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Central limit theorems for long range dependent spatial linear processes

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Central limit theorems are established for the sum, over a spatial region, of observations from a linear process on a d-dimensional lattice. This region need not be rectangular, but can be irregularly-shaped. Separate results are established for the cases of positive strong dependence, short range dependence, and negative dependence. We provide approximations to asymptotic variances that reveal differential rates of convergence under the three types of dependence. Further, in contrast to the one dimensional (i.e., the time series) case, it is shown that the form of the asymptotic variance in dimensions $d > 1$ critically depends on the geometry of the sampling region under positive strong dependence and under negative dependence and that there can be non-trivial edge-effects under negative dependence for $d > 1$. Precise conditions for the presence of edge effects are also given.

Keywords: central limit theorem; edge effects; increasing domain asymptotics; long memory; negative dependence; positive dependence; sampling region; spatial lattice

1. Introduction

The presence of long range dependence in spatial data has been noted in various empirical studies but a suitable formulation and systematic study of such spatial processes is lacking. For example, the "law of environmental variation" of Fairfield Smith [12], based on "Agricultural Field Trials" data, posits that the covariance function of the yield in the plane decays as the inverse of the Euclidean distance. Thus, the covariance functions of such spatial processes are not absolutely summable and may exhibit long range dependence. More recently, the effect of spatial long range dependence has been noted in Atmospheric sciences (cf. Kashyap and Lapsa [19], Gneiting [13]), Economics (Leonenko and Taufer [25]), Oceanography (cf. Percival *et al.* [27]) and Solid State Physics (cf. Carlos-Davila *et al.* [7]), among others. See also Lavancier [23] for some specific examples and other applications of spatial long range dependence. The traditional approach of quantifying spatial dependence through various notions of mixing is inadequate for dealing with long range dependence. In this paper, we consider a class of stationary spatial linear processes that allow for long range dependence, as well as the properties of short range- and negative-dependence, and establish central limit theorems for the sum over the entire range of such dependence.

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The dependence structure of a real-valued stationary process on a d-dimensional spatial lattice can be non-parametrically modeled by the linear process

$$Z(\mathbf{i}) = \mu + \sum_{\mathbf{j}\in\mathbb{Z}^d} \alpha(\mathbf{i}-\mathbf{j})\varepsilon(\mathbf{j}), \qquad \mathbf{i}\in\mathbb{Z}^d, \tag{1.1}$$

where the collection of real numbers $\{\alpha(\mathbf{i}), \mathbf{i}\in\mathbb{Z}^d\}$ satisfies

$$\sum_{\mathbf{i}\in\mathbb{Z}^d} \alpha(\mathbf{i})^2 < \infty, \tag{1.2}$$

and $\{\varepsilon(\mathbf{i}), \mathbf{i}\in\mathbb{Z}^d\}$ is a collection of independent homoscedastic random variables with zero mean and finite variance. If the $\varepsilon(\mathbf{i})$ are only uncorrelated, (1.1) and (1.2) represent the class of purely non-deterministic processes on $\mathbb{Z}^d$, for which $Z(\mathbf{i})$ has a more parsimonious "half-plane" representation (see, e.g., Whittle [38]), to generalize the one-sided Wold representation in the time series case $d = 1$. However, we impose independence in order to establish central limit theorems (CLTs) for

$$S_n = \sum_{\mathbf{i}\in\mathcal{D}_n} Z(\mathbf{i})$$

as $n\to\infty$. Here it is supposed that we observe $Z(\mathbf{i})$ within a spatial region $R_n\subset\mathbb{R}^d$ (to be described in detail subsequently) whose volume is regarded as increasing with the integer $n\ge 1$, the data sites being given by

$$\mathcal{D}_n = R_n\cap\mathbb{Z}^d.$$

Extending time series notions (cf. Robinson [30]), we consider three different sub-classes of (1.1), (1.2), broadly described as

$$\begin{aligned}
&\text{negatively dependent (ND):} && \sum_{\mathbf{i}\in\mathbb{Z}^d}\left|\alpha(\mathbf{i})\right| < \infty, && \sum_{\mathbf{i}\in\mathbb{Z}^d}\alpha(\mathbf{i}) = 0,\\
&\text{short-range dependent (SRD):} && \sum_{\mathbf{i}\in\mathbb{Z}^d}\left|\alpha(\mathbf{i})\right| < \infty, && \sum_{\mathbf{i}\in\mathbb{Z}^d}\alpha(\mathbf{i}) \neq 0,\\
&\text{positively strongly dependent (PSD):} && \sum_{\mathbf{i}\in\mathbb{Z}^d}\left|\alpha(\mathbf{i})\right| = \infty,
\end{aligned}$$

though our results rest on conditions that respectively, imply these. Denoting by $f(\lambda)$ the spectral density of $Z(\mathbf{i})$, for ND processes $f(0) = 0$, for SRD processes $f(0)\in(0,\infty)$, and for PSD processes f may diverge at frequency 0. The three sub-classes are also associated with different rates of increase of

$$\sigma_n^2 = \mathrm{Var}(S_n),$$

where we assume

$$\sigma_n^2\to\infty \qquad \text{as } n\to\infty, \tag{1.3}$$

which is a necessary condition for a CLT. Define $N_n = |\mathcal{D}_n|$, where $|B|$ denotes the size (i.e., number of elements) of a finite set B, so N_n denotes sample size. Under additional conditions, we have

$$N_n^{-1}\sigma_n^2 \to 0 \qquad \text{as } n \to \infty \text{ when } Z(\mathbf{i}) \text{ is ND,}$$
$$N_n^{-1}\sigma_n^2 \to \sigma_0^2 \in (0,\infty) \qquad \text{as } n \to \infty \text{ when } Z(\mathbf{i}) \text{ is SRD,}$$
$$N_n^{-1}\sigma_n^2 \to \infty \qquad \text{as } n \to \infty \text{ when } Z(\mathbf{i}) \text{ is PSD.}$$

The condition of PSD is also referred to as *long range dependence* in the literature.

Under (1.1), (1.2) and (1.3), and for $d = 1$ with $\mathcal{D}_n = (1, 2, \dots, n)$, Ibragimov and Linnik [17], pages 359–360, established that

$$[S_n - ES_n]/\sigma_n \xrightarrow{d} \mathcal{N}(0, 1) \qquad \text{as } n \to \infty. \tag{1.4}$$

Their main achievement was to allow arbitrarily slowly increasing σ_n^2, in particular to cover all ND $Z(\mathbf{i})$, as well as SRD and PSD ones. Again for $d = 1$, Hannan [16] relaxed independence of the $\varepsilon(\mathbf{i})$ to a martingale difference assumption, but only covered SRD and PSD $Z(\mathbf{i})$. On the other hand, Rosenblatt [32], Taqqu [35] and others established non-central limit theorems when $d = 1$ and $Z(\mathbf{i})$ does not satisfy (1.1) but is a non-linear function of a PSD Gaussian process, and more generally. Mention must also be made of the many CLTs for $d = 1$, where (1.1) is replaced by mixing conditions, following Rosenblatt [31], implying $Z(\mathbf{i})$ is SRD, and extended to $d > 1$ by a number of authors; see, for example, Bolthausen [6], Doukhan [10], Guyon [15], the latter two authors also discussing the mixing properties of linear processes. CLTs for SRD spatial processes over irregular sampling regions are given by Lahiri [21] and El Machkouri, Volný and Wu [11], allowing more general processes than linear fields. But relatively less attention has been paid to PSD processes with $d > 1$, under either linear or other assumptions. For rectangular regions, CLTs and invariance principles for PSD spatial linear processes have been proved by Lavancier [24] and for fractional Brownian sheets by Wang [37]. Dobrushin and Major [9] and Surgailis [34] proved central- and non-central limit theorems for functionals of PSD Gaussian processes and for functionals of PSD linear fields, respectively. There is also a small body of literature on statistical inference on the mean and covariance parameters of PSD spatial processes; see the papers by Boissy *et al.* [5], Beran *et al.* [2] and Wang and Cai [36] and the monographs by Ivanov and Leonenko [18] and Bertail *et al.* [3], and the references therein. We know of no spatial work under ND.

A major limitation of the existing work on spatial PSD processes is that it deals exclusively with rectangular spatial sampling regions. In contrast to the temporal case, in most practical applications spatial sampling regions are non-rectangular, and possibly of a non-standard shape (cf. Cressie [8], Lahiri *et al.* [22]). As a result, existing results are of limited use. The present paper attempts to fill the gap by introducing a general framework for studying linear spatial processes over sampling regions of non-standard shapes. The main results of the paper establish separate CLTs for sums of observations from, respectively, LRD, ND and SRD processes over possibly non-rectangular sampling regions, replacing σ_n in (1.4) by concise approximations which indicate the differing rates of convergence in different situations, and highlighting an intricate interplay between the spatial dependence structure and the geometry of the sampling regions.

It will be observed that the asymptotic variance of the centered sum $[S_n - ES_n]$ shows a very complex pattern of interactions among (i) the (effective) rate of decay of the coefficients $\alpha(\cdot)$, (ii) the sample size N_n or equivalently, the volume of the sampling region R_n, and (iii) the shape of the sampling region R_n. For simplicity of exposition, suppose for the time being that $\alpha(\mathbf{i})$ decays as $\|\mathbf{i}\|^{-\beta}$ as $\|\mathbf{i}\| \to \infty$ and that the volume of R_n grows at rate $c_0\lambda_n^d$ for some $c_0 \in (0, \infty)$ and $\lambda_n \to \infty$. Then, square-summability of the $\alpha(\cdot)$'s implies that $\beta > d/2$ and it can be shown that $\beta \in (\frac{d}{2}, d)$ leads to the case of PSD. In this case, we show that

$$\big(\lambda_n^{3d-2\beta}\big)^{-1/2}[S_n - ES_n] \xrightarrow{d} N\big(0, \sigma_{\text{psd}}^2\big), \tag{1.5}$$

where σ_{psd}^2 depends on certain limiting characteristics of the co-efficients $\alpha(\cdot)$ and the shape of the sampling region R_n (cf. Theorem 3.2 below). Note that in the PSD case, that is, for $\beta \in (d/2, d)$, the scaling sequence is given by $\lambda_n^{(3d-2\beta)/2}$ which is of a larger order of magnitude than $c_0^{1/2}\lambda_n^{d/2}$, the square root of the volume of the sampling region R_n. Thus, in the PSD case, the variance of the sum grows at a rate faster than the usual rate $N_n^{1/2}$ (since the sample size N_n here also grows at the rate $c_0\lambda_n^d$). Further, the limiting variance of the sum does *not* depend on the values of $\alpha(\mathbf{i})$ for $\mathbf{i}$ in any given bounded neighborhood of the origin.

In the ND case, the sum shows a very different limit behavior that critically depends on β as well as on the values of the coefficients $\alpha(\mathbf{i})$, for both small as well as large values of $\|\mathbf{i}\|$. Indeed, for $\beta \in (d, d + 1/2)$, a normal limit similar to (1.5) holds, albeit with a different asymptotic variance, which now depends on properties of both R_n and R_n^c. On the other hand, for β beyond the critical level $d + 1/2$, the edge-effect of the sampling region R_n becomes asymptotically dominant in dimensions $d \geq 2$, which in turn determines the asymptotic distribution. The corresponding scaling sequence is now given by $\lambda_n^{(d-1)/2}$ which, quite surprisingly, no longer depends on the values of $\beta \in (d + 1/2, \infty)$, that is, on the rate of decay of the coefficients $\alpha(\mathbf{i})$. Thus, in dimensions $d \geq 2$, the slowest possible rate for the variance of the sum in the ND case is given by $\lambda_n^{(d-1)}$ for all $\beta > d + 1/2$. This may be contrasted with the one dimensional case, where the edge-effect is asymptotically negligible and the growth rate of the variance of the sum can be very slow (e.g., $O(\lambda_n^{3-2\beta})$ with β close to $d + 1/2 = 3/2$). See Section 4 for full details. For the sake of completeness, we also prove a CLT for the SRD case. Here the sum has the usual rate of $N^{1/2}$ and the limiting variance depends on the $\alpha(\mathbf{i})$ only through their sum A, agreeing with a familiar result in the time series case $d = 1$ (cf. Section 5). The following Table 1 summarizes different limit behavior of the sum under PSD, ND and SRD for $d \geq 2$.

The rest of the paper is organized as follows. In Section 2, we consider the spatial linear process allowing non-identically distributed (but homoscedastic) errors and describe an asymptotic framework that can accommodate a large class of sampling regions of non-standard shapes. In Section 2, we also state a regular variation condition on the coefficients that, in particular, allows the coefficients to have different rates of decay along different directions, and give some examples to illustrate the scope of the formulation. In Sections 3 and 4, we establish the limit distribution of the sum under PSD and ND, respectively. We prove the CLT in the SRD case in Section 5. Proofs of the main results are given in Section 6. Here we also present a very general version of the CLT for a spatial linear process observed on bounded regions that may be of independent interest.

Table 1. A summary of the limit behavior of the sum S_n under PSD, ND and SRD when $\alpha(\mathbf{i}) = c_1 \|\mathbf{i}\|^{-\beta}$ for $\|\mathbf{i}\| > c_2$ and when vol.$(R_n) \sim c_3 \lambda_n^d$, for some constants $c_1, c_2, c_3 \in (0, \infty)$ and for some $\lambda_n \to \infty$, where vol.(R_n) denotes the volume of R_n. Note that here the sample size $N_n \sim$ vol.(R_n) and $A = \sum_{\mathbf{i} \in \mathbb{Z}^d} \alpha(\mathbf{i})$. The cases ND-EE and ND-NEE in the first column stand for the ND case with- and without-edge effects, respectively

		Effects on limit variance	
	Growth rate of Var(S_n)	Coefficients $\alpha(\mathbf{i})$	Irregular shape of R_n
PSD	$\lambda_n^{(3d-2\beta)}, \beta \in (\frac{d}{2}, d)$	Tail behavior at infinity	Geometry of R_n
ND-NEE	$\lambda_n^{(3d-2\beta)}, \beta \in (d, d + \frac{1}{2})$	Tail behavior at infinity and $A = 0$	Geometry of R_n and R_n^c
ND-EE	$\lambda_n^{(d-1)}, \beta \in (d + \frac{1}{2}, \infty)$	$A = 0$, but not on tail behavior	Geometry of ∂R_n, the boundary of R_n
SRD	$N_n, \beta > d$	Only on A	None

2. The theoretical framework

In Section 2.1, we specify the spatial linear process and in Section 2.2, we give a formulation for the sampling regions R_n. In Section 2.3, we introduce the regularity conditions on the coefficients $\alpha(\mathbf{i})$ and give some illustrative examples in Section 2.4. Under these conditions, it is possible to determine the exact order of the variance term σ_n^2 and derive explicit expressions for the asymptotic variance.

2.1. Spatial linear processes

We define a spatial linear process $\{Z(\cdot)\}$ as:

$$Z(\mathbf{i}) = \mu + \sum_{\mathbf{j} \in \mathbb{Z}^d} \alpha(\mathbf{i} - \mathbf{j}) \varepsilon(\mathbf{j}), \qquad \mathbf{i} \in \mathbb{Z}^d, \tag{2.1}$$

where $\{\varepsilon(\mathbf{j})\colon \mathbf{j} \in \mathbb{Z}^d\}$ is a collection of independent zero mean random variables with common variance 1 (w.l.o.g.), the $\{\varepsilon(\mathbf{j})^2\colon \mathbf{j} \in \mathbb{Z}^d\}$ are uniformly integrable and $\{\alpha(\mathbf{j})\colon \mathbf{j} \in \mathbb{Z}^d\}$ is a sequence of real numbers satisfying $\sum_{\mathbf{i} \in \mathbb{Z}^d} |\alpha(\mathbf{i})|^2 < \infty$.

2.2. Sampling regions

Next, we specify the structure of the sampling region R_n. Let R_0 be an open connected subset of $(-1/2, 1/2]^d$ containing the origin. We regard R_0 as a "prototype" of the sampling region R_n. Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\lambda_n \to \infty$ as $n \to \infty$. We assume that the sampling region R_n is obtained by "inflating" the set R_0 by the scaling factor λ_n (cf. Lahiri *et al.* [22]), that is,

$$R_n = \lambda_n R_0. \tag{2.2}$$

Since the origin is assumed to lie in R_0, the shape of R_n remains the same for different values of n. To avoid pathological cases, we assume that the boundary ∂R_0 of R_0 has d-dimensional Lebesgue measure zero. A stronger version of this condition will be needed for the ND case, which is stated as condition (C.3) in Section 2.3 below. The (stronger) boundary condition holds for most regions R_n of practical interest, including common convex subsets of $\mathbb{R}^d$, such as spheres, ellipsoids, polyhedrons, as well as for many non-convex star-shaped sets in $\mathbb{R}^d$. (Recall that a set $A \subset \mathbb{R}^d$ is called *star-shaped* if for any $x \in A$, the line segment joining x to the origin lies in A.) The latter class of sets may have fairly irregular shapes (cf. Sherman and Carlstein [33] and Lahiri [20]). Some practical applications and studies involving sampling regions that satisfy the regularity conditions above are given by the wheat yield data of Mercer and Hall [26] on agricultural field trials, the coal ash data of Gomez and Hazen [14] from Mining, and the cancer mortality counts data of Riggan *et al.* [29] from Epidemiology, among others.

2.3. Regularity conditions

For $\mathbf{x} = (x_1, \ldots, x_d)' \in \mathbb{R}^d$, let $\|\mathbf{x}\| = (x_1^2 + \cdots + x_d^2)^{1/2}$ and let $\lfloor \mathbf{x} \rfloor = (\lfloor x_1 \rfloor, \ldots, \lfloor x_d \rfloor)'$ where $\lfloor y \rfloor$ denotes the integer part of a real number y. For $\delta \in (0, \infty)$, $\mathbf{x} \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$, let $A^\delta = \{\mathbf{y} \in \mathbb{R}^d \colon \|\mathbf{z} - \mathbf{y}\| \le \delta \text{ for some } \mathbf{z} \in A\}$ and $d(\mathbf{x}, A) \equiv d(A, \mathbf{x}) = \min\{\|\mathbf{z} - \mathbf{x}\| \colon \mathbf{z} \in A\}$. Let $\gamma(t) = \max\{|\alpha(\lfloor \mathbf{u}t \rfloor)| \colon \|\mathbf{u}\| = 1\}, t > 0$. With this notation, we are now ready to state the regularity conditions.

(C.1) Suppose that

$$\gamma(t) = t^{-\beta} L(t), \qquad t > 0,$$

for some $\beta > d/2$ and some function $L \colon (0, \infty) \to [0, \infty)$ that is slowly varying at infinity in the sense that $L(\cdot)$ is bounded on any bounded subinterval of $(0, \infty)$ and $\lim_{t\to\infty} \sup\{L(at)/L(t) \colon a \in [a_0, a_1]\} = 1$ for any $0 < a_0 < a_1 < \infty$ (cf. Taqqu [35]).

(C.2) Let $g_t(\mathbf{x}) = \alpha(\lfloor t\mathbf{x} \rfloor)/\gamma(t)$, $\mathbf{x} \in \mathbb{R}^d$, $t > 0$. Suppose that there exists a function $g_\infty \colon \mathbb{R}^d \to \mathbb{R}$ such that for every $\delta \in (0, \infty)$,

$$\int_{\{\mathbf{x} \in \mathbb{R}^d \colon \|\mathbf{x}\| \ge \delta\}} \left| g_t(\mathbf{x}) - g_\infty(\mathbf{x}) \right|^b \mathrm{d}\mathbf{x} \to 0 \qquad \text{as } t \to \infty,$$

where $b = b(d, \beta) = 2$ if $\beta \le d$ and $b = 1$ otherwise.

(C.3) For any measurable function $f \colon [0, \infty) \to [0, \infty)$, there exists $C_f \in (1, \infty)$ such that

$$\int_{[\partial R_0]^\varepsilon} f\big(d(\mathbf{x}, \partial R_0)\big) \nu(\mathrm{d}\mathbf{x}) \le C_f \int_0^\varepsilon f(t)\, \mathrm{d}t \qquad \text{for all } 0 < \varepsilon < C_f^{-1},$$

where ν is the Lebesgue measure on $\mathbb{R}^d$.

Condition (C.1) requires that the radial maximum of the collection of coefficients $\{\alpha(\mathbf{i})\colon \mathbf{i} \in \mathbb{Z}^d\}$ be regularly varying (at infinity). The requirement that $\beta > d/2$ in (C.1) is imposed to ensure that $\sum_{\mathbf{i}\in\mathbb{Z}^d} \alpha(\mathbf{i})^2 < \infty$. Condition (C.2) is a weak form of spatial regular variation condition on the $\alpha(\mathbf{i})$. It is weaker than assuming directional separability of the coefficients, and allows for differential rates of decay along different directions. See the examples below. It is a variant of the standard form of regular variation that requires the function $g_t(\cdot)$ to satisfy (cf. Section 5.4, Resnick [28])

$$\lim_{t\to\infty} g_t(\mathbf{x}) = \|\mathbf{x}\|^{-\beta} a\big(\mathbf{x}/\|\mathbf{x}\|\big) \qquad \text{for all } \mathbf{x} \in \mathbb{R}^d, \mathbf{x} \neq \mathbf{0}, \tag{2.3}$$

for some function $a(\cdot)$ on the unit disc $\{\mathbf{x} \in \mathbb{R}^d\colon \|\mathbf{x}\| = 1\}$. In this case, the limit function g_∞ is given by

$$g_\infty(\mathbf{x}) = \|\mathbf{x}\|^{-\beta} a\big(\mathbf{x}/\|\mathbf{x}\|\big)\mathbb{1}(\mathbf{x} \neq \mathbf{0}), \qquad \mathbf{x} \in \mathbb{R}^d,$$

where $\mathbb{1}(\cdot)$ denotes the indicator function. By comparison, condition (C.2) requires convergence of g_t to g_∞ in L^b. Conditions (C.1) and (C.2) together quantify the behavior of the function $\alpha(\mathbf{i})$ for large $\|\mathbf{i}\|$ which plays an important role in determining the form of the asymptotic variance of the sum under PSD and ND.

Condition (C.3) is a regularity condition on the boundary of the prototype set R_0 which is equivalent to requiring that

$$\int_{[\partial R_0]^\varepsilon} f\big(d(\mathbf{x}, \partial R_0)\big)\nu(\mathrm{d}\mathbf{x}) = \mathrm{O}\left(\int_0^\varepsilon f(t)\,\mathrm{d}t\right) \qquad \text{as } \varepsilon \downarrow 0$$

for each non-negative measurable f. We need this condition to hold for $f \equiv 1$ and for $f(t) = t^{-b}L^2(t)$ for certain values of $b = b(\beta)$ (cf. the proof of Theorem 4.1 below). In particular, when $f \equiv 1$, this reduces to the condition

$$\nu\big((\partial R_0)^\varepsilon\big) = \mathrm{O}(\varepsilon) \qquad \text{as } \varepsilon \downarrow 0, \tag{2.4}$$

which is satisfied by most sampling regions of common interest (cf. Section 2.2). For $d = 2$, a sufficient condition is that the boundary of R_0 is delineated by a rectifiable curve of a finite length. We shall use (C.3) for proving the results *only* in the ND case where more precise information on the bounadry is needed to determine the asymptotic variance.

Next, we give a few examples to illustrate the range of spatial dependence covered by the regularity conditions above.

2.4. Examples

Example 2.1 (Isotropic spatial linear processes). Let

$$\alpha(\mathbf{i}) = a\big(\|\mathbf{i}\|\big)\big(1 + \|\mathbf{i}\|\big)^{-\beta}, \qquad \mathbf{i} \in \mathbb{Z}^d,$$

for some bounded function $a : [0, \infty) \to \mathbb{R}$, where $\beta \in (\frac{d}{2}, \infty)$. Also, suppose that $a(t) \to c_0 \neq 0$ as $t \to \infty$. Then, using the fact that

$$\sup\bigl\{\bigl|\|\lfloor t\mathbf{u}\rfloor\| - t\|\mathbf{u}\|\bigr|: \mathbf{u} \in \mathbb{R}^d \setminus \{\mathbf{0}\}\bigr\} \leq \sqrt{d}, \tag{2.5}$$

it is easy to see that condition (C.1) holds with $\gamma(t) = t^{-\beta}L(t)$ where $L(t) \to |c_0|$ as $t \to \infty$. Further, using (2.5), one can show that for any $\eta > 0$,

$$g_t(\mathbf{x}) \to \frac{c_0}{|c_0|}\|\mathbf{x}\|^{-\beta} \equiv g_\infty(\mathbf{x}) \qquad \text{as } t \to \infty \text{ for all } \|\mathbf{x}\| > \eta$$

and

$$\bigl|g_t(\mathbf{x}) - g_\infty(\mathbf{x})\bigr| \leq C_1\|\mathbf{x}\|^{-\beta} \qquad \text{for all } t > C_1,$$

for some constant $C_1 \equiv C_1(\eta, \beta) \in (0, \infty)$. Hence, condition (C.2) holds.

This gives an example of an “isotropic” spatial linear process where the coefficients $\alpha(\mathbf{i})$ have an identical rate of decay in all directions.

Example 2.2 (A class of anisotropic spatial linear processes). Suppose that O is a $d \times d$ orthonormal matrix with rows $\mathbf{o}_i'$, $i = 1, \dots, d$. Let $\phi_i(\mathbf{x}) = |\mathbf{o}_i'\mathbf{x}|/\|\mathbf{x}\|$, $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. Let $\delta \in (0, \frac{1}{\sqrt{d}})$. Suppose that

$$\alpha(\mathbf{x}) = \prod_{i=1}^{d}\bigl\{\bigl|\mathbf{o}_i'\mathbf{x}\bigr|^{-a_i}\mathbb{1}\bigl(\phi_i(\mathbf{x}) > \delta\bigr)\bigr\} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^d \setminus \Gamma$$

for some $a_1, \dots, a_d \in [0, \infty)$ with $a_1 + \cdots + a_d > d/2$ and for some open neighborhood Γ of the origin. There is no restriction on the definition of the function $\alpha(\mathbf{x})$ on Γ.

It is easy to check that

$$\begin{aligned}
\gamma(t) &= \sup\bigl\{\bigl|\alpha\bigl(\lfloor\mathbf{u}t\rfloor\bigr)\bigr|: \|\mathbf{u}\| = 1\bigr\} \\
&= \sup\bigl\{\bigl|\alpha\bigl(\lfloor O'\mathbf{u}t\rfloor\bigr)\bigr|: \|\mathbf{u}\| = 1\bigr\} \\
&= \sup\left\{\prod_{i=1}^{d}\bigl|\lfloor u_i t\rfloor\bigr|^{-a_i}\mathbb{1}\bigl(\phi_i\bigl(\lfloor t\mathbf{u}\rfloor\bigr) > \delta\bigr): \|\mathbf{u}\| = 1\right\}\bigl(1 + \mathrm{o}(1)\bigr) \\
&= c_0 t^{-(a_1 + \cdots + a_d)}\bigl(1 + \mathrm{o}(1)\bigr) \qquad \text{as } t \to \infty,
\end{aligned}$$

for some $c_0 \in (0, \infty)$.

Next, let $D = \{\mathbf{0}\} \cup \{\mathbf{y}: \phi_i(\mathbf{y}) = \delta \text{ for } i = 1, \dots, d\}$. Note that the d-dimensional Lebesgue measure of D is zero. And, for any $\mathbf{x} \notin D$,

$$\begin{aligned}
g_t(\mathbf{x}) &= \alpha\bigl(\lfloor t\mathbf{x}\rfloor\bigr)/\gamma(t) \\
&= \gamma(t)^{-1}\prod_{i=1}^{d}\bigl|\bigl\lfloor t\mathbf{o}_i'\mathbf{x}\bigr\rfloor\bigr|^{-a_i}\mathbb{1}\bigl(\phi_i\bigl(\lfloor t\mathbf{x}\rfloor\bigr) > \delta\bigr)
\end{aligned}$$

$$= \left[c_0^{-1} \prod_{i=1}^{d} \left\{ \left| \mathbf{o}_i' \mathbf{x} \right|^{-a_i} \mathbb{1}\big(\phi_i(\mathbf{x}) > \delta\big) \right\} \right] \big(1 + \mathrm{o}(1)\big) \qquad \text{as } t \to \infty,$$

$$\equiv g_\infty(\mathbf{x})\big(1 + \mathrm{o}(1)\big) \qquad \text{as } t \to \infty.$$

Thus, the point-wise limit of the functions $g_t(\cdot)$ exists for all $\mathbf{x} \notin D$. Now using the Dominated Convergence Theorem (DCT), it is easy to check that for any $\eta > 0$,

$$\lim_{t \to \infty} \int_{\{\|\mathbf{x}\| \geq \eta\}} \left| g_t(\mathbf{x}) - g_\infty(\mathbf{x}) \right|^b \mathrm{d}\mathbf{x} = 0.$$

Thus, conditions (C.1) and (C.2) are satisfied by the coefficients generated by the function $\alpha(\cdot)$.

Note that in this example, the coefficients $\alpha(\mathbf{i})$ are *zero* whenever $|\phi_i(\mathbf{i})| \leq \delta$ for some $i \in \{1, \ldots, d\}$ but they are non-zero for all $\mathbf{x}$ such that $\phi_i(\mathbf{x}) > \delta$ for all $i = 1, \ldots, d$. Since δ is small, the latter condition is satisfied in a conic region in each of the 2^d quadrants. Thus, this gives an example of an anisotropic spatial process. The maximal rate of decay of $\alpha(\cdot)$ over the set $\{\mathbf{i} \in \mathbb{Z}^d\colon |\phi_i(\mathbf{i})| > \delta \text{ for all } i = 1, \ldots, d\}$ can vary with the choice of $a_1, \ldots, a_d$, allowing all possible types of long-range (as well as short-range) dependence. Also, note that here the ND case can be realized by a suitable choice of $\alpha(\mathbf{i})$ for $\mathbf{i} \in \Gamma$. The rate of convergence of the sum in this example depends only on a *single* parameter, namely, the combined exponent $\beta = a_1 + \cdots + a_d$; Individual a_i's do not have an impact.

Example 2.3 (Spatial linear processes with non-uniform directional decay rates). Let $\mathcal{I}$ be a finite set and let $\{\mathbf{o}_i\colon i \in \mathcal{I}\} \subset \{\mathbf{x} \in \mathbb{R}^d\colon \|\mathbf{x}\| = 1\}$. Here we suppose that the $\mathbf{o}_i$'s are distinct but they are not necessarily orthogonal. Let $\phi_i(\cdot)$ be as in Example 2.2, that is, $\phi_i(\mathbf{x}) = |\mathbf{o}_i'\mathbf{x}|/\|\mathbf{x}\|$, $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $i \in \mathcal{I}$. Define $\psi_i(\mathbf{x}) = \phi_i(\mathbf{x})\mathbb{1}(\phi_i(\mathbf{x}) > \delta_i)$ for some $\delta_i \in (0, 1)$, $i \in \mathcal{I}$. Suppose that

$$\alpha(\mathbf{x}) = \sum_{i \in \mathcal{I}} \frac{\psi_i(\mathbf{x})}{1 + \|\mathbf{x}\|^{a_i}} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^d \setminus \Gamma,$$

where $\{a_i\colon i \in \mathcal{I}\} \subset (0, \infty)$ and where Γ is an open neighborhood of the origin, as in Example 2.2. Let $a_0 = \min\{a_i\colon i \in \mathcal{I}\}$ and let $\mathcal{I}_0 = \{i \in \mathcal{I}\colon a_i = a_0\}$. Note that for any $i \in \mathcal{I}$ and any $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$,

$$\phi_i\big(\lfloor t\mathbf{x} \rfloor\big) = \phi_i(\mathbf{x})\big(1 + \mathrm{o}(1)\big) \qquad \text{as } t \to \infty, \text{provided } \phi_i(\mathbf{x}) > 0,$$

and

$$\big\| \lfloor t\mathbf{x} \rfloor \big\| = t\|\mathbf{x}\|\big(1 + \mathrm{o}(1)\big) \qquad \text{as } t \to \infty.$$

Hence, it follows that

$$\gamma(t) = c_1 t^{-a_0}\big(1 + \mathrm{o}(1)\big) \qquad \text{as } t \to \infty,$$

where $c_1 = \sup\{\sum_{i \in \mathcal{I}_0} \psi_i(\mathbf{u})\colon \|\mathbf{u}\| = 1\}$.

Next, we identify the limit function $g_\infty(\cdot)$. By arguments as above, it follows that for any $\mathbf{x} \neq \mathbf{0}$ with $\psi_i(\mathbf{x}) > 0$ for some $i \in \mathcal{I}_0$,

$$\begin{aligned} g_t(\mathbf{x}) &= \gamma(t)^{-1}\left[\sum_{i\in\mathcal{I}} \frac{\psi_i(\lfloor t\mathbf{x}\rfloor)}{1+\|\lfloor t\mathbf{x}\rfloor\|^{a_i}}\right] \\ &= \frac{\sum_{i\in\mathcal{I}_0}\psi_i(\mathbf{x})}{c_1\|\mathbf{x}\|^{a_0}}\bigl(1+\mathrm{o}(1)\bigr) \qquad \text{as } t\to\infty, \text{a.e.} \end{aligned}$$

Since for any i, the set $\{\mathbf{x}\in\mathbb{R}^d\setminus\{\mathbf{0}\}\colon\ \phi_i(\mathbf{x}) = \delta_i\}$ has d-dimensional Lebesgue measure zero, it follows that $g_t \to g_\infty$ as $t\to\infty$ (a.e.), where

$$g_\infty(\mathbf{x}) = \frac{\sum_{i\in\mathcal{I}_0}\phi_i(\mathbf{x})}{c_1\|\mathbf{x}\|^{a_0}}, \qquad \mathbf{x}\in\mathbb{R}^d\setminus\{\mathbf{0}\}.$$

Now using the DCT, one can show that for any $\eta > 0$,

$$\lim_{t\to\infty}\int_{\{\|\mathbf{x}\|\geq\eta\}} \bigl|g_t(\mathbf{x}) - g_\infty(\mathbf{x})\bigr|^b\,\mathrm{d}\mathbf{x} = 0.$$

Thus, conditions (C.1) and (C.2) are satisfied by the coefficients generated by the function $\alpha(\cdot)$.

Note that for δ_i close to 1, the ith component $\frac{\psi_i(\mathbf{x})}{1+\|\mathbf{x}\|^{a_i}}$ in $\alpha(\mathbf{x})$ takes non-zero values in a thin cone around $\mathbf{o}_i$ and it may or may not intersect with the jth cone (for any given $j\neq i$) depending on the relative magnitudes of δ_i and δ_j and the angle between $\mathbf{o}_i$ and $\mathbf{o}_j$. As a result, with different choices of $\mathbf{o}_i$, δ_i and a_i for $i\in\mathcal{I}$, the coefficients $\alpha(\mathbf{i})$ here may have different rates of decay along the directions $\mathbf{o}_i$ for $i\in\mathcal{I}\setminus\mathcal{I}_0$, allowing *any combinations of short- and long-range dependent rates along different directions*. However, the limit distribution of the sum depends only on a_0 which is the minimum of the exponents $\{a_i, i\in\mathcal{I}\}$.

In the next section, we describe the limit behavior of the sum S_n depending on the rate of decay β in (C.1).

3. Results under PSD

From the proofs given in Section 6, it follows that for $\beta\in(d/2,d)$, the variance of the sum S_n grows at the rate $\lambda_n^{3d-2\beta}L(\lambda_n)^2$ and hence, the correct scaling factor sequence is given by $\lambda_n^{(3d-2\beta)/2}L(\lambda_n)$. Since $\beta\in(d/2,d)$, this scaling sequence grows at a rate *faster* than square-root of the sample size $|N_n|^{1/2}\sim[\lambda_n^d\,\mathrm{vol.}(R_0)]^{1/2}$, where $\mathrm{vol.}(B)$ denotes the volume (i.e., the Lebesgue measure) of a Borel set B in $\mathbb{R}^d$. As a result, for $\beta\in(d/2,d)$, $N_n^{-1}\sigma_n^2\to\infty$ and the spatial process $\{Z(\cdot)\}$ exhibits PSD.

To describe the limit distribution of the centered and scaled sum under PSD, define

$$G_\infty(\mathbf{x}) = \int_{R_0} g_\infty(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{y}, \qquad \mathbf{x}\in\mathbb{R}^d. \tag{3.1}$$

The first result shows that the function G_∞ is well defined on all of $\mathbb{R}^d$ for $\beta \in (0, d)$.

Proposition 3.1. *Suppose that conditions* (C.1) *and* (C.2) *hold for some* $\beta \in (0, d)$. *Then the integral in* (3.1) *exists and is finite for all* $\mathbf{x} \in \mathbb{R}^d$.

The function G_∞ determines the asymptotic variance of the sum S_n for $\beta \in (d/2, d)$. We make this precise in the following result that gives the limit distribution of S_n under PSD.

Theorem 3.2. *Let* $\{Z(\cdot)\}$ *be the linear process given by* (2.1) *such that conditions* (C.1) *and* (C.2) *hold for some* $\beta \in (d/2, d)$. *Then* $G_\infty \in L^2(\mathbb{R}^d)$ *and*

$$\frac{[S_n - ES_n]}{[\lambda_n^{3d-2\beta} L(\lambda_n)^2]^{1/2}} \xrightarrow{d} N\left(0, \int_{\mathbb{R}^d} G_\infty^2(\mathbf{x})\, \mathrm{d}\mathbf{x}\right) \qquad \text{as } n \to \infty. \tag{3.2}$$

Theorem 3.2 shows that for $\beta \in (d/2, d)$, the growth rate of (the variance of) the sum S_n is $[\lambda_n^{3d-2\beta} L(\lambda_n)^2]^{1/2}$, which is of a larger order than the usual order $N_n^{1/2}$. Since G_∞ is continuous, the asymptotic variance is non-zero if

$$G_\infty(\mathbf{x}_0) \neq 0 \qquad \text{for some } \mathbf{x}_0 \in \mathbb{R}^d.$$

From (3.2), also note that the limiting variance of S_n depends on the prototype set R_0 as well as the function g_∞ of condition (C.2). Thus, unlike the time series case, the geometry of the sampling region plays an important role in the spatial case under PSD.

4. Results under ND

When $Z(\mathbf{i})$ is ND, it can be shown (cf. the proofs of Theorems 4.1, 4.3 and 4.4) that $N_n^{-1}\sigma_n^2 \to 0$. The limit behavior of the sum S_n in the ND case critically depends on the behavior of the terms

$$\theta_n(\mathbf{i}) = \sum_{\mathbf{j} \in [R_n - \mathbf{i}] \cap \mathbb{Z}^d} \alpha(\mathbf{j}), \qquad \mathbf{i} \in \mathbb{Z}^d,$$

in a shrinking neighborhood of the set, ∂R_n, the boundary of R_n. In the parlance of spatial statistics, this represents an instance of *edge effect* (cf. Cressie [8]) that may have a non-trivial effect on the limit behavior of the sum. Indeed, depending on the relative orders of contributions from the boundary terms and the non-boundary terms, we get different growth rates for the sum in the ND case. Further, the limiting variances are also different. For clarity of exposition, we present the two subcases of the ND case separately.

4.1. ND with asymptotically negligible edge effects

First, we consider the relatively simple case where the contribution from the boundary $\theta_n(\mathbf{i})$'s is asymptotically negligible. Suppose that $\beta > d$, so that $\sum_{\mathbf{i} \in \mathbb{Z}^d} |\alpha(\mathbf{i})| < \infty$, and that $A \equiv$

$\sum_{\mathbf{i}\in\mathbb{Z}^d}\alpha(\mathbf{i})=0$. In this case, it will be shown that the asymptotic distribution of the sum depends on the $\alpha(\mathbf{i})$ only through (an analog of) the function G_∞ of (3.1). However, Proposition 3.1 no longer holds, that is, the function G_∞ of (3.1) may not be well defined for all $\mathbf{x}\in\mathbb{R}^d$ for $\beta>d$. To appreciate why, consider the special case where $g_\infty(\mathbf{x})=\|\mathbf{x}\|^{-\beta}\mathbb{1}(\mathbf{x}\neq\mathbf{0})$ (cf. (2.3)). In this case, $\int_{\{\delta\le\|\mathbf{y}\|\le 1\}}g_\infty(\mathbf{y})=\mathrm{O}(\delta^{d-\beta})=\mathrm{o}(1)$ as $\delta\downarrow 0$ for all $\beta\in(d/2,d)$, but the integral blows up for $\beta>d$. As a result, the limit in (3.1) may not exist *for all* $\mathbf{x}\in\mathbb{R}^d\setminus\{\mathbf{0}\}$ in the case $\beta>d$. However, using the condition $A=0$, we can define $G_\infty(\mathbf{x})$ (and a suitable variant of it) for a restricted set of $\mathbf{x}$'s that would be adequate for our purpose.

To that end, note that the set $[R_0\cup\partial R_0]^c$ is open and hence, for all $\mathbf{x}\in[R_0\cup\partial R_0]^c$, there exists a $\eta=\eta(\mathbf{x})>0$ such that

$$B(\mathbf{x};\eta)\subset[R_0\cup\partial R_0]^c,$$

where $B(\mathbf{x};\eta)\equiv\{\mathbf{y}\in\mathbb{R}^d\colon\ \|\mathbf{y}-\mathbf{x}\|<\eta\}$ denotes the open ball of radius η around $\mathbf{x}$. As a consequence, for $\mathbf{x}\in[R_0\cup\partial R_0]^c$, $B(\mathbf{0};\eta)\cap[R_0-\mathbf{x}]=\varnothing$ and

$$\int_{R_0}\bigl|g_\infty(\mathbf{y}-\mathbf{x})\bigr|\,\mathrm{d}\mathbf{y}=\int_{R_0-\mathbf{x}}\bigl|g_\infty(\mathbf{y})\bigr|\,\mathrm{d}\mathbf{y}\le\biggl[\int_{\{\|\mathbf{y}\|\ge\eta\}}\bigl|g_\infty(\mathbf{y})\bigr|^2\,\mathrm{d}\mathbf{y}\biggr]^{1/2}\bigl[\mathrm{vol.}(R_0)\bigr]^{1/2}<\infty$$

whenever condition (C.2) holds with $b=2$. But, for $\beta>d$, $b=1$ in condition (C.2). Nonetheless, the square integrability of $g_\infty(\cdot)$ on sets of the form $B_\eta\equiv\{\mathbf{y}\in\mathbb{R}^d\colon\ \|\mathbf{y}\|\ge\eta\}$, $\eta>0$, follows from (C.2) and the fact that $|g_\infty(\cdot)|\le C(\eta)$ a.e. (w.r.t. the Lebesgue measure on $\mathbb{R}^d$) on B_η, for some $C(\eta)\in(0,\infty)$ (see (6.6) below). Hence,

$$G_\infty(\mathbf{x})=\int_{R_0}g_\infty(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{y}\in\mathbb{R}$$

for all $\mathbf{x}\in[R_0\cup\partial R_0]^c$. By similar arguments, the integral $\int_{[R_0\cup\partial R_0]^c}g_\infty(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{y}$ is well defined for all $\mathbf{x}\in R_0$. Since ∂R_0 has d-dimensional Lebesgue measure zero, the value of the integrals remains unchanged (with any measurable extension of $g_\infty(\cdot)$) if $[R_0\cup\partial R_0]^c$ is replaced by R_0^c (cf. Billingsley [4] or Athreya and Lahiri [1], page 49). With this convention, define the function $G_\infty^\dagger(\cdot)$ as

$$G_\infty^\dagger(\mathbf{x})=\begin{cases}\displaystyle\int_{R_0}g_\infty(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{y}, & \text{if } \mathbf{x}\in[R_0\cup\partial R_0]^c,\\ \displaystyle\int_{R_0^c}g_\infty(\mathbf{y}-\mathbf{x})\,\mathrm{d}\mathbf{y}, & \text{if } \mathbf{x}\in R_0,\\ 0, & \text{if } \mathbf{x}\in\partial R_0.\end{cases}\tag{4.1}$$

For $\beta\in(d,d+1/2)$, the asymptotic distribution of the sum depends only on the function $G_\infty^\dagger(\cdot)$, as shown by the following result.

Theorem 4.1. *Let* $\{Z(\cdot)\}$ *be the linear process given by* (2.1) *such that conditions* (C.1)–(C.3) *hold with* $\beta\in(d,d+1/2)$ *in* (C.1). *Also suppose that* $A=0$. *Then,* $G_\infty^\dagger\in L^2(\mathbb{R}^d)$ *and*

$$\frac{[S_n-ES_n]}{[\lambda_n^{3d-2\beta}L^2(\lambda_n)]^{1/2}}\xrightarrow{d}N\biggl(0,\int_{\mathbb{R}^d}\bigl[G_\infty^\dagger(\mathbf{x})\bigr]^2\,\mathrm{d}\mathbf{x}\biggr)\qquad\text{as } n\to\infty,\tag{4.2}$$

where the function $G_\infty^\dagger$ *is as defined in* (4.1).

Theorem 4.1 shows that the asymptotic variance of the centered and scaled sum depends on the coefficients $\alpha(\mathbf{i})$ only through the integral of the function $G^{\dagger}_{\infty}(\cdot)^2$ over $\mathbb{R}^d$. Thus, the behavior of the $\alpha(\mathbf{i})$ for large values of $\|\mathbf{i}\|$ determines the asymptotic variance. The exact values of the $\alpha(\mathbf{i})$ for small values of $\|\mathbf{i}\|$ have no direct effect except for the condition $A = 0$. Further, the growth rate of the sum under the ND case is $[\lambda_n^{3d-2\beta} L(\lambda_n)^2]^{1/2} = \mathrm{o}(N_n^{1/2})$, which is slower than the PSD rate and, also slower than the SRD rate, given by $N_n^{1/2}$ (cf. Theorem 5.1 below). To compare the asymptotic variances under the ND case without edge effects and the PSD case, note that the integrals of $G^{\dagger}_{\infty}(\cdot)$ and $G_{\infty}(\cdot)$ over R_0^c are the same and hence, the difference in the asymptotic variances in the ND and the PSD cases comes from the integrals of the respective functions over R_0.

4.2. ND with asymptotically non-negligible edge effects

Next, consider the case where $\beta \geq d + 1/2$. In this case, we may write the variance of the sum as the sum of two terms, one involving the sum of $\theta_n(\mathbf{i})^2\sigma^2$ for $\mathbf{i}$ near the boundary of the sampling region R_n and the other over the rest of the $\theta_n(\mathbf{i})^2\sigma^2$. It can be shown that the growth rate of the second term is of the order $\lambda_n^{3d-2\beta} L(\lambda_n)^2$. On the other hand, under condition (C.3) (cf. (2.4)), for any sequence $\{t_n\} \subset (0, \infty)$ with $t_n^{-1} + \lambda_n^{-1} t_n = \mathrm{o}(1)$ as $n \to \infty$, the volume of the t_n-enlargement of the boundary of R_n is of the order of $\lambda_n^{d-1} t_n$. It is easy to check that for $\beta \geq d + 1/2$, this boundary term can be of a *larger* order of magnitude than $\lambda_n^{3d-2\beta} L(\lambda_n)^2$. As a result, the contribution from $\theta_n(\mathbf{i})^2$ for $\mathbf{i}$ near the boundary of the sampling region R_n may become dominant and additional care must be taken to determine the exact growth rate of σ_n^2. The following example serves to illustrate such dominating "edge effects" in the ND case:

Example 4.2. Suppose that $d = 2$, $R_0 = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ and let

$$\alpha(i, j) = \begin{cases} b(i)b(j), & \text{if } i, j \in \mathbb{Z} \setminus \{0\}, \\ 0, & \text{if } ij = 0, (i, j) \neq (0, 0), \\ -4B^2 & \text{if } (i, j) = (0, 0), \end{cases}$$

where $\{b(i)\colon\ i \geq 1\} \subset (0, \infty)$, $b(-i) = b(i)$ for $i \geq 1$ and $B \equiv \sum_{i=1}^{\infty} b(i) \in (0, \infty)$. Further, suppose that $b(i) \sim c_0 i^{-\beta}$ as $i \to \infty$, for some $\beta > d + 1/2 = 2.5$. Then, $A = \sum_{\mathbf{i} \in \mathbb{Z}^2} \alpha(\mathbf{i}) = 0$ and $\gamma(t) \sim c_0^2 t^{-\beta}$ as $t \to \infty$. Further, we may write σ_n^2 as

$$\sigma_n^2 = \sum_{k=1}^{3} \sum_{\mathbf{i} \in I_{kn}} \theta_n(\mathbf{i})^2,$$

where $I_{1n} = [-\lambda_n/2 + c_n, \lambda_n/2 - c_n]^2 \cap \mathbb{Z}^2$, $I_{2n} = \mathbb{Z}^2 \setminus [-\lambda_n/2 - c_n, \lambda_n/2 + c_n]^2$, and $I_{3n} = \mathbb{Z}^2 \setminus [I_{1n} \cup I_{2n}]$, respectively denote the collections of integer vectors $\mathbf{i}$ that lie in the interior, the exterior, and the boundary parts of R_n, where c_n is a suitably chosen sequence satisfying

$c_n^{-1} + \lambda_n^{-1} c_n = \mathrm{o}(1)$. It can be shown (cf. Section 6.4 below) that for $\beta > d + 1/2 = 2.5$,

$$\sum_{k=1}^{2} \sum_{\mathbf{i} \in I_{kn}} \theta_n(\mathbf{i})^2 = \mathrm{o}(\lambda_n) \quad \text{and} \quad \sum_{\mathbf{i} \in I_{3n}} \theta_n(\mathbf{i})^2 = \sigma_0^2 \lambda_n \bigl(1 + \mathrm{o}(1)\bigr), \tag{4.3}$$

where

$$\sigma_0^2 = 16B^2 \left[B^2 + \sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} b(j) \right)^2 + \sum_{k=1}^{\infty} \left(\sum_{j=k+1}^{\infty} b(j) \right)^2 \right].$$

Hence, in this case, the contribution of the boundary part dominates the other two terms, and the scaling is given by $\lambda_n \equiv \lambda_n^{d-1}$, which is the $(d-1)$-dimensional Lebesgue measure of the boundary of the sampling region R_n. Note that the rate of convergence no longer depends on $\beta \in (d + 1/2, \infty)$.

The main reason why the edge effect dominates in the ND case as highlighted by Example 4.2 can be explained by noting the form of the constant σ_0^2 in (4.3). Although the condition $A = 0$ makes the sum of the $\alpha(\mathbf{i})$ over large open neighborhoods of the origin small, sums of the $\alpha(\mathbf{i})$ over half-planes, as determined by the $\theta_n(\mathbf{i})$ near the boundary of R_n are not small. As a result, the combined contribution of these terms near the edge of R_n determines the asymptotic behavior of the sum S_n for $\beta > d + 1/2$.

The next result proves the CLT in presence of non-trivial edge effects, for $\beta > d + 1/2$. The case $\beta = d + 1/2$ will be treated in Theorem 4.4 below.

Theorem 4.3. *Let* $\{Z(\cdot)\}$ *be the linear process given by* (2.1) *such that conditions* (C.1) *hold with* $\beta \in (d + 1/2, \infty)$. *Also suppose that* $d \geq 2$, $A = 0$ *and the following condition holds*:

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \left| \lambda_n^{-(d-1)} \sum_{\mathbf{i} \in [\partial R_n]^{\delta \lambda_n} \cap \mathbb{Z}^d} \bigl|\theta_n(\mathbf{i})\bigr|^2 - \sigma_{\mathrm{EE}}^2 \right| = 0 \tag{4.4}$$

for some $\sigma_{\mathrm{EE}}^2 \in (0, \infty)$. *Then*

$$\lambda_n^{-(d-1)/2} [S_n - ES_n] \xrightarrow{d} N\bigl(0, \sigma_{\mathrm{EE}}^2\bigr). \tag{4.5}$$

Thus, it follows that under the conditions of the theorem, only the $\theta(\mathbf{i})$ with indices $\mathbf{i}$ close to the boundary of R_n contribute to the asymptotic variance of the sum. The contribution of the $\theta(\mathbf{i})$ for the rest of $\mathbf{i}$-values becomes asymptotically negligible for $\beta > d + 1/2$. It can be shown that in Example 4.2, the limiting variance σ_{EE}^2 is given by σ_0^2. Although we do not explicitly state it, note that the boundary condition (2.4) on R_0 is implicit in the formulation of (4.4). Also, note that this edge-effect phenomenon in the ND case appears ONLY in dimensions $d \geq 2$.

Next, we consider the case where $\beta = d + 1/2$. In this case, the edge effect may or may not have a non-trivial effect on the limit distribution, depending on the growth rate of the slowly varying function $L(\cdot)$. More precisely, we have the following results.

Theorem 4.4. *Let* $\{Z(\cdot)\}$ *be the linear process given by* (2.1) *such that conditions* (C.1) *holds with* $\beta = d + 1/2$ *for some* $d \geq 2$. *Further suppose that* $A = 0$ *and that* (4.4) *holds.*

(i) *If* $L(t) = \mathrm{o}(1)$ *as* $t \to \infty$, *then* (4.5) *holds.*
(ii) *Suppose that condition* (C.2) *holds and* $G^{\dagger}_{\infty} \in L^2(\mathbb{R}^d)$ *where* $G^{\dagger}_{\infty}$ *is as in* (4.1).
   (a) *If* $L(t) = c_0(1 + \mathrm{o}(1))$ *as* $t \to \infty$ *for some* $c_0 \in (0, \infty)$, *then*

$$\lambda_n^{-(d-1)/2}[S_n - ES_n] \xrightarrow{d} N\left(0, \sigma^2_{\mathrm{EE}} + c_0^2 \int_{\mathbb{R}^d} \left[G^{\dagger}_{\infty}(\mathbf{x})\right]^2 \mathrm{d}\mathbf{x}\right).$$

   (b) *If* $L(t)^{-1} = \mathrm{o}(1)$ *as* $t \to \infty$, *then* (4.2) *holds.*

Theorem 4.4 shows that the edge effect is non-trivial under ND whenever $\lambda_n^{3d-2\beta} L^2(\lambda_n) = \mathrm{O}(\lambda_n^{(d-1)})$, that is, whenever the contribution to σ_n^2 from the $\theta_n(\mathbf{i})^2$ near the boundary is at least as large as that from the remaining $\theta_n(\mathbf{i})^2$. When the slowly varying function $L(\cdot)$ is bounded, both $\lambda_n^{3d-2\beta} L^2(\lambda_n)$ and $\lambda_n^{(d-1)}$ are of the same order and the asymptotic variance depends on both σ^2_{EE} and the function $G^{\dagger}_{\infty}(\cdot)$ of (4.1). On the other hand, when the factors $\lambda_n^{3d-2\beta} L^2(\lambda_n)$ and $\lambda_n^{(d-1)}$ are not asymptotically equivalent, the scaling sequence and the asymptotic variance of the centered sum are determined by the dominant factor.

Remark 4.5. Note that in dimensions $d \geq 2$, the slowest possible growth rate of the variance of the sum in the ND case is $\lambda_n^{(d-1)}$. This may be contrasted with the one dimensional ND case where the variance of the sum grows at rate $\lambda_n^{[3-2\beta]} L(\lambda_n)^2$ which, in turn, can grow very slowly for β close to $d + 1/2 = 3/2$. The main reason for this unusual behavior of the sum in higher dimensions is the presence of the edge effect which is not rate adaptive, that is, it does not become asymptotically smaller even when the coefficients $\alpha(\mathbf{i})$ or the function $\gamma(t)$ have a faster rate of decay.

Remark 4.6. For the one dimensional ND case, the variance of the sum σ_n^2 does not necessarily go to infinity for $\beta \geq 3/2$ and hence, rate adaptivity of the variance for $d = 1$ is meaningful only when $\beta < 3/2$, which is covered by Theorem 4.1. It can be shown that when $\beta = d + 1/2$ and $d = 1$, part (ii)(b) of Theorem 4.4 holds. However, no analog of parts (i) and (ii)(a) holds for $d = 1$, as the edge effects are asymptotically negligible in the one dimensional case. Also, for $\beta \in (d + 1/2, \infty)$, CLTs for the sum are not available for $d = 1$ in the ND case (as $\sigma_n^2 \not\to \infty$), but they are available in dimensions $d \geq 2$ (cf. Theorem 4.3).

5. Result under SRD

For completeness, we also give the result in the SRD case. Suppose that $\beta \in [d, \infty)$ with $\int_0^{\infty} t^{d-1}\gamma(t)\,\mathrm{d}t < \infty$. Then it follows that $A = \sum_{\mathbf{i} \in \mathbb{Z}^d} \alpha(\mathbf{i}) \in \mathbb{R}$. If $A \neq 0$, then the spatial process is SRD and we have the following result.

Theorem 5.1. *Let $\{Z(\cdot)\}$ be the linear process given by* (2.1) *such that condition* (C.1) *holds with $\beta \in [d, \infty)$ and that $\int_0^\infty t^{d-1}\gamma(t)\,\mathrm{d}t < \infty$ and $A \neq 0$. Then, as $n \to \infty$,*

$$N_n^{-1/2}[S_n - ES_n] \xrightarrow{d} N\big(0, A^2\big). \tag{5.1}$$

Thus, under the conditions of Theorem 5.1, the sum S_n is asymptotically normal and the asymptotic variance grows at the standard rate, namely, the square root of the sample size. Note that in this case, we only assume condition (C.1), but not (C.2) or (C.3). The asymptotic variance of the sum depends on the coefficients $\alpha(\cdot)$ only through the sum A, but not on the relative behavior of the $\alpha(\cdot)$ and $\gamma(\cdot)$ at infinity.

Remark 5.2. The asymptotic variance in the SRD case is determined by the $\alpha(\mathbf{i})$ for $\mathbf{i}$ in arbitrarily large compact neighborhoods of the origin, while in the PSD case, it is determined by the relative behavior of $\alpha(\cdot)$ and $\gamma_t(\cdot)$ near infinity – the values of $\alpha(\mathbf{i})$ for any fixed compact neighborhood of the origin has no effect on the asymptotic variance. In the ND case, the asymptotic variance depends on the $\alpha(\mathbf{i})$ for both – (i) for smaller $\mathbf{i}$ through the condition $A = 0$ and (ii) for large $\mathbf{i}$ through the relative behavior of $g_t(\cdot)$ and $\gamma(\cdot)$ near infinity, in absence of the edge-effect. For $d \geq 2$, in the ND case with non-trivial edge effects, the asymptotic variance depends on the $\alpha(\mathbf{i})$ for $\mathbf{i}$ in arbitrarily large compact neighborhoods of the origin, as in the SRD case, but not on the relative behavior of $\alpha(\cdot)$ and $\gamma(\cdot)$.

6. Proofs

6.1. Notation

Let $\mathcal{C} = [0, 1)^d$ denote the unit cube in $\mathbb{R}^d$. Let L^2 denote the collection of all square integrable functions (with respect to the Lebesgue measure on $\mathbb{R}^d$) from $\mathbb{R}^d$ to $\mathbb{R}$, and let $\|\cdot\|$ denote the L^2 norm, that is, $\|f\|^2 = \int f^2(\mathbf{x})\,\mathrm{d}\mathbf{x}$, $f \in L^2$. Let $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively denote the ℓ^1 and ℓ^∞-norms on $\mathbb{R}^d$, that is, for $(x_1, \ldots, x_d)' \in \mathbb{R}^d$,

$$\big\|(x_1, \ldots, x_d)'\big\|_1 = \sum_{i=1}^d |x_i| \quad \text{and} \quad \big\|(x_1, \ldots, x_d)'\big\|_\infty = \max\big\{|x_i|\colon 1 \leq i \leq d\big\}.$$

Recall that for any set $A \subset \mathbb{R}^d$ and $\delta \in (0, \infty)$, let $A^\delta = \{\mathbf{x} \in \mathbb{R}^d\colon \|\mathbf{x} - \mathbf{y}\| \leq \delta \text{ for some } \mathbf{y} \in A\}$ denote the δ-enlargement of A. Similarly, define the set $A^{-\delta} = \{\mathbf{x} \in A\colon B(\mathbf{x}; \delta) \subset A\}$ where $B(\mathbf{x}; \delta) = \{\mathbf{y} \in \mathbb{R}^d\colon \|\mathbf{x} - \mathbf{y}\| < \delta\}$. Let ∂A denote the boundary of A. Let $C, C(\cdot)$ denote generic constants that do not depend on n. Unless otherwise specified, all limits (including those in the order symbols) are taken by letting $n \to \infty$. Also, for notational simplicity, we set $\mu = 0$ for the rest of this section (except in cases where there is a chance of confusion).

6.2. Auxiliary results

Here we prove a general version of the CLT for spatial linear processes without structural conditions on the coefficients $\alpha(\mathbf{i})$ and the sampling regions. This result forms the basis for proving the results from Sections 3–5, and may be of independent interest.

Theorem 6.1. *Let $\{Z(\mathbf{j})\colon \mathbf{j} \in \mathbb{Z}^d\}$ be the spatial linear process in (2.1) with $\mu = 0$. Let Λ_n be a finite subset of $\mathbb{Z}^d$ such that $|\Lambda_n| \to \infty$ as $n \to \infty$. Let $S_n = \sum_{\mathbf{j}\in\Lambda_n} Z(\mathbf{j})$ and $\sigma_n^2 = \sum_{\mathbf{i}\in\mathbb{Z}^d} \theta_n(\mathbf{i})^2$, where $\theta_n(\mathbf{i}) = \sum_{\mathbf{j}\in\Lambda_n} \alpha(\mathbf{j}-\mathbf{i})$, $\mathbf{i} \in \mathbb{Z}^d$. Suppose that as $n \to \infty$,*

$$\frac{1}{\sigma_n} + \frac{\max\{|\theta_n(\mathbf{i})|\colon \mathbf{i} \in \mathbb{Z}^d\}}{\sigma_n} \to 0. \tag{6.1}$$

Then $S_n/\sigma_n \to^d N(0,1)$ as $n \to \infty$.

Proof. First, we shall show that there exists a sequence of integers $m_n \to \infty$ such that

$$\sigma_n^{-2} \sum_{\|\mathbf{i}\|>m_n} \theta_n(\mathbf{i})^2 = \mathrm{o}(1). \tag{6.2}$$

To that end, note that $|\theta_n(\mathbf{i})| \le [\sum_{\mathbf{j}\in(\Lambda_n-\mathbf{i})} \alpha(\mathbf{j})^2]^{1/2}|\Lambda_n|^{1/2}$ for all $\mathbf{i} \in \mathbb{Z}^d$. Since $\sum_{\mathbf{i}\in\mathbb{Z}^d} \alpha(\mathbf{i})^2 < \infty$, there exists $m_{1n} \to \infty$ such that $\sigma_n^{-2}|\Lambda_n|^2 \sum_{\|\mathbf{i}\|>m_{1n}} \alpha(\mathbf{i})^2 = \mathrm{o}(1)$. Define $m_n = \max\{\|\mathbf{j}\|\colon \mathbf{j} \in \Lambda_n\} + m_{1n}$. Then, it follows that

$$\sigma_n^{-2} \sum_{\|\mathbf{i}\|>m_n} \theta_n(\mathbf{i})^2 \le \sigma_n^{-2}|\Lambda_n| \sum_{\|\mathbf{i}\|>m_n} \sum_{\mathbf{j}\in\Lambda_n} \alpha(\mathbf{j}-\mathbf{i})^2 \le \sigma_n^{-2}|\Lambda_n|^2 \sum_{\|\mathbf{i}\|>m_{1n}} \alpha(\mathbf{i})^2 = \mathrm{o}(1),$$

proving (6.2).

Next, define $U_n = \{\mathbf{i} \in \mathbb{Z}^d\colon \ \|\mathbf{i}\| \le m_n\}$ and $\bar{U}_n = \{\mathbf{i} \in \mathbb{Z}^d\colon \ \|\mathbf{i}\| > m_n\}$. Define $\tilde{S}_n = \sum_{\mathbf{i}\in U_n} \theta_n(\mathbf{i})\varepsilon(\mathbf{i})$ and $\tilde{\sigma}_n^2 = \sum_{\mathbf{i}\in U_n} \theta_n(\mathbf{i})^2$. Then, by (6.2), $\sigma_n^{-2}[\sigma_n^2 - \tilde{\sigma}_n^2] = \mathrm{o}(1)$ and hence,

$$\begin{aligned} \sigma_n^{-1} S_n - \tilde{\sigma}_n^{-1}\tilde{S}_n &= \sigma_n^{-1}\left[\sum_{\mathbf{i}\in\bar{U}_n} \theta_n(\mathbf{i})\varepsilon(\mathbf{i})\right] + \frac{[\tilde{\sigma}_n - \sigma_n]}{\sigma_n\tilde{\sigma}_n}\tilde{S}_n \\ &= \mathrm{o}_p(1), \end{aligned} \tag{6.3}$$

provided $\tilde{\sigma}_n^{-1}\tilde{S}_n = \mathrm{O}_p(1)$. Hence, it is enough to show that $\tilde{\sigma}_n^{-1}\tilde{S}_n \to^d N(0,1)$ as $n \to \infty$.

By Lindeberg's CLT, this would follow if for all $\delta > 0$,

$$\lim_{n\to\infty} \sum_{\mathbf{i}\in U_n} EY_n(\mathbf{i})^2 \mathbb{1}\big(|Y_n(\mathbf{i})| > \delta\big) = 0, \tag{6.4}$$

where $Y_n(\mathbf{i}) = \varepsilon(\mathbf{i})\theta_n(\mathbf{i})/\tilde{\sigma}_n$, $\mathbf{i} \in U_n$. Now, by uniform integrability of $\{\varepsilon(\mathbf{i})^2\colon \mathbf{i} \in \mathbb{Z}^d\}$ and (6.1), for any $\delta > 0$,

$$\begin{aligned}\sum_{\mathbf{i}\in U_n} EY_n(\mathbf{i})^2 \mathbb{1}\big(\big|Y_n(\mathbf{i})\big| > \delta\big) &= \tilde{\sigma}_n^{-2} \sum_{\mathbf{i}\in U_n} \theta_n(\mathbf{i})^2 E\varepsilon(\mathbf{i})^2 \mathbb{1}\big(\big|\theta_n(\mathbf{i})\varepsilon(\mathbf{i})\big| > \delta\tilde{\sigma}_n\big) \\ &\le \max_{\mathbf{i}\in\mathbb{Z}^d} E\varepsilon(\mathbf{i})^2 \mathbb{1}\left(\big|\varepsilon(\mathbf{i})\big| > \delta \frac{\tilde{\sigma}_n}{\max_{\mathbf{j}\in\mathbb{Z}^d} |\theta_n(\mathbf{j})|}\right) \\ &= \mathrm{o}(1).\end{aligned}$$

Hence, (6.4) holds and the result is proved. □

Corollary 6.2. *Let Λ_n be as in Theorem 6.1. Then, (6.1) holds if either of the following two conditions holds*:

(i) $\max\{|\theta_n(\mathbf{i})|\colon \mathbf{i} \in \mathbb{Z}^d\} = \mathrm{O}(1)$ *and* $\sigma_n{}^2 \to \infty$ *as* $n \to \infty$.
(ii) $\liminf_{n\to\infty} \sigma_n^2/|\Lambda_n| > 0$.

Proof. Sufficiency of (i) for (6.1) is trivial. Consider (ii). Fix a sequence $\{p_n\} \subset (0, \infty)$ such that $p_n^{-1} + |\Lambda_n|^{-1/d} p_n = \mathrm{o}(1)$. Then, by the Cauchy–Schwarz inequality (and (ii)),

$$\begin{aligned}\max_{\mathbf{j}\in\mathbb{Z}^d} \frac{|\theta_n(\mathbf{j})|}{\sigma_n} &\le \max_{\mathbf{j}\in\mathbb{Z}^d} \sigma_n^{-1}\left[\sum_{\|\mathbf{i}-\mathbf{j}\|\le p_n, \mathbf{i}\in\Lambda_n} \big|\alpha(\mathbf{i}-\mathbf{j})\big| + \sum_{\|\mathbf{i}-\mathbf{j}\|> p_n, \mathbf{i}\in\Lambda_n} \big|\alpha(\mathbf{i}-\mathbf{j})\big|\right] \\ &\le \sigma_n^{-1}\left[C(d)p_n^{d/2}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d} \alpha(\mathbf{i})^2\right)^{1/2} + |\Lambda_n|^{1/2}\left(\sum_{\|\mathbf{i}\|> p_n} \alpha(\mathbf{i})^2\right)^{1/2}\right] \\ &= \frac{|\Lambda_n|^{1/2}}{\sigma_n}\big[\mathrm{O}\big(|\Lambda_n|^{-1/2} p_n^{d/2}\big) + \mathrm{o}(1)\big] = \mathrm{o}(1).\end{aligned}$$

This completes the proof of the corollary. □

6.3. Proofs of the results from Section 3

Proof of Proposition 3.1. Note that by definition, $|g_t(\mathbf{z})| \le 1$ for all $\|\mathbf{z}\| = 1$ and $t > 0$. For any $t > 0$ and for any $\mathbf{x} \ne \mathbf{0}$, writing $\mathbf{x} = r\mathbf{z}$ with $\|\mathbf{z}\| = 1$ and $r = \|\mathbf{x}\|$, we have

$$\big|g_t(\mathbf{x})\big| = \big|g_{tr}(\mathbf{z})\big| \frac{\gamma(tr)}{\gamma(t)} \le \frac{\gamma(tr)}{\gamma(t)} = \|x\|^{-\beta} \frac{L(t\|\mathbf{x}\|)}{L(t)}. \tag{6.5}$$

Next, using condition (C.2) and a subsequence argument (cf. page 92, Athreya and Lahiri [1]), we have

$$\big|g_\infty(\mathbf{x})\big| \le \|x\|^{-\beta} \qquad \text{almost everywhere } (m), \mathbf{x} \ne \mathbf{0}, \tag{6.6}$$

where m is the Lebesgue measure on $\mathbb{R}^d$. Hence, $\int_{R_0} \|g_\infty(\mathbf{y}-\mathbf{x})\|\,d\mathbf{y} \le \int_{R_0} \|\mathbf{y}-\mathbf{x}\|^{-\beta}\mathbb{1}(\mathbf{y} \neq \mathbf{x})\,d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}^d$ and for all $\beta \in (0, d)$. This completes the proof. □

Proof of Theorem 3.2. First, we shall show that $G_\infty \in L^2(\mathbb{R}^d)$. Note that $R_0 \subset B(\mathbf{0}, 2^{-1}\sqrt{d})$ and hence, by (6.6), $|G_\infty(\mathbf{x})| \le \int_{\|\mathbf{y}\|\le 2\sqrt{d}} |g_\infty(\mathbf{y})|\,d\mathbf{y} \le C(d,\beta)$ for all $\|\mathbf{x}\|^2 \le d$ while $|G_\infty(\mathbf{x})| \le \int_{\|\mathbf{y}-\mathbf{x}\|\le\sqrt{d}/2} |g_\infty(\mathbf{y})|\,d\mathbf{x} \le C(d,\beta)\|\mathbf{x}\|^{-\beta}$ for all $\|\mathbf{x}\|^2 > d$, implying that $G_\infty \in L^2(\mathbb{R}^d)$.

Next, we apply Corollary 6.2 to establish Theorem 3.2. Note that by condition (C.2), there exists a sequence $\eta_n \downarrow 0$ such that

$$\int_{\{2\|\mathbf{x}\|>\eta_n\}} \left|g_\infty(\mathbf{x}) - g_{\lambda_n}(\mathbf{x})\right|^2 d\mathbf{x} = \mathrm{o}(1) \qquad \text{as } n \to \infty. \tag{6.7}$$

W.l.o.g, suppose that $\lambda_n\eta_n \gg \lambda_n^\delta$ for some $\delta \in (0, 1/2)$. Next, for $\mathbf{i} \in \mathbb{Z}^d$, write

$$\begin{aligned}\theta_n(\mathbf{i}) &= \sum_{\mathbf{j}\in(R_n-\mathbf{i})\cap\mathbb{Z}^d,\|\mathbf{j}\|>\lambda_n\eta_n} \alpha(\mathbf{j}) + \sum_{\mathbf{j}\in(R_n-\mathbf{i})\cap\mathbb{Z}^d,\|\mathbf{j}\|\le\lambda_n\eta_n} \alpha(\mathbf{j}) \\ &\equiv \theta_{1n}(\mathbf{i}) + \theta_{2n}(\mathbf{i}), \qquad \text{say.}\end{aligned}$$

We shall first show that the contribution from the $\theta_{2n}(\mathbf{i})$-terms to σ_n^2 is negligible. To that end, note that by definition, $\theta_{2n}(\mathbf{i}) = 0$ for all $\mathbf{i} \notin [-2\lambda_n, 2\lambda_n]^d$. Hence,

$$\begin{aligned}\sum_{\mathbf{i}\in\mathbb{Z}^d} \theta_{2n}(\mathbf{i})^2 &\le \sum_{\mathbf{i}\in[-2\lambda_n,2\lambda_n]^d} \left(\sum_{\|\mathbf{j}\|\le\lambda_n\eta_n} \left|\alpha(\mathbf{j})\right|\right)^2 \\ &\le (4\lambda_n)^d \left(\sum_{\|\mathbf{j}\|\le\lambda_n\eta_n} \left|\alpha(\mathbf{j})\right|\right)^2 \\ &\le C(d)\lambda_n^d\big([\lambda_n\eta_n]^{d-\beta}L(\lambda_n\eta_n)\big)^2 \\ &= \mathrm{o}\big(\lambda_n^{3d-2\beta}L^2(\lambda_n)\big).\end{aligned} \tag{6.8}$$

Next, consider the $\theta_{1n}(\mathbf{i})$-terms. Note that by definition, for any $t > 0$ and $\mathbf{k} \in \mathbb{Z}^d$, $g_t(\mathbf{x}) = g_t(\mathbf{k})$ for all $\mathbf{x} \in t^{-1}(\mathbf{k} + \mathcal{C})$. Hence,

$$\begin{aligned}\theta_{1n}(\mathbf{i}) &= \sum_{\mathbf{j}\in(R_n-\mathbf{i})\cap\mathbb{Z}^d,\|\mathbf{j}\|>\lambda_n\eta_n} \alpha(\mathbf{j}) \\ &= \sum_{\mathbf{j}\in(R_n-\mathbf{i})\cap\mathbb{Z}^d,\|\mathbf{j}\|>\lambda_n\eta_n} g_{\lambda_n}(\mathbf{j}/\lambda_n)\gamma(\lambda_n) \\ &= \gamma(\lambda_n)\lambda_n^d \int g_{\lambda_n}(\mathbf{x})\mathbb{1}\big(\mathbf{x} \in [\![R_0 - \lambda_n^{-1}\mathbf{i}]\!]_n\big)\,d\mathbf{x} \\ &\equiv \gamma(\lambda_n)\lambda_n^d G_n(\mathbf{i}/\lambda_n), \qquad \text{say,}\end{aligned}$$

where $[\![R_0-\mathbf{y}]\!]_n=\bigcup\{\lambda_n^{-1}(\mathbf{k}+\mathcal{C})\colon \mathbf{k}\in\mathbb{Z}^d, \|\mathbf{k}\|>\lambda_n\eta_n, \frac{\mathbf{k}}{\lambda_n}\in R_0-\mathbf{y}\}$ and $G_n(\mathbf{y})=\int g_{\lambda_n}(\mathbf{x})\times \mathbb{1}(\mathbf{x}\in[\![R_0-\mathbf{y}]\!]_n)\,\mathrm{d}\mathbf{x}$, $\mathbf{y}\in\mathbb{R}^d$. Note that

$$\begin{aligned}\sigma_{1n}^2&\equiv\sum_{\mathbf{i}\in\mathbb{Z}^d}\theta_{1n}(\mathbf{i})^2\\&=\gamma(\lambda_n)^2\lambda_n^{2d}\sum_{\mathbf{i}\in\mathbb{Z}^d}G_n^2(\mathbf{i}/\lambda_n)\\&=\gamma(\lambda_n)^2\lambda_n^{3d}\int\tilde{G}_n^2(\mathbf{x})\,\mathrm{d}\mathbf{x},\qquad\text{say,}\end{aligned}$$

where $\tilde{G}_n(\mathbf{x})=\sum_{\mathbf{i}\in\mathbb{Z}^d}G_n(\mathbf{i}/\lambda_n)\mathbb{1}(\mathbf{x}\in\lambda_n^{-1}(\mathbf{i}+\mathcal{C}))$, $\mathbf{x}\in\mathbb{R}^d$. We shall now show that

$$\|\tilde{G}_n-G_\infty\|_2\to 0\qquad\text{as } n\to\infty. \tag{6.9}$$

To that end, write $a_n=\sqrt{d}\lambda_n^{-1}$ and note that $\sup\{\|\lambda_n^{-1}\mathbf{i}-\mathbf{z}\|\colon \mathbf{z}\in\lambda_n^{-1}(\mathbf{i}+\mathcal{C})\}\le a_n$ and that $[\![R_0-\lambda_n^{-1}\mathbf{i}]\!]_n\subset[R_0^{a_n}-\lambda_n^{-1}\mathbf{i}]\cap\{\mathbf{x}\in\mathbb{R}^d\colon \|\mathbf{x}\|>\eta_n-a_n\}$. Write $g_t^{(n)}(\mathbf{x})=g_t(\mathbf{x})\mathbb{1}(\|\mathbf{x}\|>\eta_n-a_n)$, $t\in(0,\infty]$ and $\mathbf{x}\in\mathbb{R}^d$. Then,

$$\begin{aligned}&\sum_{\mathbf{i}\in\mathbb{Z}^d}\int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})}\biggl(\int[g_{\lambda_n}(\mathbf{x})-g_\infty(\mathbf{x})]\mathbb{1}\bigl(\mathbf{x}\in[\![R_0-\lambda_n^{-1}\mathbf{i}]\!]_n\bigr)\,\mathrm{d}\mathbf{x}\biggr)^2\mathrm{d}\mathbf{z}\\&\quad\le\sum_{\mathbf{i}\in\mathbb{Z}^d}\int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})}\biggl(\int|g_{\lambda_n}(\mathbf{x})-g_\infty(\mathbf{x})|\mathbb{1}\bigl(\mathbf{x}\in R_0^{2a_n}-\mathbf{z}\bigr)\mathbb{1}\bigl(\|\mathbf{x}\|>\eta_n-a_n\bigr)\,\mathrm{d}\mathbf{x}\biggr)^2\mathrm{d}\mathbf{z}\\&\quad=\iiint|g_{\lambda_n}^{(n)}(\mathbf{x}-\mathbf{z})-g_\infty^{(n)}(\mathbf{x}-\mathbf{z})||g_{\lambda_n}^{(n)}(\mathbf{y}-\mathbf{z})-g_\infty^{(n)}(\mathbf{y}-\mathbf{z})|\\&\qquad\quad\times\mathbb{1}\bigl(\mathbf{x}\in R_0^{2a_n}\bigr)\mathbb{1}\bigl(\mathbf{y}\in R_0^{2a_n}\bigr)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}\\&\quad\le\iint\biggl(\int|g_{\lambda_n}^{(n)}(\mathbf{x}-\mathbf{z})-g_\infty^{(n)}(\mathbf{x}-\mathbf{z})|^2\,\mathrm{d}\mathbf{z}\biggr)^{1/2}\\&\qquad\quad\times\biggl(\int|g_{\lambda_n}^{(n)}(\mathbf{y}-\mathbf{z})-g_\infty^{(n)}(\mathbf{y}-\mathbf{z})|^2\,\mathrm{d}\mathbf{z}\biggr)^{1/2}\\&\qquad\quad\times\mathbb{1}\bigl(\mathbf{x}\in R_0^{2a_n}\bigr)\mathbb{1}\bigl(\mathbf{y}\in R_0^{2a_n}\bigr)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\\&\quad=\|g_{\lambda_n}^{(n)}-g_\infty^{(n)}\|_2^2\biggl[\int\mathbb{1}\bigl(\mathbf{x}\in R_0^{2a_n}\bigr)\,\mathrm{d}\mathbf{x}\biggr]^2\\&\quad=\mathrm{o}(1).\end{aligned}\tag{6.10}$$

Next, note that the symmetric difference of the sets $[\![R_0-\lambda_n^{-1}\mathbf{i}]\!]_n$ and $R_0-\mathbf{z}$ is contained in $(\partial R_0-\mathbf{z})^{2a_n}=(\partial R_0)^{2a_n}-\mathbf{z}$ for all $\mathbf{z}\in\lambda_n^{-1}(\mathbf{i}+\mathcal{C})$ and for all $\mathbf{i}\in\mathbb{Z}^d$ with $\|\mathbf{i}\|>C(d)\lambda_n$, while it is contained in $[(\partial R_0)^{2a_n}-\mathbf{z}]\cup\{\mathbf{x}\colon \|\mathbf{x}\|\le\eta_n+a_n\}$ for all $\mathbf{z}\in\lambda_n^{-1}(\mathbf{i}+\mathcal{C})$, for $\|\mathbf{i}\|\le C(d)\lambda_n$.

Now, using the set inclusion relations given above (for the first inequality), arguments similar to (6.10) above (for the first term of the last inequality), and the bounded convergence theorem and the regularity conditions on the boundary of R_0 (that the d-dimensional Lebesgue measure of ∂R_0 is zero) (for the second and the third terms in the last inequality), we have

$$
\begin{aligned}
&\sum_{\mathbf{i}\in\mathbb{Z}^d}\int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})}\left(\int g_\infty(\mathbf{x})\big[\mathbb{1}\big(\mathbf{x}\in[\![R_0-\lambda_n^{-1}\mathbf{i}]\!]_n\big)-\mathbb{1}(\mathbf{x}\in R_0-\mathbf{z})\big]\,\mathrm{d}\mathbf{x}\right)^2\mathrm{d}\mathbf{z}\\
&\quad\le\sum_{\|\mathbf{i}\|>C(d)\lambda_n}\int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})}\left\{\int\big|g_\infty(\mathbf{x})\big|\mathbb{1}\big(\mathbf{x}\in(\partial R_0)^{2a_n}-\mathbf{z}\big)\,\mathrm{d}\mathbf{x}\right\}^2\mathrm{d}\mathbf{z}\\
&\qquad+\sum_{\|\mathbf{i}\|\le C(d)\lambda_n}\int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})}\left[\int\big|g_\infty(\mathbf{x})\big|\big\{\mathbb{1}\big(\mathbf{x}\in(\partial R_0)^{2a_n}-\mathbf{z}\big)+\mathbb{1}\big(\|\mathbf{x}\|\le\eta_n+a_n\big)\big\}\,\mathrm{d}\mathbf{x}\right]^2\mathrm{d}\mathbf{z}\\
&\quad\le\big\|g_\infty(\mathbf{x})\mathbb{1}\big(\|\mathbf{x}\|>C(d)\big)\big\|_2^2\left\{\int\mathbb{1}\big(\mathbf{x}\in(\partial R_0)^{2a_n}\big)\,\mathrm{d}\mathbf{x}\right\}^2 \qquad (6.11)\\
&\qquad+2\int_{\|\mathbf{z}\|\le C(d)}\left\{\int\big|g_\infty(\mathbf{x})\big|\mathbb{1}\big(\mathbf{x}\in\partial R_0^{2a_n}-\mathbf{z}\big)\,\mathrm{d}\mathbf{x}\right\}^2\mathrm{d}\mathbf{z}\\
&\qquad+C(d)\left\{\int\big|g_\infty(\mathbf{x})\big|\mathbb{1}\big(\|\mathbf{x}\|\le\eta_n+a_n\big)\,\mathrm{d}\mathbf{x}\right\}^2\\
&\quad=\mathrm{o}(1),
\end{aligned}
$$

as $\int|g_\infty(\mathbf{x})|\mathbb{1}(\|\mathbf{x}\|\le C)\,\mathrm{d}\mathbf{x}+\int g_\infty(\mathbf{x})^2\mathbb{1}(\|\mathbf{x}\|\ge C)\,\mathrm{d}\mathbf{x}<\infty$ for all $C\in(0,\infty)$ and for all $\beta\in(d/2,d)$. This completes the proof of (6.9). Note that (6.9) implies that $\int\tilde{G}_n^2(\mathbf{x})\,\mathrm{d}\mathbf{x}\to\int G_\infty^2(\mathbf{x})\,\mathrm{d}\mathbf{x}$ as $n\to\infty$. Hence, by Corollary 6.2(ii), Theorem 3.2 follows. □

6.4. Proofs of the results from Section 4

Proof of Theorem 4.1. By Corollary 6.2(i), it is enough to show that $G_\infty^\dagger\in L^2$ and

$$\sigma_n^2=\left[\lambda_n^{3d-2\beta}L(\lambda_n)^2\int G_\infty^\dagger(\mathbf{x})^2\,\mathrm{d}\mathbf{x}\right]\big(1+\mathrm{o}(1)\big). \qquad (6.12)$$

Note that by (6.6),

$$\big|G_\infty^\dagger(\mathbf{x})\big|\le C(d,\beta)\begin{cases}d(\mathbf{x},R_0)^{d-\beta}, & \text{if }\mathbf{x}\in[R_0\cup\partial R_0]^c\text{ and }\|\mathbf{x}\|\le 1,\\ \|\mathbf{x}\|^{-\beta}, & \text{if }\|\mathbf{x}\|>1\end{cases}$$

and $|G_\infty^\dagger(\mathbf{x})|\le C(d,\beta)d(\mathbf{x},R_0^c)^{d-\beta}$ for $\mathbf{x}\in R_0$. Since $\beta\in(d,d+1/2)$, by the boundary condition (C.3), it follows that $G_\infty^\dagger\in L^2$.

Next, consider (6.12). For $\beta \in (d, d+1/2)$, by condition (C.2), (6.5) and (6.6), there exists a sequence $\eta_n \downarrow 0$ such that

$$\int_{\{2\|\mathbf{x}\|>\eta_n\}} \left|g_\infty(\mathbf{x}) - g_{\lambda_n}(\mathbf{x})\right|^p \,\mathrm{d}\mathbf{x} = \mathrm{o}(1) \qquad \text{as } n \to \infty, \tag{6.13}$$

for $p = 1, 2$. For $p = 1$, this follows directly from (C.2), as $\beta > d$. As for $p = 2$, we use the trivial bound "$f(x)^2 \le |f(x)| \sup_{x \in B}\{|f(x)|\}$ for a function $f : B \to \mathbb{R}$" in conjunction with the $p = 1$ relation and the bounds (6.5) and (6.6), which may require replacing the η_n for the $p = 1$ case by a possibly coarser sequence that still decreases to zero. Hence, (6.13) holds for both $p = 1, 2$.

W.l.o.g., suppose that $\eta_n \gg \lambda_n^{-\delta}$ (i.e., $\lambda_n^\delta \eta_n \to \infty$) for some $\delta \in (0, 1)$. Let $t_n = \lambda_n \eta_n$ and let $u_n = \lambda_n^{[2d-2\beta+1]/2}$. Then $u_n^{-1} + t_n^{-1} = \mathrm{o}(1)$ and $u_n = \mathrm{o}(\lambda_n^{[2d-2\beta+1]} L_n(\lambda_n)^2)$.

Also, define

$$\begin{aligned} V_{1n} &= \left\{\mathbf{i} \in \mathbb{Z}^d\colon\ B(\mathbf{i}; t_n) \subset R_n\right\}, \\ V_{2n} &= \left\{\mathbf{i} \in \mathbb{Z}^d\colon\ B(\mathbf{i}; t_n) \subset R_n^c\right\}, \\ V_{3n} &= \mathbb{Z}^d \setminus [V_{1n} \cup V_{2n}]. \end{aligned} \tag{6.14}$$

First, consider the sum of $\theta_n(\mathbf{i})^2$ for $\mathbf{i} \in V_{3n}$. Note that $V_{3n} \subset \{\mathbf{i} \in \mathbb{Z}^d\colon\ \mathbf{i} \in (\partial R_n)^{t_n}\}$. By (C.3) (with $f \equiv 1$), $\nu((\partial R_0)^\varepsilon) = \mathrm{O}(\varepsilon)$ as $\varepsilon \to 0$. This implies $\nu((\partial R_n)^{u_n}) = \mathrm{O}(\lambda_n^{d-1} u_n)$ and hence,

$$\sum_{\mathbf{i} \in \partial R_n^{u_n} \cap \mathbb{Z}^d} \left|\theta_n(\mathbf{i})\right|^2 \le C(d) \lambda_n^{d-1} u_n \left[\sum_{\mathbf{i} \in \mathbb{Z}^d} \left|\alpha(\mathbf{i})\right|\right]^2 = \mathrm{o}\big(\lambda_n^{[3d-2\beta]} L(\lambda_n)^2\big). \tag{6.15}$$

If $u_n > t_n$, then this shows that $\sum_{\mathbf{i} \in V_{3n}} \theta_n(\mathbf{i})^2 = \mathrm{o}(\lambda_n^{[3d-2\beta]} L(\lambda_n)^2)$. Hence, w.l.o.g., suppose that $u_n \le t_n$. Let $v_n(\mathbf{i}) = d(\mathbf{i}, \partial R_n)$, $\mathbf{i} \in [(\partial R_n)^{u_n}]^c$. By the condition $A = 0$, uniformly in $\mathbf{i} \notin (\partial R_n)^{u_n}$, we have

$$\left|\theta_n(\mathbf{i})\right| \le \sum_{\|\mathbf{l}\| > v_n(\mathbf{i})} \left|\alpha(\mathbf{l})\right| \le \sum_{\|\mathbf{l}\| > v_n(\mathbf{i})} \|\mathbf{l}\|^{-\beta} L\big(\|\mathbf{l}\|\big) \le C(d, \beta) v_n(\mathbf{i})^{d-\beta} L\big(v_n(\mathbf{i})\big).$$

Hence, by condition (C.3), it follows that

$$\begin{aligned} &\sum_{\mathbf{i} \in [\partial R_n^{t_n} \setminus \partial R_n^{u_n}] \cap \mathbb{Z}^d} \theta_n(\mathbf{i})^2 \\ &\quad \le C(d, \beta) \sum_{\mathbf{i} \in [\partial R_n^{t_n} \setminus \partial R_n^{u_n}] \cap \mathbb{Z}^d} \left\{v_n(\mathbf{i})^{d-\beta} L\big(v_n(\mathbf{i})\big)\right\}^2 \\ &\quad \le C(d, \beta) \max\left\{L(\mathbf{i})^2\colon\ u_n \le \|\mathbf{i}\| \le t_n\right\} \cdot \lambda_n^{3d-2\beta} \int_{\partial R_0^{t_n/\lambda_n}} d(\mathbf{x}, \partial R_0)^{2d-2\beta} \,\mathrm{d}\mathbf{x} \qquad (6.16) \\ &\quad \le C(d, \beta) \max\left\{L(\mathbf{i})^2\colon\ u_n \le \|\mathbf{i}\| \le t_n\right\} \cdot \lambda_n^{3d-2\beta} \int_0^{t_n/\lambda_n} t^{2d-2\beta} \,\mathrm{d}t \\ &\quad = \mathrm{o}\big(\lambda_n^{3d-2\beta} L(\lambda_n)^2\big). \end{aligned}$$

Hence, by (6.15) and (6.16), it follows that

$$\sum_{\mathbf{i}\in V_{3n}} \theta_n(\mathbf{i})^2 = \mathrm{o}\big(\lambda_n^{[3d-2\beta]} L(\lambda_n)^2\big).$$

Next, using arguments similar to those in the proof of Theorem 3.2, we have

$$\begin{aligned}\sum_{\mathbf{i}\in V_{1n}} \theta_n(\mathbf{i})^2 &= \sum_{\mathbf{i}\in V_{1n}} \Big[\sum_{\mathbf{j}\in R_n-\mathbf{i}} \alpha(\mathbf{i})\Big]^2 \\ &= \sum_{\mathbf{i}\in V_{1n}} \Big[\sum_{\mathbf{j}\in R_n^c-\mathbf{i}} \alpha(\mathbf{i})\Big]^2 \qquad (\text{as } A=0) \\ &= \lambda_n^{2d}\gamma(\lambda_n)^2 \sum_{\mathbf{i}\in V_{1n}} \Big[\int g_{\lambda_n}(\mathbf{x})\mathbb{1}\big(\mathbf{x}\in [\![R_0^c-\lambda_n^{-1}\mathbf{i}]\!]_n\big)\,\mathrm{d}\mathbf{x}\Big]^2,\end{aligned}$$

and similarly,

$$\sum_{\mathbf{i}\in V_{2n}} \theta_n(\mathbf{i})^2 = \lambda_n^{2d}\gamma(\lambda_n)^2 \sum_{\mathbf{i}\in V_{2n}} \Big[\int g_{\lambda_n}(\mathbf{x})\mathbb{1}\big(\mathbf{x}\in [\![R_0-\lambda_n^{-1}\mathbf{i}]\!]_n\big)\,\mathrm{d}\mathbf{x}\Big]^2,$$

where $[\![R_0-\lambda_n^{-1}\mathbf{i}]\!]_n$ is as defined in the proof of Theorem 3.2. Define the function $\check{G}_n(\cdot)$ by

$$\check{G}_n(\mathbf{x}) = \begin{cases} \int g_{\lambda_n}(\mathbf{y})\mathbb{1}\big(\mathbf{y}\in [\![R_0^c-\lambda_n^{-1}\mathbf{i}]\!]_n\big)\,\mathrm{d}\mathbf{y}, & \text{if } \mathbf{x}\in\lambda_n^{-1}(\mathbf{i}+\mathcal{C}), \mathbf{i}\in V_{1n}, \\ \int g_{\lambda_n}(\mathbf{y})\mathbb{1}\big(\mathbf{y}\in [\![R_0-\lambda_n^{-1}\mathbf{i}]\!]_n\big)\,\mathrm{d}\mathbf{y} & \text{if } \mathbf{x}\in\lambda_n^{-1}(\mathbf{i}+\mathcal{C}), \mathbf{i}\in V_{2n}, \\ 0, & \text{otherwise}. \end{cases} \tag{6.17}$$

Then, it follows that

$$\begin{aligned}\sigma_n^2 &= \lambda_n^{3d}\gamma(\lambda_n)^2 \int \check{G}_n(\mathbf{x})^2\,\mathrm{d}\mathbf{x} + \sum_{\mathbf{i}\in V_{3n}} \big|\theta_n(\mathbf{i})\big|^2 \\ &= \lambda_n^{3d}\gamma(\lambda_n)^2 \int \check{G}_n(\mathbf{x})^2\,\mathrm{d}\mathbf{x} + \mathrm{o}\big(\lambda_n^{3d}\gamma(\lambda_n)^2\big).\end{aligned} \tag{6.18}$$

It now remains to show that $\int \check{G}_n(\mathbf{x})^2\,\mathrm{d}\mathbf{x} \to \int G_\infty^\dagger(\mathbf{x})^2\,\mathrm{d}\mathbf{x}$, or equivalently, that $\int [\check{G}_n(\mathbf{x}) - G_\infty^\dagger(\mathbf{x})]^2\,\mathrm{d}\mathbf{x} \to 0$. To that end, define $\Gamma_n = \bigcup\{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})\colon \mathbf{i}\in V_{2n}\}$ and recall that $a_n = \sqrt{d}/\lambda_n$. Then, repeating the arguments leading to (6.10), we get

$$\begin{aligned}&\sum_{\mathbf{i}\in V_{2n}} \int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})} \Big(\int [g_{\lambda_n}(\mathbf{x}) - g_\infty(\mathbf{x})]\mathbb{1}\big(\mathbf{x}\in [\![R_0-\lambda_n^{-1}\mathbf{i}]\!]_n\big)\,\mathrm{d}\mathbf{x}\Big)^2\,\mathrm{d}\mathbf{z} \\ &\quad\le \sum_{\mathbf{i}\in V_{2n}} \int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})} \Big(\int \big|g_{\lambda_n}^{(n)}(\mathbf{x}) - g_\infty^{(n)}(\mathbf{x})\big|\mathbb{1}\big(\mathbf{x}\in R_0^{2a_n}-\mathbf{z}\big)\,\mathrm{d}\mathbf{x}\Big)^2\,\mathrm{d}\mathbf{z}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Gamma_n} \left(\int \left| g_{\lambda_n}^{(n)}(\mathbf{x}-\mathbf{z}) - g_{\infty}^{(n)}(\mathbf{x}-\mathbf{z}) \right| \mathbb{1}\left(\mathbf{x} \in R_0^{2a_n}\right) \mathrm{d}\mathbf{x} \right)^2 \mathrm{d}\mathbf{z} \\
&= \iiint \left| g_{\lambda_n}^{(n)}(\mathbf{x}-\mathbf{z}) - g_{\infty}^{(n)}(\mathbf{x}-\mathbf{z}) \right| \left| g_{\lambda_n}^{(n)}(\mathbf{y}-\mathbf{z}) - g_{\infty}^{(n)}(\mathbf{y}-\mathbf{z}) \right| \\
&\qquad \times \mathbb{1}\left(\mathbf{x} \in R_0^{2a_n}\right) \mathbb{1}\left(\mathbf{y} \in R_0^{2a_n}\right) \mathbb{1}(\mathbf{z} \in \Gamma_n) \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z} \\
&\le \iint \left(\int_{\Gamma_n} \left| g_{\lambda_n}^{(n)}(\mathbf{x}-\mathbf{z}) - g_{\infty}^{(n)}(\mathbf{x}-\mathbf{z}) \right|^2 \mathrm{d}\mathbf{z} \right)^{1/2} \left(\int_{\Gamma_n} \left| g_{\lambda_n}^{(n)}(\mathbf{y}-\mathbf{z}) - g_{\infty}^{(n)}(\mathbf{y}-\mathbf{z}) \right|^2 \mathrm{d}\mathbf{z} \right)^{1/2} \\
&\qquad \times \mathbb{1}\left(\mathbf{x} \in R_0^{2a_n}\right) \mathbb{1}\left(\mathbf{y} \in R_0^{2a_n}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&= \left\| \left(g_{\lambda_n}^{(n)} - g_{\infty}^{(n)} \right) \right\|_2^2 \left[\int \mathbb{1}\left(\mathbf{x} \in R_0^{2a_n}\right) \mathrm{d}\mathbf{x} \right]^2 \\
&= \mathrm{o}(1).
\end{aligned}
$$

By similar arguments,

$$
\begin{aligned}
&\sum_{\mathbf{i} \in V_{1n}} \int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})} \left(\int \left[g_{\lambda_n}(\mathbf{x}) - g_{\infty}(\mathbf{x}) \right] \mathbb{1}\left(\mathbf{x} \in \left[\!\left[R_0^c - \lambda_n^{-1}\mathbf{i} \right]\!\right]_n \right) \mathrm{d}\mathbf{x} \right)^2 \mathrm{d}\mathbf{z} \\
&\quad \le C(d) \cdot \left[\int_{\{\|\mathbf{y}\| \ge \eta_n\}} \left| g_{\lambda_n}(\mathbf{y}) - g_{\infty}(\mathbf{y}) \right| \mathrm{d}\mathbf{y} \right]^2 = \mathrm{o}(1).
\end{aligned}
$$

Next, note that $\partial[R_0^c] = \partial R_0$. By repeating the arguments in (6.11), one can conclude that

$$
\begin{aligned}
&\sum_{\mathbf{i} \in V_{1n}} \int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})} \left(\int g_{\infty}(\mathbf{x}) \left[\mathbb{1}\left(\mathbf{x} \in \left[\!\left[R_0^c - \lambda_n^{-1}\mathbf{i} \right]\!\right]_n \right) - \mathbb{1}\left(\mathbf{x} \in R_0^c - \mathbf{z}\right) \right] \mathrm{d}\mathbf{x} \right)^2 \mathrm{d}\mathbf{z} \\
&\qquad + \sum_{\mathbf{i} \in V_{2n}} \int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})} \left(\int g_{\infty}(\mathbf{x}) \left[\mathbb{1}\left(\mathbf{x} \in \left[\!\left[R_0 - \lambda_n^{-1}\mathbf{i} \right]\!\right]_n \right) - \mathbb{1}(\mathbf{x} \in R_0 - \mathbf{z}) \right] \mathrm{d}\mathbf{x} \right)^2 \mathrm{d}\mathbf{z} \\
&\quad = \mathrm{o}(1).
\end{aligned}
$$

Finally, using the boundary condition (C.3) and the bounds on $|G_{\infty}^{\dagger}(\mathbf{x})|$ for $\mathbf{x} \in (\partial R_0)^{\eta_n}$ (from the proof of $G_{\infty}^{\dagger} \in L^2(\mathbb{R}^d)$), one gets $\sum_{\mathbf{i} \in V_{3n}} \int_{\lambda_n^{-1}(\mathbf{i}+\mathcal{C})} cGi^2(\mathbf{x}) \,\mathrm{d}\mathbf{x} \le C(d) \int_0^{\eta_n} t^{2(d-\beta)} \,\mathrm{d}t = \mathrm{o}(1)$. Hence, it follows that

$$
\sigma_n^2 = \left[\gamma(\lambda_n) \right]^2 \lambda_n^{3d} \int \left[G_{\infty}^{\dagger}(\mathbf{x}) \right]^2 \mathrm{d}\mathbf{x} \left(1 + \mathrm{o}(1) \right).
$$

This completes the proof of Theorem 4.1. □

Proofs of claims in Example 4.2. Let $c_n = \lfloor \log \lambda_n \rfloor$, $n \ge 1$. Also, let $\|(x, y)'\|_{\infty} = \max\{|x|, |y|\}$, $x, y \in \mathbb{R}$. For a set $A \subset \mathbb{R}^2$ and $\delta > 0$, write $A_{\infty}^{\delta} = \{\mathbf{x} \in \mathbb{R}^2 \colon \|\mathbf{x} - \mathbf{y}\|_{\infty} \le \delta\}$ and $A_{\infty}^{-\delta} =$

$\{\mathbf{x} \in A\colon \|\mathbf{x} - \mathbf{y}\|_\infty < \delta \text{ implies } \mathbf{y} \in A\}$ for the δ-enlargement and the δ-interior of a set A in the $\|\cdot\|_\infty$-norm. As before, set $\theta_n(\mathbf{i}) = \sum_{\mathbf{j}+\mathbf{i}\in R_n} \alpha(\mathbf{j})$. Let $\mathcal{I}_n(i_0) \equiv \{i \in \mathbb{Z}\colon -\frac{\lambda_n}{2} - i_0 < i < \frac{\lambda_n}{2} - i_0\}$, $i_0 \in \mathbb{Z}$. For $\mathbf{i} = (i_0, j_0) \in R_{n,\infty}^{-c_n}$, it is easy to verify that both $\mathcal{I}_n(i_0)$ and $\mathcal{I}_n(j_0)$ contain the set $\{i \in \mathbb{Z}\colon |i| < c_n\}$. Hence, using the fact that $\sum_{\mathbf{i}\in\mathbb{Z}^2} \alpha(\mathbf{i}) = 0$, one gets

$$\begin{aligned}
|\theta_n(\mathbf{i})| = |\theta_n(i_0, j_0)| &= \Bigl|\sum_{i\in\mathcal{I}_n(i_0)\times\mathcal{I}_n(j_0)} \alpha(i, j)\Bigr| \\
&= \Bigl|\sum_{(i,j)\notin\mathcal{I}_n(i_0)\times\mathcal{I}_n(j_0)} \alpha(i, j)\Bigr| \\
&= \Bigl|\sum_{(i,j)\notin\mathcal{I}_n(i_0)\times\mathcal{I}_n(j_0)} b(i)b(j)\mathbb{1}(ij \neq 0)\Bigr| \\
&\le \Bigl[\sum_{|i|>\lambda_n/2-|i_0|} b(i)\Bigr] \cdot \Bigl[\sum_{|i|>\lambda_n/2-|j_0|} b(i)\Bigr].
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\sum_{\mathbf{i}\in R_{n,\infty}^{-c_n}} \theta_n(\mathbf{i})^2 &\le \Biggl[\sum_{i_0=-\lambda_n/2+c_n}^{\lambda_n/2-c_n} \Bigl\{\sum_{|i|>\lambda_n/2-|i_0|} b(i)\Bigr\}^2\Biggr]^2 \\
&\le C(\beta, B)\Biggl[\sum_{j=0}^{[\lambda_n/2]-c_n} \Bigl|\frac{\lambda_n}{2} - j\Bigr|^{2-2\beta}\Biggr]^2 \qquad (6.19) \\
&\le C(\beta, B)\bigl(c_n^{3-2\beta}\bigr)^2
\end{aligned}$$

for some (generic) constant $C(\beta, B) \in (0, \infty)$. By similar arguments, it can be shown that

$$\sum_{\mathbf{i}\notin R_{n,\infty}^{c_n}} \theta_n(\mathbf{i})^2 = \mathrm{o}(\lambda_n).$$

Hence, it remains to determine the contribution of the boundary terms to σ_n^2. For $|i_0| \le \lambda_n/2 - c_n$ and $-c_n \le k \equiv \lfloor 2^{-1}\lambda_n - j_0\rfloor \le c_n$ (this corresponds to a part of the upper boundary line of R_n), note that

$$\begin{aligned}
\theta_n(i_0, j_0) &= \sum_{|i+i_0|<\lambda_n/2} \sum_{j=-\lambda_n/2-j_0}^{k} \alpha(i, j) \\
&= \Biggl[\sum_{i\in\mathbb{Z}} \sum_{j=-\lambda_n/2-j_0}^{k} \alpha(i, j)\Biggr]\bigl(1 + \mathrm{o}(1)\bigr)
\end{aligned}$$

$$= \mathrm{o}(1) + \begin{cases} -4B^2 + (2B)\left[\sum_{j=1}^{k} b(j) + B\right], & \text{if } k > 0, \\ -2B^2, & \text{if } k = 0, \\ (2B)\left[\sum_{j=-\infty}^{k} b(j)\right], & \text{if } k < 0 \end{cases}$$

uniformly in (i_0, j_0). Note that by absolute summability of the $\alpha(\mathbf{i})$,

$$\sum_{|i_0+\lambda_n/2|\le c_n} \sum_{|j_0-\lambda_n/2|\le c_n} \theta_n(i_0, j_0)^2 = \mathrm{O}\big(c_n^2\big).$$

Hence, it follows that

$$\sum_{|i_0|\le\lambda_n/2-c_n} \sum_{|j_0-\lambda_n/2|\le c_n} \theta_n(i_0, j_0)^2 = \big[\lambda_n\sigma_0^2/4\big]\big(1 + \mathrm{o}(1)\big).$$

Now using similar arguments for the other three boundary arms, one gets the results of Example 4.2. □

Proof of Theorem 4.3. Using (4.4), one can show that there exists $\eta_n \to 0+$ such that

$$\limsup_{n\to\infty}\left|\lambda_n^{-(d-1)} \sum_{\mathbf{i}\in[\partial R_n]^{\eta_n\lambda_n}\cap\mathbb{Z}^d} \big|\theta_n(\mathbf{i})\big|^2 - \sigma_{\mathrm{EE}}^2\right| = 0. \tag{6.20}$$

By (4.4) and the monotonicity of the sum of $\theta_n(\mathbf{i})^2$ over increasing index sets, we may always replace η_n by a coarser sequence going to zero. Hence, w.l.o.g. assume that $\eta_n \ge [\log(\lambda_n)]^{-1}$. Next set $t_n = \lambda_n\eta_n$ and (re-)define the sets V_{kn}, $k = 1, 2, 3$ in (6.14) with this choice of t_n. Further, write $V_{21n} = V_{2n} \cap \{\mathbf{i} \in \mathbb{Z}^d\colon \|\mathbf{i}\| \le \lambda_n\sqrt{d}\}$ and $V_{22n} = V_{2n} \setminus V_{21n}$.

Next, note that uniformly in $\mathbf{i} \notin R_n^{t_n}$,

$$\begin{aligned} \big|\theta_n(\mathbf{i})\big| &\le \sum_{\mathbf{j}\in R_n-\mathbf{i}} \big|\alpha(\mathbf{j})\big| \\ &\le \sum_{\mathbf{j}\colon \|\mathbf{j}\|\ge d(\partial R_n,\mathbf{i}), \mathbf{j}\in[R_n-\mathbf{i}]\cap\mathbb{Z}^d} \gamma\big(\|\mathbf{j}\|\big) \\ &\le C(d,\beta)\min\big\{d(\partial R_n,\mathbf{i})^{d-\beta}L\big(d(\partial R_n,\mathbf{i})\big), \lambda_n^d d(\partial R_n,\mathbf{i})^{-\beta}L_n^*(\mathbf{i})\big\}, \end{aligned} \tag{6.21}$$

where $L_n^*(\mathbf{i}) \equiv \max\{L(\|\mathbf{j}\|)\colon \mathbf{j} \in [R_n - \mathbf{i}] \cap \mathbb{Z}^d\}$. And, using the fact that $A = 0$, one can similarly show that uniformly in $\mathbf{i} \in R_n^{-t_n}$,

$$\big|\theta_n(\mathbf{i})\big| = \left|-\sum_{\mathbf{j}\in R_n^c-\mathbf{i}} \alpha(\mathbf{j})\right| \le C(d,\beta)d\big(\partial R_n^c, \mathbf{i}\big)^{d-\beta}L\big(d\big(\partial R_n^c, \mathbf{i}\big)\big).$$

Hence, it follows that

$$\begin{aligned}\sum_{\mathbf{i}\in V_{1n}} \theta_n(\mathbf{i})^2 + \sum_{\mathbf{i}\in V_{21n}} \theta_n(\mathbf{i})^2 &\\ \le C(d)\lambda_n^d \cdot \max\{\theta_n(\mathbf{i})^2\colon \mathbf{i}\in V_{1n}\cup V_{21n}\} &\\ \le C(d,\beta)\lambda_n^d t_n^{2(d-\beta)} \max\{L^2(t)\colon t_n \le t \le d\lambda_n\} = \mathrm{o}(\lambda_n^{d-1}). &\end{aligned} \tag{6.22}$$

Next, note that for all $\mathbf{i}\in V_{22n}$,

$$\|\mathbf{i}\|/2 \le \|\mathbf{i}\| - \lambda_n\sqrt{d}/2 \le d(\partial R_n, \mathbf{i}) \le \|\mathbf{i}\|$$

and

$$\max\{\|\mathbf{j}\|\colon \mathbf{j}\in [R_n - \mathbf{i}]\cap \mathbb{Z}^d\} \subset [\|\mathbf{i}\|/2, 3\|\mathbf{i}\|/2\}.$$

Hence, by (6.21)

$$\begin{aligned}\sum_{\mathbf{i}\in V_{22n}} \theta_n(\mathbf{i})^2 &\le C(d)\lambda_n^{2d} \sum_{\|\mathbf{i}\|>\sqrt{d}\lambda_n} \|\mathbf{i}\|^{-2\beta} L^2(\|\mathbf{i}\|) \\ &\le C(d,\beta)\lambda_n^{3d-2\beta} L^2(\lambda_n) = \mathrm{o}(\lambda_n^{d-1})\end{aligned} \tag{6.23}$$

for $\beta > d + 1/2$. Hence, from (4.4), (6.22) and (6.23), it follows that

$$\sigma_n^2 = \sum_{k=1}^{3} \sum_{\mathbf{i}\in V_{kn}} \theta_n(\mathbf{i})^2 = \lambda_n^{d-1}\sigma_{\mathrm{EE}}^2(1+\mathrm{o}(1)). \tag{6.24}$$

By Corollary 6.2(i), Theorem 4.3 follows. □

Proof of Theorem 4.4. Theorem 4.4 follows from Corollary 6.2(i), by comparing the orders of the terms $\sum_{\mathbf{i}\in V_{kn}} \theta_n(\mathbf{i})^2$, $k = 1, 2, 3$. Specifically, for part (i), we use the arguments in the proof of Theorem 4.3, and the bounds from (6.20), (6.22) and (6.23) to conclude that (6.24) holds. For part (ii), note that for $\beta = d + 1/2$, the conditions and the arguments in the proof of Theorem 4.1 no longer ensure that $\int_{\|\mathbf{x}\|\le \eta_n} G_\infty^\dagger(\mathbf{x})^2\,\mathrm{d}\mathbf{x} = \mathrm{O}(1)$, which is why we need to make the assumption that $G_\infty^\dagger \in L^2$. However, under (C.1) and (C.2), the arguments in the proof of Theorem 4.1 leading to the convergence of $\int_{\|\mathbf{x}\|\ge \eta_n} |\check{G}_n(\mathbf{x}) - G_\infty^\dagger(\mathbf{x})|\,\mathrm{d}\mathbf{x}$ to zero still holds. Both parts of (ii) now follow by deriving the limits of the terms $\sum_{\mathbf{i}\in V_{kn}} \theta_n(\mathbf{i})^2$, using (6.20) for $k = 3$ and using the steps from the proof of Theorem 4.1 for $k = 1, 2$. We omit the routine details. □

6.5. Proof of the results from Section 5

Proof of Theorem 5.1. In view of Corollary 6.2(ii), it is enough to show that

$$\sigma_n^2 = N_n A^2(1+\mathrm{o}(1)). \tag{6.25}$$

Let c_n be a sequence of positive real numbers satisfying

$$c_n^{-1} + \lambda_n^{-1} c_n = \mathrm{o}(1). \tag{6.26}$$

Also, let

$$\begin{aligned} U_{1n} &= \left\{ \mathbf{i} \in \mathbb{Z}^d \colon \ \left\| \lambda_n^{-1} \mathbf{i} - \mathbf{x} \right\| \le c_n/\lambda_n \text{ for some } \mathbf{x} \in \partial R_0 \right\}, \\ U_{2n} &= \left\{ \mathbf{i} \in \mathbb{Z}^d \colon \ B(\mathbf{i}; c_n) \subset R_n \right\}, \\ U_{3n} &= \left\{ \mathbf{i} \in [-2d\lambda_n, 2d\lambda_n]^d \colon \ \mathbf{i} \notin [U_{1n} \cup U_{2n}] \right\} \quad \text{and} \\ U_{4n} &= \mathbb{Z}^d \setminus U_{3n}. \end{aligned} \tag{6.27}$$

Then σ_n^2 can be written as

$$\sigma_n^2 = \sum_{i=1}^{4} \sum_{\mathbf{i} \in U_{in}} \theta_n(\mathbf{i})^2 \equiv I_{1n} + I_{2n} + I_{3n} + I_{4n}, \qquad \text{say.} \tag{6.28}$$

Note that by (6.26), the boundary condition on R_0 (that $\nu(\partial R_0) = 0$) and the absolute summability of $\alpha(\mathbf{i})$'s,

$$\begin{aligned} I_{1n} &\le \left[\sum_{\mathbf{j} \in \mathbb{Z}^d} \left| \alpha(\mathbf{j}) \right| \right]^2 \left| \left\{ \mathbf{i} \colon \ \lambda_n^{-1} \mathbf{i} \in [\partial R_0]^{c_n/\lambda_n} \right\} \right| \\ &= \lambda_n^d \cdot \mathrm{O}\bigl(\mathrm{vol.}\bigl([\partial R_0]^{c_n/\lambda_n} \bigr) \bigr) \\ &= \mathrm{o}\bigl(\lambda_n^d \bigr). \end{aligned}$$

Next, consider I_{2n}. By definition of U_{2n}, $B(\mathbf{0}; c_n) \subset R_n - \mathbf{i}$ for all $\mathbf{i} \in U_{2n}$. Hence, it follows that

$$\sup\left\{ \left| \theta_n(\mathbf{i}) - A \right| \colon \ \mathbf{i} \in U_{2n} \right\} \le \sum_{\mathbf{j} \in \mathbb{Z}^d \colon \ \|\mathbf{j}\| \ge c_n} \left| \alpha(\mathbf{j}) \right| = \mathrm{o}(1). \tag{6.29}$$

Next, note that for any $\mathbf{i} \in U_{3n}$, $B(\mathbf{i}; c_n/2)$ is contained in the set R_n^c, and hence, $\varnothing = [R_n - \mathbf{i}] \cap B(\mathbf{0}; c_n/2) = [R_n \cap B(\mathbf{i}; c_n/2)] - \mathbf{i}$. Hence, it follows that

$$I_{3n} \le C(d) \lambda_n^d \sup\left\{ \theta_n(\mathbf{i})^2 \colon \ \mathbf{i} \in U_{3n} \right\} \le C(d) \lambda_n^d \sum_{\mathbf{j} \in \mathbb{Z}^d \colon \ 2\|\mathbf{j}\| \ge c_n} \left| \alpha(\mathbf{j}) \right| = \mathrm{o}\bigl(\lambda_n^d \bigr).$$

Finally, by condition (C.1) and the definition of U_{4n},

$$\begin{aligned} &\sum_{\mathbf{i} \in U_{4n}} \theta_n(\mathbf{i})^2 \\ &\quad \le \sum_{\mathbf{i} \in U_{4n}} \left[\left(\sum_{\mathbf{j} \in [R_n - \mathbf{i}] \cap \mathbb{Z}^d} \alpha(\mathbf{j})^2 \right) \times N_n \right] \end{aligned}$$

$$
\begin{aligned}
&\le N_n \sum_{\mathbf{i}\in U_{4n}} \sum_{\mathbf{j}\in\mathbb{Z}^d:\ \|\mathbf{j}\|_1\ge\|\mathbf{i}\|_1-d\lambda_n} \alpha(\mathbf{j})^2, \qquad \Bigl(\text{since } \sup_{\mathbf{x}\in R_n} \|\mathbf{x}\|_1 \le d\lambda_n/2\Bigr)\\
&\le N_n \sum_{\mathbf{i}\in U_{4n}} \sum_{k\ge\|\mathbf{i}\|_1-d\lambda_n} \bigl|\bigl\{\mathbf{j}\in\mathbb{Z}^d:\ \|\mathbf{j}\|_1=k\bigr\}\bigr| \cdot \sup_{\mathbf{j}\in\mathbb{Z}^d:\ \|\mathbf{j}\|_1=k} \alpha(\mathbf{j})^2\\
&\le C(d)N_n \sum_{\mathbf{i}\in U_{4n}} \sum_{k\ge\|\mathbf{i}\|_1-d\lambda_n} k^{d-1}\sup\bigl\{\gamma(t)^2:\ t\in[k/\sqrt{d},k]\bigr\}\\
&\le C(d)N_n \sum_{\mathbf{i}\in U_{4n}} \int_{\|\mathbf{i}\|_1-d\lambda_n}^{\infty} t^{d-1-2\beta}L(t)^2\,\mathrm{d}t\\
&\le C(d)N_n \int_{2d\lambda_n}^{\infty} u^{d-1}\int_{u-d\lambda_n}^{\infty} t^{d-1-2\beta}L(t)^2\,\mathrm{d}t\,\mathrm{d}u\\
&\le C(d)N_n\lambda_n^{2d-2\beta}L(\lambda_n)^2 = \mathrm{o}(N_n),
\end{aligned}
$$

since $\lambda_n^{d-\beta}L(\lambda_n) = \mathrm{o}(1)$ for all $\beta > d$ and also for $\beta = d$ by the integrability condition $\int_1^\infty \gamma(t)\,\mathrm{d}t < \infty$. Since $|U_{2n}| = |\mathcal{D}_n| - \mathrm{O}(|U_{1n}|) = N_n - \mathrm{o}(\lambda_n^d) = N_n(1+\mathrm{o}(1))$, (6.25) follows from (6.28), (6.29) and the bounds for I_{1n}, I_{3n} and I_{4n} above. This completes the proof of the theorem. □

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