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The Sound of Silence: equilibrium filtering and optimal censoring in financial markets
by

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Abstract. Following the approach of standard filtering theory, we analyse investor-valuation of firms, when these are modelled as geometric-Brownian state processes that are privately and partially observed, at random (Poisson) times, by agents. Tasked with disclosing forecast values, agents are able purposefully to withhold their observations; explicit filtering formulas are derived for downgrading the valuations in the absence of disclosures. The analysis is conducted for both a solitary firm and m co-dependent firms.

Key word: disclosure, filtering, public filtration, predictable valuation, optimal censor, asset-price dynamics.

AMS Classification: 91G50, 91G80; 93E11, 93E35; 60G35, 60G25.

1 Introduction

Motivated by applications from financial mathematics, we assume below that the flow of information reaching a market is influenced strategically by the agents responsible for making and subsequently disclosing observations, simply because observations made by a self-interested agent can be suppressed. For instance, suppose the agent's self-interest manifests itself as wanting only to report good news, and so to suppress bad news. If the cutoff that defines whether an observation is good or bad news is defined relative to prior expectations, then the agent will disclose only observations above the prior. A model of strategic reporting needs specifically to incorporate how potential recipients of disclosed observations (investors) can rationally respond to *silence* from the agent, when absence of observation cannot credibly be indicated by the agent, and disclosure is not mandatory. We assume below that at Poisson arrival times the agent is able to observe information, and has then the option to disclose the observed information, but only truthfully, or to suppress it (i.e. hide the 'bad' news). Then the question arises: what is the optimal level of suppression. We answer below in Theorem 1 and Theorem 1_m with explicit formulas in a multi-agent multi-period context. We work in a continuous Black-Scholes framework introducing the novel concept of an *optimal censor* (§2.2). This combines censoring with filtering (for which see

[1], [2], [11], [24]). It is the key here. In particular we derive closed-form *exponential decay* solutions for the optimal time-varying censor in periods of silence; a new insight here is that penalization of silence is harshest at the beginning (see §3.2 Corollary for details). These are the pleasing consequences of moving to continuous time: simplification of the natural equilibrium conditions (e.g. (1) in §2.3 and in §3.1.1; cf. (10)) to a first-order differential equation (see (6) in §3.1); a transparent narrative at the single agent level; a rich joint asset-price dynamic, via the repercussions on each other, of a stream of disclosures from the multiplicity of correlated agents. This paper is a sequel to our previous work, where such questions were pursued in a discrete two-period setting (in [27] and [16] for the case of one firm, and in [15] where further reality is added via a communication game with multiple competitors in an industry correlated by common operating conditions). We were motivated by the static-setting literature of costly state verification (e.g. by Townsend [30] in 1979 – see the later literature in [20]), and of corporate disclosure introduced by Dye [13] in 1985 (and the associated paper [19]).

The additional results in §4 highlight consequences of our main theorems in [15] for the bandwagon and quality effects in the current setting. These qualitative results extend our findings in the two-period model of [15]. For instance: when competing managers are endowed with different observation noise, those managers that observe with most noise use a lower censor and hence suppress bad news less. But then this means that, when investors see *slightly-below-mean* observations being disclosed by such managers, they rationally interpret this as being from a more noisy source, and so discount its importance when updating.

A broadly similar class of models arises in the engineering literature studying alternation between observation ('measurement') and control, typically in a discrete-time setting, but there the alternation is the result of a trade-off between the two actions, dictated by a *single* indicator of overall performance of a system (i.e. a suitable objective function); a related class considers intermittent receipt of measurements/observations, but there the suppression is caused by random transmission failures – see e.g. [7], [18], [29].

In discrete time, Shin [28] introduced (in 2004) a model of a firm, engaged in a flow of projects, receiving information with a Poisson distribution concerning the status (success or failure) of projects completed to date. By endowing the firm with the opportunity at each date of a full or partial disclosure of that information, he created an asset-pricing framework in which such disclosures are endogenously determined, and so could study equilibrium patterns of corporate disclosure.

In continuous time, Brody, Hughston and Macrina [6] introduced (in 2007) an asset-pricing framework based on noisy market-observation of continuous information that is generated from the remaining future disclosures, occuring at pre-determined ('mandatory') dates. Similarly to Shin, their disclosures are modelled as a discrete series of random variables; the latter give rise to a stream of uncertain payouts (cash flows) corresponding to the mandatory dates. The individual cash values are taken to be deterministic functions of independent random variables called 'market-factors' (not unlike our terminal-time output variable Z_1), but with the inclusion only of those that are capable of being observed by the modeller at and prior to the date of disclosure. Typically the market's noisy observation is a linear combination of the future payouts, each weighted by a coefficient that gains increasing prominence over time, and Brownian-bridge components, which vanish at the corresponding payout-dates. However, in contrast to Shin, all the dividend-disclosures are mandatory and the possibility of voluntary intermittent disclosures (say via a zero dividend) is not studied.

In a recent development, Marinovic and Varas [25] (in 2014) consider a voluntary disclosure model in continuous time, in which a single agent uninterruptedly observes an asset following a $\{0, 1\}$-valued random walk. (The binary aspect leads to market-price decay, which like ours is exponential.)

In creating a multi-asset Black-Scholes asset-pricing framework in which corporate disclosures are endogenously determined, our work is closest in spirit to Shin [28]. It is also closer in spirit with some of the earlier literature of portfolio/consumption analysis under incomplete information, e.g. Feldman's model of 1992 ([14]; cf. literature cited there) of a production exchange economy, where realized outputs are observed (whereas in the model below a noisy version of the output process Z_t is observed), and provide via a nonlinear filter an information flow on the underlying *economic state process* (a 'productivity' factor, following an Ornstein–Uhlenbeck mean-reverting process).

The rest of this paper is structured as follows. In §2, abstracting away from the market motivation, we exploit in §2.1 some of the ideas of standard filtering theory to describe scalar observations by m individuals ('agents') that are made intermittently in the interval $(0, 1)$ privately (i.e. in secret), and either kept secret or disclosed (reported) to the other agents and investors. The observation processes have a co-dependance, as do also certain other scalar processes which assign a valuation to each agent. This co-dependance is determined by a single state-process called the *common* *effect*. The shared information, together with the initial mandatory observation values of time $t = 0$, provides the basis upon which to forecast each individual's valuation at the terminal time $t = 1$ of the next mandatory disclosure.

Agents disclose their observations in order to enhance at each moment the forecast of their terminal valuation, which is contingent on the mandatory disclosures at the terminal time. This leads to an optimization problem stated in §2.2. If each agent discloses only observations above their *censoring* threshold, keeping secret lower observed values, silence at any date in $(0, 1)$ can mean either the absence of observation, or its censoring (since the arrival of an undisclosed observation remains secret, and the agent is not able to assert credibly the absence of a current observation). Since periods of silence bring precautionary valuation downgrades, the threshold drops over time, pari passu, to elicit a disclosure; additional tensions arise from potential disclosures from other agents, so subgame perfect Nash equilibria considerations enter the argument – see §3.

Section 2.3 interprets the individual valuation as the market value of the firm obtained from information disclosed to the market by managers making discretionary (i.e. non-mandatory) disclosures in between the two mandatory disclosure times of $t = 0$ and $t = 1$.

Our two main results are stated and discussed in §3 and proved in §5.2, §5.3. In §4 the interpretation of §2.3 is used to describe, as immediate corollaries of earlier work, qualitative features of a multi-agent correlated sector.

We focus on valuations inferred from periods of silence hence the title.

2 Model

We first formalize the disclosure framework in a series of steps.

2.1 Processes and filtrations

We fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, with $\mathbb{F} := \{\mathcal{F}_t : t \in [0, 1]\}$.

2.1.1. State process setup. The production of $m \geq 1$ scalar processes from one (vector) process, and the uncertainties connected with this, will be modelled in terms of the fixed ($\mathbb{F}$-$\mathbb{Q}$)-Wiener process $W_t = (W_t^0, W_t^1, ..., W_t^m)$, more precisely in terms of exponential martingales X and M^i defined from the independent component processes of W, employing correspondingly the entries of the real vector $\sigma = (\sigma_0^X, \sigma_1^M, \ldots, \sigma_m^M) \geq 0$ as follows[1]:

$$\begin{aligned} dX_t/X_t &= \sigma_0^X \, dW_t^0, \quad t \in [0,1], \\ dM_t^i/M_t &= \sigma_i^M \, dW_t^i, \quad t \in [0,1], \end{aligned}$$

for every $i \in I := \{1, \ldots, m\}$; here we assume as known the respective distributions at time-0 of these processes.

The process X has the dual roles of *(economic) state* process and *signal* process, and is regarded as representing a *common effect* that influences the individual internal histories via two processes as follows.

2.1.2 Ouput process. The *output process* $Z_t = (Z_t^1, ..., Z_t^m)$, whose terminal state at time $t = 1$ is to be forecast, encodes the production of m signals from the state process as its single input; taking values in $\mathbb{R}^m$, it is defined componentwise by a weighted power-law:

$$Z_t^i = \zeta^i(X_t), \quad \text{where} \quad \zeta^i(x) = f^i x^{\alpha_i},$$

for i in $\{1, \ldots, m\}$; here f^i is the *size constant* of the i-th output and α_i is termed the i-th *loading exponent (loading factor)*, both positive, yielding the *terminal output* vector Z_1. The exponent, which measures the exposure of a firm to common effects, enables empirical interpretation but is mathematically of little consequence (as though it had the value 1). To add a technical remark, it will be convenient in the proofs to rescale such a process to have size unity at some date, contingent on the available information, a procedure we call *common-sizing.*

2.1.3 Internal history process. The *internal history process* $Y_t = (Y_t^1, \ldots, Y_t^m)$ provides the source of intermittent noisy observations (which are to be selectively disclosed) of the output process Z_t; as this is to be a partially observable process, the construction uses the martingale $M = (M^1, \ldots, M^m)$ as a linear noise to modify the output process:

$$Y_t = Z_t M_t,$$

that is, componentwise: $Y_t^i = Z_t^i M_t^i = f^i X_t^{\alpha_i} M_t^i$.

[1] Our choice of working with exponential martingales, indeed with Doléans-Dade exponentials, is natural in a setting where a critical equation involves equity values rather than returns: see Prop. 1 below and especially the equity valuation equation (3), as well as the interplay of the key equations: (7), (8) [equivalently, (9)], and likewise (10).

2.1.4 Regression parameters. We collect regression parameters associated with the processes defined so far. Put $\sigma_i := \sigma_i^M/\alpha_i$, and define the associated *precisions*

$$p_i := 1/\sigma_i^2 \text{ and } p = p_{\text{agg}} := p_0 + p_1 + \ldots + p_m;$$

for convenience, we use dual notation for the regression coefficients (relative precisions):

$$\kappa_i \text{ or } \kappa_m^i := p_i/p \text{ and } \kappa_1^i = p_i/(p_0 + p_i) \text{ for } m = 1 \text{ with } i \text{ the solitary firm.}$$

2.1.5 Private observation process and private filtration. The observation of the economic realities embodied in the processes X, Y, and Z is modelled as an *intermittent partial observation* arising for an individual economic agent with label in $\{1, \ldots, m\}$, and expressed as a *private observation process* $Y^{i\text{-obs}}$. The constructions of these concepts below starts from a (càdlàg) Poisson process $N_t = (N_t^1, ..., N_t^m)$ which is independent of the Wiener processes W_t of §2.2.1 and whose vector of intensity functions $\lambda_t = (\lambda_t^1, \ldots, \lambda_t^m)$ is inhomogeneous and also known. The arrival times in $(0, 1)$ of each component process N^i are regarded as consecutively numbered, with $\theta_n^{i\text{-obs}}$ denoting the n-th of these; that is, on setting $\theta_0^{i\text{-obs}} = 0$,

$$\theta_n^{i\text{-obs}} := \inf\{t > \theta_{n-1}^{i\text{-obs}} : N^i(t) > N^i(t-)\}.$$

The intended meaning of this is that the i-th agent observes the process $Y^{i\text{-obs}}$ privately, and only at these arrival time in $(0, 1)$; the last observation time at or prior to t and the corresponding last observed value are

$$\begin{aligned} \theta_-^{i\text{-obs}}(t) &= \max\{s \leq t : N^i(s) > N^i(s-)\}, \\ Y^{i\text{-obs}}(t) &= Y^i(\theta_-^{i\text{-obs}}(t)). \end{aligned}$$

The resulting process $Y^{i\text{-obs}}$ is the *private observation process* of agent i. It is piecewise constant, and defines the *private filtration* of the i-th agent. This is the (time indexed) family $\mathbb{Y}_i^{\text{priv}}$ of σ-algebras generated by the jumps at or before time t, and by the observations at or before time t, here regarded as *space-time* point-processes (i.e. with their dates – for background see [9, esp. II Ch. 15]); it is formally given by

$$\mathbb{Y}_i^{\text{priv}} := \{\mathcal{Y}_t^i : t \in [0, 1]\}, \text{ where } \mathcal{Y}_t^i := \sigma(\{(s, Y^{i\text{-obs}}(s), N^i(s)) : 0 < s \leq t\}),$$

for $t \in (0, 1)$, where we let $\mathcal{Y}_0^i$ contain all the null sets of $\mathcal{F}$.

2.1.6 Public filtration. The *public filtrations* $\mathbb{G}^{\text{pub}}$, to be constructed next, formalize the notion of 'information disclosed via the publicly observed history of Y'. Their construction begins with a fixed marked point-process[2] (MPP) comprising the following list of items (i) to (iv).

(i) The underlying càdlàg counting process N_t^{pub} ('the disclosure time process'),
(ii) The functions $\theta_{\pm}^{\text{pub}}$, where $\theta_{-}^{\text{pub}}(t)$ is the last arrival time of N_t^{pub} less than or equal to t, and $\theta_{+}^{\text{pub}}(t)$ is the first arrival time of N_t^{pub} bigger than or equal to t, these relations holding almost surely on $\mathcal{F}_t$, for every t in $[0,1]$.
(iii) The point process $J_t \subseteq I := \{1,\ldots,m\}$ ('the disclosing agent set of time t').
(iv) The marks $Y_t^{j\text{-pub}} = Y^{j\text{-obs}}(\theta_{-}^{\text{pub}}(t))$ for $j \in J_t$ ('their corresponding observations of time t').

In terms of the MPP as above we obtain the *publicly observed history*, or *public filtration*, which is right-continuous and constructed in the three steps below:

$$\mathbb{G} = \mathbb{G}^{\text{pub}} := \{\mathcal{G}_t^+ : t \in [0,1]\},$$

where $\mathcal{G}_1^+ = \sigma(\mathcal{G}_1, \{(1, Y^i(1))\}_{i\in I})$ and for each t in $[0,1)$, conventionally

$$\mathcal{G}_t^+ = \bigcap\nolimits_{s>t} \mathcal{G}_s,$$

with the σ-algebras $\mathcal{G}_t$ generated as the join of the three σ-algebras corresponding to the items (i), (iii) and (iv) from §2.1.6, dates included, namely:

$$\mathcal{G}_t = \vee_{s\in[0,t)} \sigma\big(((s, N^{\text{pub}}(s)), J_s, \{(s, Y^{j\text{-pub}}(s)\}_{j\in J_s})\big).$$

2.1.7 Disclosure filtration. The key concept for the paper is that of a public filtration $\mathbb{G}$ (in the sense of §2.1.6) which is a particular kind of subfiltration of the join $\vee_i \mathbb{Y}_i^{\text{priv}}$ of the private filtrations of §2.1.5. Its definition hinges on disclosure of observations at or above the current value of a 'reference process' γ which, *crucially,* is required to be predictable with respect to the public information that it itself generates.

We define the *disclosure filtrations consistently generated from* the (private) filtrations $\{\mathbb{Y}_i^{\text{priv}} : i \in I\}$ *via* the $\mathbb{G}$-predictable *censoring filter* $\gamma =$

[2] As the Referee points out, it possible to construct the public filtration entirely from a vector of Poisson processes (with appropriate thinning), so that 'disclosers' are identified by coincidence of certain arrival times.

$(\gamma_t^i : i \in I)$, to be public filtrations satisfying additionally the following three conditions for each $0 < t < 1$. These say that at each disclosure time there are disclosing agents as in (v), with their observations made public, as in (vi), because, as in (vii), these are above their censoring thresholds:

(v) The set J_t is non-empty iff $N^{\text{pub}}(t) > N^{\text{pub}}(t-)$.
(vi) If $N^{\text{pub}}(t) > N^{\text{pub}}(t-)$, then $Y^{j\text{-pub}}(t) = Y^{j\text{-obs}}(t)$, for each $j \in J_t$.
(vii) If $N^i(t) > N^i(t-)$ and $Y_t^{i\text{-obs}} > \gamma_t^i$, for some $i \in I$,
then $N^{\text{pub}}(t) > N^{\text{pub}}(t-)$ and $i \in J_t$.

For such a filtration $\mathbb{G}^{\text{pub}}$, the counting-process arrival times occuring in $(0,1)$ in (ii) of §2.1.6 will be termed the *voluntary disclosure event times* (or just *disclosure times* if $m = 1$): $\theta_0^{\text{pub}}, \theta_1^{\text{pub}}, \ldots$.

2.2 Optimal censoring problem

Associated with a fixed disclosure filtration $\mathbb{G} = \{\mathcal{G}_t^+\}$, consistently generated via γ as in §2.1.7, is the process obtained by taking contingent expectations of the time-1 output Z_1 with respect to the time-t information subsets:

$$t \mapsto \mathbb{E}[Z_1^i | \mathcal{G}_t]\,;$$

this process is interpreted as a $\mathbb{G}$-*predictable valuation process* or $\mathbb{G}$-*forecasting process*. The *optimal censoring problem* addressed below calls for the construction of a filtration $\mathbb{G}_*$, necessarily unique, whose associated forecasting process is the *left-sided-in-time* pointwise supremum over all $\mathbb{G}$-forecasting processes, that is, for each time $t \in (0,1)$ and each agent i,

$$\mathbb{E}[Z_1^i | \mathcal{G}_{*,t}] \;= \sup\nolimits_{\mathbb{G}} \mathbb{E}[Z_1^i | \mathcal{G}_t], \tag{OC}$$

where the supremum ranges over the disclosure filtrations $\mathbb{G} = \{\mathcal{G}_t^+\}$ of §2.1.7. In (OC) both sides of the equation depend only the public information available to the *left* of the date[3] t. By definition such a $\mathbb{G}_*$, if it exists, is unique. The main results of the paper give the solution in §3 of the optimal censoring problem for the geometric Brownian signal processes X of §2.1; Theorem 1 corresponds to the one-dimensional case $m = 1$, Theorem 1_m to the case of arbitrary integer dimension $m \geq 1$.

[3]It is highly significant to the analysis that the date t is inferable from the conditioning σ-algebra $\mathcal{G}_t$ (likewise for $\mathcal{Y}_t^i$), hence its inclusion in the construction.

Remarks 1. Informally, the censoring problem requires agent i to reach disclosure/suppression decisions using only the history of all prior public information.
2. With the filtering theory language above, the optimal censoring problem amounts to the construction of a censoring-filter (process) γ_t with the following properties (i) to (iii):
(i) the disclosure subfiltration $\mathbb{G}$ of $\vee_{i\in I}\mathbb{Y}_i^{\text{priv}}$, generated through suppression of observations by reference to the process γ, is consistently generated from the private filtrations $\{\mathbb{Y}^i : i = 1, ..., m\}$;
(ii) γ is $\mathbb{G}$-predictable (this *is* crucial);
(iii) for each $i \in I$ and for each time instant $t \in (0, 1)$, γ_t^i is chosen to be a cutoff *maximizing* the expected value of Z_1^i, given only past public information, *and* best response to the simultaneous *censoring choices* of γ_t^j for $j \neq i$ (which is why γ needs to be $\mathbb{G}$-predictable). Of course, this yields an individual *private valuation* for each agent.
3. The role of $\mathbb{G}$ above is to formalize the censoring of the private observation processes relative to information 'public' *before* any time-t disclosure, so in general distinct from the *optional valuation process* of the next section (§2.3)

$$\mathbb{E}[Z_1^i|\mathcal{G}_t^+],$$

which models the later right-continuous *public valuation* at time t.
4. In Remark 2 the agent is a maximizer of an instantaneous objective linked to the terminal output (via its estimator). The alternative approach is to establish a single overall performance indicator for the entire trajectory of the estimator. Optimality of overall economic behaviour induced by instantaneous (sometimes called 'myopic') objectives is established for a class of models related to ours in Feldman [14].

2.3 Application to asset-price modelling

We consider a single 'firm' Z_t^i in isolation, so omitting i when convenient. The manager of the firm (agent i) makes a mandatory declaration at time $t = 1$ of its (fundamental) value which, taking a Bayesian stance, is the manager's estimate/forecast of the firm's economic state Z_1 using Y_1, namely

$$\tilde{\gamma}_1 := \mathbb{E}[Z_1|\mathcal{G}_1, Y_1] = \mathbb{E}[Z_1|\mathcal{G}_1^+],$$

since $\mathcal{G}_1^+ = \sigma((1, Y_1), \mathcal{G}_1)$. Below we use a standard martingale construction to create an asset-price process under $\mathbb{Q}$ (cf. [4]); we then note the option values that censoring introduces, and observe that a censor, which induces indifference between disclosure and non-disclosure, preserves the risk-neutral character of the asset-price under $\mathbb{Q}$.

Given a public filtration $\mathbb{G}$, the associated forecasting process of §2.2. yields an analogue S of an asset-price process with

$$S_t := \mathbb{E}[\mathbb{E}[Z_1|\mathcal{G}_1^+]|\mathcal{G}_t^+], \qquad \text{for } 0 \le t \le 1.$$

This construction also turns the reference measure $\mathbb{Q}$ (of the stochastic basis) into a risk-neutral valuation measure, a fact implied by the conditional mean formula, which for $t < s$ asserts that

$$\mathbb{E}[S_s|\mathcal{G}_t^+] = \mathbb{E}[\mathbb{E}[\mathbb{E}[Z_1|\mathcal{G}_1^+]|\mathcal{G}_s^+]|\mathcal{G}_t^+] = \mathbb{E}[\mathbb{E}[Z_1|\mathcal{G}_1^+]|\mathcal{G}_t^+] = S_t.$$

The disclosure cutoff γ is uniquely determined below by the asset-price process S, as follows. Fix $t < s$. At time t suppose that agent i is committed to using a cutoff γ_t and is to choose a disclosure cutoff γ for use at the later date s. Assume, until further notice, absence of any public disclosures in the interval (t, s). Let $D_s(\gamma)$ be the disclosure event of time s corresponding to the agent i observing a value of Y_s at or above γ, so that in terms of indicator functions

$$\mathbf{1}_{D_s(\gamma)} = \mathbf{1}_{N(s)>N(s-)} \cdot \mathbf{1}_{Y(s)\ge\gamma}. \tag{D}$$

Equation (D) suggests that the Black-Scholes value of the *one-or-nothing binary option on* Y_s with strike γ (represented by $\mathbf{1}_{Y(s)\ge\gamma}$) will emerge at the heart of our line of reasoning.

Next consider the complementary event $ND_s(\gamma)$ in which, given γ, the market computes a forecast for Z_1 as being $\tilde{\gamma}_s = \mathbb{E}[Z_1|ND_s(\gamma), \mathcal{G}_t^+]$. If the agent wishes to maximise the asset valuation S at time s the choice γ will induce indifference between disclosure and non-disclosure when $Y_s = \gamma$ is observed iff[4]

$$\mathbb{E}[Z_1|Y_s = \gamma, \mathcal{G}_t^+] = \mathbb{E}[Z_1|ND_s(\gamma), \mathcal{G}_t^+].$$

This observation and conditioning on knowledge at time t of the value[5]

$$\tilde{\gamma}_t := \mathbb{E}[Z_1|\mathcal{G}_t^+]$$

[4]For simplicity, here and below we adopt the *equational convention* that conditioning on an equation $Y_t = y$ is to be read as implying its disclosure.

[5]This equals $\mathbb{E}[Z_1|Y_t = \gamma_t, \mathcal{G}_t]$, absent any disclosure.

induces the following simple relation between the equilibrium value $\tilde{\gamma}_s$ and $\tilde{\gamma}_t$; an even simpler limiting form arises in Theorem 1 in §3.1. The relation involves the distribution of $Z_s^{\text{est}} := \mathbb{E}[Z_1|Y_s, \mathcal{G}_t^+]$, i.e of the time-$s$ estimator of the terminal output Z_1, conditional on the observation of Y_s by the agent. (In (2) below, direct substitution causes a spurious inner conditioning; the Landau notation below has the sense: $o(h)/h \to 0$, as $h \downarrow 0$.)

Proposition 1 (Conditional Bayes' formula; cf. [19]**).** *In the single agent setting, conditional on there being no disclosure in the interval* (t, s), *with* $\tilde{\gamma}_t := \mathbb{E}[Z_1|\mathcal{G}_t^+]$ *and* $\tilde{\gamma}_s := \mathbb{E}[Z_1|Y_s = \gamma_s, \mathcal{G}_t^+]$, *the equation*

$$\tilde{\gamma}_s = \mathbb{E}[Z_1|ND_s(\gamma_s),\ \mathcal{G}_t^+] \tag{1}$$

is equivalent for $q_{ts} := (s-t)\lambda_t$ *to*

$$\begin{aligned}(1-q_{ts})(\tilde{\gamma}_t - \tilde{\gamma}_s) + o(s-t) &= q_{ts}\int_{z\le\tilde{\gamma}_s}(z-\tilde{\gamma}_s)\mathrm{d}\mathbb{Q}(Z_s^{\text{est}} \le z|\mathcal{G}_t^+) \\ &= q_{ts}\int_{z\le\tilde{\gamma}_s}(z-\tilde{\gamma}_s)\mathrm{d}\mathbb{Q}(\mathbb{E}[Z_1|Y_s,\mathcal{G}_t^+] \le z|\mathcal{G}_t^+) \quad (2)\end{aligned}$$

The proof is in §5.1. The 'indifference choice' of γ is of significance: we cite our earlier result here as:

Proposition 2 (Risk neutrality, [27] – cf. [16]**).** *In the setting of Proposition 1, with* $D = D_s(\gamma)$ *and* $\tau_D^t := \mathbb{Q}[D|\mathcal{G}_t^+]$, *its market probability (conditional at time* t*), equation* (1) *is equivalent to*

$$S_t = \mathbb{E}[S_1|\mathcal{G}_t^+] = \tau_D^t \cdot \mathbb{E}[S_1|D_s(\gamma_s), \mathcal{G}_t^+] + (1-\tau_D^t)\tilde{\gamma}_s. \tag{3}$$

Consequently, the computation of the (unique) solution for $\tilde{\gamma}_s$ reduces in the Black-Scholes setting to a simple application of the Black-Scholes formulas, using the model parameters of §2.1; see the discussion of Theorem 1 and the equivalent equation (7) below in §3.1.1.

Remark. From a market-valuation perspective on asset prices, the value of the future cutoff γ_s (as above) must be impounded in the market measure, but this is exactly what (3) describes; so we may validly regard the risk-neutral measure here as a summary of an underlying equilibrium market-model, such as is described by [10].

3 Intra-period valuation: non-disclosure decay

In this section a consistently generated filtration $\bar{\mathbb{G}}$ with its censoring filter γ as defined in §2.1.7 is given and assumed to be an *optimal* censoring filter in the sense of (OC). The latter is seen to be uniquely characterized as a piecewise-deterministic Markov process (inevitably so – see [7]) capable of generating a public filtration from $\{\mathbb{Y}_i^{\text{priv}} : i \in I\}$ under which the censoring process is predictable. First, we consider the simpler case $m = 1$ (leaving the general case $m > 1$ to §3.2), and state the theorem asserting an *explicit* solution to the filtering problem (arising from a 'cutoff equation' characterizing γ_s, for $t < s < \theta_+$, which takes the form of a simple differential equation). The proof of the theorem is in **§**5.2, but we comment after the statement that the optimal censor satisfies a Bayesian updating rule at time s, from which our differential equation follows.

3.1 Single agent case

The situation when $m = 1$, and only the i-th agent is involved, follows. As in §2.3, between public disclosure dates, we must refer not only to the censor γ_t which is applied to the observation Y_t, but also to the image process

$$\tilde{\gamma}_t := \mathbb{E}[Z_1|Y_t = \gamma_t, \mathcal{G}_t] = \mathbb{E}[Z_1|ND_t(\gamma_t), \mathcal{G}_t].$$

In the Black-Scholes framework the time-t regression function $\gamma \mapsto \mathbb{E}[Z_1|Y_t = \gamma, \mathcal{G}_t]$ is monotonic. In the *single* agent context, since the single observer is the only source of any expansion of the public filtration, it is in principle possible to work in the language of disclosure/censoring solely of the forecast $\mathbb{E}[Z_1|Y_t, \mathcal{G}_t]$ by reference to $\tilde{\gamma}_t$, rather than of the disclosure/censoring of the observation Y_t. However, in the multiple agent setting this cannot readily be done, since other agents may expand the public filtration and so the connection (in equilibrium) between $\tilde{\gamma}_t^i$ and γ_t^i is more complicated. So one simply has to chase both sets of variables. For technical reasons connected with 're-starting' the Wiener process (at t and at $\theta_-^{\text{pub}}(t)$), the censors need to be re-scaled to unity at the re-starting date, hence the appearance also of a further process $\hat{\gamma}_t$ in the Corollary below. (See §2.1.4 for the relevant parameters.) In Theorem 1 reference is made to a fixed public filtration $\bar{\mathbb{G}}$ and also to general public filtrations $\mathbb{G}$ as in (OC) in §2.2 above; moreover, a connection is made between the *optional valuation* $\mathbb{E}[Z_1|\bar{\mathcal{G}}_t^+]$ at the disclosure in (i), and the predictable valuation process $\mathbb{E}[Z_1|\bar{\mathcal{G}}_t]$ in (ii) – cf. §2.2.

Theorem 1 (Decay Rule for $m = 1$; *with only the i-th agent present*)**.** *For the model of §2, suppose that* γ_t *is a càdlàg optimal observation-censoring filter, generating a disclosure filtration* $\bar{\mathbb{G}} = \{\bar{\mathcal{G}}_t^+\}$ *with associated sequence of* $\bar{\mathbb{G}}$*-disclosure arrival times* $0 = \theta_0^{\text{pub}} < \theta_1^{\text{pub}} < \theta_2^{\text{pub}} < \ldots < \theta_\ell^{\text{pub}} < 1$, *of random (finite) length* $\ell \leq N(1)$ *at which disclosures occur.*

The corresponding output-forecast process has the following properties:
(i) *disclosure updating condition ('re-initialization')*

$$g_*(\theta_-^{\text{pub}}(t)) := \mathbb{E}[Z_1|\bar{\mathcal{G}}_t^+] \equiv \mathbb{E}[Z_1|\bar{\mathcal{G}}_t, Y_t], \text{ if } t = \theta_-^{\text{pub}}(t), \tag{4}$$

or explicitly here, for $t = \theta_-^{\text{pub}}(t)$,

$$\mathbb{E}[Z_1|\bar{\mathcal{G}}_t^+] = \mathbb{E}[Z_1|Y_t, \bar{\mathcal{G}}_t] = kY_t^\kappa,$$

with

$$k = f^{1-\kappa}, \ \kappa = \kappa_1^i = p_i/(p_i + p_0);$$

(ii) *in each inter-arrival interval (i.e. between disclosure times)*

$$\sup\nolimits_{\mathbb{G}} \mathbb{E}[Z_1|\mathcal{G}_t] = \mathbb{E}[Z_1|\bar{\mathcal{G}}_t] = g_*((\theta_-^{\text{pub}}(t)) \exp\left(-\int_{\theta_-}^t \nu_s ds\right), \text{ for } \theta_-^{\text{pub}}(t) \leq t < \theta_+^{\text{pub}}(t),$$

where, on the extreme left of the display above, the left-sided-in-time supremum (as in (OC)*) ranges over all public filtrations* $\mathbb{G} = \{\mathcal{G}_t^+\}$ *and, on the extreme right:* g_* *is as in* (*i*) *above, while in the valuation formula there is a thinned decay-intensity given by*

$$\nu_t = \lambda_t[2\Phi(\hat{\sigma}_t/2) - 1] > 0 \text{ with } \hat{\sigma}_t^2 = \alpha_i^2(\sigma_0^2 + \sigma_i^2)(1-t); \tag{5}$$

(iii) *in each inter-arrival interval, the cutoff* γ_t *for the disclosure of an observation of* Y_t *satisfies*

$$\sup\nolimits_{\mathbb{G}} \mathbb{E}[Z_1|\mathcal{G}_t] = k\beta_t {\gamma_t}^\kappa,$$

where $\beta = \tilde{\beta}^i : [0, 1] \to \mathbb{R}$ (*with* $\beta(1) = 1$) *is a decreasing deterministic weighting function of time, identified explicitly in Lemma 2 of §5.2.*

In particular, the optimal censor γ_t *is a unique, piecewise-deterministic Markov process.*

The proof is in §5.2. The cutoff for disclosure of the output-forecast obtained in Theorem 1 has the prescribed explicit form, since $\tilde{\gamma}_t = \sup_{\mathbb{G}} \mathbb{E}[Z_1|\mathcal{G}_t]$ satisfies the *censoring differential equation*

$$\tilde{\gamma}'_t = -\tilde{\gamma}_t \nu_t, \text{ for } \theta^{\text{pub}}_{n-1} < t < \theta^{\text{pub}}_n, \tag{6}$$

referred to in §1. It may be interpreted in the context of §2.3 as expressing the *risk premium* of information suppression, in a way which hints at generalization to a broader class of models, one where regime shifts are accompanied by optional, so strategic, 'protective' activity, their exercise rates balancing the marginal protective-option value. The emergence of a piecewise-deterministic process, in the sense of [11], is not a surprise – see [23]. The following result explains how 'silence' (non-disclosure) is penalized less and less as time plays out.

Corollary. *In the setting of Theorem 1, the output-forecast process $\tilde{\gamma}_t$ for $0 < t < 1$ has the following properties:*

(a) *its jumps occur at the disclosure times θ^{pub}_n and are upward;*

(b) *between jumps, the output-valuation process has the representation $\tilde{\gamma}_t = \hat{\gamma}_t g_*(\theta^{\text{pub}}_{n-1})$, where the (rescaled cutoff) deterministic function $\hat{\gamma}_t$ satisfies:*

$$\hat{\gamma}'_t = -\hat{\gamma}_t \nu_t, \text{ for } \theta^{\text{pub}}_{n-1} < t < \theta^{\text{pub}}_n, \text{ with } \hat{\gamma}(\theta^{\text{pub}}_{n-1}) = 1;$$

(c) *(decreasing thinning) in any interval between disclosure times the relative decay-intensity ν_t/λ_t is decreasing;*

(d) *between consecutive non-disclosure intervals the intra-period relative decay intensity ν_t/λ_t decreases (to zero as $t \to 1$).*

Proof. This is immediate – the routine proof is omitted. □

3.1.1 Game-theoretic Aspects of Theorem 1

We stress the role of game-theoretic principles at work: the indifference principle, Bayesian updating, equilibrium, underlying the proof. Given information at time $0 < t < 1$, the cutoff value γ_s is characterized at any time s with $t < s < \theta_+(t) \le 1$ by the observer's indifference at time s, when observing Y_s, between disclosing the observed value if $Y_s = \gamma_s$ and not disclosing it; this is because the public valuation is identical in both circumstances. Indeed, the valuation is identical, because in the time interval $(t, s]$ with probability

$1-(s-t)\lambda_t+o(s-t)$ (as $s\downarrow t$) the agent has not observed Y_s, and this event cannot be distinguished by outsiders from the event that the agent observed Y_s, but did not disclose the observed value of Y_s (since it was below γ_s). This yields the indifference (equilibrium) condition as *conditional Bayes' formula:*

$$\tilde{\gamma}_s=\mathbb{E}[Z_1|ND_s(\gamma_s),\mathcal{G}_t^+], \tag{7}$$

where, as earlier $\tilde{\gamma}_s:=\mathbb{E}[Z_1|Y_s=\gamma_s,\mathcal{G}_t^+]$ and $\mathbb{G}=\{\mathcal{G}_t^+\}$ here denotes the public filtration.

The significance of (7) is that it is a special case of the *Nash Equilibrium* condition (10) below; the equation (7) first appears in the static model of Dye [13] to model rationality of partial (voluntary) disclosures, as a contrast to the total disclosure principle ('unravelling') of Grossman and Hart [17].

3.1.2 Cutoffs and the Black-Scholes formula

A consequence of (7) (and of (10) below) and is a Black-Scholes formula for the cutoff $\gamma=\tilde{\gamma}_s$ applied to the forecast (rather than the observation). The calculation goes back at least to [19] (cf. [15]), where (7) in the present context reduces to:

$$\begin{aligned}\mu_F-\gamma &= \frac{q}{1-q}H_F(\gamma),\ \text{where} \qquad (8)\\ H_F(\gamma) &= \mathbb{E}[(\gamma-F)^+]=\int(\gamma-x)^+d\mathbb{Q}(F\le x)=\int_{x\le\gamma}\mathbb{Q}(F\le x)dx,\end{aligned}$$

(cf. Prop. 1 and §5.1). Here F denotes the random variable $\mathbb{E}[Z_1|Y_s,\mathcal{G}_t^+]$, with mean μ_F (conditional at time $t+$), $q=(s-t)\lambda_t$ is the probability with which an observation occurs (independently of F) by time s, and $H_F(\gamma)$ is the 'lower first partial moment below a target γ', well-known in risk management[6], briefly termed (in view of its key role) the *hemi-mean function.*

The log-normal F above, prompts a standardization in terms of the parameters λ,σ for the solution $\gamma=\gamma_{\mathrm{LN}}(\lambda,\sigma)$ of

$$1-\gamma=\lambda H_{\mathrm{LN}}(\gamma;\sigma). \tag{9}$$

The behaviour of the solution derives from the properties of the Black-Scholes put-formula (with strike γ). The put formula for strike γ and expiry at time

[6]See for example McNeil, Frey and Embrechts [26], Section 2.2.4.

1, conditional on an initial asset valuation of $\tilde{\gamma}_t$ at time $t < 1$, yields

$$H_{\mathrm{LN}}(\gamma) = \gamma\Phi\left(\frac{\log(\gamma/\tilde{\gamma}_t) + \frac{1}{2}\sigma^2(1-t)}{\sigma\sqrt{1-t}}\right) - \gamma_t\Phi\left(\frac{\log(\gamma/\tilde{\gamma}_t) - \frac{1}{2}\sigma^2(1-t)}{\sigma\sqrt{1-t}}\right),$$

with Φ the standard normal distribution.

Consequently, for any time t in the intra-period (between voluntary disclosures) setting the strike in the formula above to be $\gamma = \tilde{\gamma}_s$ yields in the limit as $s \downarrow t$, $\tilde{\gamma}_s$ identical with $\tilde{\gamma}_t$; then the two terms add up to give, as in (5), $\tilde{\gamma}_t 2\Phi(\sigma_t/2)$.

3.1.3 Other Comments

The decay-intensity rate ν_t is composed of two factors both having economic significance. First, the Poisson intensity λ measures the instantaneous opportunity cost of the arrival of an observation of information about the output valuation, and impounds the chance both for a valuation upgrade (through a good observation), and for the suppression of poor observation value. Secondly, the intensity λ is thinned by the probability of suppressing poor valuation, the effect of a 'protective put'. Over time the protective put loses value and tends to zero as the precision improves (i.e. the volatility goes to zero).

To see the details of the formula intuitively, recall that, as above, it is assumed that the observer chooses to achieve the maximum valuation, and thus makes a voluntary disclosure according to the rule: find the cutoff function $t \mapsto \gamma_t$ such that for $t = \theta_n^{\mathrm{obs}}$:
(i) disclose credibly the value of Y, when $Y_t \geq \gamma_t$, or, equivalently, the public output-forecast $Z_t^{\mathrm{est}} := \mathbb{E}_t[Z_1|Y(\theta_n^{\mathrm{obs}})]$; and
(ii) make no disclosure, when $Y_t < \gamma_t$.
Replacing Y_t by $Z_t^{\mathrm{est}} := \mathbb{E}[Z_1|\mathcal{G}_t^+]$, one may restate the Y-cutoff problem (of choosing γ_t) in isomorphic terms as a cutoff problem for Z_t^{est} (i.e. of choosing its corresponding cutoff $\tilde{\gamma}_t$), assuming the regression function $\tilde{m}_t(y) := \mathbb{E}_t[Z_1|Y_t = y]$ is monotonic.

3.2 Multiple agent case

We now consider the general case $m > 1$. Here the starting point is a system of equations that relates the values $\{\gamma_t^i : i = 1, ..., m\}$ to each other by reference to a general Bayesian updating rule having the form of a system of (subgame) *Nash equilibrium* conditions for $i = 1, ..., m$:

$$\mathbb{E}[Z_1^i|(\forall j)ND_t^j(\gamma_t^j), \mathcal{G}_t] = \mathbb{E}[Z_1^i|ND_t^j(\gamma_t^j) \text{ for all } j \neq i, Y_i = \gamma_t^i, \mathcal{G}_t], \qquad (10)$$

with $ND_t^j(\gamma)$ being the non-disclosure event (of time t). The intended meaning is that, contingent on the information available prior to time t, in the event that all of the agents make no disclosures (for lack of observations, or because observations lie at or below their respective filtering-censor value) and the i-th agent's observation is identical to the value of the respective filtering-censor γ_t^i, the corresponding output estimate value (i.e. Z_1^i in expectation) is the same whether, or not, that agent chooses to disclose the observation. (Recall the equational convention in the footnote of §2.3.) The conditions (10) generalize (3) of §2.3 – see also §3.1.1 below.

Note that $ND_t^j(\gamma)$ is complementary to $D_t(\gamma)$, as earlier defined in §2.3 (though here in respect of the j-th agent).

In the inter-arrival period the absence of any disclosure from all of the m observation processes will influence the decay rate of each of the optimal observation filtering censors differentially, i.e. the filtering equation is bound to express the interdependence flowing from the Nash Equilibrium conditions (10) above. To express the explicit form, we need a number of parameters derived from the volatilities of §2.1. Using an abbreviating *tilde notation*, put

$$\sigma_{0i}^2 := \sigma_0^2 + \sigma_i^2 \text{ and } \tilde{\sigma}_{0i}^2(t) := (1-t)\sigma_{0i}^2,$$

and analogously:

$$\begin{aligned} \tilde{p}_i(t) \quad &: \quad = 1/[(1-t)\sigma_i^2], \ \tilde{p}(t) := \sum\nolimits_{i=0}^m \tilde{p}_i(t), \quad \tilde{\kappa}_i(t) := \tilde{p}_i(t)/\tilde{p}(t) \equiv \kappa_i, \\ \tilde{\kappa}_{-i}(t) \quad &: \quad = \tilde{p}_i(t)/(\tilde{p}(t) - \tilde{p}_i(t)) \equiv \kappa_{-i} := p_i/(p - p_i). \end{aligned}$$

Of particular significance is the function $\tilde{\rho}_i(t)$, which denotes for given t the (conditional) partial covariance (cf. [8]) of the i-th component of $(..., \sigma_0\tilde{W}_{1-t}^0 + \sigma_i\tilde{W}_{1-t}^i, ...)$ on the remaining components, where $\tilde{W}$ denotes the Wiener process W re-started at time t.

We may now state the general theorem; use of the subscript 'hyp' here is explained in the discussion below. The function $\tilde{\beta}_m^i$ corresponds to $\tilde{\beta}^i$ in Theorem 1, and is derived in Lemma 2_m of §5.4. Below $\tilde{\gamma}_{it} = \sup_{\mathbb{G}} \mathbb{E}[Z_1^i|\mathcal{G}_t]$, is as earlier, but y_{it} replaces γ_{it} to allow $\tilde{y}_{it}$ to have another meaning, corresponding to a rescaling of Y_s^i to $\tilde{Y}_s^i$ in the proof. The notation here of $\bar{\mathbb{G}}$ and $\mathbb{G}$ is as in Theorem 1.

Theorem 1_m (Filtering rule during continued non-disclosure: $m \geq 1$). *For the model of §2, suppose $y_t = (..., y_{it}, ...)$ is a càdlàg optimal censoring filter generating a disclosure filtration $\bar{\mathbb{G}}$ with associated sequence of $\bar{\mathbb{G}}$-disclosure-arrival times $0 = \theta_0^{\text{pub}} < \theta_1^{\text{pub}} < \theta_2^{\text{pub}} < ... < \theta_\ell^{\text{pub}} < 1$ of random (finite) length $\ell \leq \sum_{i=1}^m N^i(1)$, at which disclosure events occur.*

Then, for $0 \leq n \leq \ell$, the corresponding output-forecast process has the following properties:

(i) *disclosure updating condition (the i-th agent's 're-initialization')*

$$g_*^i(t) := \mathbb{E}[Z_1^i|\mathcal{G}_t^+] = \mathbb{E}[Z_1^i|\{Y_t^j : j \in J_t\}], \text{ if } t = \theta_-^{\text{pub}}(t); \tag{11}$$

(ii) *in each inter-arrival interval $\theta = \theta_-^{\text{pub}}(t) \leq t < \theta_+^{\text{pub}}(t)$*

$$\sup\nolimits_{\mathbb{G}} \mathbb{E}[Z_1^i|\mathcal{G}_t] = \mathbb{E}[Z_1^i|\bar{\mathcal{G}}_t] = k_m^i \tilde{\beta}_m^i g_*^i(\theta) \exp\left(-\int_\theta^t \nu_{\text{agg}}(s)ds\right),$$

where on the extreme left, the left-sided-in-time supremum ranges over public filtrations $\mathbb{G} = \{\mathcal{G}_t^+\}$ and, on the extreme-right, the correlation-aggregated decay-intensity v_{agg} is

$$\nu_{\text{agg}}(t) := \sum\nolimits_j \frac{\kappa_j}{\kappa_{-j}} \left(1 + \sum\nolimits_h \frac{\alpha_h}{\alpha_j} \frac{\kappa_h}{\kappa_0}\right) \nu_{j\text{hyp}}(t),$$

and

$$\nu_{i\text{hyp}}(t) := \Phi\left(\frac{1}{2}\alpha_i \kappa_i \tilde{\sigma}_{0i} \sqrt{1 - \tilde{\rho}_i^2}\right) \tilde{\lambda}_i,$$

and $\tilde{\beta}_m^i = \tilde{\beta}_{\text{indiv}}^i \cdot \tilde{\beta}_{\text{agg}}$ with

$$\tilde{\beta}_{\text{indiv}}^i := \mu(\alpha_i, (1 - \kappa_0)\tilde{\sigma}_0^2)\mu(\kappa_1^i, (1 - \kappa_0)\tilde{\sigma}_i^2), \text{ and } \tilde{\beta}_{\text{agg}} = \prod\nolimits_j \mu(\kappa_j, \tilde{\sigma}_{0j}^2);$$

(iii) *in each inter-arrival interval between disclosure event times, the disclosure censor y_t^i of the i-th agent is given for $\theta = \theta_{n-1}^{\text{pub}} \leq t < \theta_n^{\text{pub}}$ by:*

$$\begin{aligned} \frac{1}{\alpha_i} \log y_t^i &= -\frac{1}{\alpha_i \kappa_{-i}} \int_\theta^t \nu_{i\text{hyp}}(s)ds \\ &\quad -\frac{1}{\kappa_0}\left(\frac{\kappa_1}{\alpha_1 \kappa_{-1}} \int_\theta^t \nu_{1\text{hyp}}(s)ds + \frac{\kappa_2}{\alpha_2 \kappa_{-2}} \int_\theta^t \nu_{2\text{hyp}}(s)ds + ...\right). \end{aligned}$$

In particular, the optimal censor y_t is a unique, piecewise-deterministic Markov process.

Note that for $m = 1$ this reduces to Theorem 1. The proof is in §5.3 and depends (as does Theorem 1) on the factorization of the process $M = M^i$ (with σ_M for σ_{M_i}) for fixed t in the self-evident form

$$M_1 = M_0 \exp(\sigma_M W_1 - \frac{1}{2}\sigma_M^2) = M_t \exp(\sigma_M \tilde{W}_{1-t} - \frac{1}{2}\sigma_M^2(1-t)),$$

where $\tilde{W}^i$ is W^i re-started at time t.

Discussion of Theorem $\mathbf{1}_m$. This result builds on Theorem 1, hence bears appropriate similarities (e.g. updating at disclosure dates), and relies on the conditional Bayes' formula (7), so we now comment only on what is most significantly different here for $m > 1$, namely the need to disaggregate the co-dependance. To define the observation cutoffs of the m observing agents, one first constructs m corresponding agents, termed *hypothetical* agents, each of whom faces a suppressed-observation problem of the kind considered in Theorem 1, but in isolation. This introduces two features: firstly, an *amended-mean factor* $L_{-i}(t)$, multiplying the current conditional mean of the observation process (defined in Theorem M of §5.3 and reflecting incremental effects of a single agent, that are based only on loading and precision factors) – and, secondly, the *partial covariance* $\tilde{\rho}_i$, arising from the use of the Schur complement (see [3, Note 4.27, p.120], cf. [21, Ch. 27], [22, §§46.26-28]). The corresponding hypothetical observation-cutoffs are later aggregated to yield observation cutoffs of the original m-agent problem. From these observation cutoffs a current forecast $\tilde{\gamma}_t^i$ of the i-th output Z_1^i is derived. Recall that in Theorem 1 the formula for $\tilde{\gamma}_t$ required the use of a rescaled function $\hat{\gamma}_t$ (with unit value at $t = \theta$, the latest disclosure date). Likewise in Theorem 1_m we also see a function $\hat{\gamma}_t^i$ used in similar fashion to construct $\tilde{\gamma}_t^i$; here the size-constant k_m^i corresponds to f^i, and reflects relative output magnitude, just as κ_i reflects relative precision.

4 Application: Market valuation with censored voluntary disclosures

The differential equations of Theorem 1_m can be used to derive comparative statics of price formation in an asset market, in which investors expect voluntary disclosures (with positive probability) at all times between the two fixed mandatory disclosure dates. These comparative statics are concerned with intervals of time during which the agents are known to be privately and intermittently observing noisy signals of the asset values, arising from the common effect. The interpretation of §2.3 extends here to

$$S_t^i := \mathbb{E}[\mathbb{E}[Z_1^i|\mathcal{G}_1^+]|\mathcal{G}_t^+]$$

as an asset-price process (with expectations under $\mathbb{Q}$ as the market's risk-neutral valuation measure).

Agent i is endowed intermittently with private information about the evolution of the i-th asset price via the martingale Y_t^i, and can voluntarily disclose information about the asset to the market. At times t strictly between consecutive public disclosure arrival times, conditioning on $\mathcal{G}_t^+$ or on $\mathcal{G}_t$ yields identical forecasts of the time-1 valuation $\mathbb{E}[Z_1^i|\mathcal{G}_1^+]$, so correspondingly the asset price S_t^i is identical with $\tilde{\gamma}_t^i = \mathbb{E}[\mathbb{E}[Z_1^i|\mathcal{G}_1^+]|\mathcal{G}_t]$, where $\tilde{\gamma}_t^i$ is the unique valuation-cutoff given by Theorem 1_m. Furthermore, §2.3 describes disclosure behaviour as being incentivized so that the asset price gives the current stock-holders (investors) asset valuations that are maximal given the available public information.

The effects on price formation may thus be studied by recourse to the observation cutoffs employed by the respective agents in terms of the parameters of the model: the precisions p_i, the loading factors α_i, and the private observation arrival intensities λ_i. It is interesting to note the predictions about suppression. Since the correlation between firms and the environment factor are positive, the formulas established above imply that a good-news *bandwagon effect* holds: ceteris paribus, agents *all* choose a higher cutoff (relative to the single agent case $m = 1$), reducing the probability that they will release private observations. Additionally, there is an intuitively clear *estimator-quality effect,* which leads to agents being *partitioned* into below- and above-'average precision' (over the m-agent population), as in the theorems that follow. Those with below-average precision are shown to adopt a lower cutoff (relative to the single agent case), and ceteris paribus increase the probability that they will release private observations, with the reverse holding for the above-average.

Bandwagon Theorem. *In any intra-period the presence of correlation increases the precision parameter of the cutoff and hence raises the cutoff:*

$$\hat{\gamma}_{\text{LN}}(\tilde{\lambda}_i, \tilde{\sigma}_{0i}) < \hat{\gamma}_{\text{LN}}(\tilde{\lambda}_i, \kappa_i\tilde{\sigma}_{0i}) < \hat{\gamma}_{\text{LN}}\left(\tilde{\lambda}_i, \kappa_i\tilde{\sigma}_{0i}\sqrt{1-\tilde{\rho}_i^2}\right).$$

where $\hat{\gamma}_{\text{LN}}(\lambda, \sigma)$ denotes the unique solution of the following equation in y :

$$(1 - y) = \lambda H_{\text{LN}}(y, \sigma),$$

and represents the normalized cutoff in the single agent case, as in (9).

Proof. This is immediate from the static model of [15, §6].

When the correlation is positive, there is also a counter-vailing precision effect on the related hypothetical agent's cutoff, arising from the amended mean factor L_{-i} (see the discussion of Theorem 1_m above), when the actual agent has below-average precision, as defined below.

Estimator-Quality Theorem. *Suppose that $m \geq 2$ and $\alpha_i > 0$ for all i. The amended mean $L_{-i}(t)$ of the hypothetical firm i increases with p_i and*

$$\exp\left(-\frac{\alpha_i(1-t)}{2(p-p_i)}\right) \quad < \quad L_{-i}(t) < \exp\left(\frac{\alpha_i\left(1+\frac{\alpha_i-1}{n-1}\right)(1-t)}{2p_{\text{av},-i}}\right),$$

$$\text{where } p_{\text{av},-i} \quad : \quad = \frac{p-p_i}{n-1+\alpha_i}.$$

In particular, if the loading index is identical for all firms, then

$$L_{-i}(t) < L_{-j}(t) \text{ iff } p_i < p_j.$$

Otherwise, if $0 < \alpha_i < \alpha_j$ and $p_i < p_j$, then also $L_{-i}(t) < L_{-j}(t)$.

The amended mean is a strict deflator, i.e. $L_{-i}(t) < 1$, iff p_i is below the loading-adjusted competitor average, i.e.

$$p_i < \frac{p}{n-1+\alpha_i} := p_{\text{av,-}i}$$

so that for $\alpha_i = 1$ one has $p_{\text{av,-}i} = p/n$.

Proof. This again is immediate from the static model of [15, §6].

Thus low-precision managers are more likely to make a disclosure, but by definition the disclosure will be less precise. Hence, in terms of giving it weighting in the updating rules, investors will give less weight to disclosure of bad news by such imprecise managers.

5 Proofs

We begin in §5.1 with a Proof of Proposition 1 and use it in §5.2 to prove Theorem 1, but only after we have prepared the ground with the calculations in two lemmas. In §5.3 we prove Theorem 1_m; the argument will require generalizations of the lemmas of §5.2, and these are relegated to §5.4.

5.1 Proof of Proposition 1

In the notation of §2, we have

$$\begin{aligned}\tilde{\gamma}_s &= \frac{\mathbb{Q}(N(s)=N(t))\cdot\mathbb{E}[Z_1|\mathcal{G}_t^+]+\mathbb{Q}(N(t)<N(s))\cdot\int_{z_1\le\tilde{\gamma}_s} z_1 d\mathbb{Q}(\mathbb{E}[Z_1|Y_s,\mathcal{G}_t^+]\le z_1|\mathcal{G}_t^+)}{\mathbb{Q}(N(s)=N(t))+\mathbb{Q}(N(t)<N(s))\mathbb{Q}(\mathbb{E}[Z_1|Y_s,\mathcal{G}_t^+]\le\tilde{\gamma}_s|\mathcal{G}_t^+)}\\ &= \frac{(1-(s-t)\lambda_t)\cdot\tilde{\gamma}_t+\mathbb{Q}(N(t)<N(s))\cdot\int_{z_1\le\tilde{\gamma}_s} z_1 d\mathbb{Q}(\mathbb{E}[Z_1|Y_s,\mathcal{G}_t^+]\le z_1|\mathcal{G}_t^+)}{(1-(s-t)\lambda_t)+(s-t)\lambda_t\mathbb{Q}(\mathbb{E}[Z_1|Y_s,\mathcal{G}_t^+]\le\tilde{\gamma}_s|\mathcal{G}_t^+)}.\end{aligned}$$

So putting $q_{ts}=(s-t)\lambda_t$, cross-multiplying and re-arranging one has

$$\begin{aligned}(1-q_{ts})(\tilde{\gamma}_t-\tilde{\gamma}_s)+o(s-t) &= q_{ts}\int_{z_1\le\tilde{\gamma}_s}(z_1-\tilde{\gamma}_s)\mathrm{d}\mathbb{Q}(\mathbb{E}[Z_1|Y_s,\mathcal{G}_t^+]\le z_1|\mathcal{G}_t^+)\\ &= -q_{ts}\int_{z_1\le\tilde{\gamma}_s}\mathbb{Q}(\mathbb{E}[Z_1|Y_s,\mathcal{G}_t^+]\le z_1|\mathcal{G}_t^+)\mathrm{d}z_1,\end{aligned}$$

where the last line is from integrating by parts. □

5.2 Proof of Theorem 1

This section is devoted to a proof of Theorem 1. Although $m=1$ below, having in mind the notational needs of §5.3, where $m>1$, it is helpful to employ here a general index i for the single agent of interest (rather than to specialize $i=1$), thereby anticipating the later general case. Also, as $m=1$, the index i can be omitted at will, whenever convenient. We begin with some auxiliary results stated as Lemmas 1 and 2, which will require the notation

$$\mu(\kappa,\sigma^2):=e^{\frac{1}{2}\kappa(\kappa-1)\sigma^2}.$$

The exponent $\frac{1}{2}\kappa(\kappa-1)\sigma^2$ in μ above corresponds to the second-order term of Itô's Lemma for the power function $t\to t^\kappa$, a recurring feature of the regression formulas (from Lemma 1). For other parameters refer to §2.1.

Recall the tilde notation of §3.2 that, for any constant σ, we write $\tilde{\sigma}$ for the function $\tilde{\sigma}^2(t) := \sigma^2 \cdot (1-t)$, with value corresponding to re-starting the model at time t. In particular, with $\bar{\sigma}^2_{0i} := \alpha_i^2 \sigma_0^2 + (\alpha_i \sigma_i)^2$, we write

$$\tilde{\sigma}_i^2(t) = \sigma_i^2(1-t) \text{ and } \bar{\sigma}^2_{0i}(t) = \alpha_i^2[\sigma_0^2 + \sigma_i^2](1-t) = \alpha_i^2 \tilde{\sigma}^2_{0i}.$$

For convenience, we may denote X also by M^0. For the Wiener processes W^i, recall the corresponding Wiener process re-started at time t by

$$\tilde{W}^i_s := W^i_{t+s} - W^i_t.$$

(So $\bar{W}_0 = 0$.) Omitting suffices and writing σ_W for σ_i^M,

$$M_{t+s} = M_t \exp(\sigma_W \tilde{W}_s - \frac{1}{2}\sigma_W^2 s). \tag{12}$$

We need two lemmas.

Lemma 1 (Valuation of Z_1 given observation Y_1 for $m = 1$). *Put*

$$\kappa = \kappa_1^i = p_i/(p_0 + p_i).$$

Then

$$\mathbb{E}[Z_1 | Y_1^i = y, \mathcal{G}_1] = k y^\kappa \text{ for } k = k_1^i = (f^i)^{1-\kappa}.$$

Proof. We cite and apply a formula from [15, Prop. 10.3]. To distinguish notational contexts, we use overbars on letters when citing formulas from there. The noisy observation there, $\bar{T}$, of a random (state) variable $\bar{X}$ takes the form $\bar{T} = \bar{X}\bar{Y}$, where $\bar{X}$ and $\bar{Y}$ are independent random variables, with their log-normal distributions having underlying normal precision parameters $\bar{p}_X, \bar{p}_Y$ respectively. The required formula is

$$\mathbb{E}[\bar{X}^\alpha | \bar{T}] = \bar{K}_\alpha (\bar{T})^{\alpha\kappa},$$

where $\kappa = \bar{p}_Y / \bar{p}$ for $\bar{p} = \bar{p}_X + \bar{p}_Y$ and

$$\bar{K}_\alpha = \exp\left(\frac{\alpha + \alpha(\alpha - 1)}{2\bar{p}}\right).$$

We will take $\bar{X} = X_1$ and $\bar{Y} = M_1^{1/\alpha}$, with $M = M^i$ and $\alpha = \alpha_i$ below. We first compute the corresponding constants $\bar{p}_X, \bar{p}_Y$ and $\bar{K}_\alpha$. Substituting $s = 1 - t = \Delta t$ in (12) above, gives

$$X_1 = X_{1-\Delta t} \exp(\sigma_0 \tilde{W}^0_{\Delta t} - \frac{\sigma_0^2}{2}\Delta t),$$

and

$$M_1^{1/\alpha} = M_{1-\Delta t}^{1/\alpha} \exp(\frac{\sigma_i^M}{\alpha_i} \tilde{W}_{\Delta t}^i - \frac{(\sigma_i^M)^2}{2\alpha_i} \Delta t) = M_{1-\Delta t}^{1/\alpha} \exp(\sigma_i \tilde{W}_{\Delta t}^i - \frac{1}{2} \alpha_i \sigma_i^2 \Delta t).$$

Conditional on the realizations of $X_{1-\Delta t}$ and $M_{1-\Delta t}^{1/\alpha}$, $\bar{X}$ and $\bar{Y}$ have respective underlying conditional variances of $\sigma_0^2 \Delta t$ and $\sigma_i^2 \Delta t$, as $\sigma_i^M/\alpha_i = \sigma_i$. So the corresponding regression coefficient κ for $\bar{X}$ on $\bar{Y}$ is

$$\frac{\alpha^2/(\alpha_i \sigma_i)^2}{\alpha^2/(\alpha_i \sigma_i)^2 + 1/\sigma_0^2} = \frac{1/(\sigma_i)^2}{1/(\sigma_i)^2 + 1/\sigma_0^2} = \frac{p_i}{p_i + p_0} = \kappa_1^i,$$

having cancelled $\Delta t > 0$ from numerator and denominator; this remains constant as Δt varies, and so also in the passage as $\Delta t \to 0$. Also

$$1/\bar{p} = \frac{\Delta t}{1/\sigma_i^2 + 1/\sigma_0^2},$$

so $\bar{K}_\alpha = \bar{K}_\alpha(\Delta t) \to 1$ as $\Delta t \to 0$. Finally, since $Y_t^i = Z_t^i M_t^i = f^i X_t^{\alpha_i} M_t^i$,

$$\bar{T} = (Y_1^i/f^i)^{1/\alpha} = X_1 M_1^{1/\alpha} = \bar{X}\bar{Y},$$

and so, conditioning on $Y_1^i = y$, and setting $k = f^{1-\kappa}$,

$$\mathbb{E}[Z_1^i|\bar{T}] = \mathbb{E}[f^i X_1^\alpha|\bar{T}] = f^i(\bar{T})^{\alpha\kappa} = f^i((Y_1^i/f^i)^{1/\alpha})^{\alpha\kappa} = f^i(y/f^i)^\kappa = ky^\kappa. \qquad \square$$

Below the deterministic functions $\tilde{\beta}^i$ factor into $\tilde{\beta}^i_{\text{indiv}}, \tilde{\beta}_{\text{agg}}$ to anticipate m-fold versions which 'separate' individual and aggregate effects of agents.

Lemma 2 (Time-t conditional law of the valuation of Z_1, given observation Y_t – for $m = 1$). *Conditional on $Y_t^i = y$, the time-t distribution of the time-1 valuation $\mathbb{E}[Z_1|Y_1, \mathcal{G}_1]$ is that of*

$$k\tilde{\beta}^i y^\kappa \hat{Z}_t := k\tilde{\beta}^i_{\text{indiv}} \tilde{\beta}_{\text{agg}} y^\kappa \hat{Z}_t,$$

with $k = k_1^i$ as in Lemma 1, and:
(i)

$$\begin{aligned} \kappa &= \kappa_1^i, \ \tilde{\beta}^i_{\text{indiv}} := (\mu_t^0(\alpha_i)\mu_t^i)^\kappa, \ \tilde{\beta}_{\text{agg}} := \mu(\kappa, \alpha_i^2 \tilde{\sigma}_{0i}^2), \\ \mu_t^i &:= \mu(\alpha_i, \alpha_i^2 \tilde{\sigma}_i^2) \text{ and } \mu_t^0(\alpha_i) = \mu(\alpha_i, \tilde{\sigma}_0^2); \end{aligned}$$

(ii) $\hat{Z}_t$ *log-normal, its underlying mean-zero normal of variance* $\hat{\sigma}_t^2 = \kappa\alpha_i^2\tilde{\sigma}_{0i}^2$. *In particular, this time-t distribution has mean given by*

$$\mathbb{E}[Z_1|Y_t^i = y, \mathcal{G}_t] = k\tilde{\beta}_{\text{indiv}}^i\tilde{\beta}_{\text{agg}}y^\kappa.$$

Proof. From (12) with $M = M^i$ and *any* $\delta > 0$

$$M_{t+s}^\delta = M_0^\delta \exp(\delta\sigma_M W_{t+s} - \frac{1}{2}\delta\sigma_M^2(t+s)) = M_t^\delta \exp(\delta\sigma_M \tilde{W}_s - \frac{1}{2}\delta\sigma_M^2 s).$$

So, for $s = 1 - t$,

$$M_1^\delta = \mu_t(\delta)M_t^\delta \exp(\delta\sigma_M\tilde{W}_{1-t} - \frac{1}{2}\delta^2\sigma_M^2(1-t)),$$

where the last term has unit-mean and

$$\mu_t(\delta) = \mu(\delta, \tilde{\sigma}_M^2) = \mu(\delta, \alpha_i^2\tilde{\sigma}_i^2).$$

In particular, for $\delta = \kappa = \kappa_1^i$ (i.e. for κ as in Lemma 1) and $M = M^i$,

$$M_1^\kappa = \mu_t^i \cdot M_t^\kappa \cdot \exp(\kappa\alpha_i\sigma_i W_t^i(1-t) - \frac{1}{2}\kappa^2\alpha_i^2\sigma_i^2(1-t)),$$

where $\mu_t^i = \mu(\kappa, \alpha_i^2\tilde{\sigma}_i^2) = \mu(\kappa_1^i, \alpha_i^2\tilde{\sigma}_i^2)$; likewise, for $M = X$ and $\delta = \alpha_i$,

$$Z_1^i = f^i X_1^{\alpha_i} = \mu_t^0(\alpha_i) \cdot Z_t^i \cdot \exp(\alpha_i\sigma_0\tilde{W}_{1-t}^0 - \frac{1}{2}\alpha_i^2\sigma_0^2(1-t)),$$

where $\mu_t^0(\alpha_i) = \mu(\alpha_i, \tilde{\sigma}_0^2)$. Combining, as $Y_t^i = Z_t^i M_t^i$, for any $\delta > 0$:

$$(Y_1^i)^\delta = (\mu_t^0(\alpha_i)\mu_t^i \cdot Z_t^i M_t^i)^\delta \cdot \exp(\alpha_i\delta\sigma_0\tilde{W}_{1-t}^0 - \delta\alpha_i^2\sigma_0^2(1-t)/2) \cdot \exp(\delta\alpha_i\sigma_i\tilde{W}_{1-t}^i - \delta\alpha_i^2\sigma_i^2(1-t)/2).$$

But

$$\delta[\alpha_i\sigma_0\tilde{W}_{1-t}^0 + \alpha_i\sigma_i\tilde{W}_{1-t}^i] = \delta\alpha_i[\sigma_0 W_t^0(1-t) + \sigma_i\tilde{W}_{1-t}^i]$$

has variance $\delta^2\alpha_i^2\tilde{\sigma}_{0i}^2$, where $\tilde{\sigma}_{0i}^2 = [\sigma_0^2 + \sigma_i^2](1-t)$. So taking

$$\hat{Z}_t(\delta) := \exp\left(\delta\alpha_i[\sigma_0\tilde{W}_{1-t}^0 + \sigma_i\tilde{W}_{1-t}^i] - \frac{1}{2}\delta^2\alpha_i^2\tilde{\sigma}_{0i}^2\right),$$

which has unit-mean and variance $\delta^2\alpha_i^2\tilde{\sigma}_{0i}^2$, gives

$$(Y_1^i)^\delta = (\mu_t^0\mu_t^i \cdot Y_t^i)^\delta \mu(\delta, \alpha_i^2\tilde{\sigma}_{0i}^2)\hat{Z}_t(\delta). \tag{13}$$

In particular, condition on $Y_t^i = y$, take $\delta = \kappa$ above, and set

$$\hat{Z}_t := \hat{Z}_t(\kappa);$$

then, by (13) and with k as in Lemma 1 above, the conditional time-t distribution of $\mathbb{E}[Z_1|Y_1, \mathcal{G}_1]$ is that of the variable Z_t^{est} given by

$$Z_t^{\text{est}} := k(Y_1^i)^\kappa = k(\mu_t^0 \mu_t^i \cdot y)^\kappa \mu(\kappa, \alpha_i^2 \tilde{\sigma}_{0i}^2)\hat{Z}_t. \;\Box$$

Proof of Theorem 1. Below we suppress reference to the unique agent i. We condition on the event $\theta_-^{\text{pub}}(t) \le t < s < \theta_+^{\text{pub}}(t)$, i.e. that there has been no subsequent disclosure in $(t, s]$. Denote by γ_t the càdlàg censor assumed in Theorem 1. For $t \le u \le s$, let $ND_u(\gamma)$ denote the event that, at time u, either $\Delta N(u) = 0$ or the agent observes Y_s to be below γ, and let $\tilde{\gamma}_u$ be the time-u evaluation of the random variable $\mathbb{E}[Z_1|Y_1, \mathcal{G}_1]$; then

$$\tilde{\gamma}_u = \mathbb{E}[Z_1|ND_u(\gamma_u), \mathcal{G}_t^+].$$

As in §2.3, by the Indifference Principle (cf. [15, §11 (Appendix 8)]), the unique cutoff value γ_u for Y_u and the time-u evaluation $\tilde{\gamma}_u$ of $\mathbb{E}[Z_1|Y_1, \mathcal{G}_1]$ are related by

$$\tilde{\gamma}_u = \mathbb{E}[Z_1|ND_u(\gamma_u), \mathcal{G}_t^+] = \mathbb{E}[Z_1|Y_u = \gamma_u, \mathcal{G}_t^+] = k\tilde{\beta}_u \gamma_u^\kappa,$$

with $\tilde{\beta}_u = \tilde{\beta}_u^i$ denoting $\tilde{\beta}^i$ evaluated at u, as in Lemma 2 above, so that

$$\tilde{\gamma}_u = k\tilde{\beta}_u \gamma_u^\kappa, \text{ or } \log \tilde{\gamma}_u = \kappa \log \gamma_u + \log k\tilde{\beta}_u.$$

For $u > t$, put $\hat{\gamma}_u := \tilde{\gamma}_u/\tilde{\gamma}_t$ (so that $\hat{\gamma}_t = 1$), thus rescaling $\mathbb{E}[\mathbb{E}[Z_1|Y_1, \mathcal{G}_1]|\mathcal{G}_t^+]$, the time-$t$ valuation of $\mathbb{E}[Z_1|Y_1, \mathcal{G}_1]$ to unity; we now work as though $\tilde{\gamma}_t = 1$. Let the corresponding time-u valuation be the random variable $\hat{Z}_u$ of Lemma 2. The underlying zero-mean normal random variable of the Lemma has variance $\hat{\sigma}_u^2 := \kappa^2 \tilde{\sigma}_{0i}^2(u)$. In the notation of Lemma 2, with $q_{ts} := (s - t)\lambda_t$ the formula of Prop. 1 gives

$$\begin{aligned} (1 - q_{ts})(1 - \hat{\gamma}_s) + o(s - t) &= q_{ts} \int_{z_s \le \hat{\gamma}_s} (z_s - \hat{\gamma}_s) \mathrm{d}\mathbb{Q}(\hat{Z}_s \le z_s | \mathcal{G}_t, \tilde{\gamma}_t = 1) \\ &= -q_{ts} \int_{z_s \le \hat{\gamma}_s} \mathbb{Q}(\hat{Z}_s \le z_s | \mathcal{G}_t, \tilde{\gamma}_t = 1) \mathrm{d}z_s. \end{aligned}$$

Dividing by $-q_{ts}$ and rearranging the differential term, by the Black-Scholes formula

$$\begin{aligned} -(1-q_{ts})\frac{1}{\lambda_t}\frac{\hat{\gamma}_s-1}{(s-t)} &= \int_{z_s\leq\tilde{\gamma}_s}\mathbb{Q}(\hat{Z}_s\leq z_s|\mathcal{G}_t,\tilde{\gamma}_t=1)\mathrm{d}z_s \\ &= \hat{\gamma}_s\Phi\left(\frac{\log(\hat{\gamma}_s)+\frac{1}{2}\hat{\sigma}^2(1-t)}{\hat{\sigma}\sqrt{1-t}}\right)-\Phi\left(\frac{\log(\hat{\gamma}_s)-\frac{1}{2}\hat{\sigma}^2(1-t)}{\hat{\sigma}\sqrt{1-t}}\right), \end{aligned}$$

up to $o(s-t)/[(s-t)\lambda_t]$. Rearranging once more and, using the abbreviating notation $\hat{\sigma}_t^2$ for the variance,

$$(1-q_{ts})\frac{1}{\lambda_t}\frac{\hat{\gamma}_s-1}{(s-t)}=-\hat{\gamma}_s\Phi\left(\frac{\log(\hat{\gamma}_s)+\frac{1}{2}\hat{\sigma}_t^2}{\hat{\sigma}_t}\right)+\Phi\left(\frac{\log(\hat{\gamma}_s)-\frac{1}{2}\hat{\sigma}_t^2}{\hat{\sigma}_t}\right)+\frac{o(s-t)}{(s-t)\lambda_t}.$$

Because y_u is càdlàg, so also are $\tilde{\gamma}_u$ and $\hat{\gamma}_u$; so the terms on the right have a limit as $s\downarrow t$. As $q_{ts}\to 0$, the function $\hat{\gamma}(u)$ is seen to be right-differentiable at t, and, since $\Phi(-u)=1-\Phi(u)$,

$$\frac{1}{\lambda_t}\hat{\gamma}'(t)=-[2\Phi(\hat{\sigma}_t/2)-1], \tag{14}$$

or equivalently, recalling that $\hat{\sigma}_t^2=\alpha_i^2(\sigma_0^2+\sigma_i^2)(1-t)$,

$$\frac{\tilde{\gamma}'(t)}{\tilde{\gamma}(t)}=-\lambda_t[2\Phi(\hat{\sigma}_t/2)-1]. \tag{15}$$

Now unfixing t, we permit t to vary over an interval during which there is no disclosure. Solving the differential equation, by integrating from the last disclosure date $\theta_-(t)\geq 0$ to the date t, and conditioning on t being prior to the next disclosure $\theta_+(t)$, gives the following:

$$\begin{aligned} \log\left(\tilde{\gamma}(t)/\tilde{\gamma}(\theta_-(t))\right) &= -\int_{\theta_-}^t\lambda_u[2\Phi(\hat{\sigma}_u/2)-1]du, \\ \tilde{\gamma}(t) &= \tilde{\gamma}(\theta_-(t))\exp\left(-\int_{\theta_-}^t\lambda_u[2\Phi(\hat{\sigma}_u/2)-1]du\right). \end{aligned}$$

Note that as $\hat{\sigma}_u\geq 0$, the factor $[2\Phi(\hat{\sigma}_s/2)-1]$ is non-negative.

So the conditional time-t evaluation of $\mathbb{E}[Z_1|Y_1]$ is given explicitly by

$$\tilde{\gamma}_t=\mathbb{E}[\mathbb{E}_1[Z_1|Y_1]|ND_t(\gamma_t),\mathcal{G}_t]=\mathbb{E}[Z_1|ND_t(\gamma_t),\mathcal{G}_t]=\hat{\gamma}(t)\mathbb{E}[Z_1|Y_\theta,\mathcal{G}_\theta],$$

for $\theta = \theta_-(t)$ (and with t unfixed), where now

$$\hat{\gamma}(t) := \exp\left(-\int_{\theta_-}^{t} \lambda_u[2\Phi(\hat{\sigma}_u/2) - 1]du\right),$$

by abuse of notation, as this function satisfies (14), but with $\hat{\gamma}(\theta_-(t)) = 1$.

To obtain the explicit form, apply Lemma 2 with $\theta = \theta_-$ to give

$$\mathbb{E}[Z_1|Y_\theta, \mathcal{G}_\theta] = k\beta_\theta Y_\theta^\kappa,$$

where, re-instating the index i,

$$\kappa = \kappa_1^i = p_i/(p_0 + p_i), \qquad k = k_1^i = f_i^{1-\kappa_1^i}, \qquad \beta_t = (\mu_t^0 \mu_t^i)^{\kappa_1^i} \mu(\kappa_1^i, \tilde{\sigma}_{0i}^2),$$

with $\mu_t^i := \mu(\alpha_i, \alpha_i^2 \tilde{\sigma}_i^2)$ and $\mu_t^0(\alpha_i) = \mu(\alpha_i, \tilde{\sigma}_0^2)$. Finally, the cutoff for Y_u^i is given explicitly by Lemma 2 as

$$\gamma_u^i = (\tilde{\gamma}_u^i/(k_1^i \beta_u^i))^{1/\kappa_1^i}.$$

5.3 Proof of Theorem 1_m

This section is devoted to a proof of Theorem 1_m, which is like Theorem 1, but more intricate in its details on account of an application of a result from [15, Th. 14.2 (Appendix 7)], Theorem M below. Appropriate substitutions (of the parameter values used here for the parameter values used there) are needed; justification of these is routine, but cumbersome, so shown as a tabulation in the arXiv version only; their basis comes from some calculations deferred to §5.4. In view of similarities, as well as differences of context, between the present and the source paper, we follow the convention of §5.2 of *overbarring* variables cited from [15, Th. 14.2 (Appendix 7)]. This allows Theorem M to be read as applicable in either of the two contexts according as parameters are over-barred, or not. We note that $\bar{\lambda} = 1/\lambda$ – the λ variables of the two papers are reciprocals, by Prop. 1.

Theorem M (Multi-firm Cut-off Equations, [15, Th. 14.2 (Appendix 7)]**).** *In the setting of this section, after rescaling so that $Y_t^i = 1$, with observations Y_s^i of time $s > t$ replaced by their re-scaled versions $\tilde{Y}_s^i$, the simultaneous conditional Bayes equations determining the cutoffs for $\tilde{Y}_s^i$ may be reduced to a non-singular system of linear equations relating the log-cutoffs* *to the hypothetical cutoffs g_i defined below. Furthermore, the unique solution for the disclosure cutoff $\tilde{y}^i$ for the observation of $\tilde{Y}^i_s$ is given by*

$$\log \tilde{y}^i = \frac{\log g_i}{\alpha_i \kappa_{-i}} + \frac{1}{\kappa_0}\left(\frac{\kappa_1}{\alpha_1\kappa_{-1}}\log g_1 + \frac{\kappa_2}{\alpha_2\kappa_{-2}}\log g_2 + ... + \frac{\kappa_m}{\alpha_m\kappa_{-m}}\log g_m\right), \tag{16}$$

where

$$\begin{aligned} g_i &= g_i(s) = \hat{\gamma}_{\mathrm{L\ N}}(\tilde{\lambda}_i, \alpha_i\kappa_i\tilde{\sigma}_{0i}\sqrt{1-\tilde{\rho}_i^2})L_{-i} \text{ with } \tilde{\lambda}_i = \lambda_i(s), \text{ the } N^i_s \text{ intensity,} \\ L_{-i} &= L_{-i}(s) = \exp\left(\frac{\alpha_i(m-1)+\alpha_i(\alpha_i-1)}{2(\tilde{p}-\tilde{p}_i)}\right)\exp\left(-\frac{m\alpha_i+\alpha_i(\alpha_i-1)}{2\tilde{p}}\right), \\ &\quad \text{(the `amended mean' -- an adjustment coefficient for the cutoff),} \end{aligned}$$

and where:
$\hat{\gamma}_{\mathrm{LN}}(\lambda, \sigma)$ *denotes the unique solution of the following equation in* y :

$$(1-y) = \lambda H_{\mathrm{LN}}(y, \sigma), \tag{17}$$

$\kappa_i = \bar{p}_i/\bar{p}$ *(the standard regression coefficient),*
$\kappa_{-i} = \bar{p}_i/(\bar{p}-\bar{p}_i)$ *(removing agent-i's contribution from the aggregate precision),*
$1-\tilde{\rho}_i^2$ *is the partial covariance of* $\tilde{w}_i$ *on the remaining variates* $\tilde{w}_j$*, with* $w_j(t) := \sigma_0\tilde{W}^0_{1-t} + \sigma_i\tilde{W}^j_{1-t}$.

Proof of Theorem $\mathbf{1}_m$. We proceed stepwise.
1. (Hypothetical cutoff dynamics). We consider s, t with $\theta = \theta_-^{\mathrm{pub}} \leq t < s < \theta_+^{\mathrm{pub}} \leq 1$. We apply Theorem M above (with t as the ex-ante date and s the interim date) to an agent i. Theorem M sets the observation cutoffs for $\tilde{Y}^i_u$ in terms of the cutoffs $y_i^{\mathrm{hyp}}(u)$ of a correspondingly defined 'hypothetical' observer, the latter being defined implicitly via equation (17). As in §5.2, we perform a variational analysis to derive explicitly the cutoffs $\hat{z}_i(u)$ of the corresponding hypothetical observation.

Since Theorem M applies to variables that have been common-sized to unity at the ex-ante date t, put $G_i(u) = \hat{z}_i(u)/\hat{z}_i(t)$, so that $G_i(t) = 1$ for each i. Then, corresponding to a common-sized process, there is a hypothetical process (g-process) with adjusted value $G_i(u)L_{-i}(u)$ as at time u, and with a hypothetical volatility per unit time of

$$\tilde{\sigma}_{\mathrm{hyp},i} = \sigma_{\mathrm{hyp},i}(u) := \alpha_i\kappa_i\tilde{\sigma}_{0i}\sqrt{1-\tilde{\rho}_i^2}.$$

The solitary hypothetical process, observed intermittently by agent i, is now subjected to the variational analysis of §5.2, as follows.

As in §3, by the Indifference Principle of [15, §15 (Appendix 8)] applied at time t, the unique cutoff value $\tilde{y}_{is}$ for $\tilde{Y}_s^i$ and the time-s evaluation of $\mathbb{E}[Z_1^i|Y_1,...,Y_m]$ are related by

$$\begin{aligned}\mathbb{E}[Z_1^i|(\forall j)ND_j(\tilde{y}_{jt}),\mathcal{G}_t] &= \mathbb{E}[Z_1^i|(\forall j\neq i)ND_j(\tilde{y}_{jt}),\bar{T}_i=\tilde{y}_{it},\mathcal{G}_t]=\mathbb{E}[Z_1^i|(\forall j)[\bar{T}_j=\tilde{y}_j],\mathcal{G}_t]\\ &= k_m^i\beta_t^i\tilde{y}_{1t}^{\alpha\kappa_1}...\tilde{y}_{mt}^{\alpha\kappa_m},\end{aligned}$$

for $\alpha=\alpha_i$, and $\kappa_i=p_i/p$ as above; here the last regression formula is quoted from Lemma 2_m (cf. [15, §10.3.3 (Appendix 3)].)

Dropping subscripts, as in §5.2, one has by the conditional Bayes formula (Prop. 1, §2.3) for the hypothetical output valuation:

$$G(t)L(t)-G(s)L(s)=\lambda_t(s-t)\int_{z_1\leq G(s)L(s)}\mathbb{Q}(\mathbb{E}[Z_1^{\text{hyp}}|Y_s^{\text{hyp}},\mathcal{G}_t^+]\leq z_1|\mathcal{G}_t^+)dz_1+o(s-t),$$

where $Z_1^{\text{hyp}}=Y_1^{\text{hyp}}$, by definition of the hypothetical agent. Now argue as in §5.2 with $G(t)L(t)$ in place of $\tilde{\gamma}(t)$ to deduce the analogue of (15):

$$-\frac{1}{\lambda_t}\cdot\frac{d}{du}\left[G(u)L(u)\right]\bigg|_{u=t}=G(t)L(t)[2\Phi(\tilde{\sigma}_{\text{hyp}}/2)-1].$$

(This uses the fact that the equations connecting the log-cutoffs and the hypothetical log-cutoffs have non-singular matrix – details in the arXiv version.) Performing the differentiation, we obtain the following

$$\begin{aligned}G'(t)L(t)+G(t)L'(t) &= -\lambda_tG(t)L(t)[2\Phi(\tilde{\sigma}_{\text{hyp}}/2)-1],\\ \frac{G'(t)}{G(t)}+\frac{L'(t)}{L(t)} &= -\lambda_t[2\Phi(\tilde{\sigma}_{\text{hyp}}/2)-1].\end{aligned}$$

But $\hat{z}(u)=\hat{z}(t)G(u)$ [and $\hat{z}'(u)=\hat{z}(t)G'(u)$, so $\hat{z}'(u)/\hat{z}(u)=G'(u)/G(u)$], so

$$\frac{\hat{z}'(t)}{\hat{z}(t)}+\frac{L'(t)}{L(t)}=-\nu_{\text{hyp}}(t):=-\lambda_t[2\Phi(\tilde{\sigma}_{\text{hyp}}/2)-1].$$

So for the hypothetical agent i we obtain explicitly that

$$\log\left(\hat{z}_i(t)/\hat{z}_i(\theta)\right)+\log\left(L_{-i}(t)/L_{-i}(\theta)\right)=-\int_\theta^t\nu_{i\text{hyp}}(s)ds,$$

by integration from the date $\theta = \theta_-$ up to any time t prior to θ_+ (and re-instating subscripts). Rearranging and using $y_i^{\text{hyp}}(u) = \hat{z}_i(u)L_{-i}(u)$, gives

$$\log y_i^{\text{hyp}}(u) = \log y_i^{\text{hyp}}(\theta) - \int_\theta^t \nu_{i\text{hyp}}(s)ds. \tag{18}$$

2. (Actual correlated observation dynamics). We apply the formula of Theorem M to obtain the cutoffs $\tilde{y}_t^i$ for $\tilde{Y}_t^i$. To have common-sizing at time θ, set $\hat{\gamma}_i(t) := \tilde{y}^i(u)/\tilde{y}^i(\theta)$ for $\theta = \theta_- < t < \theta_+$. Returning to the corresponding actual agent i, substitution from (18) into (16) gives

$$\log \hat{\gamma}_i(t) = \frac{1}{\alpha_i \kappa_{-i}} \left(\log y_i^{\text{hyp}}(\theta) - \int_\theta^t \nu_{i\text{hyp}}(s)ds \right)$$
$$+\frac{1}{\kappa_0} \left(... + \frac{\kappa_j}{\alpha_j \kappa_{-j}} \left(\log(y_j^{\text{hyp}}(\theta)) - \int_\theta^t \nu_{j\text{hyp}}(s)ds \right) + ... \right).$$

We rearrange this to separate out two components displayed below. The first, a continuous "inter-arrival discounting" term, generalizes the single observer case, as a regression-weighted average over the hypothetical counterparts:

$$\begin{aligned} & \frac{1}{\alpha_i \kappa_{-i}} \int_\theta^t \nu_{i\text{hyp}}(s)ds + \frac{1}{\kappa_0} \left(\frac{\kappa_1}{\alpha_1 \kappa_{-1}} \int_\theta^t \nu_{1\text{hyp}}(s)ds + \frac{\kappa_2}{\alpha_2 \kappa_{-2}} (...) + ... \right) \\ = & \frac{1}{\alpha_i \kappa_{-i}} \int_\theta^t \nu_{i\text{hyp}}(s)ds + \frac{1}{\kappa_0} \left(\frac{\kappa_1}{\alpha_1 \kappa_{-1}} \int_\theta^t \nu_{1\text{hyp}}(s)ds + \frac{\kappa_2}{\alpha_2 \kappa_{-2}} \int_\theta^t \nu_{2\text{hyp}}(s)ds + ... \right) \\ = & \int_\theta^t \nu_i(s)ds. \end{aligned}$$

The second, a shift term, arises from adjustment of the initial mean in the move from actual to hypothetical agent:

$$\frac{1}{\alpha_1 \kappa_{-1}} \log y_i^{\text{hyp}}(\theta) + \frac{1}{\kappa_0} \left(... + \frac{\kappa_i}{\alpha_i \kappa_{-i}} \log y_i^{\text{hyp}}(\theta) + ... \right),$$

necessarily equivalent (after factoring through by α_i, since$(\log y^i(u))/\alpha_i = \log \tilde{y}^i(u)$) to $(\log y_i(\theta))/\alpha_i$ for $y_i(\theta)$ the re-initialization value at time θ. So

$$\begin{aligned} \frac{1}{\alpha_i} \log y^i(u) &= \log \tilde{y}^i(u) = -\frac{1}{\alpha_i \kappa_{-i}} \int_\theta^t \nu_{i\text{hyp}}(s)ds \\ & -\frac{1}{\kappa_0} \left(\frac{\kappa_1}{\alpha_1 \kappa_{-1}} \int_\theta^t v_{1\text{hyp}}(s)ds + \frac{\kappa_2}{\alpha_2 \kappa_{-2}} \int_\theta^t v_{2\text{hyp}}(s)ds + ... \right). \end{aligned}$$

So

$$\begin{aligned}\log y_t^i &= -\frac{1}{\kappa_{-i}}\int_\theta^t \nu_{i\text{hyp}}(s)ds \\ &\quad -\frac{1}{\kappa_0}\left(\kappa_1\frac{\alpha_i}{\alpha_1}\frac{1}{\kappa_{-1}}\int_\theta^t v_{1\text{hyp}}(s)ds+\kappa_2\frac{\alpha_i}{\alpha_2}\frac{1}{\kappa_{-2}}\int_\theta^t v_{2\text{hyp}}(s)ds+...\right).\end{aligned}$$

3. (Actual correlated valuation dynamics). From Lemma 2_m

$$\tilde{\beta}^i=\mu(\alpha_i,\kappa\tilde{\sigma}_0^2)\mu(\kappa_1^i,\kappa\tilde{\sigma}_i^2),\text{ and }\tilde{\beta}_m=\prod\nolimits_j\mu(\kappa_j,\tilde{\sigma}_{0j}^2),$$

$$\mathbb{E}[Z_1^h|Y_t=y_t,\mathcal{G}_t]=k_m^h\tilde{\beta}^h\cdot\tilde{\beta}_m\cdot y_{1t}^{\kappa_1}...y_{mt}^{\kappa_m},$$

using notation established there. Put

$$\tilde{\gamma}_t=y_{1t}^{\kappa_1}...y_{mt}^{\kappa_m}.$$

Take logarithms, substitute for $\kappa_i\log y_{it}$, next relabel j for i in the first term and change order of summation in the second, to obtain

$$\begin{aligned}\log\tilde{\gamma}_t &:= -\sum\nolimits_i\frac{\kappa_i}{\kappa_{-i}}\int_\theta^t \nu_{i\text{hyp}}(s)ds+\sum\nolimits_i\sum\nolimits_j\frac{\kappa_i}{\kappa_0}\frac{\alpha_i}{\alpha_j}\frac{\kappa_j}{\kappa_{-j}}\int_\theta^t v_{j\text{hyp}}(s)ds \\ &= -\sum\nolimits_j\frac{\kappa_j}{\kappa_{-j}}\int_\theta^t \nu_{j\text{hyp}}(s)ds+\sum\nolimits_j\sum\nolimits_i\frac{\kappa_i}{\kappa_0}\frac{\alpha_i}{\alpha_j}\frac{\kappa_j}{\kappa_{-j}}\int_\theta^t v_{j\text{hyp}}(s)ds \\ &= -\sum\nolimits_j\left(1+\sum\nolimits_i\frac{\alpha_i}{\alpha_j}\frac{\kappa_i}{\kappa_0}\right)\frac{\kappa_j}{\kappa_{-j}}\int_\theta^t \nu_{j\text{hyp}}(s)ds.\end{aligned}$$

So

$$\mathbb{E}[Z_1^i|Y_t=y_t,\mathcal{G}_t]=k_m^i\tilde{\beta}^i\cdot\tilde{\beta}_m\cdot\tilde{\gamma}_t\cdot g_*^i[\theta_-].$$

5.4 Extending the two lemmas of §5.2

We extend Lemmas 1 and 2 of §5.2 to general m to establish:

$$\tilde{\gamma}_t^i=\mathbb{E}[Z_1^i|Y_t=y_t,\mathcal{G}_t]=k_m^i\beta_t^i y_{1t}^{\kappa_1}...y_{mt}^{\kappa_m}.$$

We need some additional notation chosen so as to have the general case appear typographically similar to the $m=1$ case. Treating m-vectors $y,z\in$

$\mathbb{R}^m_+$ as functions on $\{1, 2, 3, ..., m\}$, define $y{\cdot}z$ (and so y/z) and the exponential y^z in the pointwise sense; also write the *product* of the components of y^z as

$$\langle y^z \rangle := (y_1{}^{z_1})...(y_m{}^{z_m}),$$

by analogy with inner products, so that

$$\log\langle y^z \rangle := \langle z, \log y \rangle = z_1 \log y_1 + ... + z_m \log y_m.$$

In particular, identifying $\alpha > 0$ with the vector all of whose components are α (qua function constantly α), $\langle y^\alpha \rangle = (y_1...y_m)^\alpha$. Finally, for convenience:

$$\kappa_i \text{ or } \kappa^i_m := p_i/p \qquad (i = 0, 1, ..., m), \qquad \kappa^i_1 := p_i/(p_0{+}p_i) \qquad (i = 1, ..., m).$$

Lemma 1_m (Valuation of Z_1 given observation $Y_1 = y$). *Put* $\kappa = (..., \kappa^j_m, ...)$. *Then*

$$\mathbb{E}_1[Z^i_1|Y_1 = y,] = k^i\langle y^\kappa \rangle \text{ for } k^i = k^i_m = f^i\langle (1/f)^\kappa \rangle.$$

Proof. With the overbar notations as in Lemma 1, note from [15, Prop. 10.3] that if $\bar{T}$ has components $\bar{T}_i = \bar{X}\bar{Y}_i$, where $\bar{X}$ and $(...\bar{Y}_i...)$ are independent, with precision parameters $\bar{p}_i$, then for $\kappa_i = \bar{p}_i/(\bar{p}_0 + \bar{p}_1 + ... + \bar{p}_m)$ and $\delta > 0$

$$\mathbb{E}[\bar{X}^\delta|\bar{T} = t] = K_\delta t_1^{\delta\kappa_1}...t_m^{\delta\kappa_m} = K_\delta\langle t^{\delta\kappa}\rangle.$$

As before $\bar{X}$ and $\bar{Y}_i$ have conditional variances $\sigma^2_0\Delta t$ and $\sigma^2_i\Delta t$. Take $K_\delta = 1$ (its limiting value as $\Delta t \to 0$; see Lemma 1) and note that $\bar{p}_i = (1/\sigma_i)^2$. For $\delta = \alpha_i$, conditioning on $Y_1 = y$, read off $k^i_m = f^i\langle (1/f)^\kappa \rangle$ from

$$\mathbb{E}[Z^i_1|\bar{T}] = \mathbb{E}[f^i X^\delta_1|\bar{T}] = f^i\langle \bar{T}^{\delta\kappa}\rangle = f^i\langle ((Y_1/f)^{1/\delta})^{\delta\kappa}\rangle = f^i\langle (y/f)^\kappa\rangle. \qquad \square$$

Lemma 2_m (Time-t conditional law of the valuation of Z_1, given observation Y_t). *Conditional on $Y_t = y$, the time-t distribution of the time-1 valuation $\mathbb{E}[Z^i_1|Y_1, \mathcal{G}_1]$ is that of*

$$k^i\tilde{\beta}^i_m\langle y^\kappa\rangle\hat{Z}^i_t,$$

where, as in Lemma 1_m and Lemma 2:
(i) $\kappa := (\kappa^1_m, ..., \kappa^m_m)$;

(ii) *for* $\bar{\kappa} = 1 - \kappa_0 = (p - p_0)/p$, *and* $\kappa_1^i = p_i/(p_0 + p_i)$,

$$\tilde{\beta}_m^i := \tilde{\beta}_{\text{indiv}}^i \cdot \tilde{\beta}_{\text{agg}} : \qquad \tilde{\beta}_{\text{indiv}}^i := \mu(\alpha_i, \bar{\kappa}\tilde{\sigma}_0^2)\mu(\kappa_1^i, \bar{\kappa}\tilde{\sigma}_i^2), \qquad \tilde{\beta}_{\text{agg}} = \prod\nolimits_j \mu(\kappa_j, \tilde{\sigma}_{0j}^2);$$

(iii) $\hat{Z}_t^i$ *is log-normal, its underlying mean-zero normal of variance* $\sum_j \kappa_j^2 \tilde{\sigma}_{0j}^2$. *In particular, this time-t distribution has mean given by*

$$\mathbb{E}[Z_1^i | Y_t = y, \mathcal{G}_t] = k_m^i \tilde{\beta}_{\text{indiv}}^i \cdot \tilde{\beta}_{\text{agg}} \langle y^\kappa \rangle.$$

Proof. From Lemma 1_m we have for $\kappa = (\kappa_m^1, ..., \kappa_m^j, ...)$

$$\mathbb{E}_1[Z_1^i | Y_1] = \mathbb{E}_1[f^i X_1^{\alpha_i} | Y_1] = k_m^i \langle (Y_1)^\kappa \rangle.$$

Conditional on $Y_t = (...Y_t^j...)$, by (13) of Lemma 2, with $\delta := \kappa_m^j$ for each j, there is $\hat{Z}_{jt} = \hat{Z}_{jt}(\kappa^j)$ of unit mean and variance $\kappa_j^2 \tilde{\sigma}_{0j}^2$, with

$$\hat{Z}_{jt} = \exp\left(\kappa_j [\alpha_j \sigma_0 \tilde{W}_{1-t}^0 + \sigma_j \tilde{W}_{1-t}^j] - \frac{1}{2} \kappa_j^2 \tilde{\sigma}_{0j}^2 \right),$$

such that

$$(Y_1^j)^{\kappa_j} = \beta^j (Y_t^j)^{\kappa_j} \hat{Z}_{jt}, \text{ where } \beta^j = (\mu_t^0(\alpha_j)\mu_t^j)^{\kappa_j} \mu(\kappa_j, \alpha_j^2 \tilde{\sigma}_{0j}^2),$$

for $\mu_t^j := \mu(\alpha_j, \alpha_j^2 \tilde{\sigma}_j^2)$. So by substitution

$$\mathbb{E}_1[Z_1^i | Y_1] = k_m^i \prod\nolimits_{j \geq 1} \beta^j (Y_t^j)^{\kappa_j} \hat{Z}_{jt}.$$

Now

$$\begin{aligned} \prod\nolimits_{j \geq 1} (\mu_t^0 \mu_t^j)^{\kappa_j} \mu(\kappa_j, \alpha_j^2 \tilde{\sigma}_{0j}^2) &= \prod\nolimits_j \mu(\alpha_j, \tilde{\sigma}_0^2)^{\kappa_j} \mu(\kappa_1^j, \tilde{\sigma}_i^2)^{\kappa_j} \mu(\kappa_j, \alpha_j^2 \tilde{\sigma}_{0j}^2) \\ &= \mu(\alpha_i, \kappa \tilde{\sigma}_0^2) \mu(\kappa_1^i, \kappa \tilde{\sigma}_i^2) \prod\nolimits_j \mu(\kappa_j, \alpha_j^2 \tilde{\sigma}_{0j}^2), \end{aligned}$$

where $\bar{\kappa} := \sum_{j \geq 1} \kappa_m^j := \sum_{j \geq 1} p_j/p = (p - p_0)/p = 1 - \kappa_0$. Put

$$\begin{aligned} \hat{Z}_t^i \quad : &= \prod\nolimits_{j \geq 1} \hat{Z}_{jt}^i = \exp \sum\nolimits_j \left(\kappa_j [\alpha_j \sigma_0 \tilde{W}_{1-t}^0 + \sigma_j \tilde{W}_{1-t}^j] - \frac{1}{2} \kappa_j^2 \tilde{\sigma}_{0j}^2 \right) \\ &= \exp \sum\nolimits_{j \geq 1} \left(\kappa_j \alpha_j \sigma_0 \tilde{W}_{1-t}^0 - \frac{1}{2} \kappa_j^2 \alpha_j^2 \tilde{\sigma}_0^2 \right) \exp \sum\nolimits_{j \geq 1} \left(\kappa_j \sigma_j \tilde{W}_{1-t}^j - \frac{1}{2} \kappa_j^2 \tilde{\sigma}_j^2 \right) \end{aligned}$$

(since $\tilde{\sigma}_{0j}^2 = \alpha_j^2 \tilde{\sigma}_0^2 + \tilde{\sigma}_j^2$). This is a product of independent terms each of unit expectation. The product has variance $\sum_j \kappa_j^2 [\tilde{\sigma}_0^2 + \tilde{\sigma}_j^2] = \sum_j \kappa_j^2 \tilde{\sigma}_{0j}^2$. We arrive at

$$\mathbb{E}[Z_1^i | Y_t, \mathcal{G}_t] = k_m^i \beta^i \langle Y_t^\kappa \rangle \hat{Z}_t^i. \ \square$$

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