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A caching game with infinitely divisible hidden material

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Abstract

We consider a *caching game* in which a unit amount of infinitely divisible material is distributed among $n \geq 2$ locations. A Searcher chooses how to distribute his search effort r about the locations so as to maximize the probability she will find a given minimum amount $\bar{m} = 1 - m \leq r$ of the material. If the search effort y_i invested by the Searcher in a given location i is at least as great as the amount of material x_i located there she finds all of it, otherwise the amount she finds is only y_i. In other words she finds $\min\{x_i, y_i\}$ in location i. We seek the randomized distribution of search effort that maximizes the probability of success for the Searcher in the worst case, hence we model the problem as a zero-sum win-lose game between the Searcher and a malevolent Hider who wishes to keep more than m of the material. We show that in the case $r = \bar{m}$ the game has a geometric interpretation that for $n = 2$ corresponds to a problem posed by W. H. Ruckle in his monograph, *Geometric Games and Their Applications*. We give solutions for the geometric game when $n = 3$ for certain values of m, and bounds on the value for other values of m. In the more general case $r \geq \bar{m}$ we show that for $n = 2$ the game reduces to Ruckle's game.

A caching game with infinitely divisible hidden material

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1 Introduction

Suppose some material is hidden, or *cached*, in a finite number of locations. The material may be food hidden by an animal or arms hidden by a terrorist organization. A *Searcher* (a pilferer or the military, respectively) looks in some of the locations and confiscates the material that she finds. The amount of effort the Searcher can invest in searching is limited by a fixed resource constraint, and her aim is to capture a certain minimum amount of the material. The problem is modeled as a zero-sum game between the Searcher and a *Hider*, which may be the squirrel or a terrorist organisation. The Hider wins if the Searcher captures less than a certain given amount of material, so in the case of the squirrel, it has enough food to survive the winter, or in the case of the terrorist organisation, it has enough arms to carry out a planned terrorist attack.

Games of this type were introduced by Kikuta and Ruckle ([16], [17] and [18]) who called them *accumulation games*, and have been furthered studied for example in [2] and [4]. In the most general definition of accumulation games the Hider accumulates resources in stages over several time periods, but in practice most of the work in this area has focused on single stage games, which have come to be called *caching games*. Usually in such accumulation games, it is assumed that when a Searcher looks in a location she confiscates all the material hidden there. However, in this paper we assume that the amount of material found in a location depends not only on how much has been hidden there, but also on how much effort the Searcher invests in searching the location. This is a natural assumption, and generalizes the Kikuta-Ruckle model.

We will define the model formally in Section 2, but we first illustrate it with an example. Suppose the Hider has an amount of material which we normalize to 1. He distributes it about three locations (locations 1, 2 and 3); that is, he chooses a vector $x = (x_1, x_2, x_3)$ of non-negative numbers summing to 1. On the left of Figure 1 we illustrate the choice of vector $x = (0.5, 0.25, 0.25)$ with dotted lines of corresponding lengths. The Searcher then searches for the material using a total 'search effort' of $r = 0.7$, which we think of as the maximum amount of material the Searcher can find. The Hider wins if he if left with more than $m = 0.5$ of the material; the Searcher wins if she finds at least $\bar{m} = 1 - m = 0.5$. A strategy for the Searcher is some vector $y = (y_1, y_2, y_3)$ of non-negative numbers summing to at most r, which indicates the amount of material she looks for in each location. On the right of Figure 1 we illustrate the choice of $y = (0.3, 0, 0.4)$ with the thickened lines at locations 1 and 3.

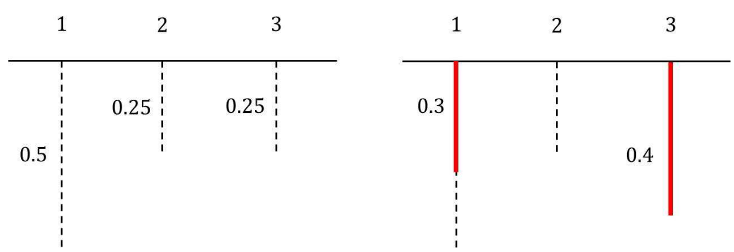

Figure 1. An example of the players' strategies.

The amount found by the Searcher in each location depends on both the amount hidden there, and the effort she allocates to that location. Thus in location 1, she looks for an amount of 0.3, and since there is at least this much material this is the amount she finds. In location 2 she looks for no material, so she does not find any, and in location 3 she looks for 0.4, but since the amount of material there is only 0.25, this is all she finds. In other words, the amount of material she finds in a location is the minimum of the amount she looks for and the amount that is hidden there. So in total she finds $0.3+0+0.25 = 0.55 \geq \bar{m}$, and thus wins the game.

We offer another interpretation of the model, which is a variant on the classic newsvendor problem [8]. Suppose a newsvendor sells newspapers in a number of locations. The total demand for newspapers is known but what is unknown is how the demand is distributed amongst the locations. The newsvendor has a fixed number of newspapers she wants to sell and has to decide how to distribute them amongst the locations. She has a target for the number of newspapers she wants to sell and wishes to maximize the probability she reaches that target. We think of this as a game against Nature, where the malevolent Nature (the Hider) chooses how the demand is split between the various locations. The newsvendor (the Searcher) decides how to allocate the newspapers to the locations; if she allocates more newspapers than the demand then they will be wasted and if she allocates too few she will miss out on the opportunity of sales. Thus her total sales in any location are equal to the minimum of the demand and the number of newspapers she allocates to the location.

This example clearly departs from the classic newsvendor model in several ways, in part because we assume the newsvendor wishes to maximize the probability of reaching some target rather than to maximize her total expected revenue. In some circumstances this choice of utility function for a business may be justified, as argued in [11]. Indeed, in [15] the authors consider a newsvendor model with a target profit. We might imagine that the newsvendor has been hired for a trial period and will be given a permanent job if she reaches some sales target. The idea of modeling several locations for the newsvendor is also not without precedent, for example in [19].

The idea that the amount of material the Searcher captures in a location should depend on her search effort stems from the caching games studied in [3] and [5], in which a Hider caches a number of discrete objects about various locations. For example, a *scatter hoarder* such as squirrel buries nuts and a pilferer digs holes in each location with the aim of finding the nuts. Thus the number of nuts found in a given location depends on both the number of nuts located there and the depth of the hole dug by the Searcher. The difference in our model is of course that the material being hidden is infinitely divisible, not discrete, which is closer in spirit to Kikuta and Ruckle's original accumulation games.

This work fits more widely into the sphere of search games. Good accounts of the main results in this field can be found in [7] and [14].

After defining our model precisely in Section 2, we go on to show how the game can be viewed as a geometric game, as studied in [22]. See [10] for an interesting discussion of some geometric games of Ruckle's that have not received much attention. In Section 3 we will see that in the case that the Searcher can only win by finding *all* the material she is searching for (so $r = \bar{m}$) and the number of locations, $n = 2$, the game is equivalent to a form of Ruckle's *Interval Hider Game*. For $n > 2$, our game generalizes Ruckle's, and we find that solving the game is far from trivial, but we give some solutions and bounds for the value when $n = 3$. Finally, in Section 4, we go on to give a full solution of our game for $n = 2$ and any value of r by reducing it to the case $r = \bar{m}$.

2 Game definition and geometric interpretation

We formalize the game as follows. A Hider distributes some material of total mass 1 about $n \geq 2$ different locations. That is, he chooses an n-vector $x = (x_1, ..., x_n)$ of non-negative numbers with $\sum_{i=1}^{n} x_i = 1$. A Searcher confiscates some of the material, but has enough resources to search for a total amount of material equal to some r. That is, she chooses an n-vector $y = (y_1, ..., y_n)$ of non-negative numbers with $\sum_{i=1}^{n} y_i \leq r$. The amount of material the Searcher captures is $M(x, y) = \sum_{i=1}^{n} \min\{x_i, y_i\}$. The Hider wins if he is left with total material greater than m, where $0 < m < 1$. Or equivalently, the Hider loses and the Searcher wins if the amount of material she finds satisfies $M(x, y) \geq \bar{m} := 1 - m$, so we assume that $r \geq \bar{m}$, otherwise the Hider trivially wins the game. This is a zero-sum game, with payoff $P(x, y) = 1$ if the Searcher wins and $P(x, y) = 0$ if the Hider wins. We denote the game by $\Gamma = \Gamma(n, m, r)$.

The Hider pure strategy set is the regular unit $(n-1)$-simplex $\Delta^{n-1} = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1\}$ and the Searcher strategy set is the convex hull of the zero vector and the regular $(n-1)$-simplex $r\Delta^{n-1}$ (We use the standard term *regular simplex* to mean a simplex whose sides all have the same length.) Since the strategy sets are infinite, we cannot use the standard minimax theorem for finite zero-sum games to establish the existence of a value. However, both sets strategy sets are compact and it is easy to see that the payoff function $P(x, y)$ is upper semicontinuous in both x and y: if $P(x, y) = 1$ then P is certainly upper semicontinuous at (x, y) since P can only decrease. On the other hand, if $P(x, y) = 0$ then $M(x, y) < \bar{m}$ and any (x', y') within a radius of $\varepsilon = \bar{m} - M(x, y)$ of (x, y) must also satisfy $M(x', y') < \bar{m}$ and hence $P(x', y') = 0$. Hence, the existence theorem below follows from Alpern and Gal's minimax theorem for zero-sum games [6].

Theorem 1 *The game Γ has a value. The Searcher has optimal strategies and the Hider has ε-optimal strategies.*

We denote the value by $V = V(n, m, r)$. Although the theorem guarantees only ε-optimal strategies for the Hider, we will see that for most of the cases we consider the Hider has optimal strategies, as is commonly the case in the search games literature. However in one case we will give only ε-optimal strategies for the Hider (see Proposition 9).

Every Searcher strategy is dominated by some $y \in r\Delta^{n-1}$, so in practice the Searcher always has an optimal strategy with support in $r\Delta^{n-1}$, and in general we will assume the Searcher only uses such strategies. For any Searcher pure strategy $y \in r\Delta^{n-1}$, we can consider the set $S(y)$ of all Hider pure strategies that it beats. That, is

$$S(y) = S_m(y) = \left\{x \in \Delta^{n-1} : P(x, y) = 1\right\} = \left\{x \in \Delta^{n-1} : M(x, y) \geq m\right\}.$$

It is easy to verify that the set $S(y)$ is closed and convex, and we denote the set of all such $S(y)$ by $\mathcal{S} = \mathcal{S}(n, m, r) = \left\{S(y) : y \in r\Delta^{n-1}\right\}$. So now we can think of the game as follows: the Hider picks a point x in the simplex Δ^{n-1}, and the Searcher selects some subset of the simplex from $\mathcal{S}$. The Searcher wins if and only if the subset contains x. The game has been transformed into a geometric game in which the Searcher is trying to 'capture' the Hider's point.

It is far from clear what the sets in $\mathcal{S}$ look like, but we will see in the next section that they have a nice structure when $r = \bar{m}$.

2.1 Related literature

The geometric interpretation of the game can be considered as a special case of a more general game in which a Hider picks a point in a set X and a Searcher picks one of a collection $\mathcal{S}$ of subsets of X whose union is the whole of X. The Searcher wins if and only if the Hider's point is contained in her subset. Such a game is considered in the context of information theory in [20] and

[21]. In particular, the latter work considers this game in the case that X is a compact metric space and the Searcher can pick any closed ball of radius r. The authors prove a relation between the value of a restricted form of the game and the absolute r-entropy, which can be computed from the covering number of Cartesian powers of X. The authors restrict themselves to the case where S consists of closed balls of a given radius, so their results do not immediate apply to our game. In information theory, solutions to such games are difficult to compute even for small parameters, as we also find in this paper. See [1] for some open problems.

The game also has a relation to packing and covering problems. One form of mixed strategy available to the Searcher in our game is an equiprobable choice between some finite collection $\mathcal{C}$ of sets $S(y)$ which cover the space Δ^{n-1}. If $\mathcal{C}$ has size k then this strategy guarantees a payoff of at least $1/k$ for the Searcher, and finding such a collection with minimal k is a covering problem. In a similar way, the Hider may try to find a strategy in which he makes an equiprobable choice between a collection of points in Δ^{n-1} at most one of which is beaten by any given pure strategy of the Searcher. Choosing a minimal such collection of points can be transformed into a packing problem.

A similar packing problem on the square is studied in a classic paper of Erdös and Graham [13]. A covering problem for squares in considered in [23] and a covering problem for equilateral triangles in [12]. The concept of the *weighted covering number* of a convex set was recently introduced [9]. This generalizes the idea of the covering number of a set and corresponds to general mixed Searcher strategies in our game in a natural way.

3 The case $r = \bar{m}$

We begin by analyzing the case where the amount of effort, r that the Searcher can use is equal to the minimum amount of material, $\bar{m}$ she must find to win the game. In other words, in order to win she has to find all the material she is searching for, so that the game we are considering is $\Gamma(n, m, \bar{m})$. We make this assumption throughout the rest of the section. It follows that the Searcher wins if and only if $y_i \leq x_i$ for $i = 1, ..., n$.

It is clear that the Hider can always guarantee that the value is at least $1/n$ by choosing randomly from the unit coordinate vectors, which we denote by $e_1, ..., e_n$. The only Searcher pure strategies that can win against this Hider strategy with positive probability must be of the form $\bar{m}e_i$ for some $i = 1, .., n$, and such a strategy wins with probability $1/n$. If the Searcher chooses randomly between the vectors $\bar{m}e_i$ then she will win with probability at least $1/n$ as long as she can be sure that there is at least $\bar{m}$ of material in at least one location. This is true if the average amount of material $1/n$ in each location is at least $\bar{m}$, so we have the following.

Proposition 2 *If $\bar{m} \leq 1/n$ then $V = 1/n$. It is optimal for the Hider to hide all the material in one randomly chosen location, and for the Searcher to look for an amount $\bar{m}$ in one randomly chosen location.*

To analyze the game in this special case $r = \bar{m}$, it is helpful to observe that the set of Searcher strategies $\mathcal{S}$ has a special structure: $\mathcal{S}$ consists of precisely all transformations of the regular simplex $m\Delta^{n-1}$ that are contained in Δ^{n-1}.

Lemma 3 *The set of Searcher strategies $\mathcal{S}$ is given by*

$$\mathcal{S} = \left\{T \subset \Delta^{n-1} : T = y + m\Delta^{n-1} \text{ for some } y \in \mathbb{R}^n\right\}.$$

Proof. Suppose $S(y) \in \mathcal{S}$. We will show that $S(y) = y + m\Delta^{n-1}$. By definition of $S(y)$,

$$\begin{aligned} S(y) &= \left\{x \in \Delta^{n-1} : x - y \geq 0\right\} \\ &= \left\{y + z \in \Delta^{n-1} : z \geq 0\right\}, \text{ writing } z = x - y, \\ &= \left\{y + z : \sum_{i=1}^{n} z_i = m,\, z \geq 0\right\} \text{ (since } \sum_{i=1}^{n} y_i = \bar{m}) \\ &= y + m\Delta^{n-1}. \end{aligned}$$

Conversely, suppose $T = y + m\Delta^{n-1}$ is contained in Δ^{n-1} for some $y \in \mathbb{R}^n$. We will show that y must be a valid Searcher strategy in $\bar{m}\Delta^{n-1}$, so that $T = S(y) \in \mathcal{S}$. Let x be some element of T. We write x as $x = y + mz$ with $z \in \Delta^{n-1}$, and note that since T is a subset of Δ^{n-1}, the coordinates of x must sum to 1. So

$$\begin{aligned} \sum_{i=1}^{n+1} y_i + (1-m)\sum_{i=1}^{n+1} z_i &= 1, \text{ or} \\ \sum_{i=1}^{n+1} y_i &= \bar{m}, \text{ since } z \in \Delta^{n-1}. \end{aligned}$$

We must also show that $y_i \geq 0$ for all i. Suppose some $y_i < 0$. Then choose some $j \neq i$ and consider the unit vector in the j direction, $e_j \in \Delta^{n-1}$. Then the vector $(1-m)\, e_j + y$ is contained in S, and therefore in Δ^{n-1}, but has ith coordinate of $y_i < 0$, a contradiction. So $y_i \geq 0$ for all i and $y \in \bar{m}\Delta^{n-1}$. ■

The significance of Lemma 3 is that the game Γ can be considered as a geometric game played on the unit regular $(n-1)$-simplex, where the Hider picks a point x in Δ^{n-1} and the Searcher picks a regular simplex $S(y) = y + m\Delta^{n-1}$ contained in Δ^{n-1}; the Searcher wins if and only if $x \in S(y)$.

We remark that the value of the game is increasing in m, since $S_m(y)$ is increasing in m with respect to set inclusion. Suppose for some m, a given Searcher mixed strategy guarantees a payoff of V against any Hider strategy in the game $\Gamma(n, m, \bar{m})$. Then for $m' > m$, the same Searcher mixed strategy is also a valid strategy in $\Gamma(n, m', \bar{m}')$ and guarantees the same payoff of V against any Hider strategy.

3.1 Ruckle's Interval Hider Game

For $n = 2$, our game translates to the following. The Hider picks a point x in the interval $[0, 1]$, the Searcher picks a subinterval of $[0, 1]$ of length m and the Searcher wins if and only if the Searcher's subinterval contains the Hider's point. This is an example of Ruckle's *Interval Hider Game* [22]. In the most general version of Ruckle's game the Hider picks not a point but a subinterval of given length and wins if the two subintervals have empty intersection.

The solution of the game in its full generality is given in [22], but we present the solution to our particular version of the game for completeness (and in fact we present slightly different optimal strategies to those given by Ruckle). In particular, the proof is a good 'warm-up' for the proofs found in the 2-dimensional version of the game (when $n = 3$).

Proposition 4 *Suppose* $n = 2$ *and* $1/(t+1) \le m < 1/t$ *for a positive integer* t. *Then the value* V *of the game* $\Gamma(n, m, \bar{m})$ *is* $1/(t+1) = \lceil 1/m \rceil$. *It is optimal for the Hider to choose equiprobably between the* $t+1$ *equally spaced points starting at* 0 *and ending at* 1. *It is optimal for the Searcher to choose the lower end of her interval equiprobabably between the* $t+1$ *equally spaced points starting at* 0 *and ending at* $1-m$.

Proof. If the Hider uses the strategy given, then since the distance between the adjacent points from which he chooses is $1/t > m$, no interval of length m can contain more than one of the points. So the Searcher can win with probability no more than $1/(t+1)$, and $V \le 1/(t+1)$.

If the Searcher uses the strategy given, since $m \ge 1/(t+1)$, it is easy to see that the $t+1$ intervals from which she chooses cover the whole interval $[0, 1]$, so whichever point the Hider chooses the Searcher finds him with probability at least $1/(t+1)$. Hence $V \ge 1/(t+1)$, and we must have equality. ■

3.2 A Triangle Hider Game

In this section we turn to the game played with $n = 3$, still assuming that $r = \bar{m}$. By Lemma 3, the game can be considered as follows. The Hider picks a point on the unit equilateral triangle Δ^2 and the Searcher picks a equilateral triangle of edge length m in Δ^2 (of the same orientation). The Searcher wins if and only if her triangle contains the Hider's point.

We already know from Proposition 2 that if $m \ge 2/3$ then the value of the game is $1/3$. The Hider's optimal strategy can be interpreted as as random choice between the vertices of Δ^2 and the Searcher optimal strategy can be interpreted as a random choice between the triangles $S_{\bar{m}e_i}$ (for $i = 1, 2, 3$). These three triangles, which are the only triangles contained in Δ^2 that contain the vertices of Δ^2 are depicted in Figure 2. It is clear from the figure that if $m \ge 2/3$, they cover the whole of Δ^2. We will refer to these triangles as the *vertex triangles*, and denote them as $E_i = E_i(m)$ for $i = 1, 2, 3$.

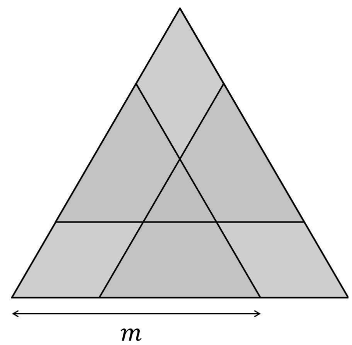

Figure 2. The vertex triangles, E_i.

We will see that when $m = 1/2$, the value of the game is $1/6$. The optimal Searcher strategy is simple, and is given in Lemma 5, and later we will give ε-optimal Hider strategies which are less straightforward to describe.

First a comment on our use of notation. Suppose $y^1, ..., y^k$ are Searcher pure strategies, and $T_j = S(y^j)$ for $j = 1, ..., k$. Then for the Searcher strategy $y = \sum \alpha_i y^i$ we will use a slight abuse of notation to write $\sum_{j=1}^k \alpha_j T^j$ for $S(y)$.

Lemma 5 *If $n = 3$ and $m = 1/2$, the value of the game is at least $1/6$. The Searcher can guarantee an expected payoff of $1/6$ by choosing a random vertex triangle E_i with probability $1/2$ and choosing the midpoint $(E_i + E_j)/2$ of two randomly chosen vertex triangles $E_i \neq E_j$ with probability $1/2$.*

Proof. To show the Searcher strategy wins with probability $1/6$ it is sufficient to show that every point $x \in \Delta^2$ is covered by one of the Searcher's 6 pure strategies. This is clear from drawing the 6 triangles corresponding to the Searcher's 6 pure strategies, and observing that their intersection covers the whole of Δ^2, as shown in Figure 3.

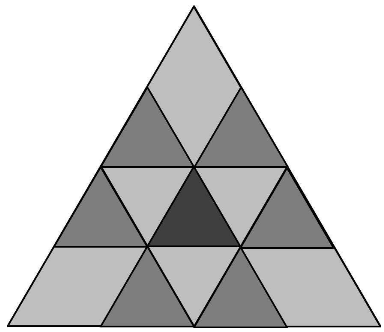

Figure 3. The Searcher's optimal strategy for $m = 1/2$.

■

For the rest of the values of m between $1/2$ and $2/3$, the solution of the game is not straightforward. In the same way that the solution of Ruckle's Interval Hider Game broke down into discrete cases, so does the 2 dimensional version of the game. We start by giving the solution for the interval $4/7 \leq m < 2/3$.

Proposition 6 *If $n = 3$ and $4/7 \leq m < 2/3$ then it is optimal for the Hider to choose with equal probability one of the three vertices of Δ^2 or the vector $(1/3, 1/3, 1/3)$ in the center of Δ^2. It is optimal for the Searcher to choose with equal probability one of the vertex triangles E_i or the triangle $(E_1 + E_2 + E_3)/3$ in the 'centre' of Δ^2. The value of the game is $1/4$.*

Proof. In Figure 4, the four Hider strategies are depicted on the left by the four small black circles and the four Searcher strategies are depicted on the right.

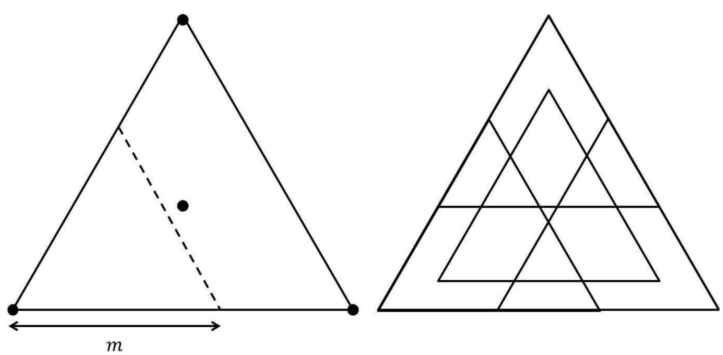

Figure 4. The optimal strategies for $4/7 \leq m < 2/3$.

It is clear from the left of Figure 4 that any Searcher strategy can only win against one Hider pure strategy, since $m < 2/3$ so any $S(y)$ of side length m can contain only one of the Hider's pure strategies. This shows that the value is at least $1/4$.

Conversely, we need to show that every point $x \in \Delta^2$ is covered by at least one of the given Searcher pure strategies. We can see informally from the right hand side of Figure 4 that this is indeed the case. More precisely, we first suppose that some coordinate i of x is at least $3/7$. Then x is covered by E_i. Now suppose all the coordinates of x are at most $3/7$, so that all the coordinates of x must be at least $1/7$. We note that $(E_1 + E_2 + E_3)/3 = S(y)$, where $y = (\bar{m}/3, \bar{m}/3, \bar{m}/3)$, and since $\bar{m} \leq 3/7$, all the coordinates of x are at least $\bar{m}/3$. So every x is covered by one of the 4 Searcher pure strategies, and the value is at most $1/4$. Combining this with the previous paragraph shows that the value is exactly $1/4$. $\blacksquare$

In order to solve the game for the remaining values of m between $1/2$ and $4/7$, we need an elementary lemma about triangles.

Lemma 7 *Suppose T and T' are equilateral triangles of edge lengths s and $s' \leq s$ where T' can be obtained from T by a scaling and rotation of π. Then the maximum length of the intersection of T with the edges of T' (given by the dotted lines in the diagram on the left of Figure 5) is $2s - s'$.*

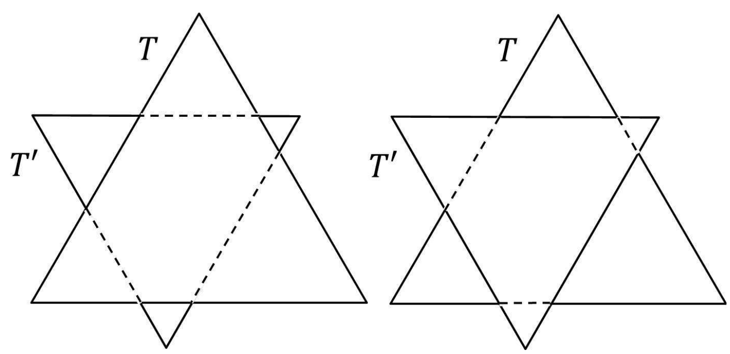

Figure 5. The triangles T and T'.

Proof. Let l_1 be the total length of the intersection of T with the edges of T' and let l_2 be the length of the intersection of T' with the sides of T, as shown by the dotted lines in the diagrams to the left and right of Figure 5, respectively. We can assume that both of these intersections are made up of three disconnected segments of positive length, as depicted in the figure, otherwise it is clear that l_1 could be increased by some translation of T in a direction parallel to one of its edges. Then it is easy to see that the length of the perimeter of T is $2l_1 + l_2$ and the length of the perimeter of T' is $2l_1 + l_2$, so

$$3s = 2l_1 + l_2 \text{ and } 3s' = 2l_2 + l_1, \text{ hence } l_1 = 2s - s'.$$

■

We now give a solution to the game for all m with $1/2 < m < 4/7$. Every such m lies in an interval $I_t := \left[\frac{2(t+1)}{4(t+1)-1}, \frac{2t}{4t-1}\right)$ for some $t = 2, 3, \ldots$, and the solution of the game depends on which of these intervals m belongs to. For a given interval I_t it turns out to be optimal for the Hider to mix between the vertices of Δ^2 and a finite subset of points on the perimeter of a particular triangle. The vertices of this triangle, which we denote by T_t are the unique three points at which the edges of the vertex triangles $E_i\left(2t/\left(4t-1\right)\right)$ intersect, as depicted in Figure 6. Thus the coordinates of the vertices of T_t are $\left(\frac{2t-1}{4t-1}, \frac{2t-1}{4t-1}, \frac{1}{4t-1}\right)$ and all permutations thereof; the edges of T_t have length $\left(2t-2\right)/\left(4t-1\right)$.

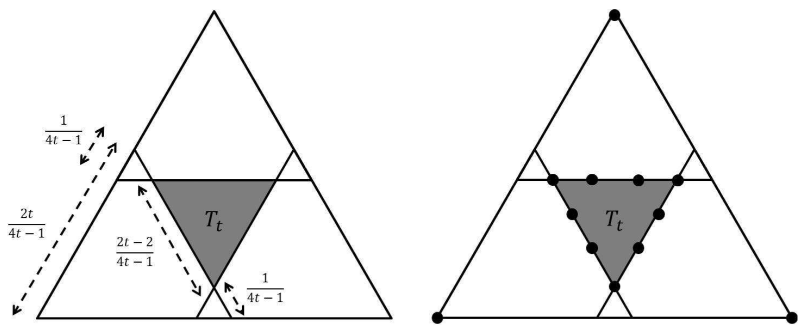

Figure 6. The triangle T_t and the Hider's optimal strategy.

We write $X_t \subset T_t$ for the union of the three sets of t equally spaced points on each edge of T_t starting at one vertex of the edge and ending at the other. So T_t contains $3t$ points in total, a third of which are on each edge. Note that the points on the vertices of T_t are 'counted' twice, so strictly speaking X_t is a multiset. We illustrate the set of Hider strategies on the right of Figure 6 for $t = 4$.

We now give the bound on the value obtained by the Hider using this strategy.

Lemma 8 *Suppose $n = 3$ and $m < \frac{2t}{4t-1}$. Then if the Hider picks a random vertex e_i with probability $p = \frac{t+2}{2t+2}$ and with probability $1-p$ makes an equiprobable choice between all points in X_t, he ensures that the payoff of the game is no more than $p/3$.*

Proof. We must show that against any Searcher pure strategy this Hider strategy guarantees the payoff is no more than $p/3$. If the Searcher's triangle $S(y)$ is a vertex triangle $E_i(m)$ then she wins with probability $p/3$ since $E_i(m)$ must contain precisely one vertex of Δ^2 and does not intersect with T_t. If $S(y)$ intersects with T_t then it cannot contain any of the vertices of Δ^2. We will show that the number of elements N of X_t that are contained in $S(y)$ can be no more than $t+2$, so that the payoff is no more than $\left(1-p\right)\left(t+2\right)/\left(3t\right) = p/3$.

Let N_1, N_2 and N_3 be the number of elements of X_t contained in $S(y)$ on each of the edges of T_t. As mentioned earlier (and depicted in Figure 6), the edges of T_t have length $2\,(t-1)\,/\,(4t-1)$ and there are t points in X_t on each edge of T_t so the distance between adjacent points on an edge is $2/\,(4t-1)$. Hence the length l of the total intersection of $S(y)$ and the edges of T_t must satisfy

$$\begin{aligned} l &\geq \sum_{i=1}^{3} (N_i - 1) \frac{2}{4t-1} \\ &= (N-3) \frac{2}{4t-1}, \text{ so} \\ N &\leq \left(\frac{4t-1}{2}\right) l + 3. \end{aligned} \tag{1}$$

By Lemma 7 we must have

$$\begin{aligned} l &\leq 2N - \frac{2t-2}{4t-1} \\ &< \frac{4t}{4t-1} - \frac{2t-2}{4t-1} \text{ (since } N < \frac{2t}{4t-1}) \\ &= \frac{2t+2}{4t-1}. \end{aligned}$$

Substituting this into (1) gives $N < t+4$. Now we note that the number of members $3t - N$ of X_t that are not contained in $S(y)$ must be even (the reader can convince herself of this by drawing an appropriate diagram). Hence N must have the same parity as t, since $3t$ has the same parity as t, so N is at most $t+2$. ■

Before we go on to define the Searcher's optimal strategies in the range $1/2 < m < 4/7$, we can now see that the value of the game when $m = 1/2$ is $1/6$, since the Searcher can guarantee a payoff of at least $1/6$ by Lemma 5 and for any $\varepsilon > 0$, the Hider can guarantee the payoff is no more than $1/6 + \varepsilon$ by Lemma 8 choosing t to be sufficiently large.

Proposition 9 *If $n = 3$ and $m = 1/2$ then the value of the game is $1/6$. The Searcher's optimal strategy is to choose one of the vertex triangles with probability $1/2$ and to choose the midpoint of two vertex triangles with probability $1/2$. The Hider has an ε-optimal strategy given in Lemma 8 for t sufficiently large.*

We turn to the Searcher's optimal strategy for m in the interval I_t. It is sufficient to assume that m takes the value of the lower bound $2\,(t+1)\,/\,(4\,(t+1)-1)$, by the remark just before Subsection 3.1. We will see it is optimal for the Searcher to pick one of the vertex triangles $E_i(m)$ with probability p, and with probability $1-p$ to choose randomly from a multiset Y_t of $3t$ triangles. In order to describe this multiset we first define the three triangles U_1, U_2 and U_3

as the unique equilateral triangles of side length m whose edges intersect with precisely two of the vertices of T_{t+1}. These triangles correspond to the Searcher strategies $S(y)$ given by $y = \left(\frac{2t-1}{4t+3}, \frac{1}{4t+3}, \frac{1}{4t+3}\right)$ and all permutations of these coordinates. One of these three triangles is depicted in Figure 7.

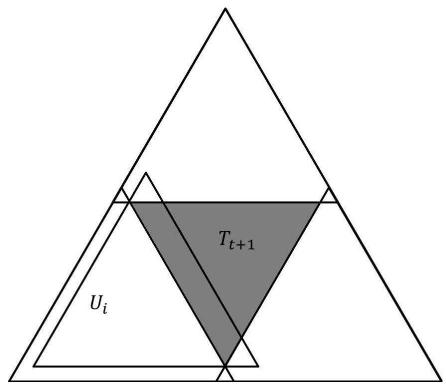

Figure 7. One of the Searcher's strategies, U_i.

The set Y_t contains the t equally spaced triangles starting from U_i and ending

at U_j for each $i \neq j$. That is, the t triangles given by $(k/t)\, U_i + (t-k)/tU_j$ for $k = 0, ..., t$. Note that Y_t contains two copies of each of U_1, U_2 and U_3.

Lemma 10 *Suppose $n = 3$ and $m = \frac{2(t+1)}{4(t+1)-1}$. Then if the Searcher chooses equiprobably between the vertex triangles $E_i(m)$ with probability $p = \frac{t+2}{2t+2}$ and with probability $1-p$ picks a random triangle from Y_t then she wins the game with probability at least $p/3$.*

Proof. If the Hider's point is located in one of the vertex triangles E_i then clearly the Searcher wins with probability $p/3$. So suppose not, and the Hider's point must be located in T_{t+1}. We need to show that every point of T_{t+1} is contained in at least $t+2$ of the triangles in Y_t. We prove this by considering the triangulation of T_{t+1} into t^2 congruent equilateral triangles, as depicted in

Figure 8 for $t = 5$. We denote the set of t^2 triangles in the triangulation by $\mathcal{T}$.

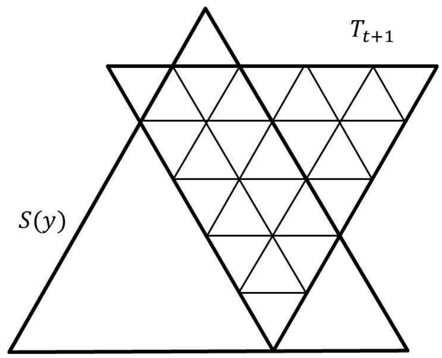

Figure 8. The triangulation of T_{t+1}.

The Searcher's strategy has been designed in such as way that each triangle in $\mathcal{T}$ is either contained in or disjoint from every triangle in Y_t, as shown in Figure 8. (We say 'disjoint' here to mean the measure of the intersection is 0.) We will show that each triangle in $\mathcal{T}$ is contained in at least $t+2$ of the triangles in Y_t. The triangles in $\mathcal{T}$ can be partitioned into two sets, the set $\mathcal{T}_1$ of triangles with the same orientation as T_{t+1} and the set $\mathcal{T}_2$ of triangles whose orientation is the same as T_{t+1} rotated by π. Consider some $R \in \mathcal{T}_2$, and let R' be a neighboring triangle in $\mathcal{T}_1$. If R' is contained in some triangle in Y_t then R is also contained in it. Thus it is sufficient to show that all triangles in $\mathcal{T}_1$ are contained in at least $t+2$ of the triangles in Y_t.

The position of each triangle $R \in \mathcal{T}_1$ can be specified by its 'row' with respect to each of the edges of T_{t+1}, where a row consists of all the triangles in $\mathcal{T}_1$ with the same perpendicular distance to that edge. Thus there are t rows with respect to any given edge of T_{t+1}, and we label them 1 to t, where row 1 is the closest to the edge. In fact, the position of some $R \in \mathcal{T}_1$ can be exactly specified by which row it is in with respect to any two of the edges of T_{t+1}, since the sum of the three rows that R is contained in must be $t+2$. Now we simply observe that if a triangle $R \in \mathcal{T}_1$ is contained in rows i, j and k with respect to the three edges of T_{t+1} then R is contained in exactly $i+j+k = t+2$ of the triangles in Y_t, which completes the proof. ■

We now put together Lemma 8 and Lemma 10.

Theorem 11 *Suppose $n = 3$ and $\frac{2(t+1)}{4(t+1)-1} \leq m < \frac{2t}{4t-1}$, where t is a positive integer. Then the value of the game is $\frac{t+2}{6t+6}$, and optimal strategies for the players are given by Proposition 2, Proposition 6, Lemma 8 and Lemma 10.*

In Figure 9 below we show a plot of the value of the game for all values of m between 1/2 and 1.

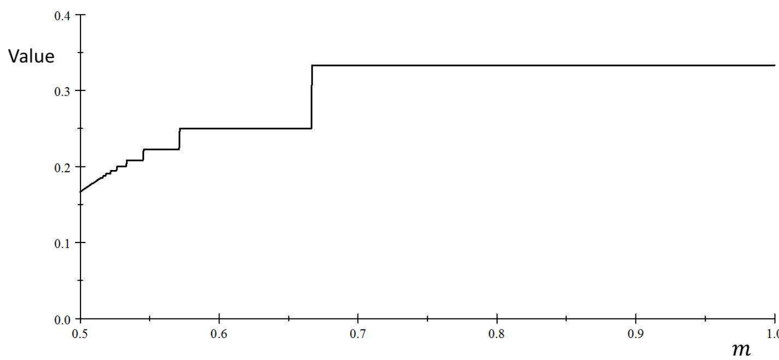

Figure 9. The value of the game for $n = 3$ and $1/2 \leq m \leq 1$.

We observe that in order to find the solution of the game in these cases, we have relied heavily on our geometric interpretation of the game. It seems unlikely that the solution could have been found using any intuition in the original form of the game. For example, let us consider the game where $m = 17/32 = 0.53125$, so that $t = 4$. To calculate the Hider's optimal strategy, we first see that the coordinates of the vertices of T_t are $\left(\frac{2t-1}{4t}, \frac{2t-1}{4t}, \frac{1}{4t}\right) = (7/15, 7/15, 1/15)$ and all permutations thereof. Hence the optimal strategy for the Hider is to choose with probability $3/5$ one of the 3 permutations of $(1, 0, 0)$, with probability $1/5$ one of the 3 permutations of $(7/15, 7/15, 1/15)$ and with probability $1/5$ one of the 6 permutations of $(3/15, 5/15, 7/15)$.

For the Searcher's optimal strategy, we assume that $m = 2\,(t+1)\,/\,(4\,(t+1)-1) = 10/19$, since the Searcher is allowed to use any pure strategies y whose coordinates sum to no more than $17/32 > 10/19$. The triangles U_1, U_2 and U_3 are given by $y = (7/19, 1/19, 1/19)$ and permutations of these coordinates. So the optimal Searcher strategy is to choose with probability $3/5$ one of the 3 permutations of $(9/19, 0, 0)$, with probability $1/5$ one of the 3 permutations of $(1/19, 1/19, 7/19)$ and with probability $1/5$ one of the 6 permutations of $(1/19, 3/19, 5/19)$.

The value of the game is $(t+2)\,/\,(6t+6) = 1/5$.

3.3 Some bounds on the value

For the range $0 < m < 1/2$ we do not offer a general solution to the game, but we give some bounds on the value.

Theorem 12 *The value V of the game satisfies*

$$\frac{2}{\lceil 2/m \rceil \left(\lceil 2/m \rceil - 1\right)} \leq V \leq m^2.$$

Proof. If the Hider uses the uniform strategy, where each measurable subset of the triangle is chosen with probability proportional to its measure, then the payoff against any Searcher pure strategy is m^2, so the value of the game is at most m^2.

Since $V(m)$ is increasing in m, it is sufficient to assume that $2/m = \lceil 2/m \rceil$, so that the lower bound for the value in the statement of the theorem is $m^2/(2-m)$.

We will give $(2-m)/m^2$ Searcher strategies such that every point in Δ^2 is covered by one of the strategies. We subdivide Δ^2 into $(2/m)^2$ congruent equilateral triangles of side length $m/2$. We denote this set of smaller triangles by $\mathcal{T}$. A Searcher pure strategy $S(y) \subset \Delta^2$, if appropriately positioned, can contain exactly 4 of the triangles in $\mathcal{T}$. Consider all possible such Searcher pure strategies. The total number of these strategies is $t(k)$ where $k = 2/m - 1$ and $t(k) = \binom{1+k}{2}$ is the kth triangular number. Every triangle in $\mathcal{T}$ is contained in at least one of these pure strategies, so if the Searcher chooses each of them equiprobably then any point in Δ^2 is found with probability at least $1/t(k) = m^2/(2-m)$. This completes the proof. ■

The upper bound in Theorem 12 is only tight for $m = 1$, because the Searcher needs to be able to non-wastefully cover Δ^2 in triangles of side length m.

However, in the range of value of m for which we have solved the game, the lower bound is tight when $m = 1$, $2/3$ or $1/2$. In each of these cases $\lceil 2/m \rceil = 2/m$, and the Searcher strategy used in the proof above is optimal. We conjecture that the bound is always tight when $\lceil 2/m \rceil = 2/m$.

Conjecture 13 *If $n = 3$ and $2/m$ is an integer, then the value of the game is $1/t(k)$, where $k = 2/m - 1$. The Searcher has an optimal strategy where she chooses equiprobably between $t(k)$ strategies that cover the triangle Δ^2.*

We go one stage further. Let $P_n(k)$ be the kth simplicial $(n-1)$-polytopic number given by

$$P_n(k) = \binom{n+k-2}{k},$$

so when $n = 3$, $P_n(k) = t(k)$. Proposition 2 tells us that for $m \geq (n-1)/n$, the value of the game is $\lceil (n-1)/m \rceil = 1/P_n(k)$, where $k = \lceil (n-1)/m \rceil - n + 2$. We offer a generalized version of Conjecture 13.

Conjecture 14 *If $\lceil (n-1)/m \rceil$ is an integer then the value of the game is $1/P_n(k)$, where $k = \lceil (n-1)/m \rceil - n + 2$. The Searcher has an optimal strategy where she chooses equiprobably between $P_k(n)$ strategies that cover Δ^{n-1}.*

Note that this conjecture is certainly true when $n = 2$.

4 The case $r > \bar{m}$

We now turn to the case where r is not necessarily equal to $\bar{m}$, which we shall see is more difficult in general. For $n = 2$ we show that the game reduces to an instance of the game when $r = \bar{m}$, and so the solution is given by the previous section.

We first note that for particular choices of the parameters the game is trivial and the Searcher can always win.

Lemma 15 *If $r \geq n\bar{m}$ then $V = 1$. The Searcher can win by allocating equal weight r/n to all locations.*

Proof. Let y be the Searcher strategy described above. Suppose $x_i \geq \bar{m}$ for some i. Then since $y_i = r/n \geq \bar{m}$, we have $M(x, y) \geq \min_i \{x_i, r/n\} \geq \bar{m}$. Conversely, if $x_i < \bar{m}$ for all i then $x_i < r/n$, so $M(x, y) = \sum_{i=1}^{n} x_i = 1 \geq \bar{m}$. Either way, $P(x, y) = 1$ so the Searcher always wins and $V = 1$. ■

Unlike in the case $r = \bar{m}$, in the more general case, the sets $S(y)$ do not have a form that is easy to express. To simplify the analysis, we will define another game $\tilde{\Gamma}$ which has the same value as Γ.

Definition 16 *Let $\tilde{\Gamma} = \tilde{\Gamma}(n, m, r)$ be the zero-sum game between a Hider and Searcher in which the Hider's strategy set is Δ^{n-1} and the Searcher's strategy set is $\left\{ y \in \mathbb{R}^n : \sum_{i=1}^{n} y_i = r \right\}$. For a Hider strategy x and Searcher strategy y, the payoff is 1 if an only if $M(x, y) := \sum_{i=1}^{n} \min\{x_i, y_i\} \geq \bar{m}$, otherwise the payoff is 0.*

The game $\tilde{\Gamma}$ is the same as Γ in every respect except that the strategy set of the Searcher is larger, since the components of a Searcher strategy y are allowed to be negative. This means the value of $\tilde{\Gamma}$ must be at least the value of Γ since the Searcher has an advantage. But in fact it is easy to show that the Searcher cannot do any better in $\tilde{\Gamma}$.

Lemma 17 *The value $\tilde{V}$ of $\tilde{\Gamma}$ is the same as the value V of Γ.*

Proof. As noted above, $\tilde{V} \geq V$. To show the reverse inequality, let y be a Searcher strategy in $\tilde{\Gamma}$, and we will show that y is dominated by some strategy y' in Γ. If all the components of y are non-negative then y is already a strategy in Γ. If not, then suppose y_i is negative for some i. There must be some other set of components $J \subset \{1, \ldots, n\}$ of y whose sum is at least $-y_i$. We define y' by $y'_i = 0$ and we add a total of y_i to the components y_j with $j \in J$ in such a way that they all still remain positive. Then for any Hider strategy x, the contribution to $M(x, y)$ from the ith component is increased by $-y_i$ from y_i to 0 and the sum of the contributions from the components in J is decreased by no more than $-y_i$, so that $M(x, y') \geq M(x, y)$ and $V \geq \tilde{V}$. Hence the values are equal. ■

For the game Γ we defined the set $S(y)$ as the set of Hider strategies beaten by a Searcher strategy y. We now define analagous sets $\tilde{S}(y) = \{x \in \mathbb{R}^n : M(x, y) \geq \bar{m}\}$ for the game $\tilde{\Gamma}$. The difference here is that $\tilde{S}(y)$ is not limited to points in the

Hider's strategy set, but contains all vectors x in $\mathbb{R}^n$ with $M(x,y) \geq \bar{m}$ (note that we are extending the domain of M to the whole of $\mathbb{R}^n \times \mathbb{R}^n$). For a Searcher strategy y in $\tilde{\Gamma}$, the set $S(y)$ can be expressed as $\tilde{S}(y) \cap \Delta^{n-1}$. We can now think of the game $\tilde{\Gamma}$ geometrically as follows: the Hider picks a point $x \in \Delta^{n-1}$ and the Searcher picks a set $\tilde{S}(y)$. The Searcher wins if and only if x is contained in $\tilde{S}(y)$. We denote the collection of all sets $\tilde{S}(y)$ by $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}(n,m,r)$.

The advantage to analyzing the game Γ through the lens of $\tilde{\Gamma}$ is that the sets $\tilde{S}(y)$ are easier to describe than the sets $S(y)$. In fact, they are just translations of each other.

Lemma 18 *If y is a Searcher strategy in $\tilde{\Gamma}$ then $\tilde{\mathcal{S}}$ is the set of all translations of $\tilde{S}(y)$ by some $a \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} a_i = 0$.*

Proof. Let $a \in \mathbb{R}^n$ be such that $\sum_{i=1}^{n} a_i = 0$. Then

$$\begin{aligned}
\tilde{S}(y+a) &= \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \min\{x_i, y_i + a_i\} \geq \bar{m} \right\} \\
&= \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \left(\min\{x_i - a_i, y_i\} + a_i\right) \geq \bar{m} \right\} \\
&= a + \left\{ x' \in \mathbb{R}^n : \sum_{i=1}^{n} \min\{x'_i, y_i\} \geq \bar{m} \right\} \text{ (substituting } x' = x - a \text{ and using } \sum_{i=1}^{n} a_i = 0) \\
&= a + \tilde{S}(y).
\end{aligned}$$

Since any Searcher strategy can be expressed as $y + a$ for some such a, this proves the lemma. ■

For the remainder of this section we will consider the case $n = 2$. Lemma 15 says that $V = 1$ if $r \geq 2\bar{m}$, so we will restrict attention to the case $r < 2\bar{m}$. As in Subsection 3.1 we can consider the Hider's strategy set as the unit interval $[0,1]$, so that instead of considering a Hider strategy as a vector $(x, 1-x)$ we consider it as a point $x \in [0,1]$. Similarly, the Searcher's strategy set in Γ can be considered as the interval $[0,r]$, so that a vector $(y, 1-y)$ corresponds to a point $y \in [0,r]$; for $\tilde{\Gamma}$ the Searcher's strategy set can be considered as $\mathbb{R}$. For a given Hider strategy x and Searcher strategy y (in either Γ or $\tilde{\Gamma}$) the Searcher wins if and only if $M(x,y) = \min\{x,y\} + \min\{1-x, r-y\} \geq \bar{m}$. The sets $S(y)$ must be subintervals of $[0,1]$, since these are the only closed, convex subsets. The sets $\tilde{S}(y)$ must also be intervals in $\mathbb{R}$ for the same reason.

From this we can deduce the value of the game Γ when $n = 2$ by reducing it to the case $r = \bar{m}$.

Theorem 19 *When $n = 2$ and $r \leq 2\bar{m}$, the solution of the game $\Gamma(n,m,r)$ is given by the solution of $\Gamma(n, m', \bar{m}')$, where $m' = 2m + r - 1$ and $\bar{m}' = 1 - m'$. The value of the game is $\lceil 1/(2m + r - 1) \rceil$.*

Proof. The value of the game $\Gamma(n,m,r)$ is the same as the value of the game $\tilde{\Gamma}(n,m,r)$, by Lemma 17. If we take the Searcher strategy $y=r$ in $\tilde{\Gamma}(n,m,r)$ then $M(x,y)\geq \bar{m}$ if and only if $x\geq \bar{m}$ and $1-x\geq \bar{m}-r$, so $\tilde{S}(y)=[\bar{m},r+1-\bar{m}]$. This is an interval of length $r+1-2\bar{m}=2m+r-1$, so $\tilde{\mathcal{S}}(n,m,r)$ must consist of all intervals of this length by Lemma 18. But by the same reasoning, in the game $\tilde{\Gamma}(n,m',\bar{m}')$, the corresponding set $\tilde{\mathcal{S}}(n,m',\bar{m}')$ must consist of all intervals of length $2m'+\bar{m}'-1=m'=2m+r-1$, so $\tilde{\mathcal{S}}(n,m,r)=\tilde{\mathcal{S}}(n,m',\bar{m}')$. Hence the game $\tilde{\Gamma}(n,m,r)$ must have the same value as $\tilde{\Gamma}(n,m',\bar{m}')$, which by Lemma 17 is the same as the value of $\Gamma(n,m',\bar{m}')$, which by Proposition 4 is $\lceil 1/m'\rceil=\lceil 1/(2m+r-1)\rceil$. ■

5 Conclusion

We have defined a new caching game where a Hider caches an amount of material scaled to 1 and a Searcher uses 'search effort' of r to try and find $\bar{m}$ of the material. We showed that the game can be considered as a geometric game played on the regular n-simplex. We first considered the case where $r=\bar{m}$ so that the Searcher could only win by finding all the material she was searching for, and we found that in this case the game has a particularly attractive geometric interpretation. For $n=2$ we saw that the game had already been solved by Ruckle. For $n=3$ we solved the game for $m\geq 1/2$, giving bounds on the value for $m<1/2$, and for general n we conjectured the that the value of the game is given by the inverse of the simplicial polytopic numbers if $(n-1)/m$ is an integer. We also showed that the more general case $r\geq\bar{m}$ reduces to the case $r=\bar{m}$ for $n=2$.

There is clearly much more work that can be done on this game. In particular, it may be possible to find solutions of the game for $n=3$ and $r=\bar{m}$ when $m<1/2$, or at least verify Conjecture 13. We have concentrated on the case $r=\bar{m}$ in this paper, but another direction that could be taken is to study the game in the more general case $r\geq\bar{m}$, starting with an investigation of the structure of $S(y)$ for $n=3$, perhaps by extending the method used in Section 4 of defining sets $\tilde{S}(y)$.

We also note that the game has another interpretation as a specific case of a game played on a network with distinguished root O. The Hider chooses a connected subset of the network of measure 1 which contains O and the Searcher chooses a connected subset of the network of measure r which contains O. The Searcher wins if and only if the measure of the intersection of the sets is at least $\bar{m}$. In the case that the network is a uniform star (a set of unit arcs meeting at O), this game is equivalent to Γ. The author would like to thank Steve Alpern for this observation. Further work could investigate this more general game on other types of networks, for example the circle.

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