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Incentive Compatible Networks and the Delegated Networking Principle

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Abstract

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Keywords: Incentive Compatible Multilayered Networks, Delegated Networking Principle, Delegation Principle, Bilateral Incentive Compatibility, Mechanism Design, Catalog Games

JEL Classification: C7

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Incentive Compatible Networks and the Delegated Networking Principle[*]

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Abstract

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1 Introduction

We consider the problem faced by a principal who seeks to the structure incentives faced by a set of agents in forming a network of connections among themselves in such a way that each agent, in light of his private information, forms connections that are the best interest of the principal. Thus, the principal seeks to influence - if not control - not only who is interacting (i.e., which pairs of agent's form connections) but also how they are interacting. In many networking situations, however, the principal, in addition to not being able to observe who is interacting and how, does not have complete information concerning the agent's "type" (i.e., a parameter summarizing the agent's basic characteristics). Thus, there is an adverse selection problem.

To address the issues raised above, we construct a principal-agent game of network formation (over layered networks) with asymmetric information and we consider the following two questions: (1) is it possible for the principal to design a mechanism that links the reports of agents' about their private information and the set of connections allowed and recommended by the principal via the mechanism in such a way that players truthfully reveal their private information to the principal and follow the recommendations specified by the mechanism. (2) An even more fundamental question we address is whether or not it is possible for the principal to achieve the same outcome (as that achieved via a mechanism and centralized reporting) by instead choosing a profile of sets of allowable ways to connect (here modeled as node-pair specific sets - or catalogs - of arc types) and then delegating connection choices to each pair of players. We call this approach to network formation with incomplete information delegated networking and we show, under relatively mild conditions on our game-theoretic model of network formation, that strategic network formation with incomplete information, implemented via a mechanism and centralized reporting, is equivalent to implementation via delegated networking with monitoring. Thus, we show that the delegation principle of contracting theory holds for games of network formation with incomplete information.

2 Networks and Incomplete Information

2.1 Primitives

Assume the following:

(1) N is a finite set of agents, consisting of n agents, equipped with the discrete metric η_N, having typical elements i and j.

(2) $N^2 := N \times N$ is the set of agent pairs, consisting of n^2 pairs, each representing a player, equipped with the discrete metric $\eta_{N \times N} := \eta_N + \eta_N$, having typical elements ij (including the diagonal pairs $ii \in N^2$).

(3) C is a compact metric space of clubs equipped with metric ρ_C having typical element c, containing a special "no interaction" club c_0 (more on this below).

(4) $(S, \mathcal{B}(S))$ is a space consisting of mutually observable states, $s \in T$, where S is a complete, separable metric (Polish) space with metric ρ_S and Borel σ-field $\mathcal{B}(S)$.

(5) $(T_i, \mathcal{B}(T_i))$ is a space consisting of ith agent types, $t_i \in T_i$, where T_i is a complete, separable metric (Polish) space with metric ρ_{T_i} and Borel σ-field $\mathcal{B}(T_i)$.

(6) $T_{ij} := T_i \times T_j$ is the space of player ij's possible types, $t_{ij} := (t_i, t_j) \in T_{ij}$, equipped with the Borel product σ-field, $\mathcal{B}(T_{ij}) := \mathcal{B}(T_i) \times \mathcal{B}(T_j)$.

(7) $T = \Pi_i T_i$ is the space of player type profiles (n-tuples), $t \in T$, equipped with the Borel product σ -field, $\mathcal{B}(T) = \Pi_i \mathcal{B}(T_i)$.

(8) A is the feasible set of arc types, a, where A is a weak star compact, metrizable, convex subset of the separable norm dual, $(E^*, \|\cdot\|^*)$, of a separable Banach space, $(E, \|\cdot\|)$, equipped with compatible metric ρ_{w^*}.

(9) $A(\cdot, c)$ is club $c's$ feasible arc correspondence, a set-valued mapping from the set of all players, ij, taking values in the collection, 2^A, of ρ_{w^*} -closed subsets of A such that for each player, ij,

$$A(ijc) \subset A_c \subset A \subset E^*.$$

Alternatively, the 2^A-valued correspondence, $A(ij, \cdot)$, is player $ij's$ feasible arc correspondence across clubs.

We will refer to our list of primitives together with our assumptions as [A-1](γ), $\gamma = 1, 2, \ldots, 9$.

2.2 Agent Pairs as Players

Because the basic strategic ingredients of our game of network formation are bilateral connections, it is useful to view each agent pair, ij, as a player or a node in the connections game with a player's club choice representing the resolution of the "whether or not to connect" and "how to connect" part of the problem. From the perspective of the principal, because the asymmetric information relevant to the connections issue is with regard to the types of the two agents contemplating the connection, by the very nature of the principal's problem of incentivizing truthful revelation and connections, we are led to think of the agent pair, ij, as the player and to think of their joint information, t_{ij}, as the information of strategic interest. Moreover, by taking agent pairs as players and by viewing a player's type as the joint type of the underlying agent pair, we are able to bring to bear on the contentious problem of bilateral incentive compatibility methods from contracting and mechanism design.

2.3 Pre-Connections, Connections, and Networks

Connections are the fundamental building blocks of networks. While connections can be modeled in many different ways, all connections are made up of two basic ingredients: nodes and arcs. In our model the nodes are given by players, $ij \in N^2$ and clubs, $c \in C$. Thus, in our model of club networks, the set of nodes is given by

$$\underbrace{(N \times N)}_{\text{players}} \cup \underbrace{C}_{\text{clubs}}.$$

Our approach to the network formation problem will be to model the totality of the connections between agents as a directed club network where the connection between an agent pair is represented by that agent pair's (i.e., that player's) connection to a club. In our club network - a bipartite network - each club $c \in C \backslash \{c_0\}$ is a "venue" where players ij who are members of club c can engage in particular types of interactions, $a \in A_c \subset A$, where a is an interaction initiated or proposed by agent i and directed toward agent j. More compactly, we say that player ij joins club c and takes feasible action a. A typical connection in such a club network looks like the following:

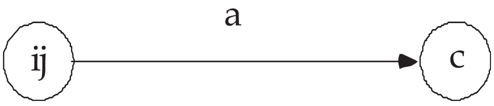

Figure 1: Player ij joins club c and takes action $a \in A(ijc) \subset A_c$

We will refer to the player and club pairing, $ijc \in N^2 \times C$ as a pre-connection. If player ij joins the special no-interaction-club, c_0, then the only action that player ij can take is $a_0 \in A(ijc_0) := \{a_0\}$. We will assume that if player ij joins a club $c \neq c_0$ and takes action $a \in A(ijc)$, then agents i and j composing player ij share their private in formation (i.e., i knows $j's$ type t_j and j knows $i's$ type t_i) and this is common knowledge. Whereas if player ij joins the no interaction club, c_0, and takes action a_0 - the only action ij can take - then agents i and j do not share their private information - and this too is common knowledge. Finally, if player ij_k is in club c and player $ij_{k'}$ is in club c', j_k does not know $j_{k'}$'s type nor does $j_{k'}$ know j_k's type - and this remains true even if c and c' are the same club.

2.4 Pre-connections and Pre-Networks

Our objective is to build a game theoretic model of endogenous network formation that makes clear how the interplay between strategic behavior and network structure - under asymmetric information - determine the payoff to the principal. Because the fundamental building blocks of such a network are bilateral interactions between agents, in our principal-agent game of network formation - as mentioned above - we will take as the set of players the set of all possible ordered pairs of agents, $ij := (i,j) \in N^2$. Thus, N^2 is the set of players.

A group of players is given by a subset $g \subset N \times N$ consisting of a subset of ordered pairs of agents. Given a set of players, g, we can then form a group g club membership *pre-network*,

$$R \subset g \times C,$$

consisting of a nonempty $\rho_{N^2 \times C}$-closed subset, R, of *pre-connections*, $ijc \in R$ (where $\rho_{N^2 \times C} := \eta_N + \eta_N \times \rho_C$). Thus, the set of all pre-networks involving members of group, g, is given by $P_f(g \times C)$, the hyperspace of all nonempty, $\rho_{N^2 \times C}$-closed subsets of $g \times C$. We will call each such pre-network, $R \in P_f(g \times C)$ a g-pre-network.

The set of all pre-connections is given by

$$N^2 \times C.$$

Thus, the hyperspace of all pre-networks, $R \subset N^2 \times C$, is given by,

$$P_f(N^2 \times C),$$

the hyperspace of all nonempty, $\rho_{N^2 \times C}$-closed subsets of $N^2 \times C$.

Here, if player ij joins club c, then agent i can engage in the interactions contained in $A(ijc)$ with agent j (e.g., see Page and Wooders (2009, 2010)). *Because there are different clubs, in a club network representation of a network, the network is layered, with each layer being specified by a club.*[1]

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[1] An important mathematical point: the sets of types of interactions that players can engage in the different clubs - i.e., A_c, are all subsets of the *same* space of arc types, namely, A. In He and Page (2014), the case in which the arc type spaces, A_c, differ across clubs (for example with respect to metric) is treated - in a model otherwise similar to the model given here. While this added degree of model flexibility seems minor, technically, it is not minor - it requires much more mathematical machinery to establish the delegated networking principle for the case in which the arc type spaces differ across clubs.

From assumptions [A-1], the feasible arc correspondence from pre-connections into arc sets is given by

$$ijc \longrightarrow A(ijc).$$

From the perspective of player ij, $c \longrightarrow A(ijc)$, is player ij's feasible arc correspondence across clubs, while from the perspective of club c, $ij \longrightarrow A(ijc)$, is club c's feasible arc correspondence across players.

2.5 Club Networks and Layered Networks

In our club network model, we will assume that each player, $ij \in N^2$, can join multiple clubs, $c \in C$, and in each club player ij takes a particular action a from a feasible set of actions, $A(ijc)$, relevant to that club. This set of relevant actions for each player-club pair is given by the feasible arc correspondence, $ijc \longrightarrow A(ijc)$.

We have the following formal definition of a club network.

Definitions 1: (Club Networks)
Given arc set A, node set, $N^2 \cup C$, and feasible arc correspondence, $ijc \longrightarrow A(ijc) \in 2^A$, a club network is a nonempty, closed subset, G, of $A \times \left(N^2 \times C\right)$ such that (i) $|G(ijc)| \leq 1$ and $|G(ijc)| = 1$ for some $c \in C$, and (ii) if for $c \in C$, $|G(ijc)| = 1$, then $(a, (ij, c)) \in G$ if and only if $a \in A(ijc)$. We will denote by $\mathbb{G}$ the collection of all feasible club networks. Thus,

$$\mathbb{G} := \left\{G \in P_f(A \times (N^2 \times C)) : \text{ satisfying (i) and (ii)}\right\}.$$

Thus, in a club network a typical connection is given by

$$(a, (ij, c)) \in A \times ((N \times N) \times C),$$

where connection, $(a, (ij, c))$, indicates that player ij is in club c and that in this club player ij takes feasible action $a \in A(ijc)$.

We will call the connection, $(a, (ij, c))$, a *c-connection.* The set of all c-connections is given by

$$K_c := A_c \times (N^2 \times \{c\})$$

We will equip K_c with the sum metric,

$$\rho_{K_c} := \rho_{w^*} + \eta_N + \eta_N.$$

2.5.1 The c-Network Decomposition

A c-layer G_c is a ρ_{K_c}-closed subset of the set of connections, K_c, such that $a, (ij, c)) \in G_c$ if and only if $a \in A(ijc)$. We will often be interested in the arc section of a c-layer, G_c, at various pre-connections. For example, at pre-connection (ij, c), the arc section of G_c at (ij, c) is given by

$$G_c(ijc) := \{a \in A(ijc) : (a, (ij, c)) \in G_c\}.$$

The arc section, $G_c(ijc)$, of c-layer G_c at (ij, c), lists the feasible set of arcs used in connecting agent pair (player) ij to club c in layer G_c of club network G. The cardinality of the arc section, $G_c(ijc)$, $|G_c(ijc)|$, gives the number of arcs used in connecting agent pair ij to club c in layer G_c. The *domain* of c-layer, $G_c \subset A_c \times (N^2 \times \{c\})$, is given by

$$\mathcal{D}(G_c) := \{ij \in N^2 : G_c(ijc) \neq \varnothing\}.$$

Note that $\mathcal{D}(G_c)$ is the subset of players who belong to club c. It is possible for a c-layer has no members (i.e., in club network G, club c has no members).

The domain of a club network G is given by

$$\mathcal{D}(G) := \{ijc \in N^2 \times C : G(ijc) \neq \varnothing\}.$$

While the domain of a c-layer is a set of players (possibly empty), the domain of a club network is a set of pre-connections - or a pre-network.

Note that a club network, G, can be uniquely decomposed into the union of its c-layers,

$$G := \cup_{c \in C} G_c.$$

Equipped with the Hausdorff metric $h_{\rho_{K_c}}$, the collection of all possible ρ_{K_c}-closed subsets of c-connections, 2^{K_c}, is a compact metric space. Thus, $(2^{K_c}, h_{\rho_{K_c}})$ the compact metric space contains all possible c-layers (including the empty layer - and when the layer is nonempty, it is called a c-club network). We will define the distance between two club networks, $G^1 := \cup_{c \in C} G^1_c$ and $G^2 := \cup_{c \in C} G^2_c$ as the sum of the distances between the c-layers which make up G^1 and G^2. Thus, the distance between club networks G^1 and G^2 is given by

$$h_K(G^1, G^2) := \textstyle\sum_c h_{\rho_{K_c}}(G^1_c, G^2_c),$$

where $h_{\rho_{K_c}}(G^1_c, G^2_c)$ is the Hausdorff distance in 2^{K_c}.

For each player ij, the graph of the feasible arc correspondence,

$$c \longrightarrow A(ijc),$$

is given by

$$GrA(ij, \cdot) := \{(c, a) \in C \times A : a \in A(ijc)\}.$$

If $(c, a) \in GrA(ij, \cdot)$, then action a can be taken in club c by player ij. Given club network G, the arc section of G at (ij, c), given by

$$G(ijc) := \{a \in A(ijc) : (a, (ij, c)) \in G\},$$

is such that for all (ij, c),

$$G(ijc) \subset A(ijc).$$

Moreover,

$$GrG(ij, \cdot) \subset GrA(ij, \cdot).$$

2.5.2 The ij-Network Decomposition

Besides the c-layer representation, another useful representation of a club network G is the ij-network representation. We can think of a club network G as being composed of each individual player's club network. For example, player $ij's$ network, $G_{ij} \subset A \times (\{ij\} \times C)$. Because each player ij can join the no interaction club, c_0, (i.e., if agents i and j choose not to interact), then the ij-network G_{ij} is given by

$$G_{ij} = \{(1, (ij, c_0))\}.$$

Thus, for each player, ij, G_{ij} is a nonempty, closed subset of $A \times (\{ij\} \times C)$. Letting

$$K_{ij} := A \times (\{ij\} \times C),$$

each ij-network, G_{ij}, is contained in $P_f(K_{ij})$, the collection of all nonempty, closed subsets of K_{ij} (as opposed to being contained in $2^{K_{ij}}$, the collection of all closed subsets of K_{ij} - including the empty set). The representation of club network G, via its constituent ij-networks, is given by

$$G = \cup_{ij} G_{ij}.$$

We will sometimes write G as an n^2-tuple of ij-networks as follows,

$$G = (G_{11}, G_{12}, \ldots, G_{1n}, \ldots, G_{n1}, G_{n2}, \ldots, G_{nn})$$

rather than as a union of ij-networks

$$G = \cup_{ij} G_{ij}.$$

Let $P_f(ij \times C)$ denote the collection of all nonempty closed subsets of $\{ij\} \times C$ where $\{ij\} \times C$ is the set of all pre-connections for player ij in the collection of all preconnections, $N^2 \times C$.

Given any ij-network, $G_{ij} \subset A \times (\{ij\} \times C)$, the domain of G_{ij} is given by

$$\mathcal{D}(G_{ij}) := \{c \in C : G_{ij}(ijc) \neq \varnothing\}.$$

Because the set of clubs, C, includes the no-interaction club, c_0, each ij-network has a nonempty domain. Thus, $\mathcal{D}(G_{ij}) \neq \varnothing$ for all players ij. In fact, we have for all ij-networks,

$$1 \leq |\mathcal{D}(G_{ij})| \leq k + 1.$$

The collection of all feasible ij-networks that can be formed by player ij is given by,

$$\mathbb{G}_{ij} := \{G_{ij} \in P_f(K_{ij}) : |G_{ij}(ijc)| \leq 1 \text{ for all } c\}.$$

We note that $G_{ij}(ijc) \in 2^{A(ijc)}$ the collection of all ρ_{w^*}-closed subsets of $A(ijc)$ (including the empty set).

2.6 Information as States and the Network Representation of Information

We will think of the state space as being given by $\Omega := T \times S$, with typical element,

$$(t, s) := (t_1, t_2, \ldots, t_n, s).$$

If at time point $k = 0, 1, 2, \ldots$, the prevailing state is $(t_k, s_k) := (t_{1k}, t_{2k}, \ldots, t_{nk}, s_k)$, we will assume that each agent i knows (t_{ik}, s_k) - and that this is common knowledge. Thus, agent i knows his piece of the t-state at time point k, while all agents know the s-state at time point k. Moreover, if two agents, i and j, are connected at time point k, then the agent pair (i.e., the player), ij, knows, (t_{ik}, t_{jk}, s_k) - but the pair, ij, does not share the information about the types of those agents to whom i and j are directly connected at time point k. Thus, if j and j' are connected and i and i' are connected, then the pair, ij, does not know $t_{i'k}$ or $t_{j'k}$.

Each player (agent pair), ij, has a type given by

$$t_{ij} := (t_i, t_j) \in T_i \times T_j := T_{ij}.$$

While each agent pair knows their types, the principal only knows the agent type profile,

$$t := (t_1, t_2, \ldots, t_n) \in \Pi_i T_i,$$

up to a probability measure, $\lambda(\cdot)$.

We will represent the state information possessed by each agent as a diagonal loop network $\mathbb{I} := \mathbb{T} \cup \mathbb{S}$, where

$$\mathbb{T}_i = T_i \times (\{i\} \times \{i\})$$

is the collection of all t_i-connections, with typical element $\tau_i \in \mathbb{T}_i$. A t-network is an n-tuple

$$\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{T} := \Pi_i \mathbb{T}_i.$$

The collection of all s-connections, with typical element $\sigma_i \in \mathbb{S}$,

$$\mathbb{S} := S \times (\{i\} \times \{i\}).$$

An s-network is an n-tuple

$$\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{S},$$

such that $\sigma_i = s \in S$ for all i.

It is clear that without confusion, we can represent each state information network

$$\tau \cup \sigma \in \mathbb{I} := \mathbb{T} \cup \mathbb{S}$$

as $\omega = (t, s)$. Equip $\mathbb{T}$ with the sum metric, $\rho_{\mathbb{T}} := \sum_i \rho_{T_i}$ and equip $\mathbb{S}$ with the metric, $\rho_{\mathbb{S}} := \rho_S$.

3 Mechanism Games vs Catalog Games

3.1 Games Over Mechanisms

Let $\omega_{ij} = (t_{ij}, s) = (t_i, t_j, s)$ denote the part of the state, $\omega = (t_1, t_2, \ldots, t_n, s)$ knows to player ij (i.e., to agent pair ij). We begin with a formal definition.

Definition 2 (Direct ij-Network Formation Mechanism)
A direct (network formation) mechanism is a $(\mathcal{B}(\Omega_{ij}), \mathcal{B}(\mathbb{G}_{ij}))$-measurable mapping

$$M_{ij} : \Omega_{ij} \longrightarrow \mathbb{G}_{ij}$$

from player ij's state space, Ω_{ij}, into the collection of all ij-networks, $\mathbb{G}_{ij}$, that player ij can form.

For any ij's state $\omega_{ij} \in \Omega_{ij}$, there exists some $G_{ij} \in \mathbb{G}_{ij}$, such that $M_{ij}(\omega_{ij}) = G_{ij}$. Note that for $(ij, c) \notin \mathcal{D}(M_{ij}(\omega_{ij}))$, the set,

$$M_{ij}(\omega_{ij})(ijc) := \{a \in A(ijc) : (a, (ij, c)) \in G_{ij}\},$$

is empty.

3.1.1 Incentive Compatible Mechanisms with Voluntary Nonparticipation

Let $G := (G_{ij}, G_{-ij}) \in \mathbb{G}$ be a feasible club network and let

$$u_{ij}(\omega, G) := u_{ij}(\omega_{ij}, \omega_{-ij}, G_{ij}, G_{-ij})$$

be the payoff to player ij of type t_{ij} if other player types are given by t_{-ij} and if the profile of player club networks is given by $G := (G_{ij}, G_{-ij}) \in \mathbb{G}$.

We will assume that all *payoff functions are Caratheodory* - meaning that for each player ij, the payoff function

$$(\omega, G) \longrightarrow u_{ij}(\omega, G)$$

is measurable in states on Ω and continuous in networks on $\mathbb{G}$. We will also assume that for any profile of ij-network formation mechanisms, $\omega \longrightarrow M(\omega) := (M_{ij}(\omega_{ij}))_{ij \in N \times N}$, player $ij's$ payoff function,

$$\omega \longrightarrow u_{ij}(\omega, M(\omega)),$$

is $\mathcal{B}(\omega)$-measurable.

We will assume the following concerning each player's payoff function, $u_{ij}(\cdot, \cdot)$ (see Balder, 1997, and Bloch and Jackson, 2007)

[A-2] (Payoff functions are additively coupled)
We will assume that for each $(\omega, G) \in \Omega \times \mathbb{G}$,

$$u_{ij}(\omega, G) = v_{ij}(\omega_{ij}, G_{ij}) + \textstyle\sum_{i'j' \neq ij} \bar{v}_{ij}(\omega_{ij}, \omega_{i'j'}, G_{i'j'}),$$

for some functions $v, \bar{v}$, *where* v *is Caratheodory.*

We have the following formal definition of an incentive compatible mechanism (or an IC mechanism).

Definition 3 (Incentive Compatible Network Formation Mechanism with Voluntary Nonparticipation)
We say that a network formation mechanism, $\omega \longrightarrow M(\omega)$, *is incentive compatible if for each player,* ij, $ij's$ *part of the mechanism,* $\omega_{ij} \longrightarrow M_{ij}(\omega_{ij})$, *is such that for all* ω_{ij} *and* ω'_{ij}

$$\begin{aligned} &\max\left\{u_{ij}(\omega_{ij}, \omega_{-ij}, M_{ij}(\omega_{ij}), M_{-ij}(\omega_{-ij})), R_{ij}(\omega_{ij}, \omega_{-ij}, M_{-ij}(\omega_{-ij}))\right\} \geq \\ &\quad \max\left\{u_{ij}(\omega_{ij}, \omega_{-ij}, M_{ij}(\omega'_{ij}), M_{-ij}(\omega_{-ij})), R_{ij}(\omega_{ij}, \omega_{-ij}, M_{-ij}(\omega_{-ij}))\right\}, \end{aligned} \tag{1}$$

where $R_{ij}(\omega_{ij}, \omega_{-ij}, M_{-ij}(\omega_{-ij})) := u_{ij}(\omega_{ij}, \omega_{-ij}, (1, (ij, c_0)), M_{-ij}(\omega_{-ij}))$ *is player* $ij's$ *aggregate payoff when player* ij *chooses not to participate in the mechanism.*

Denote by $\mathbb{M}(\Omega, \mathbb{G})$ the set of all $(\mathcal{B}(\Omega), \mathcal{B}(\mathbb{G}))$-measurable functions and denote by $\mathcal{IC}$ the subset of $\mathbb{M}(\Omega, \mathbb{G})$ consisting of functions that satisfy the $\mathcal{IC}$ inequalities (1).

Note that under the assumption that player payoff functions are additively coupled, in order for a mechanism,

$$M(\cdot) \in \mathbb{M}(\Omega, \mathbb{G})$$

to be incentive compatible, it suffices that each player's part of the mechanism, $M_{ij}(\cdot) \in \mathbb{M}(\Omega_{ij}, \mathbb{G}_{ij})$ be incentive compatible while allowing voluntary nonparticipation. In particular, under additively coupled payoffs, a mechanism, $M(\cdot) := (M_{ij}(\cdot), M_{-ij}(\cdot))$ is incentive compatible (allowing voluntary nonparticipation) provided that for each player, ij, and for each ω_{ij} and ω'_{ij},

$$\max\left\{v_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})), r_{ij}(\omega_{ij})\right\} \geq \max\left\{v_{ij}(\omega_{ij}, M_{ij}(\omega'_{ij})), r_{ij}(\omega_{ij})\right\}. \tag{2}$$

where $r_{ij}(\omega_{ij}) := v_{ij}(\omega_{ij}, (1, (ij, c_0)))$ is player $ij's$ aggregate payoff - without the additive spillovers from other players' interactions - when player ij chooses not to participate in the mechanism and there is no information sharing between agents i and j.

Let

$$U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})) := \max\left\{v_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})), r_{ij}(\omega_{ij})\right\},$$

We will assume that player ij chooses nonparticipation if and only there is a positive gain from doing so. Therefore, if $M_{ij}(\cdot)$ satisfies the IC constraints with voluntary nonparticipation (2) and if for some $\overline{\omega}_{ij}$,

$$U_{ij}(\overline{\omega}_{ij}, M_{ij}(\overline{\omega}_{ij})) = r_{ij}(\overline{\omega}_{ij})$$

then

$$r_{ij}(\overline{\omega}_{ij}) > v_{ij}(\overline{\omega}_{ij}, M_{ij}(\overline{\omega}_{ij})) \Longrightarrow r_{ij}(\overline{\omega}_{ij}) \geq v_{ij}(\overline{\omega}_{ij}, M_{ij}(\omega'_{ij})) \text{ for all } \omega'_{ij}.$$

Therefore, no amount of misreporting will make participation attractive.

If player ij reports his private information to be ω'_{ij} then the mechanism recommends to player ij that ij form the ij-club network $M_{ij}(\omega'_{ij})$. Thus, if an ij-mechanism satisfies expression (1), then player ij will participate in the mechanism and will have nothing to gain by misreporting his private information to the mechanism. The mechanism is incentive compatible with voluntary nonparticipation.

Assuming for the moment that all players report their information truthfully and choose the profile of ij-networks recommended by the mechanism, then the profile of ij-mechanisms will give rise to a feasible club network,

$$M(\omega) := (M_{ij}(\omega_{ij}), M_{-ij}(\omega_{-ij})) \in \mathbb{G},$$

with typical connection $(a_{ij}(\omega_{ij}), (ij, c))$ where pre-connection, (ij, c), is in the set of pre-connections, $\mathcal{D}(M_{ij}(\omega_{ij}))$, specified by the mechanism and where for all $\omega_{ij} \in \Omega_{ij}$,

$$a_{ij}(\omega_{ij}) \in A(ijc) \text{ for all } (ij, c) \in \mathcal{D}(M_{ij}(\omega_{ij})).$$

We close this subsection on IC network formation mechanisms by noting that if for some type of player ij, with truthfully reported type, ω_{ij}, $\mathcal{D}(M_{ij}(\omega_{ij})) = \{(ij, c_0)\}$ so that $M_{ij}(\omega_{ij})(ijc_0) = \{a_{ij}(\omega_{ij})\} = \{1\}$, then player ij of type $\omega_{ij} := (t_{ij}, s)$ will not interact and player ij will have a payoff of

$$\begin{gathered} u_{ij}(\omega_{ij}, \omega_{\_ij}, \{\underbrace{(1, (ij, c_0))}_{M_{ij}(\omega_{ij})}\}, M_{-ij}(\omega_{-ij})) \\ = \\ v_{ij}(\omega_{ij}, (1, (ij, c_0))) + \textstyle\sum_{-ij} \bar{v}_{ij}(\omega_{ij}, \omega_{-ij}, M_{-ij}(\omega_{-ij})). \end{gathered}$$

Note that even though player ij engages in no interactions, player $ij's$ reservation payoff level,

$$u_{ij}(\omega_{ij}, \omega_{\_ij}, \{(1, (ij, c_0))\}, M_{-ij}(\omega_{-ij})),$$

is still a function of the interactions of the other players via the other ij-networks, $M_{-ij}(\omega_{-ij})$.

3.1.2 The Principal's Problem over IC Mechanisms

Assume that the principal has payoff given by

$$(\omega, M) \longrightarrow V(\omega, M) := V(\omega, (M_{ij}, M_{-ij})),$$

over profiles of player types and profiles of player club networks. Under an incentive compatible network formation mechanism,

$$(\omega_{ij}, \omega_{-ij}) \longrightarrow (M_{ij}(\omega_{ij}), M_{-ij}(\omega_{-ij}))$$

the principal's payoff becomes

$$V(\omega, M(\omega)),$$

where

$$M(\omega) := \begin{pmatrix} M_{11}(\omega_{11}) & \cdots & M_{1j}(\omega_{1j}) & \cdots & M_{1n}(\omega_{1n}) \\ \vdots & & \vdots & & \vdots \\ M_{i1}(\omega_{i1}) & \cdots & M_{ij}(\omega_{ij}) & \cdots & M_{in}(\omega_{in}) \\ \vdots & & \vdots & & \vdots \\ M_{n1}(\omega_{n1}) & \cdots & M_{nj}(\omega_{nj}) & \cdots & M_{nn}(\omega_{nn}) \end{pmatrix}_{n\times n}$$

In this section we will analyze the network formation problem under incomplete information as a principal-agent network formation game with adverse selection, assuming that the principal is allowed to design a profile of network formation mechanisms,

$$\omega_{ij} \longrightarrow (M_{ij}(\omega_{ij}))_{ij\in N\times N},$$

so as to induce players to reveal their types and to follow the connection recommendations of the mechanism.

Given the principal's probability beliefs, λ, defined on the measurable state space, $(\Omega, \mathcal{B}(\omega))$, the mechanism design problem, P1, faced by the principal is given by

$$\left.\begin{array}{c} \max_{M(\cdot)\in\mathbb{M}(\Omega,\mathbb{G})} \int_{\Omega} V(\omega,(M(\omega)))d\lambda(\omega) \\ \\ \text{such that for all } ij, \text{ and for all } \omega_{ij} \text{ and } \omega'_{ij} \\ \\ U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})) \geq U_{ij}(\omega_{ij}, M_{ij}(\omega'_{ij}))). \end{array}\right\} \tag{3}$$

Problem P1 can be restated as the following problem P2.

$$\max_{M(\cdot)\in\mathcal{IC}} \int_{\Omega} V(\omega, M(\omega))d\lambda(\omega) \tag{4}$$

3.2 Games over Catalogs

Our objective now is to characterize *all* incentive compatible network formation mechanisms via catalogs of ij-club networks. We call this characterization result the *Delegated Networking Principle.*[2] The importance of the delegated networking principle in proving existence of an optimal network formation mechanism is that it allows us to convert the principal-agent network formation game over network-valued mechanisms with incentive compatibility constraints into an *equivalent unconstrained principal-agent game* over catalogs of ij-networks. With this conversion, we are able to avoid the difficult problem of searching for a topology for the function space of network-valued mechanisms making the subset of incentive compatible mechanisms compact, players' payoff functions continuous, and the principal's payoff function upper semicontinuous. Instead, our reformulation of the network formation game as a principal-agent game over catalogs of networks allows us to utilize the topology already present in the space of ij-networks to establish existence.

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[2] The delegation principle for principal-agent games in contracting situations was proved in Page (1992, 1997).

3.2.1 Catalogs of ij-Network

Preliminaries To begin, suppose that the principal, rather than offering each player, ij, a mechanism, $M_{ij}(\cdot)$, and requiring a report from player ij concerning his type, ω_{ij}, to determine the ij-network recommendation, instead offers each player ij a catalog, $\mathcal{G}_{ij}$, of ij-networks and then observes the player $ij's$ choice from the ij-network catalog.

An ij-catalog, $\mathcal{G}_{ij}$, is a closed set of ij-networks G_{ij}. We must make precise what we mean by closed - as well as make precise what we mean by the distance between two catalogs. To begin, suppose $\mathcal{G}^1_{ij}$ and $\mathcal{G}^2_{ij}$ are two ij-network catalogs. Just as is the case with a club network G, any ij-network, G_{ij}, can be written as the union of its c-layers. Thus, we have

$$G_{ij} = \cup_c G_{ijc}$$

where for each c, G_{ijc} is a subset of $A(ijc) \times (\{ij\} \times \{c\})$. Thus, the c-layer in any ij-network, G_{ij}, is of the form

$$G_{ijc} \subset A(ijc) \times (\{ij\} \times \{c\}).$$

Because a player may not be a member of all clubs in network G, for some c', player $ij's$ network, $G_{ijc'}$, may be empty. Hence,

$$G_{ijc} \in 2^{A(ijc)\times(\{ij\}\times\{c\})}.$$

Given two ij-networks, G^1_{ij} and G^2_{ij}, the distance between them is given by

$$h_K(G^1_{ij}, G^2_{ij}) := \sum_c h_{K_c}(G^1_{ijc}, G^2_{ijc}), \tag{5}$$

and for each $c \in C$,

$$h_{K_c}(G^1_{ijc}, G^2_{ijc}) = \left\{ \begin{array}{cll} +\infty & \text{if} & G^1_{ijc} = \varnothing, G^2_{ijc} \neq \varnothing, \\ \rho_{w^*_c}(a^1_{ijc}, a^2_{ijc}) & \text{if} & G^1_{ijc} \neq \varnothing, G^2_{ijc} \neq \varnothing, \\ 0 & \text{if} & G^1_{ijc} = \varnothing, G^2_{ijc} = \varnothing, \end{array} \right. \tag{6}$$

A ij-catalog $\mathcal{G}_{ij}$ is a h_K-closed, subset of ij-networks, where recall, the collection of all feasible ij-networks is given by,

$$\mathbb{G}_{ij} := \{G_{ij} \in P(K_{ij}) : |G_{ij}(ijc)| \leq 1 \text{ for all } c \text{ and } |G_{ij}(ijc)| = 1 \text{ for some } c\}.$$

The Hyperspace of Catalogs of ij-Networks Recall that for each player ij, $P_{h_K f}(\mathbb{G}_{ij})$ denotes the collection of nonempty h_K-closed sets, $\mathcal{G}_{ij}$, where each set $\mathcal{G}_{ij} \subset \mathbb{G}_{ij}$ consists of a collection of ij-networks, $G_{ij} \in \mathbb{G}_{ij}$, where each ij-network is the union, $\cup_c G_{ijc}$, of ρ_{K_c}-closed subsets of $A(ijc) \times (\{ij\} \times \{c\})$. The catalog, $\mathcal{G}_{ij}$, is closed in the sense that if $\{G^n_{ij}\}_n \subset \mathcal{G}_{ij}$ is a sequence such that

$$G^n_{ij} \xrightarrow[h_K]{} G^*_{ij}$$

then $G^*_{ij} \in \mathcal{G}_{ij}$. We want to equip the hyperspace of catalogs, $P({}_{h_K f}(\mathbb{G}_{ij}))$, with typical element $\mathcal{G}_{ij}$, with the Hausdorff metric, H_{h_K}, induced by the metric h_K on $\mathbb{G}_{ij}$. Thus, in the hyperspace of catalogs $P({}_{h_K f}(\mathbb{G}_{ij}))$ if the catalog sequence $\{\mathcal{G}^n_{ij}\}_n$ is such that,

$$\mathcal{G}^n_{ij} \xrightarrow[H_{h_K}]{} \mathcal{G}^*_{ij} \text{ then } \mathcal{G}^*_{ij} \in P({}_{h_K f}(\mathbb{G}_{ij})).$$

The Hausdorff metric H_{h_k} is defined as follows: for $\mathcal{G}^1_{ij}$ and $\mathcal{G}^2_{ij}$ in $P({}_{h_K f}(\mathbb{G}_{ij}))$,

$$H_{h_K}(\mathcal{G}^1_{ij}, \mathcal{G}^2_{ij}) := \max\left\{ e_{h_k}(\mathcal{G}^1_{ij}, \mathcal{G}^2_{ij}), e_{h_k}(\mathcal{G}^2_{ij}, \mathcal{G}^1_{ij}) \right\},$$

where the excess of $\mathcal{G}^1_{ij}$ over $\mathcal{G}^2_{ij}$ is given by

$$\begin{gathered} e_{h_k}(\mathcal{G}^1_{ij}, \mathcal{G}^2_{ij}) := \sup\nolimits_{G^1_{ij} \in \mathcal{G}^1_{ij}} dist_{h_k}(G^1_{ij}, \mathcal{G}^2_{ij}) \\ \text{and} \\ dist_{h_k}(G^1_{ij}, \mathcal{G}^2_{ij}) := \inf\nolimits_{G^2_{ij} \in \mathcal{G}^2_{ij}} h_K(G^1_{ij}, G^2_{ij}). \end{gathered}$$

Convergence in $P({}_{h_K f}(\mathbb{G}_{ij}))$ can be described as follows. Let $\{\mathcal{G}^n_{ij}\}_n$ be a sequence in $P({}_{h_K f}(\mathbb{G}_{ij}))$.

The limit inferior, $Li\{\mathcal{G}^n_{ij}\}$, of the sequence $\{\mathcal{G}^n_{ij}\}_n$ is defined as follows: $G^*_{ij} \in Li\{\mathcal{G}^n_{ij}\}$ if and only if there is a sequence $\{G^n_{ij}\}$ in $\mathbb{G}_{ij}$ such that $G^n_{ij} \in \mathcal{G}^n_{ij}$ for all n and $G^n_{ij} \underset{h_K}{\longrightarrow} G^*_{ij}$.

The limit superior, $Ls\{\mathcal{G}^n_{ij}\}$, of the sequence $\{\mathcal{G}^n_{ij}\}_n$ is defined as follows: $G^*_{ij} \in Ls\{\mathcal{G}^n_{ij}\}$ if and only if there is a subsequence, $\{G^{n_k}_{ij}\}$ in $\mathbb{G}_{ij}$ such that $G^{n_k}_{ij} \in \mathcal{G}^{n_k}_{ij}$ for all k and $G^{n_k}_{ij} \underset{h_K}{\longrightarrow} G^*_{ij}$.

The ij-catalog, $\mathcal{G}^*_{ij}$, is said to be the limit of the sequence, $\{\mathcal{G}^n_{ij}\}$, if

$$Li\{\mathcal{G}^n_{ij}\} = \mathcal{G}^*_{ij} = Ls\{\mathcal{G}^n_{ij}\}.$$

Moreover, $H_{h_k}(\mathcal{G}^n_{ij}, \mathcal{G}^*_{ij}) \longrightarrow 0$ if and only if $Li\{\mathcal{G}^n_{ij}\} = \mathcal{G}^*_{ij} = Ls\{\mathcal{G}^n_{ij}\}$ (see Aliprantis and Border, 2006).

3.2.2 The Player's Catalog Problem

If the principle offers catalog $\mathcal{G}_{ij}$ to agents ij (i.e., player ij), then player $ij's$ problem is to choose an optimal ij-network, G_{ij}, from the catalog $\mathcal{G}_{ij}$. Thus, player $ij's$ problem is given by

$$U^*_{ij}(\omega_{ij}, \mathcal{G}_{ij}) := \max_{G_{ij} \in \mathcal{G}_{ij}} U_{ij}(\omega_{ij}, G_{ij})$$

Under [A-1] and [A-2] it follows from Page (1992) that because $\mathcal{G}_{ij}$ is h_k-compact, $U_{ij}(\omega_{ij}, \cdot)$ is h_k-continuous on $\mathcal{G}_{ij}$ (and measurable in ω_{ij}) for each ω_{ij}, there exists an optimal ij-network, $M^*_{ij}(\omega_{ij})$. In fact, there exists an optimal measurable selection, $\omega_{ij} \longrightarrow M^*_{ij}(\omega_{ij}) \in \mathbb{G}_{ij}$, such that

$$U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})) = U^*_{ij}(\omega_{ij}, \mathcal{G}_{ij}) \text{ for all } \omega_{ij}.$$

By Proposition 4.1 in Page (1992), for each ω_{ij}, the ij^{th} player's catalog payoff function, $U^*_{ij}(\omega_{ij}, \cdot)$ is h_k-continuous on $P({}_{h_K f}(\mathbb{G}_{ij}))$ (i.e., is h_k-continuous in catalogs), and for each catalog, $\mathcal{G}_{ij}$, $U^*_{ij}(\cdot, \mathcal{G}_{ij})$ is $\mathcal{B}(\Omega_{ij})$-measurable. Moreover, by Proposition 4.2 in Page (1992), for each ω_{ij}, the ij^{th} player's best response correspondence

$$(\omega_{ij}, \mathcal{G}_{ij}) \longrightarrow \Phi_{ij}(\omega_{ij}, \mathcal{G}_{ij})$$

is $\mathcal{B}(\Omega_{ij}) \times \mathcal{B}(\mathbb{G}_{ij})$-measurable and for each ω_{ij},

$$\mathcal{G}_{ij} \longrightarrow \Phi_{ij}(\omega_{ij}, \mathcal{G}_{ij}),$$

is h_k-upper semicontinuous on $P({}_{h_K f}(\mathbb{G}_{ij}))$.

3.2.3 The Principal's Catalog Problem

The principal's problem has two parts. First, for any given choice of a catalog profile

$$\mathcal{G} := (\mathcal{G}_{ij}, \mathcal{G}_{-ij}) \in \prod_{ij} P_{h_K f}(\mathbb{G}_{ij}),$$

together with players' best response mappings,

$$\Phi(\omega, \mathcal{G}) := \prod_{ij} \Phi_{ij}(\omega_{ij}, \mathcal{G}_{ij})$$

the principal will first make ij-network recommendations to the players. The principal's recommendation list is constructed by solving type by type for a given catalog profile, $\mathcal{G}$, the problem,

$$V^*(\omega, \mathcal{G}) := \max_{(G_{ij}, G_{-ij}) \in \Phi(\omega, \mathcal{G})} V(\omega_{ij}, \omega_{-ij}, G_{ij}, G_{-ij}).$$

By Proposition 4.3 in Page (1992), $V^*(\cdot, \cdot)$ is $\mathcal{B}(\Omega_{ij}) \times \mathcal{B}(\mathbb{G}_{ij})$-measurable and for each ω,

$$\mathcal{G} \longrightarrow V^*(\omega, \mathcal{G}),$$

is $h_k^{n \times n}$-upper semicontinuous on $\prod_{ij} P_{h_K f}(\mathbb{G}_{ij})$.

With these technical preliminaries out of the way, the principal's problem of finding an optimal catalog profile, $\mathcal{G}^*$, reduces to the following unconstrained problem:

$$\max \left\{ \int_\Omega V^*(\omega, \mathcal{G}) d\lambda(\omega) : \mathcal{G} \in \prod_{ij} P_{h_K f}(\mathbb{G}_{ij}) \right\}. \tag{7}$$

By Proposition 4.4 in Page (1992), there exists an optimal catalog profile, $\mathcal{G}^*$, such that

$$\int_\Omega V^*(\omega, \mathcal{G}^*) d\lambda(\omega) \geq \int_\Omega V^*(\omega, \mathcal{G}) d\lambda(\omega)$$

for all $\mathcal{G} \in \prod_{ij} P_{h_K f}(\mathbb{G}_{ij})$.

It is easy to see that if $\mathcal{G}^* \in \prod_{ij} P_{h_K f}(\mathbb{G}_{ij})$ solves the catalog problem, then the incentive compatible mechanism,

$$\omega \longrightarrow M^*(\omega) := (M^*_{ij}(\omega_{ij}), M^*_{-ij}(\omega_{-ij})) \in \Phi(\omega, \mathcal{G}^*)$$

solves the principal's problem over incentive compatible mechanisms given by

$$\max_{M(\cdot) \in \mathcal{IC}} \int_\Omega V(\omega, M(\omega)) d\lambda(\omega). \tag{8}$$

By taking the h_K-closure of the ranges of the ij-mechanisms, $\omega_{ij} \longrightarrow M^*_{ij}(\omega_{ij})$, making up the profile of mechanisms solving the principal's mechanism problem, we can construct an ij-catalog profile that solves the principal's problem over catalog profiles.[3] Thus, if $\omega \longrightarrow M^*(\omega)$ solves the principal's mechanism problem, the the catalog profile given by

$$\left(\overline{\left\{M^*_{ij}(\omega_{ij}) : \omega_{ij} \in \Omega_{ij}\right\}}^{h_K}\right)_{ij} := \left(\overline{M^*_{ij}(\Omega_{ij})}^{h_K}\right)_{ij}$$

solves the principal's catalog problem. Also note that for each ω_{ij}, $G^*_{ij}(\omega_{ij})$ can be rewritten in the following way,

$$M^*_{ij}(\omega_{ij}) := \cup_{c \in C} M^*_{ijc}(\omega_{ij}) := (M^*_{ij0}(\omega_{ij}), M^*_{ij1}(\omega_{ij}), \ldots, M^*_{ijk}(\omega_{ij})). \tag{9}$$

Note that if $(ij, c') \notin \mathcal{D}(M^*_{ij}(\omega_{ij}))$, then for the c'^{th} component of the $k+1$-tuple in expression (9), we have

$$M^*_{ijc'}(\omega_{ij}) = \varnothing,$$

while for all other c (i.e., those such that $(ij, c) \in \mathcal{D}(M^*_{ij}(\omega_{ij}))$), $M^*_{ijc}(\omega_{ij})$ is a nonempty, ρ_{K_c}-closed subset of

$$A(ijc) \times (\{ij\} \times \{c\}),$$

namely, $(a^*_{ijc}(\omega_{ij}), (ij, c))$.

4 The Delegated Networking Principle

We now formally state and prove the *Delegated Networking Principle* for a principal-multi-agent games of network formation with adverse selection (also see Page, 1992 and 1997, and Page and Wooders, 2010).

Theorem 2 (The Delegated Networking Principle)
Suppose assumptions [A-1] and [A-2] hold. The following statements are equivalent.
(1) $M_{ij}(\cdot) \in \mathbb{G}(\Omega_{ij}, \mathbb{G}_{ij})$, *is incentive compatible, that is, for all* ω_{ij} *and* ω'_{ij}

$$U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})) \geq U_{ij}(\omega_{ij}, M_{ij}(\omega'_{ij})).$$

(2) $M_{ij}(\cdot) \in \mathbb{M}(\Omega_{ij}, \mathbb{G}_{ij})$ *is such that there exists a unique, minimal (by set inclusion) ij-catalog,* $\mathcal{G}_{ij} \in P_{h_K f}(\mathbb{G}_{ij})$ *satisfying*

$$M_{ij}(\omega_{ij}) \in \Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}_{ij}) \textit{for all } \omega_{ij}.$$ [4]

Let $\Sigma^e_{\Phi_{U_{ij}}}(\mathcal{G}_{ij})$ denote the collection of all (everywhere) measurable selections of the best response mapping, $\Phi_{U_{ij}}(\cdot, \mathcal{G}_{ij})$, with ij-catalog $\mathcal{G}_{ij}$, the delegated networking principle can be stated compactly as follows:

$$M_{ij}(\cdot) \in \mathcal{IC}_{ij} \text{ if and only if } M_{ij}(\cdot) \in \Sigma^e_{\Phi_{U_{ij}}}(\mathcal{G}_{ij})$$
$$\text{for some } ij\text{-catalog } \mathcal{G}_{ij}.$$

---

[3] Under the mechanism, $M^*(\cdot)$, if player ij reports his private information truthfully, say ω_{ij}, then his intended ij-network is given by

$$M^*_{ij}(\omega_{ij}) := \left\{(a^*_{ijc}(\omega_{ij}), (ij, c)) \in A(ijc) \times (\{ij\} \times \{c\}) : (ij, c) \in \mathcal{D}(M^*_{ij}(\omega_{ij}))\right\}.$$

[4] By definition of Φ, $M_{ij}(\omega_{ij}) \in \Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}_{ij})$ if and only if $U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})) = \max_{G_{ij} \in \mathcal{G}_{ij}} U_{ij}(\omega_{ij}, G_{ij})$ for all ω_{ij}.

Proof of the Delegated Networking Principle:

(1) $\Longrightarrow$(2) Let $M_{ij}(\cdot)$ be an incentive compatible ij-mechanism and consider the ij-catalog $\overline{M_{ij}(\Omega_{ij})}^{h_K}$.

First, note that for each $\omega_{ij} \in \Omega_{ij}$,

$$U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})) \geq U_{ij}(\omega_{ij}, G_{ij}) \text{ for all } G_{ij} \in \overline{M_{ij}(\Omega_{ij})}^{h_K}.$$

Suppose not. Then for some ij, there exists $G^*_{ij} \in \overline{M_{ij}(\Omega_{ij})}^{h_K}$, such that,

$$U_{ij}(\omega_{ij}, G^*_{ij}) > U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})).$$

Because $G^*_{ij} \in \overline{M_{ij}(\Omega_{ij})}^{h_K}$, there exists a sequence $\{\omega^n_{ij}\}_n$ in Ω_{ij} such that $M_{ij}(\omega^n_{ij}) \underset{h_K}{\longrightarrow} G^*_{ij}$. But now because

$$U_{ij}(\omega_{ij}, G^*_{ij}) > U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij}))$$

by the h_K-continuity of $U_{ij}(\omega_{ij}, \cdot)$ the fact that $M_{ij}(\omega^n_{ij}) \underset{h_K}{\longrightarrow} G^*_{ij}$ implies that for n large enough,

$$U_{ij}(\omega_{ij}, M_{ij}(\omega^n_{ij})) > U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})),$$

contradicting the fact that $M_{ij}(\cdot)$ is incentive compatible. Therefore, because

$$U_{ij}(\omega_{ij}, M_{ij}(\omega_{ij})) \geq U_{ij}(\omega_{ij}, G_{ij}) \text{ for all } G_{ij} \in \overline{M_{ij}(\Omega_{ij})}^{h_K},$$

we conclude that

$$M_{ij}(\omega_{ij}) \in \Phi_{U_{ij}}(\omega_{ij}, \overline{M_{ij}(\Omega_{ij})}^{h_K}) \text{ for all } \omega_{ij} \in \Omega_{ij}.$$

Let $\mathcal{G}_{ij} := \overline{M_{ij}(\Omega_{ij})}^{h_K}$. It remains to show $\mathcal{G}_{ij}$ is the minimum ij-catalog.

Now suppose that there exists another ij-catalog, $\mathcal{G}'_{ij}$, such that

$$M_{ij}(\omega_{ij}) \in \Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}'_{ij}) \text{ for all } \omega_{ij} \in \Omega_{ij}$$

but that for some $G^*_{ij} \in \overline{M_{ij}(\Omega_{ij})}^{h_K}$,

$$G^*_{ij} \notin \mathcal{G}'_{ij}.$$

Again, because $G^*_{ij} \in \overline{M_{ij}(\Omega_{ij})}^{h_K}$, there exists a sequence of player types $\{\omega^n_{ij}\}_n$ in Ω_{ij} such that $M_{ij}(\omega^n_{ij}) \longrightarrow G^*_{ij}$. But now we have

$$M_{ij}(\omega^n_{ij}) \in \Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}'_{ij}) \text{ for all } n \text{ and } \Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}'_{ij}) \subseteq \mathcal{G}'_{ij}.$$

Because $\mathcal{G}'_{ij}$ is h_K-closed and $M_{ij}(\omega^n_{ij}) \longrightarrow G^*_{ij}$, we must conclude that $G^*_{ij} \in \mathcal{G}'_{ij}$, a contradiction.

(2)$\Rightarrow$(1). The proof is straightforward.

5 The Equivalence of Mechanism Games and Catalog Games

Given ij-catalog, $\mathcal{G}_{ij}$, the type ω_{ij} player is indifferent over the networks in $\Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}_{ij})$, the principal will not be. In order to resolve the principal's lack of indifference over $\Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}_{ij})$, the principal will suggest or recommend a particular ij-network choice from $\Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}_{ij})$. Any such recommendation will be followed by the type ω_{ij} player because it is incentive compatible for the agent to do so provided the player is type ω_{ij} and the ij-catalog offered is $\mathcal{G}_{ij}$. The principal's optimal recommendation list is found by solving pointwise (i.e., player type by player type) the following problem

$$\max_{(G_{ij})_{ij\in N\times N}\in\Pi_{ij\in N\times N}\Phi_{U_{ij}}(\omega_{ij},\mathcal{G}_{ij})} V(\omega_{ij},\omega_{-ij},(G_{ij})_{ij\in N\times N}).$$

Let

$$V^{*}(\omega_{ij},\omega_{-ij},(\mathcal{G}_{ij})_{ij\in N\times N})$$

$$=\max_{(G_{ij})_{ij\in N\times N}\in\Pi_{ij\in N\times N}\Phi_{U_{ij}}(\omega_{ij},\mathcal{G}_{ij})} V(\omega_{ij},\omega_{-ij},(G_{ij})_{ij\in N\times N}).$$

By Theorem 2 in Himmelberg, Parthasarathy, and VanVleck (1976), given ij-catalogs, $(\mathcal{G}_{ij})_{ij\in N\times N}$, there exists for each player ij a selection from $\Phi_{U_{ij}}(\cdot,\mathcal{G}_{ij})$, say

$$M^{*}_{ij}(\cdot)\in\Sigma^{e}_{\Phi_{U_{ij}}}(\mathcal{G}_{ij}),$$

such that

$$V(\omega_{ij},\omega_{-ij},(M^{*}_{ij}(\omega_{ij}))_{ij\in N\times N})$$

$$=V^{*}(\omega_{ij},\omega_{-ij},(\mathcal{G}_{ij})_{ij\in N\times N}).$$

Moreover, because

$$\Phi(\cdot,\cdot):=\Pi_{ij\in N\times N}\Phi_{U_{ij}}(\cdot,\cdot)$$

is $[\mathcal{B}(\Omega)\times\Pi_{ij\in N\times N}\mathcal{B}(P_{h_K f}(\mathbb{G}_{ij}))]$-measurable and h_K-compact-valued in $\Pi_{ij\in N\times N}P_{h_K f}(\mathbb{G}_{ij})$ and $V(\cdot,\cdot)$ is measurable and h_K-upper semicontinuous on

$$\Pi_{ij\in N\times N}P_{h_K f}(\mathbb{G}_{ij})$$

for each $(\omega_{ij})_{ij\in N\times N}$, it follows from Theorem 2 in Himmelberg, Parthasarathy, and Van Vleck (1976) that $V^{*}(\cdot,\cdot)$ is measurable. Moreover, because

$$\Phi(\cdot,\cdot):=\Pi_{ij\in N\times N}\Phi_{U_{ij}}(\omega_{ij},\cdot)$$

is h_K-upper semicontinuous on $\Pi_{ij\in N\times N}P_{h_K f}(\mathbb{G}_{ij})$, it follows from Theorem 2 in Berge (1963) that $V^{*}(\omega_{ij},\omega_{-ij},\cdot)$ is h_K-upper semicontinuous on

$$\Pi_{ij\in N\times N}P_{h_K f}(\mathbb{G}_{ij}).$$

The catalog game can now be stated very compactly as

$$\max_{(\mathcal{G}_{ij})_{ij\in N\times N}\in\Pi_{ij\in N\times N}P_{h_K f}(\mathbb{G}_{ij})}\int_{\Omega}V^{*}(\omega_{ij},\omega_{-ij},(\mathcal{G}_{ij})_{ij\in N\times N})d\lambda(\omega). \tag{10}$$

The mechanism game can also be stated very compactly as

$$\max_{(M_{ij}(\cdot))_{ij\in N\times N}\in\Pi_{ij\in N\times N}\mathcal{IC}_{ij}}\int_{\Omega}V(\omega_{ij},\omega_{-ij},(M_{ij}(\omega_{ij}))_{ij\in N\times N})d\lambda(\omega). \tag{11}$$

Also, recall that

$$\Sigma^{e}_{\Phi_{U_{ij}}}(\mathcal{G}_{ij}):=\left\{M_{ij}(\cdot):M_{ij}(\omega_{ij})\in\Phi_{U_{ij}}(\omega_{ij},\mathcal{G}_{ij})\text{ for all }\omega_{ij}\in\Omega_{ij}\right\} \tag{12}$$

denotes the set of all measurable selections from $\Phi_{U_{ij}}(\cdot,\mathcal{G}_{ij})$ for a given ij-catalog $\mathcal{G}_{ij}\in P_{h_K f}(\mathbb{G}_{ij})$ and define

$$\Sigma^{e}_{\Phi_{U_{ij}}}=\cup_{\mathcal{G}_{ij}\in P_{h_K f}(\mathbb{G}_{ij})}\Sigma^{e}_{\Phi_{U_{ij}}}(\mathcal{G}_{ij}). \tag{13}$$

An alternative statement of the Delegation Principal is that

$$\mathcal{IC}_{ij}=\Sigma^{e}_{\Phi_{U_{ij}}}. \tag{14}$$

We now have our main result on the equivalence of mechanism games and catalog games. This result is an immediate consequence of the *Delegated Networking Principle*.

Theorem 3 (The Equivalence of Mechanism Games and the Catalog Games)
Suppose assumptions [A-1] and [A-2] hold. Then the following statements are true.
(1) If $(M^*_{ij}(\cdot))_{ij\in N\times N} \in \Pi_{ij\in N\times N}\mathcal{IC}_{ij}$ *solves the network formation game over mechanisms given by*

$$\max_{(M_{ij}(\cdot))_{ij\in N\times N}\in \Pi_{ij\in N\times N}\mathcal{IC}_{ij}} \int_\Omega V(\omega_{ij}, \omega_{-ij}, (M_{ij}(\omega_{ij}))_{ij\in N\times N})d\lambda(\omega),$$

then

$$(\overline{M^*_{ij}(\Omega_{ij})}^{h_K})_{ij\in N\times N} \in \Pi_{ij\in N\times N} P_{h_K f}(\mathbb{G}_{ij})$$

solves the network formation game over catalogs $\Pi_{ij\in N\times N} P_{h_K f}(\mathbb{G}_{ij})$ *given by*

$$\max_{(\mathcal{G}_{ij})_{ij\in N\times N}\in \Pi_{ij\in N\times N} P_{h_K f}(\mathbb{G}_{ij})} \int_\Omega V^*(\omega_{ij}, \omega_{-ij}, (\mathcal{G}_{ij})_{ij\in N\times N})d\lambda(\omega).$$

(2) If $(\mathcal{G}^*_{ij})_{ij\in N\times N} \in \Pi_{ij\in N\times N} P_{h^*_K f}(\mathbb{G}_{ij})$ *solves the network formation game over catalogs given by*

$$\max_{(\mathcal{G}_{ij})_{ij\in N\times N}\in \Pi_{ij\in N\times N} P_{h^*_K f}(\mathbb{G}_{ij})} \int_\Omega V^*(\omega_{ij}, \omega_{-ij}, (\mathcal{G}_{ij})_{ij\in N\times N})d\lambda(\omega),$$

then

$$(M^*_{ij}(\cdot))_{ij\in N\times N} \in \Pi_{ij\in N\times N}\Sigma^e_{\Phi_{U_{ij}}}(\mathcal{G}^*_{ij})$$

such that

$$V(\omega_{ij}, \omega_{-ij}, (M^*_{ij}(\omega_{ij}),)_{ij\in N\times N})$$

$$= \max_{(G_{ij})_{ij\in N\times N}\in \Pi_{ij\in N\times N}\Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}^*_{ij})} V(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij\in N\times N}).$$

solves the network formation game over mechanisms given by

$$\max_{(M_{ij}(\cdot))_{ij\in N\times N}\in \Pi_{ij\in N\times N}\mathcal{IC}_{ij}} \int_\Omega V(\omega_{ij}, \omega_{-ij}, (M_{ij}(\omega_{ij}))_{ij\in N\times N})d\lambda(\omega).$$

6 Existence

Because the principal's optimal payoff function $V^*(\omega, \cdot)$ is upper semicontinuous on $\Pi_{ij\in N\times N} P_{h_K f}(\mathbb{G}_{ij})$ for all ω and because $\Pi_{ij\in N\times N} P_{h_K f}(\mathbb{G}_{ij})$ is an h_K-compact metric space, we easily obtain the following existence result.

Theorem 4 (Existence of an Optimal Catalog)
Suppose assumptions [A-1] and [A-2] hold. Then there exists an ij-catalog profile, $(\mathcal{G}^*_{ij})_{ij\in N\times N} \in \Pi_{ij\in N\times N} P_{h_K f}(\mathbb{G}_{ij})$*, solving the catalog game, that is, a catalog profile,* $(\mathcal{G}^*_{ij})_{ij\in N\times N}$*, such that*

$$\int_\Omega V^*(\omega_{ij}, \omega_{-ij}, (\mathcal{G}^*_{ij})_{ij\in N\times N})d\lambda(\omega)$$

$$= \max_{(\mathcal{G}_{ij})_{ij\in N\times N}\in \Pi_{ij\in N\times N} P_{h_K f}(\mathbb{G}_{ij})} \int_\Omega V^*(\omega_{ij}, \omega_{-ij}, (\mathcal{G}_{ij})_{ij\in N\times N})d\lambda(\omega).$$

Suppose that $(\mathcal{G}^*_{ij})_{ij\in N\times N} \in \Pi_{ij\in N\times N} P_{h_K f}(\mathbb{G}_{ij})$ solves the catalog game (10) and

$$(M^*_{ij}(\cdot))_{ij\in N\times N} \in \Pi_{ij\in N\times N}\Sigma^e_{\Phi_{U_{ij}}}(\mathcal{G}^*_{ij})$$

is such that

$$V(\omega_{ij}, \omega_{-ij}, (M^*_{ij}(\omega_{ij}))_{ij\in N\times N})$$

$$= \max_{(G_{ij})_{ij\in N\times N}\in \Pi_{ij\in N\times N}\Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}^*_{ij})} V(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij\in N\times N}).$$

For all $(\mathcal{G}_{ij})_{ij\in N\times N} \in \Pi_{ij\in N\times N} P_{h_K f}(\mathbb{G}_{ij})$ and for all $(M_{ij}(\cdot))_{ij\in N\times N} \in \Pi_{ij\in N\times N}\Sigma^e_{\Phi_{U_{ij}}}(\mathcal{G}_{ij})$

$$\textstyle\int_\Omega V^*(\omega_{ij},\omega_{-ij},(\mathcal{G}^*_{ij})_{ij\in N\times N})d\lambda(\omega) = \int_\Omega V(\omega_{ij},\omega_{-ij},(M^*_{ij}(\omega_{ij}))_{ij\in N\times N})d\lambda(\omega)$$

$$\geq \textstyle\int_\Omega V^*(\omega_{ij},\omega_{-ij},(\mathcal{G}_{ij})_{ij\in N\times N})d\lambda(\omega) = \int_\Omega V(\omega_{ij},\omega_{-ij},(M_{ij}(\omega_{ij}))_{ij\in N\times N})d\lambda(\omega),$$

where $(M^*_{ij}(\cdot))_{ij\in N\times N} \in \Pi_{ij\in N\times N}\Sigma^e_{\Phi_{U_{ij}}}(\mathcal{G}^*_{ij})$ is such that

$$V((\omega_{ij})_{ij\in N\times N},(M^*_{ij}(\omega_{ij}))_{ij\in N\times N})$$

$$= \max_{(G_{ij})_{ij\in N\times N}\in\Pi_{ij\in N\times N}\Phi_{U_{ij}}(\omega_{ij},\mathcal{G}^*_{ij})} V(\omega_{ij},\omega_{-ij},(G_{ij})_{ij\in N\times N}).$$

Thus, we can conclude via the Delegated Networking Principle (i.e., via the fact that $\mathcal{IC}_{ij} = \Sigma^e_{\Phi_{U_{ij}}}$) that

$$\textstyle\int_\Omega V^*(\omega_{ij},\omega_{-ij},(\mathcal{G}^*_{ij})_{ij\in N\times N})d\lambda(\omega) = \int_\Omega V(\omega_{ij},\omega_{-ij},(M^*_{ij}(\omega_{ij}))_{ij\in N\times N})d\lambda(\omega)$$

$$= \max_{(M_{ij}(\cdot))_{ij\in N\times N}\in\Pi_{ij\in N\times N}\mathcal{IC}_{ij}} \textstyle\int_\Omega V(\omega_{ij},\omega_{-ij},(M_{ij}(\omega_{ij}))_{ij\in N\times N})d\lambda(\omega).$$

We have the following Corollary.

Corollary (Existence of an Optimal Network Formation Mechanism)
Suppose assumptions [A-1] and [A-2] hold. Then there exists a network formation mechanism $(M^*_{ij}(\cdot))_{ij\in N\times N} \in \Pi_{ij\in N\times N}\mathcal{IC}_{ij}$ *solving the mechanism game, that is, a mechanism* $(M^*_{ij}(\cdot))_{ij\in N\times N}$ *such that*

$$\textstyle\int_\Omega V(\omega_{ij},\omega_{-ij},(M^*_{ij}(\omega_{ij}))_{ij\in N\times N})d\lambda(\omega)$$

$$= \max_{(M_{ij}(\cdot))_{ij\in N\times N}\in\Pi_{ij\in N\times N}\mathcal{IC}_{ij}} \textstyle\int_\Omega V(\omega_{ij},\omega_{-ij},(M_{ij}(\omega_{ij}))_{ij\in N\times N})d\lambda(\omega).$$

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