TRANSIENT FLOWS THROUGH AN INFINITE

SATURATED AQUIFER OF ZERO SLOPE

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ABSTRACT

Prediction of the water table position and the amount of withdrawal where ground water is flowing from an aquifer to a reservoir has not been exact due, in part, to the difficulty of solving the nonlinear partial differential equation which describes the flow.

Approximate solutions obtained by two different methods for the nonlinear equation are presented. Both solutions give better agreement with experimental results than does the solution of the simplified linear equation for the flow.

The solution of one-dimensional boundary value problems in ground water flow has many interesting applications. For example, the increment of water added to bank storage when filling a reservoir or the decrement taken from bank storage

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when emptying a reservoir can be a considerable amount and its prediction is desirable. The same solution also applies to certain cases of transients caused by the digging of drains and to bank storage adjacent to streams that undergo changes in water level. This paper presents four approximate solutions for the problem. Solutions I and III are first approximations in which the effect of the drawdown on the area available for the flow of ground water is neglected. Solutions II and IV are second approximations which take the effect of drawdown into account.

The solutions presented in this paper are based upon the following assumptions:

1. The saturated aquifer is of infinite extent and overlays an impermeable layer of zero slope.
2. The Dupuit-Forchheimer assumptions hold.
3. The change in reservoir water level is instantaneous.
4. The aquifer is composed of an isotropic and homogeneous material.

Solution I

Consider the flow into a reservoir as shown in figure 1.

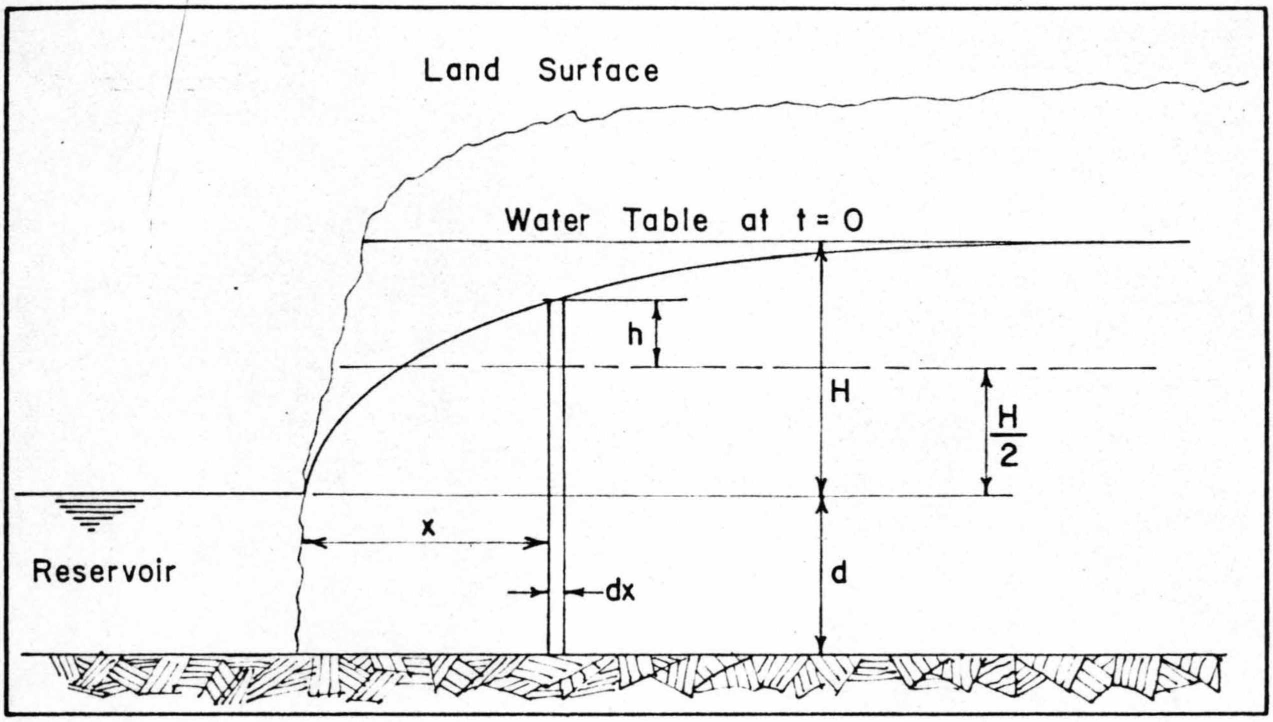

Fig. 1 Flow into a Reservoir

The flow through a cross section of unit width at the distance x from the origin is

$$Q = K\left(d + \frac{H}{2} + h\right)\frac{\partial h}{\partial x} \tag{1}$$

where: K is the permeability of the aquifer.

If we let

$$D = d + \frac{H}{2} \tag{2}$$

then

$$Q = K(D + h)\frac{\partial h}{\partial x} \tag{3}$$

The equation of continuity for the strip of width dx and time increment dt is

$$\frac{\partial Q}{\partial x} \, dx \, dt = V \frac{\partial h}{\partial t} \, dt \, dx \tag{4}$$

where: V is the specific yield of the aquifer.

Or, by use of equation (3)

$$K(D+h)\frac{\partial^2 h}{\partial x^2} + K\left(\frac{\partial h}{\partial x}\right)^2 = V\frac{\partial h}{\partial t} \tag{5}$$

By letting $\alpha = \frac{KD}{V}$, equation (5) becomes

$$\alpha\left(\frac{\partial^2 h}{\partial x^2}\right) + \frac{\alpha}{D}\left(\frac{\partial h}{\partial x}\right)^2 = \frac{\partial h}{\partial t} - \frac{\alpha}{D} h \frac{\partial^2 h}{\partial x^2} \tag{6}$$

This differential equation is nonlinear in form. If d is much larger than H, h can be discarded from equation (3). The continuity equation (4) then gives the following linear differential equation.

$$\alpha \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t} \tag{7}$$

A solution of this equation as presented by Glover[4] is

$$h_1 = -\frac{H}{2} + H\phi \tag{8}$$

where:

$$\phi = \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4\alpha t}} e^{-u^2} \, du \tag{9}$$

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[4] Glover, R. E. Personal communication. July 1958.

ϕ is the error function. The integral is tabulated in Reference 3.

This is solution I. When the drawdown is small compared to the thickness of the aquifer, Solution I is an approximate solution of equation (6). It satisfies the boundary and initial conditions:

for	$x = 0$	$t > 0$	$h_1 = -\frac{H}{2}$
	$x \to \infty$		$h_1 = \frac{H}{2}$
	$t = 0$	$x > 0$	$h_1 = \frac{H}{2}$

Solution II

A second approximation to the true solution of equation (6) may be obtained by applying the method of Picard (1). This method uses each successive approximation to approach a closer approximation. To obtain the second approximation, the non-linear terms of equation (6) are computed from the first approximate solution, equation (8), and substituted into equation (6) as known functions. The differential equation, as thus modified, is then solved again subject to the initial and boundary conditions. The process of solution requires that particular integrals of the known functions be found.

A particular integral p_1, for the term $-\frac{\alpha}{D}\left(\frac{\partial h_1}{\partial x}\right)^2$ is:

$$p_1 = -\frac{h_1^2}{2D} \tag{10}$$

Similarly a particular integral p_2, for the term $-\frac{\alpha}{D} h_1 \frac{\partial^2 h_1}{\partial x^2}$ is:

$$p_2 = -\frac{1}{2D}\frac{\partial h_1}{\partial x}\int h_1\,dx \tag{11}$$

The expanded form of this particular integral is:

$$p_2 = -\frac{H}{2}\Big[-\frac{H}{2D}\frac{2}{\sqrt{\pi}}\frac{x}{\sqrt{4\alpha t}}\,e^{-\frac{x^2}{4\alpha t}} + \frac{H}{D}\frac{2}{\pi}\,e^{-\frac{2x^2}{4\alpha t}}$$

$$+\frac{H}{D}\frac{2}{\sqrt{\pi}}\frac{x}{\sqrt{4\alpha t}}\,e^{-\frac{x^2}{4\alpha t}}\,\phi\Big] \tag{12}$$

An approximate solution of the modified form of differential equation (6) which satisfies the initial and boundary conditions is:

$$h_2 = p_1 + p_2 + q \tag{13}$$

This is solution II; where q is the first approximation altered to correct h_2 for the initial and boundary conditions.

$$q = -\frac{H}{2}\left[1-\left(\frac{H}{4D}+\frac{2}{\pi}\frac{H}{D}\right)\right]\left[1-\phi\right]+\frac{H}{2}\left[1+\frac{H}{4D}\right]\phi$$

A plot of equation (13) is shown in figure 2 with the dimensionless parameter h/H as ordinate and $x/\sqrt{4\alpha t}$ as abscissa. A complete drawdown is assumed so that the solution can be compared with experimental data. The solution for the linear case, equation (8), is also shown in figure 2.

Keller and Robinson (2) obtained measurements of the drawdown curve in a laboratory experiment in a sand filled flume. Water was drained from the sand through a 4-inch perforated pipe located at the floor of the flume. The sand used had values of $K = 0.034$ ft/sec, and $V = 25.5$ percent. The data

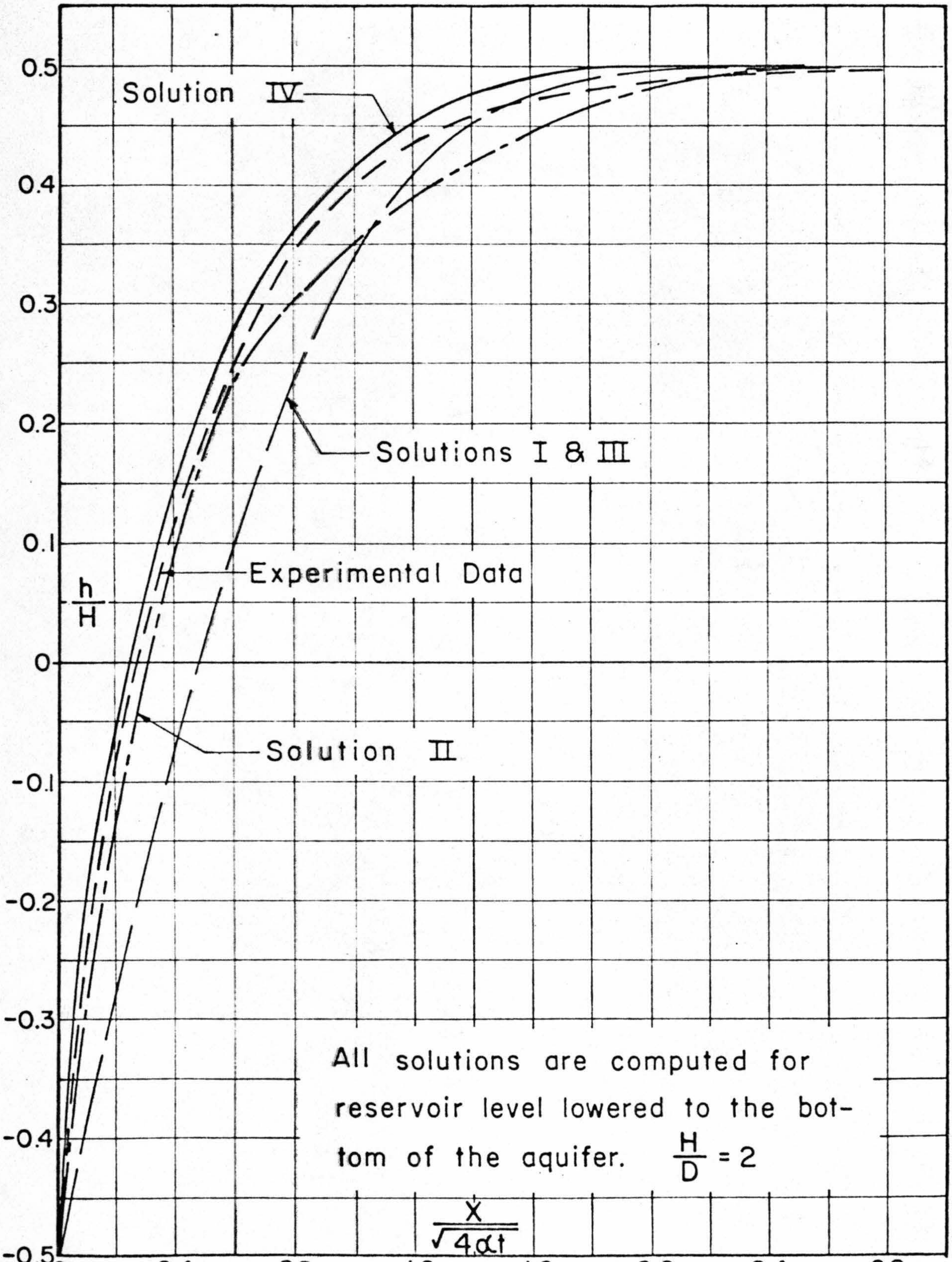

Fig. 2 Approximate Solutions of Ground Water Equations and Experimental Data

of Keller and Robinson are compared with the theoretical approximation, equation (13), in figure 2.

Solution III

An alternative second approximation for the bank storage case may be obtained by a method of equating flows. A first approximation is obtained as before and the flow Q is computed from it. This flow is then substituted into equation (3) and an improved solution is obtained by integrating the resulting expression. The first approximation will be solution III. It is obtained by solving the linearized differential equation

$$\alpha \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t} \tag{14}$$

subject to the conditions

$$\begin{array}{lll} x = 0 & t > 0 & h = 0 \\ t = 0 & x > 0 & h = H \\ x \to \infty & & h \to H \end{array}$$

The significance of the notation used in this case is shown in figure 3.

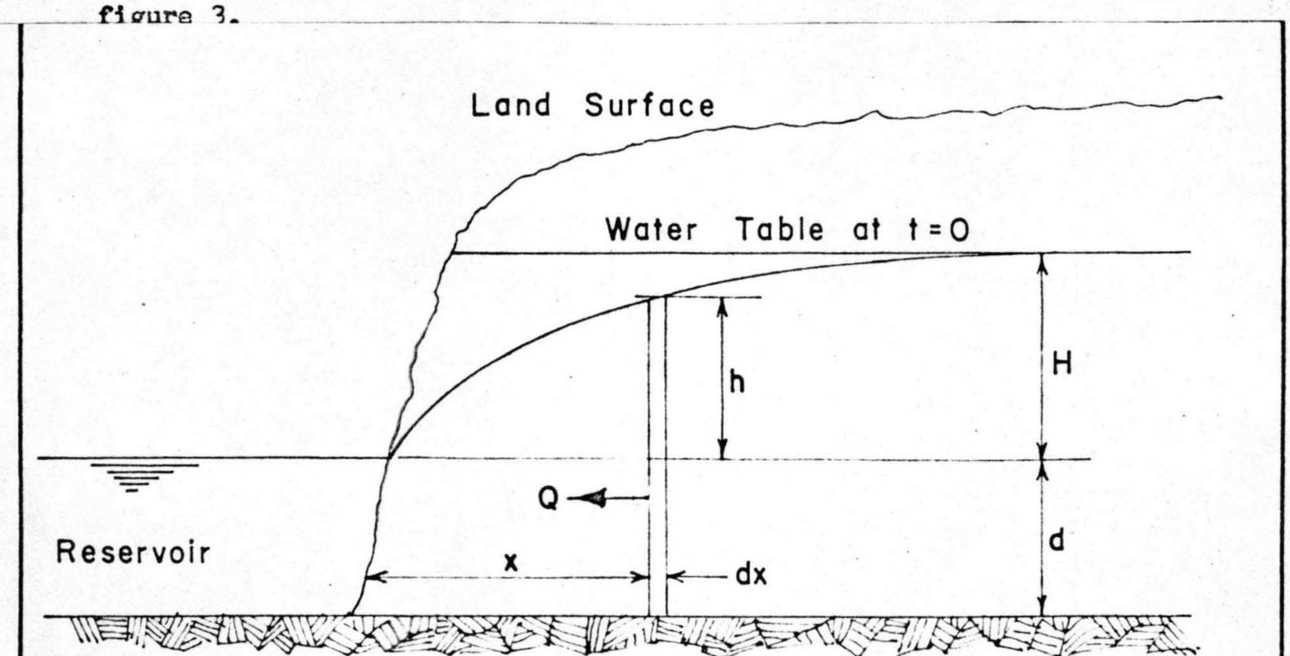

Fig. 3 Flow into a Reservoir

The required first approximation, solution III, is then

$$h_3 = H\phi \tag{15}$$

Solutions I and III satisfy the same equations and boundary conditions; only the reference point has changed. These solutions therefore are identical.

Solution IV

A first approximation to the flow Q_1 through a cross section of unit width at x is

$$Q_1 = KD\,\frac{\partial h_3}{\partial x} \tag{16}$$

where

$D = d + \frac{H}{2}$ and h_3 is given by equation (15).

If the drawdown is taken into account, the expression for flow is

$$Q = K(d+h)\,\frac{\partial h}{\partial x} \tag{17}$$

To obtain a closer approximation of h, equations (16) and (17) are arbitrarily equated:

$$K(d+h)\frac{\partial h}{\partial x} = KD\,\frac{\partial h_3}{\partial x} \tag{18}$$

An integration with respect to x yields

$$\frac{(d+h)^2}{2} = D h_3 + C_1 \tag{19}$$

The constant C_1 may be evaluated for the boundary condition that $h = 0$ when $x = 0$. The value so obtained is

$$C_1 = \frac{d^2}{2} \tag{20}$$

Then, by substitution

$$(d+h)^2 = d^2 + 2Dh_3 \tag{21}$$

Solution IV is obtained from this relation in the form:

$$h_4 = \sqrt{2Dh_3 + d^2} - d \tag{22}$$

This solution satisfies the initial condition $h = H$ when $t = 0$. If h_4 is measured from the same datum as h_2 so that they may be compared directly, equation (21) becomes:

$$h_4' = \sqrt{2Dh_3 + d^2} - d - 0.5H$$

For the case of a complete drawdown it takes the form:

$$h_4' = \sqrt{Hh_3} - 0.5H = H\sqrt{\phi} - 0.5H \tag{23}$$

This equation is shown on figure 2 for comparison with experimental data.

A comparison of these solutions shows that the second approximations II and IV agree better with the experimental data than do the first approximations I and III. Fortunately, the best agreement is obtained from solution IV, which is the easiest to use. The comparison is made here with the extreme case of complete drawdown.

All the solutions are approximations based on the assumption that d is very large compared to H. This assumption is least valid for the case of complete drawdown. Therefore, the results from using this solution with partial drawdowns should be good.

Solutions I through IV were derived for the case of inflow into a reservoir upon lowering of the reservoir water level. However, the same equations and solutions apply to the case of outflow if Q is considered to be negative for flow into the aquifer.

Computation of α

Curves similar to those of figure ~~3~~ 2 can be used to compute α for a given aquifer if values of h, H, d, x, and t have been measured in the field. From these values H/D can be computed and curves drawn for solution I or solution IV using h/H as ordinate and $x/\sqrt{4\alpha t}$ as abscissa. For a given h/H, α can be calculated from the measured value of $x/\sqrt{t}$ and the value of $x/\sqrt{4\alpha t}$ obtained from the curve.

Solutions I and IV were used to determine α by the above method for several of the points from Keller and Robinson's data. These values were compared with the value of α computed from laboratory measurements of K, D, and V. Table 1 indicates the percentage error in α as determined by these two solutions.

TABLE 1

Error in Determination of α

h/H	Solution I	Solution IV
	Percent	Percent
- 0.41	88	84
- 0.12	77	36
+ 0.06	60	24
+ 0.42	21	17

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