

Optimal asymptotic bounds on the oracle use in computations from Chaitin's Omega*

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Abstract. Chaitin's number Ω is the halting probability of a universal prefix-free machine, and although it depends on the underlying enumeration of prefix-free machines, it is always Turing-complete. It can be observed, in fact, that for every computably enumerable (c.e.) real α , there exists a Turing functional via which Ω computes α , and such that the number of bits of Ω that are needed for the computation of the first n bits of α (i.e. the *use* on argument n) is bounded above by a computable function $h(n) = n + o(n)$.

We characterise the asymptotic upper bounds on the use of Chaitin's Ω in oracle computations of halting probabilities (i.e. c.e. reals). We show that the following two conditions are equivalent for any computable function h such that $h(n) - n$ is non-decreasing: (1) $h(n) - n$ is an information content measure, i.e. the series $\sum_n 2^{n-h(n)}$ converges, (2) for every c.e. real α there exists a Turing functional via which Ω computes α with use bounded by h . We also give a similar characterisation with respect to computations of c.e. sets from Ω , by showing that the following are equivalent for any computable non-decreasing function g : (1) g is an information-content measure, (2) for every c.e. set A , Ω computes A with use bounded by g . Further results and some connections with Solovay functions (studied by a number of authors [Sol75, BD09, HKM09, BMN11]) are given.

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1 Introduction

Chaitin's constant is the most well-known algorithmically random number. It is the probability that a universal prefix-free machine U halts, when a random sequence of 0s and 1s is fed into the input tape. In symbols:

$$\Omega_U = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}.$$

While Ω_U clearly depends on the choice of underlying universal prefix-free machine U , all versions of it are quite similar with respect to the properties we shall be concerned with here. For this reason, when the choice of underlying machine U is irrelevant, we shall simply refer to Ω_U as an Ω -number, or denote it by Ω . The choice of U is essentially equivalent to the choice of a programming language. Although Ω is uncomputable, it is the limit of an increasing computable sequence of rational numbers. Such reals are known as computably enumerable reals (c.e. reals, a more general class than the c.e. *sets*), and can be viewed as the halting probabilities of (not necessarily universal) prefix-free machines.

The binary expansion of Ω is an algorithmically unpredictable sequence of 0s and 1s, which packs a lot of useful information into rather short initial segments. In the words of Charles H. Bennett (also quoted by Martin Gardner in [Gar79]),

[Chaitin's constant] embodies an enormous amount of wisdom in a very small space . . . inasmuch as its first few thousands digits, which could be written on a small piece of paper, contain the answers to more mathematical questions than could be written down in the entire universe. . .

Charles H. Bennett [Ben79]

Being of the same Turing degree, Ω may be seen as containing the same information as the halting problem, but it contains this same information in a much more compact way. It is Turing complete, so it computes all c.e. sets and all c.e. reals. As one might expect, given these remarkable properties, it is very hard to obtain initial segments of Ω . The modulus of convergence in any monotone computable approximation to Ω , for example, dominates all computable functions. Chaitin's incompleteness theorem [Cha75] (also independently proved by Muchnik) tells us that any computably axiomatizable theory interpreting Peano arithmetic can determine the value of only finitely many bits of Ω . Solovay [Sol00] exhibited a specific universal prefix-free machine U with respect to which ZFC cannot determine *any* bit of Ω_U . Despite these extreme negative results, Calude, Dinneen and Shu [CDS02, CD07] computed a finite number of bits for certain versions of Ω (i.e. corresponding to certain canonical enumerations of the prefix-free Turing machines).

Given the Turing completeness of Ω and the extreme difficulty of obtaining the bits of its binary expansion, it is interesting to consider the lengths of the initial segments of Ω that are used in oracle computations of c.e. sets and c.e. reals. How many bits of Ω , for example, are needed in order to compute n bits of a given c.e. real or a given c.e. set? The goal of this paper is to answer these questions, as well raising and discussing a number of related issues that point to an interesting research direction, with clear connections to contemporary research in algorithmic randomness.

1.1 Our main results, in context

We review some standard notation and terminology. The positions in the binary expansion of a real (to the right of the decimal point) are numbered $1, 2, 3, \dots$ from left to right. Given a real α , we let $\alpha(t)$ denote the value of the digit of α at position $t + 1$ of its binary expansion. We identify infinite binary sequences with binary expansions of reals in $[0, 1]$ and with subsets of the natural numbers \mathbb{N} . Subsets of \mathbb{N} are denoted by upper-case latin letters while lower-case Greek letters are used for the reals in $[0, 1]$ (the fact that dyadic rationals have two expansions will not be an issue here). The *use*-function corresponding to an oracle computation of a set A , is the function whose value on argument n is the length of the initial segment of the oracle tape that is read during the computation of the first n bits of A . Given $A \subseteq \mathbb{N}$, we let $A \upharpoonright_n$ denote the initial segment of the characteristic function of A of length n . The n -bit prefix of a real number α is also denoted by $\alpha \upharpoonright_n$. Some background material on algorithmic randomness is given in Section 1.3.

Definition 1.1 (Use of computations). *Given a function h , we say that a set A is computable from a set B with use bounded by h if there exists an oracle Turing machine which, given any n and $B \upharpoonright_{h(n)}$ in the oracle tape, outputs the first n bits of A .*

Use functions are typically assumed to be nondecreasing. We say that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is right-c.e. if there is a computable function $f_0 : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ which is non-increasing in the second argument and such that $f(n) = \lim_s f_0(n, s)$ for each n . Our results connect use-functions with *information content measures*, as these were defined by Chaitin [Cha87]. These are right-c.e. functions f with the property that $\sum_i 2^{-f(i)}$ converges. This notion is fundamental in the theory of algorithmic randomness, and is the same as the c.e. discrete semi-measures of Levin [Lev74], which were also used in Solomonoff [Sol64]. For example, prefix-free complexity can be characterized as a minimal information content measure (by [Cha87, Lev74]). Throughout this paper the reader may consider $\log n$ as a typical example of a computable non-decreasing function f such that $\sum_i 2^{-f(i)}$ diverges, and $2 \log n$ or $(\epsilon + 1) \log n$ for any $\epsilon > 0$ as an example of a function f such that $\sum_i 2^{-f(i)}$ converges.

Our results address computations of (a) c.e. sets and (b) c.e. reals. In a certain sense c.e. reals can be seen as compressed versions of c.e. sets. Indeed, as we explain at the beginning of Section 2, a c.e. real α can be coded into a c.e. set X_α in such a way that $X_\alpha \upharpoonright_{2^n}$ can compute $\alpha \upharpoonright_n$ and $\alpha \upharpoonright_n$ can compute $X_\alpha \upharpoonright_{2^n}$. So it is not surprising that our results for the two cases are analogous. Computations of c.e. sets (from Ω) correspond to use-functions g which are information content measures, while computations of c.e. reals correspond to use-functions of the form $n + g(n)$, where g is an information content measure.

Theorem 1.2 (Computing c.e. reals). *Let h be computable and such that $h(n) - n$ is non-decreasing.*

- (1) *If $\sum_n 2^{n-h(n)}$ converges, then for any Ω -number Ω and any c.e. real α , α can be computed from Ω with use bounded by h . This also holds without the monotonicity assumption on $h(n) - n$.*
- (2) *If $\sum_n 2^{n-h(n)}$ diverges then no Ω -number can compute all c.e. reals with use $h + \mathbf{O}(1)$. In fact, in this case there exist two c.e. reals such that no c.e. real can compute both of them with use $h + \mathbf{O}(1)$.*

A considerable amount of work has been devoted to the study of computations with use-function $n \mapsto n + c$ for some constant c . Note that these are a specific case of the second clause of Theorem 1.2. These types of reductions are called *computably-Lipschitz*, and were introduced by Downey, Hirschfeldt and Laforte in [DHL04] and have been studied in [DY04, Bar05, BL06a, BL06c, BL06b, BL07] as well as more recently in [BDG10, Day10, ASDFM11, Los15]. The second clause of Theorem 1.2 generalises the result of Yu and Ding in [DY04] which assumes that $h(n) = n + \mathbf{O}(1)$. It would be an interesting project to generalise some

of the work that has been done for computably-Lipschitz reductions to Turing-reductions with computable use h such that $\sum_i 2^{n-h(n)}$ diverges. We believe that in many cases this is possible. The second clause of Theorem 1.2 can be viewed as an example of this larger project.

If, as we discussed earlier, c.e. reals are just compressed versions of c.e. sets, then what are the c.e. sets that correspond to Ω numbers? In Section 1.2 (with the proofs deferred to Section 2) we discuss this duality and point out that the linearly-complete c.e. sets could be seen as the analogues of Ω in the c.e. sets. Recall that the linearly complete c.e. sets are those c.e. sets that compute all c.e. sets with use $\mathbf{O}(n)$.¹ A natural example of a linearly-complete set is the set of non-random strings, $\{\sigma \mid C(\sigma) < |\sigma|\}$ (where $C(\sigma)$ denotes the plain Kolmogorov complexity of the string σ) which was introduced by Kolmogorov [Kol65]. This latter set can be viewed as a set of numbers with respect to the bijection that corresponds to the usual ordering of strings, first by length and then lexicographically. In [BHL13] it was shown that a c.e. set A is linearly-complete if and only if $C(A \upharpoonright_n) \geq \log n - c$ for some constant c and all n .

Theorem 1.3 (Computing c.e. sets). *Let g be a computable function.*

- (1) *If $\sum_n 2^{-g(n)}$ converges, then every c.e. set is computable from any Ω -number with use g .*
- (2) *If $\sum_n 2^{-g(n)}$ diverges and g is nondecreasing, then no Ω -number can compute all c.e. sets with use g . In fact, in this case, no linearly complete c.e. set can be computed by any c.e. real with use g .*

Theorem 1.3 is an analogue of Theorem 1.2 for computations of c.e. sets from Ω . Beyond the apparent analogy, the two theorems are significantly connected. Theorem 1.3 will be used in the proof of Theorem 1.2, and this justifies our approach of studying c.e. reals as compressed versions of c.e. sets.

A natural class of a linearly complete c.e. sets are the halting sets, with respect to a *Kolmogorov numbering*. A numbering (M_e) of plain Turing machines is a Kolmogorov numbering if there exists a computable function f and a machine U such that $U(f(e, n)) \simeq M_e(n)$ for each e, n (where ‘ \simeq ’ means that either both sides are defined and are equal, or both sides are undefined) and for each e the function $n \mapsto f(e, n)$ is $\mathbf{O}(n)$. Kolmogorov numberings, in a sense, correspond to optimal simulations of Turing machines and are the basis for the theory of plain Kolmogorov complexity, and have been studied in [Lyn74, Sch75].

The following result can be viewed in the context of Solovay functions, which stem from Solovay’s work [Sol75] on Ω -numbers and were extensively studied by Bienvenu and Downey [BD09], Hölzl, Kräling and Merkle [HKM09], and Bienvenu, Merkle and Nies [BMN11]. A Solovay function is a computable information content measure g such that $\sum_i 2^{-g(i)}$ is 1-random. As we discussed above, every information content measure is an upper bound on prefix-free Kolmogorov complexity (modulo an additive constant). By [BMN11], Solovay functions g are exactly the computable *tight upper bounds* of the prefix-free complexity function $n \mapsto K(n)$, i.e. a computable function g is a Solovay function if and only if $K(n) \leq g(n) + \mathbf{O}(1)$ for all n and $g(n) \leq K(n) + \mathbf{O}(1)$ for infinitely many n .

Theorem 1.4 (Solovay functions as tight upper bounds on oracle use). *Let g be a nondecreasing computable function such that $\sum_n 2^{-g(n)}$ converges to a 1-random real. Then a c.e. real is 1-random if and only if it computes the halting problem with respect to a Kolmogorov numbering with use g .*

This result is another way to see the halting sets with respect to Kolmogorov numberings as analogues of Chaitin’s Ω in the c.e. sets. Moreover, it gives a characterization of monotone Solovay functions in terms

¹We note that this is different than what was defined to be linear reducibility in [BL07], which was later more commonly known as computably Lipschitz reducibility.

of the oracle-use for certain computations between c.e. sets and c.e. reals.

Finally we obtain a gap theorem characterizing the array computable degrees, in the spirit of Kummer [Kum96]. Recall from [DJS96] that a Turing degree \mathbf{a} is array computable if there function f which is computable from the halting problem with a computable upper bound on the use of the oracle, and which dominates every function g which is computable in \mathbf{a} . Array computable degrees play a significant role in classical computability theory. Kummer showed that they also relate to initial segment complexity. He showed that the initial segment plain complexity $C(A \upharpoonright_n)$ of every c.e. set A in an array computable degree is bounded above (modulo an additive constant) by any monotone function f such that $\lim_n(f(n) - \log n) = \infty$. On the other hand, he also showed that every array noncomputable degree contains a c.e. set A such that $C(A \upharpoonright_n)$ is not bounded above by any function f such that $\lim_n(2 \log n - f(n)) = \infty$. This is known as Kummer's gap theorem (see [DH10, Section 16.1]) since it characterizes array computability of the c.e. degrees in terms of a logarithmic gap on the initial segment plain complexity. The following is another logarithmic gap theorem characterizing array computability in the c.e. degrees, but this time the gap refers to the oracle use in computations from Chaitin's omega.

Theorem 1.5. *Let \mathbf{a} be a c.e. Turing degree and let Ω be (an instance of) Chaitin's halting probability.*

- (1) *If \mathbf{a} is array computable, then every real in \mathbf{a} is computable from Ω with identity use.*
- (2) *Otherwise there exists a c.e. real in \mathbf{a} which is not computable from Ω with use $n + \log n$.*

This result is largely a corollary of the work in [BDG10, Sections 4.2, 4.3] applied to the construction in the proof of Theorem 1.2 (1).

1.2 Omega numbers and completeness

This paper concerns certain aspects of completeness of Ω , so it seems appropriate to discuss the completeness of Ω a little more generally. The fact that Ω computes all c.e. reals is referred to as the *Turing-completeness* of Ω (with respect to c.e. reals), and has been known since Ω was first defined. Calude and Nies observed in [CN97] that there is a computable bound on the oracle use in computations of c.e. reals and c.e. sets from Ω . The results of Section 1.1 give a sharp characterisation of these computable bounds. It is a reasonable question as to whether Ω -numbers can be characterised as the complete c.e. reals with respect to Turing reductions with appropriately bounded use. In terms of the strong reducibilities of classical computability theory, this question has a negative answer. Figueira, Stephan, and Wu showed in [FSW06], for example, that there are two Ω -numbers which do not have the same truth-table degree. Stephan (see [BDG10, Section 6] for a proof) showed that given any Ω -number, there is another Ω -number which is not computable from the first with use $n + \mathbf{O}(1)$.

Putting this question aside, there are a number of characterisations of the Ω -numbers as the complete elements of the set of c.e. reals with respect to certain (weak) reducibilities relating to algorithmic randomness. Downey, Hirschfeldt and Laforte studied and compared these and other reducibilities in [DHL04], and a good presentation of this study can also be found in the monograph of Downey and Hirschfeldt [DH10, Chapter 9]. One of these reducibilities that pertains to the following discussion is the rK reducibility, denoted by $A \leq_{rK} B$ which is the relation that $K(A \upharpoonright_n \mid B \upharpoonright_n) = \mathbf{O}(1)$ for all n . We note that by Downey, Hirschfeldt and Laforte [DHL04], this relation implies that A is computable in B , in the sense of Turing oracle computability. The following fact highlights the correspondence between Ω -numbers and linearly

complete c.e. sets. Here $K(\Omega \upharpoonright_n \mid H \upharpoonright_{g(n)})$ denotes the prefix-free complexity of $\Omega \upharpoonright_n$ relative to $H \upharpoonright_{g(n)}$, and $K(\Omega \upharpoonright_n \mid H \upharpoonright_{g(n)}) = \mathbf{O}(1)$ can be seen as a nonuniform computation of Ω from H (see Section 1.3 for definitions relating to complexity).

Theorem 1.6. *Consider a 1-random c.e. real Ω , a c.e. set H , a computable function g and a right-c.e. function f such that $\sum_i 2^{-f(i)}$ is finite.*

- (i) *If $K(\Omega \upharpoonright_n \mid H \upharpoonright_{g(n)}) = \mathbf{O}(1)$ then $2^n = \mathbf{O}(g)$.*
- (ii) *If $g = \mathbf{O}(2^n)$ and $K(\Omega \upharpoonright_{n+f(n)} \mid H \upharpoonright_{g(n)}) = \mathbf{O}(1)$ then H is linearly-complete.*
- (iii) *If $g = \mathbf{O}(n)$ and $K(\Omega \upharpoonright_{f(n)} \mid H \upharpoonright_{g(n)}) = \mathbf{O}(1)$ then H is linearly-complete.*
- (iv) *If H is linearly-complete then it computes Ω (and any c.e. real) with use $2^{n+\mathbf{O}(1)}$.*

In order to understand this fact better, note that if Ω is computed by a c.e. set H with computable use h , then we have $K(\Omega \upharpoonright_n \mid H \upharpoonright_{h(n)}) = \mathbf{O}(1)$ for all n . So the first clause of Theorem 1.6 says that the tightest use in a computation of Ω from a c.e. set is exponential, even when the computation is done non-uniformly. The fourth clause asserts that the linearly-complete c.e. sets achieve this lower bound on the use in computations of Ω . In this sense, they can be viewed as analogues of Ω . There are more facts supporting this suggestion. In [BHL13], for example, it was shown that a c.e. set H is linearly complete if and only if $K(W \upharpoonright_n \mid H \upharpoonright_n) = \mathbf{O}(1)$ for all n and all c.e. sets W . By the basic properties of Kolmogorov complexity, prefix-free complexity in the previous statement and in Theorem 1.6 can be replaced by plain Kolmogorov complexity (e.g. see [DHL04]).

It is also instructive to compare clauses (ii) and (iii) of Theorem 1.6 with an older result of Solovay. For each n let D_n be the set of strings of length at most n , in the domain of the universal prefix-free machine. Solovay [Sol75] (see [DH10, Section 3.13]) showed that the cardinality of D_n is proportional to $2^{n-K(n)}$, that $K(\Omega \upharpoonright_n \mid D_{n+K(n)}) = \mathbf{O}(1)$ and that $K(D_n \mid \Omega \upharpoonright_n) = \mathbf{O}(1)$.

We conclude this section with a brief discussion of some results from Tadaki [Tad09] which came to our attention only recently. Tadaki's results are incomparable with those presented here, but relate to the results from this paper concerning reductions between Ω and c.e. sets (rather than c.e. reals). He considered the problem of how many bits of Ω are needed in order to compute the domain of the universal prefix-free machine U up to the strings of length n ; and vice-versa, what is the least number m such that, in general, we can compute the first n bits of Ω from the domain of U restricted to the strings of length at most m . Tadaki showed that, if we restrict the question to computable use-functions,

- (a) the answer to the first question is given by the computable functions of the type $n - f(n) + \mathbf{O}(1)$ such that $\sum_n 2^{-f(n)}$ is finite.
- (b) the answer to the second question is given by the computable functions of the type $n + f(n) + \mathbf{O}(1)$ such that $\sum_n 2^{-f(n)}$ is finite.

These optimal results are quite pleasing, and were subsequently used by Tadaki [Tad11a] in order to obtain a statistical mechanics interpretation of algorithmic information theory, which was further developed in [Tad11b, Tad12]. The main difference here is that the c.e. sets considered by Tadaki are the domains of universal prefix-free machines, and these are not linearly-complete (a fact which is not hard to verify).

1.3 Preliminaries

We assume the reader is familiar with the basic concepts of Kolmogorov complexity and computability theory. We use $C(\sigma)$ and $K(\sigma)$ to denote the plain and prefix-free Kolmogorov complexity of a string σ respectively. Moreover we use $C(\sigma \mid \rho)$, $K(\sigma \mid \rho)$ to denote the relative plain and prefix-free complexities of σ with respect to ρ respectively. Recall that A is 1-random if there exists a constant c such that $K(A \upharpoonright_n) \geq n - c$ for all n . The cumulative work of Solovay [Sol75], Calude, Hertling, Khoussainov and Wang [CHKW01] and Kučera and Slaman [KS01] has shown that the 1-random c.e. reals are exactly the halting probabilities of (optimal) universal machines. Most of this work revolves around a reducibility between c.e. reals that was introduced in [Sol75]. We say that a c.e. real α is Solovay reducible to c.e. real β if there are nondecreasing computable rational approximations $(\alpha_s), (\beta_s)$ to α, β respectively, and a constant c such that $\alpha - \alpha_s \leq c \cdot (\beta - \beta_s)$ for all s . Downey, Hirschfeldt and Nies [DHN02] showed that

$$\alpha \text{ is Solovay reducible to } \beta \text{ iff there are nondecreasing computable rational approximations } (\alpha_s), (\beta_s) \text{ to } \alpha, \beta \text{ respectively, and a constant } c \text{ such that } \alpha_{s+1} - \alpha_s \leq c \cdot (\beta_{s+1} - \beta_s) \text{ for all } s. \quad (1.3.1)$$

Moreover it follows from [Sol75, CHKW01, KS01] that the 1-random c.e. reals are exactly the c.e. reals that are complete (i.e. they are above all other c.e. reals) with respect to Solovay reducibility. In particular, if α is a 1-random c.e. real which is Solovay reducible to another c.e. real β , then β is also 1-random. We use these facts in the proof of Theorem 1.3 in Section 2.3. Without loss of generality, all c.e. reals considered in this paper belong to $[0, 1]$. The following convergence test will be used throughout this paper.

Lemma 1.7 (Condensation). *If f is a nonincreasing sequence of positive reals then the series $\sum_i f(i)$ converges if and only if the series $\sum_i (2^i \cdot f(2^i))$ converges. Moreover, if f is computable and the two sums converge, the first sum is 1-random if and only if the second sum is 1-random.*

Proof. The first part is a standard series convergence test known as the Cauchy condensation test. Its proof usually goes along the lines of Oresme's proof of the divergence of the harmonic series which shows that

$$\sum_{i=2^t}^{\infty} f(i) \leq \sum_{i=t}^{\infty} (2^i \cdot f(2^i)) = 2 \cdot \sum_{i=t}^{\infty} (2^{i-1} \cdot f(2^i)) \leq 2 \cdot \sum_{i=2^{t-1}}^{\infty} f(i) \quad \text{for all } t \in \mathbb{N}^+.$$

The second part of the lemma follows from the above bounds, which show that the two sums are c.e. reals, each one Solovay reducible to the other. So one is 1-random if and only if the other is. \square

Algorithmic randomness for reals was originally defined in terms of effective statistical tests by Martin-Löf in [ML66]. We assume that the reader is familiar with this classical definition. Another kind of test that can be used for the definition of 1-randomness is the Solovay test. A Solovay test is a uniform sequence (I_i) of Σ_1^0 classes (often viewed as a uniformly c.e. sequence of sets of binary strings) such that the sum of the measures of the sets I_i is finite. We say that a real passes a Solovay test (I_i) if there are only finitely many indices i such that I_i contains a prefix of X . Solovay [Sol75] showed that a real is 1-random if and only if it passes all Solovay tests.

Some of the following proofs involve the construction of prefix-free machines. There are certain notions and tools associated with such constructions which are standard in the literature, and which we briefly review now. The *weight* of a prefix-free set S of strings is defined to be the sum $\sum_{\sigma \in S} 2^{-|\sigma|}$. The *weight* of a prefix-free machine M is defined to be the weight of its domain. Prefix-free machines are most often built in terms of *request sets*. A request set L is a set of pairs $\langle \rho, \ell \rangle$ where ρ is a string and ℓ is a positive integer.

A ‘request’ $\langle \rho, \ell \rangle$ represents the intention of describing ρ with a string of length ℓ . We define the *weight of the request* $\langle \rho, \ell \rangle$ to be $2^{-\ell}$. We say that L is a *bounded request set* if the sum of the weights of the requests in L is less than 1. The Kraft-Chaitin theorem (see e.g. [DH10, Section 2.6]) says that for every bounded request set L which is c.e., there exists a prefix-free machine M with the property that for each $\langle \rho, \ell \rangle \in L$ there exists a string τ of length ℓ such that $M(\tau) = \rho$. So the dynamic construction of a prefix-free machine can be reduced to a mere description of a corresponding c.e. bounded request set.

For more background in algorithmic information theory and computability theory we refer to the standard monographs [LV08, DH10, Nie09]. Since we have included a number of citations to the unpublished manuscript of Solovay [Sol75], we note that every result in this manuscript has been included, with a proof, in the monograph by Downey and Hirschfeldt [DH10].

2 Omega and the computably enumerable sets

We start with the proof of Theorem 1.6 and continue with the proof of Theorem 1.3. The proof of the last clause of Theorem 1.6 in Section 2.2 describes a compact coding of a c.e. real into a c.e. set, which will also be used in later arguments in this paper.

2.1 Proof of Theorem 1.6, clauses (i) and (ii)

Recall the following theorem.

Theorem 1.6 (Clauses (i) and (ii)). *Consider a 1-random c.e. real Ω , a c.e. set H , a computable function g and a right-c.e. function f such that $\sum_i 2^{-f(i)}$ is finite.*

(i) *If $K(\Omega \upharpoonright_n \mid H \upharpoonright_{g(n)}) = \mathbf{O}(1)$ then $2^n = \mathbf{O}(g)$.*

(ii) *If $g = \mathbf{O}(2^n)$ and $K(\Omega \upharpoonright_{n+f(n)} \mid H \upharpoonright_{g(n)}) = \mathbf{O}(1)$ then H is linearly-complete.*

For the first clause, it suffices to show that for each c.e. real α , each c.e. set H and each computable function g :

if $K(\alpha \upharpoonright_n \mid H \upharpoonright_{g(n)}) = \mathbf{O}(1)$ and $2^n \neq \mathbf{O}(g)$ then α is not 1-random.

Assuming the above properties regarding H , α and g , we construct a Martin-Löf test (V_i) such that $\alpha \in \cap_i V_i$. Let d be a constant such that $K(\alpha \upharpoonright_n \mid H \upharpoonright_{g(n)}) < d$ for all n . Since $2^n \neq \mathbf{O}(g)$, for each constant c there exists some n such that $g(n) < 2^{n-c}$. Let (n_i) be an increasing sequence with $g(n_i) < 2^{n_i-i-d}$ for each i . We describe a construction enumerating the sets V_k . We say that V_k requires attention at stage $s+1$ if $K_{s+1}(\alpha_{s+1} \upharpoonright_{n_k} \mid H_{s+1} \upharpoonright_{g(n_k)}) < d$ and either $V_k[s]$ is empty or the last string enumerated into V_k is not a prefix of α_{s+1} . At each stage $s+1$ we pick the least k such that V_k requires attention and enumerate $\alpha_{s+1} \upharpoonright_{n_k}$ into V_k . In order to see that (V_i) is a Martin-Löf test, note that each time we enumerate into V_k , we are guaranteed a change in $\alpha \upharpoonright_{n_k}$. Moreover, after at most 2^d such consecutive enumerations, there must be a change in $H \upharpoonright_{g(n_k)}$. This is true, because after each such enumeration, V_k can only require attention if the underlying universal machine issues a new description of $\alpha \upharpoonright_{n_k}$ relative to $H \upharpoonright_{g(n_k)}$, of length less than d ; moreover the underlying machine can only issue at most 2^d such descriptions as long as the approximation to $H \upharpoonright_{g(n_k)}$ does not change. Since $g(n_k) < 2^{n_k-k-d}$ and H is a c.e. set, there can be at most 2^{n_k-k-d} changes in $H \upharpoonright_{g(n_k)}$. It follows that there can be at most $2^d \cdot 2^{n_k-k-d} = 2^{n_k-k}$ many enumerations into V_k , the last

enumeration into this set being a prefix of α . Since each string in V_k has length n_k , the measure of V_k is at most 2^{-k} . So (V_k) is a Martin-Löf test with $\alpha \in \cap_i V_i$, demonstrating that α is not 1-random.

For the second clause, let W be a c.e. set and assume the hypothesis of the second clause of the theorem. Since $g = \mathbf{O}(2^n)$ it suffices to show that for all but finitely many k , and for all $i < 2^k$ we can compute $W(2^k + i)$ uniformly from $H \upharpoonright_{g(k)}$. Let c be a constant such that $K(\Omega \upharpoonright_{f(k)+k} \mid H \upharpoonright_{g(k)}) < c$ for all k . We enumerate a Solovay test (I_k) as follows. We define the sets $I_k(i)$ for each $k \in \mathbb{N}$ and $i < 2^k$, and for each k we let I_k be the union of all $I_k(i)$, $i < 2^k$. In what follows, when we write $\Omega_s \upharpoonright_{f(k)+k}$, it is to be understood that the value $f(k)$ referred to, is in fact the approximation to $f(k)$ at stage s . At any stage s of the construction, for each k and each $i < 2^k$, let $m_k(i)$ be the first stage $\leq s$ at which a string was enumerated into $I_k(i)$ if such a stage exists, and let $m_k(i)$ be undefined otherwise. We say that $I_k(i)$ requires attention at stage s if $i < 2^k$, $2^k + i \in W_s$, $K_s(\Omega_s \upharpoonright_{f(k)+k} \mid H_s \upharpoonright_{g(k)}) < c$ and the following two conditions hold:

- $H_s \upharpoonright_{g(k)} = H_{m_k(i)} \upharpoonright_{g(k)}$ or $m_k(i)$ is undefined;
- $I_k = \emptyset$ or the last string enumerated into I_k is not a prefix of Ω_s .

We say that I_k requires attention at stage s if $I_k(i)$ requires attention at stage s for some $i < 2^k$.

At stage $s + 1$ let k be the least number such that I_k requires attention. Let i be the least number which is less than 2^k and such that $I_k(i)$ requires attention, and enumerate $\Omega_{s+1} \upharpoonright_{k+f(k)}$ into $I_k(i)$ (or do nothing if no I_k requires attention). This concludes the enumeration of (I_k) .

First we show that (I_k) is a Solovay test. Fix k . If $s_0 < s_1$ are two stages at which enumerations are made into I_k , we have that $\Omega_{s_0} \upharpoonright_{k+f(k)} \neq \Omega_{s_1} \upharpoonright_{k+f(k)}$ (if the approximation to f changes, then these are distinct strings). Now let $i < 2^k$. If $s_0 < s_1$ are two stages at which enumerations into $I_k(i)$ are made then $K_{s_0}(\Omega_{s_0} \upharpoonright_{k+f(k)} \mid H_{s_0} \upharpoonright_{g(k)}) < c$ and $K_{s_1}(\Omega_{s_1} \upharpoonright_{k+f(k)} \mid H_{s_0} \upharpoonright_{g(k)}) < c$. So for each new enumeration into $I_k(i)$, we have an additional description in the universal machine with oracle $H_{m_k(i)} \upharpoonright_{g(k)}$ (where $m_k(i)$ is the first stage where an enumeration occurred in $I_k(i)$) of length less than c . Since there are at most 2^c such descriptions, for each k and $i < 2^k$ there can be at most 2^c enumerations into $I_k(i)$. So the measure of $I_k(i)$ is at most $2^{c-k-f(k)}$. Hence the measure of I_k is at most $2^{c-f(k)}$. From our hypothesis concerning f , it follows that (I_k) is a Solovay test.

Now we show that W is linearly reducible to H . Since $g = \mathbf{O}(2^n)$, it suffices to show that for all but finitely many k , if $i < 2^k$ and $2^k + i$ is enumerated into W at some stage s_0 , then there exists a stage $s_1 > s_0$ such that $H_{s_0} \upharpoonright_{g(k)} \neq H_{s_1} \upharpoonright_{g(k)}$. Since Ω is 1-random and each of the sets $I_k(i)$ is a finite set of strings, there exist some k_0 such that for each $k > k_0$ and each $i < 2^k$ the set $I_k(i)$ does not contain any prefix of Ω . Now suppose that some number $2^k + i$ with $k > k_0$ and $i < k$ is enumerated into W at some stage s_0 , and towards a contradiction suppose that $H_{s_0} \upharpoonright_{g(k)} = H \upharpoonright_{g(k)}$. Then $I_k(i)$ will require attention at every sufficiently late stage s at which Ω_s does not have a prefix in I_k . Eventually $\Omega \upharpoonright_{k+f(k)}$ will be enumerated into $I_k(i)$, which contradicts the choice of k_0 . This concludes the proof that H is linearly-complete, and the proof of the second clause.

2.2 Proof of Theorem 1.6, clauses (iii) and (iv)

Recall the following theorem.

Theorem 1.6 (Clauses (iii) and (iv)). *Consider a 1-random c.e. real Ω , a c.e. set H , a computable function g and a right-c.e. function f such that $\sum_i 2^{-f(i)}$ is finite.*

(iii) *If $g = \mathbf{O}(n)$ and $K(\Omega \upharpoonright_{f(n)} \mid H \upharpoonright_{g(n)}) = \mathbf{O}(1)$ then H is linearly-complete.*

(iv) *If H is linearly-complete then it computes Ω (and any c.e. real) with use $2^{n+\mathbf{O}(1)}$.*

For clause (iii) the proof is similar to clause (ii). Let c be a constant such that $g(n) \leq c \cdot n$ and $K(\Omega \upharpoonright_{f(n)} \mid H \upharpoonright_{g(n)}) < c$ for all n . We enumerate a Solovay test (I_k) as follows. At any stage s of the construction, for each k let m_k be the first stage $\leq s$ at which a string was enumerated into I_k if such a stage exists, and let m_k be undefined otherwise. We say that I_k requires attention at stage s if $k \in W_s$, $K_s(\Omega_s \upharpoonright_{f(k)} \mid H_s \upharpoonright_{g(k)}) < c$, $H_s \upharpoonright_{g(k)} = H_{m_k} \upharpoonright_{g(k)}$ or m_k is undefined, and:

$$I_k = \emptyset \text{ or the last string enumerated in } I_k \text{ is not a prefix of } \Omega_s.$$

At stage $s + 1$ let k be the least number such that I_k requires attention and enumerate $\Omega_{s+1} \upharpoonright_{f(k)}$ into I_k (or do nothing if no I_k requires attention). This concludes the enumeration of (I_k) .

First we show that (I_k) is a Solovay test. If $s_0 < s_1$ are two stages at which enumerations into I_k are made, we have $K_{s_0}(\Omega_{s_0} \upharpoonright_{f(k)} \mid H_{s_0} \upharpoonright_{g(k)}) < c$, $K_{s_1}(\Omega_{s_1} \upharpoonright_{f(k)} \mid H_{s_0} \upharpoonright_{g(k)}) < c$ and $\Omega_{s_0} \upharpoonright_{f(k)} \neq \Omega_{s_1} \upharpoonright_{f(k)}$. Hence for each new enumeration into I_k , we have an additional description in the universal machine with oracle $H_{m_k} \upharpoonright_{g(k)}$ (where m_k is the first stage where an enumeration occurred in I_k) of length less than c . Since there are at most 2^c such descriptions, for each k there can be at most 2^c enumerations into I_k . So the measure of I_k is at most $2^{c-f(k)}$. By our hypothesis concerning f it follows that (I_k) is a Solovay test.

Now we show that W is linearly reducible to H . It suffices to show that for all but finitely many n , if n is enumerated into W at some stage s_0 then there exists a stage $s_1 > s_0$ such that $H_{s_0} \upharpoonright_{g(n)} \neq H_{s_1} \upharpoonright_{g(n)}$. Since Ω is Martin-Löf random, there exist some n_0 such that for each $n > n_0$ the set I_n does not contain any prefix of Ω . Suppose that some $n > n_0$ is enumerated into W at some stage s_0 , and towards a contradiction suppose that $H_{s_0} \upharpoonright_{g(n)} = H \upharpoonright_{g(n)}$. Then I_n will require attention at every sufficiently late stage s at which Ω_s does not have a prefix in I_n . Eventually $\Omega \upharpoonright_{f(n)}$ will be enumerated into I_n , which contradicts the choice of n_0 . This concludes the proof that H is linearly-complete, and the proof of the third clause.

For clause (iv), let us start by examining how Ω may be coded into a c.e. set. Given a c.e. real α , there is a computable increasing sequence of rational numbers (α_s) that converges to α . For each $t \in \mathbb{N}$ let:

$$\begin{aligned} p_\alpha(t) &= |\{s \mid \alpha_s(t) \neq \alpha_{s+1}(t) \wedge \alpha_{s+1}(t) = 1\}| \leq 2^t \\ \sigma_\alpha(t) &= 1^{p_\alpha(t)} * 0^{2^t - p_\alpha(t)}. \end{aligned}$$

The c.e. real α can be coded into a c.e. set X_α as follows:

$$X_\alpha = \sigma_\alpha(0) * \sigma_\alpha(1) * \dots \quad (2.2.1)$$

Then clearly α is computable from X_α with use bounded by 2^n , and X_α is computable from α with use bounded by $\lfloor \log n \rfloor + 1$. From this coding, we can deduce the fourth clause of the theorem. Let α be a c.e. real and consider X_α as above. Since H is linearly-complete, $X_\alpha \upharpoonright_{2^n}$ is uniformly computable from $H \upharpoonright_{2^{n+\mathbf{O}(1)}}$. Since $\alpha \upharpoonright_n$ is uniformly computable from $X_\alpha \upharpoonright_{2^n}$, it follows that it is also uniformly computable from $H \upharpoonright_{2^{n+\mathbf{O}(1)}}$.

2.3 Proof of Theorem 1.3

Recall the following theorem.

Theorem 1.3. *Let g be a computable function.*

- (1) *If $\sum_n 2^{-g(n)}$ converges, then every c.e. set is computable from any Ω -number with use g .*
- (2) *If $\sum_n 2^{-g(n)}$ diverges and g is nondecreasing, then no Ω -number can compute all c.e. sets with use g . In fact, in this case, no linearly complete c.e. set can be computed by any c.e. real with use g .*

For clause (1), let A be a c.e. set and let c be a constant such that $\sum_{i>c} 2^{-g(i)} < 1$. We construct a prefix-free machine N using the Kraft-Chaitin theorem as follows. At stage $s + 1$, for each $n \in A_{s+1} - A_s$ which is larger than c , we enumerate an N -description of $\Omega_{s+1} \upharpoonright_{g(n)}$ (chosen by the online Kraft-Chaitin algorithm) of length $g(n)$. This completes the definition of N . Note that the weight of the domain of N is at most $\sum_{n>c} 2^{-g(n)} < 1$, so that the machine N is well defined by the Kraft-Chaitin theorem. Since Ω is 1-random, there exists some $n_0 > c$ such that for all $n > n_0$ we have $K_N(\Omega \upharpoonright_{g(n)}) > g(n)$. Hence if some $n > n_0$ enters A at some stage $s + 1$, we have that $\Omega_{s+1} \upharpoonright_{g(n)}$ is not a prefix of Ω . Since Ω is a c.e. real, this means that Ω computes A with use g , which concludes the proof of clause (1) of Theorem 1.3.

We first prove clause (2) of Theorem 1.3 for the case when the linearly-competent set is the halting set with respect to some Kolmogorov numbering. We do this because the proof of Theorem 1.4 is based on the same ideas. On the other hand, the general case can easily be obtained with some modifications of the argument which we lay out at the end of this section. So suppose that A is the halting problem with respect to a Kolmogorov numbering (M_e) of Turing machines. Let (Φ_e, α_e) be an effective list of all Turing functionals Φ_e with use g and c.e. reals α_e . This effective list comes with nondecreasing computable rational approximations $(\alpha_e[s])$ to α_e and effective enumerations $\Phi_e[s]$ of Φ_e , which are based on the underlying universal machine. Also let U be the universal machine whose halting problem is A , and let f be a computable function such that for each e the function $n \mapsto f(e, n)$ is $\mathbf{O}(n)$ and such that $U(f(e, n)) \simeq M_e(n)$ for all e, n . By definition, we have $A = \{f(e, n) \mid M_e(n) \downarrow\}$. We will define a Turing machine M such that the following requirements are met:

$$\mathcal{R}_e : A \neq \Phi_e^{\alpha_e}.$$

Note that M is not directly mentioned in the requirement \mathcal{R}_e , but is implicit in the definition of A . By the recursion theorem we can assume given b such that $M = M_b$. During the construction of M we check the enumeration of A to ensure that there does not exist any n with $A(f(b, n)) = 1$ and for which we have not yet defined M on argument n . If such an n is found, then we terminate the construction of M (so that, in fact, no such n can be found at any stage for the correct index b). It is also convenient to speed up the enumeration of A , so that whenever we define M on argument n , $f(b, n)$ is enumerated into A at the next stage.

Since $\sum_i 2^{-g(i)} = \infty$, by Lemma 1.7 we also have $\sum_n 2^{n-g(2^n)} = \infty$. Let $I_t = [2^t - 1, 2^{t+1} - 1)$ so that $|I_t| = 2^t$. Let $(J_c(e))$ be a computable partition of \mathbb{N} into consecutive intervals (i.e. such that $J_c(e)$ and $J_{c'}(e')$ are disjoint if $e \neq e'$ or $c \neq c'$), such that:

$$\sum_{t \in J_c(e)} 2^{(t+c)-g(2^{t+c})} > 2^c \quad \text{for each } e, c. \quad (2.3.1)$$

Such a partition exists by the hypothesis for g and Lemma 1.7. Let $M[s]$ denote the machine M as defined by the end of stage s , i.e. $M(n)[s] \downarrow$ precisely if an axiom to that effect is enumerated prior to the end of stage s . For each pair c, e let $j_{c,e}$ be the largest number j such that there exists $t \in J_c(e)$ with $j \in I_t$. We

say that \mathcal{R}_e requires attention with respect to (c, t) at stage $s + 1$, if t is the least number in $J_c(e)$ such that $M(i)[s]$ is not defined for some $i \in I_t$, and if also $\Phi_e^{\alpha_e}(j)[s + 1] = A(j)[s + 1]$ for all $j \leq e \cdot j_{c,e}$. We say $\mathcal{R}_e(c)$ requires attention with respect to t at some stage if \mathcal{R}_e requires attention at that stage with respect to (c, t) . Let $\langle c, e \rangle$ be an effective bijection between $\mathbb{N} \times \mathbb{N}$ and \mathbb{N} . The definition of M is as follows. At stage $s + 1$ we look for the least $\langle c, e \rangle$ such that $\mathcal{R}_e(c)$ requires attention. If there exists no such $\langle c, e \rangle < s$, we go to the next stage. Otherwise let $\langle c, e \rangle$ be this tuple and let t be the least number such that $\mathcal{R}_e(c)$ requires attention with respect to t . Let k be the least element of I_t such that $M_s(k)$ is undefined and define $M_{s+1}(k) = 0$. We say that $\mathcal{R}_e(c)$ received attention with respect to t at stage $s + 1$. This concludes the definition of M .

It remains to verify that each \mathcal{R}_e is met. Towards a contradiction, suppose that \mathcal{R}_e is not met. Let b be an index of the machine M and fix c to be a constant such that $f(b, n) \leq 2^{c-1} \cdot n$ for all n . By the padding lemma we can assume that $e > 2^{c-1}$, which means that if $\mathcal{R}_e(c)$ requires attention at stage $s + 1$ then:

$$\Phi_e^{\alpha_e}(f(b, i))[s + 1] \downarrow = A(f(b, i))[s + 1] \text{ for all } i \in I_t \text{ and all } t \in J_c(e). \quad (2.3.2)$$

Fix $t \in J_c(e)$ and let $s_0 < s_1$ be stages at which $\mathcal{R}_e(c)$ receives attention with respect to t . Since all elements of I_t are less than 2^{t+1} , g is nondecreasing, and since we define M on $i \in I_t$ at stage s_0 , causing $f(b, i)$ to be enumerated into A , it follows that we must see an increase in $\alpha_e \upharpoonright_{g(2^{c-1} \cdot 2^{t+1})} = \alpha_e \upharpoonright_{g(2^{t+c})}$ between stages s_0 and s_1 . If \mathcal{R}_e is not satisfied, as we have assumed, then for each $t \in J_c(e)$ it will receive attention $|I_t| = 2^t$ many times, causing a total increase in $\alpha_e \upharpoonright_{g(2^{t+c})}$ of $2^{t-g(2^{t+c})}$. Summing over all $t \in J_c(e)$ this means that ultimately:

$$\alpha_e \geq \sum_{t \in J_c(e)} 2^{t-g(2^{t+c})} = 2^{-c} \cdot \sum_{t \in J_c(e)} 2^{t+c-g(2^{t+c})} > 2^{-c} \cdot 2^c = 1.$$

This gives the required contradiction, since $\alpha_e \leq 1$. We can conclude that each \mathcal{R}_e is met, which completes the proof of clause (2) of Theorem 1.3 for the case of halting sets.

For the more general case, assume that A is merely a linearly-complete c.e. set. Then for all c.e. sets W we have $K(W \upharpoonright_n \mid A \upharpoonright_n) = \mathbf{O}(1)$ (see [BHL13]). As before, let (α_e) be an effective enumeration of all c.e. reals in $[0, 1]$, with nondecreasing rational computable approximations $(a_e[s])$ respectively. We want to show that for each e , the real α_e does not compute A with oracle-use bounded above by g . If α_e did compute A in this way, then we would have $K(A \upharpoonright_n \mid \alpha_e \upharpoonright_{g(n)}) = \mathbf{O}(1)$ for all n , and since A is linearly complete, it follows from the remark above that for each c.e. set W we would have $K(W \upharpoonright_{2^n} \mid \alpha_e \upharpoonright_{g(2^n)}) = \mathbf{O}(1)$ for all n . So it suffices to enumerate a c.e. set W such that the following conditions are met for each e, c :

$$\mathcal{R}_e(c) : \quad \text{If for all } n, \quad K(W \upharpoonright_{2^n} \mid \alpha_e \upharpoonright_{g(2^n)}) \leq c, \quad \text{then} \quad \alpha_e > 1.$$

As before we let $I_t = [2^t - 1, 2^{t+1} - 1)$ for each t , so that $|I_t| = 2^t$ and all elements of I_t are less than 2^{t+1} . We also define an appropriate version of the intervals $J_c(e)$ as follows. We consider a computable partition of \mathbb{N} into finite intervals $J_c(e)$ for $e, c \in \mathbb{N}$ such that

$$\sum_{t \in J_c(e)} 2^{t-g(2^{t+1})} > 2^c \quad \text{for each } e, c \in \mathbb{N}. \quad (2.3.3)$$

As before, such a partition of \mathbb{N} exists by Lemma 1.7. We say that $\mathcal{R}_e(c)$ requires attention with respect to $t \in J_c(e)$ at stage $s + 1$, if $I_t - W_s \neq \emptyset$ and if $K_{s+1}(W_s \upharpoonright_{2^{t'+1}} \mid \alpha_e[s + 1] \upharpoonright_{g(2^{t'+1})}) \leq c$ for all $t' \in J_c(e)$. We say that $\mathcal{R}_e(c)$ requires attention at stage $s + 1$ if it requires attention with respect to some $t \in J_c(e)$. The enumeration of W takes place as follows. At stage $s + 1$ we look for the least $\langle c, e \rangle$ such that $\mathcal{R}_e(c)$ requires attention. If there exists no such $\langle c, e \rangle < s$, we go to the next stage. Otherwise let $\langle c, e \rangle$ be this tuple and

let t be the least number such that $\mathcal{R}_e(c)$ requires attention with respect to t . Let k be the least element of $I_t - W_s$ and enumerate k into W_{s+1} . We say that $\mathcal{R}_e(c)$ received attention with respect to t at stage $s + 1$. This concludes the enumeration of W .

It remains to verify that each \mathcal{R}_e is met. Towards a contradiction, suppose that \mathcal{R}_e is not met. Fix $t \in J_c(e)$ and for each $i \leq 2^{t-c}$ let s_i be the stage where $\mathcal{R}_e(c)$ receives attention with respect to t for the $(2^c \cdot i)$ -th time. Since whenever $\mathcal{R}_e(c)$ requires attention with respect to t at some stage $s + 1$ we have $K_{s+1}(W_s \upharpoonright_{2^{t+1}} \mid \alpha_e[s + 1] \upharpoonright_{g(2^{t+1})}) \leq c$ and there are at most 2^c many descriptions of length c , between each s_i and s_{i+1} we must see an increase in $\alpha_e \upharpoonright_{g(2^{t+1})}$. If \mathcal{R}_e is not satisfied, as we have assumed, then for each $t \in J_c(e)$ it will receive attention $|I_t| = 2^t$ many times, meaning an increase in $\alpha_e \upharpoonright_{g(2^{t+1})}$ of at least $2^{t-c-g(2^{t+1})}$. Summing over all $t \in J_c(e)$ we have that

$$\alpha_e \geq \sum_{t \in J_c(e)} 2^{t-c-g(2^{t+1})} = 2^{-c} \cdot \sum_{t \in J_c(e)} 2^{t-g(2^{t+1})} > 2^{-c} \cdot 2^c = 1.$$

This is the required contradiction, since $\alpha \leq 1$. We conclude that each $\mathcal{R}_e(c)$ is met, which completes the proof of clause (2).

2.4 Proof of Theorem 1.4

Recall the following theorem.

Theorem 1.4. *Let g be a nondecreasing computable function such that $\sum_n 2^{-g(n)}$ converges to a 1-random real. Then a c.e. real is 1-random if and only if it computes the halting problem with respect to a Kolmogorov numbering with use g .*

According to the hypothesis of Theorem 1.4 and Lemma 1.7 we have that $\sum_i 2^{i-g(2^i)}$ is 1-random. Suppose that A is the halting problem with respect to a Kolmogorov numbering (M_e) of Turing machines, and α is a c.e. real which computes A with use g . Let U be the universal machine whose halting problem is A , and let f be a computable function such that $n \mapsto f(e, n)$ is $\mathbf{O}(n)$ and such that $U(f(e, n)) \simeq M_e(n)$ for all e, n . By definition, we have $A = \{f(e, n) \mid M_e(n) \downarrow\}$. Moreover let Φ be a Turing functional with oracle use uniformly bounded by $g(n)$ on each argument n , such that $A = \Phi^\alpha$. Fix computable enumerations $(\Phi_s), (A_s)$ of Φ, A respectively. The proof proceeds much as in the first proof we gave for clause (2) of Theorem 1.3 previously. Once again we construct a machine M , and assume by the recursion theorem that we are given b such that $M = M_b$. During the construction of M we check the enumeration of A as before, so as to ensure that there is no n for which $f(b, n)$ is enumerated into A but for which we have not yet defined M on argument n , etc. The machine M that we construct will be very simple, and it is the timing of enumerations into the domain of M which is key. Fix c to be a constant such that $f(b, n) \leq 2^{c-1} \cdot n$ for all n . Again we define $I_t = [2^t - 1, 2^{t+1} - 1)$ so that $|I_t| = 2^t$. This time, however, we say that I_t requires attention at stage $s + 1$, if $M(i)[s]$ is not defined for some $i \in I_t$, and if also $\Phi^\alpha(j)[s + 1] = A(j)[s + 1]$ for all $j \leq 2^{c-1}2^{t+1} = 2^{c+t}$. For the least t which requires attention at stage $s + 1$ (if any), we find the least $i \in I_t$ such that $M(i)[s] \uparrow$ and we define $M(i) \downarrow = 0$.

The rough idea is now to define a sequence of stages s_t such that if we define $\gamma_t = \alpha[s_t]$ and $\delta_t = \sum_{n=0}^{t-1} 2^{n+c-g(2^{n+c})}$, then

$$2^c \cdot (\gamma_{t+1} - \gamma_t) \geq (\delta_{t+1} - \delta_t). \quad (2.4.1)$$

According to the characterisation (1.3.1) of Solovay reducibility, this means that the limit δ of (δ_t) is Solovay reducible to the limit α of (γ_t) . Since δ is 1-random, it then follows that α is 1-random, as required. We define $s_0 = 0$ and, for each $t > 0$, we define s_t to be the first stage at which I_t receives attention. Each time I_t receives attention, we must see an increase in $\alpha \upharpoonright_{g(2^{t+c})} = \alpha \upharpoonright_{g(2^{t+c})}$. Since I_t will receive attention $|I_t| = 2^t$ many times, this means a total increase in $\alpha \upharpoonright_{g(2^{t+c})}$ of at least $2^{t-g(2^{t+c})} = 2^{-c} 2^{t+c-g(2^{t+c})}$. We shall therefore have that (2.4.1) holds, if we define $\gamma_t = \alpha \upharpoonright_{g(2^{t+c})} \upharpoonright_{[s_t]}$.

3 Computing c.e. reals from Omega numbers

This section is devoted to the proof of Theorem 1.2. In Section 3.1 we derive the first part of this result from Theorem 1.3, while Section 3.2 contains a more sophisticated argument for the proof of the second part.

3.1 Proof of Theorem 1.2, clause (1)

Recall the following theorem.

Theorem 1.2 (Clause (1)). *Let h be computable. If $\sum_n 2^{n-h(n)}$ converges, then for any Ω -number Ω and any c.e. real α , α can be computed from Ω with use bounded by h .*

Let Ω be an omega number, let α be a c.e. real and let $g = h(n) - n$ be as in the statement of the first clause. By the proof of clause (iv) of Theorem 1.6 in Section 2.2, there exists a c.e. set A which computes α with use 2^n . By Lemma 1.7

$$\sum_i 2^{-g(i)} < \infty \Rightarrow \sum_i 2^i \cdot 2^{-(\log 2^i + g(\log 2^i))} < \infty \Rightarrow \sum_i 2^{-(\log i + g(\log i))} < \infty \quad (3.1.1)$$

so by Theorem 1.3 the set A is computable from Ω with use bounded by $\log n + g(\log n)$. By composing the two reductions we conclude that α is computable from Ω with use bounded by $\log 2^n + g(\log 2^n)$ i.e. $n + g(n)$.

We can use a more direct argument to prove the same fact, without the hypothesis that $h(n) - n$ is non-decreasing. Let g, Ω, α be as above and let $(\alpha_s), (\Omega_s)$ be computable nondecreasing dyadic rational approximations that converge to α, Ω respectively. We construct a Solovay test along with a c.e. set I , as follows. At each stage $s + 1$ we consider the least n such that $\alpha_s(n) \neq \alpha_{s+1}(n)$, if such exists. If there exists such an n we define $\sigma_s = \Omega_{s+1} \upharpoonright_{n+g(n)}$ and enumerate s into I . We verify that the set of strings $\sigma_s, s \in I$ is a Solovay test. Note that for every n , the number of stages s such that n is the least number with the property that $\alpha_s(n) \neq \alpha_{s+1}(n)$ is bounded above by the number of times that $\alpha_s(n)$ can change from 0 to 1 in this monotone approximation to α . Hence this number is bounded above by 2^n . So:

$$\sum_{s \in I} 2^{-|\sigma_s|} \leq \sum_n 2^n \cdot 2^{-g(n)-n} = \sum_n 2^{-g(n)} < \infty.$$

Since Ω is Martin-Löf random, there exists some s_0 such that none of the strings σ_s for $s \in I$ and $s > s_0$ are prefixes of Ω . This means that whenever our construction enumerates s in I because we find n with $\alpha_s(n) \neq \alpha_{s+1}(n)$, there exists some later stage where the approximation to $\Omega \upharpoonright_{n+g(n)}$ changes. Hence with oracle $\Omega \upharpoonright_{s+g(s)}$ we can uniformly compute $\alpha(n)$, so α is computable from Ω with oracle use h .

3.2 Proof of Theorem 1.2, clause (2)

Recall the following theorem.

Theorem 1.2 (Clause (2)). *Let h be computable and such that $h(n) - n$ is non-decreasing. If $\sum_n 2^{n-h(n)}$ diverges then no Ω -number can compute all c.e. reals with use $h + \mathbf{O}(1)$. In fact, in this case there exist two c.e. reals such that no c.e. real can compute both of them with use $h + \mathbf{O}(1)$.*

Given a computable function h such that $h(n) - n$ is non-decreasing and $\sum_i 2^{n-h(n)}$ diverges, we need to construct two c.e. reals such that no c.e. real can compute both of them with use $h + \mathbf{O}(1)$. Our presentation will be neater, however, if we consider the following elementary fact, which allows one to ignore the constants in the previous statement.

Lemma 3.1 (Space lemma). *If g is computable, non-decreasing and $\sum_n 2^{-g(n)} = \infty$ then there exists a computable non-decreasing function f such that $\lim_n (f(n) - g(n)) = \infty$ and $\sum_n 2^{-f(n)} = \infty$.*

Proof. Consider a computable increasing sequence (n_i) such that $n_0 = 0$ and

$$\sum_{n \in I_k} 2^{-g(n)} > 2^k \quad \text{for all } k$$

where $I_k = [n_k, n_{k+1})$. Then for each k and each $i \in [n_k, n_{k+1})$ define $f(i) = g(i) + k$. Then

$$\sum_n 2^{-f(n)} \geq \sum_k \left(\sum_{n \in I_k} 2^{-f(n)} \right) = \sum_k \left(2^{-k} \cdot \sum_{n \in I_k} 2^{-g(n)} \right) \geq \sum_k (2^{-k} \cdot 2^k) = \infty$$

which concludes the proof. \square

By Lemma 3.1, for the proof of the second clause of Theorem 1.2 it suffices to show that, given a computable non-decreasing function h with $\sum_i 2^{n-h(n)} = \infty$, there exist two c.e. reals such that no c.e. real can compute both of them with use h . We need to construct two c.e. reals α, β such the requirement

$$\mathcal{R}(\Phi, \Psi, \gamma) : \alpha \neq \Phi^\gamma \quad \vee \quad \beta \neq \Psi^\gamma \quad (3.2.1)$$

is met for every triple (Φ, Ψ, γ) such that Φ, Ψ are Turing functionals with use h and γ is a c.e. real. For the simple case where $h(n) = n + \mathbf{O}(1)$, this was done by Yu and Ding in [DY04], and a simplification of their argument appeared in [BL06c]. The underlying method for this type of argument involves an amplification game, which we discuss in Section 3.2.1. Then in Section 3.2.2 we build on these ideas in order to produce a more sophisticated argument which deals with an arbitrary choice of h satisfying the hypothesis of the theorem.

3.2.1 Amplification games

The reals γ in requirements (3.2.1) will be given with a non-decreasing computable rational approximation (γ_s) . Our task is to define suitable computable approximations $(\alpha_s), (\beta_s)$ to α, β respectively, so that (3.2.1) is met. The idea is that if (the approximation to) $\alpha \upharpoonright_n$ changes at a stage where Φ^γ is defined to be an extension of (the previous version of) $\alpha \upharpoonright_n$, then either $\alpha \neq \Phi^\gamma$ or (the approximation to) $\gamma \upharpoonright_{h(n)}$ needs to

change at a later stage (and similarly for β). This gives us a way to drive γ to larger and larger values, if Φ, Ψ insist on mapping the current approximations of γ to the current approximations of α, β respectively.

In order to elaborate on this approach, consider the following game between two players, which monotonically approximate two c.e. reals α, γ respectively. Each of these numbers has some initial value, and during the stages of the game values can only increase. If α increases and i is the leftmost position where an α -digit change occurs, then γ has to increase in such a way that some γ -digit at a position $\leq h(i)$ changes. This game describes a Turing computation of α from γ with use h . If γ has to code two reals α, β then we get a similar game (where, say, at each stage only one of α, β can change). In order to block one of the reductions in (3.2.1) the idea is to devise a strategy which forces γ to either stop computing α, β in this way, or else grow to exceed the interval $(0, 1)$ to which it is assumed to belong.

It turns out that in such games, there is a best strategy for γ , in the sense that it causes the least possible increases in γ while responding to the requests of the opponent(s). We say that γ follows the *least effort strategy* if at each stage it increases by the least amount needed in order to satisfy the requirements of the game.

Lemma 3.2 (Least effort strategy). *In a game where γ has to follow instructions of the type ‘change a digit at position $\leq n$ ’, the least effort strategy is a best strategy for γ . In other words, if a different strategy produces γ' then at each stage s of the game $\gamma_s \leq \gamma'_s$.*

Proof. We use induction on the stages s . We have that $\gamma_0 \leq \gamma'_0$. If $\gamma_s = \gamma'_s$ then it is clear from the definition of the least effort strategy that the induction hypothesis will hold at stage $s + 1$. So suppose otherwise. Then $\gamma_s < \gamma'_s$ so that there will be a position n such that $0 = \gamma_s(n) < \gamma'_s(n) = 1$ and $\gamma_s \upharpoonright n = \gamma'_s \upharpoonright n$. Suppose that γ, γ' are forced to change at a position $\leq t$ at stage $s + 1$. If $t < n$ it is clear that $\gamma_{s+1} \leq \gamma'_{s+1}$. Otherwise the leftmost change γ can be forced to make is at position n . Once again $\gamma_{s+1} \leq \gamma'_{s+1}$. \square

Lemma 3.2 allows us to assume a fixed strategy for the opponent approximating γ , which reduces our analysis to the assessment of a deterministic process, once we specify our strategy for the requests that the approximations to α, β generate. The following observation is also useful in our analysis.

Lemma 3.3. (Accumulation) *Suppose that in some game (e.g. like the above) γ has to follow instructions of the type ‘change a digit at position $\leq n$ ’. Although $\gamma_0 = 0$, some γ' plays the same game while starting with $\gamma'_0 = \sigma$ for a finite binary expansion σ . If γ and γ' both use the least effort strategy and the sequence of instructions only ever demands change at positions $> |\sigma|$ then $\gamma'_s = \gamma_s + \sigma$ at every stage s .*

Proof. By induction on s . For $s = 0$ the result is obvious. Suppose that the induction hypothesis holds at stage s . Then γ'_s, γ_s have the same expansions after position $|\sigma|$. At $s + 1$, some demand for a change at some position $> |\sigma|$ appears and since γ, γ' look the same on these positions, γ'_s will need to increase by the same amount that γ_s needs to increase. So $\gamma'_{s+1} = \gamma_{s+1} + \sigma$ as required. \square

We are now ready to describe the strategy for the approximation of α, β restricted to an interval $(t - n, t]$. This strategy automatically induces a deterministic response from γ according to the least effort strategy.

Definition 3.4. *The h -load process in $(t - n, t]$ with (α, β, γ) is the process which starts with $\alpha_0 = \beta_0 = \gamma_0 = 0$, and at each stage 2^{-t} is added alternately in α and β . Moreover at each stage $s + 1$, if k is the least number such that $\alpha_{s+1} \upharpoonright k \neq \alpha_s \upharpoonright k$ or $\beta_{s+1} \upharpoonright k \neq \beta_s \upharpoonright k$, we add to γ the least amount which can change $\gamma \upharpoonright_{h(k)}$.*

In more detail, the h -load process in $(t-n, t]$ with (α, β, γ) is defined by the following instructions. Assume that α, β, γ have initial value 0. Repeat the following instructions until $\alpha(i) = \beta(i) = 1$ for all $i \in (t-n, t]$.

- (1) If s odd then, let $\alpha = \alpha + 2^{-t}$ and let k equal the leftmost position where a change occurs in α . Also add to γ the least amount which causes a change in $\gamma \upharpoonright_{h(k)}$.
- (2) If s even, let $\beta = \beta + 2^{-t}$ and let k equal the leftmost position where a change occurs in β . Also add to γ the least amount which causes a change in $\gamma \upharpoonright_{h(k)}$.

It is not hard to see that the above procedure describes how γ evolves when it tries to code α, β via Turing reductions with use h and it uses the *least effort strategy*. Player γ follows the *least effort strategy* when it increases by the least amount which can rectify the functionals holding its computations of α and β . Note that in a realistic construction, the steps of the above strategy are supposed to happen at stages where γ currently computes the first $t+1$ bits of both α, β via Φ, Ψ respectively.

The following lemma says that in the special case where $h(n) = n + \mathbf{O}(1)$, the h -load process is successful at forcing γ to grow significantly.

Lemma 3.5 (Atomic attack). *Let $n > 0$ and $h(x) = x + c$ for some constant c . For any $k > 0$ the h -load process in $(k, k+n]$ with (α, β, γ) ends with $\gamma = n \cdot 2^{-k-c}$.*

Proof. By induction: for $n = 1$ the result is obvious. Assume that the result holds for n . Now pick $k > 0$ and consider the attack using $(k-1, k+n]$. It is clear that up to a stage s_0 this will be identical to the procedure with attack interval $(k, k+n]$. By the induction hypothesis $\gamma_{s_0} = n2^{-k-c}$ and $\alpha(i) = \beta(i) = 1$ for all $i \in (k, k+n]$, while $\alpha(k) = \beta(k) = 0$. According to the next step α changes at position k and this forces γ to increase by 2^{-k-c} since γ has no 1s to the right of position $k+c$. Then β does the same and since γ still has no 1s to the right of position $k+c$, γ has to increase by 2^{-k-c} once again. So far

$$\gamma = n2^{-k-c} + 2^{-k-c} + 2^{-k-c} = n2^{-k-c} + 2^{-(k-1)-c}$$

and $\alpha(i) = \beta(i) = 0$ for all $i \in (k, k+n]$ while $\alpha(k) = \beta(k) = 1$. By applying the induction hypothesis again together with Lemma 3.3, the further increase of γ will be exactly $n2^{-k-c}$. So

$$\gamma = n2^{-k-c} + 2^{-(k-1)-c} + n2^{-k-c} = (n+1) \cdot 2^{-(k-1)-c}$$

as required. □

The analysis we just presented is sufficient for a construction of α, β which proves the second clause of Theorem 1.2 for the specific case that $h(n) = n + \mathbf{O}(1)$. One only has to assign attack intervals to different versions of requirement (3.2.1) and implement the h -load process in a priority fashion, gradually satisfying all conditions. In the next section we build on these ideas in order to deal with the general case for h , and in Section 3.2.3 we give the formal construction of α, β .

3.2.2 Generalized amplification games

We wish to obtain a version of Lemma 3.5 which does not depend on a fixed choice of h . For ease of notation, let $g(n) = h(n) - n$ and assume that g is non-decreasing.

It is tempting to think that perhaps, in this general case, the h -load process in $(k, k+n]$ with (α, β, γ) ends up with

$$\gamma = 2^{-k} \cdot \sum_{i \in (k, k+n]} 2^{-g(i)} \quad (\text{false amplification lower bound}). \quad (3.2.2)$$

With a few simple examples, the reader can be convinced that this is not the case. Luckily, the h -load process does give a usable amplification effect, but in order to obtain the factor in this amplification we need to calibrate the divergence of the series $\sum_i 2^{-g(i)}$, by viewing g as a step-function.

Definition 3.6 (Signature of a non-decreasing function). *Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing function. The signature of g is a (finite or infinite) sequence (c_j, I_j) of pairs of numbers c_j and intervals I_j (for $j \geq 0$), such that (I_j) is a partition of \mathbb{N} , $g(x) = c_j$ for all $x \in I_j$ and all j , and $c_j > c_{j'}$ for $j > j'$. The length of the sequence (c_j, I_j) is called the length of the signature of g .*

Note that if (c_j, I_j) is the signature of g then $\sum_i 2^{-g(i)} = \sum_j |I_j| \cdot 2^{-c_j}$, where in the latter sum the indices run over the length of the sequence (c_j, I_j) . More generally, if J is an interval of numbers which are less than the length of the signature of g we have $\sum_{i \in J} 2^{-g(i)} = \sum_{j \in J} |I_j| \cdot 2^{-c_j}$ where $I = \cup_{j \in J} I_j$. We define a truncation operation that applies to such partial sums, in order to replace the false amplification lower bound 3.2.2 with a correct lower bound.

Definition 3.7 (Truncation). *Given a real number $x \in (0, 1)$ and an increasing sequence (c_j) , for each t let*

$$T(x, c_t) = \sum_{i \leq t} n_i \cdot 2^{-c_i} \quad \text{where} \quad x = n_0 \cdot 2^{-c_0} + n_1 \cdot 2^{-c_1} + \dots = \sum_i n_i \cdot 2^{-c_i}$$

is the unique representation of x as a sum of multiples of 2^{-c_j} such that $n_{i+1} \cdot 2^{-c_{i+1}} < 2^{-c_i}$ for all i .

We now define a sequence of truncated sums that can be used in replacing the false lower bound (3.2.2) with a valid one. These quantities depend on the sequence (c_j, I_j) , which in turn depends on the function g .

Definition 3.8 (Truncated sums). *Given the finite or infinite sequence (c_j, I_j) and k such that c_j and I_j are defined, the sequence of truncated sums with respect to k is defined inductively as follows ($i < k$):*

$$\begin{aligned} S_k(0) &= T(|I_k| \cdot 2^{-c_k}, c_{k-1}) \\ S_k(i) &= T(|I_{k-i}| \cdot 2^{-c_{k-i}} + S_k(i-1), c_{k-i-1}) \quad \text{for } i > 0. \end{aligned}$$

Before we show that the h -load strategy with (α, β, γ) forces γ to grow to a suitable truncated sum, we show that the truncated sums grow appropriately, assuming that $\sum_i 2^{-g(i)}$ is unbounded. Note that for increasing i , the sum $S_k(i)$ corresponds to later stages of the h -load strategy, and intervals I_{k-i} which are closer to the decimal point. This explains the decreasing indices in the above definition and the following lemma.

Lemma 3.9 (Lower bounds on truncated sums). *For each $t < k$ we have $S_k(t) \geq \sum_{i \leq t} |I_{k-i}| \cdot 2^{-c_{k-i}} - 1$.*

Proof. Since $\sum_{i \in (c_0, c_k]} 2^{-i} < 1$ it suffices to show that for each $t < k$ we have

$$\sum_{i \leq t} |I_{k-i}| \cdot 2^{-c_{k-i}} \leq S_k(t) + \sum_{i \in (c_{k-t-1}, c_k]} 2^{-i}. \quad (3.2.3)$$

We use finite induction on $t < k$. For $t = 0$, we have $|I_k| \cdot 2^{-c_k} - S_k(0) \leq \sum_{i \in (c_{k-1}, c_k]} 2^{-i}$. Now inductively assume that (3.2.3) holds for some t such that $t < k - 1$. Then using the induction hypothesis we have:

$$\sum_{i \leq t+1} |I_{k-i}| \cdot 2^{-c_{k-i}} = \sum_{i \leq t} |I_{k-i}| \cdot 2^{-c_{k-i}} + |I_{k-t-1}| \cdot 2^{-c_{k-t-1}} \leq S_k(t) + \sum_{i \in (c_{k-t-1}, c_k]} 2^{-i} + |I_{k-t-1}| \cdot 2^{-c_{k-t-1}}.$$

But $S_k(t) + |I_{k-t-1}| \cdot 2^{-c_{k-t-1}}$ is clearly bounded above by $T(S_k(t) + |I_{k-t-1}| \cdot 2^{-c_{k-t-1}}, c_{k-t-2}) + \sum_{i \in (c_{k-t-2}, c_{k-t-1}]} 2^{-i}$. So:

$$\sum_{i \leq t+1} |I_{k-i}| \cdot 2^{-c_{k-i}} \leq T(S_k(t) + |I_{k-t-1}| \cdot 2^{-c_{k-t-1}}, c_{k-t-2}) + \sum_{i \in (c_{k-t-2}, c_k]} 2^{-i} = S_k(t+1) + \sum_{i \in (c_{k-t-2}, c_k]} 2^{-i},$$

which concludes the induction step and the proof. \square

Now, in general, the signature of g may be finite or infinite. The case in which this signature is finite, however, corresponds to the situation $h(n) = n + \mathbf{O}(1)$, which was previously dealt with by Yu and Ding in [DY04], as discussed earlier. For ease of notation, we therefore assume from now on that the signature of g is infinite.

Suppose that we apply the h -load process in $I = \bigcup_{j \in [k-t, k]} I_j$. Then we want to show that γ will be larger than $2^{-m} \cdot S_k(t)$ by the end of the process, where m is the least element of I . We will derive this (see Corollary 3.11) from the following more general result, which can be proved by induction. We note that in (3.2.5), in the trivial case where $t = 0$ we let $S_k(t-1) := 0$.

Lemma 3.10. *Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing computable function with signature (c_j, I_j) (of infinite length), let $h(x) = x + g(x)$ and suppose $t \leq k$. If m is either inside I_{k-t} or is the largest number which is less than all numbers in I_{k-t} , then the final value of γ after the h -load process with (α, β, γ) in the interval*

$$(m, \max I_{k-t}] \cup \left(\bigcup_{i \in (k-t, k]} I_i \right) \quad (3.2.4)$$

has the property that

$$T(2^m \cdot \gamma, c_{k-t}) = S_k(t-1) + (\max I_{k-t} - m) \cdot 2^{-c_{k-t}}. \quad (3.2.5)$$

Proof. We use finite induction on the numbers $t \leq k$ and the numbers m in $I_{k-t} \cup \{\min I_{k-t} - 1\}$. For the case when $t = 0$, the h -load process occurs in the interval $(m, \max I_k]$, where m is either $\min I_k - 1$ or any number in I_k . In this case, by Lemma 3.5 we have $T(2^m \cdot \gamma, c_k) = (\max I_k - m) \cdot 2^{-c_k}$. Therefore in this case (3.2.5) holds.

Now consider some $t > 0$ (and, of course, $t \leq k$) and any $m \in [\min I_{k-t} - 1, \max I_{k-t})$, and inductively assume that when the attack occurs in the interval

$$(m+1, \max I_{k-t}] \cup \left(\bigcup_{i \in (k-t, k]} I_i \right) \quad (3.2.6)$$

the final value of γ satisfies

$$T(2^{m+1} \cdot \gamma, c_{k-t}) = S_k(t-1) + (\max I_{k-t} - m - 1) \cdot 2^{-c_{k-t}}. \quad (3.2.7)$$

Note that this conclusion follows from the induction hypothesis, even in the case that $m = \max I_{k-t} - 1$, since then the induction hypothesis gives:

$$T(2^{m+1} \cdot \gamma, c_{k-(t-1)}) = S_k(t-2) + |I_{k-(t-1)}| \cdot 2^{-c_{k-(t-1)}}. \quad (3.2.8)$$

Now if we let the r.h.s. equal δ , then by definition $T(\delta, c_{k-t}) = S_k(t-1)$, giving 3.2.7, as required.

There are two qualitatively different cases regarding the value of $m+1$. The first one is when $m+1 = \max I_{k-t}$, which means that $m+1$ is the first number in the latest interval I_{k-t} . The other case is when $m+1 < \max I_{k-t}$ and $m+1 \geq \min I_{k-t}$. What is special in the first case is that half-way through the h -load process in the interval (3.2.4), when $\alpha(m+1)$ changes, this is the first time that some $\alpha(i)$ change requires γ to change on digit $i + c_{k-t}$ (or before that). All previous requests on γ required a change at $i + c_{k-t+s}$ for $s > 0$. In other words, the case $m+1 = \max I_{k-t}$ is special because it is when we cross into a new interval. Despite this apparent difference between the two cases, they can be dealt with uniformly, as we now show.

The attack on the interval (3.2.4) can be partitioned into three phases. Let ℓ be the maximum of the interval (3.2.4). Recall that each stage of this attack consists of adding $2^{-\ell}$ alternately to α and β . The first phase of the attack consists of those stages up to the point where $\alpha(m+1) = \beta(m+1) = 0$ and $\alpha(i) = \beta(i) = 1$ for all i in the interval (3.2.6). According to the induction hypothesis, at the end of the first phase of the attack 3.2.7 holds. By the definition of $S_k(t-1)$, this number is an integer multiple of $2^{-c_{k-t}}$. At the next stage of the attack, $\alpha(m+1)$ will turn from 0 to 1, forcing γ to change at position $m+1 + c_{k-t}$ or higher. This means that $T(\gamma, m+1 + c_{k-t})$ will increase by $2^{-m-1-c_{k-t}}$, because it does not have any 1s after position $m+1 + c_{k-t}$. In other words, $T(2^{m+1} \cdot \gamma, c_{k-t})$ will increase by $2^{-c_{k-t}}$. Then $\beta(m+1)$ will turn from 0 to 1, forcing γ to change again at position $m+1 + c_{k-t}$ or higher. This means that $T(2^{m+1} \cdot \gamma, c_{k-t})$ will increase by another $2^{-c_{k-t}}$. The latter two stages make up the second phase of the attack. The third and final phase consists of all remaining stages of the attack. At the end of the second phase $T(2^{m+1} \cdot \gamma, c_{k-t})$ has increased by

$$2 \cdot 2^{-c_{k-t}} + S_k(t-1) + (\max I_{k-t} - m - 1) \cdot 2^{-c_{k-t}}$$

compared to its initial value (which was 0). By applying the induction hypothesis and the accumulation lemma (Lemma 3.3), at the end of the attack $T(2^{m+1} \cdot \gamma, c_{k-t})$ then equals:

$$2 \cdot 2^{-c_{k-t}} + 2 \cdot (S_k(t-1) + (\max I_{k-t} - m - 1) \cdot 2^{-c_{k-t}}) = 2 \cdot (S_k(t-1) + (\max I_{k-t} - m) \cdot 2^{-c_{k-t}}).$$

By definition $S_k(t-1)$ is an integer multiple of $2^{-c_{k-t}}$. The equation above therefore establishes (3.2.5) as required, since if $T(2^{m+1} \cdot \alpha, c) = 2 \cdot C$ where C is an integer multiple of 2^{-c} , then $T(2^m \cdot \alpha, c) = C$. This concludes the induction step and the proof of the lemma. \square

The main construction employs h -load processes in unions of consecutive intervals I_j from the signature of g . So we only need the following special case of Lemma 3.10.

Corollary 3.11 (Lower bound). *Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing computable function with infinite signature (c_j, I_j) , and let $h(x) = x + g(x)$. The final value of γ after the h -load process with (α, β, γ) in the interval $I = \bigcup_{j \in [k-t, k]} I_j$ is at least $2^{-m} \cdot S_k(t)$, where m is the largest number less than all elements of I .*

Proof. Since m is the largest number which is less than all numbers in I_{k-t} , we have $|I_{k-t}| = \max I_{k-t} - m$, so by the definition of the reduced sums and Lemma 3.10 we get

$$T(2^m \cdot \gamma, c_{k-t}) = S_k(t-1) + |I_{k-t}| \cdot 2^{-c_{k-t}}$$

so

$$T(2^m \cdot \gamma, c_{k-t-1}) \geq T(T(2^m \cdot \gamma, c_{k-t}), c_{k-t-1}) = T(S_k(t-1) + |I_{k-t}| \cdot 2^{-c_{k-t}}, c_{k-t-1}) = S_k(t),$$

which means that $\gamma \geq 2^{-m} \cdot S_k(t)$. \square

We are ready to describe an explicit construction of two reals α, β establishing the second clause of Theorem 1.2 for the (remaining) case that g has infinite signature.

3.2.3 Construction of the two reals and verification

Let g be a non-decreasing computable function with infinite signature such that $\sum_i 2^{-g(i)} = \infty$, and let $h(x) = x + g(x)$. Let $(\Phi_e, \Psi_e, \gamma_e)$ be an effective enumeration of all triples of Turing functionals Φ_e, Ψ_e , and c.e. reals γ_e . According to the discussion at the beginning of Section 3.2, for the proof of the second clause of Theorem 1.2, it suffices to construct two c.e. reals α, β such that the following requirements are met.

$$\mathcal{R}_e : \alpha \neq \Phi_e^{\gamma_e} \quad \vee \quad \beta \neq \Psi_e^{\gamma_e}. \quad (3.2.9)$$

Let (c_j, I_j) be as specified in Definition 3.6. The construction consists of assigning appropriate intervals J_e to the requirements \mathcal{R}_e and implementing the h -load process with $(\alpha, \beta, \gamma_e)$ in J_e independently for each e , in a priority fashion. We start with the definition of the intervals J_e , which is informed by the lower bound established in Corollary 3.11. We define a computable increasing sequence (n_j) inductively and define:

$$J_e = \bigcup_{j \in (n_e, n_{e+1}]} I_j.$$

Let $n_0 = 1$. Given n_e , let n_{e+1} be the least number which is greater than n_e and such that if m is the largest number in I_{n_e} then:

$$2^{-m} \cdot S_{n_{e+1}}(n_{e+1} - n_e) > 1.$$

According to our hypothesis concerning g and Lemma 3.9, such a number n_{e+1} exists, so the definition of (n_j) is sound. Fix a universal enumeration with respect to which we can approximate the Turing functionals Φ_e, Ψ_e and the c.e. reals γ_e . The suffix ‘[s]’ on a formal expression means the approximation of the expression at stage s of the universal enumeration. The stages of the construction are the stages of the universal enumeration. Recall from Definition 3.4 that each step in the h -load process on J_e consists of adding $2^{-n_{e+1}}$ alternately to α and β . According to this process, we will only allow \mathcal{R}_e to act at most $2^{n_{e+1}-n_e} - 1$ times for each of α, β . We say that strategy \mathcal{R}_e is *active* at stage s if:

- (a) $\alpha(t)[s] = \Phi_e^{\gamma_e}(t)[s]$ and $\beta(t)[s] = \Psi_e^{\gamma_e}(t)[s]$ for all $t \in \cup_{i \leq e} J_i$.
- (b) strategy \mathcal{R}_e has acted less than $2 \cdot (2^{n_{e+1}-n_e} - 1)$ many times in previous stages.

Construction. At each stage s , if there exists no e such that \mathcal{R}_e is active, then go to the next stage. Otherwise, let e be the least such. We say that s is an ‘ e -stage’. If there has not been a previous e -stage, or if β was increased at the most recent e -stage, then add $2^{-n_{e+1}}$ to α and say that \mathcal{R}_e acts on α at stage s . Otherwise add $2^{-n_{e+1}}$ to β and say that \mathcal{R}_e acts on β . Go to the next stage.

We now verify that the construction produces approximations to reals α, β such that requirement \mathcal{R}_e is met for each e . Note that \mathcal{R}_e is only allowed to act at most $2^{n_{e+1}-n_e} - 1$ times for each of α, β . This means that no action of \mathcal{R}_e can change a digit of α or β which is outside J_e . In particular, the approximations to α and β that are defined in the construction converge to two reals in $[0, 1]$.

Finally, we show that for each e , requirement \mathcal{R}_e is met and is active at only finitely many stages. Inductively suppose that this is the case for \mathcal{R}_i , $i < e$. Note that \mathcal{R}_e can only be active when it has acted less than $2 \cdot (2^{n_{e+1}-n_e} - 1)$ many times. Therefore, by the construction and the induction hypothesis, \mathcal{R}_e is only active at finitely many stages. It remains to show that \mathcal{R}_e is met. Towards a contradiction, suppose that this is not the case. Then for some c.e. real γ'_e (the one that follows the least effort strategy with respect to α, β) throughout a subsequence $s_i, i < k$ of the stages of the construction, the triple $(\alpha, \beta, \gamma'_e)$ follows a complete h -load process in J_e . By Corollary 3.11, it follows that $\gamma'_e \geq 2^{-n_e} \cdot S_{n_{e+1}}(n_{e+1} - n_e)$ and by the choice of J_e we have $\gamma'_e > 1$. Since the functionals Φ_e, Ψ_e have use-function $x + g(x)$, if $\gamma_e \neq \gamma'_e$ then the approximations to γ_e in the stages $s_i, i < k$ can be seen as a suboptimal response to the changes of the bits of α, β in J_e according to the amplification games of Section 3.2.1. In particular, according to Lemma 3.2 and Lemma 3.3, we have $\gamma_e[s_i] \geq \gamma'_e[s_i]$ for each $i < k$. This means that $\gamma_e > 1$, which contradicts the fact that $\gamma_e \in [0, 1]$. We conclude that \mathcal{R}_e is satisfied.

4 Proof of Theorem 1.5

Recall the following theorem.

Theorem 1.5. *Let \mathbf{a} be a c.e. Turing degree and let Ω be (an instance of) Chaitin's halting probability.*

- (1) *If \mathbf{a} is array computable, then every real in \mathbf{a} is computable from Ω with identity use.*
- (2) *Otherwise there exists a c.e. real in \mathbf{a} which is not computable from Ω with use $n + \log n$.*

Theorem 1.5 can be obtained directly by an application of the methods in [BDG10, Sections 4.2 and 4.3] to the construction in the proof of Theorem 1.2 (b). Since there are no new ideas involved in this application, and in the interest of space, we merely sketch the argument. In [BDG10, Section 4.2] it was shown that if α is in a c.e. array computable degree, then it is computable from $\sum_n g(n) \cdot 2^{-n}$ with identity use, where $g = \mathbf{O}(n)$ is a function with a nondecreasing computable approximation. It is not hard to see that this implies that α is computable from Ω with identity use, which is clause (1) of Theorem 1.5.

In [BDG10, Section 4.3] it was shown that if \mathbf{a} is an array noncomputable c.e. degree, then there exist two c.e. reals in such that no c.e. real can compute both of them with use $n + \mathbf{O}(1)$. This result was obtained by the application of array noncomputable permitting to the construction in [DY04] (which obtained two c.e. reals with the above property). The same can be done in entirely similar way to the construction in the proof of Theorem 1.2 (2), which is structurally similar to the construction in [DY04]. This extension of our argument in Section 3.2 shows that the two c.e. reals of the second clause of Theorem 1.2 (1) can be found in any array noncomputable c.e. degree. If we choose $h(n) = n + \log n$ in this theorem, we obtain the second clause of Theorem 1.5.

5 Conclusion and a conjecture

Chaitin's omega numbers are known to compute the solutions to many interesting problems, and to do so with access only to a short initial segments of the oracle. Although Ω contains the same information as the halting problem, this information is much more tightly packed into short initial segments. Despite these well-known facts, little was known about the number of bits of Ω that are needed to compute halting probabilities or c.e. sets, and in particular the asymptotically optimal use of the oracle in such computations.

In this work we provide answers to these questions, and expose various connections with current research in algorithmic randomness. Moreover, our results point to several other open problems which appear to be an interesting direction of research.

Barmpalias and Lewis-Pye [BL06a] showed that there exists a c.e. real which is not computable from any omega number with use $n + \mathbf{O}(1)$. Another proof of this result (which was used to obtain a characterisation of the degrees of the c.e. reals with this property) was given in [BDG10, Section 4.3]. We conjecture that this holds more generally in the spirit of this paper. In particular, if g is a computable non-decreasing function such that $\sum_i 2^{-g(i)} = \infty$, we conjecture that there exists a c.e. real which is not computable by any omega number with use $g(n) + n$.

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