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Axel Gandy and Luitgard A. M Veraart

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A Bayesian methodology for systemic risk assessment in financial networks

Axel Gandy *
Imperial College London

Luitgard A. M. Veraart †
London School of Economics and Political Science

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Abstract

We develop a Bayesian methodology for systemic risk assessment in financial networks such as the interbank market. Nodes represent participants in the network and weighted directed edges represent liabilities. Often, for every participant, only the total liabilities and total assets within this network are observable. However, systemic risk assessment needs the individual liabilities. We propose a model for the individual liabilities, which, following a Bayesian approach, we then condition on the observed total liabilities and assets and, potentially, on certain observed individual liabilities. We construct a Gibbs sampler to generate samples from this conditional distribution. These samples can be used in stress testing, giving probabilities for the outcomes of interest. As one application we derive default probabilities of individual banks and discuss their sensitivity with respect to prior information included to model the network. An R-package implementing the methodology is provided.

Key words: Financial network, unknown interbank liabilities, systemic risk, Bayes, MCMC, Gibbs sampler, power law.

1 Introduction

Assessing systemic risk in financial systems is a key concern for regulators and policy makers. We think of systemic risk as the risk that some external or economic shock causes a participant in the financial system to default on their obligations which then leads to severe knock-on effects to other participants, such as their default; see (Hurd, 2015, Section 1.2) for a discussion of different definitions of systemic risk.

Participants in these financial systems consist of various types of institutions, e.g. banks, hedge funds, pension funds or insurance companies. As banks are key players for systemic risk assessment we will, for simplicity, refer to all participants in the financial system as banks.

*Imperial College London, Department of Mathematics, South Kensington Campus, London, SW7 2AZ, UK, a.gandy@imperial.ac.uk

†London School of Economics and Political Science, Department of Mathematics, Houghton Street, London WC2A 2AE, UK, l.veraart@lse.ac.uk

Systemic risk is often assessed using a network model (Elsinger et al., 2013), in which nodes represent banks and weighted directed edges represent liabilities. Stress tests apply shocks to the network and analyse their consequences. Banks who survive the initial shock might default due to contagion caused by other banks no longer satisfying their obligations. Such balance sheet spill-over effects are one main channel of systemic risk. We focus on balance sheet contagion in this paper, but our methodology can also be used in stress tests that include additional channels of systemic risk such as fire sales (Upper, 2011; Hurd, 2015).

Often, the full network of interbank liabilities is not available. Aggregates for each bank, such as the total interbank liabilities and assets, are more readily available (Upper & Worms, 2004; Elsinger et al., 2013; Anand et al., 2014). These aggregates can for example be obtained from public balance sheet information. Looking beyond public information, it is not the case that regulators can observe financial networks in full. They usually only have limited data on financial institutions not required to report to them, such as banks outside their jurisdiction[1] or non-bank financial institutions such as insurance companies, hedge funds etc. Typically also among the data that they do obtain only *large* exposures are reported (Langfield et al., 2014). We provide more details on this in Section 6 of the Supporting Document.

As of 2015, there are still large gaps in the data available on bilateral exposures and interlinkages between financial systems[2] [3]. Hence, finding a rigorous and tractable way to deal with the lack of data when performing stress tests is of paramount importance.

We focus on the situation where the total interbank liabilities and assets are known but not all of the individual liabilities are known. To be precise, we describe interbank liabilities in a network with n banks by a liabilities matrix $L \in [0, \infty)^{n \times n}$ where L_{ij} represents the nominal liability of bank i to bank j. Knowing the total interbank liabilities and assets means that the row and column sums of L are known. Often, all or some of the L_{ij} are not observable.

The number and severity of contagious defaults in financial markets is strongly dependent on the often unobserved bilateral liabilities L_{ij}. Any stress testing results will therefore significantly depend on the method used for filling in the missing information. We will provide a new methodology to do this.

The formal setting where the row and column sums of a matrix describing a financial network are known but the matrix itself is not known has been recognised as a key problem by several major central banks.

[1] Some additional information beyond national data on so-called global systemically important banks is starting to become available due to the G-20 Data Gaps Initiative (Financial Stability Board & International Monetary Fund, 2015).

[2] "Even among the world's largest banks, data on their bilateral exposures to one another remains partial and patchy, especially for off balance sheet positions and securities holdings. That means large parts of the core of the international banking map remain, essentially, uncharted territory", (Haldane, 2015, p. 14)

[3] The missing data problem was also acknowledged in the recent report on the Bank of England's approach to stress testing the UK banking system "However, models need good data and there are big gaps in the data on interlinkages between different parts of the financial system and common exposures across the financial system. The lack of data makes it difficult to build up a point-in-time picture of the interlinkages between different parts of the financial system and calibrate quantitative models" (Bank of England, 2015, p. 31).

Currently, an international study involving several central banks and their proprietary data on interbank, payment, Repo, FXS, CDS, equity and derivatives networks is being conducted to test network reconstruction methods from the row and column aggregates for these data, see Anand (2015). Initial results suggest that the performance of the tested methods depends strongly on the similarity measure used and the sparsity of the underlying network. Since our new methodology is more flexible than existing methods, it can cope with a wide range of underlying network structures. Furthermore, we consider a more general information setting, by allowing some entries of the matrix to be known. This allows for an improved data situation on parts of the network that either exists already or might become available in the future. The classical situation where all entries of the network are unknown is included in our model as a special case.

Our paper makes two main contributions. First, we provide a Bayesian framework for a distribution on the bilateral exposures L conditional on observed balance sheet data. We start by proposing a probabilistic model for the liabilities matrix L which we then condition on the observed row and column sums as well as possibly on some observed individual entries. We construct a Gibbs sampler for this distribution. Our methodology is flexible enough to include a wide range of possible network structures (adjacency matrices), such as complete networks, Erdős-Rényi networks, tiered networks or scale-free networks. We can also model a wide range of probability distributions for the weights (i.e., the liabilities), such as light or heavy-tailed distributions. We are not aware of any other approach that achieves this. An implementation is available as an R-package (systemicrisk) on the Comprehensive R Archive Network (CRAN).

Second, we illustrate how our framework can be applied to assess systemic risk in financial networks. The key novelty is that we can give probabilities for the outcomes of stress tests. These probabilities are based on stochastic assumptions that are consisted with stylised facts observed in financial networks. The variability in the outcomes clearly shows that only using a point estimate for the liabilities matrix gives a very restricted view of the systemic risk.

The current standard approach for deriving individual interbank liabilities from aggregates is to minimise the Kullback-Leibler (KL) divergence between the liabilities matrix and a previously specified input matrix (Upper & Worms, 2004). This approach has been used to estimate interbank exposures from balance sheet data for banks in Germany (Upper & Worms, 2004), UK (Wells, 2004; Elsinger et al., 2006) and Belgium (Degryse & Nguyen, 2007). One drawback of the KL method is that the resulting interbank liabilities usually form a complete network, meaning that all entries of L (except the diagonal) are positive. However, empirical research (Craig & Von Peter, 2014) shows that such networks are usually sparse. Furthermore, using the KL method can underestimate systemic risk, as Mistrulli (2011) illustrate using bilateral exposures from the Italian interbank market. Parts of the problem can be overcome by using different input matrices for the KL method, see e.g. Chen et al. (2014).

Mastromatteo et al. (2012) have proposed a *message-passing algorithm* for estimating interbank exposures. Their aim is to fix a global level of sparsity for the network, i.e., the adjacency matrix. Once this is given the weights on the existing links are distributed similarly to the KL-method. Our methodology is more general since we can control sparsity on an individual exposures level and have an additional methodology for distributing the weights that is consistent with empirical properties of these weights (such as heavy tails).

Anand et al. (2014) have proposed the *minimum density (MD)* method. They minimise the total number of edges that is consistent with the aggregated interbank assets and liabilities. They argue that the MD method tends to overestimate contagion and therefore can together with the KL method be used to provide upper and lower bounds for stress test results. They also provide an extension to derive less sparse results for the underlying network structure. We will see in Section 5.2, however, that general monotonicity arguments of the type that more links result in more stable network and fewer links result in more systemic risk are not true in general. Hence, any attempts to derive bounds on systemic risk by purely optimising over the degree of completeness is unlikely to provide the full picture.

All previously mentioned methods produce a point estimate for the financial network and this is treated as the *true* network when performing stress tests. No additional information accounting for the uncertainty inherent in all estimation procedures such as confidence intervals etc. is considered. This is already problematic, when one wants to assess the health of the financial system as a whole, but it is even more problematic for deriving conclusions about individual financial institutions.

A simulation-based approach to reconstructing financial networks has been proposed by Hałaj & Kok (2013). They randomly generate different network structures. In contrast to our approach, the probabilistic model and the distribution from which the samples are generated, however, is not completely characterised. In particular, it is not explicitly designed to reproduce stylised facts observed in financial networks.

Another simulation-based approach (Moussa, 2011) samples financial networks from an externally given random graph model that does not satisfy the balance sheet constraints. The balance sheet constraints are only met in expectation, by weighting the samples appropriately. In our approach, however, every sample satisfies the balance sheet constraints individually. Furthermore, our approach is Bayesian, making it possible to learn properties of the network from observed information.

Musmeci et al. (2013) consider the problem of reconstructing topological properties from limited information. They use a bootstrapping approach and decide on link existence via a fitness model. Their method estimates global properties of the underlying network, but does not try to reconstruct individual links.

There is also a substantial body of literature on related mathematical and statistical problems that deal with the problem of sampling or counting matrices with given marginals whose entries are (non-negative) integer values or binary values (i.e., 0 or 1). For some recent overview and results see e.g. Barvinok (2012),

Miller et al. (2013). In statistics this problem arises when evaluating tests for independence in contingency tables, for which the sampling needs to be done on the uniform distribution on all matrices. One approach is to start with one matrix that is consistent with the given row and column sums and apply a series of transformations that preserves row and column sums, see e.g. Ryser (1960). He proves an interchange theorem, see (Ryser, 1960, Theorem 1.2), that shows that all matrices within the class of interest can be obtained via suitable swaps/ changes of a subset of elements of the matrix. Monte Carlo methods have been proposed to enumerate and simulate such matrices with fixed marginals, see e.g. Snijders (1991) for binary matrices or Diaconis et al. (1998) who propose an algorithm based on a Markov chain with Metropolis step for contingency tables. When swapping methods are applied within a Monte Carlo framework, one needs to be careful to choose admissible matrices with the correct weights, see e.g. Zaman & Simberloff (2002). Nonnegative matrices with fixed marginals also occur as solutions to transportation problems and are often referred to as transportation polytopes, see e.g. (Brualdi, 2006, Section 8.1).

Considering a liabilities matrix is substantially different from this strand of the literature and its methods are not directly applicable in our context: the elements of the liabilities matrix are not restricted to either 0-1 or nonnegative integers and we have the additional constraint that the diagonal of the liabilities matrix has to be zero (and that potentially some additional entries are known).

Glasserman & Young (2015) study the likelihood of contagion in financial networks based on aggregates available from balance sheet data. They compared expected losses within a given financial network to the losses in the situation where all liabilities from the interbank market were replaced by liabilities to external entities. They find that for some stylised default mechanisms the difference in expected losses can be quite small. However, if more advanced default mechanisms are used, accounting e.g. for market frictions such as bankruptcy costs, they find that network structure does matter. This supports the idea that filling in the missing network structure is one of the key problems of systemic risk assessment.

2 The network of interbank liabilities

We consider a financial system consisting of $n \in \mathbb{N}$ banks with indices $\mathcal{N} = \{1, \dots, n\}$. We describe liabilities between the banks through a *liabilities matrix* which we define to be an $n \times n$ matrix L with nonnegative entries. For notational convenience we only consider square liabilities matrices. L_{ij}, $i, j \in \mathcal{N}$, represents the nominal liability of bank i to bank j, i.e., the payment that is due from bank i to bank j.

We can think of the banks representing the nodes in a network. If $L_{ij} > 0$ then there is a directed edge from node i to node j with weight L_{ij}. To indicate whether there is a link between two nodes we also use the adjacency matrix $\mathcal{A} = (\mathcal{A}_{ij}) \in \mathbb{R}^{n \times n}$ defined by $\mathcal{A}_{ij} = 1$ whenever $L_{ij} > 0$ and $\mathcal{A}_{ij} = 0$ otherwise.

We consider the situation in which the liabilities matrix is not fully known. We allow certain fixed elements of the liabilities matrix to be known or observed. For this, we introduce the matrix $L^* \in \mathcal{L}^* :=$

$(\{*\} \cup [0,\infty))^{n\times n}$ where $L^*_{ij} = *$ means that the liability between i and j is unknown. We let $\mathcal{F} = \{(i,j) : L^*_{ij} \neq *\}$ be the positions where the liabilities matrix is known, i.e. fixed to a given value.

Our canonical example is that there are no liabilities from a bank to itself, i.e. $L^*_{ii} = 0$ for all $i \in \mathcal{N}$ but that all other liabilities are unknown, i.e. $L^*_{ij} = *$ for all $i \neq j$. We call this the *minimal observation setting*. Other examples include the view-point from an individual market participant, who knows their own liabilities and assets, but only the aggregates for other participants and the view-point of a regulator, who is able to observe individual liabilities from banks reporting to them, but only aggregates from other banks.

Definition 2.1 (Liabilities matrix)**.** *A matrix $L = (L_{ij}) \in [0,\infty)^{n\times n}$ is called a* liabilities matrix *respecting L^* if $\forall i,j \in \mathcal{N}$: $L^*_{ij} \neq *$ implies $L_{ij} = L^*_{ij}$. We write $L \equiv L^*$.*

We assume that we observe the total nominal interbank liabilities, the row sums of L, and the total nominal interbank assets of each bank, the column sums of L. We often need the row and column sums of a given matrix, including potentially matrices with unknown values (which will be ignored in the sums). For this we define $r, c : \mathcal{L}^* \to \mathbb{R}^n$ by $r(M) := (r_1(M), \dots, r_n(M))^\top$, $c(M) := (c_1(M), \dots, c_n(M))^\top$ and $r_i(M) := \sum_{j=1, M_{ij}\neq *}^n M_{ij}$, $c_i(M) := \sum_{j=1, M_{ji}\neq *}^n M_{ji}$.

Definition 2.2 (Admissible liabilities matrix)**.** *We call a matrix $L \in \mathbb{R}^{n\times n}$ an* admissible liabilities matrix *for $a, l \in [0,\infty)^n$ respecting $L^* \in \mathcal{L}^*$ if it is a liabilities matrix respecting L^* and satisfies*

$$c(L) = a, \quad r(L) = l. \tag{1}$$

Knowing the row and column sums of the liability matrix only (and some other fixed values) leaves considerable degrees of freedom for the liabilities matrix L. For example, in the minimal observation setting the liabilities matrix L has $n^2 - n$ unknown entries. Assuming that the row and column sums of L are known gives $2n$ linear constraints on L. At least one of these is redundant, as the sum of column sums equals the sum of row sums. Thus we have $n^2 - n - (2n-1) = n^2 - 3n + 1$ degrees of freedom when estimating L.

Under what conditions does an admissible liabilities matrix exist? The following theorem gives a necessary and sufficient condition.

Theorem 2.3 (Existence of an admissible liabilities matrix)**.** *Consider two vectors $a \in [0,\infty)^n, l \in [0,\infty)^n$ and $L^* \in \mathcal{L}^*$ satisfying $\sum_{i=1}^n a_i = \sum_{i=1}^n l_i$, $r(L^*) \leq l$ and $c(L^*) \leq a$. Then the following are equivalent:*

1. *There exists an admissible liabilities matrix L for a and l respecting L^*.*
2. *$\forall I \subset \mathcal{N}, J \subset \mathcal{N}$ with $L^*_{ij} \neq *\forall i \in I, j \in J$ we have*

$$\sum_{i\in I} \tilde{l}_i + \sum_{j\in J} \tilde{a}_j \leq A \tag{2}$$

where $\tilde{l} = l - r(L^*)$ and $\tilde{a} = a - c(L^*)$ and $A = \sum_{i=1}^n \tilde{l}_i$.

The proof in Appendix A transforms the problem into a maximum flow problem, for which there are efficient algorithms, see e.g. Cormen et al. (1990). In the minimal observation setting, (2) is equivalent to

$$a_i \leq \sum_{j \neq i} l_j \quad \forall i \in \mathcal{N}, \tag{3}$$

which requires that the assets of any bank are bounded by the total liabilities of all other banks.

Example 2.4. Consider a network with $n = 3$ banks with the minimal observation setting. The system of linear equations given by (1) has one degree of freedom. Letting $x = L_{32}$ be the free parameter, we see that the resulting general liabilities matrix $L = L(x)$ is of the form

$$L(x) = \begin{pmatrix} 0 & a_2 - x & -a_2 + l_1 + x \\ a_1 - l_3 + x & 0 & l_2 - a_1 + l_3 - x \\ l_3 - x & x & 0 \end{pmatrix}.$$

An x satisfying the nonnegativity constraint $L_{ij} \geq 0 \ \forall i, j \in \{1, 2, 3\}$ exists iff a and l satisfy condition (3). As an example, assume $l = a = (1, 1, 1)$. Then nonnegativity of L is equivalent to $0 \leq x \leq 1$ and

$$L(x) = \begin{pmatrix} 0 & 1 - x & x \\ x & 0 & 1 - x \\ 1 - x & x & 0 \end{pmatrix}. \tag{4}$$

For $x = 1/2$ we obtain a complete network with the same weight of $1/2$ for all edges. This is the network one would obtain using KL divergence minimisation (with input matrix given by $\tilde{L} = (\tilde{L}_{ij}) \in \mathbb{R}^{n \times n}$, where $\tilde{L}_{ij} = a_i l_j / \sum_{\nu=1}^{n} a_\nu \, \forall i \neq j$, $\tilde{L}_{ii} = 0 \, \forall i$). For $x \in (0, 1) \setminus \{1/2\}$ the network is also complete but with inhomogeneous weights. Setting $x = 0$ or $x = 1$ gives a cycle and the network is no longer complete. The minimum number of possible edges in this example is 3 which corresponds to the two cycle networks. The maximum number of edges is 6 which corresponds to the complete networks.

3 A Bayesian methodology for estimating interbank liabilities

In this section, we describe probabilistic models for the liabilities matrix. We first describe a basic model, which we then use as building block in hierarchical models. Throughout we assume that we have an underlying matrix $L^* \in \mathcal{L}^*$ describing the observed elements.

3.1 Basic model

The basic model first constructs an adjacency matrix $\mathcal{A} = (\mathcal{A}_{ij})$ through a generalised version of the Erdős-Rényi model (Erdős & Rényi, 1959): directed edges from i to j are generated via independent Bernoulli trials with success probabilities $p_{ij} \in [0, 1]$ $i, j \in \mathcal{N}$. Second, weights (i.e., liabilities) are attached to the existing directed edges using exponential distributions. To be precise, the model is as follows.

$$\begin{aligned} \mathbb{P}(\mathcal{A}_{ij} = 1) &= p_{ij}, \\ L_{ij} | \{\mathcal{A}_{ij} = 1\} &\sim \text{Exponential}(\lambda_{ij}). \end{aligned} \tag{5}$$

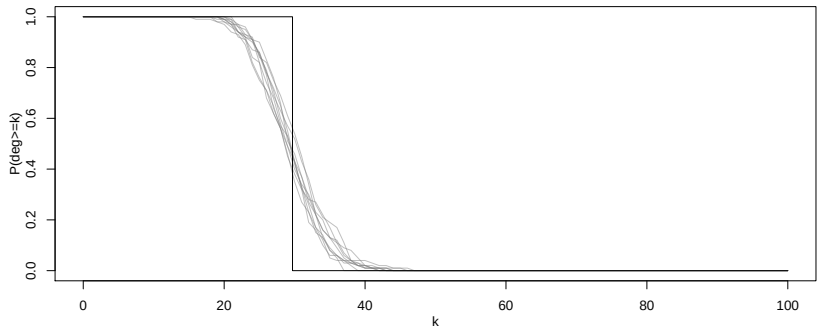


Figure 1: Survival function of the out-degree distribution of 10 simulated instances of the unconditional basic model (5) with $n = 100$ and $p_{ij} = 0.3\mathbb{I}(i \neq j)$.

Model parameters are two matrices: $p \in [0,1]^{n\times n}$, where p_{ij} is the probability of the existence of a directed edge from i to j, and $\lambda \in (0,\infty)^{n\times n}$, which governs the distribution of the weights given that an edge exists. Figure 1 shows an illustration of the out-degree distribution for a homogeneous choice of p.

We assume that we observe the following from the basic model (5):

$$r(L) = l, \quad c(L) = a, \quad L \equiv L^*, \tag{6}$$

i.e., we only observe the row sums and column sums (l, a) and some elements of the liabilities matrix given by L^*, the liabilities matrix L is not directly observable.

Our main interest lies in the distribution of some functional h of L ($h(L)$) conditional on l, a, L^*. For example, we consider in Section 5.1 functions h that compute the vector of default indicators.

The distribution of $h(L)|l, a, L^*$ it not available in closed form; however it can be approximated using MCMC methods. Indeed, the relatively simple form of the model (including the choice of an exponential distribution of the existing liabilities) allows us to construct a Gibbs sampler (Section 4.1), for which we need to compute certain conditional distributions.

It is a very flexible model. We will consider two setups in later examples, but other choices are possible: In Section 5.3.1 we consider a homogeneous setup in which $p_{ij} = \tilde{p}\mathbb{I}(i \neq j)$ and $\lambda_{ij} = \tilde{\lambda}$ for fixed $\tilde{p}$ and $\tilde{\lambda}$; the resulting adjacency matrix is then essentially a classical Erdős-Rényi graph. In Section 5.3.2 we will consider core-periphery models where the elements of the parameter matrices are not identical.

3.2 Identifiability

The matrices p and λ cannot be fully identified merely by observing the row and column sums. Specifically, the following proposition shows that in the minimal observation setting p cannot be identified from row and column sums by showing that for any (reasonable) matrix p there exits a matrix of rate parameters $\lambda = (\lambda_{ij})$ s.t. $\sum_{i=1}^n \mathbb{E}(L_{ij}) = a_j$ and $\sum_{j=1}^n \mathbb{E}(L_{ij}) = l_j$.

Proposition 3.1. *Let $l, a \in (0,\infty)^n$ with $A = \sum_{i=1}^n l_i = \sum_{i=1}^n a_i$ and $a_i + l_i < A\ \forall i$. Let $p \in [0,1]^{n\times n}$*

with $p_{ij} > 0 \; \forall i \neq j$, *diag*$(p) = 0$. *Then*

$$\exists(\lambda_{ij}) \textit{ such that } \forall j : \sum_{i=1}^{n} \mathbb{E}(L_{ij}) = a_j \textit{ and } \forall i : \sum_{j=1}^{n} \mathbb{E}(L_{ij}) = l_i.$$

A proof is given in Appendix A. The consequence of this lack of identifiability is that one needs to impose some structure on p and λ, e.g. by setting these to specific values for the basic model or by assuming a hierarchical model that makes assumptions about the network structure. This choice will be specific to the application under consideration.

3.3 Hierarchical models

The basic model can be embedded into a larger Bayesian model, where the matrices p and λ are themselves random quantities. This allows incorporation of various features that one might observe in empirical data (heavy or light tailed distribution etc.). Making p random allows us to model a wide range of degree distributions including power laws. Making λ random in such a larger model gives us great flexibility on the distribution of the liabilities. Indeed, a large range of important probability distributions can be characterized by exponential mixture models. An important result in this context is Bernstein's theorem, which states that every completely monotone probability density function is a mixture of exponential distributions (Feldmann & Whitt, 1997, Theorem 3.1.). For example, the Pareto II distribution (which can be heavy-tailed) is a Gamma mixture of exponential distributions (Harris, 1968), see also Section 4.2 of the Supporting Document.

We consider the following hierarchical model in which we assume that p and λ are influenced by some underlying random parameter θ. In addition to (5), we assume

$$\begin{aligned} \theta &\sim \pi(\theta), \\ (p_{ij}, \lambda_{ij})_{i,j \in \mathcal{N}} &= f(\theta), \end{aligned} \tag{7}$$

where π is an a-priori distribution on θ and f is a given function. We will explore two examples of such hierarchical models in the following.

3.3.1 Conjugate distribution model

This model assumes that all components of p (apart from the diagonal) and all components from λ are equal but random. More precisely, we set $\theta = (\tilde{p}, \tilde{\lambda})$ and assume

$$\begin{aligned} \tilde{p} \sim \text{Beta}(\alpha, \beta), \quad \tilde{\lambda} \sim \text{Gamma}(\gamma, \delta), \\ p_{ij} = \tilde{p}\mathbb{I}(i \neq j), \quad \lambda_{ij} = \tilde{\lambda}, \quad i, j \in \mathcal{N}, \end{aligned}$$

for some parameters $\alpha, \beta, \gamma, \delta$. The prior on $\tilde{p}$, $\tilde{\lambda}$ is flexible and brings advantages in constructing samplers as the conditional distribution of $\tilde{p}|L$ follows a Beta distribution and $\tilde{\lambda}|L$ follows a Gamma distribution.

A tiered financial network in which subsets of banks are assumed to be similar can be defined by partitioning the matrix L and using independent models of the above type for each element of the partition.

3.3.2 Fitness model

Fitness models (see e.g. Caldarelli et al., 2002; Servedio et al., 2004) have mainly been used to model undirected non-weighted networks. Typically, each node i is equipped with a random "fitness" x_i (whose pdf we denote by ρ) and links between nodes with fitness x_i and x_j are formed with probability $f(x_i, x_j)$ for a given function f. Several choices for f and ρ have been proposed, giving rise to different degree distributions. In particular, power laws for the degree distributions can be obtained (Servedio et al., 2004). A popular choice for f is the product of the (potentially transformed) fitnesses, i.e. $f(x_i, x_j) = g(x_i)g(x_j)$ for some function g. This structure limits the expected degree of the largest bank. Since we want to have the possibility of one bank having connections to a large number of banks we do not use a product structure.

Our model will be a fitness model in which every bank i has an underlying fitness x_i which influences both its propensity to generate links (via p) as well as the size of its links (via λ). The model allows for a power law for the number of links of banks (the degree distribution) as well as for the interbank liabilities of a bank. This is motivated by empirical studies of interbank networks which have found power law distributions for the degree distribution and the weights (see e.g. Boss et al., 2004).

Our model assumes that the fitness has an Exponential distribution with rate 1, i.e. $\rho(x) = \exp(-x)\mathbb{I}(x \geq 0)$. Furthermore, we assume that the link function depends on the two fitnesses only through the sum of the two fitnesses, i.e. $f(x_i, x_j) = \tilde{f}(x_i + x_j)$ for a suitable function $\tilde{f}$. The precise model is as follows.

$$\begin{aligned} X_i &\sim \text{Exp}(1), i \in \mathcal{N}, \\ p_{ij} &= \tilde{f}(X_i + X_j)\mathbb{I}_{\{i \neq j\}}, \quad i \neq j \in \mathcal{N}, \\ \lambda_{ij} &= G^{-1}_{\zeta,\eta}(\exp(-X_i)) + G^{-1}_{\zeta,\eta}(\exp(-X_j)) \quad i, j \in \mathcal{N}, \\ (\zeta, \eta) &\sim \pi(\zeta, \eta), \end{aligned} \tag{8}$$

where π is a suitable prior distribution, $G^{-1}_{\zeta,\eta}$ is the quantile function of a Gamma distribution with shape parameter $\zeta > 0$ and scale parameter $\eta > 0$ and

$$\tilde{f}(x) := \begin{cases} \beta \left(\frac{\gamma}{\beta}\right)^{1-\exp(-x)} \left(1 - \log\left(\frac{\gamma}{\beta}\right)\exp(-x)\right), & \text{if } \alpha = -1, \\ \beta \left(\xi + (1-\xi)e^{-x}\right)^{\frac{1}{\alpha+1}} \left\{1 + \frac{1}{\alpha+1}\frac{1-\xi}{\xi e^x + 1 - \xi}\right\}, & \text{if } \alpha \neq -1, \end{cases} \tag{9}$$

where $\xi := (\gamma/\beta)^{\alpha+1}$, $0 < \beta < \gamma \leq 1$ and $\alpha < 0$. We need to ensure $0 \leq \tilde{f}(x) \leq 1$. Straightforward calculations show that this is satisfied for $\alpha \leq -2$, but that for $\alpha > -2$ we get the additional constraints:

$$\text{if } \alpha \in (-2, 0) \setminus \{-1\} : \gamma/\beta \leq (\alpha + 2)^{\frac{1}{\alpha+1}}; \quad \text{if } \alpha = -1 : \gamma/\beta \leq e.$$

The link function $\tilde{f}$ has been constructed such that the degree distribution exhibits a power law in the sense that the pdf of the expected out-degree of a node at k is proportional to k^α, where the distribution of the expected out-degree is the distribution of the random variable $d^{\text{out}}(X)$, with $X \sim \text{Exp}(1)$ and

$$d^{\text{out}}(x) = (n-1)\int_0^\infty \tilde{f}(x+z)e^{-z}dz.$$

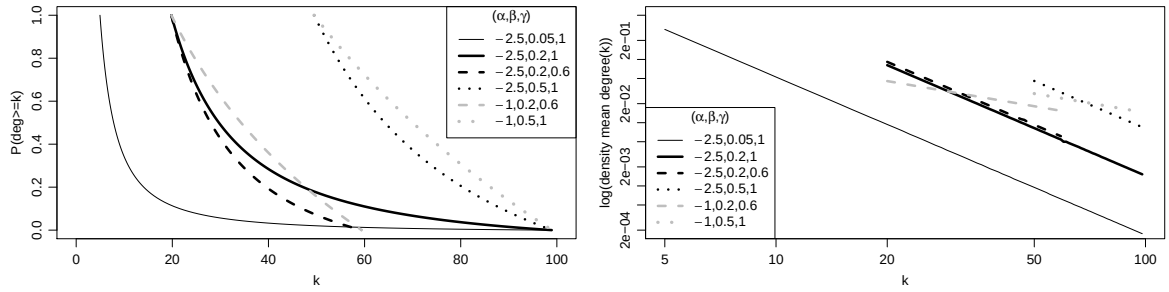


Figure 2: Survival function (left) and log-log plot of the pdf (right) of the distribution of the expected out-degree for the fitness model with $n = 100$ nodes and various choices of parameters.

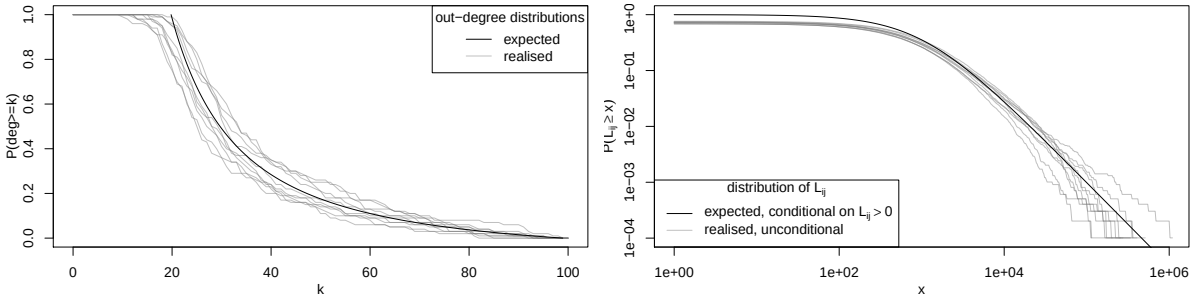


Figure 3: Survival function of the degree distribution (left) and of the entries of L (right) in simulated instances of the fitness model with $n = 100$, $\alpha = -2.5$, $\beta = 0.2$, $\gamma = 1$ and fixed $\zeta = 0.75$ and $\eta = 1000$.

$d^{\text{out}}(x)$ can be interpreted as the expected out-degree of a node with fitness x. The range of the power law is $[\beta(n-1), \gamma(n-1)]$. See Section 4.1 of the Supporting Document for details on the derivation of $\tilde{f}$, which is based on ideas in Servedio et al. (2004). Formula (15) of Servedio et al. (2004) is identical to (9) with $\alpha = -2$.

Since, $\lim_{x\to\infty} \tilde{f}(x) = \gamma$, we obtain $\lim_{x\to\infty} d^{\text{out}}(x) = \gamma(n-1)$ showing that banks with large fitness have liabilities to roughly $\gamma(n-1)$ other banks. Furthermore, $d^{\text{out}}(0) = \beta(n-1)$, showing that a bank with fitness 0 has liabilities to roughly $\beta(n-1)$ other banks. Figure 2 illustrates the flexibility of the model by showing the distribution of the expected out-degree for various parameter choices. Figure 3 shows how the realised out-degrees and the realised entries of L_{ij} scatter around the expected values. The fitness model allows a much wider range of degrees than the basic model (see Figure 1).

The parts of the model concerning the adjacency matrix were originally proposed by Servedio et al. (2004) for undirected, non-weighted networks and they derived specific formulae for the case $\alpha = -2$. We adapt their methodology to the directed/weighted network case and derive formulae for all $\alpha < 0$. The extension to using the fitness to also model the weights of the edges in this form is new.

The model for λ_{ij} is constructed such that we obtain economically reasonable behavior for the positive entries of L. First, higher values of the fitnesses will imply smaller values of λ and hence higher expected

liabilities. This is because $G^{-1}_{\zeta,\eta}(e^{-x})$ is decreasing in x.

Second, our model for L_{ij} conditional on being positive is a mixture of exponentials as the distribution on the fitness induces a mixture distribution on the λ_{ij}. Our particular choice leads to heavy tails for this distribution. Conditional on the parameters (ζ, η), we have $G^{-1}_{\zeta,\eta}(e^{-X}) \sim \text{Gamma}(\zeta, \eta)$ for $X \sim \text{Exp}(1)$ because e^{-X} has uniform distribution on $(0, 1)$ and because applying the quantile function to a $U(0, 1)$ distributed random variable gives a random variable having this quantile function. Thus the rate parameter λ_{ij} is the sum of two independent Gamma distributed random variables and hence a Gamma distributed random variable with shape parameter 2ζ and scale parameter η. A Gamma mixture of Exponentials has Pareto II distribution (Harris, 1968). Hence, the distribution of a randomly selected L_{ij} is for large arguments close to the cdf of a Pareto II distribution with shape parameter 2ζ and scale parameter $1/\eta$. I.e. for such an L_{ij} and $x > 0$, $\mathbb{P}(L_{ij} > x) \approx (1 + \eta x)^{-2\zeta}$ (if $\beta = \gamma = 1$, this is the exact distribution). We provide details on this in Section 4.2 of the Supporting Document. Boss et al. (2004) observe a power law in the interbank liabilities of the Austrian interbank market with a power around -1.87. We can choose ζ to match any power of interest. Hence, the priori model assumption is reasonable in the context of empirical data.

Third, when considering pairs of banks with very different fitnesses, the large bank essentially determines the link existence probability (to be high), and the small bank determines the weight (to be small). Indeed, consider two banks, one with a high fitness x_i and one with a low fitness x_j. The high fitness x_i will lead to a high link existence probability. The small fitness x_j leads to the transformed value e^{-x_j} being large, implying a large corresponding quantile of the Gamma distribution. Hence, there is a large summand in the definition of the corresponding λ_{ij} which is resulting in a low expected weight. We would not obtain this behaviour if we were to add the fitnesses before transforming to the desired Gamma distribution.

We have designed the specific fitness model such that it has many desirable economic features. Our Bayesian framework can easily accommodate alternative choice for the distribution of the fitness ρ, the link existence probabilities f or the relationship between the fitnesses and the rate parameter λ.

4 Sampling

We show how an MCMC algorithm (more precisely, a Gibbs-sampler) can be used to sample from the conditional distribution of the liabilities matrix given its row and column sums (l, a) and known entries. These samples can then be plugged into functionals of interest for systemic risk assessment, e.g., to work out the distribution of the number of banks defaulting or the probability of the default of a specific bank. We consider the basic model before extending the resulting methodology for hierarchical models.

In addition to the theoretical derivation of the samplers, we provide a simulation study in Section 1 of the Supporting Document which shows that our samplers are sampling from the correct distribution and that the implementation provided in the R package *systemicrisk* works.

To diagnose convergence of MCMC samplers, a large number of methods have been developed, see e.g. Plummer et al. (2006). We demonstrate some of these methods in Section 2 of the Supporting Document.

4.1 A Gibbs sampler for the basic model

A large number of algorithms for sampling from matrices conditionally on their row and column sums have been proposed; the introduction contains a short overview. Most algorithms are constructed for count data, i.e., all entries are either assumed to be binary or to be nonnegative integers. These methods are not directly applicable in our setting for two reasons. First, we are assuming that our entries are coming from absolutely continuous random variables with a point mass at 0. Second, certain elements can be set to fixed values - in the minimal observation setting the diagonal is set to 0.

We develop a Gibbs sampler to generate samples from the liabilities matrix conditional on the observed row and column sums. The key idea of a Gibbs sampler is to iteratively update one or several components of the entire parameter vector by sampling them from their joint conditional distribution given the other components of the parameter vector. Often, components of the parameter vector are updated individually, but sometimes it is necessary or more efficient to update several components of the parameter vector jointly. By repeating these updating steps one constructs a Markov Chain whose distribution converges to the target distribution as the number of updating steps tends to infinity.

In practice, the results are reported based on a single run of the chain. Often a burn-in period is used, i.e., one discards the first few samples. Afterwards one often does not use every sample but only every δth sample for a suitable nonnegative integer δ. This thinning reduces the autocorrelation of the retained samples. See Robert & Casella (2004) for background regarding MCMC methods and the Gibbs sampler.

In our case the parameter vector is the matrix L. The MCMC sampler will then produce a sequence of matrices $L^1, L^2, \ldots$ and for our quantity of interest we use the approximation

$$\mathbb{E}[h(L)|l, a] \approx \frac{1}{N} \sum_{i=1}^{N} h(L^{i\delta+b}),$$

where N is the number of samples to use in the estimation, b is the length of the burn-in period and $\delta \in \mathbb{N}$ defines the amount of thinning employed. For a specific application, a reasonable ambition is to ensure that the N dependent samples being produced from the MCMC chain are equivalent to an independent sample of size, say, $N/10$. An appropriate amount of thinning can be set using pilot runs of the chain together with estimates of the componentwise effective sample size (using e.g. the CODA package (Plummer et al., 2006)). Our R-package includes a function that determines the thinning in this way. As default burn-in we use b equivalent to 10% of the overall effort.

4.1.1 Components that can be updated

For the construction of a Gibbs sampler, updating individual components of L is not useful, as conditionally on the row and column sums and the other values of L, the individual components are uniquely

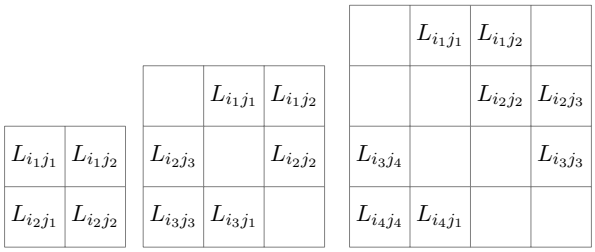


Figure 4: Illustration of submatrices to be updated via a Gibbs step (left-to-right): $k = 2, 3, 4$.

determined. Thus we need to jointly update a subset of the components of L.

The smallest possible update where we can hope to have some degrees of freedom would be based on updating a 2x2 submatrix, see the left matrix in Figure 4. Conditioning on the row and column sums and all other components of L is equivalent to conditioning on the row and column sum of the original 2x2 submatrix of L. This gives $4 - 1 = 3$ side conditions (the total row and column sums automatically match), leaving one degree of freedom if none of the elements of the submatrix are fixed.

Performing Gibbs updates of 2x2 submatrices will not be enough to iterate through the space of possible liabilities matrices. To see this, consider Example 2.4. Here, any 2x2 submatrix contains at least one element of the diagonal. Thus Gibbs updates of 2x2 matrices leave L unchanged.

This is why we use more general updates - as subcomponents to update we use cycles given by an integer $k \in \{2, \ldots, n\}$ and mutually disjoint row indices $(i_1, i_2, \ldots, i_k)$ and mutually disjoint column indices $(j_1, j_2, \ldots, j_k)$. We update L at the indices

$$\eta := ((i_1, j_1), (i_1, j_2), (i_2, j_2), \ldots, (i_k, j_k), (i_k, j_1))$$

conditional on all other values of L. We refer to such a cycle as a cycle of length k. A cycle of length k contains the indices of $2k$ elements of L. Figure 4 illustrates submatrices that can be updated along a cycle.

Conditioning on the row and column sums of the current subelements of L is equivalent to conditioning on other values of L. Thus we can implement a Gibbs sampler if we are able to sample from such a subset of values with given row and column sums. We discuss how to do this in Section 4.1.2.

We need to initialise the chain with a matrix L that satisfies $r(L) = l$ and $c(L) = a$. To generate such a matrix we could use a maximum flow algorithm on the flow network constructed in the proof of Theorem 2.3. This would result in a very sparse matrix. This would potentially lead to the need to use a long burn-in phase. To reduce this problem, we first sample an Erdős-Rényi random matrix $\tilde{L}$ with degree existence probability roughly equal to the the average degree distribution of the unconditional model and with $r(\tilde{L}) \leq l$ and $c(\tilde{L}) \leq a$ (which is easy to ensure). We then use the maximum flow-algorithm on the remaining row and column sum and add the resulting matrix to $\tilde{L}$.

Our implementation of the Gibbs sampler uses the above update steps as follows. Based on a given

matrix $L^{(t)}$ satisfying all constraints (1), we derive a new matrix $L^{(t+1)}$ again satisfying (1) by selecting a cycle of the above type and replacing the corresponding values of $L^{(t)}$ with conditionally sampled new values and by leaving the remaining values untouched.

There is considerable flexibility in how to choose which cycle to update. From a theoretical point of view it suffices to ensure that all cycles can be selected with positive probability. From a practical point of view the efficiency of the resulting MCMC sampler will be affected by this choice. We use the following method: We first sample the cycle length k from the discrete distribution which assumes that the probability of selecting a cycle of length k is given by $2^{n-k}/(2^{n-1}-1)$ for $k \in 2, \ldots, n$. This choice implies that for large n we roughly select $k = 2$ with probability $1/2$ and $k = 3$ with probability $1/4$. After that we sample $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ uniformly from $\mathcal{N}$ without replacement.

As the expected number of elements updated in an individual step is constant, the number of individual updating steps (i.e. the thinning) should be proportional to n^2, the number of elements in L - this is why we will describe the amount of thinning as multiples of n^2.

4.1.2 Conditional distribution for the Gibbs sampler

We now derive the conditional distribution of the elements of L contained in a cycle given all other values. For this, suppose we want to update L using a cycle of length $k \in \{2, \ldots, n\}$ and mutually disjoint row indices $(i_1, i_2, \ldots, i_k)$ and mutually disjoint column indices $(j_1, j_2, \ldots, j_k)$. In other words we update L at the indices $\eta = ((i_1, j_1), (i_1, j_2), (i_2, j_2), \ldots, (i_k, j_k), (i_k, j_1))$ conditional on all other values of L. Let L_η denote the vector obtained by extracting the $2k$ values of L along the cycle into a vector.

Updating L along the cycle conditional on all other values of L is equivalent to updating L along η conditional on consecutive row and column sums. To formalise this, let $g : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ be a function that takes the sum of consecutive elements, i.e.,

$$g(x) = (x_1 + x_2, x_2 + x_3, \ldots, x_{2k-1} + x_{2k}, x_{2k} + x_1) \quad \forall x \in \mathbb{R}^{2k}.$$

Let $\tilde{L}_\eta \in \mathbb{R}^{2k}$ have the same distribution as L_η. We are interested in the distribution of $\tilde{L}_\eta$ conditional on $g(\tilde{L}_\eta) = g(L_\eta)$. The only $\tilde{L}_\eta$ satisfying this are

$$\tilde{L}_{\eta_i} = L_{\eta_i} + (-1)^{i+1}\Delta,$$

for some $\Delta \in \mathbb{R}$. Thus it suffices to derive the distribution of Δ.

Next, we derive constraints on Δ and discuss how it can be sampled. If the cycle contains an observed element, i.e. $\eta_i \in \mathcal{F}$ for some i then $\Delta = 0$ is the only available choice and hence the entries along the cycle will not get changed. Otherwise, as the elements of $\tilde{L}$ need to be nonnegative, we have $\Delta \geq -L_{\eta_i}$ for i odd and $\Delta \leq L_{\eta_i}$ for i even. Putting these together gives

$$\Delta \in [\Delta_{\text{low}}, \Delta_{\text{up}}] := [-\min_{i \text{ odd}} L_{\eta_i}, \min_{i \text{ even}} L_{\eta_i}].$$

The unconditional density of $\tilde{L}_\eta$ (with respect to the sum of the Lebesgue measure and the counting

measure at 0) is

$$f(\tilde{L}_\eta) = \prod_{i=1}^{2k} \Big((1-p_{\eta_i}) \mathbb{I}(\tilde{L}_{\eta_i} = 0) + p_{\eta_i} \mathbb{I}(\tilde{L}_{\eta_i} > 0) \lambda_{\eta_i} \exp(-\lambda_{\eta_i} \tilde{L}_{\eta_i}) \Big). \tag{10}$$

When conditioning on $g(\tilde{L}_\eta) = g(L_\eta)$ we need to be careful as we are working with a measure consisting of both continuous and discrete parts. The discrete part will come into play only for $\Delta \in \{\Delta_{\text{low}}, \Delta_{\text{up}}\}$ and the continuous part for $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$.

It turns out (Appendix B.1) that if only one zero is contained in the boundary cases $\tilde{L}_{\text{low},i} = L_{\eta_i} + (-1)^{i+1}\Delta_{\text{low}}$, $i \in \{1, 2, \ldots, 2k\}$ or $\tilde{L}_{\text{up},i} = L_{\eta_i} + (-1)^{i+1}\Delta_{\text{up}}$ $i \in \{1, 2, \ldots, 2k\}$ then the conditional distribution of Δ is supported on both $\{\Delta_{\text{low}}, \Delta_{\text{up}}\}$ as well as on the continuous part $(\Delta_{\text{low}}, \Delta_{\text{up}})$. However, if more than one zero is contained in $\tilde{L}_{\text{low}}$ or $\tilde{L}_{\text{up}}$ then the entire probability mass is concentrated on the boundary case with the highest number of zeros and the probability of the intermediate case $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$ is zero. If the number of zeros is identical for both boundary cases then the conditional distribution is split between these cases.

Conditional on the intermediate case $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$, using density transformation, we show in Section B.2 that the distribution of Δ conditional on $g(\tilde{L}_\eta) = g(L_\eta)$ and $L_{\eta_i} \neq 0, i = 1, \ldots, 2k$ is given by what we call a "restricted generalised exponential distribution" with rate parameter $\sum_{i=1}^{2k}(-1)^{i+1}\lambda_{\eta_i}$ meaning that its density on the support $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$ is proportional to

$$\exp\left[-\left(\sum_{i=1}^{2k} (-1)^{i+1} \lambda_{\eta_i} \right) \Delta \right].$$

The generalisation compared to the conventional exponential distribution is that the rate parameter need not be positive. This is possible because Δ has bounded support $(\Delta_{\text{low}}, \Delta_{\text{up}})$.

One of the consequences of the above is that an updating step can delete edges as well as create edges and change weights of existing edges.

These considerations together with working out all conditional probabilities via Bayes formula lead to Algorithm 1. Some remarks on it follow. The function f^* defined in line 2 is related to the density f defined in (10) in the sense that $f^*(\Delta) = f(L_\eta + s\Delta)$. If, for at least one component of the cycle, the probability of a link is exactly 0 then there is no flexibility in the update and thus the original matrix needs to be returned; this is done in line 4. Line 7 computes the number of zeros arising in the boundary cases. If this is at most one, then both the boundary cases $\{\Delta_{\text{low}}, \Delta_{\text{up}}\}$ as well as the intermediate case $(\Delta_{\text{low}}, \Delta_{\text{up}})$ are part of the possible support. In lines 10–12 we sample among these cases with the appropriate probabilities. Lines 18–21 deal with the case that more than one value of zero is possible in this update step. In this case the boundary case with the highest number of 0s dominates the other cases, see Appendix B.

Within our R-package, Algorithm 1 is implemented in C++. This allows for fast execution of a large number of steps. For example, in a setup with roughly 30% of the elements of L being positive in a network

Algorithm 1: Sampling of a cycle conditional on row and column sums in the model (5)

Input: $k \in \mathbb{N}$, $i, j \in \{1, \dots, n\}^k$, $L \in [0,\infty)^{n\times n}$, $p \in [0,1]^{n\times n}$, $\lambda \in (0,\infty)^{n\times n}$

$\eta = ((i_1, j_1), (i_1, j_2), (i_2, j_2), \dots, (i_k, j_k), (i_k, j_1))$; $s = (1, -1, 1, \dots, 1, -1) \in \mathbb{R}^{2k}$

Let $f^* : \mathbb{R} \to \mathbb{R}, \Delta \mapsto \prod_{i=1}^{2k} \left[\mathbb{I}(L_{\eta_i} + s_i\Delta = 0)(1 - p_{\eta_i}) + \mathbb{I}(L_{\eta_i} + s_i\Delta > 0)p_{\eta_i}\lambda_{\eta_i}e^{-\lambda_{\eta_i}(L_{\eta_i}+s_i\Delta)}\right]$.

For every $\Delta \in \mathbb{R}$, let $L^\Delta \in \mathbb{R}^{n\times n}$ be defined by $L^\Delta_{\eta_i} = L_{\eta_i} + s_i\Delta$, $i = 1, \dots, 2k$ and by being equal to L for all other components.

if $\exists i : p_{\eta_i} = 0$ **then return** L

$\Delta_{\text{up}} = \min_{i=2,4,\dots,2k} L_{\eta_i}$; $\Delta_{\text{low}} = -\min_{i=1,3,\dots,2k-1} L_{\eta_i}$

if $\Delta_{low} = \Delta_{up}$ **then return** L

$n^0_{\text{up}} = \#\{i \text{ even} : L_{\eta_i} = \Delta_{\text{up}}\}$; $n^0_{\text{low}} = \#\{i \text{ odd} : L_{\eta_i} = -\Delta_{\text{low}}\}$

if $n^0_{up} \le 1$ *and* $n^0_{low} \le 1$ **then**

- $\tilde\lambda = \sum_i \lambda_{\eta_i} s_i$
- **if** $\tilde\lambda = 0$ **then** $p^* = (\Delta_{\text{up}} - \Delta_{\text{low}}) f^*(\frac{1}{2}(\Delta_{\text{up}} + \Delta_{\text{low}}))$
- **else** $p^* = \tilde\lambda^{-1}\left(\exp\left[\tilde\lambda\frac{\Delta_{\text{up}}-\Delta_{\text{low}}}{2}\right] - \exp\left[-\tilde\lambda\frac{\Delta_{\text{up}}-\Delta_{\text{low}}}{2}\right]\right) f^*(\frac{1}{2}(\Delta_{\text{up}} + \Delta_{\text{low}}))$
- Sample x from $\{1, 2, 3\}$ with probabilities $\xi f^*(\Delta_{\text{low}})$, ξp^*, $\xi f^*(\Delta_{\text{up}})$ where $\xi = (f^*(\Delta_{\text{low}}) + p^* + f^*(\Delta_{\text{up}}))^{-1}$
- **if** *x=1* **then return** $L^{\Delta_{\text{low}}}$
- **if** *x=2* **then**
  - Sample Δ from the extended exponential distribution restricted to $(\Delta_{\text{low}}, \Delta_{\text{up}})$ with rate $\tilde\lambda$
  - **return** L^Δ
- **if** *x=3* **then return** $L^{\Delta_{\text{up}}}$

else

- **if** $n^0_{up} > n^0_{low}$ **then return** $L^{\Delta_{\text{up}}}$
- **if** $n^0_{low} > n^0_{up}$ **then return** $L^{\Delta_{\text{low}}}$
- **if** $n^0_{low} = n^0_{up}$ **then** with probability $\frac{f^*(\Delta_{\text{low}})}{f^*(\Delta_{\text{low}})+f^*(\Delta_{\text{up}})}$ **return** $L^{\Delta_{\text{low}}}$; otherwise **return** $L^{\Delta_{\text{up}}}$

with $n = 100$ nodes, $5 \cdot 10^6$ cycle updates take about one second on a single core of an Intel Core i7-860 CPU with a clock speed of 2.8 GHz.

4.1.3 Gibbs moves can reach all admissible matrices

By construction, a Gibbs sampler automatically has the correct invariant distribution. Full convergence proofs usually rely on showing Harris recurrence (Robert & Casella, 2004), which is essentially the requirement that the entire space is explored repeatedly. In this section we make a step towards such a proof by showing that, under conditions, the Gibbs sampler can move from a matrix L^1 to any other matrix L^2.

Theorem 4.1. *Let $a, l \in [0,\infty)^n$. Suppose that $L^1, L^2 \in [0,\infty)^{n\times n}$ are such that for $L = L^1$ and $L = L^2$ the following conditions are satisfied:*

(i) (correct row/column sums) $r(L) = l$, $c(L) = a$.

(ii) (L consistent with p) $\forall i, j \in \mathcal{N} : L_{ij} > 0 \Rightarrow p_{ij} > 0$.

(iii) (L connected along rows/columns) The undirected graph $G = (V, E)$ with vertices

$V = \{r_1, \ldots, r_n, c_1, \ldots, c_n\}$ *and edges* $E = \{(r_i, c_j) : (i,j) \notin \mathcal{F}, L_{ij} > 0\}$ *is connected.*

(iv) (no positive ties) $\forall i, j, i^*, j^* \in \mathcal{N} : L_{ij} = L_{i^*j^*} \neq 0$ *implies* $i = i^*$, $j = j^*$.

Then there exists a sequence of moves by the Gibbs sampler in Algorithm 1 that transforms L^1 *to* L^2.

The proof can be found in Appendix A. Condition (iii) considers the bipartite graph with nodes corresponding to all rows and all columns in which a row is connected to a column if the corresponding entry in L is positive. Connectedness of this graph, i.e. that all nodes can be reached from any node, can be efficiently checked for a given matrix L using e.g. depth first search (Cormen et al., 1990).

One can see that condition (iii) is implied for all L satisfying (i) by the following condition on a and l:

$$\forall I, J \subseteq \mathcal{N} : \left(\sum_{i \in I} l_i = \sum_{j \in J} a_j \implies I = J = \mathcal{N} \text{ or } I = J = \emptyset \right). \tag{11}$$

Fully checking (11) is usually only feasible for small n as it is similar to the NP-hard equal-subset-sum problem, see e.g. Woeginger & Yu (1992). Condition (iv) is satisfied with probability one in the models that we consider, e.g. for models (5) and (8).

4.2 MCMC samplers for hierarchical models

In the hierarchical model the unknown parameters are L and θ. To sample from $(\theta, L)|a, l, L^*$ in this joint model we can again use a Gibbs sampler (or a Metropolis within Gibbs sampler). As θ directly defines the matrices p and λ, the updates used in the basic model can be used to update components of L. Updating components of θ will depend on the specific choice of θ and f. Below we give some examples.

The model of Section 3.3.1 has been specifically set up such that the priors for $\tilde{p}$ (a Beta distribution) and $\tilde{\lambda}$ (a Gamma distribution) are conjugate priors for the model generating L (the indicators $\mathbb{I}(L_{ij} > 0)$ are independent Bernoulli random variables with parameter $\tilde{p}$ and L_{ij} conditional on $L_{ij} > 0$ follows an Exponential distribution with parameter $\tilde{\lambda}$). Hence, $(\tilde{p}, \lambda)|L$ is given by independent Beta and Gamma distributions with appropriately updated parameter values.

For the fitness model in Section 3.3.2 we use Metropolis-Hastings updates to update θ. As these are straightforward extensions of our basic sampler we present the details in Section 3 of the Supporting Document.

5 Applications to systemic risk assessment

5.1 Stress testing a financial network

We now use our methodology for stress testing. We assume that in addition to the row and column sums of L we also observe the *external assets* $a^{(e)} \in [0, \infty)^n$ and the *liabilities to entities outside the interbank network* $l^{(e)} \in [0, \infty)^n$. Table 1 illustrates a simple balance sheet based on this. The vector of *total liabilities* is $l^{\text{all}} := l^{(e)} + r(L) \in \mathbb{R}^n$ and the vector of *net worths* is $w = w(L, a^{(e)}, l^{(e)}) = a^{(e)} + c(L) - l^{\text{all}}$. If $w_i \geq 0$, the net worth corresponds to the book value of bank i's equity.

Table 1: Balance sheet of bank i.

Assets		Liabilities	
external assets	$a_i^{(e)}$	external liabilities	$l_i^{(e)}$
interbank assets	$a_i := c_i(L)$	interbank liabilities	$l_i := r_i(L)$
		net worth	w_i

We consider deterministic proportional shocks defined by $s \in [0,1]^n$: bank i's external assets are reduced to $s_i a_i^{(e)}$. The banks in the set $\mathbb{D}_0 := \{i \in \mathcal{N} \mid w_i(L, s_i a_i^{(e)}, l_i^{(e)}) < 0\}$ are *fundamentally insolvent*. These *fundamental defaults* are purely caused by the external shocks under the assumption that all banks in the network are meeting their obligations. Fundamental defaults depend on the liabilities matrix only via its row and column sums l and a but not on the individual entries L_{ij}.

Default may also be caused by other banks no longer being able to repay their debt, called *contagious defaults*. As mechanism of contagion we use the clearing payment approach (Eisenberg & Noe, 2001) and its generalisation to include default costs (Rogers & Veraart, 2013). Other generalisations could also be used, e.g. to fire sales (Cifuentes et al., 2005), to combinations of default costs, fire sales and cross-holdings (Awiszus & Weber, 2015), or alternatives such as the default cascade algorithm (Amini et al., 2013) and systemic risk measures (Chen et al., 2013; Kromer et al., 2015; Biagini et al., 2015; Feinstein et al., 2015).

The main idea of the clearing approach, consistent with the assumption that all liabilities have equal priority, is that banks repay their creditors in the same *proportions* as their original debt was distributed. We define the *relative liabilities matrix* $\Pi \in \mathbb{R}^{n \times n}$ by setting $\Pi_{ij} := L_{ij}/l_i^{\text{all}}$, if $l_i^{\text{all}} > 0$ and 0 otherwise. A *clearing vector* $c^\star(s) \in [0, l^{\text{all}}]^n$ is a solution to $c^\star(s) = \Phi(c^\star(s))$, where

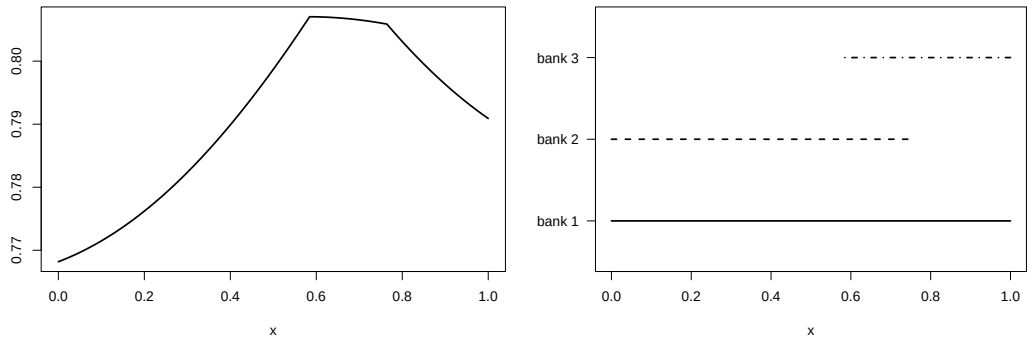
$$\Phi(c(s))_i := \begin{cases} l_i^{\text{all}}, & \text{if } l_i^{\text{all}} \leq \sum_{j=1}^n \Pi_{ji} c_j(s) + s_i a_i^{(e)}, \\ \beta^{(c)} \sum_{j=1}^n \Pi_{ji} c_j(s) + \alpha^{(c)} s_i a_i^{(e)}, & \text{else.} \end{cases}$$

Here $\alpha^{(c)}, \beta^{(c)} \in [0,1]$ are constants modeling default costs (Rogers & Veraart, 2013). For $\alpha^{(c)} = \beta^{(c)} = 1$, this is the classical clearing vector introduced by Eisenberg & Noe (2001) corresponding to no default costs. A bank i is in *default* if $c_i^\star(s) < l_i^{\text{all}}$; this includes both contagious and fundamental defaults.

5.2 Toy example

To illustrate the sensitivity of defaults and clearing payments on the underlying network, we consider a simple situation in which we can easily analytically characterize all networks that are consistent with given row and column sums. Specifically, we consider the network with three banks from Example 2.4 with liabilities matrix (4).

In addition to the row and column sums, we choose $a^{(e)} = \left(\frac{1}{2}, \frac{5}{8}, \frac{3}{4}\right)^\top$ for the external assets and $l^{(e)} = \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)^\top$ for the external liabilities. Hence, the total liabilities are $l^{\text{all}} = \left(\frac{5}{2}, \frac{3}{2}, \frac{3}{2}\right)^\top$ and the net worth is $w = \left(-1, \frac{1}{8}, \frac{1}{4}\right)^\top$. Therefore, only bank 1 is in fundamental default.



(a) Sum of clearing payments / sum of total liabilities

(b) Default range

Figure 5: Dependence of stress test outcomes on the network structure: Figure 5(a) shows $\sum_{i=1}^{3} c_i^\star(0) / \sum_{i=1}^{3} l_i^{\text{all}}$ as a function of x. The lines in Figure 5(b) show for which x each bank defaults.

We use the clearing mechanism without default costs. Figure 5 shows the dependence of the stability of the interbank market on the network configuration. Bank 2 defaults due to contagion for all $x \in [0, \frac{6}{5} - \frac{1}{10}\sqrt{19}) \approx [0, 0.76)$. Bank 3 defaults due to contagion for all $x \in (-\frac{1}{20} + \frac{1}{20}\sqrt{161}, 1] \approx (0.58, 1]$. Hence, just from the marginals a, l we cannot determine the outcomes for bank 2 and bank 3. In some network constellations both suffer a contagious default, in others only one of them. Even in the situations where only one of them defaults due to contagion the total clearing payments differ quite substantially. Furthermore, the parameter range for which all banks default corresponds to the highest value of clearing payments. This can be explained by the losses being (partially) absorbed by the net worth of two banks rather than the net worth of only one bank.

Applying the standard KL method to this example gives the liabilities matrix $L(1/2)$. For this situation only bank 2 suffers a contagious default, but this is not the worst possible outcome for the network, hence contagion can indeed be underestimated by the KL-method.

The circular networks with either $x = 0$ or $x = 1$ have the fewest number of edges. However, they do not represent the worst possible network scenarios. This indicates that the sparsest network consistent with the marginals is not necessarily the network with the highest contagion risk. Hence, using such a network as an upper bound for systemic risk as suggested by Craig & Von Peter (2014) will not always work.

When applying our basic model to this example (using e.g. $p_{ij} = 0.3\mathbb{I}(i \neq j)$ and $\lambda_{ij} = \mathbb{I}(i \neq j)$) then once the sampler has reached one of the extreme cases $x = 0$ or $x = 1$ it iterates between these two cases only, visiting each with the same probability (using a cycle move of length 3). This is consistent with the model - similarly to the arguments in Appendix B.1, one can show that these extreme cases are more probable than the intermediate case.

5.3 Empirical example

In the following we demonstrate how our methodology can be applied to empirical data. To do this, we use data from banks that took part in the European Banking Authority's (EBA) 2011 stress test[4].

For every bank, the total assets $a_i + a_i^{(e)}$, the interbank assets a_i and the net worth w_i (i.e., the Tier 1 capital) are available. Hence, the external assets $a_i^{(e)}$ are available as the difference between the total assets and the interbank assets. The interbank liabilities l_i are not provided in the data set. Some empirical studies based on these data assumed $l_i = a_i \; \forall i \in \mathcal{N}$, see e.g. Chen et al. (2014) and Glasserman & Young (2015). We will assume that l_i is a slightly perturbed version of a_i to ensure that condition (11) is satisfied. In particular, we set $l_i := \mathrm{r}\left((a_i + \epsilon_i)\frac{\sum_{j=1}^n a_j}{\sum_{j=1}^n (a_j+\epsilon_j)}\right)$, $i \in \{1, \ldots, n-1\}$, and $l_n := \sum_{i=1}^n a_i - \sum_{i=1}^{n-1} l_i$, where $r(\cdot)$ is the rounding function to 1 decimal place and $\epsilon_1, \ldots, \epsilon_n$ are independent realisations from a normal distribution with mean 0 and standard deviation 100. Throughout this example we use one fixed realisation for the l_i. Using this assumption, the external liabilities are given by $l_i^{(e)} = (a_i + a_i^{(e)}) - w_i - l_i$.

We look at a subset of the EBA stress test data containing eleven German banks as in Chen et al. (2014). The corresponding balance sheet data are provided in Section 5 of the Supporting Document. The EBA stress test data set contains the total interbank assets for each bank and total interbank assets that a given bank has from banks in a specific country. To construct a closed network between the eleven banks, we only use interbank assets from other German banks (rather than the total interbank assets across all countries). We use the simplifying assumption that these interbank assets are from banks within the network, whereas they will partially be from other German banks besides the 11 banks that we are considering. All other assets are considered to be external assets in our analysis, even though parts of them are interbank assets from banks that are not part of the network that we consider.

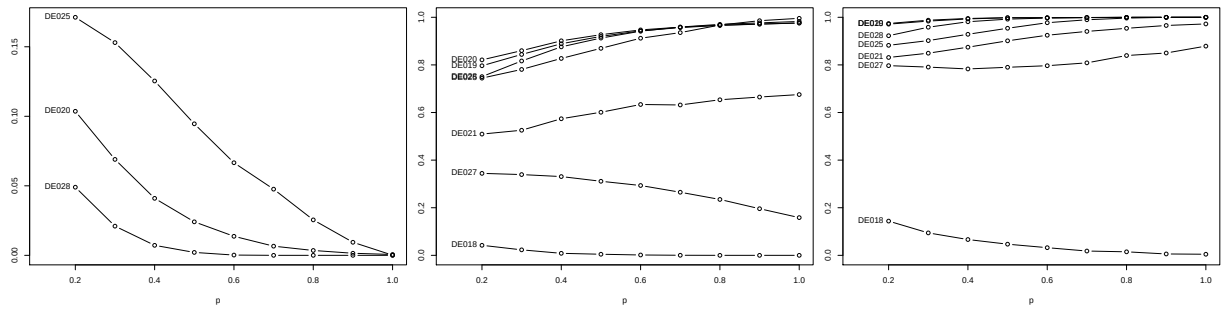
The data set allows us to study default behaviour in a heterogeneous network, which is rarely done in the existing literature. Nevertheless, given the above simplifying assumptions, the results should be considered an illustration of our methodology rather than specific information about the banks involved.

Initially all banks are solvent. We then apply a deterministic shock to the external assets of all eleven banks in the network by considering the shocked external assets $s_i a_i^{(e)}$ with $s_i = 0.97 \, \forall i \in \mathcal{N}$. This shock causes the fundamental default of four banks: DE017, DE022, DE023, DE024. We then use our new methodology to determine the posteriori default probabilities for the remaining seven banks.

5.3.1 Basic model - homogeneous network assumption

Initially we assume that all link probabilities p_{ij} are identical, i.e., $p_{ij} = p \, \forall i \neq j \in \mathcal{N}$ and $p_{ii} = 0 \forall i \in \mathcal{N}$. We consider probabilities $p \in \{0.2, 0.3, \ldots, 0.9, 1\}$. We do not consider probabilities smaller than 0.2 since to satisfy the $2n - 1 = 21$ side conditions we need at least a fraction of $21/(121 - 11) \approx 0.191$ of the

[4] See http://www.eba.europa.eu/risk-analysis-and-data/eu-wide-stress-testing/2011/results



(a) Clearing with $\alpha^{(c)} = 1, \beta^{(c)} = 1$ (b) Clearing with $\alpha^{(c)} = 1, \beta^{(c)} = 0.7$ (c) Clearing with $\alpha^{(c)} = 0.9, \beta^{(c)} = 1$

Figure 6: Posteriori default probabilities as a function of p in the basic model for different default costs parameters $\alpha^{(c)}$ and $\beta^{(c)}$ in the clearing mechanism. Plots have different scale on y-axis.

non-diagonal entries of the liabilities matrix to be positive.

We assume that the parameters of the exponential distribution modelling the weights on all existing edges are all identical, i.e., $\lambda_{ij} = \lambda \, \forall i \neq j \in \mathcal{N}$. We set $\lambda := pn(n-1)/A$, where again $A = \sum_{i=1}^{n} l_i = \sum_{i=1}^{n} a_i$, to ensure that the expected total of the entries of the liabilities matrix is equal to the observed total, i.e.,

$$\mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} L_{ij}\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{p_{ij}}{\lambda_{ij}} = n(n-1)\frac{p}{pn(n-1)/A} = A. \tag{12}$$

With this parameter choice we run the Gibbs sampler. We discarded the first 10000 samples as burn-in. We then retained a sample every 5000th step of the chain, producing a total of 10000 samples.

Figure 6 shows the (posteriori) probability of default for the various banks that are not fundamentally insolvent if we use the clearing approach with or without default costs. The banks that do not appear in the figure are either fundamentally insolvent or never default in our simulations.

The default probabilities clearly depend on the network structure characterised by the edge probability p. In the situation without default costs (Figure 6(a)) we see that the probability of default is decreasing in p. Figures 6(b) and 6(c) show that default probabilities are in general higher in the presence of default costs. For our choice of default costs all 7 remaining banks can in principle default. Default probabilities of banks that are quite small (here less than 0.5) for $p = 0.2$ tend to decrease in p, whereas default probabilities of banks that are larger (here greater than 0.5) for $p = 0.2$ tend to increase in p. Hence we see, that one cannot say in general that more connected networks are more stable or sparser networks are more fragile.

More connected networks spread losses more evenly, so banks get a relatively small loss from the interbank network with relatively high probability. In sparse networks, banks receive, with small probability, a relatively large loss from the interbank network. Thus, if a bank is already close to default (the probability for a bank to default is high) then a small amount of additional loss from interbank liabilities is already sufficient to force the bank into default. As more connected networks increase the probability of this happening we see the increase of default probabilities in p. If a bank is better capitalised (i.e. the probability

of it defaulting is small) then a relatively large loss from the interbank network is needed to cause default. More connected networks only lead to a relatively small loss for each bank from the interbank network - this is why for better capitalised banks the default probability is decreasing in p.

For comparison purpose we have run the three different clearing mechanisms presented in Figure 6 on the network that one obtains from minimising the Kullback-Leibler divergence. We used the code provided in Temurshoev et al. (2013). The resulting network is complete. Without default costs (as in Figure 6(a)) all remaining 7 banks survive, for $\alpha^{(c)} = 1, \beta^{(c)} = 0.7$ (as in Figure 6(b)) only bank DE018 and DE027 survive and for $\alpha^{(c)} = 0.9, \beta^{(c)} = 1$ (as in Figure 6(c)) only bank DE018 survives. Here those banks whose default probabilities tend to (almost) zero in the basic model as p goes to 1 turn out to be the survivors under the KL method. Those banks that have higher default probabilities under the basic model assumption are categorised as defaults under the KL method.

5.3.2 Basic model - tiered network assumption

Several empirical studies have found that financial networks exhibit tiering or some core-periphery structure, see e.g. Craig & Von Peter (2014) for the German interbank market. Motivated by this we now consider situations in which not all edge probabilities p_{ij} are identical.

Following (Nier et al., 2007, Section 6), we assume that there are $n^{(l)}$ large banks and $n^{(s)}$ small banks such that $n^{(l)} + n^{(s)} = n$. The probability $p^{(l)}$ that a large bank is connected to other banks (both large or small) is assumed to be higher than the probability $p^{(s)}$ that small banks are connected to each other, i.e. $0 \leq p^{(s)} \leq p^{(l)} \leq 1$. Nier et al. (2007) argue that to make a meaningful comparison between an Erdős-Rényi network with link probability $p^{(\text{ER})} \in [0, 1]$ and a tiered network with $p^{(s)} \leq p^{(l)}$ one should require that the expected number of links stays the same. This can be obtained by fixing $n^{(l)}$, $n^{(s)}$, $p^{(l)}$, $p^{(ER)}$ and setting

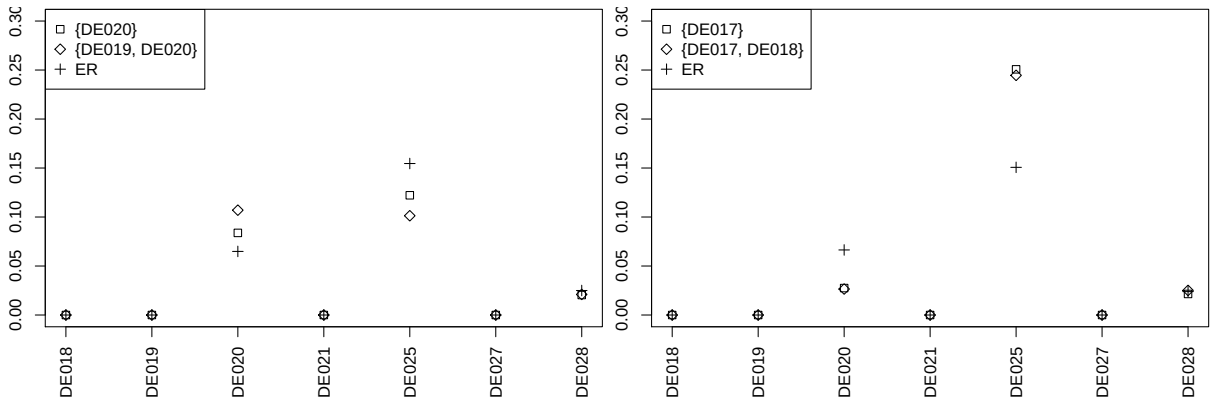
$$p^{(s)} = \frac{n(n-1)p^{(ER)} - n^{(l)}(n-1)p^{(l)} - n^{(l)}n^{(s)}p^{(l)}}{n^{(s)}(n^{(s)}-1)}. \tag{13}$$

Let $\mathcal{I}^{(s)}, \mathcal{I}^{(l)} \subseteq \mathcal{N}$ denote the sets of indices corresponding to *small* and *large* banks, respectively, let $p_{ii} = 0$ $\forall i \in \mathcal{N}$ and for all $i \neq j$ let

$$p_{ij} = \begin{cases} p^{(l)}, & \text{if } \{i,j\} \cap \mathcal{I}^{(l)} \neq \emptyset, \\ p^{(s)}, & \text{if } \{i,j\} \subseteq \mathcal{I}^{(s)}. \end{cases}$$

For the parameters of the exponential distribution we choose $\lambda = \lambda_{ij} = \frac{\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} p_{\mu\nu}}{A} \forall i \neq j$, which is consistent with the considerations in (12).

In the following we use two different criteria to decide on the tiered structure: total interbank assets and total assets. The results are presented in Figure 7. The first network has bank DE020 as the only large bank. The second has banks DE019 and DE020 as the only large banks (according to interbank assets). Figure 7(a) contains the corresponding default probabilities. The third tiered network uses bank DE017 as the only large bank. The fourth uses DE017 and DE018 as large banks (according to total assets). Figures 7(b) and



(a) Tiering according to interbank assets

(b) Tiering according to total assets

Figure 7: Default probabilities in tiered networks with one large bank (either DE020 or DE017), in tiered networks with two large banks (either {DE019, DE020} or {DE017, DE018}) and in a homogeneous Erdős-Rényi network for the clearing mechanism without default costs.

shows the corresponding default probabilities. We have $n = 11$ banks and use $p^{(\text{ER})} = 0.3$ $p^{(l)} = 0.5$. Hence from (13) we obtain that $p^{(s)} \approx 0.26$ and $p^{(s)} \approx 0.20$ for $n^{(l)} = 1$ and $n^{(l)} = 2$ respectively.

In some additional simulation studies not reported here, the gap between default probabilities in tiered networks compared to those in the Erdős-Rényi network increases in $p^{(l)} - p^{(s)}$ as one would expect.

Grouping banks according to their interbank assets seems natural. However, it is not absolutely necessary to enforce this through a tiered network structure. Even in a homogeneous Erdős-Rényi network, the corresponding posteriori degrees of the nodes will adjust for inhomogeneity in the interbank assets. This is confirmed by the results in Table 2, which shows the mean out-degrees, i.e., $\mathbb{E}\left[\sum_{j=1}^{n} \mathcal{A}_{ij} \mid a, l\right]$ with $\mathcal{A}$ the adjacency matrix, for all 11 nodes in the network where we assumed a homogeneous Erdős-Rényi network with different link probabilities $p^{(ER)}$. The in-degrees, not presented here, are similar due to the near symmetry of our example.

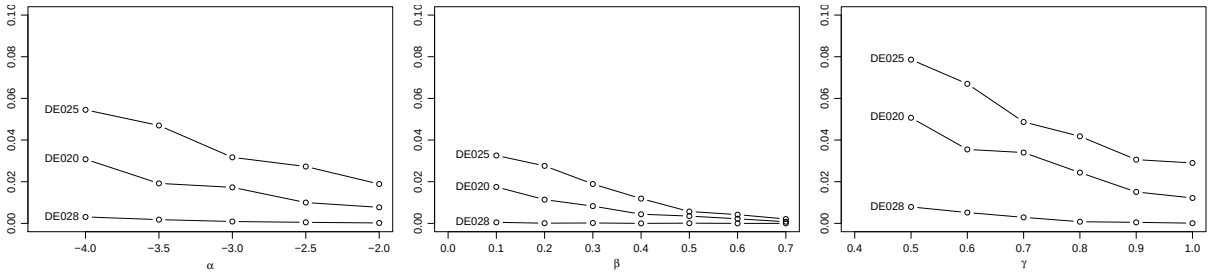
In a homogeneous Erdős-Rényi network the expected in-degree is equal to the expected out-degree for all nodes and it is given by $(n-1)p$, hence in our example it is $10p$. When averaging over all nodes in Table 2, we get quite close to $10p$ as out-degree, but the out-degrees of the individual nodes are clearly different. The mean out-degrees of all 11 nodes in the network are increasing in the interbank assets (shown in the second column of Table 2).

Consistent with the results in Table 2, the difference between tiered and non-tiered, i.e., Erdős-Rényi networks is rather small when we choose total interbank assets to decide on the grouping as in Figure 7(a).

Figure 7(b) illustrates how default probabilities can change if grouping decisions are made based on information that are not part of the liabilities matrix L itself - such as using external assets. In larger networks one might consider grouping banks together according to countries etc. Furthermore, one might

Table 2: Mean out-degree of banks, i.e., $\mathbb{E}[\sum_j \mathcal{A}_{ij} \mid a, l]$, for different p^{ER} in the Erdős-Rényi network and different α, β, γ in the fitness model.

model		ER		Fitness			
model parameters		p^{ER}		α, β, γ			
name	interbank liab l	0.5	0.9	-2.5, 0.2, 1	-2.5, 0.2, 0.6	-2.5, 0.5, 1	-1, 0.5, 1
DE020	99936	6.20	9.00	8.80	6.10	9.40	9.60
DE019	91314	6.00	8.90	8.50	5.80	9.20	9.40
DE021	66494	5.50	8.80	7.50	5.30	8.70	9.00
DE022	54907	5.30	8.80	6.90	4.90	8.40	8.70
DE018	49864	5.10	8.70	6.70	4.80	8.30	8.60
DE017	46989	5.10	8.70	6.60	4.70	8.20	8.60
DE028	30963	4.50	8.40	5.70	4.20	7.60	8.00
DE027	27679	4.30	8.30	5.50	4.00	7.40	7.80
DE024	23971	4.10	8.20	5.30	3.90	7.30	7.70
DE023	8023	2.80	6.90	4.00	3.10	6.30	6.60
DE025	4841	2.40	6.10	3.60	2.70	5.90	6.30
posteriori mean out-deg		4.66	8.25	6.30	4.50	7.90	8.20
apriori mean out-deg		5.00	9.00	3.60	3.10	6.80	7.20



(a) Sensitivity wrt α (b) Sensitivity wrt β (c) Sensitivity wrt γ

Figure 8: Default probabilities in the fitness model for the clearing mechanism without default costs.

want to use more than two groups in tiered networks. This can be easily accommodated in our framework.

5.3.3 Hierarchical model - fitness model

We now analyse the default probabilities under the fitness model assumption for the clearing approach without default costs. We choose $\alpha = -2.5$, $\beta = 0.2$, $\gamma = 1$ as default parameters for the power law of the degree distribution. For the prior distribution π of the parameters of the Gamma distribution used to model the weights we assume that $\zeta \sim U(0.5, 2)$ and $\eta \sim \text{Exp}(1000)$.

The MCMC sampler updates θ, which consists of the fitness vector x and the parameters ζ, η, via standard Metropolis-Hastings steps. Details of this are in Section 3 of the Supporting Document. For every update of θ, we perform $n^2 = 121$ updates of the matrix. We thin the resulting chain, reporting one sample for every 200 updates of θ. We produce a total of 10000 samples. The number of steps of the burn-in is set to equal 10% of the steps used in the main part of the simulation. We computed the effective sample size for

all components of L as well as well as for all elements of θ using the CODA package in R (Plummer et al., 2006): the effective sample size was above 1000 for all simulation runs of this section. Further diagnostic assessments, not reported here, did not flag any convergence problems.

Figure 8 shows the sensitivities of the default probabilities with respect to the parameters modelling the degree distribution α, β, γ. The same banks that are at risk of defaulting (DE025, DE020, DE028) under the basic model are also at risk of defaulting under the fitness model. Their default probabilities are in a similar range under both modelling assumption. Default probabilities are decreasing as the power parameter α of the degree distribution increases (note that $\alpha < 0$). As we can see in Table 2 the smaller α the lower is the corresponding mean out-degree in the network. This corresponds to higher default probabilities in this example as it was the case under the basic modelling assumption. A similar effect can be observed when we change the range parameters of the power law β and γ. Higher values of β imply a higher mean out-degree of the network, see Table 2, and again we observe a decrease in default probabilities. Also higher values of γ imply higher mean out-degrees and lower default probabilities in this example.

The ordering of the banks stays the same under both the basic model as well as the fitness model. The larger the total interbank liabilities the larger the out-degree (and the larger the corresponding posteriori fitness). Under the power law assumption we can influence the variability of the out-degrees with a small number of parameters. If the range of the power law is quite narrow, i.e. β, γ are close together we obtain a more homogeneous network; otherwise the out-degrees between the banks can vary substantially. This can be further influenced by the choice of the power parameter α.

6 Conclusion

We have described mechanisms for generating liabilities matrices that are consistent with the marginal constraints on the total liabilities and total assets of each bank in the network and satisfy some additional properties as well. These additional properties are describing properties of the network structure that we assume to be known or at least estimated from empirical data.

Instead of a fixed shock we could use a random shock. This can be easily accommodated in our methodology by sampling one or several different shocks for each sample of a liabilities matrix.

In practice, the data from different banks may be based on (slightly) different time points or use slightly different accounting assumptions. Even the basic consistency condition $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} l_i$ will often not be satisfied. To accommodate this our model could be extended by allowing for random errors in the observed total interbank liabilities/assets. Details of this are outside the scope of this paper.

A Proofs

Proof of Theorem 2.3. We transform the problem of finding a liabilities matrix into a max-flow problem. We use the terminology defined in Cormen et al. (1990). We may assume that the observed values in L^* are

0; otherwise we subtract the known parts from l and a. Consider the flow network (V, E, c) with vertices $V = \{s, r_1, \ldots, r_n, c_1, \ldots, c_n, t\}$, where s is the source and t the sink, edges

$$E = \{(s, r_1), \ldots, (s, r_n), (c_1, t), \ldots, (c_n, t)\} \cup \{(r_i, c_j) : (i, j) \notin \mathcal{F}\}$$

and capacities $c : V \times V \to [0, \infty)$ being 0 except for

$$c(s, r_i) = l_i, i = 1, \ldots, n, \quad c(c_j, t) = a_j, j = 1, \ldots, n, \quad \text{and } \forall (i, j) \notin \mathcal{F} : c(r_i, c_j) = \infty.$$

A cut is a partition of V into two sets S, T with $s \in S$ and $t \in T$. The capacity $c(S, T) = \sum_{a \in S, b \in T} c(a, b)$ of a cut is an upper bound for the value of any flow (Cormen et al., 1990, Cor 27.6). Using the cut $(\{s\}, V \setminus \{s\})$, which has capacity A, we see that the value of any flow is at most A.

Let $\mathcal{L}$ be the set of liabilities matrices L with $r(L) = l$, $c(L) = a$ and $\forall (i, j) \in \mathcal{F} : L_{ij} = 0$. The existence of a matrix in $\mathcal{L}$ is equivalent to the existence of a maximal flow with value A in (V, E, c). This is because there is a one-to-one correspondence between $\mathcal{L}$ and the flows with value A in the above network given by mapping $L \in \mathcal{L}$ to the flow f given by

$$\forall i, j : \quad f(s, r_i) = l_i, \quad f(r_i, c_j) = L_{ij}, \quad f(c_j, t) = a_j.$$

We already know that there is a cut with capacity A. Thus, by the Max-flow min-cut theorem (Cormen et al., 1990, Theorem 27.7), the existence of a maximum flow of size A is equivalent to all cuts having capacity greater or equal to A . Next, we show that this is equivalent to 2. in the statement of the theorem.

Suppose 2. holds. Let (S, T) be a cut. If there are $(i, j) \notin \mathcal{F}$ with $r_i \in S$, $c_j \in T$ then $c(S, T) \geq c(r_i, c_j) = \infty \geq A$. Otherwise, let $I = \{i : r_i \in S\}$ and $J = \{j : c_j \in T\}$. Then $I \times J \subset \mathcal{F}$. Using 2.,

$$c(S, T) = \sum_{i \notin I} l_i + \sum_{j \notin J} a_j = 2A - \left(\sum_{i \in I} l_i + \sum_{j \in J} a_j \right) \geq A.$$

Suppose that the capacity of every cut is at least A and let I, J be s.t. $I \times J \subset \mathcal{F}$. Define the cut (S, T) by $S = \{s\} \cup \{r_i : i \in I\} \cup \{c_j : j \notin J\}$ and $T = V \setminus S$. Then (2) is implied by

$$A \leq c(S, T) = \sum_{i \notin I} l_i + \sum_{j \notin J} a_j = 2A - \left(\sum_{i \in I} l_i + \sum_{j \in J} a_j \right). \quad \square$$

Proof of Proposition 3.1. The condition $a_i + l_i < A$ ensures that (l, a) is in the interior for the admissible row/column sums given by (3).

Let $\epsilon > 0$ s.t. $\tilde{l} = l - \epsilon 1$ and $\tilde{a} = a - \epsilon 1$ are admissible row/column sums. By Theorem 2.3 we can find M with $r(M) = \tilde{l}$, $c(M) = \tilde{a}$, $M \geq 0$, $\text{diag}(M) = 0$. For $i, j \in \mathcal{N}$, let $\lambda_{ij} = \frac{p_{ij}}{\epsilon/(n-1) + M_{ij}}$. Then

$$\mathbb{E}[\sum_{i=1}^{n} L_{ij}] = \sum_{i \neq j} p_{ij} \mathbb{E}[L_{ij} | L_{ij} > 0] = \sum_{i \neq j} \frac{p_{ij}}{\lambda_{ij}} = \sum_{i \neq j} \left(\frac{\epsilon}{n-1} + M_{ij} \right) = \epsilon + \sum_{i \neq j} M_{ij} = \epsilon + \tilde{a}_j = a_j.$$

The other statement can be shown similarly. $\square$

Proof of Theorem 4.1. We only need the moves from Algorithm 1 for the case $n^0_{\text{low}} =\leq 1$ and $n^0_{\text{up}} \leq 1$. In

this case moves exist to all L^Δ for $\Delta \in [\Delta_{\text{low}}, \Delta_{\text{up}}]$. These moves are reversible in the sense that if one can move from L to L^Δ then one can move from L^Δ to L by using $-\Delta$ as increment.

Let $S(L) = \{(i,j) : L_{ij} > 0\}$ be the support of a matrix L. Let $\mathcal{S} = S(L^1) \cup S(L^2)$ be the joint support of L^1 and L^2. Let C be the set of matrices satisfying (i), (ii), (iii), (iv).

We first show that we can transform any $L \in C$ with $S(L) \subset \mathcal{S}$ to a matrix in C with $S(L) = \mathcal{S}$ using cycle moves. Suppose $(i,j) \in \mathcal{S}$ with $L_{ij} = 0$. We construct a cycle move that gives a matrix in C with $S(L) \subset \mathcal{S}$ where this entry is positive without creating additional zeros. Iteratively applying this construction transforms L to a matrix that is positive on $\mathcal{S}$.

As $(i,j) \in \mathcal{S}$ we have $L^1_{ij} > 0$ or $L^2_{ij} > 0$ implying $r_i(L) = r_i(L^1) = r_i(L^2) > 0$. Hence, there exists j^* such that $L_{ij^*} > 0$. Similarly, there exists i^* such that $L_{i^*j} > 0$. By the connectedness of L, there exists a simple path between (r_i, c_{j*}) and (r_{i^*}, c_j) along G. Together with (i,j) this defines a cycle that only has one zero. As there are no tied positive value in L there is a unique smallest positive value ξ of L along this cycle and we can use the cycle move with $\Delta \in (0, \xi)$ chosen such that no positive ties are being created. This will transform L into a matrix in C with $(i,j) \in \mathcal{S}$.

Thus we can transform L^1 into a positive matrix on $\mathcal{S}$. As all of the steps are reversible, any matrix $L \in C$ with $S(L) = \mathcal{S}$ can be transformed into L^2. It remains to construct transformations between matrices in C with support $\mathcal{S}$. Let L^1 and L^2 be two such matrices. Let $D = L^2 - L^1$. The following step reduces the number of nonzero entries in D by at least 1. Repeated application removes the differences between L^1 and L^2, as there are only finitely many differences and as the step does not create new differences.

Let $\eta^1 = (i,j)$ be such that $D_{ij} < 0$. Starting from η^1, construct a path with $\eta^i \in \mathcal{N}^2$ and $\eta^i_1 = \eta^{i+1}_1$ for odd i and $\eta^i_2 = \eta^{i+1}_2$ for even i such that $D_{\eta^i} < 0$ for odd i and $D_{\eta^i} > 0$ for even i. As the row and column sums of D are 0 it is always possible to extend this path by an additional element. Extend the path until, for the first time, the first component or the second component repeats itself in non-adjacent elements of the path i.e., if m is the current length of the path then for some $j \leq m-2$ we have ($\eta^j_1 = \eta^m_1$, $\eta^j_1 \neq \eta^{j+1}_1$) or ($\eta^j_2 = \eta^m_2$ and $\eta^j_2 \neq \eta^{j+1}_2$). Then use the cycle $(\eta^j, \ldots, \eta^m)$ to remove the smallest value (in absolute values) in D (denote the index of this element by b). To do this choose this element as the first element of the cycle (reversing the cycle if necessary) and use $\Delta = -D_b$. □

B Conditional distribution on a cycle

This section provides details on the derivation of Algorithm 1. Suppose the current liabilities matrix is L. Suppose we have a fixed cycle of length k with η determining the index pairs of the elements to be updated. To ease notation, we use $L_i = L_{\eta_i}$, $\tilde{L}_i = \tilde{L}_{\eta_i}$, and similarly for p, λ. Let $s = (1, -1, \ldots, 1, -1) \in \mathbb{R}^{2k}$.

B.1 Probability of the boundary cases and the intermediate case

We work out the probability of the boundary cases $\Delta \in \{\Delta_{\text{low}}, \Delta_{\text{up}}\}$ and the intermediate case $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$. For this we compute the probability of a realisation in a neighbourhood of $(L_i + s_i\Delta)_{i=1,\dots,2k}$ for $\Delta = \Delta_{\text{low}}$,$\Delta = \Delta_{\text{up}}$ and $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$ before letting the size of this neighbourhood go to 0.

First we consider the boundary case. Suppose $\Delta \in \{\Delta_{\text{low}}, \Delta_{\text{up}}\}$, then the exact value for $\tilde{L}_i$ would be $x_i = L_i + s_i\Delta$, at least one of which is 0. The probability of all $\tilde{L}_i$ being at most $\epsilon > 0$ away from x_i is

$$\mathbb{P}(|\tilde{L}_i - x_i| < \epsilon, i = 1, \dots, 2k) = \prod_{i=1}^{2k} \left((1-p_i)\mathbb{I}(|x_i| < \epsilon) + p_i \int_{(x_i-\epsilon)^+}^{x_i+\epsilon} \lambda_i e^{-\lambda_i y} dy \right) = O\left(\epsilon^{\#\{i:x_i \neq 0\}}\right)$$

as $\epsilon \to 0$, where $x^+ = \max(x, 0)$ and $\#B$ is the number of elements in the set B. In particular, $\#\{i : x_i \neq 0\} \leq 2k - 1$ and we might even sometimes have a strict inequality.

Next we consider the intermediate case $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$. Suppose $\Delta_{\text{low}} \neq \Delta_{\text{up}}$, then for all $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$, all $x_i(\Delta) = L_i + s_i\Delta$, $i = 1, \dots, 2k$ will be positive, implying that only the absolutely continuous part of the random variable is relevant in this case. Integrating out the one degree of freedom given by Δ, one sees that the probability of $\tilde{L}_\eta$ being close to $(x_1(\Delta), \dots, x_{2k}(\Delta))$ for some Δ is

$$\begin{aligned}
&\mathbb{P}(\tilde{L}_1 - L_1 \in (\Delta_{\text{low}}, \Delta_{\text{up}}), \tilde{L}_i \in B_i^{\tilde{L}_1 - L_1}(\epsilon), i = 2, \dots, 2k) = \\
&= \int_{L_1+\Delta_{\text{low}}}^{L_1+\Delta_{\text{up}}} \prod_{i=2}^{2k} \left(\int_{B_i(x_1 - L_1, \epsilon)} \lambda_i e^{-\lambda_i x_i} dx_i \right) \lambda_1 e^{-\lambda_1 x_1} dx_1 = O(\epsilon^{2k-1})
\end{aligned}$$

as $\epsilon \to 0$ where $B_i^{\Delta}(\epsilon) = (\max(L_i + s_i\Delta - \epsilon, 0), L_i + s_i\Delta + \epsilon)$.

Hence, if two or more 0s are produced in $\tilde{L}_\eta$ for $\Delta = \Delta_{\text{low}}$ or $\Delta = \Delta_{\text{up}}$ then the boundary case dominates the intermediate case (hence we have lines 18–21 in Algorithm 1). Otherwise, the probability contributions of the boundary case and the intermediate case are of the same order and we sample among these cases in lines 10–12 of Algorithm 1; for details of this see the next subsection.

B.2 The case where there is at most one 0 at the boundary

The conditional distribution of $\tilde{L}$ along the cycle η is concentrated on matrices of the form $L_\eta + s\Delta$ with $\Delta \in [\Delta_{\text{low}}, \Delta_{\text{up}}]$. This section considers the case $\#\{i : L_i + s_i\Delta_{\text{up}} = 0\} = \#\{i : L_i + s_i\Delta_{\text{low}} = 0\} = 1$.

We first work out the probability mass for the three cases $\Delta = \Delta_{\text{low}}$, $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$ and $\Delta = \Delta_{\text{up}}$. The probability mass for the boundary case $\Delta = \Delta_{\text{low}}$ is $f(L_\eta + s\Delta_{\text{low}})$ and for $\Delta = \Delta_{\text{up}}$ is $f(L_\eta + s\Delta_{\text{up}})$. The total probability mass of the intermediate case $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$ is

$$\begin{aligned}
\int_{(\Delta_{\text{low}}, \Delta_{\text{up}})} f(L_\eta + s\Delta)\mathrm{d}\Delta &= \int_{(\Delta_{\text{low}}, \Delta_{\text{up}})} \prod_{i=1}^{2k} p_i \lambda_i e^{-\lambda_i(L_i + s_i\Delta)} \mathrm{d}\Delta \\
&= \prod_{i=1}^{2k} p_i \lambda_i e^{-\lambda_i(L_i + s_i\tilde{\Delta})} \int_{(\Delta_{\text{low}}, \Delta_{\text{up}})} e^{-\tilde{\lambda}(\Delta - \tilde{\Delta})} \mathrm{d}\Delta = f(L_\eta + s\tilde{\Delta}) \int_{(\Delta_{\text{low}}, \Delta_{\text{up}})} e^{-\tilde{\lambda}(\Delta - \tilde{\Delta})} \mathrm{d}\Delta,
\end{aligned}$$

where $\tilde{\Delta} = \frac{\Delta_{\text{low}}+\Delta_{\text{up}}}{2}$, $\tilde{\lambda} = \sum_{i=1}^{2k} \lambda_i s_i$. If $\tilde{\lambda} = 0$ then this is $f(L_\eta + s\tilde{\Delta})(\Delta_{\text{up}} - \Delta_{\text{low}})$, otherwise it is

$$f(L_\eta + s\tilde{\Delta}) - \tilde{\lambda}^{-1} e^{-\tilde{\lambda}(\Delta-\tilde{\Delta})}|_{\Delta_{\text{low}}}^{\Delta_{\text{up}}} = f(L_\eta + s\tilde{\Delta})\tilde{\lambda}^{-1}\left[e^{\tilde{\lambda}\frac{\Delta_{\text{up}}-\Delta_{\text{low}}}{2}} - e^{-\tilde{\lambda}\frac{\Delta_{\text{up}}-\Delta_{\text{low}}}{2}}\right].$$

Next, we work out the distribution of Δ conditional on the intermediate case $\Delta \in (\Delta_{\text{low}}, \Delta_{\text{up}})$. Consider the mapping $g : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$, $g_1(x) = x_1 - L_1$, $g_i(x) = x_{i-1} + x_i$. Then $g(\tilde{L}) = (\Delta, \tilde{L}_1 + \tilde{L}_2, \ldots, \tilde{L}_{2k-1} + \tilde{L}_{2k})$. We are interested in the distribution of $\Delta = g_1(\tilde{L})$ conditional on $g_i(\tilde{L}) = g_i(L)$, $i = 2, \ldots, 2k$.

Using standard density transformation, and the fact that g is a linear mapping where the matrix has determinant 1, we get that the density of $M = g(\tilde{L})$ is $f(g^{-1}(M))$.′ Hence, conditionally on $g_i(\tilde{L}) = g_i(L)$, $i = 2, \ldots, 2k$, the density of $\Delta = g_1(\tilde{L})$ is proportional to

$$f(g^{-1}(\Delta, L_1 + L_2, \ldots, L_{2k-1} + L_{2k})) = f(L + s\Delta) = \prod_{i=1}^{2k} p_i \lambda_i e^{-\lambda_i(L_i + s_i\Delta)} \propto e^{-(\sum_{i=1}^{2k} \lambda_i s_i)\Delta} = e^{-\tilde{\lambda}\Delta},$$

which is an extended exponential distribution with parameter $\tilde{\lambda}$ restricted to the interval $(\Delta_{\text{low}}, \Delta_{\text{up}})$.

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