\leq\lambda\leq\lambda_D(p)$, where

$\lambda_D(p)$ is the fractional entrainment rate of the sub-ensemble which detrains at level p. The subgrid scale fluxes of water vapor and liquid water above p_B are analogous to that of dry static energy except that there is no vertical flux of liquid water in the environment since the environment contains no liquid water.

Certain combinations of the three basic fluxes given in (1) - (3) are useful. Thus, let us define the subgrid scale fluxes of virtual dry static energy, moist static energy, total water content, and liquid water static energy as

$$F_{sv}(p) \equiv F_s(p) + \delta\varepsilon(p) L F_q(p), \tag{4}$$

$$F_h(p) \equiv F_s(p) + L F_q(p), \tag{5}$$

$$F_{q+\ell}(p) \equiv F_q(p) + F_\ell(p), \tag{6}$$

$$F_{s-L\ell}(p) \equiv F_s(p) - L F_\ell(p). \tag{7}$$

In (4), $\delta = 0.608$ and $\varepsilon(p) \equiv c_p \bar{T}(p)/L$. The liquid water static energy ($s-L\ell$) is the static energy analog of the liquid water potential temperature introduced by Betts (1973a). A discussion of the liquid water static energy is given in Betts (1973b).

The governing equations for the large-scale environment are derived from the heat and moisture budgets for the region above the mixed layer, for the infinitisimally thin transition layer, and for the mixed layer. These budgets are

$$\frac{\partial \bar{s}}{\partial t} = - \bar{\mathbb{V}} \cdot \nabla \bar{s} - \bar{\omega}\frac{\partial \bar{s}}{\partial p} + g\frac{\partial}{\partial p} F_{s-L\ell} + LR + Q_R, \tag{8}$$

$$\frac{\partial \bar{q}}{\partial t} = - \bar{\mathbb{V}} \cdot \nabla \bar{q} - \bar{\omega}\frac{\partial \bar{q}}{\partial p} + g\frac{\partial}{\partial p} F_{q+\ell} - R, \tag{9}$$

$$-g^{-1}\left(\frac{\partial p_B}{\partial t} + \mathbb{V}_B\cdot\nabla p_B - \bar{\omega}_B\right) = + \frac{\Delta F_{s-L\ell}}{\Delta s}, \quad (10)$$

$$-g^{-1}\left(\frac{\partial p_B}{\partial t} + \mathbb{V}_B\cdot\nabla p_B - \bar{\omega}_B\right) = + \frac{\Delta F_{q+\ell}}{\Delta q}, \quad (11)$$

$$\frac{\partial s_M}{\partial t} = - \mathbb{V}_M\cdot\nabla s_M + \frac{g}{p_o - p_B}\left[(F_s)_o - (F_s)_B\right] + (Q_R)_M, \quad (12)$$

$$\frac{\partial q_M}{\partial t} = - \mathbb{V}_M\cdot\nabla q_M + \frac{g}{p_o - p_B}\left[(F_q)_o - (F_q)_B\right]. \quad (13)$$

In addition to large-scale advection terms, the heat and moisture budgets above the mixed layer (equations (8) and (9)) contain subgrid scale flux divergence terms and the subgrid scale liquid water sink term R, defined by

$$R(p) \equiv \int_o^{\lambda_D(p)} \eta(p,\lambda)r(p,\lambda)m_B(\lambda)d\lambda. \quad (14)$$

Since $\eta(p,\lambda)r(p,\lambda)m_B(\lambda)d\lambda$ is the sub-ensemble sink of liquid water, R(p) is the total ensemble sink of liquid water. Q_R is the radiational heating. The detrainment forms of the heat and moisture budgets above the mixed layer were derived in I. The relation of (8) and (9) to the detrainment forms is discussed in Section 6.

Since the transition layer is assumed to be infinitesimally thin, the heat and moisture budgets for this layer (equations (10) and (11)) turn out to be conditions on the discontinuities across the layer. In (10) and (11) the symbol delta represents the jump of a quantity across the transition layer, e.g., $\Delta s \equiv \bar{s}(p_B-) - s_M$ and $\Delta F_{s-L\ell} = F_{s-L\ell}(p_B-) - (F_s)_B$. The left hand side of (10) or (11) is the large-scale mass flux into the mixed layer, i.e., the large-scale mass flux relative to the moving p_B surface. Equations (10) and (11) show that discontinuities in the relative large-scale fluxes of s and q must be balanced by discontinuities in the subgrid scale fluxes of s and q.

The heat and moisture budget equations for the mixed layer (equations (12) and (13)) are similar to those above the mixed layer except for the absence of vertical advection terms and precipitation terms. $(Q_R)_M$ is the vertically averaged radiational heating of the mixed layer.

Equations (8) through (13) have several interesting integral properties. Integrating (8) with respect to p from zero to p_B- and combining the result with (10) and (12) we obtain

$$\frac{\partial}{\partial t}\left\{s_M g^{-1}(p_o-p_B)+\int_o^{p_B}\bar{s}(p)g^{-1}dp\right\} + \nabla\cdot\left\{\mathbb{V}_M s_M g^{-1}(p_o-p_B)+\int_o^{p_B}\bar{\mathbb{V}}(p)\bar{s}(p)g^{-1}dp\right\}$$

$$= (F_s)_o + L\int_o^{p_B}R(p)g^{-1}dp + \left\{(Q_R)_M g^{-1}(p_o-p_B)+\int_o^{p_B}Q_R(p)g^{-1}dp\right\} . \quad (15)$$

The quantity within the first brackets in (15) is the total dry static energy per unit area in a column extending from the surface to the top of the atmosphere. Equation (15) shows that cumulus convection increases the total dry static energy in the column only if there is precipitation from the column.

Similarly, integrating (9) and combining the result with (11) and (13) we obtain

$$\frac{\partial}{\partial t}\left\{q_M g^{-1}(p_o-p_B)+\int_o^{p_B}\bar{q}(p)g^{-1}dp\right\}+\nabla\cdot\left\{\mathbb{V}_M q_M g^{-1}(p_o-p_B)+\int_o^{p_B}\bar{\mathbb{V}}(p)\bar{q}(p)g^{-1}dp\right\}$$

$$=(F_q)_o-\int_o^{p_B}R(p)g^{-1}dp. \qquad (16)$$

The quantity within the first brackets in (16) is the total mass of water vapor per unit area in a column extending from the surface to the top of the atmosphere. Equation (16) shows that cumulus convection decreases the total mass of water vapor in the column only if there is precipitation from the column. Equations (15) and (16) can be combined to give

$$\frac{\partial}{\partial t}\left\{h_M g^{-1}(p_o-p_B)+\int_o^{p_B}\bar{h}(p)g^{-1}dp\right\}+\nabla\cdot\left\{\mathbb{V}_M h_M g^{-1}(p_o-p_B)+\int_o^{p_B}\mathbb{V}(p)\bar{h}(p)g^{-1}dp\right\}$$

$$=(F_h)_o+\left\{(Q_R)_M g^{-1}(p_o-p_B)+\int_o^{p_B}Q_R(p)g^{-1}dp\right\}, \qquad (17)$$

which shows that the total moist static energy per unit area in the column is unaffected by cumulus convection. The cumulus convection simply transports moist static energy from the lower levels to the higher levels.

The fluxes at p_B+ can be written in terms of the surface fluxes by considering the turbulent energy balance of the mixed layer. This yields

$$(F_{sv})_B = -\,k(F_{sv})_o, \qquad (18)$$

where k is an empirical constant with a value of approximately 0.20.

Defining

$$M_B = \int_o^{\lambda max} m_B(\lambda) d\lambda, \tag{19}$$

(8) through (13) and (18) can be reduced to

$$\frac{\partial \bar{s}}{\partial t} = - \bar{\mathbb{V}} \cdot \nabla \bar{s} - \bar{\omega} \frac{\partial s}{\partial p} + g \frac{\partial}{\partial p} F_{s-L\ell} + LR + Q_R, \tag{20}$$

$$\frac{\partial \bar{q}}{\partial t} = - \bar{\mathbb{V}} \cdot \nabla \bar{q} - \bar{\omega} \frac{\partial \bar{q}}{\partial p} + g \frac{\partial}{\partial p} F_{q+\ell} - R, \tag{21}$$

$$\frac{\partial s_M}{\partial t} = - \mathbb{V}_M \cdot \nabla s_M + \frac{g}{p_o - p_B} [(F_s)_o + k \frac{\Delta s}{\Delta s_v} (F_{sv})_o] + (Q_R)_M, \tag{22}$$

$$\frac{\partial q_M}{\partial t} = - \mathbb{V}_M \cdot \nabla q_M + \frac{g}{p_o - p_B} [(F_q)_o + k \frac{\Delta q}{\Delta s_v} (F_{sv})_o], \tag{23}$$

$$\frac{\partial p_B}{\partial t} = - \mathbb{V}_B \cdot \nabla p_B + \bar{\omega}_B + g M_B - \frac{gk}{\Delta s_v} (F_{sv})_o . \tag{24}$$

Thus, the temperature and moisture fields above and below p_B and the pressure at the top of the mixed layer p_B can be predicted if we can somehow determine $F_{s-L\ell}$, $F_{q+\ell}$, R, and M_B. The cumulus ensemble transport terms $F_{s-L\ell}$, $F_{q+\ell}$, M_B and the cumulus ensemble source/sink term R constitute the feedback loop shown in Figure 1.

From (1), (2), (3), (6), (7), (14), and (19) we can see that this is equivalent to determining $\eta(p,\lambda)$, $s_c(p,\lambda)$, $q_c(p,\lambda)$, $\ell(p,\lambda)$, $r(p,\lambda)$, $\lambda_D(p)$, and $m_B(\lambda)$. All except $m_B(\lambda)$ are determined from the static control loop of the theory, which is discussed in Section 3. $m_B(\lambda)$ is determined from the dynamic control loop of the theory, which is discussed in Section 4.

3. STATIC CONTROL

The sub-ensemble normalized mass flux $\eta(p,\lambda)$, the sub-ensemble moist static energy $h_c(p,\lambda)$, and the sub-ensemble total water content $q_c(p,\lambda) + \ell(p,\lambda)$ are determined from the sub-ensemble mass, moist static energy, and total water budget equations. These are

$$\frac{\partial \eta(p,\lambda)}{\partial p} = - \frac{\lambda \bar{H}(p)}{p} \eta(p,\lambda), \tag{25}$$

$$\frac{\partial}{\partial p}[\eta(p,\lambda)h_c(p,\lambda)] = - \frac{\lambda \bar{H}(p)}{p} \eta(p,\lambda)\bar{h}(p), \tag{26}$$

$$\begin{aligned}\frac{\partial}{\partial p} \left\{\eta(p,\lambda)\,[q_c(p,\lambda) + \ell(p,\lambda)]\right\} = &- \frac{\lambda \bar{H}(p)}{p} \eta(p,\ )\bar{q}(p) \\ &+ \frac{\bar{H}(p)}{p} \eta(p,\lambda) r(p,\lambda),\end{aligned} \tag{27}$$

where $\bar{H}$ is the scale height $R\bar{T}/g$. Between the top of the mixed layer p_B and the condensation level p_c, $q_c(p,\lambda)$ is determined from

$$\frac{\partial}{\partial p}[\eta(p,\lambda)q_c(p,\lambda)] = - \frac{\lambda \bar{H}(p)}{p} \eta(p,\lambda)\bar{q}(p), \tag{28a}$$

while above p_c the air inside the clouds is saturated at a temperature only slightly different from the environment, allowing us to write

$$q_c(p,\lambda) = \bar{q}^*(p) + \frac{\gamma(p)}{1+\gamma(p)} \frac{1}{L} [h_c(p,\lambda) - \bar{h}^*(p)], \tag{28b}$$

where $\gamma \equiv \frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p$.

$\lambda_D(p)$ is given implicitly by

$$s_{vc}(p,\lambda_D(p)) = \bar{s}_v(p), \tag{29}$$

a statement that level p is a level of vanishing buoyancy (in terms of virtual dry static energy) for sub-ensemble $\lambda_D(p)$.

If $r(p,\lambda)$ is regarded as a known function of $\ell(p,\lambda)$, (25), (26), (27), (28), and (29) constitute a set of five equations in the five unknowns $\eta(p,\lambda)$, $s_c(p,\lambda)$, $q_c(p,\lambda)$, $\ell(p,\lambda)$, and $\lambda_D(p)$[1]. Thus, the sub-ensemble budgets (25) through (28a), the saturation relation (28b), and the condition of vanishing buoyancy at the detrainment level (29) constitute the static control loop as shown in Figure 1.

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[1]Equations (25), (26), (27), and (28a) are differential equations which are solved from p_B upward. The boundary conditions at p_B are simply $\eta(p_B,\lambda) = 1$, $h_c(p_B,\lambda) = h_M$, $q_c(p_B,\lambda) = q_M$, and $\ell(p_B,\lambda)=0$. An iterative procedure for solving (25) through (29) in a vertically discrete model is discussed in II.

4. DYNAMIC CONTROL

In order to predict the large-scale fields from (20) through (24) there remains only the problem of determining $m_B(\lambda)$.

Let us define the cloud work function as

$$A(\lambda) = \int_{p_D(\lambda)}^{p_B} \eta(p,\lambda)[s_{vc}(p,\lambda) - \bar{s}_v(p)]\frac{dp}{p} . \tag{30}$$

$A(\lambda)$ is an integral measure of the buoyancy force. It is also a measure of the efficiency of kinetic energy generation for sub-ensemble λ. Since $A(\lambda)$ is actually a property of the large-scale, its time derivative can be written in terms of the time derivatives of s_M, q_M, p_B, $\bar{s}(p)$, and $\bar{q}(p)$. Thus,

$$\frac{dA(\lambda)}{dt} = \frac{a(p_B,\lambda)}{\bar{H}_B}\frac{\partial s_{vM}}{\partial t} + \frac{b(p_B,\lambda)}{\bar{H}_B}\frac{\partial h_M}{\partial t}$$

$$+ \frac{\partial p_B}{\partial t}\frac{1}{p_B}\left\{[-1 + \lambda a(p_B,\lambda)]\Delta s_v + \lambda b(p_B,\lambda)\Delta h + \lambda \bar{H}_B A(\lambda)\right\}$$

$$+ \int_{p_D(\lambda)}^{p_B} \eta(p,\lambda)\left\{[-1 + \lambda a(p,\lambda)]\frac{\partial \bar{s}_v(p)}{\partial t} + \lambda b(p,\lambda)\frac{\partial \bar{h}(p)}{\partial t}\right\}\frac{dp}{p} . \tag{31}$$

$\bar{H}_B$ is the scale height at p_B. $a(p,\lambda)$ and $b(p,\lambda)$ are known weighting functions. The terms on the right hand side of (31) can be divided into two classes: those which depend on $m_B(\lambda)$ and those which do not. Thus, (31) can be written

$$\frac{dA(\lambda)}{dt} = \int_0^{\lambda max} K(\lambda,\lambda')m_B(\lambda')d\lambda' + F(\lambda), \tag{32}$$

where the kernel $K(\lambda,\lambda')$ and the forcing function $F(\lambda)$ are known.

The quasi-equilibrium assumption discussed in I is as follows. When absolutely no convection exists the temperature and moisture fields are free to change in a manner which is unconstrained by cumulus convection. When some convection exists (i.e., $m_B(\lambda)>o$ for some λ), then the large-scale tendencies are constrained such that for each λ, we must have

$$\frac{dA(\lambda)}{dt} = o \quad \text{if} \quad m_B(\lambda)>o$$

$$\frac{dA(\lambda)}{dt} < o \quad \text{if} \quad m_B(\lambda) = o.$$

Thus, for each λ, $m_B(\lambda)$ must satisfy either

$$m_B(\lambda)>o \quad \text{and} \quad \int_o^{\lambda max} K(\lambda,\lambda')m_B(\lambda')d\lambda' + F(\lambda) = o \tag{33a}$$

or

$$m_B(\lambda) = o \text{ and } \int_o^{\lambda max} K(\lambda,\lambda')m_B(\lambda')d\lambda' + F(\lambda)<o \tag{33b}$$

Since the kernel and the forcing function are known, (33) is an integral equation for $m_B(\lambda)$. The condition on whether to apply the integral equality or inequality is in terms of the unknown function $m_B(\lambda)$. This makes solving (33) a somewhat difficult task. An iterative method of solution is discussed in II.

Equation (33) constitutes the dynamic control loop shown in Figure 1. The parameterization theory is now closed.

5. DIAGNOSTIC USE OF THE THEORY

The theory described in the preceeding sections was developed for use in large-scale prognostic models. However, the feedback and static control portions of the theory can be used for diagnostic studies, as has been done by Ogura and Cho (1973), Nitta (1974), and Yanai, Chu, and Stark (1974). Perhaps the easiest way to understand the diagnostic use of the theory is to combine (8) and (9) to obtain

$$\frac{\partial \bar{h}}{\partial t} + \bar{\mathbb{V}}\cdot\nabla\bar{h} + \bar{\omega}\frac{\partial \bar{h}}{\partial p} - Q_R = g\frac{\partial F_h}{\partial p}, \tag{34}$$

where

$$F_h(p) = \int_o^{\lambda_D(p)} \eta(p,\lambda)[h_c(p,\lambda) - \bar{h}(p)]m_B(\lambda)d\lambda. \tag{35}$$

The left hand side of (34) has been called Q_1-Q_2-Q_R by Yanai, Esbensen, and Chu (1973). From (34) we can see that the vertical integration of Q_1-Q_2-Q_R from a level p_T where F_h is zero downward to a level p will yield

$$F_h(p) = \int_{p_T}^{p} (Q_1-Q_2-Q_R)g^{-1}dp. \tag{36}$$

Thus, if the local tendency of h, the horizontal and vertical advection of h, and the radiational heating are known, $F_h(p)$ can be computed. Once $F_h(p)$ has been computed from (36), and once $\lambda_D(p)$, $\eta(p,\lambda)$, and $h_c(p,\lambda)$ have been computed from the equations in Section 3, (35) can be solved as a Volterra integral equation for $m_B(\lambda)$[1]. Once $m_B(\lambda)$ has been computed, either (8) or (9) can be solved for R(p), and some properties of $r(p,\lambda)$ can be determined from (14).

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[1]The discrete version of (35) is a triangular matrix equation.

6. SUBGRID SCALE FLUX FORM vs. DETRAINMENT FORM

The forms of the equations for $\bar{s}(p)$ and $\bar{q}(p)$ given in Section 2 are the subgrid scale flux forms. Differentiating (1a), (2a), and (3a) with respect to p, using (6), (7), and the sub-ensemble budgets of Section 3, (20) and (21) can be written

$$\frac{\partial \bar{s}}{\partial t} + \mathbb{V}\cdot\nabla\bar{s} + \bar{\omega}\frac{\partial \bar{s}}{\partial p} = D(\hat{s}-\bar{s}-L\hat{\ell}) + gM_c\frac{\partial \bar{s}}{\partial p} + Q_R, \qquad (37)$$

$$\frac{\partial \bar{q}}{\partial t} + \mathbb{V}\cdot\nabla\bar{q} + \bar{\omega}\frac{\partial \bar{q}}{\partial p} = D(\hat{q}^{*}-\bar{q}+\hat{\ell}) + gM_c\frac{\partial \bar{q}}{\partial p}, \qquad (38)$$

where the total detrainment and total cloud mass flux are given by

$$D(p) = gm_B(\lambda_D(p))\eta(p,\lambda_D(p))\frac{d\lambda_D(p)}{dp}, \qquad (39)$$

and

$$M_c(p) = \int_0^{\lambda_D(p)} \eta(p,\lambda)m_B(\lambda)d\lambda, \qquad (40)$$

and where

$$\hat{s}(p) = s_c(p,\lambda_D(p)) \qquad (41a)$$

$$\hat{q}^{*}(p) = q_c(p,\lambda_D(p)) \qquad (41b)$$

$$\hat{\ell}(p) = \ell(p,\lambda_D(p)). \qquad (41c)$$

Thus, the detrainment forms (37) and (38), which were used in I and II, are equivalent to (37) and (38). The subgrid scale flux forms appear to have certain advantages, especially in models which are discrete in the vertical.

7. SUMMARY

Let us now summarize the cloud ensemble/large-scale interaction model in terms of Figure 1.

The properties of the large-scale environment are given by the dry static energy above and within the mixed layer, $\bar{s}(p)$ and s_M, by the water vapor mixing ratio above and within the mixed layer, $\bar{q}(p)$ and q_M, and by the pressure at the top of the mixed layer p_B.

The properties of the cloud ensemble are given by the sub-ensemble profiles of normalized mass flux $\eta(p,\lambda)$, dry static energy $s_c(p,\lambda)$, water vapor mixing ratio $q_c(p,\lambda)$, liquid water mixing ratio $\ell(p,\lambda)$, by the detrainment pressure $p_D(\lambda)$, and by the mass flux distribution function $m_B(\lambda)$.

The feedback of the cloud ensemble onto the large-scale occurs through the cumulus terms in the prognostic equations for $\bar{s}(p)$, $\bar{q}(p)$, and p_B, i.e., in equations (20), (21), and (24). s_M and q_M are unaffected by cumulus convection but are affected by the subgrid scale turbulent transports of the mixed layer.

The control of the cloud ensemble by the large-scale can be divided into two parts: static control and dynamic control. All the properties of the cloud ensemble except $m_B(\lambda)$ can be determined from the static control, which consists of the subensemble budgets (25), (26), (27), (28a), the saturation relation (28b), and the condition of vanishing buoyancy at the detrainment level (29). $m_B(\lambda)$ can be determined from the dynamic control, which consists of the integral equation (33).

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