SIGNIFICANCE AND APPLICATION OF FROUDE AND REYNOLDS NUMBERS AS CRITERIA FOR SIMILITUDE

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CORRECTIONS

Page	Line	Read
3	9 down	The subscript m denotes the model and the subscript p denotes the ...
8	8 up	Leonhard Euler
17	10 down	The celerity of the gravity wave
	9 up	(density) currents
19	3 down	Shear velocity V_*
	10 down	~~(15)~~ (34)
	6 up	Lacey
20	4 up	involve the parameter
	19 down	Straub and others (27)
	20 down	2,000 to 3,000
25	4 up	~~Danel~~ Craya
31	10 up	The forces of gravity predominate

SIGNIFICANCE AND APPLICATION OF FROUDE AND REYNOLDS NUMBERS AS CRITERIA FOR SIMILITUDE*

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Many phenomena in fluid mechanics, including the stress and strain relationship for fluid motion and its accompanying energy loss, are so complex that mathematical analysis is normally unattainable except in special cases. Hence, practical solutions for hydraulic engineering problems have relied heavily upon experimental investigations and model studies. The development of experimental studies of hydraulic problems has taken place almost entirely since the Froude's law of similitude and that of Reynolds' were discovered. Since then these laws of similitude have been extended to many phases of hydraulic problems, and a considerable number of variations of the laws have been formulated. Numerous textbooks of fluid mechanics, scientific papers, and technical reports have been written which include discussion of the basic principles and applications of these laws of similitude. Although the basic concepts are in general simple and straight forward, applications are sometimes confusing and debatable. The purpose of this paper is to examine the laws of hydraulic similitude together with certain applications in an attempt to help in their ultimate clarification.

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Dimensional Considerations

Since fluid motion is basically a problem of mechanics, it is sufficient to restrict the study to the four fundamental dimensions of mechanics--that is, length, L, time, T, mass, M, and force, F. Furthermore, these four dimensions can be related by the use of Newton's second law of motion:

$$\bar{F} = \alpha M \bar{a} \tag{1}$$

in which $\bar{F}$ is a vector denoting the resultant of all forces acting on a body which has a total mass of M, $\bar{a}$ is a vector denoting the acceleration of the body, and α is a universal positive constant--the magnitude of which depends only on the units employed and the dimensions of which are $\left[\frac{FT^2}{ML}\right]$. Leaving the units of mass, M, length, L, and time, T unchanged, the magnitude of α can be made equal to the pure number 1, by suitable choice of the unit of force; thus:

$$\bar{F} = M\bar{a} \tag{2}$$

According to Eq 2 the dimensions of force, length, time and mass can be interrelated

$$\left[\frac{FT^2}{ML}\right] = 1 \tag{3}$$

In which any one of L,T,F, or M can be regarded as the dependent dimension. It is customary to choose either L,T,F, or L,T,M, as the three fundamental dimensions for problems of mechanics.

Principles of Hydraulic Similitude

Complete similitude of two flow systems requires that the systems in consideration be geometrically, kinematically, and dynamically similar.

Geometric similarity exists between two systems with the ratios of all corresponding linear dimensions equal. This relationship only involves similarity in shape and form. That is,

$$L_r = \frac{L_p}{L_m} = \frac{D_p}{D_m} = \frac{B_p}{B_m} = \text{----} \tag{4}$$

in which L_r denotes the geometric ratio, L the horizontal length, D the flow depth, and B the flow width. The subscript p denotes the model and the subscript m denotes the prototype.

Kinematic similarity is a similarity of motion. When the ratios of the components of velocity and acceleration of all homologous points in two geometrically similar systems are equal, the two states of motion are kinematically similar. The paths of homologous Particles will then also be geometrically similar. That is,

$$V_r = \frac{V_p}{V_m} = \frac{u_p}{u_m} = \frac{v_p}{v_m} = \frac{w_p}{w_m}$$

and (5)

$$a_r = \frac{a_p}{a_m}$$

in which V_r denotes the velocity ratio, a_r the acceleration ratio, V the mean velocity, u, v, and w the velocity components along x, y and z directions respectively.

Dynamic similarity between two geometrically and kinematically similar systems requires that ratios of all homologous forces in the two systems be the same. In the case of flow of real fluids, the forces acting on an element of the fluid are $\bar{F_p}$ due to pressure variation, $\bar{F_f}$ due to viscosity, and $\bar{F_g}$ due to gravity. (In this paper the forces due to surface tension and elastic compression are neglected.) Accordingly, the equation of Motion can be written as

$$\bar{F_p} \dotplus \bar{F_g} \dotplus \bar{F_f} = M\bar{a} \tag{6}$$

Eq 6 can be changed into

$$\overline{F}_p \mathrel{+\!\!>} \overline{F}_g \mathrel{+\!\!>} \overline{F}_f \mathrel{+\!\!>} \overline{F}_i = 0 \tag{7}$$

which is the condition for dynamic equilibrium, in which $\overline{F}_i$ denotes the inertia force and is equal to $(-M\bar{a})$.

Eq 7 can be represented by Fig. 1a which is a force polygon. The condition of <u>dynamic similarity</u> for incompressible homogenous fluid motion is

$$\frac{(\overline{F}_i)_p}{(\overline{F}_i)_m} = \frac{(\overline{F}_p)_p}{(\overline{F}_p)_m} = \frac{(\overline{F}_g)_p}{(\overline{F}_g)_m} = \frac{(\overline{F}_f)_p}{(\overline{F})_m} = F_r \tag{8}$$

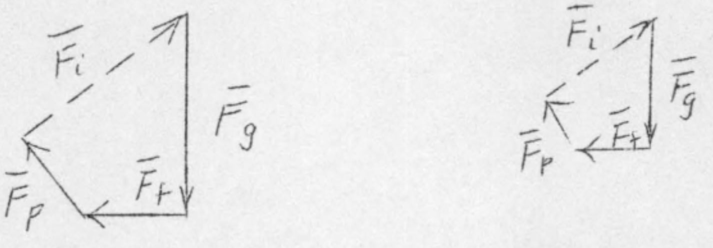
The force polygon for the model is shown as Fig. 1b. It can be shown that if

$$\frac{(\overline{F}_i)_p}{(\overline{F}_i)_m} = \frac{(\overline{F}_g)_p}{(\overline{F}_g)_m} \quad \text{or} \quad \frac{(\overline{F}_i)_p}{(\overline{F}_i)_m} = \frac{(\overline{F}_f)_p}{(\overline{F}_f)_m} \tag{9}$$

then

$$\frac{(\overline{F}_i)_p}{(\overline{F}_i)_m} = \frac{(\overline{F}_p)_p}{(\overline{F}_p)_p} \tag{10}$$

The proof of which is based upon the condition that similarity of two force systems means not only similarity in magnitude but also similarity in direction. F_r denotes the ratio of corresponding individual forces.



(a) Prototype Fig. 1 (b) Model

The foregoing illustration indicates that the pressure force is not an independent quantity. In other words, if the gravitational force and viscous force are similar in two systems and the resulting motion is also similar, then the force due to pressure variation also must be similar. In the case of compressible fluid flow, such as flow of air or other gases, the pressure is affected by the thermal condition as well as the density of the fluid. Therefore, for compressible fluids, the statement that the pressure force depends upon the viscous force, gravitational force, and inertia force must by modified. In addition to the geometrical similarity, kinematic similarity, and dynamic similarity, a thermal similarity is also required. (1)

The inertia force on the fluid element, by definition, is equal to the negative of the product of mass and acceleration; that is,

$$-\bar{F}_i = M\frac{D\bar{u}}{Dt} = \rho\forall\left[\frac{\partial\bar{u}}{\partial t} + u\frac{\partial\bar{u}}{\partial x} + v\frac{\partial\bar{u}}{\partial y} + \omega\frac{\partial\bar{u}}{\partial z}\right] \tag{11}$$

in which u,v,w are velocity components along the x,y,z directions, $\forall$ the elementary volume ($dx\,dy\,dz$), and ρ the density of the fluid. For steady motion, the term of local acceleration $\frac{\partial\bar{u}}{\partial t}$ is zero, and the inertia force can be written approximately as

$$-\bar{F}_i = \rho\forall\, u\frac{\partial\bar{u}}{\partial x} \tag{11a}$$

The other two convective acceleration terms have been neglected for simplicity. The ratio of the inertia force between the two systems can be written as

$$\frac{(F_i)_p}{(F_i)_m} = \frac{\left(\rho\forall\, u\frac{\partial u}{\partial x}\right)_p}{\left(\rho\forall\, u\frac{\partial u}{\partial x}\right)_m} = \frac{\left(\rho\forall\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right)_p}{\left(\rho\forall\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right)_m}$$

Since

$$u \propto V,\quad \forall \propto L^3 \quad \text{and} \quad dx \propto L$$

the foregoing equation becomes

$$\frac{(F_i)_p}{(F_i)_m} = \frac{(\rho L^2 V^2)_p}{(\rho L^2 V^2)_m} = \rho_r L_r^2 V_r^2 \tag{12}$$

in which ρ_r is the density ratio. Eq 12 has been considered by Birkoff (2), as the principle of inertial modelling.

The viscous force per unit volume can be represented as

$$\mu \frac{\partial^2 u}{\partial y^2}$$

therefore, the ratio of the viscous force between the two systems can be written as

$$\frac{(F_f)_p}{(F_f)_m} = \frac{(\forall \mu \frac{\partial^2 u}{\partial y^2})_p}{(\forall \mu \frac{\partial^2 u}{\partial y^2})_m} = \mu_r V_r L_r \tag{13}$$

The gravitational force per unit volume is

$$\gamma = \rho g$$

The ratio of the gravitational force between the two systems is

$$\frac{(F_g)_p}{(F_g)_m} = \frac{(\forall \gamma)_p}{(\forall \gamma)_m} = \gamma_r L_r^3 \tag{14}$$

The pressure force per unit volume along the x-direction is $\frac{\partial p}{\partial x}$

The ratio of the pressure force between the two systems is

$$\frac{(F_p)_p}{(F_p)_m} = \frac{(\forall \frac{\partial p}{\partial x})_p}{(\forall \frac{\partial p}{\partial x})_m} = p_r L_r^2 \tag{15}$$

According to the definition of "dynamic similarity," (see Eq 8) the inertia force ratio can be put equal to the viscous force ratio,

$$\frac{(F_i)_p}{(F_i)_m} = \frac{(F_f)_p}{(F_f)_m}, \text{ or } \left(\frac{F_i}{F_f}\right)_p = \left(\frac{F_i}{F_f}\right)_m \tag{16}$$

Equating Eq 12 and Eq 13

$$\rho_r L_r^2 V_r^2 = \mu_r L_r V_r$$

which means

$$\frac{\rho_r V_r L_r}{\mu_r} = 1 \tag{17a}$$

and

$$\left(\frac{\rho V L}{\mu}\right)_p = \left(\frac{\rho V L}{\mu}\right)_m \tag{17b}$$

The dimensionless term

$$\frac{\rho V L}{\mu}$$

is known as the Reynolds number, which is a dimensionless number introduced by Osborne Reynolds in connection with his study of turbulence in pipe flow (3).

On the other hand, if the ratio of the inertia force is put equal to that of the gravitational force

$$\frac{(F_i)_p}{(F_i)_m} = \frac{(F_g)_p}{(F_g)_m}, \text{ or } \left(\frac{F_i}{F_g}\right)_p = \left(\frac{F_i}{F_g}\right)_m \tag{18}$$

Equating Eq 12 and Eq 14

$$\rho_r L_r^2 V_r^2 = \gamma_r L_r^3, \qquad \frac{\rho_r V_r^2}{\gamma_r L_r} = 1$$

or

$$\left(\frac{\rho V^2}{\gamma L}\right)_p = \left(\frac{\rho V^2}{\gamma L}\right)_m \tag{19}$$

the dimensionless term

$$\sqrt{\frac{\rho V^2}{\gamma L}}$$

is called the Froude number in honor of William Froude who first discovered its importance in his model studies of ships (4).

If the inertia force ratio is put equal to the pressure force ratio

$$\frac{(F_i)_p}{(F_i)_m} = \frac{(F_p)_p}{(F_p)_m}, \text{ or } \left(\frac{F_i}{F_p}\right)_p = \left(\frac{F_i}{F_p}\right)_m \tag{20}$$

Equating Eq 12 and Eq 15

$$\rho_r L_r^2 V_r^2 = p_r L_r^2,$$

or

$$\frac{p_r}{\rho_r V_r^2} = 1 \tag{21}$$

which, in terms of model and prototype is

$$\left(\frac{p}{\rho V^2}\right)_p = \left(\frac{p}{\rho V^2}\right)_m$$

This dimensionless parameter $\left(\frac{p}{\rho V^2}\right)$ is knows as the pressure coefficient (5).

With an incompressible fluid, the change of pressure will not affect the density of the fluid and the whole pressure field can be increased or decreased without changing the motion. In other words, it is not necessarily the absolute pressure intensity but rather the pressure gradient which is related to the flow pattern--therefore, Δp can be used in Eq 21, that is

$$\left(\frac{\Delta p}{\rho V^2}\right)_p = \left(\frac{\Delta p}{\rho V^2}\right)_m \tag{21a}$$

According to Fig. 1, the pressure force will affect the inertia force. On the other hand, once the inertia force, the gravitational force, and the viscous force are given, then the corresponding pressure is fixed. This explains why the pressure force is not considered as an independent variable in hydraulic similitude.

The inverse of the pressure coefficient $\sqrt{\frac{\rho V^2}{\Delta p}}$ has been termed the "local Euler Number" by Rouse (6) in honor of Leonlard Euler, a Swiss mathematician of the eighteenth century. According to Rouse the discharge coefficient of an orifice and the resistance coefficient of pipe flow are particular forms of the Euler number. The coefficient of drag for a submerged body such as sphere was considered by Rouse as a modified Euler number. From Eqs 9 and 10, the "pressure coefficient" depends on the Reynolds number and Froude number. In case the effect due to viscosity and gravity is nil, then the Euler number alone will determine the flow pattern, as is the case for ideal fluid flow. (Note that the Euler number does not involve fluid properties other than the density.)

Sometimes the cavitation number is eroneously considered as a type of "Euler number". This is not justified because the cavitation number is related to the difference between the minimum pressure intensity, which depends upon the absolute pressure in the flow, and the vapor pressure which is a property of the liquid and depends upon its temperature. In modelling cavitation phenomena, the model size, the free stream velocities, the dissolved air content, the free nuclei content, the roughness of the body surface, and the cavitation number are all considered to assume similarity.

The foregoing dimensionless numbers--such as the pressure coefficient, the Froude number, and the Reynolds number--can be obtained also from dimensional analysis by use of the π-theorem. There are many references written on the π-theorem and dimensional analysis (5). Therefore, it is not presented in this paper. However, the method of dimensional analysis is not always adequate because in order to use dimensional analysis most effectively, considerable experience with physical interpretation is needed. This requirement can cause considerable difficulty.

The dimensionless numbers, Froude number, Reynolds number, and Pressure coefficient, can also be obtained (7) through the application of the Navier-Stokes equation of motion. The Navier-Stokes equation for prototype flow in vector form is

$$\left(\rho \frac{D\bar{V}}{Dt}\right)_p = (\bar{G})_p - (\nabla p)_p + (\mu \nabla^2 \bar{V})_p \tag{22}$$

in which $\bar{V}$ is the velocity vector, $\bar{G}$ the gravitational force, $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$, and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Let

$$\begin{aligned}
L_p &= L_r\, L_m && \text{for length} \\
\bar{V}_p &= V_r\, \bar{V}_m && \text{for velocity} \\
p_p &= p_r\, p_m && \text{for pressure} \\
\bar{G}_p &= G_r\, \bar{G}_m && \text{for gravitational force} \\
\rho_p &= \rho_r\, \rho_m && \text{for density} \\
t_p &= t_r\, t_m && \text{for time}
\end{aligned} \tag{23}$$

$$\mu_p = \mu_r \, \mu_m$$ for dynamic viscosity

for undistorted model

$$t_p = \frac{L_p}{V_p} = \frac{L_r L_m}{V_r V_m} = t_r L_m$$

therefore

$$t_r = \frac{L_r}{V_r} \tag{24}$$

Substituting Eqs 23 into Eq 22 yields

$$\frac{\rho_r V_r}{t_r}\left(\rho \frac{D\bar{V}}{Dt}\right)_m = G_r(\bar{G})_m - \frac{p_r}{L_r}(\nabla p)_m + \frac{\mu_r V_r}{L_r^2}(\mu \nabla^2 \bar{V})_m \tag{25a}$$

or

$$\left(\rho \frac{D\bar{V}}{Dt}\right)_m = \frac{G_r t_r}{\rho_r V_r}(\bar{G})_m - \frac{p_r t_r}{\rho_r V_r L_r}(\nabla p)_m + \frac{\mu_r V_r t_r}{L_r^2 \rho_r V_r}(\mu \nabla^2 \bar{V})_m \tag{25b}$$

In order for the model flow to be dynamically similar to the prototype flow, the coefficients of the gravitational force, the pressure force, and the viscous force must all be unity.

For the <u>gravitational force</u>

$$\frac{G_r t_r}{\rho_r V_r} = 1 \tag{26}$$

Since $t_r = \frac{L_r}{V_r}$ and G_r can be considered as the ratio of the specific weight of the fluid, that is γ_r, then

$$\frac{G_r t_r}{\rho_r V_r} = \frac{\gamma_r L_r}{\rho_r V_r^2} = 1$$

which becomes

$$\left(\frac{\rho V^2}{\gamma L}\right)_p = \left(\frac{\rho V^2}{\gamma L}\right)_m \tag{19}$$

Therefore, the Froude criterion of model law, as given by Eq 19 previously, is again obtained. It should by noted that the Froude law of similitude is required wherever a body force is present in the system.

For the viscous force:

$$\frac{\mu_r t_r V_r}{\rho_r V_r L_r^2} = 1 \tag{27}$$

or

$$\frac{\mu_r}{\rho_r V_r L_r} = 1$$

which is the relationship between the Reynolds number for the prototype and the model

$$\left(\rho \frac{VL}{\mu}\right)_p = \left(\rho \frac{VL}{\mu}\right)_m \tag{17}$$

For the <u>pressure</u> <u>force</u>

$$\frac{p_r t_r}{\rho_r V_r L_r} = 1 \tag{28}$$

or

$$\frac{p_r}{\rho_r V_r^2} = 1 \tag{21}$$

$$\left(\frac{p}{\rho V^2}\right)_p = \left(\frac{p}{\rho V^2}\right)_m$$

In the case of an incompressible fluid, Eq 21 can be written as

$$\left(\frac{\Delta p}{\rho V^2}\right)_p = \left(\frac{\Delta p}{\rho V^2}\right)_m \tag{21a}$$

for reasons given previously.

Since the model flow is considered a miniature of the prototype flow, the specific forms of equations necessary for the prototype flow--such as, the equation of motion, the equation of continuity, and the boundary conditions--must also describe the model flow. Since energy is a product of force and distance, it can be shown that if the two systems of flow are geometrically, kinematically, and dynamically similar, the energy equation pertaining to the prototype flow should also pertain to the model flow. In case the compressibility of the flow is important, the equation of state and the energy equation are also needed in formulating conditions of similitude (2).

In case the equation of state follows the adiabatic relationship, that is,

$$p \propto \rho^{k} \tag{29}$$

the pressure coefficient can be changed into a Mach number as follows (7):

$$\log p - k \log \rho = \text{constant}$$

$$\frac{dp}{p} - k\frac{d\rho}{\rho} = 0 \tag{30}$$

$$\frac{dp}{p} = k\frac{d\rho}{\rho}$$

or

$$\frac{dp}{\frac{d\rho}{\rho}} = k\,dp = E \tag{31}$$

in which E is the modulus of elasticity

Hence

$$\frac{kp}{\rho} = \frac{E}{\rho} = C^2 \tag{32}$$

in which C is wave celerity

or

$$\frac{p}{\rho} = \frac{C^2}{k} \tag{33}$$

Substituting Eq 33 into Eq 21 yields

$$\left(\frac{V^2}{C^2}\right)_p = \left(\frac{V^2}{C^2}\right)_m \tag{34}$$

The Mach number is also known as the "Cauchy number." It is also the ratio of the kinetic energy to the intrinsic energy (1). An introduction of the Mach number here is for the purpose of discussing the Froude number in the next section. The complete similarity principle of compressible fluid flow is omitted here because it is not essential to the problem of hydraulic similitude.

Froude Criterion

The Froude number as shown in the foregoing pages can be obtained by combining the inertia force and the gravitational force. In case the flow is unlimited; that is, there is no interface, and the density of the fluid is homogenous, the gravitational force is counter-balanced by the bouyance force due to hydrostatic pressure. Therefore, the fluid motion can be considered as independent of the effect of gravity. Also, in case of confined flow, such as flow in pipes, the flow pattern is determined by the boundary conditions. The effect of fluid weight is only in the determination of the piezometric head--that is, the effect of gravity is to change the pressure intensity from point to point in direct proportion to the change in elevation. Under these conditions, the Froude number plays no part.

If the fluid is unconfined in any zone, the form of the free surface, and therefore the form of the entire flow pattern, will be subjected to gravitational influence. It can be seen readily that the extent of the influence is a relative matter. This point has been well illustrated by Rouse (6):

> "Consider, for instance, the efflux of the fluid from the boundaries shown in Fig. 51 (Authors' remark - See Fig. 2). The inertia of the fluid tends to make it continue in the longitudinal direction after leaving the orifice, but the effect of gravity is to deflect it in the vertical direction; evidently, the greater the density and the velocity, the smaller the deflection in a given distance; whereas, the greater the difference in specific weight between the moving fluid and the surrounding medium, the greater the deflection will tend to be. A jet of air emerging into the atmosphere (or a submerged jet of water) would thus remain symmetrical around the longitudinal axis, regardless of how small the velocity or density might be, as indicated by profile A in Fig. 51, that is, the gravitational effect must be nil so long as there is no difference in specific weight. But a jet of a fluid of greater specific weight than that of the surrounding medium, would not remain symmetrical, the asymmetry becomes more pronounced (profiles B and C) the lower the velocity and density or the greater the relative specific weight of the moving fluid."

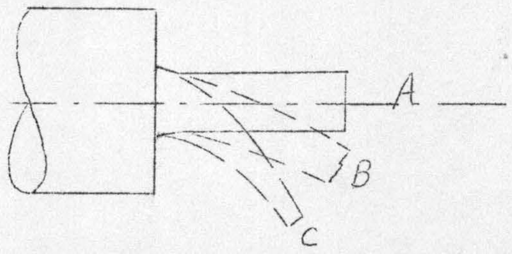


Fig. 2

In view of the fact that the gravitational force is necessary for consideration only if there exists an interface, such as a free surface, the gravitational force is best represented by $\Delta\gamma$ instead of γ . Therefore, the Froude number can be written as

$$F_r = \frac{V}{\sqrt{\frac{\Delta\gamma}{\rho} L}} \tag{35}$$

in which L is a characteristic length. In case of flow having a free surface the Froude number is commonly written as

$$F_r = \frac{V}{\sqrt{gL}} \qquad \Delta\gamma \doteq \gamma = \rho g \tag{36}$$

As mentioned earlier, in case the influence of the discharge of the flow is only the fluid density, velocity, and pressure difference, the Euler number is constant. Similarly in case of the discharge coefficient for flow over a weir, the gravitational effect is important. Any one of the weir equations is reducible to a dimensionless number of a combination of velocity, length, and acceleration of gravity, which is similar to Froude number. In this case a fixed numerical value of Froude number corresponds to a fixed coefficient of discharge, regardless of boundary scale.

Following the initial formulation of the Froude parameter, various forms of Froude number have appeared in the literature. In the following, certain of these forms of Froude number are discussed in connection with their common usage.

Ship models

In the study of ship models the Froude number used is $\frac{V}{\sqrt{gL}}$, in which V is the velocity of the ship and L is the length of the ship L. This parameter was originally used by Froude in his ship model study. Since L is along the direction of motion, which can be considered proportional to dx of Eq 11a, it is very clear that this form of Froude number can be considered as the ratio of the inertia force to the gravitational force. Froude reasoned that for a ship traveling on the water surface, there are several kinds of energy losses:

1. Those due to boundary resistance
2. Those due to eddy formation
3. Those due to wave formation

The boundary resistance loss and the eddy loss can be considered as a function of the Reynolds number and the form of the ship, and the loss due to wave formation can be considered as a function of the Froude number and the form of the ship. Therefore, Froude proposed the model law for ships as $V_r \propto \sqrt{L_r}$. Detail of ship model technique can be found eleswhere. The name Reech is often associated with Froude number. Reech (8) proposed the Froude law in 1831 and Froude proposed the same law in 1874. However, Froude separated the energy loss according to the boundary resistance loss, eddy formation loss, and wave formation loss, which Reech did not do. Therefore, the name of Froude law is preferred(2).

Open channel flow - using depth

Study of flow in open channels involves the following Froude number

$$F_r = \frac{V}{\sqrt{gh}}$$

in which V is the mean velocity of flow, h is the depth of flow. This form of Froude number is sometimes written as $\frac{V^2}{gh}$ which Bakhmeteff(9) has called the "kinetic flow factor". Besides being recognized as a form of Froude number, $\frac{V}{\sqrt{gh}}$ can also be interpreted as proportional to the square root of the ratio of kinetic energy to potential energy of the flow. The energy-ratio concept can be shown as follows:

$$\frac{E_i}{E_g} \propto \frac{\overline{F_i} \cdot \overline{\Delta x}}{\overline{F_g} \cdot \overline{\Delta y}} \propto \frac{\rho \forall u \frac{\partial u}{\partial x} \Delta x}{\forall \gamma \Delta y} \propto \frac{\rho \frac{\partial}{\partial x} (\frac{u^2}{2}) \Delta x}{\gamma h} \propto \frac{V^2}{gh} \tag{37}$$

Since

$$u \propto V, \text{ and } \quad \Delta y \propto h$$

The conditions of geometric and kinematic similarity are used in changing the sign of equality to proportionality. The form $\frac{V^2}{gh}$ can be called a "Bernoulli number" which indicates an energy ratio.

It is well known that the denominator $\sqrt{gh}$ in the Froude number is the celerity of a gravity wave of small amplitude in shallow flow. The significance of this parameter in open channel flow is that when the parameter is equal to 1, the flow is called critical flow--that is, for a given discharge, the specific energy of the flow at the critical stage is a maximum. When this number is equal to 1 the mean velocity of flow is equal to the celerity of the disturbance. Therefore, the disturbance is stationary. If the Froude number is less than 1, the flow is classified as tranquil or subcritical. In tranquil flow the disturbance can travel upstream. If the Froude number is greater than 1, it is called rapid flow, shooting flow, or supercritical flow. In this case, the disturbance cannot travel upstream but will be swept downstream.

Since the term $\sqrt{gh}$ is the celerity of the surface wave, this form of Froude number can be considered as a velocity ratio. In case of a compressible fluid, this ratio is known as the Mach number $\frac{V}{c}$ which is also the ratio of the kinetic energy to the intrinsic energy. There is a striking similarity between the supercritical flow and the supersonic flow.

It should be noted that the greater the Froude number, the greater the inertia effect and the kinetic energy. On the other hand, the smaller the Froude number the larger the relative effect of gravity and the potential energy. For example in the case of a hydraulic jump, the greater the Froude number of the approaching flow, the greater the energy loss through the hydraulic jump. When the Froude number is between 1 and 2, on the other hand, undular waves develop on the water surface instead of the hydraulic jump. In this case, the loss of energy is due mainly to formation of surface waves and gravitational effects are relatively important.

The hydraulic jump can be created also by relative flow between two fluids of different density such as demonstrated by Rouse and others (10). This is an excellent example of the significance of the difference in specific weight $\Delta\gamma$ of the two fluids,

Hydraulic structures

In model and prototype studies of hydraulic structures, the Froude number is $\frac{V}{\sqrt{gy_o}}$, in which V is the characteristic velocity and y_o the characteristic depth. The characteristic length in this case is designated as any length characterizing the flow condition. For instance, in case of a hydraulic jump in a sloping channel the depth y of approaching flow measured perpendicular to the apron is often used as the characteristic length. Or in case of a submerged jet from a sluice gate, the height of the gate opening is often used as the characteristic length . The Froude number based on these characteristic lengths is not necessarily the energy ratio, or the velocity ratio. Furthermore, it is difficult to visualize these as ratios of the inertia force to the gravitational force. As long as the model and the prototype are not distorted, however, this form of Froude number will give a correct similarity criterion.

The depth of flow perpendicular to the apron cannot be used in computing the celerity of the approaching flow. This can be shown in the following way. If the inclination of the apron is 90 degrees, obviously the celerity of the small gravity has no meaning since they cannot travel vertically. Should the celerity of the waves be used as the denominator, the length parameter is $y\cos\theta$ in which θ is the inclination angle of the sloping channel.

<u>Gravity currents</u>

Flow of one fluid under another fluid of smaller specific gravity requires a special Froude number

$$\frac{\Delta V}{\sqrt{\frac{\Delta\gamma}{\rho}\delta}}$$

in which ΔV is a differential velocity at the interface δ is the thickness of the interface or the amplitude of the waves. This form of Froude number used in modelling gravity density currents is well known. Extensive work on gravity currents has been done by Ippen and Harleman (11), and Keulegan (12) and Yih (12).

<u>Open channel flow--using hydraulic radius</u>

In the study of energy loss for flow in open channel a dimensionless number $\frac{V}{\sqrt{gR}}$, has often been used, in which V is the mean velocity of flow and R the hydraulic radius. This form of Froude number was first proposed by Engel (13), he called this the Boussinesq number which he used in correlating his data on flow in a venturi flume. The argument (14) against this form of Froude number is that it does not have the physical meaning as an energy ratio, a force ratio, or a velocity ratio. Engel (15) argued that by the introduction of hydraulic radius, the effect of nonuniforn velocity distribution on the kinetic energy term can be corrected. Such an effect is more logically related to the Reynolds number.

The difficulty of interpreting this dimensionless number is due chiefly to the hydraulic radius which is a derived and hypothetical length. However, the use of a hypothetical length in Froude number is not new. For example, in case the channel is not rectangular, the average depth of flow is commonly used as the characteristic length in forming the Froude number. The average depth is defined as the total flow area divided by the top width. Whether the celerity of surface waves based upon an average depth has any physical meaning is not known.

A comparison between the use of hydraulics radius and the use of mean depths for flow in pipes is interesting. In the limit for pipe flowing full the Froude number based upon the mean depth is zero, because the mean depth of pipe flow is infinity since the top width is zero. However, it is not conceivable that the celerity of surface waves of flow in a pipe not flowing full depends upon only the top width od the flow, and is independent of the shape of the conduit. The hydraulic radius of a pipe flowing full is $\frac{1}{4}$ of the diameter. In case the hydraulic radius is used as the characteristic length in the Froude number, the Froude number of the pipe flow has a definite value but no physical significance.

It should be noted that the Froude criterion $\frac{V}{\sqrt{gL}}$ is based upon the condition that there exists geometric and kinematic similarity. Otherwise, the Froude criterion may not be adequate for dynamic similarity. For example, if the width-depth ratios of two open channels are not the same, the Froude criterion may not be sufficient to describe the similarity completely because the velocity distribution in the two systems may not be the same. The Boussinesq number has been introduced to correct the boundary effect on the velocity distribution. Hence, on the one hand, the Froude law is not exact and on the other hand the Boussinesq number is not truly a Froude criterion but may serve a purpose of correlating flow conditions of two or more systems.

The use of the Boussinesq number is to replace the Froude number. The physical meaning of the Boussinesq number for flow in open channels might be more appealing if it is written as the conbination of velocity V shear velocity and energy gradient,

$$\frac{V}{\sqrt{gR}} = \frac{V}{\sqrt{gRS}}\sqrt{S} = \frac{V}{V_*}\sqrt{S} \qquad (38)$$

which shows that the surface waves ffect not only the energy dissipation, but also on the velocity distribution, and the boundary shear. In case the bed roughness is not the same as the wall roughness, the wall effect on the flow can be approximately eliminated by substituting R_b for R in Eq 38, in which R_b is the hydraulic radius pertaining to the bed. This concept has been used by Liu and Hwang in their paper "A discharge formula for flow in straight alluvial channels"(15).

The use of the Boussinesq number is mainly in connection with flow energy loss. The application of Eq 38 can be shown by the following example. The energy loss for flow in pipes is normally written as

$$h_f = f\frac{L}{D}\frac{V^2}{2g} = \frac{f}{8}\frac{V^2}{gR}L \qquad (39)$$

in which f is a function of $\frac{VR}{\nu}$, and relative roughness. Substitution of $\frac{V^2}{gR}$ by $\frac{V^2}{gRS}\cdot S$ into Eq 39 does not introduce any new variable, since $S = \frac{h_f}{L}$. On the other hand, the coefficient f for flow in open channels is a function of $\frac{VR}{\nu}$, $\frac{V^2}{gR}$ and relative roughness. Substitution of $\frac{V^2}{gR}$ by $\frac{V^2}{gRS}\cdot S$ will give f as a function of $\frac{VR}{\nu}$ relative roughness, and s in which s is a factor representing the gravitational force.

Froude number of this type is quite often used for flow in open channels. Lacy called $\frac{V^2}{gR}$, a silt factor (16), and Blench called $\frac{V^2}{gD}$ (17) the bed factor.

<u>Alluvial channels</u>

Studies of alluvial channels involve the following parameter $\frac{\tau_b}{\Delta\gamma_s d}$ in which τ_b is the shear force per unit area pertaining to the movable bed $\Delta\gamma_s$ the difference in specific weight between sediment and fluid, and d the sediment size. This dimensionless number is used very often in studying sediment transport. It is similar to the type of Froude number mentioned above if the boundary shear is written

$$\tau_b = \rho V_*^2 \tag{40}$$

If the boundary condition and the velocity distribution is similar, the motion having the same value of this number will have similar relative motion between the fluid and the sediment. This number has been called the coefficient of tractive force by Shields (18), flow intensity by Einstein (19), and the "channel stability factor" by Bogardi (20). It has been used extensively in modelling movable beds.

<u>Fall velocity</u>

Study of sediment transported by fluid involves the following parameter $\frac{\rho W^2}{\Delta \gamma_s d}$ in which W is the terminal velocity of a falling particle, and d is the particle size. On account of the fact that the fall velocity is a function of the specific gravity and the size of the material, this parameter should be considered as a drag coefficient.

Reynolds Criterion

The Reynolds number normally involves only the molecular viscosity of the flow, the mean steady-state velocity, and a characteristic length. A more complete consideration of the Reynolds criterion, however, shows it to have two important areas of significance:

1. When the Reynolds number is not large, the dynamic similarity of flow can be obtained by maintaining the Reynolds number of the flow the same.

2. At large Reynolds numbers for which the flow is not only fully turbulent, but the Reynolds stresses also are large compared with the mean viscous stresses. (Similarity of flow under these conditions is considered by Birkhoff (2) to be inertial modelling.)

   At large Reynolds numbers, the mean motion, and the motion of the energy-containing components of the turbulence, are determined by the boundary conditions of the flow alone and are independent of the fluid viscosity. This principle involves the hypothesis that the flow structure is similar at all large Reynolds numbers.

The general form of the Reynolds number $\frac{VL}{\nu}$ indicates that it may be similar to the ratio of the eddy viscosity to the molecular viscosity. This is reasonable because the intensity of turbulence $\sqrt{\overline{v'^2}}$ can be considered proportional to the mean velocity of flow V, and the scale of turbulence ℓ depends upon the scale of model L.

It is not easy to model large Reynolds numbers on a small scale. Difficulty arises because, for a given fluid, reducing the scale requires increasing velocities in the same ratio. Unvortunately, there seems to be no liquid for which kinematic viscosity is much less than that of water although many have much greater kinematic viscosity. Hence, wind tunnels are known to provide the only economical Reynolds models of phenomena of water flow.

In using the Reynolds criterion for modelling, two precautions must be taken.

1. The models must have a similar surface roughness to insure complete geometric similarity. The onset of turbulent flow, and the transition of the boundary layer from laminar to turbulent flow, are greatly influenced by this factor. For example, the critical Reynolds number of the drag of a sphere can be greatly reduced by roughening it suitably.

2. The turbulence of the free stream must be the same in the model as in the

prototype. For example, it has been found that the critical Reynolds number of the sphere in wind tunnels, can vary by a factor of two depending on the turbulence of the wind tunnel.

One important phenomenon in Reynolds modelling is separation. When separation occurs, the approaching flow separates from the boundary. Consequently the velocity distribution and drag along the boundary are affected appreciably. Changes in the point of separation generally cause the pressure drag to be increased and the shear drag to be reduced, or vice-versa. Fully-developed separation involves the formation of eddies in the zone of separation which are associated with increased energy loss. For turbulent flow along a boundary, the separation phenomenon is responsible for changing the smooth boundary to a rough boundary.

In view of the foregoing facts, it is important in Reynolds modelling to maintain complete similarity of the shape, the location, and the flow pattern of the separation zone. For flow around sharp corners of a body, the point or points of separation together with the associated flow are rather fixed and easily predicted. Therefore, the modelling of separation under these conditions does not present a great problem. For flow around a blunt rounded body, however, the shape and location of the separation zone may change appreciably with Reynolds number--so that modelling special techniques must be employed to overcome such difficulties.

As indicated in the foregoing, a complete understanding of the Reynolds similarity requires a knowledge of turbulence structure and the boundary layer theory. Without such information, the investigator frequently finds himself in a very confusing position. The following is a discussion of transitions and velocity distributions involved in modelling. The reader is referred to standard references on these subjects as a supplement. (1) (6) (22)

Transitions involved in modelling problems

There are three types of transition problems involved in Reynolds modelling.

a. Zone of flow establishment

b. Laminar to turbulent flow

c. Hydraulically smooth to hydraulically rough boundaries.

The first involves the development of the boundary layer with respect to distance in the direction of flow, the second involves the nature of the flow within the boundary layer, and the third depends upon the relationship between the flow and the boundary. These three transitions are interrelated with each other and the transition problem is further related to velocity distribution and ultimately to resistance and drag coefficients.

Zone of flow establishment: - The zone of flow establishment is the region in which the characteristics of flow such as velocity distribution, turbulence and energy loss are changing along the flow direction. For complete dynamic similarity in the model, both the zone of flow establishment and that of established flow must be made similar to those of the prototype. In general, the characteristics of flow in the zone of flow establishment depend upon the history of the flow and the entrance condition. Specifically, they depend upon the boundary geometry, and scale, the velocity and the properties of the fluid. Modelling of the zone of flow establishment is usually more difficult than modelling of the established flow. Frequently in the prototype the length of flow is so long that the zone of flow establishment is insignificant compared with the length of the zone of established flow. In this case the model design is usually concerned with modifying the influence of the zone of flow establishment. Sometimes this can be achieved to some extent by increasing the roughness of the approach boundary and the scale and intensity of the turbulence in the approaching flow.

For pipe flow, the zone of established flow begins at the section where the boundary layer reaches the center line of the pipe. The boundary layer can be either laminar or turbulent. For flow in wide open channels, the zone of established flow begins at the section where the boundary layer reaches the water surface. In order to predict the zone of flow establishment and the zone of established flow, a thorough knowledge of the boundary layer growth is necessary.

Laminar to turbulent flow: - In general, prototype flow encountered in hydraulic engineering is turbulent. Therefore, because of the small scale of the model, one of the important requirements of a model study is to insure dynamically similar flow conditions. The critertion for transition from laminar to turbulent flow in a pipe was found by Reynolds (3). If the pipe diameter D is used as the characteristic length in the Reynolds number, the critical Reynolds number is approximately 2,000. If the hydraulic radius is used as the characteristic length, then the critical number is about 500. For Reynolds numbers less than these, the flow is laminar and turbulent is not possible. However, the upper limit of Reynolds number, beyond which the flow is alway turbulent and laminar flow is not possible, is extremely variable depending upon the internal movement within the fluid. Reynolds numbers as large as 75,000 (25) have been found possible for laminar flow--largely as the result of increasing the stilling time, improving the rounding of the inlet, and eliminating all possible vibrations. This upper limit of Reynolds number is important because it shows that the flow is not necessarily turbulent even when R_e is greater than 2,000.

Although various conclusions about the critical Reynolds number for pipe flow have been reported, the transition problem from laminar to turbulent flow in open channels has not been thoroughly investigated. For two-dimensional open-channel flow, the depth of flow is the same as the hydraulic radius. Therefore, the hydraulic radius used in pipe flow can be substituted for the depth in two-dimensional open-channel flow.

Owen (26) found that the lower critical Reynolds number for open-channel flow is 4,000. Straub (27) and others reported that the range of transition of the Reynolds number for a rectangular channel is from 2,800 to 3,000. Other experimenters have found somewhat different values and have conclusively shown that the precise value of the Reynolds number at which turbulent flow becomes established is variable--depending upon the shape of the cross-section and the free turbulence or movement within the fluid.

The characterisitcs of boundary-layer development of flow along a flat plate have been studied extensively. The results of these studies can be used as a guide to turbulent flow along a rigid boundary. It has been found(22) that in a laminar boundary layer along a plate with zero pressure gradient turbulent motion is incipient at certain Reynolds number $\left(\frac{V_\infty x}{\nu}\right)_c$ in which V_∞ is the velocity of the free stream outside the boundary layer, and δ is the thickness of the laminar layer. Prandtl (21) has shown that the Reynolds number $(\frac{V_\infty \delta}{\nu})_c$ in the case of pipe flow is of the same order of magnitude as the critical value $(\frac{V_o D}{\nu})_c$ for transition from laminar to turbulent flow. It can be reasoned that the mean velocity of a laminar flow Vo is equal to one half its maximum velocity which corresponds to the free stream velocity V_α $(V_o = \frac{1}{2} V_\alpha)$ and the maximum thickness of laminar boundary layer is equal to the pipe radius $(D = 2\delta)$

Because the growth of the laminar boundary layer depends upon the distance from the entrance, there is a distance Reynolds number $(\frac{V_\alpha x}{\nu})_c$ corresponding $(\frac{V_\alpha \delta}{\nu})_c$ at which laminar flow becomes turbulent. In this number, x is the distance from the leading edge. According to Schlichting $(\frac{V_\alpha x}{\nu})$ varies from about 3×10^5 to 5×10^5. Prandtl (21) has shown how to estimate the distance x which is the required transition length for a given Reynolds number $\frac{V_o D}{\nu}$ of the flow.

<u>Transition from hydraulically smooth to hydraulically rough boundaries</u>:--The criterion $\frac{V_o D}{\nu}$ can be used to determine whether the main flow along a boundary is laminar or turbulent, but it cannot be used to determine the effect of the laminar sub-layer on the velocity distribution. In case of turbulent flow near a boundary there may exist a laminar layer known as the laminar sub-layer δ'. The thickness of the laminar sub-layer can be estimated by use of the following equation.

$$\delta' = 11.6 \frac{\nu}{V_*} \tag{41}$$

According to Nikuradse's data for pipe flow (6);

1. The boundary is hydraulically smooth is $\frac{V_* k}{\nu} < 3.5$ that is, approximately $\frac{k}{\delta'} < \frac{1}{4}$ in which k is the roughness height.
2. The boundary is hydrualically rough if $\frac{V_* k}{\nu} > 70$, that is approximately $\frac{k}{\delta'} > 6$.

This concept of boundary roughness is based upon Karman's hypothesis of turbulent flow near a boundary. Therefore the Reynolds number $\frac{V_* k}{\nu}$ has been called by Danel (23) the Karman number. In modelling the boundary roughness, the Karman number $\frac{V_* k}{\nu}$ should be used. The boundary of prototype flow can usually be considered as hydraulically rough. Therefore, precaution should be taken to make the boundary condition in the model also hydraulically rough. In this case, the boundary should be modelled according to the relative roughness $\frac{K}{D}$.

Velocity distributions near a boundary in model and prototype

Reynolds law of similitude is necessary in modelling velocity distribution near a boundary. In laminar flow there are many cases in which the velocity distribution can be found by solving the Navier-stokes equation by approximation for given boundary conditions. In the case of turbulent flow, the velocity distribution for certain boundary conditions has been found mostly by laboratory investigation with the aid of dimensional analysis, this has resulted in the "Wall-law" and the "Velocity-defect law" (24) In any turbulent flow near a solid boundary, there is a region adjacent ot the wall within which the total shear stress is nearly constant and the motion is determined almost entirely by the shear stress and the fluid viscosity. This is known as the "Wall law" which is due to Prandtl by use of dimensional analysis, and can be written as

$$\frac{u}{V_*} = f_1\left(\frac{V_* y}{\nu}, \frac{y}{K}\right) \qquad y \to 0 \tag{42}$$

in which K is the height of the roughness, y is measured from the boundary. For a smooth boundary, it reduces to

$$\frac{u}{V_*} = f_2\left(\frac{V_* y}{\nu}\right) \qquad y \to 0 \tag{43}$$

The velocity-defect law is applicable near the outer edge of the boundary layer. The general form of the velocity defect law is

$$\frac{u - V}{V_*} = f_3\left(\frac{y}{\delta_t}\right) \tag{44}$$

in which u is the local velocity measured at a distance y from the boundary, δ_t the thickness of the turbulent boundary layer. This law has been attributed to Karman (1).

In the over-lapping zone in which both the "Wall-law" and the "velocity-defect law" are applicable, the logarithmic law of velocity distribution is known to be applicable that is

$$\frac{u}{V_*} = \frac{1}{K} \ln \frac{y}{y'} \tag{45}$$

in which K is the so called universal constant and y' depends upon the boundary roughness, Although the logarithmic law was first discovered by Prandtl by use of his mixing length hypothesis, it has been shown by Clauser (24) that it can also be derived by comparing the "Wall-law" and the "velocity-deficit law". Various forms of formulas have been proposed to suit various velocity measurements. However, there is no single formula which can be used to decide the velocity distribution for turbulent flow near a boundary. Fortunately, the logarithmic law can be used for a large part of the flow near a boundary. Equation (45) can be expanded to various forms depending upon the boundary roughness which can be classified as smooth, transition, and rough according to the Karman number $\frac{V_* k}{\nu}$. Detail discussion of the logarithmic law can be found elsewhere (6).

Particular attention should be given to the resistance coefficient f as a function of Reynolds number $\frac{VR}{\nu}$ and relative roughness $\frac{k}{D}$. A thorough understanding of this relationship is helpful for interpreting data concerning resistance of flow in conduits.

The constants involved in the logarithmic law for turbulent flow in pipes have been determined by Nikuradse and accepted by scientists as standard values. Although the application of Equation (45) to flows in open channels has been partly successful,(28) there are some uncertainties about the constants and coefficients. Uncertainties are due to many factors, such as the presence of a free surface, surface waves and irregular and sometimes movable boundary conditions for example, sand dunes and vegetable growth along the banks. Despite the fact that the great strides are being made on the velocity distribution of open-channel flow, study should be not only continued but also intensified.

Drag coefficient

Another type of flow in which the Reynolds similarity is important is related to the drag of submerged bodies. Considerable experimental and theoretical analyses have been done for two kinds of immersed bodies, namely cylinders and spheres. In these cases, the velocity of the approaching flow, V_o, the size of the cylinder or sphere d, and kinematic viscosity of the flow ν are often used to obtain the Reynolds number $\frac{V_o d}{\nu}$. For very small Reynolds numbers the interial effect is negligible and the flow is purely laminar. For laminar flow around spheres, the Stokes law is valid which can be written as

$$F = 3\pi d \mu V_o = C_D A \frac{1}{2} \rho V_o^2 \tag{46}$$

therefore

$$C_D = \frac{24}{\frac{V_o d}{\nu}} \tag{47}$$

For very large Reynold's numbers between 2×10^4 and 2×10^5 the drag coefficient is approximately constant at $C_D = 0.5$. At the Reynold's number of approximately 2×10^5 the drag coefficient drops abruptly. Prandtl has demonstrated that such change of drag coefficient is due to the free turbulence, and the roughness of the surface. Considerable discussion on drag coefficient of spheres and other forms of bodies has been presented in textbooks on Fluid Mechanics and hence will not be represented here. Again, however, a thorough understanding of the fundamental principles related to such items as:

1. Distribution of shear and pressure, their relation to drag, and their relative importance for a given flow system.
2. Formation and influence of separation zones
3. Types and influence of boundary roughness are absolutely essential to the proper design and interpretation of model studies

Simultaneous Froude and Reynolds Criteria

According to the Reynolds criterion, a reduction of the scale requires an increase of velocity. On the other hand, according to the Froude criterion a reduction of scale requires a reduction of velocity. It is evident that the simultaneous fulfillment of the two similitude criteria with the same fluid is impossible. That is, for the same fluid there cannot exist a law of similarity considering inertia forces, frictional forces, and gravity forces simultaneously. Using two fluids of different kinematic viscosity, however, it is possible theoretically to make both laws of similarity valid as shown in the following:

Froude law $$V_r = \sqrt{\frac{\gamma_r L_r}{\rho_r}} \qquad (19a)$$

Reynolds law $$V_r = \frac{\mu_r}{L_r \rho_r} \qquad (17a)$$

Equating Eq. 19a and 17a $$\frac{\mu_r}{L_r \rho_r} = \sqrt{\frac{\gamma_r L_r}{\rho_r}}$$

$$\frac{\mu_r^2}{\rho_r} = g_r L_r^3$$

Since the gravitational conditions for model and prototype are usually the same, the gravitational factor becomes one and can be dropped from the foregoing equation. Therefore, the dynamic viscosities and the densities of the two fluids bear the relationship

$$\frac{\mu_r^2}{\rho_r} = L_r^3 \qquad (41)$$

Despite this relationship which shows that modelling by both the Froude and Reynolds laws is theoretically possible, it is practically impossible because fluids of the required viscosity and density do not exist.

Various techniques can be employed in a given model study to meet the difficult problem of Reynolds number and Froude number being significant simultaneously. The first step is to establish the ranges of conditions in both the model and the prototype for which:

1. Reynolds number is of primary importance.

2. Froude number is of primary importance, and
3. Both Reynolds and Froude numbers are important

Ranges 1 and 2 cause relatively little difficulty. In Range 3 an attempt can be made to:

1. Compute by theory the influence of Reynolds number and subtract this from the total combined effect.
2. Eliminate or reduce the Reynolds effect by destroying the laminar sublayer with excessive roughness or turbulence, or by bleeding it off just upstream from the test area.
3. Create the same viscous effect in the model by other means such as distorting certain parts of the model or installing special control systems.
4. Extrapolate Ranges 1 and 2 into the transition Range 3 to determine the order of magnitude of effect of Reynolds number and Froude number separately.

The following are a few examples for which both the Reynolds and Froude criteria are significant.

Ship models:- A ship experiences both shear drag and pressure drag. The shear drag follows very closely with that which is predicted from the boundary layer theory. The pressure drag, however, arises from two sources--the shape of the ship and the influence of the waves. The pressure caused by waves readily follows the Froude criterion, but the shear drag depends upon the Reynolds number. Since these cannot both be made the same in the model as in the prototype, the drag on the model is determined for the combination of the shear and pressure drag and then the shear drag is calculated and subtracted from the total drag to determine the pressure drag only. This pressure drag can then be changed to the pressure drag for the prototype and a new shear drag, computed for the prototype, can be added to the prototype pressure drag to determine the prototype total drag. This technique has been developed so that total drag can be determined.

Sediment transport: In this case, the surface waves in natural channels and the interface between the movable bed and the flow are usually important; therefore, factors containing gravitational effect $\frac{V}{\sqrt{gR}}$, $\frac{\tau_b}{\Delta\gamma_s d}$ becomes involved. On the other hand, there may exist a laminar sublayer along the stream bottom (for example along the upstream face of a long sand bar) so that the viscous effect also may be important. Furthermore, in the case of fine sediment transported, the Reynolds number of the sediment is also important. The model technique for studying the sediment transport and scour problems for natural streams are very complicated. Additional research is badly needed on this subject.

Conduits associated with free-surface flow: Frequently, structures (such as a dam) need to be studied for which both free-surface and closed conduit flow are involved. In this case, the Froude number is usually assumed to be of greatest significance and special steps must be taken to compensate for Reynolds number effects. This is usually done by adjusting the length and size of the conduit in accordance with computations based on standard information regarding flow in closed conduits. Consequently, steps must be taken to compensate for this situation. Generally, the forces of gravity and predominate in free-surface flow and Froude number governs. In the prototype the surface energy is usually insignificant because of the large size of flow system. Furthermore, the Reynolds number is frequently insignificant in the prototype because the laminar sublayer is destroyed as a result of the large Reynolds number and/or the extreme roughness. In a model study, the model must be sufficiently large so that the influence of surface energy is not significant. Eliminating the influence of the Reynolds number in the model, however, is not as easy as eliminating the Weber number. Furthermore, reproducing the same relative roughness in the model as in the prototype is not always convenient. These various factors should be considered with respect to specific model studies.

SUMMARY

Models in which either the Reynolds criterion or the Froude criterion or both are significant are the most common in hydraulic engineering. Success in coping with problems associated with such models depend not only upon familarity of similitude principles but in large measure upon a sound and complete understanding of the basic principles of fluid mechnics involved. These include principles of free-surface flow, wave motion, interfacial flow, laminar and turbulent flow, laminar and turbulent boundary layers, smooth and rough boundaries, transitions from one condition to another, separation zones, closed conduit flow, open channel flow, sediment transport, jet diffussion and wake problems.

On the basis of fundamental principles of fluid mechanics, special techniques can be employed to modify or eliminate the effect of one similarity criterion, such as Reynolds number when more than one is significant.

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