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Abstract

The Ramsey number $R(G, H)$ has been actively studied for the past 40 years, and it was determined for a large family of pairs (G, H) of graphs. The Ramsey number of paths was determined very early on, but surprisingly very little is known about the Ramsey number for the powers of paths. The r-th power P_n^r of a path on n vertices is obtained by joining any two vertices with distance at most r.

We determine the exact value of $R(P_n^2, P_n^2)$ for n large and discuss some related questions.

Keywords: Ramsey Theory, Regularity Method, Stability, Powers of graphs.

1 Introduction

For graphs G and H, the Ramsey number $R(G, H)$ is the least natural number N such that any colouring of the edges of the complete graph K_N on N vertices with blue and red contains either a blue copy of G or a red copy of H. One

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of the earliest sets of Ramsey numbers to be determined were those of paths for which Gerencsér and Gyárfás [3] proved the following.

Theorem 1.1 *For $n \geqslant m \geqslant 2$ we have $R(P_m, P_n) = n + \lfloor m/2 \rfloor - 1$.*

The r-th power of a path P_n^r is the graph with vertex set $[n] = \{1, ..., n\}$ in which ij forms an edge if and only if $|i-j| \leqslant r$. The *bandwidth* of an n-vertex graph G is the smallest integer k such that G is a subgraph of P_n^k. The Ramsey numbers of powers of paths were studied by Allen, Brightwell and Skokan in [2] while investigating the impact of bandwidth on the Ramsey numbers of graphs with bounded degree. In particular, they proved the following upper bound on Ramsey numbers of graphs with bounded degree and sublinear bandwidth.

Theorem 1.2 *Given $\Delta \geqslant 1$, there exist n_0 and β such that, whenever $n \geqslant n_0$ and G is an n-vertex graph with maximum degree at most Δ and bandwidth at most βn, we have $R(G, G) \leqslant (2\chi(G) + 4)n$.*

The proof of Theorem 1.2 requires good estimates for $R(P_n^k, P_n^k)$. For n a large multiple of $k+1$, Allen, Brightwell and Skokan obtained the bounds

$$(k+1)n - 2k \leqslant R(P_n^k, P_n^k) \leqslant \left(2k + 2 + \frac{2}{k+1}\right) n + o(n)$$

and conjectured the lower bound to be correct. They also conjectured the following strengthening of Theorem 1.2.

Conjecture 1.3 *For any $\Delta \geqslant 1$, there exist n_0, β and C such that whenever $n \geqslant n_0$ and G is an n-vertex graph with maximum degree at most Δ and bandwidth at most βn we have $R(G, G) \leqslant (\chi(G) + C)n$.*

It is suggested in [2] that a strong form of this conjecture may even be true: we can take $C = \varepsilon$ for any $\varepsilon > 0$. Determining $R(P_n^k, P_n^k)$ would be an important step towards proving Conjecture 1.3. For $k = 2$, we slightly alter the construction from [2], by adding an extra vertex, and then prove a matching upper bound to obtain the following theorem.

Theorem 1.4 *There exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$*

$$R(P_{3n}^2, P_{3n}^2) = 9n - 3\,.$$

A natural next question is whether we can use this result to improve Theorem 1.2, obtaining $R(G, G) \leqslant \big(3 + o(1)\big)n$ for the case $\chi(G) = 3$. This is work in progress.

It would be very interesting to find $R(P_n^k, P_n^k)$ for all k (and sufficiently large n). It seems reasonable to hope that if one can prove an upper bound of $\big(k+1+o(1)\big)n$ then one could go further and prove the strong form of Conjecture 1.3 mentioned above. One attractive consequence of this conjecture is, using the fact that bounded degree planar graphs are four-colourable and have sublinear bandwidth, that given any $\Delta \geqslant 1$, if H is a planar graph with maximum degree Δ then $R(H,H) \leqslant \big(4+o(1)\big)v(H)$. This is asymptotically best possible as $R(P_n^3, P_n^3) \geqslant 4n-8$.

2 Construction

We now give a lower bound construction for $R(P_{3n}^2, P_{3n}^2)$. For sets of vertices X and Y we denote by $E(X,Y)$ the set of edges with one end in X and the other in Y. We also set $E(X) := E(X,X)$.

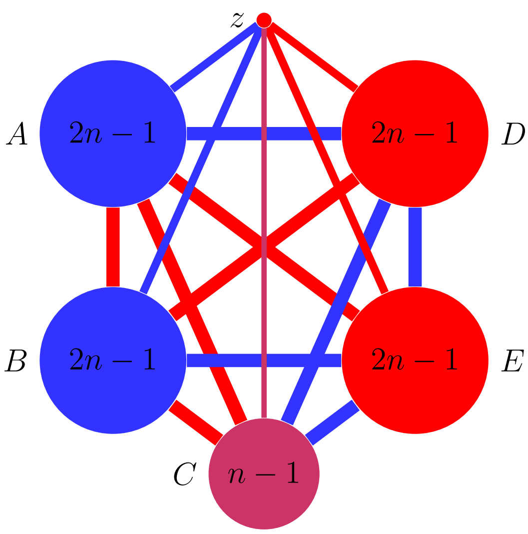

Fig. 1. Lower Bound Construction

Partition the vertices of K_{9n-4} into sets A, B, D, E, each of size $2n-1$, a set C of size $n-1$ and a single vertex z. Colour all edges of $E(A\cup\{z\}) \cup E(B\cup\{z\}) \cup E(A,D) \cup E(B,E) \cup E(C,D) \cup E(C,E) \cup E(D,E)$ by blue. Colour all edges in $E(D \cup \{z\}) \cup E(E \cup \{z\}) \cup E(A,E) \cup E(B,D) \cup E(A,C) \cup E(B,C) \cup E(A,B)$ by red. Colour edges in $E(C \cup \{z\})$ arbitrarily. The addition of the vertex z is the only difference from the construction given in [2].

3 Methods

In this section we outline the main methods of the proof of Theorem 1.4.

3.1 The Regularity Lemma

Szemerédi's regularity lemma [4] states that graphs of positive edge density can be partitioned into finitely many parts such that the edges between almost all pairs of parts are evenly distributed. We use a version that can be applied to 2-coloured graphs to say that between almost all pairs the edges are evenly distributed in both colours. First we give a definition.

Definition 3.1 Given a real number $\varepsilon > 0$ and a graph $G = (V, E)$ we say a pair (A, B) of disjoint vertex sets is *ε-regular* (with respect to G) if any $A' \subseteq A$, $B' \subseteq B$ satisfying $|A'| \geqslant \varepsilon|A|$, $|B'| \geqslant \varepsilon|B|$ also satisfy

$$|d(A', B') - d(A, B)| \leqslant \varepsilon$$

where $d(X, Y) := e(X, Y)/(|X||Y|)$ denotes the density of the pair (X, Y).

Lemma 3.2 (The Regularity Lemma [4]) *For any $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there exist $n_0, t_1 \geqslant t_0$ such that any 2-coloured complete graph G on $N \geqslant n_0$ vertices has a partition into $t + 1$ classes $V = V_0 \cup V_1 \cup \cdots \cup V_t$ satisfying:*

(i) $t_0 \leqslant t \leqslant t_1$.

(ii) $|V_0| \leqslant \varepsilon|V|$, $|V_1| = |V_2| = \ldots = |V_t|$.

(iii) *All but at most $\varepsilon\binom{t}{2}$ of the pairs (V_i, V_j), $1 \leqslant i < j \leqslant t$ are ε-regular in both colours.*

Given a 2-coloured complete graph G and an ε-regular partition of G we can define an auxiliary 2-coloured graph R called the ε-reduced graph in which the vertex set of R is $\{1, ..., t\}$ and ij form an edge if (V_i, V_j) is ε-regular in both colours. We colour edges ij of R with the majority colour of (V_i, V_j).

3.2 Embedding Lemma

We next introduce a notion of connectivity for monochromatic triangles in a 2-coloured graph. Given a graph G, we say that two blue triangles T and T' are *blue-triangle-connected* if there exists a sequence of blue triangles of the form $T = T_1, ..., T_\ell = T'$ such that consecutive triangles in $T_1, ..., T_\ell$ share an edge. We call a set of vertex disjoint blue triangles which are pairwise blue-triangle-connected to each other a *blue triangle-connected-triangle-factor* or *blue TCTF* for short, and we define a red TCTF similarly. The following lemma, which is a direct consequence of Lemma 8 from [1], shows that if we find a monochromatic TCTF on $(1 + \delta)\lambda t$ vertices in our reduced graph R then we also have a monochromatic square of a path on $3\lambda N$ vertices in G.

Lemma 3.3 (Embedding Lemma) *For all $0 < \delta, \lambda < 1$ there exist $\varepsilon > 0$, $m, n_0 \in \mathbb{N}$ such that if G is a 2-coloured K_N with $N \geqslant n_0$ and R is an*

ε-reduced graph of G with $|R| = t \leqslant m$ and R contains a monochromatic TCTF on $(1+\delta)\lambda t$ triangles then G has a monochromatic $P^2_{3\lambda N}$.

3.3 Stability Lemma

The main part of our method is proving a stability lemma, which states that very dense 2-coloured graphs without large monochromatic TCTF's look a lot like our lower bound construction.

Lemma 3.4 (Stability Lemma) *For any $\delta, \gamma > 0$ there exist $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ such that if G is a 2-coloured graph on $t \geqslant t_0$ vertices and G has at least $(1-\varepsilon)\binom{t}{2}$ edges then either G contains a monochromatic TCTF on at least $\frac{1}{9}(1+\delta)t$ triangles, or G can be turned into the lower bound construction by adding the missing edges and changing the colour of at most γt^2 edges.*

3.4 Sketch Proof of Theorem 1.4

Given an arbitrarily 2-coloured graph G on $N = 9n-3$ vertices we apply the regularity lemma to obtain a 2-coloured reduced graph R on t vertices. Applying the embedding lemma, Lemma 3.3, with $\lambda = \frac{1}{9}$, we see that if R contains a monochromatic TCTF with $\frac{1}{9}(1+\delta)t$ triangles then G has a monochromatic P^2_{3n}. However if R has no such TCTF, by the stability lemma it is of a very similar form to our lower bound construction. This in turn means G resembles the lower bound construction. We finally show, without using the Regularity Lemma, that any 2-coloured K_N that is close to the lower bound construction contains a monochromatic copy of P^2_{3n}.

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