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BIOGRAPHICAL MEMOIRS

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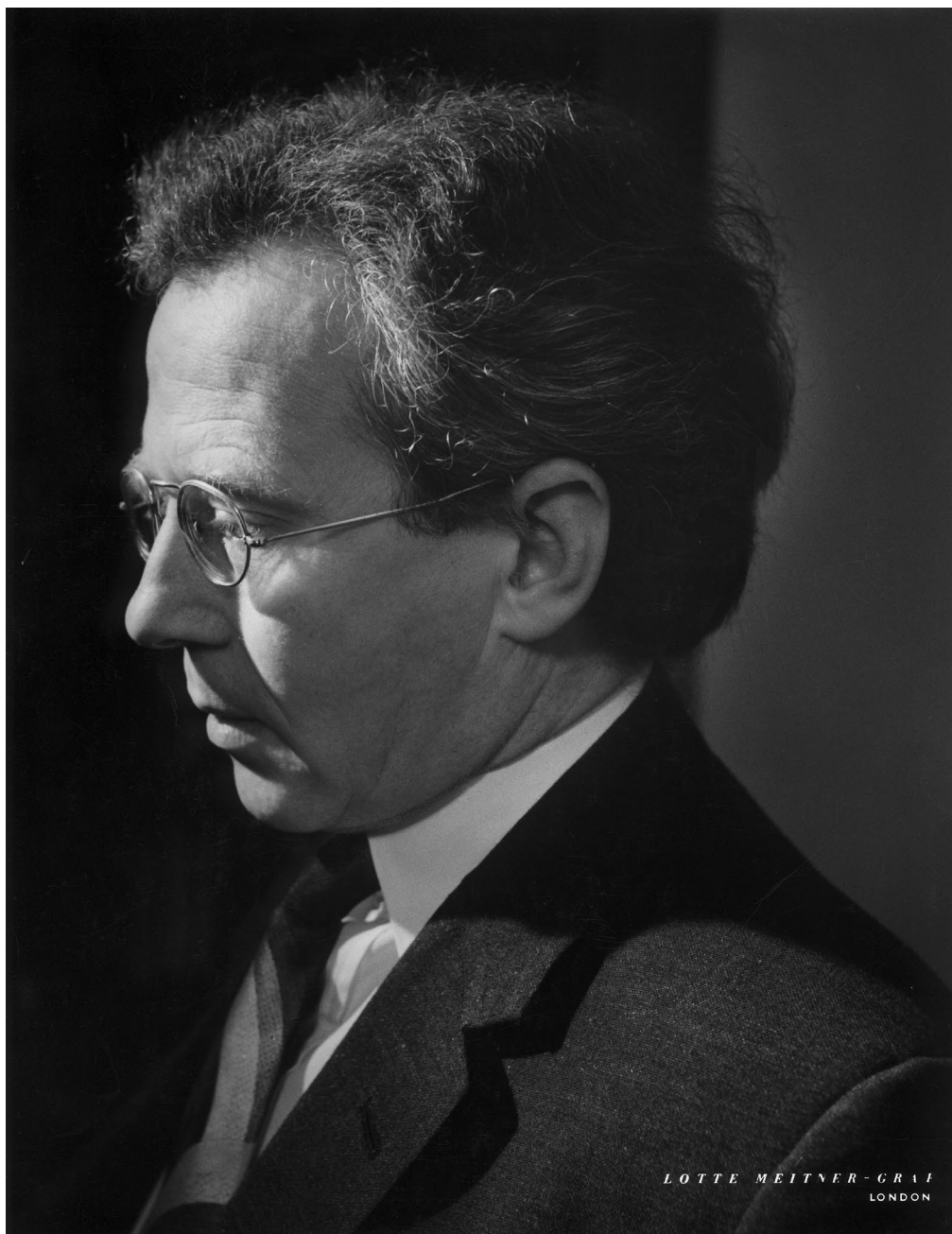
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C. A. Rogers

CLAUDE AMBROSE ROGERS

1 November 1920 — 5 December 2005

Elected FRS 1959

By Kenneth Falconer[1], Peter M. Gruber[2], Adam Ostaszewski[3]
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Claude Ambrose Rogers and his identical twin brother, Stephen Clifford, were born in Cambridge in 1920 and came from a long scientific heritage. Their great-great-grandfather, Davies Gilbert, was President of the Royal Society from 1827 to 1830; their father was a Fellow of the Society and distinguished for his work in tropical medicine. After attending boarding school at Berkhamsted with his twin brother from the age of 8 years, Ambrose, who had developed very different scientific interests from those of his father, entered University College London in 1938 to study mathematics. He completed the course in 1940 and graduated in 1941 with first-class honours, by which time the UK had been at war with Germany for two years. He joined the Applied Ballistics Branch of the Ministry of Supply in 1940, where he worked until 1945, apparently on calculations using radar data to direct anti-aircraft fire. However, this did not lead to research interests in applied mathematics, but rather to several areas of pure mathematics. Ambrose's PhD research was at Birkbeck College, London, under the supervision of L. S. Bosanquet and R. G. Cooke. Although his first paper was a short note on linear transformations of convergent series, his substantive early work was on the geometry of numbers. Later, Rogers became known for his very wide interests in mathematics, including not only geometry of numbers but also Hausdorff measures, convexity and analytic sets, as described in this memoir. Ambrose was married in 1952 to Joan North, and they had two daughters, Jane and Petra, to form a happy family.

Introduction and overview, by Trevor Stuart

This memoir was originally planned by Professor David Larman, a student and later a colleague of Ambrose Rogers, including the idea of four sections to cover Rogers's wide interests: geometry of numbers, Hausdorff measures, convexity, and analytic sets. Moreover the first of these topics, geometry of numbers, was requested by David of his friend and colleague, Peter M. Gruber, who wrote it before I was involved. However, difficulties arose over the other sections, and David therefore asked me, as Editor of *Biographical Memoirs*, to take over in organizing this memoir. I agreed to his request and consulted Professor Nick Bingham, who recommended Kenneth Falconer and Adam Ostaszewski, a former student of Ambrose Rogers. I am indebted to each of them and to Peter M. Gruber for their excellent collaborative work, and to Nick Bingham for his help and advice. I also acknowledge that the pattern of this memoir is the one planned by David Larman, and I thank him warmly.

Family background and education

Claude Ambrose Rogers had an interesting family history, as I learned first when, at the Royal Society, Ambrose showed me a portrait of his great-great-grandfather, Davies Gilbert [or Giddy], who became President of the Royal Society from 1827 to 1830; he was born with the surname Giddy but changed it to Gilbert a few years after he married Mary Ann Gilbert. Their daughter, Catherine, was Ambrose's great-grandmother (Robinson 1980). In addition, a bust of Davies Gilbert is held by the Royal Society.

Ambrose Rogers's father, Sir Leonard Rogers, was also a Fellow of the Society (see Boyd 1963) and was distinguished for his work on tropical medicine.

An Enys family tree, which includes Catherine Gilbert, was sent to me by Ambrose's nephew, Professor L. C. G. (Chris) Rogers.

A Rogers family history had been compiled by Ambrose's daughters, Jane and Petra, and his nephew, Chris. Many of the remarks that follow are taken from that history. Ambrose and his younger (twin) brother, Stephen Clifford, were born on 1 November 1920 in Cambridge, which was the location of their father's first post after returning from India. He had retired after a distinguished career in the Indian Medical Service. Their mother, Una Elsie North, had been sister in charge of surgery at the Medical College Hospital in Calcutta, where she met their father. They were married in 1914 and their eldest son, Gordon Leonard, was born in India.

The family moved to London in 1921, where Sir Leonard was appointed as a lecturer at the London School of Hygiene and Tropical Medicine.

Ambrose's father was a stern figure, but his mother was a much softer personality. The three boys, Ambrose, Clifford and Gordon, were brought up with the aid of a nanny, and after early schooling were sent at the age of 8 years to board at Berkhamsted School. At that time neither of the twins could read or write fluently, so their father summoned them and told them in clear terms that they had better start to do so. Ambrose was heard later to say to his brother, 'Well, Clifford, we will have to read.' The task took them little time. Ambrose remained throughout his life an indifferent speller, and Clifford adopted the undecipherable handwriting of a GP. In the pool at Berkhamsted the twins enjoyed a form of 'synchronized swimming': Clifford would swim a length on the surface with Ambrose swimming about half a metre below him.

After leaving school in 1938 Ambrose hoped to follow his brother Gordon and study at Cambridge. However, his father was concerned about the possibility of another war and

wanted both the twins to become qualified as soon as possible. Sir Leonard persuaded his son not to delay for a year to take the Cambridge entrance examination and so Ambrose enrolled at University College London (UCL), from which he graduated in mathematics with first-class honours, having spent part of his time in Bangor, North Wales, to which UCL had been evacuated to avoid the bombing of London.

Early work and marriage

Having finished his undergraduate studies in 1940 after two years, but before he formally graduated from the University of London in 1941, he joined the Applied Ballistics Branch of the Ministry of Supply for his war service until 1945; several others, who later became Fellows of the Royal Society, were similarly employed, including Louis Rosenhead (Stuart 1986), David Kendall (Kingman 2009), Leslie Howarth (Stuart 2009) and Rodney Hill (Sewell 2015). From remarks made to his daughters in later life, it seems that Ambrose worked on the use of radar data to direct anti-aircraft fire, but he spoke little of it. That was typical of Ambrose because, even to his closest family, he said little of his early life and claimed not to remember much of it.

After the war ended, Ambrose Rogers returned to mathematical studies, being appointed as a lecturer in mathematics at UCL in 1946 and working for his PhD at Birkbeck College, where his supervisors were L. S. Bosanquet and R. G. Cooke. Part of his work had been done as a part-time break from his war work, including his first paper, which was published in 1946 in *Journal of the London Mathematical Society*. In that paper Rogers expressed his debt 'to Dr L. S. Bosanquet and Dr R. G. Cooke for advice and encouragement during the preparation of this note'. He spent a sabbatical at the Institute for Advanced Study, Princeton, on a Commonwealth Fund Fellowship in 1949, where he met and collaborated with Aryeh Dvoretsky. After returning to UCL he was later promoted to a readership. His work at UCL was very much influenced by Harold Davenport FRS, whom Ambrose regarded as his mathematical mentor.

When his mother died in 1951, Ambrose went to live as a lodger with a cousin's family. There he promptly fell in love with his landlady's daughter (his mother's great-niece), Joan North. They were married in 1952 and subsequently had two daughters: Jane was born in 1955 and Petra in 1956. Joan wrote four children's books, which were published under her maiden name. Jane and Petra mostly knew their grandfather as Uncle Leonard, having learned of him from their mother.

In 1954 Rogers was an applicant for the Mason Chair of Pure Mathematics in Birmingham. Walter Hayman (FRS 1956) also was a candidate for that chair, and he comments as follows. As he and Ambrose entered the building for the interviews, the porter, seeing two young-looking men, called out 'Scholarship candidates this way.' However, Rogers was appointed to the Mason Chair!

According to David Larman this caused Davenport to write to the Provost as follows:

> The departure of Dr C. A. Rogers to take up the Mason Chair of Pure Mathematics at the University of Birmingham represents a serious loss to the strength of the Department. For the past few years he has been the most active research worker in the department and has been my closest collaborator in one of my research interests. From the teaching point of view also the loss of Dr Rogers is serious. He is one of those rare men who combine deep learning with adaptability and who can take any class, whatever its special needs might be, and give the class exactly what is best for it.

University College London, 1958–86

But in 1958 Rogers returned to UCL to succeed Harold Davenport as Astor Professor of Mathematics when Davenport moved to Cambridge. In the meantime, Hayman had been appointed as Professor of Pure Mathematics at Imperial College in 1956.

David Larman also wrote that in 1961 Ambrose visited Maurice Sion at the University of British Columbia (UBC) in Vancouver and wrote most of his book on Hausdorff measures while there, although it was not published until 1970 (58)*. David Larman also wrote that it was Maurice Sion who stimulated Ambrose's interest in analytic sets. Much later, when spending a year at UBC in 1972/73, on opening a drawer in his desk, Larman was astonished to find a pad of Ambrose's UCL notepaper!

One of Rogers's students, Richard Gardner, was with him from 1971 to 1974 'focusing on measure theory' while another student, Adam Ostaszewski, was studying analytic sets and a third student was working on packing and covering of convex bodies.

Richard Gardner writes:

> These three topics, measure theory, descriptive set theory and convex geometry, were Rogers's main research interests at the time. There was little small talk. He would address me, if at all, by my surname and almost never asked any personal questions. There was no social activity between us. It was a long time before he found out, quite by chance, that I was married. For my part, I never inquired about Rogers's personal life either, and was just as happy to limit our discussions to mathematics.

(Geoffrey Burton writes: 'Rogers was rumoured to believe that married research students did no work. David Larman, who married young, concealed the fact from Rogers, who was surprised to learn of it at the PhD viva when the External Examiner asked!')

Richard Gardner continues:

> Rogers was very generous with his time and would set up regular meetings of at least an hour every ten days or so. Typically I would sit in his huge office, which was equipped with two long desks covered with stacks of papers, while he would usually stand. When thinking he paced up and down, staring at the floor and jangling keys or change in his pocket. I thought Rogers was a wonderful supervisor, always helpful, and in his own formal way, caring and kind. Unable to wait for our regular meeting, I once knocked on his door to discuss something I thought I could prove. When I apologized for the interruption, he said, 'Oh, don't worry. I never consider it a waste of time to discuss mathematics.' He once said to me that if a proof is easy to see, it should be easy to write down a proof. Also he explained the importance of being open in sharing mathematics, as generosity will be amply repaid in the long run. I often passed on these excellent pieces of advice.

My sad comment, as an author, is that not everyone shares that generous view.

Adam Ostaszewski, who also was a student of Rogers from 1970 to 1973, has read Richard Gardner's remarks and has been stimulated to add to them as follows. 'Rogers was not referred to as Ambrose, an innovation due later to Larman when he became an academic.' Adam writes:

> I was fortunate to see Professor Rogers more closely, possibly because I had been his tutee as an undergraduate for three years. I remember a party at his home with small talk over glasses of wine with Ambrose, his wife and his daughter Petra. Indeed he wanted to conduct my viva at his home, hoping to make Roy Davies's visit pleasanter. However, I disagreed, feeling that I would be nervous in those unfamiliar surroundings, and it did not happen there. Roy decided not to question

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* Numbers in this form refer to the bibliography at the end of the text.

me at the viva, dismissing the matter as we had discussed things in Leicester at length and at leisure; so half-heartedly and with a brave face Rogers asked me what was the most important theorem in my thesis! Afterwards we talked about the weather. Seriously, Rogers was not above that; his sense of humour was always there.

During my first post-graduate year Rogers was invited to give a seminar by John Addison, who was then visiting Oxford, but he graciously deflected it into a first Oxford seminar for me. Probably to make the undertaking less frightening, he took me in his car along with Petra and a young friend of hers; half-way we had a surprise picnic from a hamper brought by Ambrose!

He was a most generous supervisor, caring about progress, so that when I got nowhere with the first topic he graciously shelved it. Once I recall his saying that he guessed I must be despondent, offering another problem to think about but not to be solved in a week: it was for the longer haul. Well it was, so I cut my teeth on it and eventually did well for myself.

Kenneth Falconer first met Ambrose Rogers in 1978 in Rogers's office at UCL on the occasion of his PhD viva. The experience was 'rather overwhelming, although the viva lasted only half an hour'. Falconer writes:

Over the next few years I met Ambrose fairly regularly, having common interests; he invited me to speak at UCL on several occasions and was very kind and supportive. I remember his telling me of how important it was to keep up with new methods—wise advice that I pass on to my own students. In 1998 when CUP [Cambridge University Press] published the revised edition of *Hausdorff measures* (89), I was very touched when Ambrose asked me to write the updating Foreword for the new edition.

Another aspect of Ambrose's generous and thoughtful personality is given by Peter M. Gruber, who has written:

In the 1960s Ambrose Rogers left the Geometry of Numbers and started working in Measure Theory and Analytic Sets. Rogers was well aware that he had left a beautiful field. At a conference in Vienna in honour of Edmund Hlawka, I drove several participants to my home. During the drive a prominent participant said to Ambrose: Professor Rogers, I do not understand why you left such a beautiful and attractive area as is the Geometry of Numbers for ugly Measure Theory. In an instant a painful atmosphere filled the car, but instead of exploding, Rogers, who could be choleric at times, thought for a minute or so and then said calmly: yes, there is something in what you say. The evening was saved!

Some years ago I needed advice from a mathematician with a broad and expert knowledge of many areas of mathematics. Of people I knew, the name of Ambrose Rogers sprang to mind. The reason was that I was acting as advisor to another university about academic promotions in mathematics. One particular case caused me some difficulty in deciding on a suitable referee; I was in a quandary until I realized that I should ask Ambrose, which I then did. He wrote a thoughtful and detailed assessment, which was of great help as Ambrose's care and honesty came through very strongly. His assessment 'carried the day'. I was most grateful to him, and it sealed my respect for him.

Professor Rogers had a great influence for the good on mathematics, not least through the London Mathematical Society (LMS), of which he became President during 1970–72. However, I want to discuss an aspect of his concern for mathematics, which took place earlier. In the late 1960s a very distinguished mathematician [A] had produced a report that classified by quality different areas of mathematics in the many UK departments of mathematics. Ambrose objected strongly to the results of this classification, as the following anecdote indicates. At a social party at my home in about 1968/69 the report was raised in an interchange between [A] and Keith

Stewartson FRS, who was instructed firmly and loudly to 'tell Rogers that …' The precise details are no longer with me, but [A] certainly had objected to Ambrose's views. The report seemed to die a natural death, helped I am sure by Rogers's strong opposition and probably by that of others.

The family biography says that 'although he was a man of few words, he was by no means unaware of the feelings, needs and motivations of others.' This was certainly true, as I know from those who worked with him as a researcher but also from personal knowledge. He was revered by many. There were occasional difficulties in his relationships with colleagues, but in some cases I suspect that it was, in an old phrase, 'six of one and half-a-dozen of the other'. In spite of those difficulties in relationships, Ambrose's generous qualities were always there.

After retiring in 1986, Ambrose continued to work mainly in analytic sets with John Jayne and Isaac Namioka. His last book, *Selectors* (90), which was written with John Jayne, was published in August 2002, when he was 81 years old.

London Mathematical Society

David Brannan has written extensively to me about Ambrose Rogers and about his influence through the LMS. He first met Rogers in 1971 when he visited him to discuss mathematical journals. By this time Rogers, who was Editor-in-Chief of *Mathematika*, was President of the LMS, having succeeded Sir Edward Collingwood FRS, who had died in mid-term. David Brannan writes that 'he listened politely to my ideas and suggested that I should talk to S. James Taylor, who was an LMS journal secretary'. As a result David Brannan was nominated as, and became, secretary to the LMS Council in November 1971, and then he came to know Rogers well.

David Brannan writes:

> In spite of his fierce-looking exterior, Rogers was very easy to work with. He was helpful in all sorts of ways, for example with potential teething problems which arose because I had not served on Council before. He made suggestions very discreetly beforehand to ensure that everything went smoothly.
>
> An enduring contribution to the long-term life of the world mathematical community was certainly his drive to realize the Durham Symposia. When I became Secretary, LMS Council was looking at its review document which proposed that the LMS set up a conference centre like that at Oberwolfach in Germany.

However, financial prospects were poor in the period 1970–74, whether from the UK Government or from industrial sources.

David writes:

> Council then decided that it might be able to persuade a UK University to host annual conferences under LMS auspices with funding from the then Science Research Council (SRC; later the SERC and later still the EPSRC). The LMS Council then decided to send a group of three composed of Rogers, Philip J. Higgins and David Brannan to check on facilities and university administration and to talk to local university administrators. We wrote the final factual report, which Council considered at a special Council meeting in July 1972. After a full discussion, Durham University was selected, Tom Willmore having been most enthusiastic. [Both Philip Higgins and S. James Taylor have also commented on these discussions.] Rogers guided this potentially thorny decision process discreetly through all its stages, so that Council was united in its final decision on location.

The LMS–Durham Symposia started in 1974, with Research Council support, and continue to this day, a very substantial lasting tribute to Professor Rogers.

Mathematika *and other publications*

The UCL journal *Mathematika* was founded by Harold Davenport in 1954, with Ambrose as one of the four editors together with Richard Rado (FRS 1978) and William Dean. The journal, which concentrated on areas of research strength at UCL, attracted many seminal papers. Remarkably, Ambrose remained an editor until his death in 2005. *Mathematika* was a very low-cost production and in the 1980s Ambrose's dedication to the journal led to his doing much of the copy-editing himself. After his involvement lessened, the journal started to suffer from lack of resources and by the early 2000s the publication had dropped to a single issue per year. However, the future became assured in 2010 when the LMS came to an arrangement to relaunch *Mathematika* on a commercial basis on behalf of UCL, with Cambridge University Press responsible for printing and distribution. It is fair to say that Ambrose's enormous contribution to the journal was in the mind of Kenneth Falconer, then LMS Publication Secretary, during the negotiations!

Rogers was active for the good of mathematics in other ways. In 1960 he was asked to contribute to a special edition of *New Scientist* in celebration of the Tercentenary of the Royal Society (41); other contributors included E. D. (Lord) Adrian FRS OM (PRS 1950–55), Sir Edward Bullard FRS and Dennis Gabor FRS. In 1971 he wrote the biographical memoir of Harold Davenport (62) jointly with B. J. Birch (FRS 1972), H. Halberstam and D. A. Burgess, and later (in 1977) he was a joint editor with B. J. Birch and H. Halberstam of *The collected works of Harold Davenport* (71). In 1991 he wrote the biographical memoir of Richard Rado (81).

Assessment

Ambrose Rogers had eight students and 34 mathematical descendants. He was devoted to mathematics and was revered by those associated with him in his wide range of researches. He was also admired by many in fields remote from his, for his mathematical understanding and breadth of vision. His presidency of the LMS will be remembered for his influence and drive to realize the enduring LMS Durham Symposia.

GEOMETRY OF NUMBERS AND DISCRETE GEOMETRY, BY PETER M. GRUBER

Claude Ambrose Rogers was one of the great figures in the geometry of numbers and discrete geometry in the twentieth century. This section gives an account of the major contribution of Rogers to these fields. These include in particular his results in the context of the Minkowski–Hlawka theorem and to measure theory in the geometry of numbers and on lattice and non-lattice packing and covering of convex bodies.

Introduction

The work of Rogers in the geometry of numbers and in discrete geometry began in 1946 with an extension of Blichfeldt's theorem involving successive minima (1) and ended in 1997 with a joint article with Zong (88) on the covering of a convex body by translates of another convex body. During these 50 years he contributed, in particular, to the following topics:

the Minkowski–Hlawka theorem and measure theory on the space of lattices;
packing and covering with translates of a convex body;
critical determinants and reduction of star bodies, and symmetrization;
successive minima;
product of homogeneous and inhomogeneous linear forms; and
miscellanea.

For general information on the geometry of numbers see Cassels (1997), Gruber (1979, 1993, 2007), Erdős *et al*. (1989), Gruber & Lekkerkerker (1987) and the handbooks of convex and of discrete and computational geometry edited by Gruber & Wills (1993) and Goodman & O'Rourke (2004), respectively. We assume that the reader is familiar with basic notions of the geometry of numbers such as lattice, lattice determinant, and convex body.

The Minkowski–Hlawka theorem and measure theory on the space of lattices

A version of the Minkowski–Hlawka theorem (see Hlawka 1944), says that for any Borel set S in Euclidean n-space $\mathbb{R}^n$ of measure $V(S) < 1$ there is a lattice Λ of determinant 1 that contains no point of S with the possible exception of the origin $\{o\}$; in other words, Λ is *strictly admissible* for S. If $S = K$ is an o-symmetric convex body, then $V(S) < 1$ may be replaced by $V(S) < 2\zeta(n)$, where ζ denotes the Riemann zeta function. Siegel's (1945) mean value formula says that for a certain natural probability measure μ on the space $\mathcal{L}$ of all lattices in $\mathbb{R}^n$ of determinant 1 we have the following: let $f: \mathbb{R}^n \to \mathbb{R}$ be a non-negative Borel–measurable function, then

$$\int_{\mathcal{L}} \Big(\sum_{a \in \Lambda \setminus \{o\}} f(a) \Big) \mathrm{d}\mu(\Lambda) = \int_{\mathbb{R}^n} f(x) \mathrm{d}x.$$

In particular, if f is the characteristic function of a Borel set S in $\mathbb{R}^n$, the formula reduces to

$$\int_{\mathcal{L}} \#(S \cap (\Lambda \setminus \{o\})) \mathrm{d}\mu(\Lambda) = V(S),$$

where $\#$ is the counting function, and the Minkowski–Hlawka theorem follows.

Rogers (3) and Rogers and Davenport (4) gave simple transparent proofs of the Minkowski–Hlawka theorem. For references to several other proofs see Gruber & Lekkerkerker (1987). Refinements, in particular in the sense that 1 and $2\zeta(n)$ are replaced by larger quantities, were given by Rogers (24, 26, 34), where in the latter article 1 is replaced by $\frac{1}{4} \log \frac{4}{3} n = 0.07192 \ldots n$. The best-known estimate of this type is due to Schmidt (1963), who replaced 1 by $\log \sqrt{2} n - \mathrm{const} = 0.34657 \ldots n - \mathrm{const}$. In (23) Rogers gave a version of the Minkowski–Hlawka theorem in which lattices are replaced by linear images of a given discrete set. A result dealing with spherical symmetrization of a function is contained in (27). In (24) Rogers gives extensions of Siegel's formula to functions of several vector variables and to sums over m-tuples of lattice points. Besides Siegel's method to introduce a measure on spaces of lattices, there are other ways to do it. Compare Rogers (24), the references mentioned there, and Rogers and Macbeath (25, 31, 33). Of interest is also the following variance formula of Rogers (24) ($n \geq 3$), where const > 0 is an absolute constant and $S \subseteq \mathbb{R}^n$ a Borel set:

$$\int_{\mathcal{L}} (\#(S \cap (\Lambda \setminus \{o\})) - V(S))^2 \mathrm{d}\mu(\Lambda) < \mathrm{const} \, V(S).$$

As a consequence we obtain that in the case $V(S) = +\infty$, almost every lattice $\Lambda \in \mathcal{L}$ contains infinitely many points in S. For $n = 2$, corresponding, slightly weaker, results are due to Schmidt (1960). A generalization to n-tuples of lattice points is due to Aliev & Gruber (2006). For more information see Gruber & Lekkerker (1987).

For many years it was the widespread opinion of many mathematicians that the Minkowski–Hlawka theorem is open to substantial (polynomial or even exponential) refinements. An exception was Hlawka. Many mathematicians now believe that, in essence, the theorem is the best possible. The reasons are the increasing difficulties of the minor improvements and the smaller and smaller steps by which the Minkowski–Hlawka bound was finally reached by Rush (1989), using error-correcting codes. All known proofs of the Minkowski–Hlawka theorem are based on mean value arguments. Thus, it seems that the mean value, in essence, equals the optimum, a phenomenon that appears also in the asymptotic theory of normed spaces.

Packing and covering

Let K be a convex body in $\mathbb{R}^n$. A family of translates of K is a *packing* if the translates are pairwise non-overlapping. It is a *covering* if their union equals $\mathbb{R}^n$. We speak of *lattice packing* (or *covering*) if the translation vectors are the vectors of a lattice. Without attempting to be precise, we say that the *density of a packing* is the 'proportion of $\mathbb{R}^n$ that is covered by the translates of K of the packing' and that the *density of a covering* is the 'sum of the volumes of the translates of K of the covering divided by the volume of $\mathbb{R}^n$'. The *packing density $\delta(K)$* of K is the supremum of the densities of packings by translates of K, and the *lattice packing density $\delta_L(K)$* is the supremum of the densities of the lattice packings of K. Both suprema are attained. Similarly, we define the *covering density $\vartheta(K)$* of K as the infimum of the densities of all coverings by translates of K, and the *lattice covering density $\vartheta_L(K)$* as the infimum of the densities of all lattice coverings of K. These infima are also attained. An easy proof (using precise definitions) shows that

$$\delta_L(K) \le \delta(K) \le 1 \le \vartheta(K) \le \vartheta_L(K).$$

Rogers (13) proved that, for o-symmetric K,

$$\vartheta(K) \le 2^n \, \delta(K) \le 2^n \quad \text{and} \quad \vartheta_L(K) \le 3^{n-1} \delta_L(K) \le 3^{n-1},$$

thereby improving on a result of Hlawka (1949) that $\vartheta_L(K) \le n^n \delta_L(K)$. The following is a list of successive improvements of the upper estimates for $\vartheta_L(K)$ and $\vartheta(K)$:

$$\vartheta_L(K) \le 2^n \quad (13, 32),$$

$$\vartheta_L(K) \le 1.8774^n \quad (34),$$

$$\vartheta(K) \le n \log n + n \log \log n + 5n \quad (28),$$

$$\vartheta(K) \le n \log n + n \log \log n + 4n$$

and there is a covering of this density where each point of $\mathbb{R}^n$ is covered by at most $e(n \log n + n \log \log n + 4n)$ translates of K (44),

$$\vartheta_L(K) \le n^{\log_2 \log n + \text{const}} \quad (39).$$

The latter bounds are achieved by an ingenious random method. No better estimate seems to be known.

In Rogers's last article, written jointly with Zong (88), the following result is proved: Let H, K be two convex bodies in $\mathbb{R}^n$. Then the minimum number of translates of H required to cover K is bounded above by

$$\frac{V(K \ominus H)}{V(H)} \vartheta(K),$$

where $K \ominus H = \{x : H + x \subseteq K\}$ is the *Minkowski difference* of K and H. A similar result holds in the case where K is covered by lattice translates of H.

The *star number* of a covering of K is the maximum number of translates of K that meet any given translate. Erdős and Rogers (50) proved that the star number of any covering of an o-symmetric convex body by translates has star number at least $2^{n+1} - 1$. The 2-dimensional case is due to Boltyanskii (1950).

Let B^n be the solid Euclidean unit ball in $\mathbb{R}^n$. Rogers (21) gave an upper estimate for the volume of a convex polytope in B^n with k facets. This estimate was used by Erdős and Rogers (19) to show that

$$\vartheta(B^n) \geq \frac{16}{15} - o(1) = 1.0666 \ldots - o(1) \quad \text{as} \quad n \to \infty.$$

Slightly earlier, Bambah & Davenport (1952) proved

$$\vartheta_L(B^n) \geq \frac{4}{3} - o(1) = 1.3333 \ldots - o(1) \quad \text{as} \quad n \to \infty.$$

A substantial refinement of these results is due to Coxeter, Few and Rogers (40):

$$\vartheta_L(B^n) \geq \vartheta(B^n) \geq \frac{n}{e\sqrt{e}} + o(n) = 0.22313 \ldots n + o(n) \quad \text{as} \quad n \to \infty.$$

An improvement of Rogers (36) over earlier estimates of Blichfeldt (1929) and others for packing densities says that

$$\delta_L(B^n) \leq \frac{n}{e} 2^{-0.5n} (1 + o(1)) = 0.36787 \ldots n 2^{-0.5n} (1 + o(1)) \quad \text{as} \quad n \to \infty.$$

For many years the Blichfeldt bound

$$\delta_L(B^n) \leq \frac{n+2}{2} 2^{-0.5n}$$

was believed to be, in essence, the final say. With the use of spherical harmonics, the best known upper estimate in high dimensions is due to Kabatjanskiĭ & Levenšteĭn (1978):

$$\delta_L(B^n) \leq 2^{-0.599n + o(n)} \quad \text{as} \quad n \to \infty.$$

Although the method of proof of the successive refinements of the upper bound seems to be exhausted, many number theorists see no reason why the lowering of the upper bound should not continue until the Minkowski–Hlawka bound $2^{-n+o(n)}$ as $n \to \infty$ is reached; compare the earlier remarks.

In his book *Packing and covering*, Rogers (52) presents his results on the minimum covering density. Moreover he describes Dirichlet–Voronoi tilings and their applications to the packing of balls and Delone tilings in the context of the empty ball method. The latter is applied to coverings with balls.

For first steps in extending Voronoi's theory for lattice packing of balls and the empty sphere method of Delone to lattice packing and covering of convex bodies, see the articles by Gruber (2014, 2015) and the short report by Gruber (2011). For more information we refer to Rogers's little book (52), Gruber & Lekkerkerker (1987), Erdős *et al.* (1989), Bambah (1999), Gruber (2007, 2011) and Schürmann (2009).

We next consider the planar case. Various results of L. Fejes Tóth (1972, 1983, 1985), Fáry (1950), Rogers (16), Bambah and Rogers (17), Bambah, Rogers and Zassenhaus (51), G. Fejes Tóth (1988) and others deal with the problem of for which convex discs *C* we have

$$\delta(C) = \delta_L(C) \quad \text{and} \quad \vartheta_L(C) = \vartheta(C),$$

and give estimates for these quantities in terms of circumscribed and inscribed hexagons and triangles of minimum and maximum areas, respectively. We cite a result of Bambah, Rogers and Zassenhaus (51): let $C \subseteq \mathbb{R}^2$ be a convex disc of area *A* and let *T* be the largest area of a triangle contained in *K*. Then

$$\vartheta(C) \geq A/(2T)$$

and

$$\vartheta(C) = \vartheta_L(C) = A/(2T) \quad \text{if } C \text{ is centrally symmetric.}$$

Surveys on results of this type are due to G. Fejes Tóth & Kuperberg (1993), Bambah (1999) and G. Fejes Tóth (2004). A commented English translation by G. Fejes Tóth & W. Kuperberg of L. Fejes Tóth's classical book of 1972 is forthcoming.

Critical determinants and reduction

A *star body* $S \subseteq \mathbb{R}^n$ is a closed set with *o* in its interior, such that each ray starting at *o* meets the boundary of *S* in at most one point. The *critical determinant* $\Delta(S)$ of *S* is the infimum of the determinants of the strictly admissible lattices of *S*. If $K = S$ is a convex body, then

$$\delta_L(K) = \frac{V(K)}{\Delta(K - K)},$$

where $K - K = \{x - y : x, y \in K\}$ is an *o*-symmetric convex body, the *difference body* of *K*. For the following inequality see Rogers and Shephard (30):

$$2^n V(K) \leq V(K - K) \leq \binom{2n}{n} V(K).$$

This inequality relates the lattice packing density of a convex body *K* to that of the symmetrized body $K - K = \{x - y : x, y \in K\}$. See Rogers and Shephard (35) for a related result.

Chalk and Rogers (6) proved that for a planar *o*-symmetric convex body *C* we have

$$\Delta(C) = \Delta(K), \quad \text{where } K = C \times [-1, 1] \text{ is a cylinder in } \mathbb{R}^3.$$

Before the effective algorithm of Betke & Henk (2000) was known, the critical determinants of convex 3-polytopes had only been determined for the Platonic solids, the Euclidean ball B^3 and a few 3-polytopes, for example truncated cubes.

For star bodies the situation is different. Rogers (7) specified a star body *S* in $\mathbb{R}^2$ such that

$$\Delta(S) < \Delta(S \times [-1, 1]),$$

and Davenport and Rogers (14) showed that there are planar star bodies *S* such that

$$\frac{\Delta(S)}{\Delta(S \times [-1,1])}$$

is arbitrarily small. A general result of Davenport and Rogers (12) on critical determinants yields the critical determinants of certain star bodies, including the following:

$$|xyz| \le 1 : \Delta = 7 \quad \text{and} \quad |x|(y^2 + z^2) \le 1 : \Delta = \tfrac{1}{2}\sqrt{23}.$$

A star body S is *reduced* if there is no star body $T \subsetneq S$ with $\Delta(S) = \Delta(T)$. The major problems are to characterize reduced star bodies and to find out whether a star body contains a reduced star body with the same critical determinant. Rogers (2, 5, 18) proved a series of general pertinent results, gave simple proofs of some theorems of Mahler (1946*a*, *b*), and answered a question of Mahler in the negative. Yet the great hopes in the 1950s of achieving progress in the context of critical determinants by means of reduction results on star bodies have not materialized.

Successive minima

Let K be an o-symmetric convex body and Λ a lattice. The *successive minima* $\lambda_i = \lambda_i(K, \Lambda)$ of K with respect to Λ are defined by

$$\lambda_i = \inf \{ \lambda > 0 : \dim \operatorname{lin}(\lambda K \cap \Lambda) \le i \} \text{ for } i = 1, \ldots, n.$$

Clearly,

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n < +\infty.$$

The second fundamental theorem of Minkowski says that

$$\frac{2^n d(\Lambda)}{n!} \le \lambda_1 \cdots \lambda_n V(K) \le 2^n d(\Lambda).$$

Results of Jarník (1941) and Jarník & Knichal (1946) deal with extensions where K is replaced by more general sets.

Rogers (8) considered a set $S \subseteq \mathbb{R}^n$ with positive Lebesgue measure $V(S)$ and successive minima for $S - S = \{ x - y : x, y \in S \}$ instead of K. Then

$$\lambda_1 \cdots \lambda_n V(S) \le 2^{(n-1)/2} d(\Lambda).$$

For more precise results see Rogers (8), Chabauty (1949) and Rogers (29). An unsolved conjecture claims that

$$\lambda_1 \cdots \lambda_n \Delta(K) \le d(\Lambda).$$

This is true for $n = 2$ and when $K \subseteq \mathbb{R}^n$ is an o-symmetric ellipsoid. Chalk and Rogers (10) proved this when $K = C \times [-1, 1]$ is an o-symmetric convex cylinder in $\mathbb{R}^3$. For $n = 3$ a proof is due to Woods (1956).

Product of homogeneous and inhomogeneous linear forms

Let $L_1, \cdots, L_n$ be n real linear forms in n variables of determinant 1. The *homogeneous minimum* $\lambda = \lambda(L_1, \cdots, L_n)$ of $|L_1, \cdots, L_n|$ is defined by

$$\lambda = \inf \left\{ |L_1(u) \cdots L_n(u)| : u \in \mathbb{Z}^n \setminus \{o\} \right\},$$

where $\mathbb{Z}^n$ is the integer lattice. Then hold the upper estimates,

$$\lambda^{\frac{1}{n}} \leq \frac{1}{e\sqrt{e}} + o(1) = 0.22313\ldots + o(1) \quad \text{as} \quad n \to \infty \quad (9),$$

$$\lambda^{\frac{1}{n}} \leq 0.17524\ldots + o(1) \quad \text{as} \quad n \to \infty \quad (11).$$

For other estimates and related results see Gruber & Lekkerkerker (1987). Chalk and Rogers (15) showed that for $n = 3$ there are $u, v, w \in \mathbb{Z}^3 \setminus \{o\}$ such that

$$\left| \left(L_1(u) L_2(u) L_3(u) \right) \left(L_1(v) L_2(v) L_3(v) \right) \left(L_1(w) L_2(w) L_3(w) \right) \right| \leq 1/7^3.$$

The solvability of simultaneous inequalities of the form

$$| (L_1(u) + c_1) \cdots (L_n(u) + c_n) | < 1,$$
$$(L_1(u) + c_1)^2 + \cdots + (L_n(u) + c_n)^2 < \varepsilon,$$
$$L_1(u) + c_1, \ldots, L_n(u) + c_n > 0, \text{ etc.,}$$

is studied in Rogers (22).

Some remarks

In the 1960s Ambrose Rogers left the geometry of numbers and started working in measure theory and analytic sets. Similarly, Edmund Hlawka, Wolfgang Schmidt, Enrico Bombieri, Ian Cassels, Alexander Macbeath and other workers left the field; exceptions were the schools of Bambah and Hans-Gill in Chandigarh, and Delone and Ryshkov in Moscow. Presumably it was the exceeding difficulty of the relevant problems that led to this development. Rogers was well aware that he had left a beautiful field. In recent years, progress has been achieved in several important problems of the geometry of numbers, including the subspace theorem, the conjecture on the product of inhomogeneous linear forms, Mordell's converse problem of the linear form theorem, and extensions of Voronoi's theorem. The geometry of numbers now seems to be flourishing again.

Conclusion

The contributions of Rogers to packing and covering are classical. His beautiful little Cambridge tract (52), *Packing and covering*, in which some of his more important results are elegantly presented, will long be a source of inspiration. Equally important are his contributions to measure theory on spaces of lattices. Unfortunately, a comprehensive systematic presentation of measure in the geometry of numbers is still missing.

Besides works of a systematic character such as the upper bound for the minimum covering density, Rogers produced several small pearls, among which was his concise proof of the Minkowski–Hlawka theorem.

His published work exhibits excellent mathematical taste, a clear feeling for relevance and future developments, and he had the strength to solve the chosen problems. In his work in the geometry of numbers and in discrete geometry, Rogers kept the right balance between being too specialized and too general. Many of his results have led to further work and will definitely continue to do so.

We consider that, among other results, his estimates for the covering density, his variance theorem, and the Rogers–Shephard and Rogers–Zong inequalities will remain his legacy in the far future.

HAUSDORFF MEASURES, BY KENNETH FALCONER

Hausdorff measures and dimensions

Ambrose Rogers became interested in Hausdorff measures while he was at Birmingham in the mid 1950s, where he met S. James Taylor, a former student of Besicovitch. Although Rogers wrote relatively few papers on Hausdorff measures, his influence on the area has been enormous. Not least, his beautifully written tract *Hausdorff measures* (58) remains by far the best treatment for both student and advanced researcher. The book, which was written mainly while visiting Maurice Sion at UBC in Vancouver in 1961 but was not published until 1970, introduces and develops the subject in a detailed but readable way and includes much original material. The book was reprinted in 1998 in the Cambridge Mathematical Library series of mathematical classics with a new appendix on dimension prints and an updating foreword by Kenneth Falconer (89).

Hausdorff measures are of great intrinsic interest, relating to the topology of the underlying metric space in a natural way. They are a fundamental tool in the study of the sets that are now termed 'fractals'. Hausdorff measures are not in general σ-finite on the entire underlying space, a feature that both heightens their utility and gives rise to many technical difficulties.

Let (Ω, ρ) be a metric space. A function $h : [0, \infty) \to [0, \infty]$ that is monotonic, increasing and continuous on the right with $h(t) > 0$ when $t > 0$ is termed a *gauge function*. For notational convenience we also regard h as a function on the subsets of Ω by setting $h(A) = h(d(A))$ for each non-empty $A \subseteq \Omega$, where $d(A) = \sup\{\rho(x,y) : x, y \subseteq A\}$ is the *diameter* of A, with $h(\varnothing) = 0$.

The *Hausdorff measure corresponding to h* or just *h-Hausdorff measure* is the measure μ^h obtained using Carathéodory's construction as follows. For $E \subseteq \Omega$ and $\delta > 0$ let

$$\mu^h_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} h(d(A_i)) : E \subseteq \bigcup_{i=1}^{\infty} A_i, d(A_i) \leq \delta\right\}. \qquad [1]$$

Since this infimum cannot decrease as δ decreases, the limit

$$\mu^h(E) = \lim_{\delta \to 0^+} \mu^h_\delta(E)$$

exists, either as a non-negative number or infinity, and is called the *Hausdorff (outer) measure corresponding to h* or just the *h-Hausdorff measure* of E. Then μ^h is a Borel regular measure with all Borel sets measurable and with each set containing an F_σ set of the same measure.

The most commonly used gauge functions are the dth-power functions, $h(t) = t^d$ for $d \geq 0$, in which case *h-Hausdorff measure* is referred to as *d-dimensional Hausdorff measure*, written μ^d. The *Hausdorff dimension* of a non-empty set $E \subseteq \Omega$ is then defined to be

$$\dim E = \sup\{d \geq 0 : \mu^d(E) > 0\} = \inf\{d \geq 0 : \mu^d(E) < \infty\},$$

these two values being equal. For classical sets, 1-dimensional Hausdorff measure gives the lengths of curves and 2-dimensional measure gives the area of surfaces (to within a constant multiple). However, Hausdorff measure and dimension quantify general sets, in particular fractals; for example, if $E \subset [0, 1]$ is the middle third Cantor set (consisting of those numbers expressible in base 3 using only the digits 0 and 2), then $\dim E = \log 2/\log 3$ and $\mu^{\log 2/\log 3}(E) = 1$.

Subsets of finite measure

Typically the Hausdorff measure of a metric space (Ω, ρ) is non-σ-finite—that is, it cannot be expressed as a countable union of sets of finite measure—so that many of the fundamental

techniques of measure theory cannot be used directly. One way to address this difficulty is, given a set of infinite measure, to find a subset that has positive finite measure and thus obtain a relatively large subset on which there is no problem applying the main tools of measure theory. In particular, if $E \subseteq \Omega$ has Hausdorff dimension d then $\mu^{d_1}(E) = \infty$ if $0 < d_1 < d$, where d_1 may be taken arbitrarily close to d. If there is a subset of E with positive finite μ^{d_1}-measure, properties derived by working on such subsets can often be transferred back to the original set E. This has proved very useful for certain problems in fractal geometry, such as finding the dimension of products of sets (see Marstrand 1954; Larman 1967*b*) and the dimensions of self-affine sets (see Falconer 1988).

Thus, finding subsets of finite positive measure of sets of infinite Hausdorff measure is a key problem. It was first considered in Euclidean space by Besicovitch (1952) and in more general spaces by Larman (1967*a*, *b*). A very general treatment is presented in *Hausdorff measures* (58, 89) which includes original work of Davies and Rogers. The normal approach uses *net measures* μ_N^h; that is, measures of Hausdorff type but defined using coverings from a restricted 'net' $\mathcal{N}$ of sets in equation [1]. Roughly, a *net* $\mathcal{N}$ consists of a countable collection of sets such that each point of the space is contained in arbitrarily small sets of $\mathcal{N}$ and such that any two sets of $\mathcal{N}$ are either disjoint or one is a subset of the other. If $\mu_N^h(E) = \infty$ for some compact (or, more generally, Souslin) set E then, by using the sets of $\mathcal{N}$ in a sequential manner to cut down the set E, a set of positive finite μ_N^h-measure can be constructed. Many Hausdorff measures μ^h, including those on Euclidean spaces for natural gauge functions, have *comparable* net measures μ_N^h, for which there are constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \mu_N^h(A) \leq \mu^h(A) \leq c_2 \mu_N^h(A)$$

for all A. In this case a subset of E of positive finite μ_N^h-measure automatically has positive finite μ^h-measure.

With such positive results on reasonably 'nice' spaces, a natural, but very challenging, question was whether for every gauge function h, every compact metric space of infinite μ^h-measure contains a subset of positive finite measure. A remarkable counter-example was eventually constructed by Davies and Rogers (54), depending on an ingenious graph-theoretic argument invoking the Lyusternik–Shnirel'man–Borsuk theorem on antipodal points.

Answering a related question, Rogers (46) showed that if a metric space has non-σ-finite μ^h-measure for some gauge function h, then there is another gauge function g such that $\lim_{t \to 0^+} g(t)/h(t) = 0$ such that the space has non-σ-finite μ^g-measure.

Decomposition of set functions

Around 1959, Rogers embarked on an extensive collaboration on the decomposition of countably additive set functions with S. James Taylor, a mathematician who pioneered the use of Hausdorff measures to study sample paths of Brownian and other stochastic processes.

Let F be a finite-valued, countably additive set function defined on the Borel subsets of some cube in n-dimensional Euclidean space. By classical Lebesgue theory F has a decomposition $F = F_1 + F_2 + F_3$, where F_1 is absolutely continuous with respect to Lebesgue measure, F_2 is mutually singular to Lebesgue measure and is diffuse, and F_3 is atomic. In a series of papers Rogers and Taylor (38, 43, 48) and Rogers (42) formulated and obtained analogues of this decomposition but with respect to Hausdorff measures rather than Lebesgue measure, and also showed that the diffuse component can be broken down further using a

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range of Hausdorff measures. One of the complications is that the measures involved are non-σ-finite, unlike in the Lebesgue case.

Let F be as above and let h be a gauge function that gives rise to the Hausdorff measure μ^h. The *upper h-density* of F at x is defined as

$$\bar{D}_h F(x) = \lim_{\delta \to 0+} \left[\sup_{x \in I, 0 < d(I) < \delta} \frac{|F|(I)}{h(d(I))} \right],$$

where the supremum is over the n-dimensional intervals I of diameter less than δ that contain x. Let F_1, F_2 and F_3 be the restrictions of F to the sets of x where $\bar{D}_h F(x)$ is zero, positive and finite, and infinite, respectively, so that $F = F_1 + F_2 + F_3$. These three set functions provide a decomposition with the following properties: (i) F_1 is strongly continuous with respect to μ^h in the sense that $F_1(E) = 0$ for every set E with σ-finite μ^h-measure, (ii) F_2 is absolutely continuous with respect to μ^h, so that $F_2(E) = \int_{E \cap S} f \, d\mu^h$, where $f(x) \neq 0$ on some set S with $\mu^h(S)$ non-negative and σ-finite, and (iii) F_3 is concentrated on a set K with $\mu^h(K) = 0$ so that $F_3(E) = F_3(E \cap K)$.

Such a set function F has a further 'Hausdorff dimension decomposition', in the sense that there is a finite or countable set of numbers $0 \leq d, d_1, d_2, \ldots \leq n$ such that

$$F = F^d + F^{d_1} + F^{d_2} + \cdots$$

where, if F^d is decomposed as above with respect to d-dimensional Hausdorff measure, then $F_2^d = 0$, and for each i if $\dim E < d_i$ then $F^{d_i}(E) = 0$ but there is a set K_i with $\dim K_i = d_i$ such that $F^{d_i}(E) = F^{d_i}(E \cap K_i)$. Thus the singular diffuse part of F can be decomposed into parts corresponding to different Hausdorff dimensions. Even finer decompositions of these set functions are obtained in (48).

A related paper (43) constructs a completely additive Borel set function F on $[0, 1]$ such that for every gauge function h either there is a set K with $\mu^h(K) = 0$ such that $F(E) = F(E \cap K)$ for all E, or for each set E with $\mu^h(E) < \infty$ we have $F(E) = 0$; thus the 'F_2' part of the decomposition does not occur for any h.

Hausdorff-like measures

Hausdorff measures are not very suited to 'large' spaces; for example, the Hausdorff measure of any infinite-dimensional Banach space is infinite for every gauge function. Working with Roy Johnson, Rogers introduced 'local measures' to obtain a more useful alternative when (Ω, ρ) is a large metric space (75). Given a gauge function h and any collection $\mathcal{C}$ of open balls in X, let

$$\lambda^h(\mathcal{C}) = \sup \left\{ \sum_{A \in \mathcal{A}} h(d(A)) : \mathcal{A} \subseteq \mathcal{C} \text{ and } \bigcap_{A \in \mathcal{A}} A \neq \varnothing \right\}$$

then define

$$\lambda^h(E) = \lim_{\delta \to 0^+} \inf_{\mathcal{C}} \left\{ \lambda^h(\mathcal{C}) : \mathcal{C} \text{ covers } E \text{ and } d(C) < \delta \text{ for all } C \in \mathcal{C} \right\}.$$

Although λ^h defines an outer measure, few sets of positive outer measure are measurable, in particular $\lambda^h(E \cup F) = \max \{\lambda^h(E), \lambda^h(F)\}$ if E and F are separated sets. Paper (77) considers the same definition but with covers by open balls replaced by covers of open sets, and the resulting outer measure has similar properties. The name 'local measure' is justified by the fact that if a set E is covered by any collection of open sets of local measure 0 then E itself has local measure 0. The concept is illustrated by explicit calculation of local measures of various subsets of the Banach space of all real sequences convergent to zero with the supremum norm, which can have positive finite local measure for appropriate gauge functions.

In a different direction it is natural to consider whether Hausdorff-type measures might be defined on topological spaces rather than just on metric spaces. By using coverings taken from increasingly fine finite partitions of the space, Rogers and Sion (47) extended the definition of Hausdorff measure to obtain Borel regular measures on topological spaces. As is typical of all Rogers's work this is illustrated by insightful examples.

Dimensions of specific sets

An important application of Hausdorff measures and dimensions is to estimate the size of sets that arise naturally in other areas of mathematics. For example, bounds have been obtained for the dimension of fractal attractors of systems of differential equations (see Robinson 2011).

A nice theorem of Ewald, Larman and Rogers (55) shows that, for a convex body K in n-dimensional Euclidean space, the set of directions of line segments contained in the surface of K cannot be too big; in fact it has σ-finite $(n-2)$-dimensional Hausdorff measure (as a subset of the $(n-1)$-dimensional unit sphere of direction vectors). This is considerably harder for general n than in 3 dimensions, a case that had previously been addressed by McMinn (1960). More generally, (55) also obtains bounds for the dimension of the set of orientations of r-dimensional balls that can lie in the surface of a convex body in terms of Hausdorff measures on the Grassmann manifold of r-dimensional subspaces of $\mathbb{R}^n$.

The task of finding the Hausdorff dimension of sets of real numbers defined in terms of their continued fraction expansion goes back many years. Good (1941) showed that $\dim E_2 \approx 0.532$, where E_2 is the set of numbers in $[0, 1]$ whose continued fraction expansions contain only the digits 1 and 2, and showed, in principle, how to calculate this dimension to any desired accuracy. Rogers (49) considered a related question by putting a measure on the set E_2 in a natural way corresponding to Lebesgue measure on the base 2 numbers given directly by the digits, and seeking the infimum of the Hausdorff dimensions of the subsets of E_2 of full measure (in modern parlance this is termed the upper Hausdorff dimension of the measure). With some intricate calculations involving Euler polynomials he obtained a value of about 0.514, noting in particular that this is strictly less than $\dim E_2$.

Since 1980, problems involving dimensions of continued fraction sets or measures on such sets have been unified within a very general theory of multifractal analysis of measures on self-conformal sets (see, for example, Mauldin & Urbański 1999). Indeed, the dimension of E_2 is now known to hundreds of decimal places (see Hensley 1989).

Dimension prints

Although Hausdorff dimension provides information on the fullness of a set when viewed at fine scales, two sets of the same dimension may have very different appearances. Various quantities, such as lacunarity and porosity, have been introduced to complement dimension (see Mattila 1995). In 1988 Rogers (79) proposed *dimension prints*, based on measures of Hausdorff type, to provide more information about the local affine structure of subsets of $\mathbb{R}^n$.

Let B be a box (a rectangular parallelepiped) in $\mathbb{R}^n$ with edge lengths $l_1, \ldots, l_n$, and write $\tau^{\boldsymbol{d}}(B) = l_1^{d_1} l_2^{d_2} \cdots l_n^{d_n}$ for each non-negative vector $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$. With

$$\mu_\delta^{\boldsymbol{d}}(E) = \inf\left\{\sum_{i=1}^\infty \tau^{\boldsymbol{d}}(B_i) : E \subseteq \bigcup_{i=1}^\infty B_i \text{ where } B_i \text{ are boxes with } d(B_i) \le \delta\right\},$$

the limit

$$\mu^{\boldsymbol{d}}(E) = \lim_{\delta \to 0^+} \mu_\delta^{\boldsymbol{d}}(E)$$

defines a Hausdorff-like Borel measure on $\mathbb{R}^n$. The *dimension print E* of a set $E \subseteq \mathbb{R}^n$ consists of the set of non-negative vectors $\boldsymbol{d}$ in $\mathbb{R}^n$ for which $\mu^{\boldsymbol{d}}(E) > 0$.

Dimension prints are convex and satisfy a range of natural properties such as monotonicity (if $E \subseteq F$ then $\text{print}\,E \subseteq \text{print}\,F$) and countable stability (the print of a countable union of sets is the union of their prints). One disadvantage is that prints are unstable under smooth deformation of sets. The prints of various classical and fractal sets are calculated in (79, 83, 86, 89) and succeed in distinguishing between many sets of equal dimension but of differing characters.

Convexity, by Kenneth Falconer

The 1960s and 1970s were a golden age in the study of convex bodies. (Here, for the most part, a 'convex body' will mean a bounded, compact convex set with proper interior in finite-dimensional Euclidean space.) During that period many attractive and easily understood questions were resolved, often requiring considerable ingenuity but with elegant and sometimes simple solutions once seen. Ambrose Rogers contributed widely to the area, both individually and with collaborators including Clinton Petty, Geoffrey Shephard and especially David Larman, who graduated as Rogers's PhD student in 1965 and continued to work with him at UCL. Rogers was keen on proposing and promulgating problems on convexity, and some of these appear in published collections, including those of the 1975 Durham Symposium on Convexity (68) and the 1980 Coxeter Festschrift (74). His enthusiasm for the subject was manifested in many ways, not least by his choice of 'Probabilistic and combinatorial problems in convex and other geometry' as the title of his invited lecture at the International Congress of Mathematicians in Vancouver in 1974 (67).

Although single convex bodies have fascinating properties, there are many questions involving families of convex sets, where both their individual geometry and combinatorial aspects of their arrangement are key. Rogers was able to combine these elements in his foundational work on packing and covering by convex sets, discussed elsewhere in this memoir.

Volumes that provide a broad overview of convex geometry include collections edited by Gruber (2007), Gruber & Wills (1983, 1993) and Lay (2007).

Sections and projections of convex bodies

A recurring theme in Rogers's work is the relationship between convex bodies in $\mathbb{R}^n$ and their sections by k-dimensional planes and their projections onto k-dimensional subspaces ($1 \le k \le n - 1$). For example, he showed that if two convex bodies in $\mathbb{R}^n$ ($n \le 3$) are such that their projections onto every 2-dimensional plane are directly homothetic (that is, similar and similarly situated) then the two bodies are also homothetic (53). (The special case when one of the bodies was centrally symmetric had earlier been proved by Groemer (1962).) The paper also includes the corresponding result for homothetic plane sections through a given point of each of the bodies.

A beautiful theorem, proved together with Aitchison and Petty (59), is stated simply in the title of the *Mathematika* paper 'A convex body with a false centre is an ellipsoid'. A *false centre* of a convex body K in $\mathbb{R}^n$ ($n \le 3$) is an interior point p such that every 2-dimensional section of K through p has a centre of symmetry. Clearly, every point inside an n-dimensional ellipsoid is a false centre. The paper shows, by first reducing the problem to the case of bodies

of revolution, that no other convex bodies can have a false centre. Larman (1974) removed the necessity for *p* to be interior, and a simpler proof, along with other characterizations of ellipsoids, was later given by Montejano & Morales-Amaya (2007).

The following question was posed by Busemann & Petty (1956): if a convex body *K* in $\mathbb{R}^n$ ($n \leq 3$) that is symmetric about the origin *o* is such that every $(n-1)$-dimensional section through *o* has strictly smaller $(n-1)$-dimensional volume than that of a central section of the *n*-dimensional unit ball *B*, does *K* necessarily have *n*-dimensional volume less than that of *B*? The problem was eventually solved in 1975 by Larman and Rogers (66), who used a probabilistic method to construct counter-examples in 12 and higher dimensions by removing a collection of thin spherical caps from a ball. Counter-examples have since been obtained for smaller *n*, and it is now known that the conclusion is true for $n \leq 4$ but not for $n \geq 5$ (see Gardner 2006).

One of the most notorious problems in convexity was posed by Fujiwara (1916) and also by Blaschke *et al.* (1917). The 'equichordal point problem' asks whether a planar convex set *K* can have two *equichordal points*; that is, two distinct points through which all chords of *K* have the same length. In 1981 Rogers wrote (74):

> The problem appears to be most intractable. If you are interested in studying the problem, my first advice is 'Don't,' my second is 'If you must, do study the work of Wirsing and Butler,' and third is 'You may well have to develop sophisticated techniques for obtaining extremely accurate asymptotic expansions for the solution of a certain recurrence relation ….'

Eventually, Rychlik (1997) gave a 72-page proof that no convex body with two equichordal points exists, using a combination of complex analysis and algebraic geometry. Nevertheless, Rogers obviously thought about the problem himself and proved a different equichordal theorem: if *K* is a plane convex domain with two interior points *p* and *p'* such that every chord through *p* has the same length as the parallel chord through *p'* then *K* is centrally symmetric about the midpoint of *p* and *p'* (73). His elegant method involved 'chord-chasing' along a sequence of chords of *K* alternately passing through *p* and *p'*. Larman & Tamvakis (1981) extended this result to higher dimensions and for points that are not necessarily interior to the set. The technique was subsequently generalized to address 'tomography' questions on reconstructing convex bodies knowing the lengths of all chords passing through two points (see Gardner 2006).

Extremal problems

Extremal problems arise naturally in convexity, with the isoperimetric inequality (that the sphere is the convex body of given volume with minimum surface area) the most fundamental. One of Rogers's earliest results in convexity has this flavour, with his determination of the maximum *n*-dimensional volume of a convex polytope with *N* faces that is contained in the *n*-dimensional unit sphere (20).

In a series of papers Rogers and Shephard investigated convex bodies that may be derived naturally from a given convex body *K* in $\mathbb{R}^n$, in particular comparing various measures, such as their *n*-dimensional volume V_n, with those of *K*. Such inequalities have proved valuable in applications across many branches of mathematics and science. For the *difference body* $K - K = \{x - y : x, y \in K\}$ they showed that

$$2^n V_n(K) \leq V_n(K-K) \leq \binom{2n}{n},$$

these bounds being best possible (30). Equality on the right occurs if and only if K is a simplex. To establish this they showed that K is a simplex if and only if $K \cap (K + x)$ is a non-negative homothetic image of K whenever it is non-empty, refining an earlier result of Choquet that was the starting point for Choquet theory; see Gruber (2007) for details. This was further generalized by Gruber (1970), who showed that $K \cap (K + x)$ is an affine image of K for all x for which the intersection is proper if and only if K is a finite direct sum of simplices.

Again, the *reflexion body* about the interior point a of K is the minimal convex body $R_a K$ that contains K and is centrally symmetric about a. Rogers and Shephard obtained the best possible inequalities

$$V_n(K) \leq V_n(R_a K) \leq 2^n V_n(K),$$

with equality on the left when K is centrally symmetric about a and on the right when K is a simplex with a as a vertex (35).

Rogers and Shephard also derived inequalities involving pairs of convex bodies (37); see also Rogers (45). Let $C(A)$ denote the *convex hull* of a set A; that is, the minimal convex set containing A. The following inequality gives a lower bound for the volume of the convex hull of intersecting translates of a pair of convex bodies H and K in $\mathbb{R}^n$:

$$\max_{x: B_H \cap (B_K + x) \neq \varnothing} V_n\big(C\big(B_H \cup (B_K + x)\big)\big) \leq \max_{x: H \cap (K + x) \neq \varnothing} V_n\big(C\big(H \cup (K + x)\big)\big),$$

where B_H and B_K are spherical balls of the same volumes as H and K, and $A + x$ denotes the set A translated by the vector x.

Boundary structure of convex bodies

The boundary or surface of a convex body may be smooth and rotund or may contain points of non-differentiability of various types and/or flat components, with a wide range of local forms possible. However, there are limitations to the irregularity of the boundary of a convex body; for example, it cannot contain line segments pointing in 'too many' directions. More precisely, Ewald, Larman and Rogers (55) showed that the set of directions of line segments in the boundary of an n-dimensional convex body K, considered as a subset of the $(n - 1)$-dimensional sphere, is of σ-finite $(n - 2)$-dimensional Hausdorff measure. This had previously been established by McMinn in 3 dimensions and Pepe in 4 dimensions, but the higher-dimensional cases required new ideas. Subsequently, Larman and Rogers (61) refined their conclusion by showing that the set of line segments in the boundary of K parallel to any given $(n - 1)$-dimensional plane and not contained in the pair of parallel planes touching K must have $(n - 2)$-dimensional Hausdorff measure zero.

The *1-skeleton* of a convex body K in $\mathbb{R}^n$ ($n \geq 3$) comprises those points on the boundary of K that are not the centre of any 2-dimensional disc contained in K. Although 'most' points on the boundary of a convex body must belong to the 1-skeleton, it may be topologically very awkward; for example, it need not be closed or locally connected. Nevertheless, Larman and Rogers (56) were able to show, using the aforementioned result on the Hausdorff measure of surface line segments, that any two exposed points of K can be joined by n continuous arcs, disjoint except for their end points, and lying entirely in the 1-skeleton. (A point x is an *exposed point* of K if there is a supporting hyperplane touching K solely at x.)

Not surprisingly, Rogers's interest in descriptive set theory informed some of his work on convexity. The Borel and analytic structure of a convex set that is not closed depends on which parts of the set's topological boundary lie within the set. There is some regularity; for

example, the set of exposed points and the set of extreme points of a Borel convex set in $\mathbb{R}^n$ are both Borel, as was pointed out in (57, 69). (A point is *extreme* if it does not lie on any open line segment contained in the convex set.) Indeed, relatively recently, Holický & Keleti (2005) have analysed the Borel classes of extreme and exposed points. Rogers (57) showed that the projections of convex Borel sets in $\mathbb{R}^n$ onto lower-dimensional subspaces are always Borel, a property that fails for Borel sets in general. In the same paper he showed that the convex hull of the union of two convex Borel sets is also Borel.

Questions concerning the boundaries of convex sets are also relevant in infinite-dimensional spaces. In a general topological vector space, the *r-skeleton* (*r* a non-negative integer) of a compact convex set K consists of those points of K that are not in the relative interior of an $(r + 1)$-dimensional convex subset of K. Thus, the extreme points form the 0-skeleton. Choquet (1969) had shown that, in a Hausdorff locally convex topological vector space, the set of extreme points of a compact convex set is of second category in itself, and in the case of a metrizable space it is a G_δ-set (that is, the intersection of a countable number of open sets). Using an earlier result of theirs (64) that reduced consideration to sets in normed spaces, Larman and Rogers (65) obtained the same conclusions for *r*-skeletons for all non-negative integers *r*.

The structure of the extreme and exposed sets of non-compact convex sets in Banach spaces can be very strange, as Rogers and Jayne (69) demonstrated with several striking examples. These included closed bounded convex subsets of the sequence spaces of l_1 (summable sequences) and of c_0 (sequences convergent to zero) with the sets of extreme points and of exposed points neither Borel nor analytic. In the same spirit they constructed a relatively compact convex $G_{\delta\sigma}$ subset (that is, one formed by countable unions of countable intersections of open sets) of l_2 (square summable sequences) for which neither the extreme points nor exposed points formed Borel sets.

Miscellaneous problems in Euclidean geometry

Several other highly innovative papers deserve mention.

Peano's construction of a space-filling curve—that is, a continuous map from the unit interval [0, 1] onto the unit square—is well known. Mihalik and Wieczorek asked if there was such a curve with the image of every sub-interval of [0, 1] a convex subset of the square. Pach and Rogers (76) gave an ingenious construction of a Peano curve with the property that the images of every initial segment [0, a] and also of every final segment [b, 1] are convex. The complexity of this construction suggests that the original problem, which remains unsolved, is extremely challenging.

A question in combinatorial geometry, going back at least to Kelly (1947), asked what is the greatest number of points in a finite set S in $\mathbb{R}^n$ with just two distinct distances occurring between any pair of points in S. Larman, Rogers and Seidel (70) obtained various bounds, and in particular they gave a remarkably simple demonstration that S can have at most $\frac{1}{2}(n + 1)$ $(n + 4)$ points. With r and s as the two distances, the set of polynomials $\{F_y : y \in S\}$ given by $F_y(x) = (|x - y|^2 - r^2)(|x - y|^2 - s^2)$ are linearly independent, since $F_y(x) = 0$ for all $x \in S$ unless $x = y$. On expansion, these polynomials are easily seen to be spanned by a set of $\frac{1}{2}(n + 1)(n + 4)$ primitive polynomials, so the conclusion follows.

Hadwiger (1944) showed that if $\mathbb{R}^n$ is covered $n + 1$ by closed sets, then one of the sets realizes all distances; that is, it contains a pair of points any given distance apart. Raiskii (1970) then showed that this remains true for coverings by $n + 1$ arbitrary sets. It was long

believed that the conclusion should be true for coverings by much larger numbers of sets, and this was established in 1972 in a remarkable paper by Larman and Rogers (63). Let S be a configuration of M points in $\mathbb{R}^n$ such that every subset of S of more than D points includes a pair of points distance 1 apart. A short, elegant argument establishes that if $\mathbb{R}^n$ is covered by fewer than M/D sets then at least one of the sets realizes all distances. By investigating many possible configurations, the paper obtains improved bounds in all dimensions $n \geq 5$. For example, the configuration given by the Special Leech–Conway spindle showed spectacularly that if $\mathbb{R}^n$ is covered by 101 sets for $24 \leq n \leq 76$ then one of the sets realizes all distances. For $n \geq 77$, certain spherical configurations lead to the conclusion for all coverings by fewer than $\frac{4}{3}n$ sets, with even better results when $n \geq 139$. Asymptotically, some set from any covering of $\mathbb{R}^n$ by $\sim n^2/(\log_2 n)^3$ sets realizes all distances.

Topological descriptive set theory, by Adam Ostaszewski

From 1960 to the end of his life, Rogers was fascinated by *analytic sets* (see below). He would often give talks entitled 'Which sets do we need?', his answer being: analytic sets. He, and his co-authors, contributed much to the detailed development of the field. Perhaps even more important, and influential, was the—almost messianic—proseletizing zeal that Rogers brought to his mission to bring analytic sets to the centre of the mathematical stage, and to the attention of mathematicians.

Rogers's move in 1954 to Birmingham, and so to the Midlands mathematical community, led to several fruitful collaborations in measure theory, among them with Roy Davies, who was based in Leicester. Through him Rogers learned about the significance to measure theory of *analytic sets*, manifest already in the *Hausdorff measures* book (58, 89), which marked the end of Rogers's 'measure' phase. These sets, or more accurately their context of descriptive set theory, provide an interweaving thread connecting several themes in Rogers's work (measure theory, convexity, functional analysis)—a veritable code-name for a lifelong passage from classical to functional analysis, summarized by two more milestone books of his.

The first (published in 1980), with the brief title *Analytic sets* (72), spans aspects from classical, topological and mathematical logic angles, and includes the then-new vistas from functional analysis, centring on analyticity in the weak-topology of Banach spaces that are spanned by a weakly compact set (which include all separable Banach spaces); the latter theme was inspired by the seminal paper (Corson 1961) on the weak topology. The book was forged at an important conference (in 1978), which brought together its co-authors and captured the main trends of the time, and, while serving also to integrate his own research up to that time, marked him out as the *spiritus movens* of the modern topological concept. This was inspirational, and its influence continues to be felt strongly today.

The second, published in his last years (in 2002, with John Jayne), even more briefly entitled *Selectors* (90), is concerned with wide-ranging applications of analytic sets, turning on the construction of functions of 'very good' descriptive character (such as Baire of level 1; see below), which thus 'nicely' select representative points from a family of sets also with good descriptive character, yielding what Dellacherie calls 'théorème du bon choix' ((72), p. 221). The subject, which goes back a long way to the 'uniformization' theorems of the founders of descriptive set theory ('live' to this day in the literature of mathematical logic—again, see below), received a more general topological development in the pioneering work

in 1956–57 of E. ('Ernie') Michael (1925–2013)—see, for example, Michael (1956)—and soon afterwards in Kuratowski & Ryll-Nardzewski (1965). Modern applications visited by the book involve maximal monotone maps (including subdifferentials), nearest-point maps and convex-valued maps—marrying convexity with analyticity. The book's title harks back to one of the earliest applications of analytic sets—via a measurable selection theorem proved by von Neumann—in ring theory (in 1949), not mentioned explicitly in the book because its emphasis is on non-separable Banach spaces. (The theorem itself was later central to the 'continuum of agents' models in mathematical economics; see Hildenbrand (1974).) The book is a testament to the pioneering work of Rogers and his several collaborators in the field of functional analysis, building on all themes present in the first of the two books, except on the connections here with mathematical logic (for which see, for example, Koszmider (2005)). A central concept of the book is *fragmentability*, and particularly its generalization: σ-fragmentability, a term that Rogers (and Jayne) coined at a meeting of minds with Namioka and Phelps at the 23rd Semester (On Banach Spaces) of the Stefan Banach International Mathematical Centre in Warsaw (in the spring of 1984). The concept captures the interplay between the norm and the weak topologies of a Banach space, with attendant connections to the Radon–Nikodym Property (RNP) of its dual space (a notion originally formulated by reference to representation by Bochner-integrable functions); see (78, 80, 82, 84, 85, 87). There are connections here between set-valued maps (with values that may be closed, or compact, relative to the weak topologies in play) with Rogers's other area of interest: convexity. Whereas weakly compact subsets of a Banach space are always fragmented, on the other hand, according to Namioka & Phelps (1975), a Banach space is an Asplund space (that is, its continuous convex functions are generically Fréchet differentiable) if and only if its dual, when equipped with the weak-star topology, is fragmented. So, according to Stegall (1978) or Stegall (1981), this is equivalent to the dual having RNP; there is a further connection with convexity through the notion of 'dentability'; see (80) and the references therein for definitions and background. For the contributions to this field he will be long remembered. Indeed, this new topological notion provoked a rush of papers, broadening the context (for example to groups, as in Kenderov & Moors (2012)), and would also have benefited from a book treatment with his Midas touch; sadly, however, death cut short his intentions.

The main impetus to engage with the field came during a stay in Canada, from meeting with Maurice Sion's work on analytic sets in topological spaces (Sion 1960). Rogers became an immediate convert to the need to extend classical descriptive set theory to a broader context. In lectures, he stressed how the classical programme from the turn of the twentieth century, on the heels of Cantor's introduction of sets into analysis, was motivated by a desire to avoid the logical pitfalls of naive theory, by relying on the basic building blocks of $\mathcal{G}$ (the family of open sets) and $\mathcal{F}$ (the family of closed sets, including importantly the perfect sets), and focusing on sets manufactured only via some intuitively acceptable 'positive' operations. He was particularly allied to this view. Historically, the initial focus was on the hierarchy of Borel sets (arising from the iteration of countable unions and intersections, so yielding, for example, the $\mathcal{F}_\sigma$ sets—countable unions of sets in $\mathcal{F}$, the $\mathcal{F}_{\sigma\delta}$ sets—countable intersections of sets from $\mathcal{F}_\sigma$, and likewise the $\mathcal{G}_\delta$, $\mathcal{G}_{\delta\sigma}$ sets, and so on) and on a parallel hierarchy of Baire functions, built from the continuous functions by iteration of (pointwise) sequential limits of functions (so that limits of continuous functions are Baire of level 1); Rogers invested significant effort into taking much of the 'Borelian' theory to a broader context, starting in 1965 and again later from 1979 onwards, now working with John Jayne. But the main attraction was just one step beyond.

In seeking an 'analytical' description of the Baire functions, Lebesgue (1905) overlooked the fact that the projection of a planar Borel set might fail to be Borel. Identification of the mistake, by Souslin (1917), opened the study of the broader hierarchy of projective sets, obtained by a further iteration, this time of projections and also complementation. At the first level (corresponding to a single use of projection, and no complementation) lie the *analytic sets*, with excellent structural properties that were uncovered in quick strides (within a couple of years) by Luzin*, Sierpiński and their followers in the classical phase of the subject, summed up by their respective monographs (Lusin 1930; Sierpiński 1950) (much later, but with the broader remit of the projective sets). The higher levels had to wait much longer for the marriage of several other parallel developments—the axioms of determinacy arising from the infinite games of Banach and Mazur, the study in mathematical logic of computability, and notions of recursion started by Kleene in the 1930s (linked by Addison and Mostowski to the projective hierarchy), and the consequences for classical analysis of very-large-cardinal axioms (recognized by Woodin; see Woodin (2010)). Although these mutual interconnections tell a most astonishing story of the twentieth century, they would take us beyond this appreciation of the work of Rogers; suffice it to say that this backdrop was recognized by Rogers, who was careful to include, in the formative conference and his first book, the story then unfolding from the pens of Kechris and Martin (see also the very fine Kechris (1995)). In addition, the great importance of analyticity and capacitability (see below) to probability theory and stochastic processes was developed by Dellacherie and Meyer; see Dellacherie (1972*a*, *b*) and Dellacherie's contribution to (72). All this greatly enhanced the book's influence.

Analytic sets, under a simple reinterpretation—as continuous images of the irrationals, conventionally represented as $\mathbb{N}^{\mathbb{N}}$ via continued-fraction expansion—were readily transferred with the same structural properties into the realm of Polish spaces. Their further development seemed called for, given the already important role established not only in ring theory (mentioned above) and group theory (first in Mackey (1957); later, Effros (1965) brought to bear on this area his celebrated Open Mapping Principle) but also in measure theory, where the contribution of Davies (1952) on subsets of finite measure in analytic sets was a harbinger.

The first noteworthy advance beyond Euclidean space was due to Choquet (1951), who considered sets in Hausdorff spaces which are continuous images of a more general domain but one that is still an $\mathcal{F}_{\sigma\delta}$ subset, albeit of an arbitrary compact space, just as the irrationals are an $\mathcal{F}_{\sigma\delta}$ subset of the compact interval [0,1]. The significance of his approach soon became apparent through his celebrated theorem on capacitability (Choquet 1953). This is a general 'inner regularity' property on the approximation of an analytic set by its compact subsets, already anticipated by Roy Davies (Davies 1952).

Rogers joined in this development a decade later, engaging with Frolík's apparently broader definition (Frolík 1961), which retains the classical domain $\mathbb{N}^{\mathbb{N}}$, but its image in the range space arises by sending points $\sigma \in \mathbb{N}^{\mathbb{N}}$ not to points but to compact subsets $K(\sigma)$, whose union yields a *K-analytic space*. Furthermore, continuity is relaxed to an 'outer continuity' (traditionally called 'upper semicontinuity'; that is, omitting any lower semicontinuity considerations). Despite their greater scope, such spaces adequately resemble Polish spaces by allowing as much countability as category and measure considerations require—for background on the latter see, for example, Fremlin (2003). Ultimately, this is the consequence of an underlying completeness: manifested *extrinsically* as (roughly speaking) $\mathcal{G}_{\delta}$-embeddability in some

* Lusin according to the earlier French usage.

compactification (in fact more general than topological/Čech-completeness, and pursued by Frolík, the mathematical descendent of Čech), and *intrinsically* as offering an 'analytic Cantor theorem' (see (72), Lemma 3.1.1, more recently examined in Ostaszewski (2011)). Capturing some of these aspects, although straying further from this concept, one encounters a remarkable weakening (in 1963): Arhangel'skiĭ's topological plumed-spaces (*p*-spaces) (Arhangel'skiĭ 1963), which embrace in particular all metric spaces and all locally compact spaces (see, for example, Gruenhage (1984)), although this requires reference to a domain $\kappa^{\mathbb{N}}$, with κ an arbitrary cardinal (again, compare Ostaszewski (2011)).

It was not until 1976 that Jayne proved the equivalence of the Frolík and Choquet approaches. At this stage two other useful approaches to widening the scope of classical analyticity emerged and gained acceptance, as being close in spirit. One, *non-separable analyticity*, employing the technical apparatus of general metrization theory (involving forms of discrete decomposability), was initially undertaken by A. H. Stone (Stone 1962); after a revival a decade later, this received a systematic and far-ranging development from his pupil R. W. Hansell (from 1971 onwards, and incorporating a significant collaboration with E. Michael), and here again $\kappa^{\mathbb{N}}$ plays a part. A second, *Čech-analyticity*, derives from the extrinsic formulation (above) and is due to Fremlin (in 1980, unpublished but recorded a decade later in (84) and elsewhere, for example in Hansell (1992)); in the case of a complete metric space, the analytic subsets defined by Hansell turn out to be Čech-analytic ((84), Theorem 8.2)—that is, they are embraced by Fremlin's approach.

Here again Rogers proceeded to integrate and unify these developments, collaborating also with Hansell, with functional analysis in mind. A memorable theorem, bringing together the various themes above, is the equivalence of σ-fragmentability of a Čech-analytic space with the fragmentability of all of its compact subsets ((84), Theorem 4.1), which in particular also characterizes the RNP of a dual Banach space as its being (norm) σ-fragmentable when equipped with the weak-star topology. For ramifications to Radon measures, see (80). This prompted Namioka and Pol (1996, Theorem 5.2) to strengthen the result with an innovation, replacing Čech-analytic above with a notion generalizing *almost analyticity* ('analyticity modulo category'), namely *almost Čech-analyticity*. This for a Banach space equipped with its weak topology is equivalent to (norm) σ-fragmentability, justifying the authors in commenting on the central role of the latter in topological characterizations of renormabilities, such as those that achieve locally uniformly convex, or Kadec, norms (for background see, for example, Bessaga & Pełczyński (1975), chapter 6).

In the meantime a connection was made between *K*-analyticity and de Wilde's earlier use, in solving Grothendieck's conjecture concerning generalizations of the closed-graph theorem, of a coarser and thus more flexible notion, that of a *compact resolution*; here order preservation, in the notation above, $K(\sigma) \subseteq K(\tau)$ for $\sigma \leq \tau$ (in all components), is a central theme (for example the ability to swallow (that is, cover) an arbitrary compact set with a single $K(\sigma)$) together with adjuvant weakenings of continuity. The interplay between *K*-analyticity and various forms of resolution, in which the webbings $(\sigma_1, \ldots, \sigma_n) \to \bigcup \{K(\tau) : \tau \text{ extends } (\sigma_1, \ldots, \sigma_n)\}$ play a role, is a constant theme in the descriptive theory of function spaces. These have been studied by a succession of generations and contributors too many to list here, following the pioneering work of Corson (1961) (see above) and the trail-blazing work of Rogers with Jayne and Namioka, as well as Christensen, Orihuela, Pol, Preiss, Talagrand and Valdivia; the voluminous tome by Kąkol *et al.* (2011) is a further testimony to the long-lived effects of the founding fathers. Foremost among these, Rogers stands out as the personification of modern topological analyticity and its prolific theory-builder.

Honours

1949–50	Commonwealth Fund Fellow at the Institute for Advanced Study, Princeton
1957	Berwick Prize, London Mathematical Society
1958	Astor Professor of Mathematics at UCL
1959	Fellow of the Royal Society
1964	Plenary Speaker at British Mathematical Colloquium
1966–68	Member of Council of the Royal Society
1970–72	President of the London Mathematical Society
1977	De Morgan Medal of the London Mathematical Society
1983–84	Member of the Council of the Royal Society

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