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The Dependency Diagram of a Linear Programme

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Abstract

The Dependency Diagram of a Linear Programme (LP) shows how the successive inequalities of an LP depend on former inequalities, when variables are projected out by Fourier-Motzkin Elimination. It is also explained how redundant inequalities can be removed, using the method attributed to Chernikov and to Kohler. The procedure also leads to a transparent explanation of Farkas' Lemma, LP Duality, the dual form of Caratheodory's Theorem as well as generating all vertices and extreme rays of the Dual Polytope.

1 Introduction

Fourier-Motzkin Elimination is a method of eliminating the variables in a polytope, defined by linear inequalities, by projection. It has been discovered and rediscovered, independently, a number of times and can be attributed to Fourier[3] and Motzkin[7] (in a Game Theory context). Langford[5] describes it (not using, or probably being aware of the Fourier-Motzkin name) as a method of showing that the Theory of Dense Linear Order is decidable by eliminating quantifiers. It can be used to solve LPs. Williams[9] gives an explanation of the method. Martin[6] also gives a description. The program PORTA[2] uses the method.

For any but small models the method is computationally impractical owing to the explosive growth in the number of generated inequalities, as variables are successively projected out.

In section 2 we describe the method by means of a *Dependency Diagram,* which shows how the inequalities resulting from successive eliminations of variables (projections) depend on earlier inequalities. In that section we also show how the number of resultant inequalities can be reduced using theorems attributed to Chernikov[1] and to Kohler[4].

In section 3 we show how the method gives a clear demonstration of the duality theorem of LP, as well as generating all vertices and extreme rays of the dual polytope. It also gives demonstrations of Farkas' lemma and the dual form of Caratheodory's theorem (see eg Schrijver[8]).

2 The Dependency Diagram of an LP

We explain this by means of the following numerical example (we will consider all LPs in this form as maximisations subject to '$\leq$' inequalities).

$$
\begin{array}{l}
Maximise \;\; z \\
Subject \; to
\end{array}
$$

$$
\begin{array}{rcrl}
-w - 2x + y + z & \leq & 0 & C0 \\
2w + 2x - 3y & \leq & 1 & C1 \\
w - x - y & \leq & -1 & C2 \\
-w + 2x + 3y & \leq & 4 & C3 \\
-12w \qquad + 9y & \leq & -3 & C4 \\
-w & \leq & 0 & C5 \\
-x & \leq & 0 & C6 \\
-y & \leq & 0 & C7
\end{array}
$$

The variables w, x, y can be projected out in any order. The projection (elimination) of a variable relies on the following theorem (using logical terminology as applied, for example, by Langford in terms of eliminating an $\exists$ quantifier).

Theorem 1 $\exists x_j \; \{a_{ij}x_j \geq f_i \;\; i \in I, \; -a_{kj}x_j \geq g_k \;\; k \in K\} \Longleftrightarrow 0 \geq a_{kj}f_i + a_{ij}g_k \;\; i \in I, k \in K$ *where* $a_{ij} > 0, i \in I \cup K, x_j \in \mathcal{R}$

Proof. *(i)* $\Rightarrow$ *This is obtained by adding each inequality, in the form* $x_j \geq f_i/a_{ij}$ *to each inequality, in the form* $-x_j \geq g_k/a_{kj}$ *respectively to give* $f_i/a_{ij} \leq$ *-*g_k/a_{kj}, $i \in I, k \in K$ *ie* $0 \geq a_{kj}f_i + a_{ij}g_k$ $i \in I, k \in K$.

(ii)$\Leftarrow$*Suppose* $0 \geq a_{kj}f_i + a_{ij}g_k$ *ie* $-a_{ij}g_k \geq a_{kj}f_i$. *This can expressed as* $-g_k/a_{kj} \geq f_i/a_{ij}$. *Let* $x_j = \max_i\{f_i/a_{ij}\}$ *(or* $\min_k\{-g_k/a_{kj}\}$*). Then* $a_{ij}x_j \geq f_i$ *and* $-a_{kj}x_j \geq g_k$ $i \in I, k \in K$ ■

Note that if either I or K (or both) are empty then the conclusion is tautologically true and the variable x_j (and all inequalities containing it) can be removed with no resultant inequalities. We will refer to such an elimination as 'trivial'.

Applying this theorem to the elimination of w in the example we represent the original constraints by the nodes in the top row of figure 1.

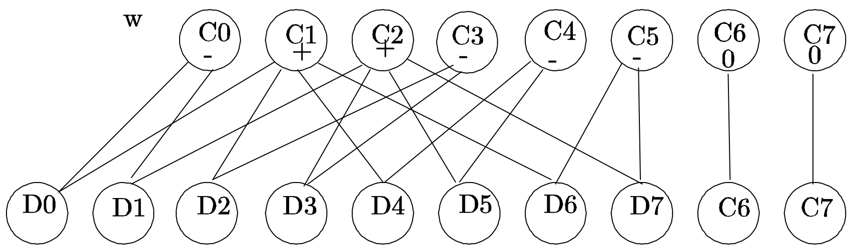

Figure 1: Dependency diagram after the elimination of variable w

The sign of the first variable, w, to be eliminated in each of these constraints is indicated.The resultant constraints are:

$$
\begin{array}{rcrl}
-2x - y + 2z & \leq & 1 & D0 = 2C0 + C1 \\
-3x + z & \leq & -1 & D1 = C0 + C2 \\
6x + 3y & \leq & 9 & D2 = C1 + 2C3 \\
x + 2y & \leq & 3 & D3 = C2 + C3 \\
12x - 9y & \leq & 3 & D4 = 6C1 + C4 \\
-12x - 3y & \leq & -15 & D5 = 12C2 + C4 \\
2x - 3y & \leq & 1 & D6 = C1 + 2C5 \\
-x - y & \leq & -1 & D7 = C2 + C5 \\
-x & \leq & 0 & C6 \\
-y & \leq & 0 & C7
\end{array}
$$

We refer to the two inequalities, from which each new inequality is derived, as the parents.Hence D0 has C0 and C1 as parents. Note that the result of carrying out successive eliminations of variables will be to produce inequalites which are **positive** combinations of some of the original inequalities (which we will refer to as the 'ancestors').

In order to reduce the number of derived constraints, we can rely on the following theorem (attributed to Kohler[4] and Chernikov[1]).

Theorem 2 *If an inequality depends on a proper, or the same, subset of the inequalities which give rise to another inequality then this latter inequality is redundant.*

Proof. *Suppose we have a system* $\mathbf{Ax} + \mathbf{By} \leq \mathbf{b}$ *where* $\mathbf{A}, \mathbf{B}$ *are matrices and* $\mathbf{x}, \mathbf{y}$ *and* $\mathbf{b}$ *vectors. Denote the rows of* $\mathbf{A}$ *by* $\mathbf{a}_1^{'}, \mathbf{a}_2^{'}, ..., \mathbf{a}_m^{'}$, *.Suppose an inequality results from eliminating* $\mathbf{x}$ *(an n-tuple) and depends (without loss of generality) on the first r inequalities in the system. We have*

$$\mu_1\mathbf{a}_1^{'} + \mu_2\mathbf{a}_2^{'} + ... + \mu_r\ \mathbf{a}_r^{'} = \mathbf{0} \qquad \textit{where } \mu_1, \mu_2, ..., \mu_r > 0 \tag{1}$$

Suppose another inequality results from eliminating $\mathbf{x}$ *and depends on the first s inequalities, where* $s \geq r$. *We have*

$$\lambda_1\mathbf{a}_1^{'} + \lambda_2\mathbf{a}_2^{'} + ... + \lambda_r\ \mathbf{a}_r^{'} + \lambda_{r+1}\mathbf{a}_r^{'} + ... + +\lambda_s\mathbf{a}_s^{'} = \mathbf{0} \qquad \textit{where } \lambda_1, \lambda_2, ..., \lambda_s > 0 \tag{2}$$

We assume, temporarily, that , in the case $s = r$ *, not all* μ_i *and* λ_i *are the same. If we choose* $\rho = Min_{i \leq r}\frac{\lambda_i}{\mu_i}$ *then we can obtain*

$$(\lambda_1 - \rho\mu_1)\mathbf{a}_1^{'} + (\lambda_2 - \rho\mu_2)\mathbf{a}_2^{'} + ... + (\lambda_r - \rho\mu_r)\mathbf{a}_r^{'} + \lambda_{r+1}\mathbf{a}_r^{'} + ... + +\lambda_s\mathbf{a}_s^{'} = \mathbf{0} \tag{3}$$

where $(\lambda_i - \rho\mu_i) \geq 0, i = 1, 2, ..., r$ *and at least one of these expressions is zero. Applying the (non-negative) multipliers* $(\lambda_i - \rho\mu_i)$ *to the system we obtain another valid (non-trivial) inequality (3) from eliminating* $\mathbf{x}$. *From theorem 1 it it follows that both this inequality and that corresponding to the multipliers in (1) are implied by the inequalities resulting from Fourier-Motzkin Elimination. Neither can be implied by that corresponding to (2) since this depends on some rows of* $\mathbf{A}$ *on which each of the other two inequalities is not dependent. But the inequality corresponding to (2) can be obtained by adding that corresponding to (1) to (3) in multiples of* ρ *and* 1 *respectively. Hence that corresponding to (2) is redundant..In the case that* $s = r$*, and not all the* μ_i *and* λ_i *are the same, the above argument shows that both are redundant. If* μ_i *and* λ_i *are the same then either one will be redundant, in the presence of the other.* ■

Theorem 3 *If, after eliminating n variables by Fourier-Motzkin Elimination, an inequality depends on more than* $n + 1$ *of the original inequalities it is redundant.*

Proof. *Consider again the system* $\mathbf{Ax} + \mathbf{By} \leq \mathbf{b}$ *and an inequality resulting from eliminating* $\mathbf{x}$ *(an n-tuple) which depends on the first* $n + r$ *rows, where* $r > 1$. *Then*

$$\mu_1 a_1^{'} + \mu_2 a_2^{'} + ... + \mu_{n+r}\ a_{n+r}^{'} = 0 \qquad \textit{where } \mu_1, \mu_2, ..., \mu_{n+r} > 0 \tag{4}$$

Since the $a_i^{'}$ *are n-tuples any* $n + 1$ *of them must be linearly dependent. Hence we have*

$$\lambda_1 a_1^{'} + \lambda_2 a_2^{'} + ... + \lambda_r\ a_r^{'} + \lambda_{n+1}a_{n+1} = 0 \qquad \textit{where at least one } \lambda_i > 0 \tag{5}$$

We can add or subtract a suitable multiple of (5) from (4) in order to eliminate at least one $a_i^{'}$ *from (4) but still keep the multipliers non-negative. The procedure can be repeated until no more than* $n + 1$ *of the* $a_i^{'}$ *are involved.The inequality resulting from applying these new, non-negative,multipliers to the system is clearly valid and so implied by the multipliers resulting from Fourier-Motzkin elimination. It cannot, however, depend on the inequality corresponding to (4) which is therefore redundant by virtue of theorem 2.* ■

This theorem can be strengthened by the following corollary.

Corollary 4 *Any non-redundant inequality, after the non-trivial elimination of n variables depends on* ***exactly*** $n + 1$ *of the original inequalities.*

By "non-trivial' we mean each elimination of a variable is between an inequality in which it has a negative coefficient and an inequality in which it has a positive coefficient. A 'trivial' elimination is that remarked on after theorem 1 where the variable has all coefficients zero,or of the same sign, resulting in the removal of the variable and all inequalities in which it occurs.

Proof. *We proceed inductively. If an inequality results from the non-trivial elimination of 1 variable it must arise from 2 original inequalities We suppose that an inequality P results from a non-trivial elimination of a variable between inequalities A and B and that A and B each result from the non-trivial elimination of* n *variables from the original inequalities. We make the inductive hypothesis that A and B each depend on* $n+1$ *of the original inequalities.* ■

Two cases will be distinguished. In case (i) A and B each depend on the same set of $n+1$ *of the original inequalities. If A and B are different inequalities they are both redundant by theorem 2 and the case does not arise. If A and B are the same inequality there can be no elimination between them and P is not derivable. In Case (ii) A and B depend on different subsets of of* $n+1$ *of the original inequalities.Should these subsets differ by more than 1 inequality P will depend on more than* $n+1$ *of the original inequalities and therefore be redundant by theorem 3. Should the subsets differ by 1 inequality then P will depend on* $n+2$ *of the original inequalities, confirming the inductive assumption.*

It is not possible to generate two inequalities (which might be the same),by different routes,by n non-trivial eliminations from exactly $n+1$ ancestors.We prove this in the theorem below.Kohler suggested this but was unable to prove it. If there are more than $n+1$ ancestors then, of course, these inequalities would be redundant by virtue of theorem 2..(This happens in the next elimination, in the example, and is shown below).

Theorem 5 *It is not possible to generate the same inequality,by different routes,by* n *non-trivial eliminations from exactly* $n+1$ *ancestors, (removing redundant inequalities by theorems 1 and 2.)*

Proof. *This is trivially true for* $n = 1, 2, 3$. *We suppose, inductively, it is true for all values up to, and including* n. *Suppose, however, that for* $n+1$ *eliminations (and* $n > 2$) *,we can generate two nodes from* $n+2$ *nodes at the top level. By the inductive assumption we need at least* $n+2$ *nodes at the next level in order to generate two final nodes. In order to do this we must have at least two nodes with a '+' and two nodes with a '−' at the top level. But the first of the final nodes (P) requires exactly* $n+1$ *ancestors at the second level* (by corollary 1). *We have the situation shown in figure 2. We name the inequalities represented by each node at top level A,B,...and give the sign of the first variable to be eliminated. At subsequent levels we name the nodes representing inequalities by the names of their predecessors, again giving the sign of the variable to be next eliminated. Without loss of generality we can name these inequalities and give the signs shown in figure 2.*

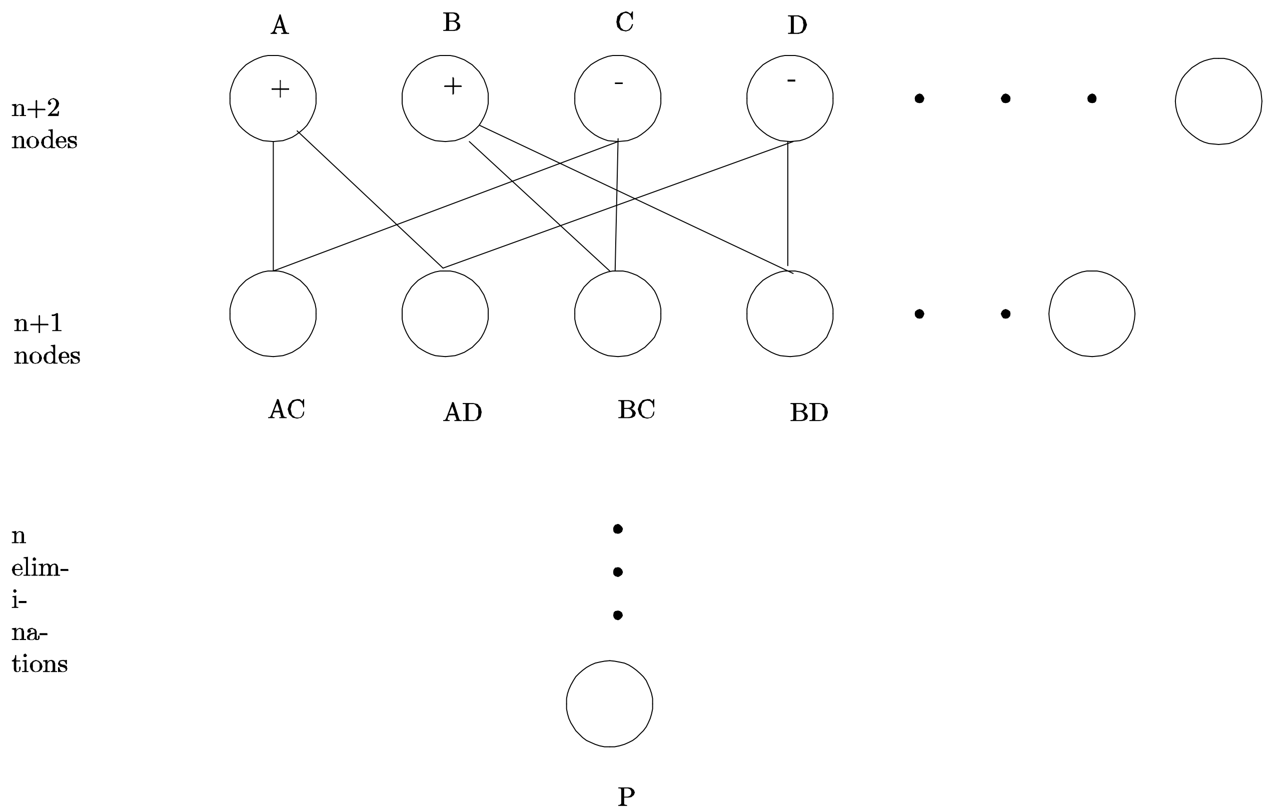

Figure 2: Impossibility of generating 2 inequalities from n+2 ancestors by n+1 eliminations

At the second level we must have at least one node with a '+' and at least one node with a '−' in order to generate nodes at the third level. Suppose we have exactly one node with a '+' or exactly one node with a '−' . Without loss of generality let AC be this node. Combining with node BD we would generate the redundant (by theorem 2) inequality ABCD, at the third stage, and therefore be unable to generate the requisite number (n) *of non-redundant inequalities at the third stage. Therefore at least two nodes at the second level have a '+' sign and at least two a '−' sign. If* $n = 3$ *and AC and BD had opposite signs (forcing AD and BC to have opposite signs) combining these pairs of inequalities would result in two redundant inequalities, making it impossible to generate the requisite number of inequalities at the next stage. Hence AC and BD must have the same sign, as each other, and also AD and BC the same sign, as each other. We can show that this sign cannot be opposite to that for AC and BD by the following argument: Writing A,B,C,D (after scaling) as*

$$\begin{aligned} x_1 + a_{12}x_2 + ... &\leq b_1 \\ x_1 + a_{22}x_2 + ... &\leq b_2 \\ -x_1 + a_{32}x_2 + ... &\leq b_3 \\ -x_1 + a_{42}x_2 + ... &\leq b_4 \end{aligned}$$

and performing the elimination shown in figure 2 gives the inequalities AC,BD,AD,BC as

$$\begin{aligned}(a_{12}+a_{32})x_2+... &\leq b_1+b_3\\(a_{22}+a_{42})x_2+... &\leq b_2+b_4\\(a_{12}+a_{42})x_2+... &\leq b_1+b_4\\(a_{22}+a_{32})x_2+... &\leq b_2+b_3\end{aligned}$$

If the sign of the coefficients of x_2 in AC and BD is opposite to that in AD and BD then we have either (if AC and BD associated with positive coefficients)

$$\begin{aligned}(a_{12}+a_{32}) &> 0\\(a_{22}+a_{42}) &> 0\\-(a_{12}+a_{42}) &> 0\\-(a_{22}+a_{32}) &> 0\end{aligned}$$

or (if AC and BD associated with negative coefficients) the four inequalities above are reversed. Adding each set of four inequalities leads to $0 > 0$ which is a contradiction. Hence the signs must all be the same. This shows that we cannot perform a non-trivial elimination from level 2 to 3 proving the theorem for $n = 3$.

If $n > 3$ then we can use the argument above to show that at successive levels we must have at least 2 nodes with a '$+$' sign and at least two nodes with a '$-$' sign. Eventually we reach a level with two '$+$' nodes and two "$-$' nodes. This corresponds to the case $n = 3$ where it is shown above that we cannot proceed to more than one final node with two eliminations. Hence the theorem is proved for all n. ■

We now proceed to the elimination of x from the example using the results of the foregoing theorems to avoid generating redundant inequalities. The result is given in figure 3.

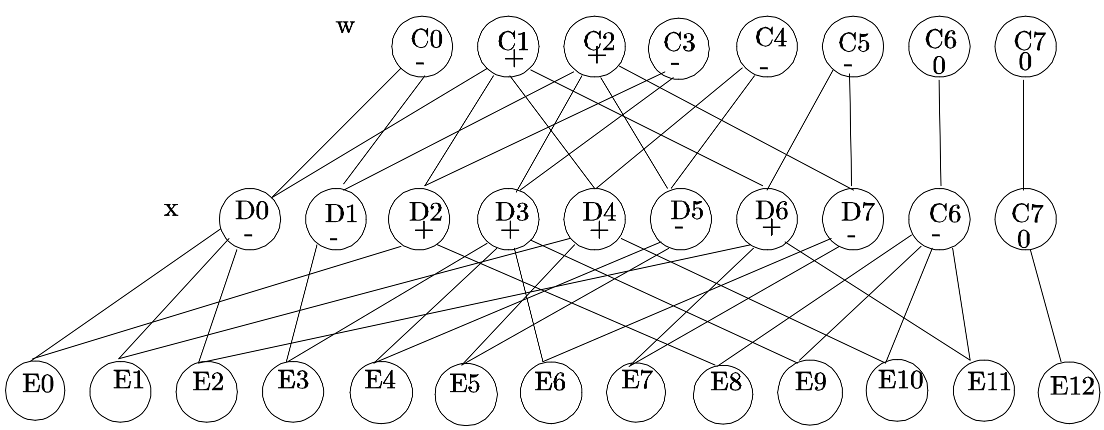

Figure 3: Dependency diagram after the elimination of variable x.

The derived inequalities are:

$6z$	$\leq$	12	$E0 = 3D0 + D2 = 6C0 + 4C1 + 2C3$
$-15y + 12z$	$\leq$	9	$E1 = 6D0 + D4 = 12C0 + 12C1 + C4$
$-4y + 2z$	$\leq$	2	$E2 = D0 + D6 = 2C0 + 2C1 + +2C5$
$6y + z$	$\leq$	8	$E3 = D1 + 3D3 = C0 + 4C2 + 3C3$
$21y$	$\leq$	21	$E4 = 12D3 + D5 = 24C2 + 12C3 + C4$
$-12y$	$\leq$	-12	$E5 = D4 + D5 = 6C1 + 12C2 + 2C4$
y	$\leq$	2	$E6 = D3 + D7 = 12C2 + C3 + C5$
$-5y$	$\leq$	-1	$E7 = D6 + 2D7 = C1 + 2C2 + 4C5$
$3y$	$\leq$	9	$E8 = D2 + 6C6 = C1 + 2C3 + 6C6$
$2y$	$\leq$	3	$E9 = D3 + C6 = C2 + C3 + C6$
$-9y$	$\leq$	3	$E10 = D4 + 12C6 = 6C1 + C4 + 12C6$
$-3y$	$\leq$	1	$E11 = D6 + 2C6 = C1 + 2C5 + 2C6$
$-y$	$\leq$	0	$C7$

Notice that we have used theorem 2 to avoid, unnecessarily, eliminating x between the following pairs of inequalities (D0,D3), (D1,D2), (D1,D4), (D1,D6), (D2,D5), (D2,D7),(D5,D6), and (D4,D7) as the resultant inequalities would all depend on 4 of the original inequalities (instead of 3). The elimination between D0 and D3 and that between D1 and D2 both produce the redundant inequality $3y + 2z \leq 7$ (which both result from $2C0 + C1 + 2C2 + 2C3$,but by different routes through the Dependency Diagram).

Proceeding with the elimination of y produces the Dependency Diagram in figure 4. Again we have not generated inequalities which would be redundant by virtue of theorem 2.

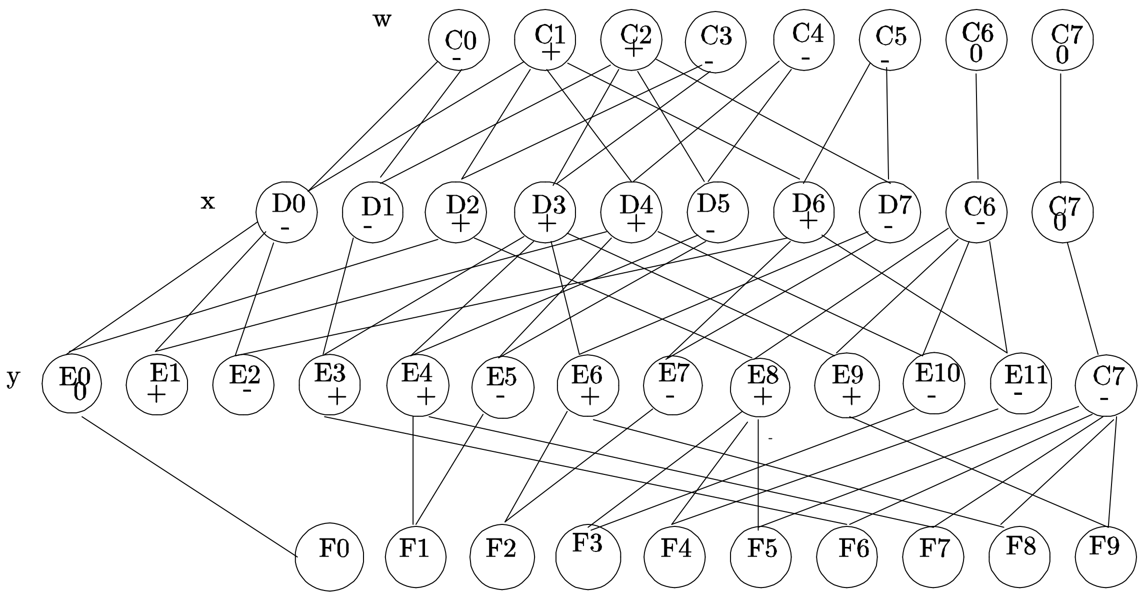

Figure 4: Dependency diagram after the elimination of variable y.

The resultant inequalities are:

$$
\begin{array}{rcll}
6z & \leq & 12 & F0 = E0 = 3D0 + D2 = 6C0 + 4C1 + 2C3 \\
0 & \leq & 0 & F1 = 12E4 + 21E5 = 144D3 + 21D4 + 33D5 = 126C1 + 540C2 + 144C3 + 54C4 \\
0 & \leq & 9 & F2 = 5E6 + E7 = 5D3 + D6 + 7D7 = C1 + 12C2 + 5C3 + 9C5 \\
0 & \leq & 30 & F3 = 3E8 + E10 = 3D2 + D4 + 30C6 = 9C1 + 6C3 + C4 + 30C6 \\
0 & \leq & 10 & F4 = E8 + E11 = D2 + D6 + 8C6 = 2C1 + 2C3 + 2C5 + 8C6 \\
0 & \leq & 9 & F5 = E8 + 3C7 = D2 + 6C6 + 3C7 = C1 + 2C3 + 6C6 + 3C7 \\
z & \leq & 8 & F6 = E3 + 6C7 = D1 + 3D3 + 6C7 = C0 + 4C2 + 3C3 + 6C7 \\
0 & \leq & 21 & F7 = E4 + 21C7 = 12D3 + D5 + 21D7 = 24C2 + 12C3 + C4 + 21C7 \\
0 & \leq & 9 & F8 = E6 + C7 = D3 + D7 + C7 = 2C2 + C3 + C5 + C7 \\
0 & \leq & 3 & F9 = E9 + 2C7 = D3 + C6 = C2 + C3 + C6 + 2C7
\end{array}
$$

The inequalities which do not involve z show that the original LP is feasible. If there were no inequalities involving z the original model would be unbounded. The optimal value of z is clearly 2 obtained by treating $F0$ as an equation. Although some of the inequalities are now redundant they would not be for other right-hand-sides of the original LP. This is shown below.

In order to obtain the values of the variables we could observe that making $F0$ an equality forces its ancestors ($C0, C1$ and $C3$), in the Dependency Diagram, to be equalities. We could therefore solve these as equations to obtain the values of the variables. Alternatively we could backtrack, from $F0$ ($E0$), through the Dependency Diagram, substituting $z = 2$ in $D0$ and $D2$ to give $y = 1$ and these values in $CO, C1$ and $C3$ to give $x = 1$ and $w = 1$.

The elimination of variables, through the Dependency Diagram, is independent of the original right-hand-side. In order to show that none of the inequalities (after applying theorem 2) generated by Fourier-Motzkin Elimination is redundant for all right-hand-sides we prove the following theorem.

Theorem 6 *The set of inequalities generated by eliminating* $\mathbf{x}$ *from* $\mathbf{Ax} + \mathbf{By} \leq \mathbf{b}$, *by Fourier-Motzkin elimination, less those which are redundant by virtue of theorem 2, will all be irredundant for certain* $\mathbf{B}$ *and* $\mathbf{b}$.

Proof. *We first show that the multiplers, which produce the the inequalities, when regarded as vectors, are independent in the sense that none is a non-negative linear combination of the others.*

Suppose this were not the case and that we had a set of multipliers $(\mu_1, \mu_2, ..., \mu_r)$ *which was a positive linear combiation of others. Any one of these others must, therefore have positive entries only in the same, or a subset of the positions in which the above vector does. By virtue of theorem 2 the inequality produced by the above set of multipliers is therefore redundant.*

We now take $\mathbf{B}$ *as the identity matrix and* $\mathbf{b}$ *as the zero vector. Eliminating* $\mathbf{x}$ *by Fourier-Motzkin elimination, and discarding those inequalities which are redundant by virtue of theorem 2, produces inequalities of the form*

$\mu_{j1}y_1 + \mu_{j2}y_2 + ... + \mu_{jr}y_r \leq 0$

$(\mu_{j1}, \mu_{j2}, +..., \mu_{jr})$ *are the associated set of multipliers. Since the set of these vecors are independent by the argument above, these inequalities are independent of each other in the sense that none can be obtained as a non-negative linear combination of the others. Hence, by theorem 2, they are all irredundant.* ■

3 Theorems relating to Linear Programming

Theorem 7 *(Dual form of Caratheodory's Theorem). For an LP model with* n *variables, at most* n *constraints will be binding in the optimal solution.*

Proof. *This follows immediately from corollary 1. The final inequality, which determines the optimal value of the objective* z *can depend on no more than* n *of the orinal inequalities together with the original inequality relating* z *to the objective function.* ■

Theorem 8 *(Farkas' Lemma). An inequality* $\sum_{j=1}^{n} c_j x_j \leq c_0$ *is implied by the inequalities* $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, ..., m$ *if and only if , for all* $y_i \geq 0$, *such that* $\sum_{i=1}^{m} a_{ij} y_i = c_j, j = 1, 2, ..., n$, $\sum_{i=1}^{m} b_i y_i \leq c_0$

Proof. *Maximise* $\sum_{j=1}^{n} c_j x_j$ *subject to* $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, ..., m$ *by means of Fourier-Motzkin elimination. This either delivers a set of* $y_i \geq 0$ *giving the maximum value of* $\sum_{j=1}^{n} c_j x_j$ *as* $\sum_{i=1}^{m} b_i y_i$ *or shows the problem to be infeasible or unbounded. Either way the theorem is proved.*

■

Theorem 9 *(Duality). If given an LP (known as the **Primal**), in the form*

$Maximise\ \ z = \sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij}x_j \leq b_i, i = 1, 2, ..., m$
$x_j \geq 0, , j = 1, 2, ..., n$

we define another LP,(known as the **Dual**), in the form
$Minimise \;\; z^{'} = \sum_{i=1}^{m} b_i y_i$

subject to $\sum_{i=1}^{m} a_{ij}y_i \geq c_j, j = 1, 2, ..., n$,

$y_i \geq 0, , i = 1, 2, ..., m$

then if (a) the Primal is not infeasible or unbounded $x_j \geq 0 = Minimum \; z^{'}$. If(b) the Primal is unbounded the Dual is infeasible. If (c) the Primal is infeasible the Dual is unbounded **or** infeasible.

Proof. Applying Fourier-Motzkin Elimination to the Primal model, in case (a) we obtain a final inequality in the form $z \leq \sum_{i=1}^{m} \lambda_i b_i$,which we then treat as an equality to give the maximum value of z, where $\lambda_i \geq 0, i = 1, 2, ..., m$, and no inequalities of the form $0 \leq a$ negative q Å$tity$.By definition the λ_i provide values for y_i which optimise the dual, showing $Maximum \; z = Minimum \; z^{'}$.If the Primal is unbounded (but feasible) no inequality of the form $z \leq \sum_{i=1}^{m} \lambda_i b_i$ will result,showing that there can be no solution to the Dual. It is, however possible (Case (c)) that no inequalities of the form $z \leq \sum_{i=1}^{m} \lambda_i b_i$ result but we do obtain inequalities of the form $0 \leq a$, where a is negative, showing that both the Primal and the Dual are infeasible. ■

Corollary 10 *(Complementarity). If the optimal solution values to the Primal (including the slack variables) are* $x_1, x_2, ..., x_n, u_1, u_2, ..., u_m$ *then an optimal solution to the Dual model* $y_1, y_2, ..., y_m, v_1, v_2, ..., v_n$ *can be found, where* v_j *are the surplus variables in the Dual, such that* $y_i > 0 \Longrightarrow u_i = 0$ *and* $v_j > 0 \Longrightarrow x_j = 0$. *This result can be summarised as* $\sum_j x_j v_j + \sum_i u_i y_i = 0$.

Proof. .The y_i are the multipliers applied to the $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$,constraints, to give the optimal value of z, and the v_j are the multipliers applied to the $x_j \geq 0$ constraints.Since the $Maximum \; z = \sum_{i=1}^{m} b_i y_i$, if $y_i > 0$ then $\sum_{j=1}^{n} a_{ij}x_j = b_i$ ie $u_i = 0$. Also if $v_j > 0$ then $x_j = 0$. ■

Theorem 11 *The multipliers which generate the non-redundant inequalities involving z correspond to all vertices of the dual polytope. Those where the final inequality does not involve z correspond to all extreme rays of the dual polytope.*

Proof. This follows from theorems 3 and 6.We first consider the non-redundant inequalities generated by the constraints of the Primal model, including a positive multiplier for the objective constraint When the multipliers are scaled, to make those for the objective constraint 1, we have solutions for the Dual model which are independent of all other solutions and therefore represent a vertices. Since the set of all possible irredundant multipliers for the Primal model apply to all possible right-hand-sides, we have solutions for all possible objectives of the Dual model ie **all** vertex solutions. For the multipliers for the Primal model which do not include a positive value for the objective constraint then we have irredundant solutions to the Dual model with zero right-hand-sides ie **all** the extreme ray solutions. ■

We illustrate this result by considering the Dual of the model given in section 2. This is:

$$Minimise\ p_1 - p_2 + 4p_3 - 3p_4$$
$$Subject\ to$$

$$\begin{array}{rcl} 2p_1 + p_2 - p_3 - 12p_4 & \geq & 1 \\ 2p_1 - p_2 + 2p_3 & \geq & 2 \\ -3p_1 - p_2 + 3p_3 + 9p_4 & \geq & -1 \\ \\ p_1, p_2, p_3, p_4 & \geq & 0 \end{array}$$

The vertices are given by the multipliers of $F0$ and $F6$ ie $(2/3, 0, 1/3, 0)$, $(0, 4, 3, 0)$.

The extreme rays (scaled) are given by the multipliers of $F1$ to $F5$ and $F7$ to $F9$ ie $(7, 30, 8, 3)$, $(1, 12, 5, 0)$, $(9, 0, 6, 1)$, $(1, 0, 1, 0)$, $(1, 0, 2, 0)$, $(0, 24, 12, 1)$, $(0, 2, 1, 0)$, $(0, 1, 1, 0)$

It is worth pointing out that the set of multipliers (including those on the n0n-negativity constaints) make up the rows of the of the 'annihilator' matrix of the original LP.

In the example we therefore have:

$$\begin{pmatrix} 1, 2/3, 0, 1/3, 0, 0, 0, 0 \\ 1, 0, 4, 3, 0, 0, 0, 6 \\ 0, 7, 30, 8, 3, 0, 0, 0 \\ 0, 1, 12, 5, 0, 9, 0, 0 \\ 0, 9, 0, 6, 1, 0, 30, 0 \\ 0, 1, 0, 1, 0, 1, 0, 4 \\ 0, 1, 0, 2, 0, 0, 6, 7 \\ 0, 0, 24, 12, 1, 0, 0, 21 \\ 0, 0, 2, 1, 0, 1, 0, 1 \\ 0, 0, 1, 1, 0, 0, 1, 2 \end{pmatrix} \begin{pmatrix} -1, -2, 1, \\ 2, 2, -3 \\ 1, -1, -1 \\ -1, 2, 3 \\ -12, 0, 9 \\ -1, 0, 0 \\ 0, -1, 0 \\ 0, 0, -1 \end{pmatrix} = \mathbf{0}$$

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