DIFFUSION OF AN INSTANTANEOUS PLANE SOURCE IN A STEADY UNIFORM OPEN CHANNEL FLOW

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A. R. Chamberlain

Colorado State University
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INTRODUCTION

One of the important problems associated with the disposal of either intentional or accidental discharges into open streams or canals of large amounts of contaminant during short intervals of time is that of estimating the maximum concentration as a function of time at any point downstream in the channel. This problem can be treated analytically as an instantaneous plane source transverse to the normal mean velocity vector.

The specific problem then appears to be one of solving the instantaneous plane source problems for various channel geometry, bed roughness, mean velocity, discharge and channel slope. Several field experiments have been conducted by USGS Surface Water Branch Personnel. Further, laboratory and theoretical work have been conducted by Taylor ( ), Orlob ( ), Hubbell and Sayre ( ), Roberts ( ), and Harleman ( ). The field work indicates that the ________-time curves for fixed stations along the stream have a substantial "tail", as illustrated by Fig. 2.

The work of these authors usually comes to two conclusions:

a) The so-called Fickian theory or heat-conduction analogy is adequate for a theoretical prediction of concentration distributions even assuming constant "diffusion coefficients."

b) The form of the concentration curve as predicted by theory should be Grussian in form.

It is thus evident that the available theoretical solutions and apparent shape of the field concentration curves do not agree. This difference has been explained phenomenologically as loss in the laminar sublayer at the channel boundary or temporary storage along the banks of the channel -- but no theoretical equation has been put forth to explain as a basis of theoretical prediction and experimental design.

This paper has as its purpose the development of a single approximate solution to the problem of an instantaneous plane source of contaminants released in a uniform open channel conveying a steady flow. The solution develops the "tail" in the concentration curves and introduces a parameter which represents the attention of maximum concentration and the nature of the boundary. Only one set of data has been available for testing the usefulness of the theory.

Discussion of the Problem

In considering what physically takes place in a straight, open, alluvial channel carrying a constant discharge, it is reasonable to assume that as an instantaneous plane source travels downstream with the mean velocity of plane u_o the concentration measured as a function of time at any position x along the stream will exhibit a tail and a steepening of the forefront of the profile due to:

a) Temporary storage in the low velocity areas near isolated roughnesses.

b) Chemical and physical absorption and absorption to the sediment materials in the channel banks.

c) Temporary storage in the near-zero velocity areas near the channel boundaries.

d) Temporary to semi-permanent losses due to conveying of some flow into and through the alluvial boundary.

As an approximation, it seems logical to lump all of these losses into a single term. Then assume that these losses, all related to the shape and material of the alluvial boundary, may be represented by putting the losses proportional to

$$hc$$

where h is a constant of the channel and c is the concentration of the contaminant. This assumes that the temporary to permanent losses at any point along the channel are directly proportional to the concentration at that same section.

This assumption then extends the classical Fickian theory, which is usually solved to yield a Grussian concentration distribution from the equation (reference frame running with the mean velocity u_o)

$$\frac{\partial c}{\partial t} + K_x \frac{\partial^2 c}{\partial x^2} = 0,$$

to the case analogans to heat conduction in a wire which is losing heat along its length in proportion to the temperature at the cross-section in question.

This paper derives the equation and provides the classical solution adapted to the problem at hand. The equation derived provides

1) a means to estimate $c(x_i t)$ downstream from an instantaneous plane source.,

2) a means to estimate losses along the channel once the appropriate value of the parameter h has been determined for the channel in question. It is assumed that K_x is a constant.

Derivation of Partial Differential Equation.

Let c be the concentration of a certain marked material in a fluid at a given point in space and at a given time t such that $c = c(x\, y_1\, \zeta_1\, t)$ in the volume τ surrounded by the surface S . Then by the conservation of mass, dropping common terms,

Time rate of change of mass = Loss of mass + Inflow of Mass. (1)

Evaluating the term:

$$\text{Time rate of change of mass} = \frac{\partial}{\partial t}\int_\tau c\, d\tau = \int_\tau \frac{\partial c}{\partial t}\, d\tau$$

$$\text{Loss of mass} = -\int_\tau (hc)\, d\tau$$

$$\text{Inflow of mass} = -\int_S c\tilde{\nu}\,.\,\hat{n}ds = -\int_\tau D.\,(\widetilde{c\nu})\, d\tau$$

The term for the Loss of mass assumes that the material is lost from the system in direct proportion to the concentration in the elemental volume involved. The constant of proportionality is the positive constant h . This term is introduced to represent a loss of marked material such that when the final equation derived herein is applied to open channels material is lost at the boundaries of the channel as if the concentration in the cross-section was a constant and each elemental volume lost a like amount to the sink at the boundary.

Combining terms:

$$\int_{\tau} \left(\frac{\partial c}{\partial t}\right) = -\int_{\tau} \nabla \cdot (c\,\tilde{\nu})\,d\tau - \int_{\tau} hc\,d\tau$$

Therefore

$$\frac{\partial c}{\partial t} = -\nabla \cdot (c\,\tilde{\nu}) - hc \qquad (2)$$

Equation 2 is the basic conservation equation. The concentration c is the instantaneous concentration -- it is not very useful for practical applications.

If it is assumed that the concentration field and velocity field can be represented by

$$c = \overline{c} + c'$$

$$u_i = \overline{u}_i + u_i \quad \left(\tilde{\nu} = \sum_{i=1}^{3} u_i \hat{c}_i\right) \qquad (3)$$

Where the bar denotes a temporal mean and the prime an instantaneous fluctuation from this mean, then Equation 2 becomes

$$\frac{\partial c}{\partial t} + \frac{\partial c'}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[(\bar{c} + c')(\bar{u}_i + u_i') \right] = -h(\bar{c} + c') \qquad (4)$$

Applying Reynold's averaging rules

$$\overline{f + g} = \bar{f} + \bar{g}$$

$$\overline{af} = a\bar{f} \quad (a = \text{constant}) \qquad (5)$$

$$\overline{\bar{f}g} = \bar{f}\bar{g}$$

$$\overline{\text{Lim}\, f_r} = \text{Lim}\, \bar{f}_r \quad (f_r = \text{sequence of function})$$

Equation 4 becomes

$$\frac{\partial \bar{c}}{\partial t} + \sum_{i=1}^{3} \frac{\partial(\bar{c}\,\bar{u}_i)}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (-\overline{c' u'_i}) - h\bar{c} \qquad (6)$$

This equation must be further simplified. A common assumption is to introduce a semi-empirical coefficient such that

$$-\overline{c' u_i'} = K_i \frac{\partial \bar{c}}{\partial x_i}$$

Where the K_i and the diagonal components of a "diffusion tensor" -- implying a specific orientation of the coordinate direction x_i. Introducing this assumption into Equation 6 plus the further assumption that the K_i are constants, and dropping the mean bars,

$$\frac{\partial c}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(c\, u_i) = \sum_{i=1}^{3} K_i \frac{\partial^2 c}{\partial x_i^2} - h\,\bar{c} \qquad (7)$$

Equation 7 can be applied to the case of diffusion of a plane source in a steady uniform open channel flow where the mean velocity u_o is in the direction of the x- coordinate and the plane source extends over the entire cross-section of the channel transverse to the x- coordinate. If the coordinate system moves with the mean velocity u_o of the stream then

$$\frac{\partial c}{\partial t} = K_x \frac{\partial^2 c}{\partial x^2} - hc \qquad (8)$$

Problems for Instantaneous Plane Source.

Equation 8 can be used as the basic equation for seeking a theoretical solution for the diffusion of an instantaneous plane source in a straight uniform channel conveying a steady flow at a mean velocity

u_o. The problem can be formulated as follows for a coordinate system moving with the velocity u_o.

$$\frac{\partial c}{\partial t} = K_x \frac{\partial^2 c}{\partial x^2} - hc \qquad (-\infty < x < \infty,\ t > 0)$$

$$c\ (x_1 + 0) = Q \qquad (0 \leq x \leq \epsilon\ ,) \qquad (9)$$

$$= 0 \qquad (-\infty < x < 0\ ;\ \ \epsilon < x < \infty)$$

In Equation 9 K_x and h are positive constants and ϵ is an arbitrarily small constant. Q is the strength of the instantaneous source.

Rearranging Equation 9 and assuming $c\ (x_1 t) = \overline{X}(x)\ T\ (t)$

one has $$\frac{1}{K_x}\left(\frac{T'}{T} + h\right) = \frac{\overline{X}''}{\overline{X}} = -\alpha^2$$

Where α is a constant and the primes denote differentiation. The problem then becomes one of solving the ordinary differential equations.

$$T' + (h + K_x \alpha^2)\ T = 0$$

$$\overline{X}'' + \alpha^2 \overline{X} = 0$$

Particular solutions which are bounded for

all x and t for $c(x_1 t) = \overline{X}(x) \cdot T(t)$

are

$$e^{-ht} \; e^{-\alpha^2 K_x t} \cos \left[\alpha(x + ctc) \right]$$

In order to use the Fourier Integral, if the above are particular solutions, so is the function

$$\frac{e^{-ht}}{\pi} \; f(x') \; e^{-\alpha^2 K_x t} \cos \left[\alpha(x' - x) \right]$$

Where x' and α are independent of x and t . Then

$$c(x_1 t) = \frac{e^{-ht}}{\pi} \int_{o'}^{\infty} d\alpha \int_{-\infty}^{+\infty} f(x') \; e^{-\alpha^2 K_x t} \cos \alpha(x' - x) \; dx' \qquad (10)$$

Equation 10 can be integrated to provide

$$c(x_1 t) = \frac{e^{-ht}}{\sqrt{2\pi} \sqrt{2 K_x t}} \int_{-\infty}^{+\infty} f(x') \; e^{-1/2 \left(\frac{x - x'}{\sqrt{2 K_x t}} \right)^2} dx' \; (t \quad o) \qquad (11)$$

Applying the boundary condition and changing coordinate systems one with its origin at the point along the stream where the instantaneous plane source was introduced,

$$c(x_1 t) = \frac{Q\,e^{-ht}}{\sqrt{2\pi}\,\sqrt{2K_x t}}\; e^{-1/2\left(\frac{x - u_0 t}{\sqrt{2K_x t}}\right)^2} \qquad (t > o) \qquad (12)$$

Equation 12 is the solution we have been seeking. The only difference from previous published solutions applied to open channel is in the addition of the term representing material temporarily or permanently accumulated along the boundary of the channel, e^{-ht}.

Discussion of the Solution.

A.) Grussian form.

B.) At fixed x .

C.) Maximums.

D.) Losses along stream channel.

E.) Comparisons to California data.

Summary

Bibliography